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Specialization of linear systems from curves to graphs

Matthew Baker
Appendix by Brian Conrad

We investigate the interplay between linear systems on curves and graphs in the context of specialization of divisors on an arithmetic surface. We also provide some applications of our results to graph theory, arithmetic geometry, and tropical geometry.

1. Introduction

1A. ***Notation and terminology.*** We set the notation which will be used throughout the paper unless otherwise noted.

- G — a graph, by which we will mean a finite, unweighted, connected multigraph without loop edges. We let $V(G)$ (respectively, $E(G)$) denote the set of vertices (respectively, edges) of G.
- Γ — a metric graph (see Section 1D for the definition).
- $\Gamma_{\mathbb{Q}}$ — the set of "rational points" of Γ (see Section 1D).
- R — a complete discrete valuation ring with field of fractions K and algebraically closed residue field k.
- $\overline{K}$ — a fixed algebraic closure of K.
- X — a smooth, proper, geometrically connected curve over K.
- $\mathfrak{X}$ — a proper model for X over R. For simplicity, we assume unless otherwise stated that $\mathfrak{X}$ is regular, that the irreducible components of $\mathfrak{X}_k$ are all smooth, and that all singularities of $\mathfrak{X}_k$ are ordinary double points.

Unless otherwise specified, by a *smooth curve* we will always mean a smooth, proper, geometrically connected curve over a field, and by an *arithmetic surface* we will always mean a proper flat scheme $\mathfrak{X}$ over a discrete valuation ring such that the generic fiber of $\mathfrak{X}$ is a smooth curve. We will usually, but not always, be

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working with *regular* arithmetic surfaces. A *model* for a smooth curve X/K is an arithmetic surface $\mathfrak{X}/R$ whose generic fiber is X. An arithmetic surface $\mathfrak{X}$ is called *semistable* if its special fiber $\mathfrak{X}_k$ is reduced and has only ordinary double points as singularities. If in addition the irreducible components of $\mathfrak{X}_k$ are all smooth (so that there are no components with self-crossings), we will say that $\mathfrak{X}$ is *strongly semistable*.

1B. ***Overview.*** In this paper, we show that there is a close connection between linear systems on a curve X/K and linear systems, in the sense of [Baker and Norine 2007b], on the dual graph G of a regular semistable model $\mathfrak{X}/R$ for X. A brief outline of the paper is as follows. In Section 2, we prove a basic inequality — the "Specialization Lemma" — which says that the dimension of a linear system can only go up under specialization from curves to graphs (see Lemma 2.8 for the precise statement) . In the rest of the paper, we explore various applications of this fact, illustrating along the way a fruitful interaction between divisors on graphs and curves. The interplay works in both directions: for example, in Section 3 we use Brill–Noether theory for curves to prove and/or conjecture some new results about graphs (compare Theorem 3.12 and Conjecture 3.9) and, in the other direction, in Section 4 we use the notion of Weierstrass points on graphs to gain new insight into Weierstrass points on curves (see Corollaries 4.9 and 4.10).

Another fruitful interaction which emerges from our approach is a "machine" for transporting certain theorems about curves from classical to tropical algebraic geometry.[1] The connection goes through the theory of arithmetic surfaces, by way of the deformation-theoretic result proved in Appendix B, and uses the approximation method introduced in [Gathmann and Kerber 2008] to pass from $\mathbb{Q}$-graphs to arbitrary metric graphs, and finally to tropical curves. As an illustration of this machine, we prove an analogue for tropical curves (see Theorem 3.20 below) of the classical fact that if g, r and d are nonnegative integers for which the Brill–Noether number

$$\rho(g,r,d) := g-(r+1)(g-d+r)$$

is nonnegative, then on every smooth curve $X/\mathbb{C}$ of genus g there is a divisor D with

$$\dim|D| = r \quad \text{and} \quad \deg(D) \le d.$$

We also prove, just as in the classical case of algebraic curves, that there exist Weierstrass points on every tropical curve of genus $g \ge 2$ (see Theorem 4.13).

We conclude the paper with two appendices which can be read independently of the rest of the paper. In Appendix A, we provide a reformulation of certain parts of Raynaud's theory [1970] of "specialization of the Picard functor" in terms of linear

[1] See, for example, [Mikhalkin 2006] for an introduction to tropical geometry.

systems on graphs. We also point out some useful consequences of Raynaud's results for which we do not know any references. Although we do not actually use Raynaud's results in the body of the paper, it should be useful for future work on the interplay between curves and graphs to highlight the compatibility between Raynaud's theory and our notion of linear equivalence on graphs. Appendix B, written by Brian Conrad, discusses a result from the deformation theory of stable marked curves, which implies that every finite graph occurs as the dual graph of a regular semistable model for some smooth curve X/K with totally degenerate special fiber. This result, which seems known to the experts but for which we could not find a suitable reference, is used several times throughout the main body of the paper.

1C. ***Divisors and linear systems on graphs.*** By a *graph*, we will always mean a finite, connected multigraph without loop edges.

Let G be a graph, and let $V(G)$ (respectively, $E(G)$) denote the set of vertices (respectively, edges) of G. We let $\mathrm{Div}(G)$ denote the free abelian group on $V(G)$, and refer to elements of $\mathrm{Div}(G)$ as *divisors* on G. We can write each divisor on G as

$$D = \sum_{v \in V(G)} a_v(v)$$

with $a_v \in \mathbb{Z}$, and we will say that $D \geq 0$ if $a_v \geq 0$ for all $v \in V(G)$. We define

$$\deg(D) = \sum_{v \in V(G)} a_v$$

to be the *degree* of D. We let

$$\mathrm{Div}_+(G) = \{E \in \mathrm{Div}(G) \;:\; E \geq 0\}$$

denote the set of *effective* divisors on G, and we let $\mathrm{Div}^0(G)$ denote the set of divisors of degree zero on G. Finally, we let $\mathrm{Div}^d_+(G)$ denote the set of effective divisors of degree d on G.

Let $\mathcal{M}(G)$ be the group of $\mathbb{Z}$-valued functions on $V(G)$, and define the Laplacian operator $\Delta : \mathcal{M}(G) \to \mathrm{Div}^0(G)$ by

$$\Delta(\varphi) = \sum_{v \in V(G)} \sum_{e = vw \in E(G)} (\varphi(v) - \varphi(w))(v).$$

We let

$$\mathrm{Prin}(G) = \Delta(\mathcal{M}(G)) \subseteq \mathrm{Div}^0(G)$$

be the subgroup of $\mathrm{Div}^0(G)$ consisting of *principal divisors*.

If $D, D' \in \mathrm{Div}(G)$, we write $D \sim D'$ if $D - D' \in \mathrm{Prin}(G)$, and set

$$|D| = \{E \in \mathrm{Div}(G) \;:\; E \geq 0 \text{ and } E \sim D\}.$$

We refer to $|D|$ as the *(complete) linear system* associated to D, and call divisors D and D' with $D \sim D'$ *linearly equivalent*.

Given a divisor D on G, define $r(D) = -1$ if $|D| = \varnothing$, and otherwise set

$$r(D) = \max\{k \in \mathbb{Z} \,:\, |D - E| \neq \varnothing \text{ for all } E \in \mathrm{Div}_+^k(G)\}.$$

Note that $r(D)$ depends only on the linear equivalence class of D, and therefore can be thought of as an invariant of the complete linear system $|D|$.

When we wish to emphasize the underlying graph G, we will sometimes write $r_G(D)$ instead of $r(D)$.

We define the *canonical divisor* on G to be

$$K_G = \sum_{v \in V(G)} (\deg(v) - 2)(v).$$

We have $\deg(K_G) = 2g - 2$, where $g = |E(G)| - |V(G)| + 1$ is the *genus* (or *cyclomatic number*) of G.

Theorem 1.1 (Riemann–Roch for graphs [Baker and Norine 2007b,Theorem 1.12]). *Let D be a divisor on a graph G. Then*

$$r(D) - r(K_G - D) = \deg(D) + 1 - g.$$

For each linear ordering $<$ of the vertices of G, we define a corresponding divisor $\nu \in \mathrm{Div}(G)$ of degree $g - 1$ by the formula

$$\nu = \sum_{v \in V(G)} \big(|\{e = vw \in E(G) \,:\, w < v\}| - 1\big)(v).$$

One of the main ingredients in the proof of Theorem 1.1, which is also quite useful for computing $r(D)$ in specific examples, is this:

Theorem 1.2 [Baker and Norine 2007b, Theorem 3.3]. *For every $D \in \mathrm{Div}(G)$, exactly one of the following holds*:

(1) $r(D) \geq 0$, *or*

(2) $r(\nu - D) \geq 0$ *for some divisor ν associated to a linear ordering $<$ of $V(G)$.*

In particular, note that $r(\nu) = -1$ for any divisor ν associated to a linear ordering $<$ of $V(G)$.

1D. ***Subdivisions, metric graphs, and $\mathbb{Q}$-graphs.*** By a *weighted graph*, we will mean a graph in which each edge is assigned a positive real number called the *length* of the edge. Following the terminology of [Baker and Faber 2006], a *metric graph* (or *metrized graph*) is a compact, connected metric space Γ which arises by viewing the edges of a weighted graph G as line segments. Somewhat more formally, a metric graph should be thought of as corresponding to an *equivalence* *class* of weighted graphs, where two weighted graphs G and G' are *equivalent* if they admit a common refinement. (A *refinement* of G is any weighted graph obtained by subdividing the edges of G in a length-preserving fashion.) A weighted graph G in the equivalence class corresponding to Γ is called a *model* for Γ. Under the correspondence between equivalence classes of weighted graphs and metric graphs, after choosing an orientation, each edge e in the model G can be identified with the real interval $[0, \ell(e)] \subseteq \Gamma$.

We let $\mathrm{Div}(\Gamma)$ denote the free abelian group on the points of the metric space Γ, and refer to elements of $\mathrm{Div}(\Gamma)$ as *divisors* on Γ. We can write an element $D \in \mathrm{Div}(\Gamma)$ as

$$D = \sum_{P \in \Gamma} a_P(P)$$

with $a_P \in \mathbb{Z}$ for all P and $a_P = 0$ for all but finitely many P. We will say that $D \geq 0$ if $a_P \geq 0$ for all $P \in \Gamma$. We let

$$\deg(D) = \sum_{P \in \Gamma} a_P$$

be the *degree* of D, we let

$$\mathrm{Div}_+(\Gamma) = \{E \in \mathrm{Div}(\Gamma) \;:\; E \geq 0\}$$

denote the set of *effective* divisors on Γ, and we let $\mathrm{Div}^0(\Gamma)$ denote the subgroup of divisors of degree zero on Γ. Finally, we let $\mathrm{Div}_+^d(\Gamma)$ denote the set of effective divisors of degree d on Γ.

Following [Gathmann and Kerber 2008], a $\mathbb{Q}$-*graph* is a metric graph Γ having a model G whose edge lengths are rational numbers. We call such a model a *rational model* for Γ. An ordinary unweighted graph G can be thought of as a $\mathbb{Q}$-graph whose edge lengths are all 1. We denote by $\Gamma_{\mathbb{Q}}$ the set of points of Γ whose distance from every vertex of G is rational; we call elements of $\Gamma_{\mathbb{Q}}$ *rational points* of Γ. It is immediate that the set $\Gamma_{\mathbb{Q}}$ does not depend on the choice of a rational model G for Γ. We let $\mathrm{Div}_{\mathbb{Q}}(\Gamma)$ be the free abelian group on $\Gamma_{\mathbb{Q}}$, and refer to elements of $\mathrm{Div}_{\mathbb{Q}}(\Gamma)$ as $\mathbb{Q}$-*rational divisors* on Γ.

A *rational function* on a metric graph Γ is a continuous, piecewise affine function $f : \Gamma \to \mathbb{R}$, all of whose slopes are integers. We let $\mathcal{M}(\Gamma)$ denote the space of rational functions on Γ. The *divisor* of a rational function $f \in \mathcal{M}(\Gamma)$ is defined[2] as

$$(f) = -\sum_{P \in \Gamma} \sigma_P(f)(P),$$

[2] Here we follow the sign conventions from [Baker and Faber 2006]. In [Gathmann and Kerber 2008], the divisor of f is defined to be the negative of the one we define here.

where $\sigma_P(f)$ is the sum of the slopes of Γ in all directions emanating from P. We let

$$\mathrm{Prin}(\Gamma) = \{(f) \ : \ f \in \mathcal{M}(\Gamma)\}$$

be the subgroup of $\mathrm{Div}(\Gamma)$ consisting of *principal divisors*. It follows from [Baker and Faber 2006, Corollary 1] that (f) has degree zero for all $f \in \mathcal{M}(\Gamma)$, that is,

$$\mathrm{Prin}(\Gamma) \subseteq \mathrm{Div}^0(\Gamma).$$

If Γ is a $\mathbb{Q}$-graph, we denote by $\mathrm{Prin}_{\mathbb{Q}}(\Gamma)$ the group of principal divisors supported on $\Gamma_{\mathbb{Q}}$.

Remark 1.3. As explained in [Baker and Faber 2006], if we identify a rational function $f \in \mathcal{M}(\Gamma)$ with its restriction to the vertices of any model G for which f is affine along each edge of G, then (f) can be naturally identified with the *weighted combinatorial Laplacian* $\Delta(f)$ of f on G.

If $D, D' \in \mathrm{Div}(\Gamma)$, we write $D \sim D'$ if $D - D' \in \mathrm{Prin}(\Gamma)$, and set

$$|D|_{\mathbb{Q}} = \{E \in \mathrm{Div}_{\mathbb{Q}}(\Gamma) \ : \ E \geq 0 \text{ and } E \sim D\}$$

and

$$|D| = \{E \in \mathrm{Div}(\Gamma) \ : \ E \geq 0 \text{ and } E \sim D\}.$$

It is straightforward using Remark 1.3 to show that if G is a graph and Γ is the corresponding $\mathbb{Q}$-graph all of whose edge lengths are 1, then two divisors $D, D' \in \mathrm{Div}(G)$ are equivalent on G (in the sense of Section 1C) if and only if they are equivalent on Γ in the sense just defined.

Given a $\mathbb{Q}$-graph Γ and a $\mathbb{Q}$-rational divisor D on Γ, define $r_{\mathbb{Q}}(D) = -1$ if $|D|_{\mathbb{Q}} = \varnothing$, and otherwise set

$$r_{\mathbb{Q}}(D) = \max\{k \in \mathbb{Z} : |D - E|_{\mathbb{Q}} \neq \varnothing \text{ for all } E \in \mathrm{Div}_{\mathbb{Q}}(\Gamma) \text{ with } E \geq 0,\ \deg(E) = k\}.$$

Similarly, given an arbitrary metric graph Γ and a divisor D on Γ, we define $r_\Gamma(D) = -1$ if $|D| = \varnothing$, and otherwise set

$$r_\Gamma(D) = \max\{k \in \mathbb{Z} \ : \ |D - E| \neq \varnothing \text{ for all } E \in \mathrm{Div}_+^k(\Gamma)\}.$$

Let k be a positive integer, and let $\sigma_k(G)$ be the graph obtained from the (ordinary unweighted) graph G by subdividing each edge of G into k edges. We call $\sigma_k(G)$ the k-th *regular subdivision* of G. A divisor D on G can also be thought of as a divisor on $\sigma_k(G)$ for all $k \geq 1$ in the obvious way. The following recent combinatorial result, which had been conjectured by the author, relates the quantities $r(D)$ on G and $\sigma_k(G)$:

Theorem 1.4 [Hladky–Král–Norine 2007]. *Let G be a graph. If $D \in \mathrm{Div}(G)$, then for every integer $k \geq 1$, we have*

$$r_G(D) = r_{\sigma_k(G)}(D).$$

When working with metric graphs, if Γ is the metric graph corresponding to G (in which every edge of G has length 1), then we will usually think of each edge of $\sigma_k(G)$ as having length $1/k$; in this way, each finite graph $\sigma_k(G)$ can be viewed as a model for the same underlying metric $\mathbb{Q}$-graph Γ.

It is evident from the definitions that $r_{\mathbb{Q}}(D)$ and $r_\Gamma(D)$ do not change if the length of every edge in (some model for) Γ is multiplied by a positive integer k. Using this observation, together with [Gathmann and Kerber 2008, Proposition 2.4], one deduces from Theorem 1.4 this result:

Corollary 1.5. *If G is a graph and Γ is the corresponding metric $\mathbb{Q}$-graph in which every edge of G has length* 1, *then for every divisor D on G we have*

$$r_G(D) = r_{\mathbb{Q}}(D) = r_\Gamma(D). \tag{1.6}$$

By Corollary 1.5, we may unambiguously write $r(D)$ to refer to any of the three quantities appearing in (1.6).

Remark 1.7. Our only use of Theorem 1.4 in this paper, other than the notational convenience of not having to worry about the distinction between $r_{\mathbb{Q}}(D)$ and $r_G(D)$, will be in Remark 4.1. In practice, however, Theorem 1.4 is quite useful, since without it there is no obvious way to calculate the quantity $r_{\mathbb{Q}}(D)$ for a divisor D on a metric $\mathbb{Q}$-graph Γ.

On the other hand, we *will* make use of the equality $r_{\mathbb{Q}}(D) = r_\Gamma(D)$ given by [Gathmann and Kerber 2008, Proposition 2.4] when we develop a machine for deducing theorems about metric graphs from the corresponding results for $\mathbb{Q}$-graphs (see Section 3D).

Finally, we recall the statement of the Riemann–Roch theorem for metric graphs. Define the *canonical divisor* on Γ to be

$$K_\Gamma = \sum_{v \in V(G)} (\deg(v) - 2)(v)$$

for any model G of Γ. It is easy to see that K_Γ is independent of the choice of a model G, and that

$$\deg(K_\Gamma) = 2g - 2,$$

where $g = |E(G)| - |V(G)| + 1$ is the *genus* (or *cyclomatic number*) of Γ.

The following result is proved in [Gathmann and Kerber 2008] and [Mikhalkin and Zharkov 2007]:

Theorem 1.8 (Riemann–Roch for metric graphs). *Let D be a divisor on a metric graph Γ. Then*

$$r_\Gamma(D) - r_\Gamma(K_\Gamma - D) = \deg(D) + 1 - g. \tag{1.9}$$

By Corollary 1.5, there is a natural "compatibility" between Theorems 1.1 and 1.8.

1E. ***Tropical curves.*** Tropical geometry is a relatively recent and highly active area of research, and in dimension one it is closely connected with the theory of metric graphs as discussed in the previous section. For the sake of brevity, we adopt a rather minimalist view of tropical curves in this paper; the interested reader should see [Gathmann and Kerber 2008; Mikhalkin 2006; Mikhalkin and Zharkov 2007] for motivation and a more extensive discussion.

Following [Gathmann and Kerber 2008, §1], we define a *tropical curve* to be a "metric graph with possibly unbounded ends". More concretely, a tropical curve $\tilde{\Gamma}$ can be thought of as the geometric realization of a pair (G, ℓ), where G is a graph and

$$\ell : E(G) \to \mathbb{R}_{>0} \cup \{\infty\}$$

is a length function; each edge e of G having finite length is identified with the real interval $[0, \ell(e)]$, and each edge of length ∞ is identified with the extended real interval $[0, +\infty]$ in such a way that the ∞ endpoint of the edge has valence 1. The main difference between a tropical curve $\tilde{\Gamma}$ and a metric graph Γ is that we allow finitely many edges of $\tilde{\Gamma}$ to have infinite length.[3] In particular, every metric graph is also a tropical curve.

One can define divisors, rational functions, and linear equivalence for tropical curves exactly as we have done in Section 1D for metric graphs; see [Gathmann and Kerber 2008, §1 and Definition 3.2] for details. (The only real difference is that one must allow a rational function to take the values $\pm\infty$ at the unbounded ends of $\tilde{\Gamma}$.) Using the tropical notion of linear equivalence, one defines $r_{\tilde{\Gamma}}(D)$ for divisors on tropical curves just as we defined $r_\Gamma(D)$ in Section 1D for divisors on metric graphs. With these definitions in place, the Riemann–Roch formula (1.9) holds in the context of tropical curves, a result which can be deduced easily from Theorem 1.8 (see [Gathmann and Kerber 2008, §3] for details).

2. The specialization lemma

In this section, we investigate the behavior of the quantity $r(D)$ under specialization from curves to graphs. In order to make this precise, we first need to introduce some notation and background facts concerning divisors on arithmetic surfaces.

[3] Unlike some other definitions in the literature, the definition of a tropical curve from [Gathmann and Kerber 2008] allows vertices of valence 1 and 2, and requires that there is a "point at infinity" at the end of each unbounded edge.

2A. ***The specialization map.*** Let R be a complete discrete valuation ring with field of fractions K and algebraically closed residue field k. Let X be a smooth curve over K, and let $\mathfrak{X}/R$ be a strongly semistable regular model for X with special fiber $\mathfrak{X}_k$ (see Section 1A). We let

$$\mathscr{C} = \{C_1, \dots, C_n\}$$

be the set of irreducible components of $\mathfrak{X}_k$.

Let G be the *dual graph* of $\mathfrak{X}_k$, that is, G is the finite graph whose vertices v_i correspond to the irreducible components C_i of $\mathfrak{X}_k$, and whose edges correspond to intersections between these components (so that there is one edge between v_i and v_j for each point of intersection between C_i and C_j). The assumption that $\mathfrak{X}$ is strongly semistable implies that G is well-defined and has no loop edges.

We let $\mathrm{Div}(X)$ (respectively, $\mathrm{Div}(\mathfrak{X})$) be the group of Cartier divisors on X (respectively, on $\mathfrak{X}$); since X is smooth and $\mathfrak{X}$ is regular, Cartier divisors on X (respectively, $\mathfrak{X}$) are the same as Weil divisors. Recall that when K is perfect, $\mathrm{Div}(X)$ can be identified with the group of $\mathrm{Gal}(\overline{K}/K)$-invariant elements of the free abelian group $\mathrm{Div}(X(\overline{K}))$ on $X(\overline{K})$.

We also let $\mathrm{Prin}(X)$ (respectively, $\mathrm{Prin}(\mathfrak{X})$) denote the group of principal Cartier divisors on X (respectively, $\mathfrak{X}$).

There is a natural inclusion $\mathscr{C} \subset \mathrm{Div}(\mathfrak{X})$, and an intersection pairing

$$\mathscr{C} \times \mathrm{Div}(\mathfrak{X}) \to \mathbb{Z}, \qquad (C_i, \mathfrak{D}) \mapsto (C_i \cdot \mathfrak{D}),$$

where

$$(C_i \cdot \mathfrak{D}) = \deg(\mathcal{O}_{\mathfrak{X}}(\mathfrak{D})|_{C_i}).$$

The intersection pairing gives rise to a homomorphism $\rho : \mathrm{Div}(\mathfrak{X}) \to \mathrm{Div}(G)$ by the formula

$$\rho(\mathfrak{D}) = \sum_{v_i \in V(G)} (C_i \cdot \mathfrak{D})(v_i).$$

(If we wish to emphasize the dependence on the ground field K, we will sometimes write $\rho_K(\mathfrak{D})$ instead of $\rho(\mathfrak{D})$.) We call the homomorphism ρ the *specialization map*. By intersection theory, the group $\mathrm{Prin}(\mathfrak{X})$ is contained in the kernel of ρ.

The Zariski closure in $\mathfrak{X}$ of an effective divisor on X is a Cartier divisor. Extending by linearity, we can associate to each $D \in \mathrm{Div}(X)$ a Cartier divisor $\mathrm{cl}(D)$ on $\mathfrak{X}$, which we refer to as the *Zariski closure* of D. By abuse of terminology, we will also denote by ρ the composition of $\rho : \mathrm{Div}(\mathfrak{X}) \to \mathrm{Div}(G)$ with the map cl. By construction, if $D \in \mathrm{Div}(X)$ is *effective*, then $\rho(D)$ is an effective divisor on G. Furthermore, $\rho : \mathrm{Div}(X) \to \mathrm{Div}(G)$ is a degree-preserving homomorphism.

A divisor $\mathcal{D} \in \mathrm{Div}(\mathfrak{X})$ is called *vertical* if it is supported on $\mathfrak{X}_k$, and *horizontal* if it is the Zariski closure of a divisor on X. If $\mathcal{D}$ is a vertical divisor, then

$$\rho(\mathcal{D}) \in \mathrm{Prin}(G),$$

which follows from the fact that $\rho(C_i)$ is the negative Laplacian of the characteristic function of the vertex v_i. Since every divisor $\mathcal{D} \in \mathrm{Div}(\mathfrak{X})$ can be written uniquely as

$$\mathcal{D}_h + \mathcal{D}_v$$

with $\mathcal{D}_h$ horizontal and $\mathcal{D}_v$ vertical, it follows that $\rho(\mathcal{D})$ and $\rho(\mathcal{D}_h)$ are linearly equivalent divisors on G.

Consequently, if $D \in \mathrm{Prin}(X)$, then although the horizontal divisor $\mathcal{D} := \mathrm{cl}(D)$ may not belong to $\mathrm{Prin}(\mathfrak{X})$, it differs from a principal divisor $\mathcal{D}' \in \mathrm{Prin}(\mathfrak{X})$ by a vertical divisor $\mathcal{F} \in \mathrm{Div}(\mathfrak{X})$ for which $\rho(\mathcal{F}) \in \mathrm{Prin}(G)$. Thus we deduce a basic fact:

Lemma 2.1. *If* $D \in \mathrm{Prin}(X)$, *then* $\rho(D) \in \mathrm{Prin}(G)$.

When D corresponds to a Weil divisor

$$\sum_{P \in X(K)} n_P(P)$$

supported on $X(K)$, there is another, more concrete, description of the map ρ. Since $\mathfrak{X}$ is regular, each point $P \in X(K) = \mathfrak{X}(R)$ specializes to a nonsingular point of $\mathfrak{X}_k$, and hence to a well-defined irreducible component $c(P) \in \mathcal{C}$, which we may identify with a vertex $v(P) \in V(G)$. Then by [Liu 2002, Corollary 9.1.32], we have

$$\rho(D) = \sum_P n_P(v(P)). \tag{2.2}$$

Remark 2.3. Since the natural map from $X(K) = \mathfrak{X}(R)$ to the smooth locus of $\mathfrak{X}_k(k)$ is surjective (see, for example, [Liu 2002, Proposition 10.1.40(b)]), it follows from (2.2) that $\rho : \mathrm{Div}(X) \to \mathrm{Div}(G)$ is surjective. In fact, this implies the stronger fact that the restriction of ρ to $\mathrm{Div}(X(K))$ (the free abelian group on $X(K)$) is surjective.

2B. *Behavior of $r(D)$ under specialization.* Let $D \in \mathrm{Div}(X)$, and let $\overline{D} = \rho(D) \in \mathrm{Div}(G)$ be its specialization to G. We want to compare the dimension of the complete linear system $|D|$ on X (in the sense of classical algebraic geometry) with the quantity $r(\overline{D})$ defined in Section 1C. In order to do this, we first need some simple facts about linear systems on curves.

We temporarily suspend our convention that K denotes the field of fractions of a discrete valuation ring R, and allow K to be an arbitrary field, with X still denoting a smooth curve over K. We let $\mathrm{Div}(X(K))$ denote the free abelian group on $X(K)$, which we can view in a natural way as a subgroup of $\mathrm{Div}(X)$. For $D \in \mathrm{Div}(X)$, let

$$|D| = \{E \in \mathrm{Div}(X) \ : \ E \geq 0,\ E \sim D\}.$$

Set $r(D) = -1$ if $|D| = \varnothing$, and otherwise put

$$r_{X(K)}(D) := \max\{\, k \in \mathbb{Z} \ : \ |D - E| \neq \varnothing \text{ for all } E \in \mathrm{Div}_+^k(X(K))\,\}.$$

Lemma 2.4. *Let X be a smooth curve over a field K, and assume that $X(K)$ is infinite. Then for $D \in \mathrm{Div}(X)$, we have*

$$r_{X(K)}(D) = \dim_K L(D) - 1,$$

where $L(D) = \{f \in K(X) \ : \ (f) + D \geq 0\} \cup \{0\}$.

Proof. It is well known that $\dim L(D - P) \geq \dim L(D) - 1$ for all $P \in X(K)$. If $\dim L(D) \geq k + 1$, it follows that for any points $P_1, \ldots, P_k \in X(K)$ we have $\dim L(D - P_1 - \cdots - P_k) \geq 1$, so

$$L(D - P_1 - \cdots - P_k) \neq (0) \quad \text{and} \quad |D - P_1 - \cdots - P_k| \neq \varnothing.$$

Conversely, we prove by induction on k that if $\dim L(D) = k$, then there exist $P_1, \ldots, P_k \in X(K)$ such that $L(D - P_1 - \cdots - P_k) = (0)$, that is, $|D - P_1 - \cdots - P_k| = \varnothing$. This is clearly true for the base case $k = 0$. Suppose $\dim L(D) = k \geq 1$, and choose a nonzero rational function $f \in L(D)$. Since f has only finitely many zeros and $X(K)$ is infinite, there exists $P = P_1 \in X(K)$ for which $f(P) \neq 0$. It follows that $L(D - P) \subsetneq L(D)$, so $\dim L(D - P) = k - 1$. By induction, there exist $P_2, \ldots, P_k \in X(K)$ such that $|D - P - P_2 \cdots - P_k| = \varnothing$, which proves what we want. $\square$

Since $\dim L(D)$ remains constant under base change by an arbitrary field extension K'/K, we conclude:

Corollary 2.5. *Let X be a smooth curve over a field K, and assume that $X(K)$ is infinite. Let K' be any extension field. Then for $D \in \mathrm{Div}(X)$, we have*

$$r_{X(K)}(D) = r_{X(K')}(D).$$

In view of Lemma 2.4 and Corollary 2.5, we will simply write $r_X(D)$, or even just $r(D)$, to denote the quantity $r_{X(K)}(D) = \dim_K L(D) - 1$.

Lemma 2.6. *Let X be a smooth curve over a field K, and assume that $X(K)$ is infinite. If $D \in \mathrm{Div}(X)$, then $r(D - P) \geq r(D) - 1$ for all $P \in X(K)$, and if $r(D) \geq 0$, then $r(D - P) = r(D) - 1$ for some $P \in X(K)$.*

Proof. Let $k = r(D)$. The result is clear for $r(D) \leq 0$, so we may assume that $k \geq 1$. If $P = P_1, P_2, \ldots, P_k \in X(K)$ are arbitrary, then since $r(D) \geq k$, we have

$$|D - P - P_2 - \cdots - P_k| \neq \varnothing,$$

and therefore $r(D - P) \geq k - 1$. Also, since $r(D) = k$, it follows that there exist $P = P_1, P_2, \ldots, P_{k+1} \in X(K)$ such that

$$|D - P - P_2 - \cdots - P_{k+1}| = \varnothing,$$

and therefore $r(D - P) \leq k - 1$ for this particular choice of P. □

The same proof shows that an analogous result holds in the context of graphs:

Lemma 2.7. *Let G be a graph, and let $D \in \mathrm{Div}(G)$. Then $r(D - P) \geq r(D) - 1$ for all $P \in V(G)$, and if $r(D) \geq 0$, then $r(D - P) = r(D) - 1$ for some $P \in V(G)$.*

We now come to the main result of this section. Returning to our conventional notation, we let R be a complete discrete valuation ring with field of fractions K and algebraically closed residue field k. We let X be a smooth curve over K, and let $\mathfrak{X}/R$ be a strongly semistable regular model for X with special fiber $\mathfrak{X}_k$ and dual graph G.

Lemma 2.8 (Specialization Lemma). *For all $D \in \mathrm{Div}(X)$, we have*

$$r_G(\rho(D)) \geq r_X(D).$$

Proof. Let $\overline{D} := \rho(D)$. We prove by induction on k that if $r_X(D) \geq k$, then $r_G(\overline{D}) \geq k$ as well. The base case $k = -1$ is obvious. Now suppose $k = 0$, so $r_X(D) \geq 0$. Then there exists an effective divisor $E \in \mathrm{Div}(X)$ with $D \sim E$, so that $D - E \in \mathrm{Prin}(X)$. Since ρ is a homomorphism and takes principal (respectively, effective) divisors on X to principal (respectively, effective) divisors on G, we have

$$\overline{D} = \rho(D) \sim \rho(E) \geq 0,$$

so $r_G(\overline{D}) \geq 0$ as well.

We may therefore assume that $k \geq 1$. Let $\overline{P} \in V(G)$ be arbitrary. By Remark 2.3, there exists $P \in X(K)$ such that $\rho(P) = \overline{P}$. Then $r_X(D - P) \geq k - 1$, so by induction we have

$$r_G(\overline{D} - \overline{P}) \geq k - 1$$

as well (and in particular, $r_G(\overline{D}) \geq 0$). Since this is true for all $\overline{P} \in V(G)$, it follows from Lemma 2.7 that $r_G(\overline{D}) \geq k$ as desired. □

Remark 2.9. In the situation of Lemma 2.8, it can certainly happen that

$$r_G(\rho(D)) > r_X(D).$$

For example, on the dual graph of the modular curve $X_0(73)$ there is an effective divisor $\overline{D}$ of degree 2 with $r(\overline{D}) = 1$, but since $X_0(73)$ is not hyperelliptic, $r(D) = 0$ for every effective divisor D of degree 2 on X with $\rho(D) = \overline{D}$ (see Example 3.6).

2C. ***Compatibility with base change.*** Let K'/K be a finite extension, let R' be the valuation ring of K', and let

$$X_{K'} := X \times_K K'.$$

It is known that there is a unique *relatively minimal* regular semistable model $\mathfrak{X}'/R'$ which dominates $\mathfrak{X} \times_R R'$, and the dual graph G' of the special fiber of $\mathfrak{X}'$ is isomorphic to $\sigma_e(G)$, where e is the ramification index of K'/K. If we assign a length of $\frac{1}{e}$ to each edge of G', then the corresponding *metric graph* is the same for all finite extensions K'/K. In other words, G and G' are different models for the same metric $\mathbb{Q}$-graph Γ, which we call the *reduction graph* associated to the model $\mathfrak{X}/R$ (see [Chinburg and Rumely 1993] for further discussion).

The discussion in [Chinburg and Rumely 1993], together with (2.2), shows that there is a unique surjective map $\tau : X(\overline{K}) \to \Gamma_{\mathbb{Q}}$ for which the induced homomorphism

$$\tau_* : \mathrm{Div}(X_{\overline{K}}) \cong \mathrm{Div}(X(\overline{K})) \to \mathrm{Div}_{\mathbb{Q}}(\Gamma)$$

is compatible with ρ, in the sense that for $D \in \mathrm{Div}(X(K'))$, we have

$$\tau_*(D) = \rho_{K'}(D).$$

Concretely, if K'/K is a finite extension and $P \in X(K')$, then $\tau(P)$ is the point of $\Gamma_{\mathbb{Q}}$ corresponding to the irreducible component of the special fiber of $\mathfrak{X}'$ to which P specializes.

Remark 2.10. In general, for $D \in \mathrm{Div}(X)$ we will not always have $\rho(D) = \tau_*(D)$, but $\rho(D)$ and $\tau_*(D)$ will at least be linearly equivalent as divisors on the metric graph Γ. (This is a consequence of standard facts from arithmetic intersection theory, see, for example [Liu 2002, Propositions 9.2.15 and 9.2.23].)

From the discussion in Section 1D, we deduce from the proof of Lemma 2.8:

Corollary 2.11. *Let* $D \in \mathrm{Div}(X_{\overline{K}})$. *Then*

$$r_{\mathbb{Q}}(\tau_*(D)) \geq r_X(D).$$

Remark 2.12. For the reader familiar with Berkovich's theory of analytic spaces [1990], it may be helpful to remark that the metric $\mathbb{Q}$-graph Γ can be identified with the *skeleton* of the formal model associated to $\mathfrak{X}$, and the map $\tau : X(\overline{K}) \to \Gamma_{\mathbb{Q}}$ can be identified with the restriction to $X(\overline{K}) \subset X^{\mathrm{an}}$ of the natural deformation retraction $X^{\mathrm{an}} \twoheadrightarrow \Gamma$, where X^{an} denotes the Berkovich K-analytic space associated to X.

3. Some applications of the specialization lemma

3A. ***Specialization of g^r_d's.*** Recall that a *complete* g^r_d on X/K is defined to be a complete linear system $|D|$ with

$$D \in \mathrm{Div}(X_{\overline{K}}), \quad \deg(D) = d, \quad \text{and} \quad r(D) = r.$$

For simplicity, we will omit the word "complete" and just refer to such a linear system as a g^r_d. A g^r_d is called *K-rational* if we can choose the divisor D to lie in $\mathrm{Div}(X)$.

By analogy, we define a g^r_d on a graph G (respectively, a metric graph Γ) to be a complete linear system $|D|$ with $D \in \mathrm{Div}(G)$ (respectively, $D \in \mathrm{Div}(\Gamma)$) such that $\deg(D) = d$ and $r(D) = r$. Also, we will denote by $g^r_{\leq d}$ (respectively, $g^{\geq r}_d$) a complete linear system $|D|$ with $\deg(D) \leq d$ and $r(D) = r$ (respectively, $\deg(D) = d$ and $r(D) \geq r$).

As an immediate consequence of Lemma 2.8 and Corollary 2.11, we obtain:

Corollary 3.1. *Let X be a smooth curve over K, and let $\mathfrak{X}/R$ be a strongly semistable regular model for X with special fiber $\mathfrak{X}_k$. Let G be the dual graph of $\mathfrak{X}_k$, and let Γ be the corresponding metric graph. If there exists a K-rational g^r_d on X, then there exists a $g^{\geq r}_d$ and a $g^r_{\leq d}$ on G. Similarly, if there exists a g^r_d on X, then there exists a $g^{\geq r}_d$ and a $g^r_{\leq d}$ on Γ.*

This result places restrictions on the possible graphs which can appear as the dual graph of some regular model of a given curve X/K.

In the particular case $r = 1$, we refer to the smallest positive integer d for which there exists a g^1_d (respectively, a K-rational g^1_d) on X as the *gonality* (respectively, *K-gonality*) of X.

Similarly, we define the *gonality* of a graph G (or a metric graph Γ) to be the smallest positive integer d for which there exists a g^1_d on G (or Γ).

As a special case of Corollary 3.1, we have:

Corollary 3.2. *The gonality of G (respectively, Γ) is at most the K-gonality (respectively, gonality) of X.*

Example 3.3. Let K_n denote the complete graph on $n \geq 2$ vertices.

Claim. *The gonality of K_n is equal to $n-1$.*

Indeed, let $D = \sum a_v(v)$ be an effective divisor of degree at most $n-2$ on K_n, and label the vertices $v_1, \ldots, v_n$ of G so that $a_{v_1} \leq \cdots \leq a_{v_n}$. Then it is easy to see that $a_{v_1} = 0$ and $a_{v_i} \leq i-2$ for all $2 \leq i \leq n$. If ν is the divisor associated to the linear ordering $v_1 < \cdots < v_n$ of $V(K_n)$, it follows that $D - (v_1) \leq \nu$, so $r(D - (v_1)) = -1$ by Theorem 1.2. In particular, we have $r(D) \leq 0$, and thus the gonality of K_n is at least $n-1$. On the other hand, for any vertex $v_0 \in V(K_n)$, the divisor

$$D = \sum_{v \in V(K_n) \setminus \{v_0\}} (v)$$

has degree $n-1$ and rank at least 1, since

$$D - (v_0) \sim (n-2)(v_0).$$

It follows from Corollary 3.2 that if X/K has K-gonality at most $n-2$, then no regular model $\mathfrak{X}/R$ for X can have K_n as its dual graph. For example, K_4 cannot be the dual graph of any regular model of a hyperelliptic curve X/K.

3B. ***Hyperelliptic graphs.*** Focusing now on the special case $d=2$, we recall from [Hartshorne 1977, §IV.5] that a smooth curve X/K of genus g is called *hyperelliptic* if $g \geq 2$ and there exists a g^1_2 on X. If such a g^1_2 exists, it is automatically unique and K-rational.

Similarly, we say that a graph G (or a metric graph Γ) of genus g is *hyperelliptic* if $g \geq 2$ and there exists a g^1_2 on G (or Γ).

Remark 3.4. One can show that if such a g^1_2 exists, it is automatically unique. Also, if G is 2-edge-connected of genus at least 2, then G is hyperelliptic if and only if there is an involution h on G for which the quotient graph $G/\langle h \rangle$ is a tree. These and other matters are discussed in [Baker and Norine 2007a].

By Clifford's theorem for graphs [Baker and Norine 2007b, Corollary 3.5], if $g \geq 2$ and D is a divisor of degree 2 on G with $r(D) \geq 1$, then in fact $r(D) = 1$, and thus G is hyperelliptic. Combining this observation with Corollary 3.1, we find:

Corollary 3.5. *If X is hyperelliptic and G has genus at least* 2, *then G is hyperelliptic as well.*

The converse is false, as the following example shows.

Example 3.6. (1) Let $G = B_n$ be the "banana graph" of genus $n-1$ consisting of 2 vertices Q_1, Q_2 connected by $n \geq 3$ edges. Then the divisor $D = (Q_1) + (Q_2)$ on G has degree 2 and $r(D) = 1$, so G is hyperelliptic. On the other hand, there are certainly nonhyperelliptic curves X possessing a regular strongly semistable model with dual graph G. For example, let $p \equiv 1 \pmod{12}$ be prime, and let $K = \mathbb{Q}_p^{\text{nr}}$ be the completion of the maximal unramified extension of $\mathbb{Q}_p$. Then the modular curve $X_0(p)$ has a regular semistable model over K (the "Deligne–Rapoport model" [1973]) whose dual graph is isomorphic to B_n with $n = \frac{p-1}{12}$. However, by a result of Ogg [1974], $X_0(p)$ is never hyperelliptic when $p > 71$.

(2) More generally, let $G = B(\ell_1, \ldots, \ell_n)$ be the graph obtained by subdividing the i-th edge of B_n into ℓ_i edges for $1 \leq i \leq n$ (so that $B(1, 1, \ldots, 1) = B_n$). Then one

easily checks that $|(Q_1)+(Q_2)|$ is still a g^1_2, so G is hyperelliptic. The dual graph of $\mathfrak{X}$ is always of this type when the special fiber of $\mathfrak{X}_k$ consists of two projective lines over k intersecting transversely at n points. For example, the modular curve $X_0(p)$ with $p \geq 23$ prime has a regular model whose dual graph G is of this type. For all primes $p > 71$, G is hyperelliptic even though $X_0(p)$ is not.

Remark 3.7. Every graph of genus 2 is hyperelliptic, since by the Riemann–Roch theorem for graphs, the canonical divisor K_G has degree 2 and dimension 1. It is not hard to prove that for every integer $g \geq 3$, there are both hyperelliptic and nonhyperelliptic graphs of genus g.

3C. *Brill–Noether theory for graphs.* Classically, it is known by *Brill–Noether theory* that every smooth curve of genus g over the complex numbers has gonality at most $\lfloor \frac{1}{2}(g+3) \rfloor$, and this bound is tight: for every $g \geq 0$, the general curve of genus g has gonality exactly equal to $\lfloor \frac{1}{2}(g+3) \rfloor$. More generally:

Theorem 3.8 (Classical Brill–Noether theory). *Fix integers $g, r, d \geq 0$, and define the* Brill–Noether number

$$\rho(g,r,d) = g - (r+1)(g-d+r).$$

Then

(1) *if $\rho(g,r,d) \geq 0$, then every smooth curve $X/\mathbb{C}$ of genus g has a divisor D with $r(D) = r$ and* $\deg(D) \leq d$;

(2) *if $\rho(g,r,d) < 0$, then on a general smooth curve of genus g, there is no divisor D with $r(D) = r$ and* $\deg(D) \leq d$.

Based on extensive computer calculations by Adam Tart (an undergraduate at Georgia Tech), we conjecture that similar results hold in the purely combinatorial setting of finite graphs:

Conjecture 3.9 (Brill–Noether conjecture for graphs). *Fix integers $g, r, d \geq 0$, and set*

$$\rho(g,r,d) = g - (r+1)(g-d+r).$$

Then

(1) *if $\rho(g,r,d) \geq 0$, then every graph of genus g has a divisor D with $r(D) = r$ and* $\deg(D) \leq d$;

(2) *if $\rho(g,r,d) < 0$, there exists a graph of genus g for which there is no divisor D with $r(D) = r$ and* $\deg(D) \leq d$.

In the special case $r = 1$, Conjecture 3.9 can be reformulated as follows:

Conjecture 3.10 (Gonality conjecture for graphs). *For each integer $g \geq 0$,*

(1) *the gonality of any graph of genus g is at most $\lfloor \frac{1}{2}(g+3) \rfloor$ and*

(2) *there exists a graph of genus g with gonality exactly* $\lfloor \frac{1}{2}(g+3) \rfloor$.

Adam Tart has verified Conjecture 3.10 (2) for $g \leq 12$, and Conjecture 3.9 (2) for $2 \leq r \leq 4$ and $g \leq 10$. He has also verified that Conjecture 3.10 (1) holds for approximately 1000 randomly generated graphs of genus at most 10, and has similarly verified Conjecture 3.9 (1) for around 100 graphs in the case $2 \leq r \leq 4$.

Although we do not know how to handle the general case, it is easy to prove that Conjecture 3.10 (1) holds for small values of g:

Lemma 3.11. *Conjecture 3.10 (1) is true for $g \leq 3$.*

Proof. For $g \leq 2$, this is a straightforward consequence of Riemann–Roch for graphs. For $g = 3$, we argue as follows. The canonical divisor K_G on any genus 3 graph G has degree 4 and $r(K_G) = 2$. By Lemma 2.7, there exists a vertex $P \in V(G)$ for which the degree 3 divisor $K_G - P$ has rank 1. (In fact, it is not hard to see that $r(K_G - P) = 1$ for *every* vertex P.) Therefore G has a g^1_3, so the gonality of G is at most 3, proving the lemma. □

For *metric $\mathbb{Q}$-graphs*, we can prove the analogue of Conjecture 3.9 (1) using the Specialization Lemma (Lemma 2.8) and classical Brill–Noether theory (Theorem 3.8):

Theorem 3.12. *Fix nonnegative integers g, r, and d for which $g \geq (r+1)(g-d+r)$. Then every metric $\mathbb{Q}$-graph Γ of genus g has a divisor D with $r(D) = r$ and* $\deg(D) \leq d$.

Proof. By uniformly rescaling the edges of a suitable model for Γ so that they all have integer lengths, then adding vertices of valence 2 as necessary, we may assume that Γ is the metric graph associated to a graph G (and that every edge of G has length 1). By Theorem B.2, there exists a strongly semistable regular arithmetic surface $\mathfrak{X}/R$ whose generic fiber is smooth of genus g and whose special fiber has dual graph G. By classical Brill–Noether theory, there exists a $g^r_{d'}$ on X for some $d' \leq d$, so according to (the proof of) Corollary 3.1, there is a $\mathbb{Q}$-rational divisor D on Γ with $\deg(D) \leq d$ and $r(D) = r$. □

In Section 3D, we will generalize Theorem 3.12 to arbitrary metric graphs, and then to tropical curves, using ideas from [Gathmann and Kerber 2008].

Remark 3.13. It would be very interesting to give a direct combinatorial proof of Theorem 3.12. In any case, we view the above proof of Theorem 3.12 as an example of how one can use the Specialization Lemma, in conjunction with known theorems about algebraic curves, to prove nontrivial results about graphs (or more precisely, in this case, *metric graphs*).

For a given graph G (or metric graph Γ) and an integer $r \geq 1$, let $D(G, r)$ (or $D(\Gamma, r)$) be the minimal degree d of a g^r_d on G (or Γ).

Conjecture 3.14. *Let G be a graph, and let Γ be the associated $\mathbb{Q}$-graph. Then for every $r \geq 1$, we have*

(1) $D(G, r) = D(\sigma_k(G), r)$ *for all* $k \geq 1$ *and*

(2) $D(G, r) = D(\Gamma, r)$.

Adam Tart has verified Conjecture 3.14 (1) for 100 different graphs with $1 \leq r, k \leq 4$, and for 1000 randomly generated graphs of genus up to 10 in the special case where $r = 1$ and $k = 2$ or 3.

Note that Conjecture 3.14, in conjunction with Theorem 3.12, would imply Conjecture 3.9 (1).

Finally, we have the analogue of Conjecture 3.9 (2) for metric graphs:

Conjecture 3.15 (Brill–Noether conjecture for metric graphs). *Fix integers g, r, $d \geq 0$, and set*

$$\rho(g, r, d) = g - (r + 1)(g - d + r).$$

If $\rho(g, r, d) < 0$, then there exists a metric graph of genus g for which there is no divisor D with $r(D) = r$ and $\deg(D) \leq d$.

Note that Conjecture 3.15 would follow from Conjecture 3.14 and Conjecture 3.9 (2).

Remark 3.16. By a simple argument based on Theorem B.2 and Corollary 3.2, a direct combinatorial proof of Conjecture 3.15 in the special case $r = 1$ would yield a new proof of the classical fact that for every $g \geq 0$, there exists a smooth curve X of genus g over an algebraically closed field of characteristic zero having gonality at least $\lfloor \frac{1}{2}(g + 3) \rfloor$.

3D. ***A tropical Brill–Noether theorem.*** In this section, we show how the ideas from [Gathmann and Kerber 2008] can be used to generalize Theorem 3.12 from metric $\mathbb{Q}$-graphs to arbitrary metric graphs and tropical curves.

The key result is the following "Semicontinuity Lemma", which allows one to transfer certain results about divisors on $\mathbb{Q}$-graphs to arbitrary metric graphs. For the statement, fix a metric graph Γ and a positive real number ϵ smaller than all edge lengths in some fixed model G for Γ. We denote by $A_\epsilon(\Gamma)$ the "moduli space" of all metric graphs that are of the same combinatorial type as Γ, and whose edge lengths are within ϵ of the corresponding edge lengths in Γ. Then

$$A_\epsilon(\Gamma) \cong \prod_{e \in E(G)} [\ell(e) - \epsilon, \ell(e) + \epsilon]$$

can naturally be viewed as a product of closed intervals. In particular, there is a well-defined notion of convergence in $A_\epsilon(\Gamma)$.

Similarly, for each positive integer d, we define

$$M = M_\epsilon^d(\Gamma)$$

to be the compact polyhedral complex whose underlying point set is

$$M := \{(\Gamma', D') \,:\, \Gamma' \in A_\epsilon(\Gamma),\ D' \in \mathrm{Div}_+^d(\Gamma')\}.$$

Lemma 3.17 (Semicontinuity Lemma). *The function $r : M \to \mathbb{Z}$ given by*

$$r(\Gamma', D') = r_{\Gamma'}(D')$$

is upper semicontinuous, that is, the set $\{(\Gamma', D') \,:\, r_{\Gamma'}(D') \geq i\}$ is closed for all i.

Proof. Following the general strategy of [Gathmann and Kerber 2008, Proof of Proposition 3.1], but with some slight variations in notation, we set

$$S := \{(\Gamma', D', f, P_1, \dots, P_d) \,:\, \\ \Gamma' \in A_\epsilon(\Gamma),\ D' \in \mathrm{Div}_+^d(\Gamma'),\ f \in \mathcal{M}(\Gamma'),\ (f) + D' = P_1 + \cdots + P_d\}.$$

Also, for each $i = 0, \dots, d$, set

$$M_i := \{(\Gamma', D', P_1, \dots, P_i) \,:\, \Gamma' \in A_\epsilon(\Gamma),\ D' \in \mathrm{Div}_+^d(\Gamma'),\ P_1, \dots, P_i \in \Gamma'\}.$$

As in [Gathmann and Kerber 2008, Lemma 1.9 and Proposition 3.1], one can endow each of the spaces S and M_i $(0 \leq i \leq d)$ with the structure of a polyhedral complex.

The obvious "forgetful morphisms"

$$\pi_i : S \to M_i,\ \ (\Gamma', D', f, P_1, \dots, P_d) \mapsto (\Gamma', D', P_1, \dots, P_i)$$

and

$$p_i : M_i \to M,\ \ (\Gamma', D', P_1, \dots, P_i) \mapsto (\Gamma', D')$$

are morphisms of polyhedral complexes, and in particular they are continuous maps between topological spaces. Following [Gathmann and Kerber 2008, Proof of Proposition 3.1], we make some observations:

(1) p_i is an *open map* for all i (since it is locally just a linear projection).

(2) $M_i \backslash \pi_i(S)$ is a union of open polyhedra, and in particular, is an open subset of M_i.

(3) For $(\Gamma', D') \in M$, we have $r_{\Gamma'}(D') \geq i$ if and only if $(\Gamma', D') \notin p_i(M_i \backslash \pi_i(S))$.

From (1) and (2), it follows that $p_i(M_i \backslash \pi_i(S))$ is open in M. So by (3), we see that the subset $\{(\Gamma', D') \,:\, r_{\Gamma'}(D') \geq i\}$ is closed in M, as desired. □

The following corollary shows that the condition for a metric graph to have a $g^r_{\leq d}$ is closed:

Corollary 3.18. *Suppose Γ_n is a sequence of metric graphs in $A_\epsilon(\Gamma)$ converging to Γ. If there exists a $g^r_{\le d}$ on Γ_n for all n, then there exists a $g^r_{\le d}$ on Γ as well.*

Proof. Without loss of generality, we may assume that $r \ge 0$. Passing to a subsequence and replacing d by some $d' \le d$ if necessary, we may assume that for each n, there exists an effective divisor $D_n \in \mathrm{Div}_+(\Gamma_n)$ with $\deg(D_n) = d$ and $r(D_n) = r$. Since M is compact, $\{(\Gamma_n, D_n)\}$ has a convergent subsequence; by passing to this subsequence, we may assume that $(\Gamma_n, D_n) \to (\Gamma, D)$ for some divisor $D \in \mathrm{Div}(\Gamma)$. By Lemma 3.17, we have $r(D) \ge r$. Subtracting points from D if necessary, we find that there is an effective divisor $D' \in \mathrm{Div}(\Gamma)$ with $\deg(D') \le d$ and $r(D) = r$, as desired. □

Corollary 3.19. *Fix nonnegative integers g, r, and d. If there exists a $g^r_{\le d}$ on every $\mathbb{Q}$-graph of genus g, then there exists a $g^r_{\le d}$ on every metric graph of genus g.*

Proof. We can approximate a metric graph Γ by a sequence of $\mathbb{Q}$-graphs in $A_\epsilon(\Gamma)$ for some $\epsilon > 0$, so the result follows directly from Corollary 3.18. □

Finally, we give our promised application of the Semicontinuity Lemma to Brill–Noether theory for tropical curves:

Theorem 3.20. *Fix integers $g, r, d \ge 0$ such that*

$$\rho(g, r, d) = g - (r+1)(g - d + r) \ge 0.$$

Then every tropical curve of genus g has a divisor D with $r(D) = r$ and $\deg D \le d$.

Proof. By Theorem 3.12, there exists a $g^r_{\le d}$ on every metric $\mathbb{Q}$-graph, so it follows from Corollary 3.19 that the same is true for *all* metric graphs. By [Gathmann and Kerber 2008, Remark 3.6], if $\tilde{\Gamma}$ is a tropical curve and Γ is the metric graph obtained from $\tilde{\Gamma}$ by removing all unbounded edges, then for every $D \in \mathrm{Div}(\Gamma)$ we have $r_{\tilde{\Gamma}}(D) = r_\Gamma(D)$. Therefore the existence of a $g^r_{\le d}$ on Γ implies the existence of a $g^r_{\le d}$ on $\tilde{\Gamma}$. □

4. Weierstrass points on curves and graphs

As another illustration of the Specialization Lemma in action, in this section we will explore the relationship between Weierstrass points on curves and (a suitable notion of) Weierstrass points on graphs. As an application, we will generalize and place into a more conceptual framework a well-known result of Ogg concerning Weierstrass points on the modular curve $X_0(p)$. We will also prove the existence of Weierstrass points on tropical curves of genus $g \ge 2$.

4A. *Weierstrass points on graphs.* Let G be a graph of genus g. By analogy with the theory of algebraic curves, we say that $P \in V(G)$ is a *Weierstrass point* if $r(g(P)) \geq 1$. We define Weierstrass points on *metric graphs* and *tropical curves* in exactly the same way.

Remark 4.1. By Corollary 1.5, if Γ is the $\mathbb{Q}$-graph corresponding to G (so that every edge of G has length 1), then $P \in V(G)$ is a Weierstrass point on G if and only if P is a Weierstrass point on Γ.

Let $P \in V(G)$. An integer $k \geq 1$ is called a *Weierstrass gap* for P if

$$r(k(P)) = r((k-1)(P)).$$

The Riemann–Roch theorem for graphs, together with the usual arguments from the theory of algebraic curves, yields the following result, whose proof we leave to the reader:

Lemma 4.2. *The following are equivalent*:

(1) *P is a Weierstrass point.*

(2) *There exists a positive integer $k \leq g$ which is a Weierstrass gap for P.*

(3) *$r(K_G - g(P)) \geq 0$.*

Remark 4.3. Unlike the situation for algebraic curves, there exist graphs of genus at least 2 with no Weierstrass points. For example, consider the graph $G = B_n$ of genus $g = n-1$ introduced in Example 3.6. We claim that B_n has no Weierstrass points if $n \geq 3$. Indeed, the canonical divisor K_G is $(g-1)(Q_1) + (g-1)(Q_2)$, and by symmetry it suffices to show that

$$r\big((g-1)(Q_2) - (Q_1)\big) = -1.$$

This follows directly from Theorem 1.2, since

$$(g-1)(Q_2) - (Q_1) \leq \nu := g(Q_2) - (Q_1)$$

and ν is the divisor associated to the linear ordering $Q_1 < Q_2$ of $V(G)$.

More generally, let $G = B(\ell_1, \ldots, \ell_n)$ be the graph of genus $g = n-1$ obtained by subdividing the i-th edge of B_n into ℓ_i edges. Let R_{ij} for $1 \leq j \leq \ell_i - 1$ denote the vertices strictly between Q_1 and Q_2 lying on the i-th edge (in sequential order). Then Q_1 and Q_2 are not Weierstrass points of G. Indeed, by symmetry it again suffices to show that $r\big((g-1)(Q_2) - (Q_1)\big) = -1$, and this follows from Theorem 1.2 by considering the linear ordering

$$Q_1 < R_{11} < R_{12} < \cdots < R_{1(\ell_1-1)} < R_{21} < \cdots < R_{n(\ell_n-1)} < Q_2.$$

Other examples of families of graphs with no Weierstrass points are given in [Baker and Norine 2007a]. The graphs in these examples are all hyperelliptic.

More recently, S. Norine and P. Whalen have discovered examples of nonhyperelliptic graphs of genus 3 and 4 without Weierstrass points. It remains an interesting open problem to classify all graphs without Weierstrass points.

By the proof of Theorem 4.13 below, given any graph G of genus at least 2, there exists a positive integer k for which the regular subdivision $\sigma_k(G)$ (see Section 1D) has at least one Weierstrass point. In particular, there are always Weierstrass points on the metric graph associated to G. From the point of view of arithmetic geometry, this is related to the fact that Weierstrass points on an algebraic curve X/K of genus at least 2 always exist, but in general they are not K-rational. So just as one sometimes needs to pass to a finite extension L/K in order to see the Weierstrass points on a curve, one needs in general to pass to a regular subdivision of G in order to find Weierstrass points.

Example 4.4. On the complete graph $G = K_n$ on $n \geq 4$ vertices, every vertex is a Weierstrass point. Indeed, if $P, Q \in V(G)$ are arbitrary, then $g(P) - (Q)$ is equivalent to the effective divisor

$$(g - (n-1))(P) - (Q) + \sum_{v \in V(G)} (v),$$

and thus $r(g(P)) \geq 1$.

The following example, due to Serguei Norine, shows that there exist metric $\mathbb{Q}$-graphs with infinitely many Weierstrass points:

Example 4.5. Let Γ be the metric $\mathbb{Q}$-graph associated to the banana graph B_n for some $n \geq 4$. Then Γ has infinitely many Weierstrass points.

Indeed, label the edges of Γ as $e_1, \dots, e_n$, and identify each e_i with the segment $[0, 1]$, where Q_1 corresponds to 0, say, and Q_2 corresponds to 1. We write $x(P)$ for the element of $[0, 1]$ corresponding to the point $P \in e_i$ under this parametrization. Then for each i and each $P \in e_i$ with $x(P) \in [\frac{1}{3}, \frac{2}{3}]$, we claim that $r(3(P)) \geq 1$, and hence P is a Weierstrass point on Γ.

To see this, we will show explicitly that for every $Q \in \Gamma$ we have

$$|3(P) - (Q)| \neq \varnothing.$$

For this, it suffices to construct a function $f \in \mathcal{M}(\Gamma)$ for which

$$\Delta(f) \geq -3(P) + (Q).$$

This is easy if $P = Q$. Otherwise we have:

Case 1(a): If $Q \in e_i$ and $x(P) < x(Q)$, let $y = \frac{1}{2}\big(3x(P) - x(Q)\big)$ and take f to be constant on e_j for $j \neq i$, and on e_i to have slope -2 on $[y, x(P)]$, slope 1 on $[x(P), x(Q)]$, and slope 0 elsewhere.

Case 1(b): If $Q \in e_i$ and $x(Q) < x(P)$, we again let $y = \frac{1}{2}\big(3x(P) - x(Q)\big)$ and take f to be constant on e_j for $j \neq i$, and on e_i to have slope -1 on $[x(Q), x(P)]$, slope 2 on $[x(P), y]$, and slope 0 elsewhere.

If $Q \in e_j$ for some $j \neq i$, take f to be constant on e_k for $k \neq i, j$. On e_j, let $z = \min(x(Q), 1 - x(Q))$, and take f to have slope 1 on $[0, z]$, slope 0 on $[z, 1-z]$, and slope -1 on $[1-z, 1]$. Finally, along e_i, we have two cases:

Case 2(a): If $x(P) \in [\frac{1}{3}, \frac{1}{2}]$, let $y = 3x(P) - 1$, and define f on e_i to have slope -1 on $[0, y]$, slope -2 on $[y, x(P)]$, and slope 1 on $[x(P), 1]$.

Case 2(b): If $x(P) \in [\frac{1}{2}, \frac{2}{3}]$, we again let $y = 3x(P) - 1$, and define f on e_i to have slope -1 on $[0, x(P)]$, slope 2 on $[x(P), y]$, and slope 1 on $[y, 1]$.

Remark 4.6. Similarly, one can show that for each integer $m \geq 2$, if $x(P) \in [\frac{1}{m}, \frac{m-1}{m}]$ then $r(m(P)) \geq 1$, and thus P is a Weierstrass point on Γ as long as the genus of Γ is at least m.

We close this section with a result which generalizes Remark 4.3, and which can be used in practice to identify non-Weierstrass points on certain graphs.

Lemma 4.7. *Let v be a vertex of a graph G of genus $g \geq 2$, and let G' be the graph obtained by deleting the vertex v and all edges incident to v. If G' is a tree, then v is not a Weierstrass point.*

Proof. Since G' is a tree, there is a linear ordering

$$v_1 < \cdots < v_{n-1}$$

of the vertices of G' such that each vertex other than the first one has exactly one neighbor preceding it in the order. Extend $<$ to a linear ordering of $V(G)$ by letting v be the last element in the order. Since the corresponding divisor ν is equal to $g(v) - (v_1)$, it follows from Theorem 1.2 that $|g(v) - (v_1)| = \varnothing$, and therefore $r(g(v)) = 0$. Thus v is not a Weierstrass point. $\square$

Remark 4.8. It is easy to see that if (G, v) satisfies the hypothesis of Lemma 4.7, then so does $(\tilde{G}, v)$, where $\tilde{G}$ is obtained by subdividing each edge e_i of G into m_i edges for some positive integer m_i.

4B. ***Specialization of Weierstrass points on totally degenerate curves.*** We say that an arithmetic surface $\mathfrak{X}/R$ is *totally degenerate* if the genus of its dual graph G is the same as the genus of X. Under our hypotheses on $\mathfrak{X}$, the genus of X is the sum of the genus of G and the genera of all irreducible components of $\mathfrak{X}_k$, so $\mathfrak{X}$ is totally degenerate if and only if all irreducible components of $\mathfrak{X}_k$ have genus 0.

Applying the Specialization Lemma and the definition of a Weierstrass point, we immediately obtain:

Corollary 4.9. *If $\mathfrak{X}$ is a strongly semistable, regular, and totally degenerate arithmetic surface, then for every K-rational Weierstrass point $P \in X(K)$, $\rho(P)$ is a Weierstrass point of the dual graph G of $\mathfrak{X}$. More generally, for every Weierstrass point $P \in X(\overline{K})$, $\tau_*(P)$ is a Weierstrass point of the reduction graph Γ of $\mathfrak{X}$.*

As a sample consequence, we have this concrete result:

Corollary 4.10. (1) *Let $\mathfrak{X}/R$ be a strongly semistable, regular, totally degenerate arithmetic surface whose special fiber has a dual graph with no Weierstrass points (for example, the graph B_n for some $n \geq 3$). Then X does not possess any K-rational Weierstrass points.*

(2) *Let $\mathfrak{X}/R$ be a (not necessarily regular) arithmetic surface whose special fiber consists of two genus 0 curves intersecting transversely at 3 or more points. Then every Weierstrass point of $X(K)$ specializes to a singular point of $\mathfrak{X}_k$.*

(3) *More generally, let $\mathfrak{X}/R$ be a (not necessarily regular) strongly semistable and totally degenerate arithmetic surface whose dual graph G contains a vertex v for which $G' := G \backslash \{v\}$ is a tree. Then there are no K-rational Weierstrass points on X specializing to the component C of $\mathfrak{X}_k$ corresponding to v.*

Proof. Part (1) follows from what we have already said. For (2), it suffices to note that X has a strongly semistable regular model $\mathfrak{X}'$ whose dual graph G' is isomorphic to $B(\ell_1, \dots, \ell_g)$ for some positive integers ℓ_i, and a point of $X(K)$ which specializes to a nonsingular point of $\mathfrak{X}_k$ will specialize to either Q_1 or Q_2 in G', neither of which is a Weierstrass point by Remark 4.3. Finally, (3) follows easily from Lemma 4.7 and Remark 4.8. □

We view Corollary 4.10 as a generalization of Ogg's argument [1978] showing that the cusp ∞ is never a Weierstrass point on $X_0(p)$ for $p \geq 23$ prime, since $X_0(p)/\mathbb{Q}_p^{\mathrm{nr}}$ has a model $\mathfrak{X}_0(p)$ of the type described in Corollary 4.10 (2) (the Deligne–Rapoport model), and ∞ specializes to a nonsingular point on the special fiber of $\mathfrak{X}_0(p)$. More generally, Corollary 4.10 shows (as does Ogg's original argument) that all Weierstrass points of $X_0(p)$ specialize to supersingular points in the Deligne–Rapoport model. Corollaries 4.9 and 4.10 give a recipe for extending Ogg's result to a much broader class of curves with totally degenerate reduction.

Remark 4.11. Corollary 4.10 has strong implications concerning the arithmetic of Weierstrass points on curves. For example, in the special case where $K = \mathbb{Q}_p^{\mathrm{nr}}$, the conclusion of Corollary 4.10 (1) implies that every Weierstrass point on X is ramified at p.

Example 4.12. The hypothesis that $\mathfrak{X}$ is totally degenerate is necessary in the statement of Corollary 4.9. For example, it follows from [Atkin 1967, Theorem 1] that the cusp ∞ is a $\mathbb{Q}$-rational Weierstrass point on $X_0(180)$. The mod 5 reduction of the Deligne–Rapoport model of $X_0(180)$ consists of two copies of $X_0(36)_{\mathbb{F}_5}$

intersecting transversely at the supersingular points, and the cusp ∞ specializes to a nonsingular point on one of these components. (This does not contradict Corollary 4.10 because $X_0(36)$ does not have genus 0.)

We conclude this section with another application of algebraic geometry to tropical geometry: we use the classical fact that Weierstrass points exist on every smooth curve of genus at least 2 to show that there exist Weierstrass points on every tropical curve of genus at least 2.

Theorem 4.13. *Let $\tilde{\Gamma}$ be a tropical curve of genus $g \geq 2$. Then there exists at least one Weierstrass point on $\tilde{\Gamma}$.*

Proof. We first consider the case of a $\mathbb{Q}$-graph Γ. By rescaling if necessary, we may assume that Γ is the metric graph associated to a finite graph G, with every edge of G having length one. By Theorem B.2, there exists a strongly semistable, regular, totally degenerate arithmetic surface $\mathfrak{X}/R$ whose generic fiber is smooth of genus g and whose special fiber has reduction graph Γ. Let $P \in X(\overline{K})$ be a Weierstrass point, which exists by classical algebraic geometry since $g \geq 2$. By Corollary 4.9, $\tau_*(P)$ is a Weierstrass point of Γ. We have thus shown that every metric $\mathbb{Q}$-graph of genus at least 2 has a Weierstrass point.

Now let Γ be an arbitrary metric graph. As in Section 3D, for $\epsilon > 0$ sufficiently small, we can approximate Γ by a sequence Γ_n of $\mathbb{Q}$-graphs within the space $A_\epsilon(\Gamma)$. Let $P_n \in \Gamma_n$ be a Weierstrass point. Passing to a subsequence if necessary, we may assume without loss of generality that

$$(\Gamma_n, P_n) \to (\Gamma, P)$$

in $M_\epsilon^1(\Gamma)$ for some point $P \in \Gamma$. Since $r_{\Gamma_n}(gP_n) \geq 1$ for all n and

$$(\Gamma_n, gP_n) \to (\Gamma, gP)$$

in $M_\epsilon^g(\Gamma)$, we conclude from the Semicontinuity Lemma that $r_\Gamma(gP) \geq 1$, that is, P is a Weierstrass point on Γ.

Finally, suppose that $\tilde{\Gamma}$ is a tropical curve, and let Γ be the metric graph obtained from $\tilde{\Gamma}$ by removing all unbounded edges. It follows from [Gathmann and Kerber 2008, Remark 3.6] that every Weierstrass point on Γ is also a Weierstrass point on $\tilde{\Gamma}$. Therefore the existence of Weierstrass points on Γ implies the existence of Weierstrass points on $\tilde{\Gamma}$. $\square$

Remark 4.14. We do not know a direct combinatorial proof of Theorem 4.13, but it would certainly be interesting to give such a proof. Also, in light of Example 4.5, it is not clear if there is an analogue for metric graphs of the classical fact that the total weight of all Weierstrass points on a smooth curve of genus $g \geq 2$ is $g^3 - g$.

4C. ***Specialization of a canonical divisor.*** Since $P \in X(\overline{K})$ is a Weierstrass point of X if and only if $r(K_X - g(P)) \geq 0$, where K_X denotes a canonical divisor on X, and since $P \in V(G)$ is a Weierstrass point of G if and only if $r(K_G - g(P)) \geq 0$, Corollary 4.9 suggests a relationship between the canonical divisor of G and the specialization of K_X when $\mathfrak{X}$ is totally degenerate. We investigate this relationship in this section.

Let $\mathfrak{X}$ be a strongly semistable, regular, totally degenerate arithmetic surface. Let $\omega_{\mathfrak{X}/R}$ be the *canonical sheaf* for $\mathfrak{X}/R$, and let $K_{\mathfrak{X}}$ be a Cartier divisor such that

$$\mathcal{O}_{\mathfrak{X}}(K_{\mathfrak{X}}) \cong \omega_{\mathfrak{X}/R}.$$

We call any such $K_{\mathfrak{X}}$ a *canonical divisor*.

Lemma 4.15. $\rho(K_{\mathfrak{X}}) = K_G$.

Proof. This is a consequence of the *adjunction formula* for arithmetic surfaces (see [Liu 2002, Theorem 9.1.37]), which tells us that

$$(K_{\mathfrak{X}} \cdot C_i) = 2g(C_i) - 2 - (C_i \cdot C_i) = -2 - (C_i \cdot C_i) \tag{4.16}$$

for all i. Since $(C_i \cdot \sum_j C_j) = 0$ for all i, we have

$$(C_i \cdot C_i) = -\sum_{j \neq i} (C_i \cdot C_j) = -\deg(v_i). \tag{4.17}$$

Combining (4.16) and (4.17) gives

$$(K_{\mathfrak{X}} \cdot C_i) = \deg(v_i) - 2$$

for all i, as desired. □

Remark 4.18. More generally, if we do not assume that X is totally degenerate, then the above proof shows that

$$\rho(K_{\mathfrak{X}}) = K_G + 2\sum_{i=1}^{n} g(C_i)(v_i),$$

where C_i is the irreducible component of $\mathfrak{X}_k$ corresponding to the vertex v_i of G.

Remark 4.19. Lemma 4.15 helps explain why there is in fact a canonical *divisor* on a graph G, rather than just a canonical *divisor class*, and also explains the connection between the canonical divisor on a graph and the canonical divisor class in algebraic geometry. This connection is implicit in the earlier work of S. Zhang [1993].

Lemma 4.20. *Let* $K_X \in \mathrm{Div}(X)$ *be a canonical divisor. Then* $\rho(K_X)$ *is linearly equivalent to* K_G.

Proof. Since the Zariski closure of K_X differs from a canonical divisor $K_{\mathfrak{X}}$ on $\mathfrak{X}$ by a vertical divisor, this follows from Lemma 4.15 and the remarks preceding Lemma 2.1. □

Remark 4.21. By a general *moving lemma* (see, for example, [Liu 2002, Corollary 9.1.10 or Proposition 9.1.11]), there exists a horizontal canonical divisor $K_{\mathfrak{X}}$ on $\mathfrak{X}$. Since $K_{\mathfrak{X}}$ is the Zariski closure of K_X in this case, it follows that there exists a canonical divisor $K_X \in \mathrm{Div}(X)$ for which $\rho(K_X)$ is *equal* to K_G (and not just linearly equivalent to it).

4D. ***An example.*** We conclude with an explicit example which illustrates many of the concepts that have been discussed in this paper.

Let p be an odd prime, and let X be the smooth plane quartic curve over $\mathbb{Q}_p$ given by $F(x, y, z) = 0$, where

$$F(x, y, z) = (x^2 - 2y^2 + z^2)(x^2 - z^2) + py^3 z. \tag{4.22}$$

By classical algebraic geometry, we have $g(X) = 3$, and the $\overline{\mathbb{Q}}_p$-gonality of X is also 3. We let $K = \mathbb{Q}_p^{\mathrm{nr}}$, and consider X as an algebraic curve over K.

Let $\mathfrak{X}'$ be the model for X over the valuation ring R of K given by (4.22). According to [Liu 2002, Exercise 10.3.10], the special fiber of $\mathfrak{X}'$ is semistable and consists of two projective lines ℓ_1 and ℓ_2 with equations $x = z$ and $x = -z$, respectively, which intersect transversely at the point $(0 : 1 : 0)$, together with the conic C defined by $x^2 - 2y^2 + z^2 = 0$, which intersects each of ℓ_1 and ℓ_2 transversely at 2 points. The model $\mathfrak{X}'$ is not regular, but a regular model $\mathfrak{X}$ can be obtained from $\mathfrak{X}'$ by blowing up the point $(0 : 1 : 0)$ of the special fiber of $\mathfrak{X}'$, which produces an exceptional divisor E in $\mathfrak{X}_k$ isomorphic to $\mathbb{P}^1_k$, and which intersects each of ℓ_1 and ℓ_2 transversely at a single point (see [Liu 2002, Corollary 10.3.25]). The special fiber $\mathfrak{X}_k$ of $\mathfrak{X}$ and the dual graph G of $\mathfrak{X}_k$ are depicted in Figure 1.

In the diagram on the right, the vertex P corresponds to the conic C, Q_i corresponds to the line ℓ_i ($i = 1, 2$), and P' corresponds to the exceptional divisor E of

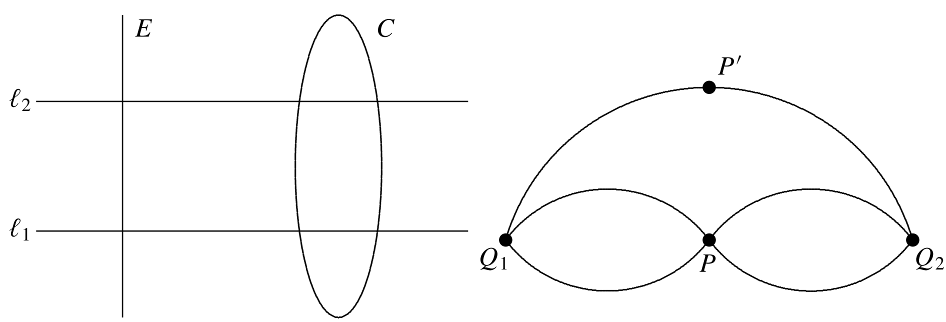

Figure 1. Left: The special fiber $\mathfrak{X}_k$. Right: The dual graph G of $\mathfrak{X}_k$.

the blowup. Note that G is a graph of genus 3, and that $\mathfrak{X}$ has totally degenerate strongly semistable reduction.

Claim. *Q_1 and Q_2 are Weierstrass points of G, while P and P' are not.*

Indeed, since $3(Q_1) \sim 3(Q_2) \sim 2(P) + (P')$, it follows easily that $r(3(Q_1)) = r(3(Q_2)) \geq 1$, and therefore Q_1 and Q_2 are Weierstrass points. On the other hand, P is not a Weierstrass point of G by Lemma 4.7, and P' is not a Weierstrass point either, since $3(P') - (P)$ is equivalent to $(Q_1) + (Q_2) + (P') - (P)$, which is the divisor associated to the linear ordering $P < Q_1 < Q_2 < P'$ of $V(G)$.

Claim. *The gonality of G is* 3.

We have already seen that $r(3(Q_1)) \geq 1$, so the gonality of G is at most 3. It remains to show that G is not hyperelliptic, that is, there is no g^1_2 on G. By symmetry, and using the fact that $(Q_1) + (Q_2) \sim 2(P')$, it suffices to show that $r(D) = 0$ for $D = (P) + (P')$ and for each of the divisors $(Q_1) + (X)$ with $X \in V(G)$. For this, it suffices to show that $|D| = \varnothing$ for each of the divisors $(Q_1) + 2(Q_2) - (P)$, $(P) + 2(Q_1) - (P')$, and $(P) + (P') + (Q_1) - (Q_2)$. But these are the ν-divisors associated to the linear orderings $P < Q_1 < P' < Q_2$, $P' < Q_2 < P < Q_1$, and $Q_1 < P < Q_2 < P'$ of $V(G)$, respectively. The claim therefore follows from Theorem 1.2.

The canonical divisor K_G on G is $2(P) + (Q_1) + (Q_2)$. We now compute the specializations of various canonical divisors in $\mathrm{Div}(X(K))$. Since a canonical divisor on X is just a hyperplane section, the following divisors are all canonical:

$$
\begin{aligned}
&K_1 : (x = z) \cap X = (0:1:0) + 3(1:0:1),\\
&K_2 : (x = -z) \cap X = (0:1:0) + 3(1:0:-1),\\
&K_3 : (z = 0) \cap X = 2(0:1:0) + (\sqrt{2}:1:0) + (-\sqrt{2}:1:0),\\
&K_4 : (y = 0) \cap X = (1:0:1) + (1:0:-1) + (1:0:\sqrt{-1}) + (1:0:-\sqrt{-1}).
\end{aligned}
$$

The specializations of these divisors under ρ (or equivalently, under τ_*) are

$$
\begin{aligned}
&\rho(K_1) : (P') + 3(Q_1),\\
&\rho(K_2) : (P') + 3(Q_2),\\
&\rho(K_3) : 2(P') + 2(P),\\
&\rho(K_4) : (Q_1) + (Q_2) + 2(P).
\end{aligned}
$$

It is straightforward to check that each of these divisors is linearly equivalent to $K_G = (Q_1) + (Q_2) + 2(P)$, in agreement with Lemma 4.20.

Finally, note that (as follows from the above calculations) $(1:0:1)$ and $(1:0:-1)$ are Weierstrass points on X, and they specialize to Q_1 and Q_2, respectively. As we have seen, these are both Weierstrass points of G, as predicted by Corollary 4.9.

On the metric graph Γ associated to G, there are additional Weierstrass points. A somewhat lengthy case-by-case analysis shows that the Weierstrass points of Γ are the four points at distance $\frac{1}{3}$ from P, together with the two intervals $[Q_i, R_i]$ of length $\frac{1}{3}$, where R_1, R_2 are the points at distance $\frac{2}{3}$ from P'. It would be interesting to compute the specializations to Γ of the remaining Weierstrass points in $X(\overline{K})$.

Appendix A. A reformulation of Raynaud's description of the Néron model of a Jacobian

In this appendix, we reinterpret in the language of divisors on graphs some results of Raynaud concerning the relation between a proper regular model for a curve and the Néron model of its Jacobian. The main result here is that the diagrams (A.5) and (A.6) below are exact and commutative. This may not be a new observation, but since we could not find a reference, we will attempt to explain how it follows in a straightforward way from Raynaud's work.

In order to keep this appendix self-contained, we have repeated certain definitions which appear in the main body of the text. Some references for the results described here are [Bertolini and Darmon 1997, Appendix; Bosch et al. 1990; Edixhoven 1998; Raynaud 1970].

Raynaud's description. Let R be a complete discrete valuation ring with field of fractions K and algebraically closed residue field k. Let X be a smooth, proper, geometrically connected curve over K, and let $\mathfrak{X}/R$ be a proper model for X with reduced special fiber $\mathfrak{X}_k$. For simplicity, we assume throughout that $\mathfrak{X}$ is regular, that the irreducible components of $\mathfrak{X}_k$ are all smooth, and that all singularities of $\mathfrak{X}_k$ are ordinary double points. We let $\mathscr{C} = \{C_1, \dots, C_n\}$ be the set of irreducible components of $\mathfrak{X}_k$.

Let J be the Jacobian of X over K, let $\mathscr{J}$ be the Néron model of J/R, and let $\mathscr{J}^0$ be the connected component of the identity in $\mathscr{J}$. We denote by

$$\Phi = \mathscr{J}_k / \mathscr{J}_k^0$$

the group of connected components of the special fiber $\mathscr{J}_k$ of $\mathscr{J}$.

Let $\mathrm{Div}(X)$ (respectively, $\mathrm{Div}(\mathfrak{X})$) be the group of Cartier divisors on X (respectively, on $\mathfrak{X}$); since X is smooth and $\mathfrak{X}$ is regular, Cartier divisors on X (respectively, $\mathfrak{X}$) are the same as Weil divisors.

The Zariski closure in $\mathfrak{X}$ of an effective divisor on X is a Cartier divisor. Extending by linearity, we can associate to each $D \in \mathrm{Div}(X)$ a Cartier divisor $\mathscr{D}$ on $\mathfrak{X}$, which we refer to as the *Zariski closure* of D.

Let $\mathrm{Div}^0(X)$ denote the subgroup of Cartier divisors of degree zero on X. In addition, let $\mathrm{Div}^{(0)}(\mathfrak{X})$ denote the subgroup of $\mathrm{Div}(\mathfrak{X})$ consisting of those Cartier divisors $\mathscr{D}$ for which the restriction of the associated line bundle $\mathscr{O}_{\mathfrak{X}}(\mathscr{D})$ to each irreducible component of $\mathfrak{X}_k$ has degree zero, that is, for which

$$\deg(\mathcal{O}_{\mathfrak{X}}(\mathcal{D})|_{C_i}) = 0 \text{ for all } C_i \in \mathcal{C}.$$

Finally, let

$$\mathrm{Div}^{(0)}(X) = \{D \in \mathrm{Div}^0(X) \ : \ \mathcal{D} \in \mathrm{Div}^{(0)}(\mathfrak{X})\},$$

where $\mathcal{D}$ is the Zariski closure of D.

Let $\mathrm{Prin}(X)$ (respectively, $\mathrm{Prin}(\mathfrak{X})$) denote the group of principal Cartier divisors on X (respectively, $\mathfrak{X}$). There is a well-known isomorphism

$$J(K) = \mathcal{J}(R) \cong \mathrm{Div}^0(X)/\,\mathrm{Prin}(X),$$

and according to Raynaud, there is an isomorphism

$$J^0(K) := \mathcal{J}^0(R) \cong \mathrm{Div}^{(0)}(X)/\,\mathrm{Prin}^{(0)}(X), \tag{A.1}$$

where

$$\mathrm{Prin}^{(0)}(X) := \mathrm{Div}^{(0)}(X) \cap \mathrm{Prin}(X).$$

The isomorphism in (A.1) comes from the fact that $\mathcal{J}^0 = \mathrm{Pic}^0_{\mathfrak{X}/R}$ represents the functor "isomorphism classes of line bundles whose restriction to each element of $\mathcal{C}$ has degree zero". (Recall that there is a canonical isomorphism between isomorphism classes of line bundles on $\mathfrak{X}$ and the Cartier class group of $\mathfrak{X}$.)

Remark A.2. In particular, it follows from the above discussion that every element $P \in J^0(K)$ can be represented as the class of D for some $D \in \mathrm{Div}^{(0)}(X)$.

There is a natural inclusion $\mathcal{C} \subset \mathrm{Div}(\mathfrak{X})$, and an intersection pairing

$$\mathcal{C} \times \mathrm{Div}(\mathfrak{X}) \to \mathbb{Z}, \quad (C_i, \mathcal{D}) \mapsto (C_i \cdot \mathcal{D}),$$

where $(C_i \cdot \mathcal{D}) = \deg(\mathcal{O}_{\mathfrak{X}}(D)|_{C_i})$.

The intersection pairing gives rise to a map

$$\begin{aligned} \alpha : &\mathbb{Z}^{\mathcal{C}} \to \mathbb{Z}^{\mathcal{C}}, \\ &f \mapsto \Big(C_i \mapsto \sum_{C_j \in \mathcal{C}} (C_i \cdot C_j) f(C_j)\Big). \end{aligned}$$

Since k is algebraically closed and the canonical map $J(K) \to \mathcal{J}_k(k)$ is surjective by [Liu 2002, Proposition 10.1.40(b)], there is a canonical isomorphism $J(K)/J^0(K) \cong \Phi$. According to Raynaud, the component group Φ is canonically isomorphic to the homology of the complex

$$\mathbb{Z}^{\mathcal{C}} \xrightarrow{\ \alpha\ } \mathbb{Z}^{\mathcal{C}} \xrightarrow{\ \deg\ } \mathbb{Z}, \tag{A.3}$$

where

$$\deg : f \mapsto \sum_{C_i} f(C_i).$$

The isomorphism

$$\phi : J(K)/J^0(K) \cong \mathrm{Ker}(\deg)/\,\mathrm{Im}(\alpha)$$

can be described in the following way. Let $P \in J(K)$, and choose $D \in \mathrm{Div}^0(X)$ such that $P = [D]$. Let $\mathfrak{D} \in \mathrm{Div}(\mathfrak{X})$ be the Zariski closure of D. Then

$$\phi(P) = [C_i \mapsto (C_i \cdot \mathfrak{D})].$$

When D corresponds to a Weil divisor supported on $X(K)$, we have another description of the map ϕ. Write

$$D = \sum_{P \in X(K)} n_P(P)$$

with $\sum n_P = 0$. Since $\mathfrak{X}$ is regular, each point $P \in X(K) = \mathfrak{X}(R)$ specializes to a well-defined element $c(P)$ of $\mathscr{C}$. Identifying a formal sum $\sum_{C_i \in \mathscr{C}} a_i C_i$ with the function $C_i \mapsto a_i \in \mathbb{Z}^{\mathscr{C}}$, we have

$$\phi([D]) = \left[\sum_P n_P c(P)\right]. \tag{A.4}$$

The quantities appearing in (A.3) can be interpreted in a more suggestive fashion using the language of graphs. Let G be the *dual graph* of $\mathfrak{X}_k$, that is, G is the finite graph whose vertices v_i correspond to the irreducible components C_i of $\mathfrak{X}_k$, and whose edges correspond to intersections between these components (so that there is one edge between v_i and v_j for each point of intersection between C_i and C_j). We let $\mathrm{Div}(G)$ denote the free abelian group on the set of vertices of G, and define $\mathrm{Div}^0(G)$ to be the kernel of the natural map $\deg : \mathrm{Div}(G) \to \mathbb{Z}$ given by

$$\deg\left(\sum a_i(v_i)\right) = \sum a_i.$$

In particular, the set $V(G)$ of vertices of G is in bijection with $\mathscr{C}$, and the group $\mathrm{Div}(G)$ is isomorphic to $\mathbb{Z}^{\mathscr{C}}$, with $\mathrm{Div}^0(G)$ corresponding to $\mathrm{Ker}(\deg)$.

Let $\mathcal{M}(G) = \mathbb{Z}^{V(G)}$ be the set of $\mathbb{Z}$-linear functions on $V(G)$, and define the Laplacian operator $\Delta : \mathcal{M}(G) \to \mathrm{Div}^0(G)$ by

$$\Delta(\varphi) = \sum_{v \in V(G)} \sum_{e = vw} (\varphi(v) - \varphi(w))\,(v),$$

where the inner sum is over all edges e of G having v as an endpoint. Finally, define

$$\mathrm{Prin}(G) = \Delta(\mathcal{M}(G)) \subseteq \mathrm{Div}^0(G),$$

and let

$$\mathrm{Jac}(G) = \mathrm{Div}^0(G)/\,\mathrm{Prin}(G)$$

be the *Jacobian* of G. It is a consequence of Kirchhoff's Matrix-Tree Theorem that $\mathrm{Jac}(G)$ is a finite abelian group whose order is equal to the number of spanning trees of G.

Since the graph G is connected, one knows that $\mathrm{Ker}(\Delta)$ consists precisely of the constant functions, and it follows from (A.3) that there is a canonical exact sequence

$$0 \longrightarrow \mathrm{Prin}(G) \xrightarrow{\gamma_1} \mathrm{Div}^0(G) \xrightarrow{\gamma_2} \Phi \longrightarrow 0.$$

In other words, the component group Φ is canonically isomorphic to the Jacobian group of the graph G.

We can summarize much of the preceding discussion by saying that the following diagram is commutative and exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Prin}^{(0)}(X) \arrow[r, "\alpha_1"] \arrow[d] & \mathrm{Div}^{(0)}(X) \arrow[r, "\alpha_2"] \arrow[d] & J^0(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Prin}(X) \arrow[r, "\beta_1"] \arrow[d] & \mathrm{Div}^0(X) \arrow[r, "\beta_2"] \arrow[d] & J(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Prin}(G) \arrow[r, "\gamma_1"] \arrow[d] & \mathrm{Div}^0(G) \arrow[r, "\gamma_2"] \arrow[d] & \mathrm{Jac}(G) \cong \Phi \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{A.5}$$

A few remarks are in order about the exactness of the rows and columns in (A.5). It is well known that the natural map from $X(K) = \mathfrak{X}(R)$ to the smooth locus of $\mathfrak{X}_k(k)$ is surjective (see for example [Liu 2002, Proposition 10.1.40(b)]); by (A.4), this implies that the natural maps

$$\mathrm{Div}(X) \to \mathrm{Div}(G) \quad \text{and} \quad \mathrm{Div}^0(X) \to \mathrm{Div}^0(G)$$

are surjective. The surjectivity of the horizontal map $\alpha_2 : \mathrm{Div}^{(0)}(X) \to J^0(K)$ follows from Remark A.2. Using this, we see from the Snake Lemma that since the vertical map $\mathrm{Div}^0(X) \to \mathrm{Div}^0(G)$ is surjective, the vertical map $\mathrm{Prin}(X) \to \mathrm{Prin}(G)$ is also surjective. All of the other claims about the commutativity and exactness of (A.5) follow in a straightforward way from the definitions.

Passage to the limit. If K'/K is a finite extension of degree m with ramification index $e \mid m$ and valuation ring R', then by a sequence of blow-ups we can obtain a regular model $\mathfrak{X}'/R'$ for X whose corresponding dual graph G' is the graph $\sigma_e(G)$ obtained by subdividing each edge of G into e edges. If we think of G as an unweighted graph and of $\sigma_e(G)$ as a weighted graph in which every edge has length $\frac{1}{e}$, then G and $\sigma_e(G)$ are different models for the same metric $\mathbb{Q}$-graph Γ, which one calls the *reduction graph* of $\mathfrak{X}/R$. The discussion in [Chinburg and Rumely 1993] shows that the various maps

$$c_{K'} : X(K') \to G'$$

are compatible, in the sense that they give rise to a specialization map $\tau : X(\overline{K}) \to \Gamma$ which takes $X(\overline{K})$ surjectively onto $\Gamma_{\mathbb{Q}}$.

It is straightforward to check that the diagram (A.5) behaves functorially with respect to finite extensions, and therefore that there is a commutative and exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Prin}^{(0)}(X(\overline{K})) \arrow[r, "\alpha_1"] \arrow[d] & \mathrm{Div}^{(0)}(X(\overline{K})) \arrow[r, "\alpha_2"] \arrow[d] & J^0(\overline{K}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Prin}(X(\overline{K})) \arrow[r, "\beta_1"] \arrow[d] & \mathrm{Div}^0(X(\overline{K})) \arrow[r, "\beta_2"] \arrow[d] & J(\overline{K}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Prin}_{\mathbb{Q}}(\Gamma) \arrow[r, "\gamma_1"] \arrow[d] & \mathrm{Div}^0_{\mathbb{Q}}(\Gamma) \arrow[r, "\gamma_2"] \arrow[d] & \mathrm{Jac}_{\mathbb{Q}}(\Gamma) \arrow[r] \arrow[d] & 0\,. \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{A.6}$$

Remark A.7. Let $\tilde{K}$ be the completion of an algebraic closure $\overline{K}$ of K, so that $\tilde{K}$ is a complete and algebraically closed field equipped with a valuation

$$v : \tilde{K} \to \mathbb{Q} \cup \{+\infty\},$$

and $\overline{K}$ is dense in $\tilde{K}$. By continuity, one can extend τ to a map $\tau : X(\tilde{K}) \to \Gamma$ and replace $\overline{K}$ by $\tilde{K}$ everywhere in the diagram (A.6).

A few explanations are in order concerning the definitions of the various groups and group homomorphisms which appear in (A.6). Since $\overline{K}$ is algebraically closed, we may identify the group $\mathrm{Div}(X_{\overline{K}})$ of Cartier (or Weil) divisors on $X_{\overline{K}}$ with $\mathrm{Div}(X(\overline{K}))$, the free abelian group on the set $X(\overline{K})$. We define $\mathrm{Prin}(X(\overline{K}))$ to be the subgroup of $\mathrm{Div}(X(\overline{K}))$ consisting of principal divisors. The group $\mathrm{Div}(X(\overline{K}))$

(respectively, $\mathrm{Prin}(X(\overline{K}))$) can be identified with the direct limit of $\mathrm{Div}(X_{K'})$ (respectively, $\mathrm{Prin}(X_{K'})$) over all finite extensions K'/K. Accordingly, we define the group $J^0(\overline{K})$ to be the direct limit of the groups $J^0(K')$ over all finite extensions K'/K, and we define $\mathrm{Div}^{(0)}(X(\overline{K}))$ and $\mathrm{Prin}^{(0)}(X(\overline{K}))$ similarly. Finally, we define $\mathrm{Jac}_{\mathbb{Q}}(\Gamma)$ to be the quotient $\mathrm{Div}^0_{\mathbb{Q}}(\Gamma)/\,\mathrm{Prin}_{\mathbb{Q}}(\Gamma)$.

That $\mathrm{Prin}_{\mathbb{Q}}(\Gamma)$, as defined in Section 1, coincides with the direct limit over all finite extensions K'/K of the groups $\mathrm{Prin}(G')$ follows easily from Remark 1.3.

With these definitions in place, it is straightforward to check using (A.5) that the diagram (A.6) is both commutative and exact.

We note the following consequence of the exactness of (A.5) and (A.6):

Corollary A.8. *The canonical maps*

$$\mathrm{Prin}(X) \to \mathrm{Prin}(G) \quad \textit{and} \quad \mathrm{Prin}(X(\overline{K})) \to \mathrm{Prin}_{\mathbb{Q}}(\Gamma)$$

are surjective.

Remark A.9. It follows from Corollary A.8 that if G is a graph, the group $\mathrm{Prin}(G)$ can be characterized as the image of $\mathrm{Prin}(X)$ under the specialization map from $\mathrm{Div}(X)$ to $\mathrm{Div}(G)$ for any strongly semistable regular arithmetic surface $\mathfrak{X}/R$ whose special fiber has dual graph isomorphic to G. (Such an $\mathfrak{X}$ always exists by Corollary B.3 below.)

Remark A.10. Another consequence of (A.6) is that there is a canonical isomorphism

$$J(\overline{K})/J^0(\overline{K}) \cong \mathrm{Jac}_{\mathbb{Q}}(\Gamma),$$

so that the group $\mathrm{Jac}_{\mathbb{Q}}(\Gamma)$ plays the role of the component group of the Néron model in this situation, even though there is not a well-defined Néron model for J over $\overline{K}$ or $\tilde{K}$, since the valuations on these fields are not discrete. One can show using elementary methods that $\mathrm{Jac}_{\mathbb{Q}}(\Gamma)$ is (noncanonically) isomorphic to $(\mathbb{Q}/\mathbb{Z})^g$ (compare with the discussion in [Grothendieck 1972, Exposé IX, §11.8]).

Appendix B. A result from the deformation theory of stable marked curves
by Brian Conrad

Recall from Section 1A that by an *arithmetic surface*, we mean an integral scheme that is proper and flat over a discrete valuation ring such that its generic fiber is a smooth and geometrically connected curve. By Stein factorization, the special fiber of an arithmetic surface is automatically geometrically connected.

In this appendix, we describe how one can realize an arbitrary graph G as the dual graph of the special fiber of some regular arithmetic surface whose special fiber C is a *totally degenerate semistable curve* (or *Mumford curve*), meaning that

C is semistable and connected, every irreducible component of C is isomorphic to the projective line over the residue field k, and all singularities of C are k-rational.

Lemma B.1. *Let G be a connected graph, and let k be an infinite field. Then there exists a totally degenerate semistable curve C/k whose dual graph is isomorphic to G.*

The proof is left to the reader.

The crux of the matter is the following theorem, whose proof is a standard application of the deformation theory of stable marked curves.

Theorem B.2. *Let C be a proper and geometrically connected semistable curve over a field k, and let R be a complete discrete valuation ring with residue field k. Then there exists an arithmetic surface $\mathfrak{X}$ over R with special fiber C such that $\mathfrak{X}$ is a regular scheme.*

Combining these two results, we obtain:

Corollary B.3. *Let R be a complete discrete valuation ring with field of fractions K and infinite residue field k. For any connected graph G, there exists a regular arithmetic surface $\mathfrak{X}/R$ whose generic fiber is a smooth, proper, and geometrically connected curve X/K, and whose special fiber is a totally degenerate semistable curve with dual graph isomorphic to G.*

Remark B.4. By [Liu 2002, Lemma 10.3.18], the genus of X coincides with the genus $g = |E(G)| - |V(G)| + 1$ of the graph G.

Proof of Theorem B.2. Let $g = \dim \mathrm{H}^1(C, \mathcal{O}_C)$ denote the arithmetic genus of C. The structure theorem for ordinary double points [Freitag and Kiehl 1988, III, §2] ensures that C^{sing} splits over a separable extension of k, so we can choose a finite Galois extension k'/k so that the locus $C_{k'}^{\mathrm{sing}}$ of nonsmooth points in $C_{k'}$ consists entirely of k'-rational points and every irreducible component of $C_{k'}$ is geometrically irreducible and has a k'-rational point. (If C is a Mumford curve then we can take $k' = k$.) In particular, each smooth component in $C_{k'}$ of arithmetic genus 0 is isomorphic to $\mathbf{P}^1_{k'}$ and so admits at least three k'-rational points. We can then construct a $\mathrm{Gal}(k'/k)$-stable étale divisor $D' \subseteq C_{k'}^{\mathrm{sm}}$ whose support consists entirely of k'-rational points in the smooth locus such that for each component X' of $C_{k'}$ isomorphic to $\mathbf{P}^1_{k'}$ we have

$$\#(X' \cap C_{k'}^{\mathrm{sing}}) + \#(X' \cap D') \geq 3.$$

In particular, if we choose an enumeration of $D'(k')$ then the pair $(C_{k'}, D')$ is a stable n-pointed genus-g curve, where $n = \#D'(k')$ and $2g - 2 + n > 0$. Let $D \subseteq C^{\mathrm{sm}}$ be the étale divisor that descends D'. We let R' be the local finite étale R-algebra with residue field k'/k.

The stack $\mathcal{M}_{g,n}$ classifying stable n-pointed genus-g curves for any $g, n \geq 0$ such that $2g-2+n > 0$ is a proper smooth Deligne–Mumford stack over Spec $\mathbb{Z}$.[4] The existence of $\mathcal{M}_{g,n}$ as a smooth Deligne–Mumford stack ensures that $(C_{k'}, D')$ admits a universal formal deformation $(\widehat{\mathscr{C}'}, \widehat{\mathscr{D}'})$ over a complete local noetherian R'-algebra A' with residue field k', and that A' is a formal power series ring over R'. Moreover, there is a relatively ample line bundle canonically associated to any stable n-pointed genus-g curve $(X, \{\sigma_1, \dots, \sigma_n\}) \to S$ over a scheme S, namely the twist

$$\omega_{X/S}\left(\sum \sigma_i\right)$$

of the relative dualizing sheaf of the curve by the étale divisor defined by the marked points. Hence, the universal formal deformation uniquely algebraizes to a pair $(\mathscr{C}', \mathscr{D}')$ over Spec A'.

Since $(C_{k'}, D') = k' \otimes_k (C, D)$, universality provides an action of the Galois group $\Gamma = \mathrm{Gal}(k'/k)$ on A' and on $(\mathscr{C}', \mathscr{D}')$ covering the natural Γ-action on $(C_{k'}, D')$ and on R'. The action by Γ on $\mathscr{C}'$ is compatible with one on the canonically associated ample line bundle $\omega_{\mathscr{C}'/A'}(\mathscr{D}')$. Since $R \to R'$ is finite étale with Galois group Γ, the Γ-action on everything in sight (including the relatively ample line bundle) defines effective descent data: $A = (A')^{\Gamma}$ is a complete local noetherian R-algebra with residue field k such that $R' \otimes_R A \to A'$ is an isomorphism, and $(\mathscr{C}', \mathscr{D}')$ canonically descends to a deformation $(\mathscr{C}, \mathscr{D})$ of (C, D) over A. This can likewise be shown to be a universal deformation of (C, D) in the category of complete local noetherian R-algebras with residue field k, but we do not need this fact.

What matters for our purposes is structural information about A and the proper flat A-curve $\mathscr{C}$. By the functorial characterization of formal smoothness, A is formally smooth over R since the same holds for A' over R', so A is a power series ring over R in finitely many variables. We claim that the Zariski-open locus of smooth curves in $\mathcal{M}_{g,n}$ is dense, which is to say that every geometric point has a smooth deformation. Indeed, since Knudsen's contraction and stabilization operations do nothing to smooth curves, this claim immediately reduces to the special cases $n = 0$ with $g \geq 2$, $g = n = 1$, and $g = 0$, $n = 3$. The final two cases are obvious and the first case was proved by Deligne and Mumford [1969, 1.9] via deformation theory. It follows that the generic fiber of $\mathscr{C}'$ over A' is a smooth curve, so the same holds for $\mathscr{C}$ over A. In other words, the Zariski-closed locus in $\mathscr{C}$ where $\Omega^1_{\mathscr{C}/A}$ is not invertible has its closed image in Spec(A) given by a closed subset Spec(A/I) for

[4] This is a standard fact: it can be found in [Deligne and Mumford 1969, 5.2] if $n = 0$ (so $g \geq 2$), in [Deligne and Rapoport 1973, IV, 2.2] if $g = n = 1$, and is trivial if $g = 0$, $n = 3$ (in which case $(\mathbf{P}^1, \{0, 1, \infty\})$ is the only such object, so the stack is Spec $\mathbb{Z}$). The general case follows from these cases by realizing $\mathcal{M}_{g,n}$ as the universal curve over $\mathcal{M}_{g,n-1}$ (due to Knudsen's contraction and stabilization operations [1983, 2.7]).

a unique nonzero radical ideal $I \subseteq A$. Since A is a power series ring over R and I is a nonzero ideal, we can certainly find a local R-algebra map $\phi : A \to R$ in which I has nonzero image. The pullback $\mathfrak{X}_\phi$ of $\mathscr{C}$ along ϕ is a proper flat semistable curve over R deforming C such that the generic fiber is smooth (as otherwise the map $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ would factor through $\mathrm{Spec}(A/I)$, a contradiction).

The only remaining problem is to show that if ϕ is chosen more carefully then $\mathfrak{X}_\phi$ is also regular. If C is k-smooth then $\mathscr{C}$ is A-smooth, so $\mathfrak{X}_\phi$ is R-smooth (and in particular regular). Thus, we may assume that C is not k-smooth. The main point is to use an understanding of the structure of $\mathscr{C}$ near each singular point $c \in C - C^{\mathrm{sm}}$. We noted at the outset that each finite extension $k(c)/k$ is separable, and if R_c is the corresponding local finite étale extension of R then the structure theory of ordinary double points [Freitag and Kiehl 1988, III, §2] provides an R_c-algebra isomorphism of henselizations

$$\mathbb{O}^{\mathrm{h}}_{\mathscr{C},\,c} \simeq A_c[u, v]^{\mathrm{h}}_{(\mathfrak{m}_c, u, v)}/(uv - a_c)$$

for some nonzero nonunit $a_c \in A_c$, where $(A_c, \mathfrak{m}_c)$ is the local finite étale extension of A with residue field $k(c)/k$; we have $a_c \neq 0$ since $\mathscr{C}$ has smooth generic fiber. For

$$B = \mathbb{Z}[t, u, v]/(uv - t),$$

the B-module $\bigwedge^2\big(\Omega^1_{B/\mathbb{Z}[t]}\big)$ is $B/(u, v) = B/(t)$. Thus, by a formal computation at each c we see that the annihilator ideal of $\bigwedge^2(\Omega^1_{\mathscr{C}/A})$ on $\mathscr{C}$ cuts out an A-finite closed subscheme of the nonsmooth locus in $\mathscr{C}$ that is a pullback of a unique closed subscheme $\mathrm{Spec}(A/J) \subseteq \mathrm{Spec}(A)$. We made the initial choice of k'/k large enough so that it splits each $k(c)/k$. Hence, the method of proof of [Deligne and Mumford 1969, 1.5] and the discussion following that result show that for each singularity c' in $C_{k'}$, the corresponding element $a_{c'} \in A'$ may be chosen so that the $a_{c'}$'s are part of a system of variables for A' as a formal power series ring over R'. The ideal JA' is the intersection of the ideals $(a_{c'})$. To summarize, A is a formal power series ring over R and we can choose the variables for A' over R' such that JA' is generated by a product of such variables, one for each singularity on $C_{k'}$. In particular, the local interpretation of each $a_{c'}$ on $\mathscr{C}'$ shows that for a local R'-algebra map $\phi' : A' \to R'$, the pullback $\mathfrak{X}'_{\phi'}$ of $\mathscr{C}'$ along ϕ' is regular if and only if $\phi'(a_{c'}) \in R'$ is a uniformizer for each c'. This condition on ϕ' is equivalent to saying that $\phi'(JA') \subseteq R'$ is a proper nonzero ideal with multiplicity equal to the number ν of geometric singularities on C.

Since $JA' = J \otimes_A A'$ is a principal nonzero proper ideal, and hence it is invertible as an A'-module, it follows that J is principal as well, say $J = (\alpha)$ for some nonzero nonunit $\alpha \in A$. We seek an R-algebra map $\phi : A \to R$ such that $\phi(I) \neq 0$ and $\phi(\alpha) \in R$ is nonzero with order ν, for then $\phi^*(\mathscr{C})$ will have smooth generic fiber (by our earlier discussion) and will become regular over R', and so it will be regular since R' is finite étale over R. The information we have about $\alpha \in A$ is that in the formal power series ring A' over R' we can choose the variables so that α is a product of ν of the variables. We are now reduced to the following problem in commutative algebra. Let $R \to R'$ be a local finite étale extension of discrete valuation rings, $A = R[\![x_1, \dots, x_N]\!]$, $I \subseteq A$ a nonzero ideal, and $\alpha \in A$ an element such that in $A' = R' \otimes_R A$ we can write $\alpha = x'_1 \cdots x'_\nu$ for some $1 \le \nu \le N$ and choice of R'-algebra isomorphism $A' \simeq R'[\![x'_1, \dots, x'_N]\!]$. Then we claim that there is an R-algebra map $\phi : A \to R$ such that $\mathrm{ord}_R(\phi(\alpha)) = \nu$ and $\phi(I) \neq 0$.

If we can choose $k' = k$, such as in the case when C is a Mumford curve, then such a ϕ obviously exists. Hence, for the intended application to Corollary B.3, we are done. To prove the claim in general, consider the expansions

$$x'_j = a'_{0j} + \sum_{i=1}^{N} a'_{ij} x_i + \cdots$$

where $a'_{0j} \in \mathfrak{m}_{R'}$ and $(a'_{ij})_{1 \le i, j \le N}$ is invertible over R'. Let $\pi \in \mathfrak{m}_R$ be a uniformizer. We seek ϕ of the form $\phi(x_i) = t_i \pi$ for $t_i \in R$. The requirement on the t_i's is that

$$(a'_{0j}/\pi) + \sum a'_{ij} t_i \in R'^{\times}$$

for $1 \le i \le \nu$ and that

$$h(t_1 \pi, \dots, t_N \pi) \neq 0$$

for some fixed nonzero power series $h \in I$. The unit conditions only depend on $t_i \bmod \mathfrak{m}_R$. Thus, once we find t_i that satisfy these unit conditions, the remaining nonvanishing condition on the nonzero power series h is trivial to satisfy by modifying the higher-order parts of the t_i's appropriately. It remains to consider the unit conditions, which is a consequence of the lemma below. □

Lemma B.5. *Let k'/k be a finite extension of fields. For $1 \le \nu \le N$ let $\{H'_1, \dots, H'_\nu\}$ be a collection of independent hyperplanes in k'^N. For any $v'_1, \dots, v'_\nu \in k'^N$, the union of the affine-linear hyperplanes $v'_i + H'_i$ in k'^N cannot contain k^N.*

Proof. The case of infinite k is trivial, but to handle finite k we have to do more work. Throughout the argument the ground field k may be arbitrary. Assuming we are in a case with $k^N \subseteq \bigcup (v'_i + H'_i)$, we seek a contradiction. Observe that $\nu \le N$ since the H'_i are linearly independent hyperplanes in k'^N. Each overlap $k^N \cap (v'_i + H'_i)$ is either empty or a translate of $V_i = k^N \cap H'_i$ by some $v_i \in V = k^N$. For $1 \le d \le N$, any d-fold intersection

$$V_{i_1} \cap \cdots \cap V_{i_d} = k^N \cap (H'_{i_1} \cap \cdots \cap H'_{i_d})$$

has k-dimension at most $N-d$. Indeed, otherwise it contains $N-d+1$ linearly independent vectors in k^N, and these may also be viewed as k'-linearly independent vectors in the overlap $H'_{i_1} \cap \cdots \cap H'_{i_d}$ of d linearly independent hyperplanes in k'^N. This contradicts the linear independence of such hyperplanes. It now remains to prove the following claim that does not involve k' and concerns subspace arrangements over k: if V is a vector space of dimension $N \geq 1$ over a field k and if $V_1, \ldots, V_m$ are linear subspaces with $1 \leq m \leq N$ such that V is a union of the translates $v_i + V_i$ for $v_1, \ldots, v_m \in V$ then there is $1 \leq d \leq m$ such that some d-fold intersection $V_{i_1} \cap \cdots \cap V_{i_d}$ has dimension at least $N-d+1$. This claim was suggested by T. Tao, and the following inductive proof of it was provided by S. Norine.

The claim is trivial for $N=1$, and in general we induct on N so we may assume $N>1$. The case $m=1$ is trivial, so with $N>1$ fixed we can assume $m>1$. The case $V_i = V$ for all i is also trivial, so we may assume V_m is contained in a hyperplane H. Let $v=0$ if $v_m \notin H$ and $v \in V-H$ if $v_m \in H$, so $(v_m+V_m)\cap(v+H)$ is empty. Since the v_i+V_i cover V, it follows that $v+H$ is covered by the v_i+V_i for $1 \leq i \leq m-1$, so H is covered by the $(v_i - v)+V_i$ for $1 \leq i \leq m-1 \leq N-1$. Each overlap $H \cap ((v_i - v)+V_i)$ for such i is either empty or a translate $w_i + W_i$ of $W_i = V_i \cap H$. Setting $w_i = 0$ if $H \cap ((v_i - v)+V_i)$ is empty, we can relabel the V_i's such that H is covered by $w_i + W_i$ for $1 \leq i \leq m-1$. By induction there is $1 \leq d \leq N-1$ so that after relabeling we have $\dim(W_1 \cap \cdots \cap W_d) \geq \dim(H) - d + 1 = N-d$. Hence, for $W = V_1 \cap \cdots \cap V_d$ we have that $\dim(H \cap W) = N-d$. In particular, $\dim(W) \geq N-d$. If $\dim(W) \geq N-d+1$ then $\{V_1, \ldots, V_d\}$ works as required in the claim we are aiming to prove, so we can assume $\dim(W) = N-d > 0$. Let $W' \subseteq V$ be a complementary subspace to W, and consider $W'_i = W' \cap V_i$ for $1 \leq i \leq d$. Obviously $\dim W' = d$ and

$$V_i = W \oplus W'_i$$

for such i. If some collection of translates $w'_i + W'_i$ for $1 \leq i \leq d$ and $w'_i \in W'$ covers W' then by induction there is $1 \leq d' \leq d$ such that (after relabeling) $\dim(W'_1 \cap \cdots \cap W'_{d'}) \geq d - d' + 1$, so

$$\begin{aligned}\dim(V_1 \cap \cdots \cap V_{d'}) &= \dim(W) + \dim(W'_1 \cap \cdots \cap W'_{d'}) \\ &= (N-d) + d - d' + 1 = N - d' + 1,\end{aligned}$$

as required.

Hence, we can now assume that no collection of translates $w'_i + W'_i$ in W' for $1 \leq i \leq d$ can cover W'. In particular, under the decomposition $V = W \oplus W'$, we may write $v_i = (w_i, w'_i)$ for $1 \leq i \leq N$ and these $w'_i + W'_i$ for $1 \leq i \leq d$ do not cover W'. Since each of $V_1, \ldots, V_d$ contains W, it follows that $v_1 + V_1, \ldots, v_d + V_d$ do not cover W'. Thus, we can choose $w' \in W'$ not in any of these d translates, so

$w' + W$ is disjoint from all of them (since the projections into W' are disjoint as well). By the initial covering hypothesis we therefore have that $w' + W$ is covered by $v_j + V_j$ for $d+1 \le j \le m$, so W is covered by translates of $V_j \cap W$ for such j. The number of such j's is $m-d \le N-d$, so since $0 < \dim W = N-d < N$ we can conclude by induction that there is some $1 \le d'' \le m-d$ so that (after relabeling) the intersection

$$(V_{d+1} \cap W) \cap \cdots \cap (V_{d+d''} \cap W) = V_1 \cap \cdots \cap V_d \cap V_{d+1} \cap \cdots \cap V_{d+d''}$$

has dimension at least $(N-d) - d'' + 1 = N - (d+d'') + 1$. This completes the induction and so proves the claim. $\square$

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Arguments des unités de Stark et périodes de séries d'Eisenstein

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Cet article décrit une construction conjecturale (dans l'esprit du 12ème problème de Hilbert) d'unités dans des extensions abéliennes de certains corps de base qui ne sont ni totalement réels ni de type CM. Ces corps de base possèdent un unique plongement complexe, et sont des extensions quadratiques d'un corps totalement réel F. On les appelle corps ATR («almost totally real»). Notre construction fait intervenir certains cycles topologiques homologiquement triviaux sur la variété modulaire de Hilbert associée à F. Les unités spéciales sont les images de ces cycles par une application qui repose sur l'intégration des séries d'Eisenstein de poids deux sur $\mathrm{GL}_2(F)$, et peut être vue comme un analogue formel des applications d'Abel–Jacobi de la théorie des cycles algébriques. On démontre que notre conjecture est compatible avec la conjecture de Stark pour les extensions ATR. Elle donne même un raffinement de la conjecture de Stark dans ce contexte, puisqu'elle fournit une formule analytique pour les arguments des unités de Stark, et pas seulement pour leurs valeurs absolues. La dernière section présente des résultats d'expériences numériques qui appuient notre conjecture.

We describe a conjectural construction (in the spirit of Hilbert's Twelfth problem) of units in abelian extensions of certain base fields which are neither totally real nor CM. These base fields are quadratic extensions with exactly one complex place of a totally real number field F, and are referred to as almost totally real (ATR) extensions. Our construction involves certain null-homologous topological cycles on the Hilbert modular variety attached to F. The special units are the images of these cycles under a map defined by integration of weight two Eisenstein series on $\mathrm{GL}_2(F)$. This map is formally analogous to the higher Abel–Jacobi maps that arise in the theory of algebraic cycles. We show that our conjecture is compatible with Stark's conjecture for ATR extensions; it is, however, a genuine strengthening of Stark's conjecture in this context since it gives an analytic formula for the arguments of the Stark units and not just their absolute values. The last section provides numerical evidence for our conjectures.

MSC2000: primary 11F67; secondary 11F75.

Mots-clefs: séries d'Eisenstein, périodes des formes modulaires de Hilbert, arguments des unités de Stark, conjectures de Stark, douzième problème de Hilbert, application d'Abel–Jacobi, cohomologie du groupe modulaire de Hilbert, fonction d'Asai, valeurs spéciales de fonctions zéta partielles, Eisenstein series, periods of Hilbert modular forms, arguments of Stark units, Stark conjectures, Hilbert twelfth problem, Abel–Jacobi map, cohomology of the Hilbert modular group, Asai function, special values of partial zeta functions.

Introduction

Soit K un corps de nombres, et soit

$$S_\infty = \{v_1, \dots, v_t\}$$

l'ensemble de ses places archimédiennes. Pour simplifier les énoncés qui suivent, on supposera (dans l'introduction seulement) que le corps K a pour nombre de classes 1 au sens restreint. On choisit, pour chaque place de S_∞, un plongement réel ou complexe associé, que l'on désignera par le même symbole, et l'on pose (pour x appartenant à $K^\times$, et $v \in S_\infty$)

$$s_v(x) := \begin{cases} \text{signe}(v(x)) \in \{-1, 1\} & \text{si } v \text{ est réelle;} \\ 1 & \text{si } v \text{ est complexe.} \end{cases}$$

Soit I un idéal de l'anneau des entiers $\mathcal{O}_K$ de K, et soit $\mathcal{O}_{K,+}^\times(I)$ le sous-groupe du groupe $\mathcal{O}_{K,+}^\times$ des unités totalement positives de $\mathcal{O}_K$ formé des éléments qui sont congrus à 1 modulo I. Pour tout $a \in (\mathcal{O}_K/I)$, on associe au choix de signes $s_{v_2}, \dots, s_{v_n}$ la fonction L partielle de Hurwitz :

$$L(a, I, s) := (\mathrm{N}I)^s \sum_{\substack{x \in \mathcal{O}_K/\mathcal{O}_{K,+}^\times(I) \\ x \equiv a \ (I)}}{}' s_{v_2}(x) \cdots s_{v_t}(x) |\mathrm{N}_{K/\mathbb{Q}}(x)|^{-s}, \tag{1}$$

où le symbole $\sum'$ indique que la somme est à prendre sur les éléments *non-nuls*. Ces fonctions L jouissent des propriétés suivantes :

1. La fonction $L(a, I, s)$ ne dépend que de l'image de a dans le quotient

$$(\mathcal{O}_K/I)/\mathcal{O}_{K,+}^\times,$$

sur lequel opère le groupe

$$\mathcal{G}_I := (\mathcal{O}_K/I)^\times/\mathcal{O}_{K,+}^\times. \tag{2}$$

Plus généralement, pour toute unité $\epsilon \in \mathcal{O}_K^\times$, on a

$$L(\epsilon a, I, s) = s_{v_2}(\epsilon)\cdots s_{v_t}(\epsilon)L(a, I, s).$$

2. La série qui définit $L(a, I, s)$ converge absolument sur le demi-plan $\mathrm{Re}(s) > 1$, et admet un prolongement méromorphe à tout le plan complexe avec au plus un pôle simple en $s = 1$. Si $t \geq 2$, cette fonction est même holomorphe, et s'annule en $s = 0$.

Pour toute place réelle $v \in S_\infty$, il existe alors une unité $\epsilon_v \in \mathcal{O}_K^\times$ telle que

$$s_v(\epsilon_v) = -1, \qquad s_{v'}(\epsilon_v) = 1 \text{ pour tout } v' \neq v.$$

De plus, la théorie du corps de classes identifie le groupe $\mathcal{G}_I$ avec le groupe de Galois d'une extension abélienne H de K, appelée le *corps de classes de rayon au sens restreint* associé à I. Soit

$$\mathrm{rec} : \mathcal{G}_I \to \mathrm{Gal}(H/K)$$

l'isomorphisme de réciprocité de la théorie du corps de classes, et pour tout $v \in S_\infty$, soit c_v la conjugaison complexe («élément de Frobenius») associée à la place v, de sorte que

$$c_v = \begin{cases} \mathrm{rec}(\epsilon_v) & \text{si } v \text{ est réelle,} \\ 1 & \text{si } v \text{ est complexe.} \end{cases}$$

On note e l'ordre du groupe des racines de l'unité dans H et l'on choisit une place $\tilde{v}_1$ de H au-dessus de la place v_1.

La conjecture de Stark [1976] (voir aussi [Tate 1984]) concerne les dérivées premières de $L(a, I, s)$ en $s = 0$, et peut s'énoncer comme suit :

Conjecture 1 (Stark). *Pour tout $a \in (\mathcal{O}_K/I)$, il existe une unité $u_a \in \mathcal{O}_H^\times$, appelée* unité de Stark *associée au couple (a, I), telle que*

1. $L'(a, I, 0) = \dfrac{1}{e} \log|\tilde{v}_1(u_a)|^2$;
2. $c_{v_1}(u_a) = u_a$;
3. *Si $t \geq 3$, alors* $c_{v_2}u_a = \cdots = c_{v_t}u_a = u_a^{-1}$;
4. *Pour tout $b \in \mathcal{G}_I$, on a* $u_{ab} = \mathrm{rec}(b)^{-1}u_a$.

On notera que les unités conjecturales u_a dépendent du choix de la place $\tilde{v}_1$ au-dessus de v_1, mais seulement à conjugaison près par $\mathrm{Gal}(H/K)$.

Si $S_\infty - \{v_1\}$ possède une place complexe, la conjecture 1 est trivialement vérifiée. En effet, quand $t = 2$, la quantité $L'(a, I, 0)$ ne dépend que de I et non de a, et s'écrit comme un multiple rationnel du logarithme d'une unité fondamentale de K. Quand $t > 2$, on a de plus $L'(a, I, 0) = 0$, de sorte que la conjecture 1 est vérifiée avec $u_a = 1$.

Par conséquent, la conjecture 1 n'a de l'intérêt que lorsque les places $v_2, \ldots, v_t$ sont toutes réelles, ce qui nous amène à distinguer deux cas.

1. Le cas totalement réel. Si la place v_1 est également réelle, le corps K est *totalement réel.* A cause de la propriété 2 dans la conjecture 1, l'expression $\tilde{v}_1(u_a)$ appartient alors à $\mathbb{R}$. La conjecture 1 permet donc d'évaluer ce nombre, du moins au signe près. On obtient ainsi, par évaluation des dérivées en $s = 0$ des séries $L(a, I, s)$, la construction analytique d'unités explicites de H. Les conjectures de Stark, une fois démontrées, fourniraient ainsi un élément de «théorie explicite du corps de classes» pour les corps de nombres totalement réels.

2. Le cas ATR. Si v_1 est une place complexe, on dit que K est un corps de nombres ATR («almost totally real») suivant la terminologie de [Darmon et Logan 2003]. Puisque $c_{v_1} = 1$, l'expression $\tilde{v}_1(u_a)$ n'a plus de raison d'être réelle *a priori*, et la conjecture 1 ne permet d'en évaluer que la *valeur absolue*. L'ambiguïté de signe du cas totalement réel s'avère donc plus sérieuse dans le contexte ATR, puisqu'elle porte sur *l'argument* de $\tilde{v}_1(u_a)$, un élément de $\mathbb{R}/(2\pi\mathbb{Z})$. On est amené à poser la question suivante qui peut servir de motivation pour cet article.

Question 1. *Existe-t-il une formule analytique explicite pour l'expression $\tilde{v}_1(u_a)$ qui apparaît dans la conjecture 1, lorsque K est ATR ?*

Une réponse affirmative à cette question fournirait une solution du douzième problème de Hilbert pour les extensions ATR.

Dans le premier cas intéressant où K est un corps cubique complexe, cette question a été considérée dans [Dasgupta 1999] et dans [Dummit et al. 2004] où la conjecture de Stark est étudiée numériquement, et un progrès décisif a été accompli dans [Ren et Sczech 2008].

Le présent article se penche sur la Question 1 lorsque le corps K est une extension quadratique d'un corps totalement réel F, et lorsque le corps de rayon H est remplacé par un certain sous-corps — le corps de classes d'anneau («ring class field») associé à I et K/F — dont la définition sera rappelée dans la section 1.

Pour motiver notre approche, examinons ce qui se passe dans le cas le plus simple où $F = \mathbb{Q}$, où K est un corps quadratique imaginaire de nombre de classes 1, et où $I = (m)$ est un idéal rationnel engendré par $m \in \mathbb{Z}$. Au lieu de porter sur les séries partielles de Hurwitz, les conjectures de cet article vont plutôt porter sur les sommes

$$\sum_{r \in (\mathbb{Z}/m\mathbb{Z})} L(ar, I, s) =: L(M, s) \;=\; (\mathrm{N}M)^s \sideset{}{'}\sum_{x \in M/\mathcal{O}_{K,+}^\times(I)} |\mathrm{N}_{K/\mathbb{Q}}(x)|^{-s}, \tag{3}$$

où

$$M = \{x \in \mathcal{O}_K,\ \text{tel qu'il existe } r \in \mathbb{Z} \text{ avec } x \equiv ar \pmod{I}\}.$$

Le module M est un réseau dans $K \subset \mathbb{C}$, et, quand a appartient à $\mathcal{G}_I$, il forme même un module projectif sur l'ordre $\mathcal{O}_I := \mathbb{Z} + I\mathcal{O}_K$. La série $L(M, s)$ ne dépend que de la classe d'homothétie de ce réseau ; elle est donc déterminée par l'invariant $\tau = \omega_2/\omega_1$, où (ω_1, ω_2) est une $\mathbb{Z}$-base de M choisie de telle manière que τ appartienne au demi-plan de Poincaré $\mathcal{H}$. La *formule limite de Kronecker* fournit les premiers termes du développement de $L(M, s)$ en $s = 0$ en la reliant au logarithme de la fonction η de Dedekind :

$$L(M, s) = -\tfrac{1}{2} - \tfrac{1}{2}\big(c_I + \log \operatorname{Im}(\tau) + 4 \log |\eta(\tau)|\big)s + O(s^2), \tag{4}$$

où c_I est une constante qui ne dépend que de I et pas de M. À des facteurs parasites près, les dérivées premières $L'(M, 0)$ sont donc fournies par l'expression $\log |\eta(\tau)|$. Or, comme τ appartient à $\mathcal{H} \cap K$, les produits d'expressions de la forme $\eta(\tau)$ donnent lieu aux unités elliptiques, que l'on sait être des unités dans des extensions abéliennes du corps K grâce à la théorie de la multiplication complexe. Plus précisément, pour $\tau_1, \tau_2 \in \mathcal{H} \cap K$, les expressions de la forme

$$u(\tau_1, \tau_2) := \eta(\tau_2)/\eta(\tau_1) \tag{5}$$

sont des nombres algébriques, et leurs puissances 24-èmes sont des unités dans des extensions abéliennes de K. C'est ainsi que les propriétés de la fonction η et la théorie de la multiplication complexe permettent non seulement de démontrer la conjecture de Stark dans le cas où K est quadratique imaginaire, mais apportent aussi une réponse à la Question 1 dans ce cas.

Dans la généralisation «traditionnelle» de la théorie de la multiplication complexe proposée par Hilbert et son école, puis développée rigoureusement par Shimura et Taniyama, on est amené à remplacer $\mathbb{Q}$ par un corps F totalement réel de degré $n > 1$ (de sorte que $F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^n$), et les formes modulaires classiques par des *formes modulaires de Hilbert*. Celles-ci correspondent à des fonctions holomorphes sur $\mathcal{H}^n$, invariantes (à un facteur d'automorphie près) sous l'action naturelle de $\mathbf{SL}_2(\mathcal{O}_F)$. Les corps quadratiques imaginaires sont remplacés par des extensions quadratiques K de F de type CM, munies d'une identification $\Phi : K \otimes_F \mathbb{R}^n \to \mathbb{C}^n$. La théorie de la multiplication complexe affirme alors que les valeurs de certaines fonctions modulaires de Hilbert (quotients de formes modulaires de même poids, possédant des développements de Fourier rationnels) en des points $\tau \in \Phi(K) \cap \mathcal{H}^n$ engendrent des extensions abéliennes du corps reflex $\tilde{K}$ associé à (K, Φ). Cette théorie possède deux inconvénients du point de vue de la Question 1 :

(a) elle ne permet d'aborder la «théorie du corps de classes explicite» que pour les corps de base de type CM ;

(b) elle ne permet pas d'obtenir facilement des unités dans des extensions abéliennes de $\tilde{K}$, les unités modulaires n'ayant pas de généralisation évidente pour les formes modulaires de Hilbert. En effet, quand $n > 1$, le faisceau structural sur l'espace complexe analytique $X := \mathbf{SL}_2(\mathbb{O}_F)\backslash\mathcal{H}^n$ ne possède pas de sections globales non-nulles. La relation entre la théorie de Shimura–Taniyama et les conjectures de Stark pour les corps CM (à supposer qu'il y en ait une) reste donc à élucider. (Voir par exemple [de Shalit et Goren 1997] et [Goren et Lauter 2007].)

Pour aborder la Question 1 lorsque K est une extension quadratique ATR d'un corps F totalement réel de degré $n > 1$, il faut relever le nombre réel $\log|\tilde{v}_1(u_a)|$ en un nombre complexe $\log \tilde{v}_1(u_a) \in \mathbb{C}/2i\pi\mathbb{Z}$. Au vu de la formule limite de Kronecker (4) et de la discussion précédente, il s'agit donc de généraliser l'expression

$$\log \eta(\tau) = \log|\eta(\tau)| + i \arg \eta(\tau) \in \mathbb{C}/2i\pi\mathbb{Z}$$

à un cadre où les formes modulaires classiques sont remplacées par des formes modulaires de Hilbert sur F. C'est ce qui a été entrepris dans la thèse du premier auteur, qui part de l'identité classique $\mathrm{d}\log\eta(z) = -i\pi E_2(z)\,dz$, où E_2 est la série d'Eisenstein définie par

$$E_2(z) = -\frac{1}{12} + 2\sum_{n\geq 1}\sigma_1(n)e^{2i\pi nz}, \quad \text{avec } \sigma_1(n) = \sum_{d|n} d.$$

La formule (5) peut donc se réécrire en prenant le logarithme complexe des deux côtés :

$$\log(u(\tau_1,\tau_2)) = -i\pi\int_{\tau_1}^{\tau_2} E_2(z)\,dz \in \mathbb{C}/2i\pi\mathbb{Z}. \tag{6}$$

Or, si les unités modulaires n'admettent pas d'analogue évident en dimension supérieure, les séries d'Eisenstein, elles, se généralisent sans difficulté à ce contexte. La section 1.1 rappelle la définition de la série d'Eisenstein E_2 de poids $(2,\dots,2)$ sur $\mathbf{SL}_2(\mathbb{O}_F)$. Celle-ci donne lieu à une n-forme différentielle ω_{E_2} holomorphe sur l'espace analytique X.

La démarche suggérée par Charollois [2004] consiste essentiellement à remplacer les valeurs de $\log\eta(\tau)$ par des intégrales de ω_{E_2} sur des cycles appropriés de dimension réelle n sur X. Plus précisément, le présent article associe à tout $\tau \in \mathcal{H}\cap v_1(K)$ un cycle fermé Δ_τ de dimension réelle $(n-1)$ sur X. En faisant abstraction pour le moment des phénomènes liés à la présence possible de torsion dans la cohomologie de X, on démontre que ces cycles sont homologues à zéro. Autrement dit, il existe une chaîne différentiable C_τ de dimension n sur X telle que

$$\partial C_\tau = \Delta_\tau.$$

Cela permet de définir un invariant canonique $J_\tau \in \mathbb{C}/2i\pi\mathbb{Z}$ en intégrant un multiple approprié de ω_{E_2} sur C_τ. La contribution la plus importante de cet article est la conjecture 4.1, qui relie les invariants J_τ à l'expression $\log \tilde{v}_1(u_a)$ dont la partie réelle apparaît dans la conjecture 1. La conjecture 4.1 apporte ainsi un élément de réponse à la Question 1.

La définition des invariants J_τ s'appuie de façon essentielle sur la thèse du premier auteur [Charollois 2004]. Elle est aussi à rapprocher de deux autres travaux antérieurs :

1. L'article [Darmon et Logan 2003], où les séries d'Eisenstein sur $\mathbf{GL}_2(F)$ du présent article sont remplacées par des formes modulaires de Hilbert *cuspidales* de poids $(2, \ldots, 2)$. Dans les cas les plus concrets qui ont pu être testés numériquement, ces formes sont associées à des courbes elliptiques définies sur F. L'invariant J_τ défini dans ce contexte semble alors permettre la construction de points sur ces courbes définis sur certaines extensions abéliennes de K.
2. L'article [Darmon et Dasgupta 2006] peut être considéré comme une variante p-adique, avec $F = \mathbb{Q}$, des constructions principales du présent article, le rôle de notre v_1 y étant joué par une place non-archimédienne p. L'extension K est alors un corps quadratique *réel* dans lequel le nombre premier p est inerte. Les séries d'Eisenstein de poids 2 sur certains sous-groupes de congruence de $\mathbf{SL}_2(\mathbb{Z})$, réinterprétées convenablement comme des «formes modulaires de Hilbert» sur $\mathcal{H}_p \times \mathcal{H}$, où $\mathcal{H}_p = \mathbf{P}_1(\mathbb{C}_p) - \mathbf{P}_1(\mathbb{Q}_p)$ est le demi-plan p-adique, donnent alors lieu à des invariants p-adiques $J_\tau \in \mathbb{C}_p$ associés à un élément $\tau \in \mathcal{H}_p \cap K$. Ces invariants correspondent conjecturalement à des p-unités dans des extensions abéliennes de K.

En se plaçant dans un cadre classique où l'on dispose de notions topologiques et analytiques générales (homologie et cohomologie singulière, théorie de Hodge), le présent article mène à une clarification conceptuelle des différentes constructions de «points et unités de Stark–Heegner» proposées jusqu'à présent dans la littérature. (Voir [Darmon et Logan 2003] et [Darmon et Dasgupta 2006], ainsi que le cadre traité originalement dans [Darmon 2004] ou les généralisations formulées dans [Trifković 2006] et [Greenberg 2008].) Le présent article peut très bien servir d'introduction aux travaux sur les points de Stark–Heegner cités en référence, bien qu'il ait été rédigé après ceux-ci.

1. Notions préliminaires

1.1. ***Séries d'Eisenstein.*** Soit F un corps totalement réel de degré $n > 1$ et $S_F := \{v_1, \ldots, v_n\}$ son ensemble de places archimédiennes. On désigne par $\mathcal{O}_F$ l'anneau des entiers de F, par d_F son discriminant, et par R_F le régulateur de F.

On supposera dans la suite de cet article que F a nombre de classes 1 au sens restreint, de sorte qu'il existe pour tout $1 \leq j \leq n$ une unité $\epsilon^{(j)} \in \mathbb{O}_F^\times$ avec

$$v_j(\epsilon^{(j)}) < 0, \qquad v_k(\epsilon^{(j)}) > 0 \quad \text{si } k \neq j.$$

Pour tout $a \in F$, on note $a_j := v_j(a)$ son image dans $\mathbb{R}$ par le plongement v_j, et $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ l'image dans $\mathbf{SL}_2(\mathbb{R})$ d'une matrice $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ de $\mathbf{SL}_2(F)$. On identifiera librement a avec le n-uplet $(a_1, \ldots, a_n)$ et A avec $(A_1, \ldots, A_n)$. On obtient ainsi une action par homographies du groupe modulaire de Hilbert $\Gamma := \mathbf{SL}_2(\mathbb{O}_F)$ sur le produit $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ de n copies du demi-plan de Poincaré. Le quotient analytique

$$X := \Gamma \backslash (\mathcal{H}_1 \times \cdots \times \mathcal{H}_n)$$

s'identifie avec les points complexes d'un ouvert de Zariski d'une variété algébrique projective lisse : la *variété modulaire de Hilbert* X_F associée au corps F.

Une *forme modulaire de Hilbert* de poids $(2, \ldots, 2)$ pour Γ est une fonction holomorphe $f(z_1, \ldots, z_n)$ sur $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ telle que la forme différentielle

$$\omega_f := f(z_1, \ldots z_n)\, dz_1 \wedge \cdots \wedge dz_n$$

soit invariante sous l'action de Γ. Autrement dit, pour tout $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, on exige que

$$f\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right) = (c_1 z_1 + d_1)^2 \cdots (c_n z_n + d_n)^2 f(z_1, \ldots, z_n).$$

Lorsque $n > 1$, une telle fonction possède, d'après le principe de Koecher, un développement en série de Fourier à l'infini de la forme

$$f(z_1, \ldots, z_n) = a_f(0) + \sum_{\mu \in \mathbb{O}_F,\ \mu \gg 0} a_f(\mu)\, e^{2i\pi\left(\frac{\mu_1}{\delta_1} z_1 + \cdots + \frac{\mu_n}{\delta_n} z_n\right)},$$

où δ désigne un générateur totalement positif de la différente de $F/\mathbb{Q}$.

La *série d'Eisenstein holomorphe* E_2 de poids 2 se définit sur $\mathcal{H}_1 \times \cdots \times \mathcal{H}_n$ par le développement en série de Fourier suivant :

$$E_2(z_1, \ldots, z_n) = \zeta_F(-1) + 2^n \sum_{\mu \in \mathbb{O}_F,\ \mu \gg 0} \sigma_1(\mu)\, e^{2i\pi\left(\frac{\mu_1}{\delta_1} z_1 + \cdots + \frac{\mu_n}{\delta_n} z_n\right)}, \tag{7}$$

où, pour un entier k donné, on a posé

$$\sigma_k(\mu) = \sum_{(\nu) | (\mu)} |N_{F/\mathbb{Q}}(\nu)|^k,$$

la sommation portant sur les idéaux (principaux) entiers (ν) qui divisent (μ). La fonction $E_2(z_1,\dots,z_n)$ est une forme modulaire de Hilbert de poids $(2,\dots,2)$ pour Γ (voir [van der Geer 1988, chap. I.6]).

On se donne des coordonnées réelles x_j, y_j sur X en posant $z_j = x_j + iy_j$, et l'on définit à partir de E_2 une forme différentielle invariante ω_{Eis} en posant

$$\omega_{\mathrm{Eis}} := \begin{cases} \dfrac{(2i\pi)^2}{\sqrt{d_F}}\omega_{E_2} + \dfrac{R_F}{2}\Big(\dfrac{dz_1\wedge d\bar{z}_1}{y_1^2} - \dfrac{dz_2\wedge d\bar{z}_2}{y_2^2}\Big) & \text{si } n=2, \\ \dfrac{(2i\pi)^n}{\sqrt{d_F}}\omega_{E_2} & \text{si } n\geq 3. \end{cases} \tag{8}$$

La n-forme différentielle ω_{Eis} est fermée et, quand $n \geq 3$, elle est holomorphe. On va s'intéresser à sa classe dans la cohomologie $H^n(X,\mathbb{C})$ formée à partir du complexe de De Rham des formes différentielles C^∞ sur X. Ce groupe de cohomologie est muni d'une action des opérateurs de Hecke T_λ, où les λ parcourent les idéaux de $\mathbb{O}_F$. La forme différentielle ω_{Eis} est vecteur propre pour ces opérateurs. Plus précisément, on a

$$T_\lambda(\omega_{\mathrm{Eis}}) = \sigma_1(\lambda)\,\omega_{\mathrm{Eis}}.$$

Pour tout $1 \leq j \leq n$, on dispose également d'une involution T_{υ_j} sur l'espace réel-analytique X associée à la place υ_j («opérateur de Hecke à l'infini»). Celle-ci se définit en posant

$$T_{\upsilon_j}(z_1,\dots,z_n) := (\epsilon_1^{(j)}z_1,\dots,\epsilon_j^{(j)}\bar{z}_j,\dots,\epsilon_n^{(j)}z_n).$$

On appelle $T_{\upsilon_j}^*$ l'involution sur $H^n(X,\mathbb{C})$ qui s'en déduit par pullback sur les formes différentielles. Les n opérateurs $T_{\upsilon_j}^*$ et les opérateurs de Hecke T_λ engendrent une algèbre commutative $\mathbf{T}$ sur $\mathbb{Z}$.

On aura besoin dans la suite de certaines fonctions qui joueront le rôle de primitives de E_2. On introduit pour cela la fonction h de Asai [1970] définie sur $\mathcal{H}^n$ par

$$\begin{aligned} h(z) = {} & \frac{4(-\pi)^n\zeta_F(-1)}{R_F\sqrt{d_F}}y_1\cdots y_n \\ & + \frac{4\sqrt{d_F}}{2^n R_F}\sum_{\mu\in\mathbb{O}_F}{}' \sigma_{-1}(\mu)\, e^{2i\pi\left(\frac{\mu_1}{\delta_1}x_1+\left|\frac{\mu_1}{\delta_1}\right|iy_1+\cdots+\frac{\mu_n}{\delta_n}x_n+\left|\frac{\mu_n}{\delta_n}\right|iy_n\right)}. \end{aligned} \tag{9}$$

Il sera plus commode de travailler avec

$$\tilde{h}(z) := \lambda_F h(z), \qquad \text{où } \lambda_F := 4^{n-1}R_F,$$

de sorte que

$$\begin{aligned} \tilde{h}(z) = {} & \frac{(-4\pi)^n\zeta_F(-1)}{\sqrt{d_F}}y_1\cdots y_n \\ & + 2^n\sqrt{d_F}\sum_{\mu\in\mathbb{O}_F}{}' \sigma_{-1}(\mu)\, e^{2i\pi\left(\frac{\mu_1}{\delta_1}x_1+\left|\frac{\mu_1}{\delta_1}\right|iy_1+\cdots+\frac{\mu_n}{\delta_n}x_n+\left|\frac{\mu_n}{\delta_n}\right|iy_n\right)}. \end{aligned} \tag{10}$$

Les fonctions $h(z)$ et $\tilde{h}(z)$ jouissent des propriétés suivantes :

1. Elles sont harmoniques par rapport à chacune des variables z_j, $1 \leq j \leq n$.
2. Elles satisfont les formules de transformation

$$h(Az) = h(z) - \log(|c_1z_1 + d_1|^2 \cdots |c_nz_n + d_n|^2), \tag{11}$$

$$\tilde{h}(Az) = \tilde{h}(z) - \lambda_F \log(|c_1z_1 + d_1|^2 \cdots |c_nz_n + d_n|^2). \tag{12}$$

3. On a

$$\frac{\partial^n \tilde{h}(z_1, \dots, z_n)}{\partial z_1 \cdots \partial z_n} dz_1 \wedge \cdots \wedge dz_n = \frac{(2i\pi)^n}{\sqrt{d_F}} \omega_{E_2}, \tag{13}$$

$$\frac{\partial^n \tilde{h}(z_1, \dots, z_n)}{\partial z_1 \cdots \partial \bar{z_j} \cdots \partial z_n} dz_1 \wedge \cdots d\bar{z}_j \cdots \wedge dz_n = T_{v_j}^* \left(\frac{(2i\pi)^n}{\sqrt{d_F}} \omega_{E_2} \right). \tag{14}$$

Toutes ces formules se vérifient par un calcul direct, sauf (11) et (12). Pour ces dernières, voir [Asai 1970], théorème 4.

Lemme 1.1. *La forme* $(\mathrm{Id} + T_{v_j}^*)\omega_{\mathrm{Eis}}$ *est exacte.*

Démonstration. Supposons que $j = 1$ pour fixer les idées et alléger les notations. À partir de la fonction $\tilde{h}$, on définit la $(n-1)$ forme différentielle sur $\mathcal{H}^n$:

$$\eta = \frac{\partial^{n-1} \tilde{h}(z_1, \dots, z_n)}{\partial z_2 \cdots \partial z_n} dz_2 \wedge \cdots \wedge dz_n. \tag{15}$$

Quand $n > 2$, la formule (12) montre que η est invariante sous Γ, et correspond donc à une $(n-1)$-forme différentielle sur X. Parce que $\tilde{h}$ est harmonique, cette forme est de plus holomorphe par rapport aux variables $z_2, \dots, z_n$, d'où la formule

$$d\eta = \left(\frac{\partial^n \tilde{h}}{\partial z_1 \cdots \partial z_n} dz_1 \wedge \cdots \wedge dz_n + \frac{\partial^n \tilde{h}}{\partial \bar{z_1} \partial z_2 \cdots \partial z_n} d\bar{z_1} \wedge dz_2 \wedge \cdots \wedge dz_n \right). \tag{16}$$

Le lemme résulte alors de (13) et de (14) avec $j = 1$.

Dans le cas où $n = 2$, la forme η définie par (15) n'est plus Γ-invariante. En effet, si $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, on a

$$A^*(\eta) = \eta - 4R_F \frac{c_2}{c_2z_2 + d_2} dz_2.$$

Il convient alors de modifier la définition de η en (15) en posant cette fois

$$\eta' := \left(\frac{\partial \tilde{h}(z_1, z_2)}{\partial z_2} - \frac{2R_F}{iy_2} \right) dz_2. \tag{17}$$

On déduit de l'identité (12) que η' est invariante sous Γ. La formule (16) s'adapte sans difficulté à condition d'ajouter la contribution de

$$d\left(\frac{-2R_F}{iy_2}dz_2\right) = -R_F\frac{dz_2 \wedge d\bar{z}_2}{y_2^2}.$$

On obtient

$$(\mathrm{Id} + T_{v_1}^*)\,\omega_{\mathrm{Eis}} = d\eta',$$

puisque la forme $dz_1 \wedge d\bar{z}_1/y_1^2$ est quant à elle dans le noyau de $\mathrm{Id} + T_{v_1}^*$. C'est ce calcul qui justifie le terme supplémentaire apparaissant dans la définition (8) de ω_{Eis} lorsque $n = 2$. □

Pour $m \geq 0$, on appelle $C_m^0(X)$ le groupe engendré par les combinaisons linéaires formelles à coefficients dans $\mathbb{Z}$ des cycles différentiables fermés de dimension réelle m sur X. On définit le groupe des périodes de ω_{Eis} par

$$\Lambda_{\mathrm{Eis}} := \left\{ \int_C \omega_{\mathrm{Eis}} \ \text{ pour } C \in C_n^0(X) \right\} \subset \mathbb{C}.$$

Proposition 1.2. *Le groupe Λ_{Eis} est un sous-groupe de $\mathbb{C}$ de rang un, commensurable avec $(2i\pi)^n\mathbb{Z}$.*

Démonstration. Cette proposition se démontre en trois parties.

(a) *Le groupe Λ_{Eis} est un sous-ensemble discret de $\mathbb{C}$.* La théorie de Harder [1975] (voir aussi le théorème 6.3, chap. III, §7 de [Freitag 1990] avec $m = n$) fournit la décomposition

$$H^n(X, \mathbb{C}) = H^n_{\mathrm{univ}}(X, \mathbb{C}) \oplus H^n_{\mathrm{Eis}}(X, \mathbb{C}) \oplus H^n_{\mathrm{cusp}}(X, \mathbb{C}), \tag{18}$$

où $H^n_{\mathrm{univ}}(X, \mathbb{C})$ provient des formes différentielles $\mathbf{SL}_2(\mathbb{R})^n$-invariantes, $H^n_{\mathrm{Eis}}(X, \mathbb{C})$ est l'espace vectoriel de dimension 1 engendré par $[\omega_{\mathrm{Eis}}]$, et $H^n_{\mathrm{cusp}}(X, \mathbb{C})$ provient des formes modulaires cuspidales de poids $(2, \ldots, 2)$ sur X. La décomposition (18) est respectée par l'algèbre de Hecke $\mathbf{T}$, et les éléments $\omega \in H^n_{\mathrm{Eis}}$ sont caractérisés par les propriétés

$$T_{v_j}^*\omega = -\omega, \qquad T_\lambda\omega = (\mathrm{N}\lambda + 1)\,\omega, \qquad \text{pour tout } \lambda \lhd \mathcal{O}_F.$$

Il en résulte que la projection naturelle $H^n(X, \mathbb{C}) \to H^n_{\mathrm{Eis}}(X, \mathbb{C})$ issue de (18) est décrite par un idempotent $\pi_{\mathrm{Eis}} \in \mathbf{T} \otimes \mathbb{Q}$. Soit Λ l'image naturelle de $H_n(X, \mathbb{Z})$ dans le dual $H^n(X, \mathbb{C})^\vee := \mathrm{Hom}(H^n(X, \mathbb{C}), \mathbb{C})$ par l'application des périodes. C'est un sous-groupe discret stable pour l'action de $\mathbf{T}$. On a de plus

$$\Lambda_{\mathrm{Eis}} = \langle \omega_{\mathrm{Eis}}, \pi_{\mathrm{Eis}}(\Lambda) \rangle,$$

où $\langle\ ,\ \rangle$ désigne l'accouplement naturel entre $H^n(X, \mathbb{C})$ et son dual. Or on a

$$\pi_{\mathrm{Eis}}(\Lambda) \subset \frac{1}{t}\Lambda \cap H^n_{\mathrm{Eis}}(X, \mathbb{C})^\vee,$$

où t est un entier tel que $t\pi_{\mathrm{Eis}}$ appartient à $\mathbf{T}$. Par conséquent $\pi_{\mathrm{Eis}}(\Lambda)$ est un sous-groupe discret de $H^n_{\mathrm{Eis}}(X, \mathbb{C})^\vee$, ce qui implique que Λ_{Eis} est lui aussi discret.

(b) *Le groupe* Λ_{Eis} *est contenu dans* $(2i\pi)^n\mathbb{R}$. En effet, le lemme 1.1 implique que $T^*_{v_1}\cdots T^*_{v_n}([\omega_{\mathrm{Eis}}]) = (-1)^n[\omega_{\mathrm{Eis}}]$. Par ailleurs, un calcul direct montre que $T^*_{v_1}\cdots T^*_{v_n}([\omega_{\mathrm{Eis}}]) = [\bar{\omega}_{\mathrm{Eis}}]$. On en déduit que $[\bar{\omega}_{\mathrm{Eis}}] = (-1)^n[\omega_{\mathrm{Eis}}]$. Les périodes de ω_{Eis} appartiennent donc bien à $(2i\pi)^n\mathbb{R}$.

(c) Les parties (a) et (b) montrent que Λ_{Eis} est de rang au plus un. Pour montrer qu'il est non-trivial et déterminer sa classe de commensurabilité, il suffit de calculer une période non-nulle de ω_{Eis}. Pour cela, on fixe $Y_1, \ldots, Y_n \in \mathbb{R}_{>0}$ et l'on considère les droites horizontales $L_j \subset \mathcal{H}_j$ formées des z_j dont la partie imaginaire est égale à Y_j. La région $R_\infty := L_1 \times \cdots \times L_n$ est préservée par le sous-groupe des translations $\Gamma_\infty \subset \Gamma$. Soit D_∞ un domaine fondamental compact pour cette action. Son image dans X est un cycle fermé de dimension n. La forme différentielle ω_{Eis} peut s'intégrer terme à terme sur D_∞ à partir de la formule (7). Seul le terme constant dans la définition de E_2 apporte une contribution non-nulle à l'intégrale, puisque les autres termes sont des multiples de caractères non-triviaux de R_∞/Γ_∞. Comme le volume de R_∞/Γ_∞ est égal à $\sqrt{d_F}$, on en déduit que

$$\int_{D_\infty} \omega_{\mathrm{Eis}} = (2i\pi)^n\zeta_F(-1).$$

Ceci achève la démonstration, puisque $\zeta_F(-1)$ appartient à $\mathbb{Q}^\times$. □

1.2. ***Extensions quadratiques et séries L associées.*** Soit K une extension quadratique de F. Pour chaque $1 \le j \le n$, la $\mathbb{R}$-algèbre $K \otimes_{F,v_j} \mathbb{R}$ est isomorphe soit à $\mathbb{C}$, soit à $\mathbb{R}\oplus\mathbb{R}$. On fixe une telle identification, que l'on appelle aussi v_j par abus de notation. Lorsque K est un corps CM, la donnée de $(v_1, \ldots, v_n)$ correspond au choix d'un type CM associé à K. On sait à quel point cette donnée supplémentaire joue un rôle important dans la théorie de la multiplication complexe pour les extensions CM de F.

On munit les $\mathbb{R}$-espaces vectoriels $\mathbb{C}$ et $\mathbb{R}\oplus\mathbb{R}$ de l'orientation standard dans laquelle une orientation positive est assignée aux bases $(1, i)$ et $((1, 0), (0, 1))$ de $\mathbb{C}$ et $\mathbb{R}\oplus\mathbb{R}$ respectivement. Une base de K (vu comme espace vectoriel sur F de dimension 2) est alors dite *positive* si ses images dans $\mathbb{C}$ et $\mathbb{R}\oplus\mathbb{R}$ par les plongements v_j sont orientées positivement. On remarque en particulier que la base $(1, \tau)$ de K sur F est positive si et seulement si :

1. on a $\tau'_j > \tau_j$ pour toute place réelle v_j de F ;
2. la partie imaginaire de τ_j est strictement positive pour toute place complexe v_j.

Soit I un idéal de l'anneau $\mathcal{O}_F$. On se permet de désigner par le même symbole l'idéal $I\mathcal{O}_K$ de K.

On maintiendra tout au long de cet article l'hypothèse que F a pour nombre de classes 1 au sens étroit. Par contre, il est souhaitable de ne pas avoir à faire d'hypothèse semblable sur le corps K. On généralise la définition (2) du groupe $\mathscr{G}_I$ de l'introduction, en le définissant comme un quotient approprié du groupe $\mathbf{A}_K^\times$ des idèles de K. Pour chaque place non-archimédienne v de K, on appelle $\mathbb{O}_v$ l'anneau des entiers du corps local K_v, et l'on pose

$$\mathscr{G}_I := \mathbf{A}_K^\times / \big(\prod_v U_v\big) K^\times,$$

avec

$$U_v = \begin{cases} \mathbb{R}_{>0}^\times & \text{si } v \text{ est réelle;} \\ \mathbb{C}^\times & \text{si } v \text{ est complexe;} \\ \mathbb{O}_v^\times & \text{si } v \nmid I; \\ 1 + I\mathbb{O}_v & \text{si } v \mid I. \end{cases}$$

Comme dans l'introduction, la loi de réciprocité du corps de classes donne un isomorphisme $\mathrm{rec} : \mathscr{G}_I \to \mathrm{Gal}(H/K)$, où H est le corps de classes de rayon de K au sens restreint associé à I.

Le sous-corps F permet d'introduire un sous-groupe $\mathscr{G}_I^+ \subset \mathscr{G}_I$, défini comme l'image naturelle dans $\mathscr{G}_I$ du groupe $\mathbf{A}_F^\times$. Le sous-corps H_I de H fixé par $\mathrm{rec}(\mathscr{G}_I^+)$ s'appelle le *corps de classes d'anneau* associé à I et K/F. On a donc l'isomorphisme de réciprocité

$$\mathrm{rec} : G_I \to \mathrm{Gal}(H_I/K), \quad \text{où } G_I := \mathbf{A}_K^\times / \big(\mathbf{A}_F^\times \prod_v U_v K^\times\big).$$

Un $\mathbb{O}_F$-ordre de K est un sous-anneau de K qui contient $\mathbb{O}_F$ et qui est localement libre de rang 2 sur $\mathbb{O}_F$ (donc libre, puisque $h(F) = 1$). On désigne par

$$\mathbb{O}_I := \mathbb{O}_F + I\mathbb{O}_K$$

l'ordre de K de conducteur I, et l'on appelle

$$\hat{\mathbb{O}}_I = \prod_{v \nmid \infty} \mathbb{O}_I \otimes \mathbb{O}_{F,v}$$

son adélisation. On a alors

$$G_I = \mathbf{A}_K^\times / \big(\hat{\mathbb{O}}_I^\times \prod_{v \mid \infty} U_v K^\times\big).$$

Ce quotient est en bijection naturelle avec le groupe $\mathrm{Pic}^+(\mathbb{O}_I)$ des modules projectifs de rang 1 sur $\mathbb{O}_I$ dans K, modulo l'équivalence au sens restreint. (Deux modules projectifs M_1 et M_2 sur $\mathbb{O}_I$ sont dits équivalents au sens restreint s'il existe un élément totalement positif $k \in K_+^\times$ tel que $M_2 = kM_1$.) On associe en effet à tout $\alpha \in G_I$ un $\mathbb{O}_I$-module $M \subset K$ en posant

$$M^\alpha := \alpha \hat{\mathbb{O}}_I \cap K.$$

L'application $\alpha \mapsto M^\alpha$ fournit une bijection naturelle :

$$G_I \xrightarrow{\sim} \left\{ \begin{array}{c} \text{classes d'équivalence} \\ \text{au sens restreint de} \\ \mathbb{O}_I\text{-modules projectifs} \end{array} \right\}. \tag{19}$$

Soit $V := \mathbb{O}_{I,1}^\times \subset \mathbb{O}_I^\times$ le groupe des unités de $\mathbb{O}_I$ de norme 1 sur $\mathbb{Q}$. Il laisse stable le module M. On a la suite exacte

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow \mathbb{O}_{F,1}^\times,$$

où V_1 désigne le sous-groupe des unités de V de norme relative 1 sur F. On note alors $\tilde{V}$ le sous-groupe de V engendré par V_1 et $\mathbb{O}_F^\times$, et l'on pose

$$\delta_I := [V : \tilde{V}]. \tag{20}$$

Cet indice est un diviseur de 2^n. On définit finalement la fonction $L(M,s)$ associée à un $\mathbb{O}_I$-module projectif M en posant

$$L(M,s) := (\mathrm{N}M)^s \delta_I \sum_{x \in M/V}{}' \operatorname{signe}(N_{K/\mathbb{Q}}(x)) |\mathrm{N}_{K/\mathbb{Q}}(x)|^{-s}. \tag{21}$$

Remarque 1.3. (a) Parce que $\operatorname{signe}(N_{K/\mathbb{Q}}(x)) = 1$ quand K est quadratique imaginaire, il en résulte que la définition (21) généralise l'équation (3) de l'introduction.

(b) Quand K est une extension ATR de F ayant v_1 pour unique place complexe, la fonction $L(M,s)$ est un multiple rationnel non-nul de la somme de fonctions L partielles de Hurwitz de l'introduction. Plus précisément, si a est un générateur de $\mathbb{O}_K/(\mathbb{O}_F, I)$ en tant que $(\mathbb{O}_F/I)$-module, et que M_a désigne le $\mathbb{O}_I$-module projectif

$$M_a := \{x \in \mathbb{O}_K \text{ tel qu'il existe } r \in \mathbb{O}_F \text{ avec } x \equiv ra \pmod{I}\},$$

alors

$$L(M_a, s) = \delta_I \sum_{r \in \mathbb{O}_F/I\mathbb{O}_F} L(ra, I, s). \tag{22}$$

On vérifie que la fonction $L(M,s)$ ne dépend que de la classe d'équivalence de M au sens restreint, puisque $L(\lambda M, s) = \operatorname{signe}(N_{K/\mathbb{Q}}(\lambda)) L(M,s)$.

Dans les sections 2 et 3 qui suivent, nous allons exprimer les valeurs spéciales $L(M,0)$ quand K est totalement réel, et les dérivées $L'(M,0)$ quand K est ATR, en fonction de périodes appropriées de la forme différentielle ω_{Eis}.

2. Extensions quadratiques totalement réelles et valeurs de fonctions L

On supposera dans cette section que l'extension quadratique K de F est totalement réelle. On veut rappeler un théorème qui apparaît dans la thèse du premier auteur et qui donne une formule explicite pour $L(M,0)$ dans ce cas.

L'hypothèse que F a nombre de classes 1 au sens restreint implique que le module M est libre de rang 2 comme module sur $\mathcal{O}_F$, et qu'il existe une $\mathcal{O}_F$-base (ω_1, ω_2) de M. On suppose que cette base est choisie de sorte que $(1, \omega_2/\omega_1)$ soit orientée positivement. L'invariant $\tau := \omega_2/\omega_1$ appartient à $K^\times$, et il ne dépend que de la classe d'équivalence de M au sens restreint, à l'action de Γ près. Le groupe $\Gamma_\tau \subset \Gamma$ formé des matrices qui fixent τ est un groupe de rang n (modulo torsion), que l'application

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c\tau + d$$

identifie avec le sous-groupe V_1 des unités de V de norme relative 1 sur F. Pour chaque $1 \leq j \leq n$, on pose

$$(\tau_j, \tau_j') := v_j(\tau) \in \mathbb{R} \times \mathbb{R},$$

et l'on appelle Υ_j la géodésique hyperbolique sur $\mathcal{H}_j$ joignant τ_j' à τ_j, orientée dans le sens allant de τ_j' à τ_j. Le produit

$$R_\tau = \Upsilon_1 \times \Upsilon_2 \times \cdots \times \Upsilon_n \subset \mathcal{H}^n$$

est un espace contractile homéomorphe à $\mathbb{R}^n$. On le munit de l'orientation naturelle héritée des Υ_j. Le groupe Γ_τ opère sur R_τ par transformations de Möbius, et le quotient $\Gamma_\tau \backslash R_\tau$ est compact, isomorphe à un tore réel de dimension n. Soit Δ_τ un domaine fondamental pour l'action de Γ_τ sur R_τ. On identifie Δ_τ avec son image dans X, qui est un cycle fermé dans ce quotient.

Théorème 2.1. *Pour tout $\mathcal{O}_I$-module projectif M dans K, on a :*

$$(-2)^n \int_{\Delta_\tau} \omega_{\mathrm{Eis}} = (2i\pi)^n L(M, 0). \tag{23}$$

Démonstration. C'est une conséquence du corollaire 7.2 qui est démontré dans la dernière partie de cet article. Puisque K est une extension quadratique totalement réelle de F, on choisit $r = n \geq 2$ et $c = 0$ dans la formule (45). Elle s'écrit alors

$$L(M, 0) = \frac{i^n}{\pi^n} \int_{\Delta_\tau} \frac{\partial^n \tilde{h}(z_1, \ldots, z_n)}{\partial z_1 \cdots \partial z_n} dz_1 \cdots dz_n. \tag{24}$$

D'après l'identité (13), on en déduit que

$$L(M, 0) = \frac{i^n}{\pi^n} \int_{\Delta_\tau} \frac{(2i\pi)^n}{\sqrt{d_F}} \omega_{E_2}. \tag{25}$$

La formule (23) en résulte immédiatement lorsque $n > 2$ vu la définition (8) de ω_{Eis}. Cette formule reste encore valable pour $n = 2$ puisque les intégrales des formes $dz_1 \wedge d\bar{z}_1/y_1^2$ et $dz_2 \wedge d\bar{z}_2/y_2^2$ sur le cycle Δ_τ sont nulles. $\square$

Corollaire 2.2. *Pour tout réseau M dans K, les valeurs spéciales $L(M,0)$ sont rationnelles. Plus précisément, il existe une constante entière e_F, ne dépendant que du corps F et pas de l'extension K ni de M, telle que $e_F L(M,0) \in \mathbb{Z}$.*

Démonstration. Cela résulte de ce que les périodes de ω_{Eis}, d'après la proposition 1.2, appartiennent à un réseau $\Lambda_{\mathrm{Eis}} \subset (2i\pi)^n \mathbb{Q}$ qui ne dépend que du corps F. □

3. Extensions quadratiques ATR et dérivées de fonctions *L*

On suppose dans cette section que l'extension quadratique K de F est ATR, et que v_1 se prolonge en une place complexe de K. On veut donner dans ce cas une formule explicite pour $L'(M,0)$, lorsque M est un $\mathcal{O}_I$-module projectif dans K.

Comme dans la section précédente, on pose $\tau := \omega_2/\omega_1$, où (ω_1, ω_2) est une $\mathcal{O}_F$-base positive de M, choisie de sorte que $(1, \tau)$ soit orientée positivement. On pose ensuite

$$\begin{cases} \tau_1 := v_1(\tau) & \in \mathcal{H}_1 \\ (\tau_j, \tau_j') := v_j(\tau) & \in \mathbb{R} \times \mathbb{R}, \quad \text{pour } j = 2, \dots, n. \end{cases}$$

Le nombre complexe τ_1 appartient alors à $\mathcal{H}_1$. Pour chaque $2 \le j \le n$, on appelle Υ_j la géodésique hyperbolique de $\mathcal{H}_j$ joignant τ_j à τ_j', orientée dans le sens allant de τ_j à τ_j'. Le produit

$$R_\tau = \{\tau_1\} \times \Upsilon_2 \times \cdots \times \Upsilon_n \subset \mathcal{H}^n$$

est un espace contractile homéomorphe à $\mathbb{R}^{n-1}$, que l'on munit de l'orientation naturelle héritée des Υ_j. Le stabilisateur Γ_τ de τ dans Γ est un groupe de rang $n-1$ modulo torsion, que l'on peut identifier avec le sous-groupe d'unités relatives V_1 introduit précédemment. Il opère sur R_τ par transformations de Möbius, et le quotient $\Gamma_\tau \backslash R_\tau$ est compact, isomorphe à un tore réel de dimension $(n-1)$. Soit Δ_τ un domaine fondamental pour l'action de Γ_τ sur R_τ. On identifie Δ_τ avec son image dans X, qui est un cycle fermé de dimension $n-1$ dans ce quotient.

Lemme 3.1. *La classe de Δ_τ dans $H_{n-1}(X,\mathbb{Z})$ est de torsion. En particulier, il existe une n-chaîne différentiable C_τ à coefficients dans $\mathbb{Q}$ telle que*

$$\partial C_\tau = \Delta_\tau. \tag{26}$$

Démonstration. Le sous-groupe de torsion de $H_{n-1}(X,\mathbb{Z})$ s'identifie avec le noyau de l'application naturelle $H_{n-1}(X,\mathbb{Z}) \to H_{n-1}(X,\mathbb{Q})$. Lorsque n est pair, le lemme 3.1 résulte de ce que $H_{n-1}(X,\mathbb{Q}) = 0$ [Freitag 1990, chap. III]. De même, lorsque $n = 2m+1$ est impair, le groupe $H^{n-1}(X,\mathbb{C})$ est engendré par les classes des $\binom{n}{m}$

formes différentielles du type

$$\eta_S := \bigwedge_{j\in S} \frac{dz_j \wedge d\bar{z}_j}{y_j^2}, \qquad S \subset \{1,\dots,n\}, \quad \#S = m.$$

Or on voit que les restrictions de ces classes sur Δ_τ (et même sur les régions R_τ) sont nulles, puisque la projection de R_τ sur chaque facteur $\mathcal{H}_j$ est de dimension réelle 0 ou 1. On en déduit par la dualité de Poincaré que l'image de Δ_τ dans $H_{n-1}(X,\mathbb{C})$ est nulle. □

On introduit $\omega_{\mathrm{Eis}}+ = \frac{1}{2}(\mathrm{Id} + T_{v_1}^*)\,\omega_{\mathrm{Eis}}$ la projection de la forme différentielle ω_{Eis} sur l'espace propre de $T_{v_1}^*$ associé à la valeur propre $+1$, autrement dit la «partie réelle pour la place v_1» de la forme ω_{Eis}.

Théorème 3.2. *Soit M un $\mathcal{O}_I$-module projectif associé à $\tau \in K$. L'intégrale de ω_{Eis}^+ sur C_τ ne dépend pas du choix de C_τ vérifiant (26), et l'on a*

$$(-2)^{n-1}\int_{C_\tau} \omega_{\mathrm{Eis}}^+ = (2i\pi)^{n-1} L'(M,0). \tag{27}$$

Démonstration. C'est encore une conséquence du corollaire 7.2 qui est démontré dans la dernière partie de cet article. La première assertion découle du fait que ω_{Eis}^+ est exacte : le lemme 1.1 montre que $\omega_{\mathrm{Eis}}^+ = d\eta/2$. Le calcul se poursuit en utilisant le théorème de Stokes pour obtenir

$$\int_{C_\tau} \omega_{\mathrm{Eis}}^+ = \int_{\Delta_\tau} \frac{\eta}{2}. \tag{28}$$

Supposons tout d'abord que $n > 2$, de sorte que η est la $(n-1)$-forme holomorphe sur X donnée par la formule (15). Comme K est une extension ATR, on choisit ici $r = n-1 \geq 2$ et $c = 1$ dans le corollaire 7.2.i. La formule correspondante s'écrit

$$L'(M,0) = \frac{i^{n-1}}{2\pi^{n-1}} \int_{\Delta_\tau} \eta, \tag{29}$$

ce qui permet de conclure.

Il reste à traiter le cas où $n = 2$ en faisant cette fois appel au corollaire 7.2.ii. On choisit $r = 1$ et $c = 1$ dans la formule (46) qui s'écrit

$$L'(M,0) = \frac{i}{2\pi}\int_{\Delta_\tau} \left(\frac{\partial \tilde{h}}{\partial z_2} - \frac{4R_F}{z_2 - \bar{z}_2} \right) dz_2.$$

Au vu de la définition (17) de la forme η pour $n = 2$, cette égalité se réduit à

$$\int_{\Delta_\tau} -\eta = (2i\pi) L'(M,0).$$

La formule de Stokes permet de nouveau de conclure à la formule souhaitée. □

4. Application d'Abel–Jacobi et unités de Stark

Quand on combine le théorème 3.2 avec la conjecture 1 de Stark, on obtient la formule conjecturale suivante pour le logarithme du module de l'unité de Stark :

$$e_I(-2)^{n-1}\int_{C_\tau}\omega_{\mathrm{Eis}}+ = 2\delta_I(2i\pi)^{n-1}\log|\tilde{v}_1(u_\tau)|, \tag{30}$$

où e_I désigne l'ordre du groupe des racines de l'unité dans H_I, et l'entier δ_I est défini dans l'équation (20). Pour relever l'invariant réel $\log|\tilde{v}_1(u_\tau)| = \mathrm{Re}\log\tilde{v}_1(u_\tau)$ en un invariant complexe bien défini modulo $2i\pi\mathbb{Z}$, il suffira de remplacer dans la formule (30) la différentielle exacte ω_{Eis}^+ par la forme différentielle ω_{Eis}.

Pour tout $m \geq 0$, on désigne par $C_m(X)$ le groupe engendré par les combinaisons linéaires formelles à coefficients dans $\mathbb{Z}$ des chaînes différentiables de dimension réelle m sur X, et l'on désigne par $C_m^0(X)$ et $C_m^{00}(X)$ les sous-groupes engendrés par les cycles différentiables fermés et homologues à zéro respectivement :

$$C_m^0(X) := \{\Delta \in C_m(X) \text{ tel que } \partial\Delta = 0\}.$$
$$C_m^{00}(X) := \{\Delta \in C_m(X) \text{ tel qu'il existe } C \in C_{m+1}(X) \text{ avec } \partial C = \Delta\}.$$

On pose aussi

$$\tilde{C}_m^{00}(X) := \{\Delta \in C_m(X) \text{ tel qu'il existe } C \in C_{m+1}(X)\otimes\mathbb{Q} \text{ avec } \partial C = \Delta\}.$$

On sait que $C_{n-1}^0(X)/C_{n-1}^{00}(X) = H_{n-1}(X,\mathbb{Z})$ est un groupe de type fini, dont le sous-groupe de torsion s'identifie avec $\tilde{C}_{n-1}^{00}(X)/C_{n-1}^{00}(X)$. Soit n_F l'exposant de ce groupe fini, et soit

$$\Lambda'_{\mathrm{Eis}} := \frac{1}{n_F}\Lambda_{\mathrm{Eis}}.$$

On peut définir à partir de la forme différentielle ω_{Eis} une «application d'Abel–Jacobi»

$$\Phi_{\mathrm{Eis}} : C_{n-1}^{00}(X) \to \mathbb{C}/\Lambda_{\mathrm{Eis}}$$

en posant

$$\Phi_{\mathrm{Eis}}(\Delta) = \int_{\partial C=\Delta}\omega_{\mathrm{Eis}} \pmod{\Lambda_{\mathrm{Eis}}},$$

l'intégrale étant prise sur n'importe quelle n-chaîne différentiable C sur X tel que $\partial C = \Delta$. Cette application Φ_{Eis} est bien définie modulo le réseau des périodes Λ_{Eis} en vertu de la proposition 1.2. Quitte à remplacer le réseau Λ_{Eis} par Λ'_{Eis}, on peut étendre Φ_{Eis} au groupe $\tilde{C}_{n-1}^{00}(X)$ tout entier, en posant

$$\Phi_{\mathrm{Eis}}(\Delta) = \frac{1}{n_F}\int_{\partial C=n_F\Delta}\omega_{\mathrm{Eis}} \pmod{\Lambda'_{\mathrm{Eis}}}, \tag{31}$$

l'intégrale étant prise sur n'importe quelle n-chaîne différentiable C sur X tel que $\partial C = n_F \Delta$. On pose ensuite

$$J_\tau := e_I(-2)^{n-1}\Phi_{\mathrm{Eis}}(\Delta_\tau),$$

élément de $\mathbb{C}/\Lambda'_{\mathrm{Eis}}$.

Soit Λ''_{Eis} le réseau de $(2i\pi)^n\mathbb{R}$ engendré par Λ'_{Eis} et $(2i\pi)^n\mathbb{Z}$. On fixe une place $\tilde{v}_1$ de H_I au-dessus de la place v_1 de K. Nous sommes maintenant en mesure d'énoncer la conjecture principale de cet article.

Conjecture 4.1. *Pour tout $\mathbb{O}_I$-module M associé à $\tau \in K$, il existe une unité $u_\tau \in \mathbb{O}_{H_I}^\times$ telle que*

$$J_\tau = 2\delta_I(2i\pi)^{n-1}\log(\tilde{v}_1(u_\tau)) \pmod{\Lambda''_{\mathrm{Eis}}}.$$

De plus, pour tout $2 \le j \le n$, l'image de u_τ par n'importe quel plongement complexe au-dessus de v_j est de module 1. *Pour tout $\alpha \in G_I$, on a $u_{\tau^\alpha} = \mathrm{rec}(\alpha)^{-1}u_\tau$, où τ^α désigne l'invariant associé au module M^α.*

5. Algorithmes

L'invariant J_τ et l'application d'Abel–Jacobi Φ_{Eis} ont l'inconvénient de ne pas être faciles à calculer numériquement a priori. Le but de la présente section est de décrire un algorithme pour le calcul de Φ_{Eis} dans le cas le plus simple où $n = 2$.

La première étape consiste à décrire la classe de cohomologie de ω_{Eis} en termes de cohomologie du groupe Γ.

On rappelle le dictionnaire bien connu entre la cohomologie de De Rham de X et la cohomologie de Γ. Si P, Q, R sont des points de $\mathcal{H}_1 \times \mathcal{H}_2 = \mathcal{H}^2$, on appelle $\Delta(P, Q, R)$ n'importe quelle 2-chaîne différentiable dont la frontière est égale au triangle géodésique de sommets P, Q et R. On munit cette région de l'orientation standard, selon les définitions usuelles de l'homologie singulière. On pose aussi, pour $P = (z_1, z_2) \in \mathcal{H}^2$ et $A, B \in \Gamma$,

$$\Delta_P(A, B) := \Delta(P, AP, ABP).$$

On associe à ω_{Eis} (plus précisément : à sa classe de cohomologie) un 2-cocycle

$$\kappa_P \in Z^2(\Gamma, \mathbb{C})$$

par la règle

$$\kappa_P(A, B) := \int_{\Delta_P(A,B)} \omega_{\mathrm{Eis}}.$$

Un calcul direct montre que κ_P satisfait la relation de 2-cocycle : $d\kappa_P = 0$, et que son image dans $H^2(\Gamma, \mathbb{C})$ ne dépend pas du choix du point base P.

On rappelle le réseau $\Lambda_{\mathrm{Eis}} \subset \mathbb{C}$ des périodes de ω_{Eis} et l'on note $\bar{\kappa}_P$ l'image de κ_P dans $Z^2(\Gamma, \mathbb{C}/\Lambda'_{\mathrm{Eis}})$.

Lemme 5.1. *La classe de $\bar{\kappa}_P$ dans $H^2(\Gamma, \mathbb{C}/\Lambda'_{\mathrm{Eis}})$ est nulle.*

Démonstration. Pour tout $A \in \Gamma$, on appelle $S_P(A)$ l'image dans X du chemin géodésique sur $\mathcal{H}^2$ allant de P à AP. Comme $S_P(A)$ est un 1-cycle fermé sur X et que $H_1(X, \mathbb{Q}) = 0$, il existe une 2-chaîne différentiable sur X à coefficients entiers, que l'on appellera $D_P(A)$, telle que

$$\partial D_P(A) = n_F S_P(A). \tag{32}$$

La région $D_P(A)$ est déterminée par cette équation modulo les 2-cycles fermés, et par conséquent l'élément de $\mathbb{C}/\Lambda'_{\mathrm{Eis}}$ défini par

$$\rho_P(A) := \frac{1}{n_F} \int_{D_P(A)} \omega_{\mathrm{Eis}} \pmod{\Lambda'_{\mathrm{Eis}}} \tag{33}$$

ne dépend pas du choix de $D_P(A)$ satisfaisant (32). On vérifie ensuite par un calcul direct que

$$d\rho_P(A, B) = \kappa_P(A, B) \pmod{\Lambda'_{\mathrm{Eis}}}. \qquad \square$$

Le lemme 5.1 permet de définir une 1-chaîne ρ_P en choisissant une solution de l'équation

$$d\rho_P = \kappa_P \pmod{\Lambda'_{\mathrm{Eis}}}. \tag{34}$$

Soit K un corps ATR et soit $\tau \in K$ un élément provenant d'une base positive d'un réseau $M \subset K$. Parce que $n = 2$, le groupe Γ_τ est de rang un modulo la torsion. On se donne un générateur γ_τ de Γ_τ modulo torsion, choisi de sorte que pour tout point z_2 de la géodésique Υ_2, le chemin allant de z_2 à $\gamma_\tau z_2$ soit orienté dans le sens positif. On choisit le point base $P \in \mathcal{H}_1 \times \mathcal{H}_2$ de manière à ce que sa première composante soit égale à $\tau_1 = v_1(\tau)$.

La proposition suivante, qui résulte directement de la formule pour ρ_P de l'équation (33), permet de calculer l'invariant numérique $\Phi_{\mathrm{Eis}}(\Delta_\tau)$ en termes de cohomologie des groupes — du moins, en admettant que l'on sache résoudre (34).

Proposition 5.2. $$\Phi_{\mathrm{Eis}}(\Delta_\tau) = \rho_P(\gamma_\tau).$$

La définition du 2-cocycle κ_P exige d'intégrer ω_{Eis} sur des régions de type $\Delta_P(A, B)$ peu commodes à paramétrer. Dans les calculs numériques, il est donc utile de remplacer ce cocycle par un représentant de la même classe de cohomologie qui ne fait intervenir que des régions rectangulaires de la forme $L_1 \times L_2 \subset \mathcal{H}_1 \times \mathcal{H}_2$ (avec L_1 et L_2 de dimension 1, bien entendu). Les intégrales de ω_{Eis} sur de telles régions s'expriment au moyen d'intégrales itérées, et sont donc plus faciles à calculer numériquement. (On se sert pour cela du développement de Fourier de ω_{Eis}.)

Si u, v appartiennent à $\mathcal{H}$, soit $\Upsilon[u, v] \subset \mathcal{H}$ le segment géodésique joignant le point u au point v. On pose

$$\square_P(A, B) = \Upsilon[z_1, A_1 z_1] \times \Upsilon[A_2 z_2, A_2 B_2 z_2],$$

et l'on définit un nouveau cocycle $\kappa_P^{\square} \in Z^2(\Gamma, \mathbb{C})$ par la règle

$$\kappa_P^{\square}(A, B) = \int_{\square_P(A,B)} \omega_{\mathrm{Eis}}.$$

Il est nécessaire de modifier légèrement $\kappa_P^{\square}$ pour qu'il représente la même classe de cohomologie que κ_P. On dispose pour cela d'un 2-cocycle classique sur $\mathbf{SL}_2(\mathbb{R})$ appelé *cocycle d'aire*, dont on rappelle la définition : étant données deux matrices $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ et $N = \begin{pmatrix} * & * \\ c' & d' \end{pmatrix}$ de $\mathbf{SL}_2(\mathbb{R})$, avec leur produit $MN = \begin{pmatrix} * & * \\ c'' & d'' \end{pmatrix}$, la formule

$$\mathrm{aire}(M,\ N) := -\mathrm{signe}(c\, c'\, c'') \tag{35}$$

(où $\mathrm{signe}(x) = x/|x|$ si $x \neq 0$, et 0 sinon) définit un 2-cocycle à valeurs entières. Par composition avec les plongements de Γ dans $\mathbf{SL}_2(\mathbb{R})$, on en déduit deux 2-cocycles sur Γ à valeurs dans $\mathbb{Z}$. On définit finalement le 2-cocycle $\tilde{\kappa}_P$ sur Γ par la formule

$$\begin{aligned}\tilde{\kappa}_P(A, B) &:= \kappa_P^{\square}(A, B) - i\pi\, R_F\, \mathrm{aire}(A_1, B_1) + i\pi\, R_F\, \mathrm{aire}(A_2, B_2)\\ &= \int_{z_1}^{A_1 z_1}\int_{A_2 z_2}^{A_2 B_2 z_2} \omega_{\mathrm{Eis}} - i\pi\, R_F\, \mathrm{aire}(A_1, B_1) + i\pi\, R_F\, \mathrm{aire}(A_2, B_2).\end{aligned}$$

Proposition 5.3. *Les cocycles κ_P et $\tilde{\kappa}_P$ représentent la même classe de cohomologie dans $H^2(\Gamma, \mathbb{C})$. Plus précisément, on a*

$$\kappa_P(A, B) - \tilde{\kappa}_P(A, B) = d\xi_P(A, B),$$

où

$$\xi_P(A) = -\int_{\Delta_P(A)} \omega_{\mathrm{Eis}}, \qquad \textit{avec}\ \ \Delta_P(A) = \Delta((z_1, z_2), (z_1, A_2 z_2), (A_1 z_1, A_2 z_2)).$$

Démonstration. On dit que deux 2-chaînes Z_1 et Z_2 sont homologues si leurs frontières sont égales, et on écrit dans ce cas $Z_1 \sim Z_2$. Un calcul direct fournit la relation

$$-\square_P(A, B) + \Delta_P(A, B) + \Delta_P(A) - \Delta_P(AB) \sim \Delta_1 + \Delta_2 - \Delta_3, \tag{36}$$

avec

$$\begin{cases} \Delta_1 = \Delta((A_1 B_1 z_1, A_2 B_2 z_2), (z_1, A_2 B_2 z_2), (A_1 z_1, A_2 B_2 z_2)),\\ \Delta_2 = \Delta((z_1, z_2), (z_1, A_2 z_2), (z_1, A_2 B_2 z_2)),\\ \Delta_3 = \Delta((A_1 z_1, A_2 z_2), (A_1 z_1, A_2 B_2 z_2), (A_1 B_1 z_1, A_2 B_2 z_2)).\end{cases}$$

Par A-invariance, on observe que $\Delta_3 = \Delta_P(B)$ dans X. En outre, l'intégrale de ω_{E_2} sur Δ_1 et Δ_2 est nulle, et par conséquent

$$\int_{\Delta_1+\Delta_2} \omega_{\mathrm{Eis}} = \frac{R_F}{2}\int_{\Delta_1} \frac{dz_1 \wedge d\bar{z}_1}{y_1^2} - \frac{R_F}{2}\int_{\Delta_2} \frac{dz_2 \wedge d\bar{z}_2}{y_2^2}.$$

Ces dernières intégrales se calculent élémentairement : on constate d'abord qu'elles ne dépendent pas du point base $P = (z_1, z_2)$, et que $dz_j \wedge d\bar{z}_j = -2i dx_j \wedge dy_j$. Or l'intégrale

$$\int_{\Delta_j} \frac{dx_j \wedge dy_j}{y_j^2}$$

n'est rien d'autre que l'aire, dans le disque de Poincaré, du triangle idéal orienté de sommets ∞, $A_j\infty$ et $A_jB_j\infty$. D'après [Kirby et Melvin 1994] formule 1.2, il en résulte que

$$\int_{\Delta_1} \frac{dz_1 \wedge d\bar{z}_1}{y_1^2} = -2i\pi \ \mathrm{aire}(A_1, B_1) \quad \text{et} \quad \int_{\Delta_2} \frac{dz_2 \wedge d\bar{z}_2}{y_2^2} = -2i\pi \ \mathrm{aire}(A_2, B_2).$$

On conclut alors de (36) que

$$\kappa_P(A,B) = \kappa_P^{\square}(A,B) + d\xi_P(A,B) - i\pi R_F \,\mathrm{aire}(A_1,B_1) + i\pi R_F \,\mathrm{aire}(A_2,B_2),$$

d'où la proposition. □

Corollaire 5.4. *Soit $\tilde{\rho}_P$ une solution de l'équation*

$$d\tilde{\rho}_P = \tilde{\kappa}_P \pmod{\Lambda'_{\mathrm{Eis}}}. \tag{37}$$

Alors on a $\Phi_{\mathrm{Eis}}(\Delta_\tau) = \tilde{\rho}_P(\gamma_\tau)$.

Démonstration. La proposition 5.3 montre que l'on peut choisir

$$\tilde{\rho}_P(A) = \rho_P(A) - \xi_P(A) \pmod{\Lambda'_{\mathrm{Eis}}},$$

où la 1-cochaîne ξ_P est définie dans l'énoncé de cette proposition. Comme la région $\Delta_P(\gamma_\tau)$ qui intervient dans la formule pour $\xi_P(\gamma_\tau)$ est contenue dans le domaine $\{\tau_1\} \times \Upsilon[z_2, \gamma_\tau z_2]$, on a

$$\xi_P(\gamma_\tau) = 0.$$

Le corollaire en résulte. □

Remarque 5.5. Dans le présent article, le cocycle $\tilde{\kappa}_P$ n'intervient que dans les algorithmes pour calculer J_τ numériquement. Signalons tout de même que la proposition 5.3 et le corollaire 5.4 sont d'un intérêt plus que pratique. Dans le contexte partiellement p-adique étudié dans [Darmon 2001] et [Darmon et Dasgupta 2006] où l'on est amené à travailler avec des formes modulaires sur $\mathcal{H}_p \times \mathcal{H}$, on ignore comment donner un sens aux régions de la forme $\Delta_P(A,B)$, ou au cocycle κ_P. Par contre, on sait définir ce qui doit jouer le rôle des intégrales «itérées» de formes modulaires (cuspidales ou Eisenstein) sur des régions «rectangulaires» de la forme $\square_P(A,B)$. Cela permet de définir un avatar p-adique de $\tilde{\kappa}_P$, et par conséquent des versions p-adiques des invariants J_τ du présent article.

Il reste finalement à calculer une solution de l'équation (34) ou (37). Le procédé étant le même qu'il s'agisse de ρ_P ou de $\tilde{\rho}_P$, on se bornera au cas de ρ_P pour alléger les notations.

L'algorithme que nous proposons pour calculer $\rho_P(\gamma)$, pour γ n'importe quel élément de Γ, se base sur l'observation suivante : lorsque $\gamma = hkh^{-1}k^{-1}$ est un commutateur dans Γ, la formule (34) permet d'exprimer $\rho_P(\gamma)$ directement en fonction de κ_P. En effet, l'identité facile $\rho_P(\mathrm{Id}) = 0$ assure que

$$\rho_P(h) + \rho_P(h^{-1}) = \kappa_P(h, h^{-1}).$$

En reportant, on en conclut que

$$\begin{aligned}\rho_P(\gamma) = -\kappa_P(h, kh^{-1}k^{-1}) - \kappa_P(k, h^{-1}k^{-1}) - \kappa_P(h^{-1}, k^{-1}) \\ + \kappa_P(h, h^{-1}) + \kappa_P(k, k^{-1}) \pmod{\Lambda'_{\mathrm{Eis}}}.\end{aligned}$$

Cette formule, avec ρ_P et κ_P remplacés par $\tilde{\rho}_P$ et $\tilde{\kappa}_P$ respectivement, donne un accès numérique à $\tilde{\rho}_P(hkh^{-1}k^{-1})$ puisque les nombres complexes $\tilde{\kappa}_P(g, g')$ se calculent grâce au développement en série de Fourier de la série d'Eisenstein.

Enfin, l'abélianisé Γ_{ab} de Γ est fini (voir [Darmon et Logan 2003, prop. 1.3]). Son ordre divise $4N_{F/\mathbb{Q}}(\epsilon^2 - 1)$, où ϵ désigne l'unité fondamentale de F. Pour calculer $\rho_P(\gamma)$ pour une matrice γ de Γ, il suffit donc de décomposer $\gamma^{|\Gamma_{\mathrm{ab}}|}$ en un produit de commutateurs.

Sous l'hypothèse que F est de nombre de classes 1, on peut procéder comme suit. L'anneau des entiers $\mathbb{O}_F$ est euclidien en k-étapes pour la norme selon la terminologie de [Cooke 1976, théorème 1]. Par conséquent, le groupe modulaire de Hilbert Γ est engendré par les matrices élémentaires de type suivant :

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T_\theta = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}.$$

Ainsi, $\gamma^{|\Gamma_{\mathrm{ab}}|}$ s'écrit comme un produit de matrices élémentaires grâce à l'algorithme d'Euclide dans $\mathbb{O}_F$, puis comme un produit de commutateurs à l'aide des relations

$$UT_\theta U^{-1}T_\theta^{-1} = T_{\theta(\epsilon^2-1)}, \qquad SUS^{-1}U^{-1} = U^2.$$

Remarque 5.6. Dans notre contexte «Eisenstein», on ne peut pas utiliser tel quel l'algorithme proposé dans [Darmon et Logan 2003, section 4]. En effet, les intégrales du type $\int^\tau \int_{c_2}^{c_1} \omega_f$ (avec $c_1, c_2 \in \mathbf{P}^1(F)$) n'ont de sens que si f est une forme modulaire de Hilbert cuspidale. Notons cependant que cet algorithme et le nôtre reposent tous deux sur l'hypothèse que $\mathbb{O}_F$ est un anneau euclidien.

6. Exemples numériques

Dans cette partie, nous présentons quelques résultats expérimentaux obtenus grâce à l'algorithme précédent. Il s'agit, pour quelques cas d'extensions ATR K/F, de tester numériquement la conjecture 4.1 de cet article et d'exhiber le polynôme minimal de l'unité attendue.

6.1. ***Corps de base*** $\mathbb{Q}(\sqrt{5})$**.** On considère d'abord la situation où $F = \mathbb{Q}(\sqrt{5})$ et l'on note $\epsilon = \frac{1}{2}(1+\sqrt{5})$ son unité fondamentale de norme -1. L'anneau des entiers $\mathcal{O}_F = \mathbb{Z}[\epsilon]$ est euclidien pour la norme. On fixe les places archimédiennes v_1 et v_2 de F de sorte que $\epsilon_1 < 0$ et $\epsilon_2 > 0$. On supposera dans cette section que $\Lambda''_{\mathrm{Eis}} = \Lambda'_{\mathrm{Eis}}$, et l'on se donne un entier $m_F > 0$ tel que $\Lambda'_{\mathrm{Eis}} \subset (2i\pi)^2 m_F \mathbb{Z}$. Les exemples ci-dessous laissent penser que $m_{\mathbb{Q}(\sqrt{5})} = 15$ convient.

Nous étudions maintenant les invariants associés à différentes extensions quadratiques ATR K de F dans lesquelles la place v_1 devient complexe.

(a) *Un exemple à groupe des classes* C_4. On considère $K = F(\sqrt{21\epsilon - 11})$. C'est une extension ATR de F, dont le groupe des classes au sens restreint est cyclique d'ordre 4. Aux quatre classes distinctes $\mathcal{C}_1, \ldots, \mathcal{C}_4$ de $\mathcal{O}_K$ au sens restreint, on associe les éléments $\tau_1, \ldots, \tau_4$ de K fixés par les matrices suivantes de Γ :

$$\gamma_1 = \begin{pmatrix} 4\epsilon+2 & -2\epsilon-5 \\ -2\epsilon-1 & 2\epsilon+1 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 13\epsilon+9 & 4\epsilon+1 \\ -32\epsilon-18 & -7\epsilon-6 \end{pmatrix},$$
$$\gamma_3 = \begin{pmatrix} -47\epsilon-17 & 9\epsilon-6 \\ -520\epsilon-308 & 53\epsilon+20 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 165\epsilon+79 & 5\epsilon-2 \\ -8512\epsilon-5160 & -159\epsilon-76 \end{pmatrix}.$$

On calcule dans $\mathbb{C}/m_F\mathbb{Z}$ les invariants $\rho_k(\gamma_k) := \tilde{\rho}_{\tau_k}(\gamma_k)/(2i\pi)^2$ associés. On trouve, avec une précision minimale de 50 décimales significatives :

$$\rho_1(\gamma_1) \approx -0.3666666666\ldots - 0.27784944302\ldots i;$$
$$\rho_2(\gamma_2) \approx 0.32623940638\ldots;$$
$$\rho_3(\gamma_3) \approx 1.83333333333\ldots + 0.27784944302\ldots i;$$
$$\rho_4(\gamma_4) \approx 17.8404272602\ldots.$$

Sans connaître la constante m_F, on doit tester l'algébricité du nombre complexe bien défini

$$u_k(m_F) = \exp(2i\pi m_F \rho_k(\gamma_k)).$$

Cependant, dans la pratique, il semble qu'il existe toujours une racine m_F-ième de $u_k(m_F)$ qui appartienne au corps de définition de $u_k(m_F)$. Pour chaque valeur de $\rho_k(\gamma_k)$ dans la liste précédente, notons

$$u_k(1) = \exp(2i\pi\rho_k(\gamma_k)).$$

Le nombre complexe $u_k(1)$ est bien défini seulement modulo les racines m_F-ièmes de l'unité. Quitte à modifier $u_k(1)$ par une racine de l'unité, on peut donc espérer tester avec succès son algébricité. Précisément, la commande Pari

$$\mathrm{algdep}(u_1(1)e^{-\frac{4}{15}i\pi}, 16)$$

suggère la relation algébrique suivante pour $u_1 := u_1(1)e^{-\frac{4}{15}i\pi}$:

$$Q_1(x) := x^8 + 4x^7 - 10x^6 + x^5 + 9x^4 + x^3 - 10x^2 + 4x + 1. \tag{38}$$

On en conclut que le nombre complexe u_1 coïncide sur 50 décimales avec la racine $-5.7303\ldots$ du polynôme Q_1. Il en va de même pour les trois autres invariants :

$u_2 := u_2(1)e^{\frac{23}{15}i\pi}$ coïncide avec la racine $0.834403847893\ldots + 0.5511535345\ldots i$ de Q_1.

$u_3 := u_3(1)e^{\frac{20}{15}i\pi}$ coïncide avec la racine $-0.17450889906\cdots = 1/u_1$.

$u_4 := u_4(1)e^{\frac{2}{15}i\pi}$ coïncide avec la racine $0.834403847893\ldots - 0.5511535345\ldots i$.

On vérifie a posteriori que Q_1 est effectivement le polynôme minimal d'une unité du corps de classes de Hilbert (au sens restreint) de K.

(b) *Un exemple à groupe des classes C_6.* On considère maintenant

$$K = F(\sqrt{26\epsilon - 37}),$$

dont le nombre de classes au sens restreint est 6. On trouve avec parfois 200 décimales de précision dans $\mathbb{C}/m_F\mathbb{Z}$:

$$\begin{aligned}
\rho_1(\gamma_1) &\approx 4.499999999999999\ldots - 0.728584512\ldots i;\\
\rho_2(\gamma_2) &\approx -1.078476376302846\ldots - 0.195385083050863\ldots i;\\
\rho_3(\gamma_3) &\approx -2.178476376302846\ldots + 0.195385083050863\ldots i;\\
\rho_4(\gamma_4) &\approx -18.61666666666666\ldots + 0.728584510266413\ldots i;\\
\rho_5(\gamma_5) &\approx -0.988190290363819\ldots + 0.195385083050863\ldots i;\\
\rho_6(\gamma_6) &\approx -2.421523623697153\ldots - 0.195385083050863\ldots i.
\end{aligned}$$

La commande $\mathrm{algdep}(u_2(1)e^{\frac{16}{15}i\pi}, 12)$ de Pari suggère la relation algébrique :

$$\begin{aligned}
Q_2(x) = x^{12} + 106x^{11} + 873x^{10} - 2636x^9 + 3040x^8 - 626x^7 - 1108x^6 - 626x^5\\
+ 3040x^4 + 2636x^3 + 873x^2 + 106x + 1.
\end{aligned}$$

Ce polynôme est effectivement le polynôme minimal d'une unité de H_K^+. En outre, les nombres complexes

$$u_1 := u_1(1) = -97.30316237461782\ldots,$$

$$
\begin{aligned}
u_2 &:= u_2(1)e^{\frac{16}{15}i\pi} \approx -3.276785825745970\ldots + 0.955188763599790\ldots i,\\
u_3 &:= u_3(1)e^{\frac{19}{15}i\pi} \approx -0.281276149057161\ldots + 0.081992486337387\ldots i,\\
u_4 &:= u_4(1)e^{\frac{5}{15}i\pi} \approx -0.010277158271074\ldots,\\
u_5 &:= u_5(1)e^{\frac{16}{15}i\pi} \approx -0.281276149057161\ldots - 0.081992486337387\ldots i,\\
u_6 &:= u_6(1)e^{\frac{16}{15}i\pi} \approx -3.276785825745970\ldots - 0.955188763599790\ldots i
\end{aligned}
$$

coïncident chacun avec une racine de Q_2 sur plusieurs dizaines de décimales.

(c) *Un exemple à groupe des classes* $C_2 \times C_4$. Le groupe des classes (au sens restreint) de $K = F(\sqrt{21\epsilon - 29})$ est d'ordre 8, isomorphe à $C_2 \times C_4$. L'algorithme décrit précédemment permet de calculer

$$
\begin{aligned}
\rho_1(\gamma_1) &\approx -1.866666666666\ldots - 0.787374943777\ldots i,\\
\rho_2(\gamma_2) &\approx 0.297896510457\ldots + 0.068709821260\ldots i,\\
\rho_3(\gamma_3) &\approx -0.300000000000\ldots + 0.161542382812\ldots i,\\
\rho_4(\gamma_4) &\approx -1.097896510457\ldots + 0.068709821260\ldots i,\\
\rho_5(\gamma_5) &\approx -0.133333333333\ldots + 0.787374943777\ldots i,\\
\rho_6(\gamma_6) &\approx -0.031229843791\ldots - 0.068709821260\ldots i,\\
\rho_7(\gamma_7) &\approx -0.900000000000\ldots - 0.161542382812\ldots i,\\
\rho_8(\gamma_8) &\approx -1.38121\ldots - 0.391304\ldots i.
\end{aligned}
$$

L'invariant le plus précis est $\rho_5(\gamma_5)$ dont on a obtenu 200 décimales significatives. La commande Pari algdep($u_5(1)e^{-\frac{19}{15}i\pi}$, 16, 200) fournit comme candidat le polynôme réciproque

$$
\begin{aligned}
Q_3(x) = x^{16} &+ 139x^{15} - 255x^{14} - 538x^{13} + 2018x^{12} - 2237x^{11}\\
&+ 1898x^{10} - 3034x^9 + 4137x^8 - 3034x^7 + 1898x^6\\
&- 2237x^5 + 2018x^4 - 538x^3 - 255x^2 + 139x + 1.
\end{aligned}
$$

On constate d'abord que Q_3 est en effet le polynôme minimal d'une unité de H_K^+. Par ailleurs, six autres invariants coïncident eux aussi avec des racines de ce polynôme, au moins pour leurs n premières décimales ($10 \leq n \leq 200$ selon les cas) :

$$
\begin{aligned}
u_1 &:= u_1(1)e^{\frac{16}{15}i\pi} \approx -140.7834195600\ldots;\\
u_2 &:= u_2(1)e^{\frac{19}{15}i\pi} \approx 0.589707772431\ldots - 0.271949567981\ldots i;\\
u_3 &:= u_3(1)e^{\frac{24}{15}i\pi} \approx -0.362402166665\ldots;\\
u_4 &:= u_4(1)e^{\frac{5}{15}i\pi} \approx 0.589707772431\ldots + 0.271949567981\ldots i;
\end{aligned}
$$

$$u_5 := u_5(1)e^{\frac{19}{15}i\pi} \approx -0.007103109180\ldots;$$
$$u_6 := u_6(1)e^{\frac{3}{15}i\pi} \approx 1.398366700490\ldots + 0.644870625513\ldots i;$$
$$u_7 := u_7(1)e^{\frac{12}{15}i\pi} \approx -2.759365401151\ldots.$$

La précision avec laquelle $\rho_8(\gamma_8)$ est obtenu se révèle insuffisante pour identifier u_8 avec une des racines de Q_3. Pour des raisons de symétrie, il doit correspondre à la racine

$$1.398366700490\ldots - 0.644870625513\ldots i.$$

6.2. *Corps de base* $\mathbb{Q}(\sqrt{2})$. L'anneau des entiers de $F' = \mathbb{Q}(\sqrt{2})$ est euclidien pour la norme. On note $\epsilon = 1 + \sqrt{2}$ son unité fondamentale de norme -1, et l'on ordonne les plongements de sorte que $\epsilon_1 < 0$ et $\epsilon_2 > 0$. La constante $m_{F'}$ optimale est vraisemblablement $m_{F'} = 6$ dans ce cas.

(a) *Un exemple à groupe des classes C_4.* L'extension ATR $K = F'(\sqrt{12\epsilon - 11})$ possède un groupe des classes au sens restreint cyclique d'ordre 4. Nous associons à chaque classe un invariant dans $\mathbb{C}/m_{F'}\mathbb{Z}$:

$$\rho_1(\gamma_1) \approx -1.333333333333333\ldots - 0.301378336840440\ldots i;$$
$$\rho_2(\gamma_2) \approx -0.274078669810665\ldots;$$
$$\rho_3(\gamma_3) \approx 0.166666666666666\ldots + 0.301378336840440\ldots i;$$
$$\rho_4(\gamma_4) \approx -3.225921330189334\ldots.$$

Tous sont obtenus avec une précision supérieure à 40 décimales. On en déduit au moyen de la commande Pari `algdep` le polynôme candidat

$$Q_4(x) = x^8 + 6x^7 - 5x^6 - 4x^5 + 5x^4 - 4x^3 - 5x^2 + 6x + 1.$$

On vérifie a posteriori que ce polynôme définit bien une unité du corps de classes de Hilbert au sens restreint de K. Par ailleurs, 4 des 8 racines de Q_4 coïncident sur leurs 40 premières décimales avec les nombres complexes

$$u_1 := u_1(1)e^{\frac{10}{6}i\pi} \approx -6.643347233735518\ldots;$$
$$u_2 := u_2(1)e^{\frac{10}{6}i\pi} \approx -0.931490243381137\ldots - 0.363766307518644\ldots i;$$
$$u_3 := u_3(1)e^{\frac{4}{6}i\pi} \approx -0.150526528994587\ldots;$$
$$u_4 := u_4(1)e^{\frac{8}{6}i\pi} \approx -0.931490243381137\ldots + 0.363766307518644\ldots i.$$

(b) *Un exemple à groupe des classes C_8.* On considère enfin l'extension quadratique ATR $K = F'(\sqrt{25\epsilon - 31})$, dont le groupe des classes au sens restreint est cyclique d'ordre 8. À chaque classe correspond une matrice $\gamma_k \in \mathbf{SL}_2(\mathcal{O}_{F'})$ et un invariant de $\mathbb{C}/m_{F'}\mathbb{Z}$:

$$\begin{aligned}
\rho_1(\gamma_1) &\approx -3.666666666666\ldots - 2.047636549497\ldots i;\\
\rho_2(\gamma_2) &\approx 1.855997078695\ldots - 0.315172999961\ldots i;\\
\rho_3(\gamma_3) &\approx -122.347\ldots + 0.625\ldots i;\\
\rho_4(\gamma_4) &\approx -87.06066958797\ldots + 0.315172999961\ldots i;\\
\rho_5(\gamma_5) &\approx -18.16666666666\ldots + 2.047636549497\ldots i;\\
\rho_6(\gamma_6) &\approx 20.060669587971\ldots + 0.315172999961\ldots i;\\
\rho_7(\gamma_7) &\approx -13.884816073095\ldots;\\
\rho_8(\gamma_8) &\approx 47.894002921304\ldots - 0.3151729999612\ldots i.
\end{aligned}$$

L'invariant le plus précis est $\rho_5(\gamma_5)$, qu'on a pu calculer avec plus de 200 décimales significatives. La commande Pari algdep($u_5(1)e^{-\frac{8}{16}i\pi}$, 16) suggère le polynôme réciproque

$$\begin{aligned}
Q_5(x) := x^{16} &+ 386792x^{15} - 5613916x^{14} + 21963312x^{13}\\
&- 13291318x^{12} + 32052888x^{11} + 15011472x^{10} + 16774296x^9\\
&+ 36336275x^8 + 16774296x^7 + 15011472x^6 + 32052888x^5\\
&- 13291318x^4 + 21963312x^3 - 5613916x^2 + 386792x + 1.
\end{aligned}$$

Il est aisé de vérifier que Q_5 définit effectivement une unité de H_K^+. À une racine de l'unité près, les exponentielles des nombres complexes précédents coïncident sur leurs premières décimales (entre 10 et 200 selon les cas) avec les racines suivantes de Q_5 :

$$\begin{aligned}
u_1 &:= u_1(1)e^{\frac{2}{6}i\pi} \approx -386806.513645927\ldots;\\
u_2 &:= u_2(1)e^{\frac{2}{6}i\pi} \approx 7.17151519909699\ldots + 1.02818667270890\ldots i;\\
u_4 &:= u_4(1)e^{\frac{1}{6}i\pi} \approx 0.136632045175690\ldots + 0.0195890608908274\ldots i;\\
u_5 &:= u_5(1)e^{\frac{8}{6}i\pi} \approx -0.00000258527187\ldots;\\
u_6 &:= u_6(1)e^{\frac{11}{6}i\pi} \approx 0.136632045175690\ldots - 0.0195890608908274\ldots i;\\
u_7 &:= u_7(1)e^{\frac{7}{6}i\pi} \approx -0.317863811003618\ldots - 0.948136381357796\ldots i;\\
u_8 &:= u_8(1)e^{\frac{1}{6}i\pi} \approx 7.17151519909699\ldots - 1.02818667270890\ldots i.
\end{aligned}$$

La précision obtenue sur les décimales de $\rho_3(\gamma_3)$ est insuffisante pour identifier u_3. Pour des raisons de symétrie, il doit correspondre à la racine suivante de Q_5 :

$$-0.317863811003618\ldots + 0.948136381357796\ldots i.$$

7. Périodes de séries d'Eisenstein.

L'objet de cette partie est d'établir une formule générale qui exprime la valeur spéciale en $s=0$ des fonctions L introduites précédemment en termes de périodes de séries d'Eisenstein pour un tore de Γ, ce qui complète la démonstration des théorèmes 2.1 et 3.2.

Des formules similaires ont déjà été obtenues dans [Haran 1987] et [Hara 1993] par exemple. On donne ici une présentation des résultats exposés dans la section 5 de la thèse [Charollois 2004] sous une forme directement utilisable dans les sections 2 et 3.

Quelques notations multi-indices standard permettront de rendre les formules plus agréables. On associe d'abord à un n-uplet $z=(z_1,\ldots,z_n)$ de nombres complexes sa partie imaginaire $y=(\operatorname{Im} z_1,\ldots,\operatorname{Im} z_n)$, sa trace $Tr(z)=z_1+\cdots+z_n$ et sa norme $N(z)=z_1\cdots z_n$. Pour un élément μ de F, on désigne par μz et $z+\mu$ les n-uplets $(\mu_1 z_1,\ldots,\mu_n z_n)$ et $(\mu_1+z_1,\ldots,\mu_n+z_n)$ respectivement. On introduit alors pour $\operatorname{Re} s>1$ la série d'Eisenstein

$$E(z,s)=\sideset{}{'}\sum_{(\mu,\nu)\in\mathbb{O}_F^2/\mathbb{O}_F^\times}\frac{N(y)^s}{|N(\mu z+\nu)|^{2s}},\tag{39}$$

où le groupe d'unités $\mathbb{O}_F^\times$ opère diagonalement sur $\mathbb{O}_F^2$. Cette série définit une forme modulaire de Hilbert non-holomorphe de poids $(0,\ldots,0)$ pour Γ. Le théorème principal de cette partie met en jeu des périodes associées à des dérivées partielles de $E(z,s)$.

Soit K une extension quadratique de F. La signature de K est de la forme $(2r,\ c)$ avec $r+c=n$. On ordonne les n places archimédiennes de F de sorte que les c premières places $v_1,\ldots,v_c$ se prolongent chacune en une place complexe de K, et que les r places suivantes $v_{c+1},\ldots,v_n$ donnent lieu à un isomorphisme de $\mathbb{R}$-algèbres $K\otimes_{F,v_j}\mathbb{R}\simeq\mathbb{R}\oplus\mathbb{R}$. On fixe une fois pour toutes de telles identifications, que l'on appelle encore v_j par abus de notation.

Comme dans la section 1.2, on fixe un idéal I de $\mathbb{O}_F$, et l'on note $\mathbb{O}_I=\mathbb{O}_F+I\mathbb{O}_K$ l'ordre de K de conducteur I. On se donne un $\mathbb{O}_I$-module projectif M de K, et l'on définit $\tau:=\omega_2/\omega_1$, où (ω_1,ω_2) est une une $\mathbb{O}_F$-base positive de M. On pose alors

$$\begin{aligned}\tau_j&:=v_j(\tau)\in\mathcal{H}_j &&\text{pour } j=1,\ldots,c,\\ (\tau_j,\tau_j')&:=v_j(\tau)\in\mathbb{R}\times\mathbb{R} &&\text{pour } j=c+1,\ldots,n.\end{aligned}$$

Pour chaque $c+1\le j\le n$, on appelle Υ_j la géodésique hyperbolique sur $\mathcal{H}_j$ joignant τ_j à τ_j', orientée dans le sens allant de τ_j' à τ_j.

Le produit

$$R_\tau=\{\tau_1\}\times\cdots\{\tau_c\}\times\Upsilon_{c+1}\times\cdots\times\Upsilon_n\subset\mathcal{H}^n$$

est un espace contractile homéomorphe à $\mathbb{R}^r$. On le munit de l'orientation naturelle héritée des Υ_j. Le stabilisateur Γ_τ de τ dans Γ est un groupe abélien de rang r (modulo la torsion), qui s'identifie avec le sous-groupe V_1 des unités de V de norme relative 1 sur F. Il opère sur R_τ par homographies, et le quotient $\Gamma_\tau\backslash R_\tau$ est compact, isomorphe à un tore réel de dimension r. Soit Δ_τ un domaine fondamental pour l'action de Γ_τ sur R_τ. On identifie Δ_τ avec son image dans X, qui est un cycle fermé de dimension r dans ce quotient.

Théorème 7.1. *Pour tout $\mathbb{O}_I$-module projectif M dans K, on a :*

$$\int_{\Delta_\tau} \frac{\partial^r E(z,s)}{\partial z_{c+1}\cdots\partial z_n} dz_{c+1}\wedge\cdots\wedge dz_n = \left(\frac{\Gamma(\frac{s+1}{2})^2}{2i\Gamma(s)}\right)^r d_F^{-s}\, L(M,s). \tag{40}$$

Démonstration. On associe à tout nombre complexe s la r-forme différentielle Γ-invariante sur $\mathcal{H}^n$

$$\omega^r_{\mathrm{Eis}}(s) := \frac{\partial^r E(z,s)}{\partial z_{c+1}\cdots\partial z_n} dz_{c+1}\wedge\cdots\wedge dz_n.$$

Lorsque $\operatorname{Re} s > 1$, un calcul direct montre que la période considérée prend la forme

$$\int_{\Delta_\tau} \omega^r_{\mathrm{Eis}}(s) = \left(\frac{s}{2i}\right)^r \int_{\Delta_\tau} \sum_{(\mu,\nu)\in\mathbb{O}_F^2/\mathbb{O}_F^\times}{}' \left(\prod_{j=1}^{c} \frac{(\operatorname{Im}\tau_j)^s}{|\mu_j\tau_j+\nu_j|^{2s}}\right) \times \left(\bigwedge_{j=c+1}^{n} \frac{y_j^{s-1}(\mu_j\bar{z}_j+\nu_j)^2}{|\mu_j z_j+\nu_j|^{2s+2}} dz_j\right). \tag{41}$$

On observe d'abord que

$$N(M) = d_F \prod_{j=1}^{c} \operatorname{Im}\tau_j \prod_{j=c+1}^{n} (\tau_j'-\tau_j).$$

On définit une action naturelle de $K^\times$ (et donc du groupe V_1) sur $(\mathbb{R}_+^\times)^r$ par la formule

$$\alpha\cdot(t_{c+1},\ldots,t_n) := \left(\left|\frac{\alpha_{c+1}}{\alpha'_{c+1}}\right| t_{c+1},\ldots,\left|\frac{\alpha_n}{\alpha'_n}\right| t_n\right).$$

Le tore compact réel $T^r : V_1\backslash(\mathbb{R}_+^\times)^r$ est muni d'une mesure de Haar canonique

$$d^\times t = \frac{dt_{c+1}}{t_{c+1}}\wedge\cdots\wedge\frac{dt_n}{t_n}.$$

On peut supposer que $(1,\tau)$ est une $\mathbb{O}_F$-base positive de M, quitte à changer M en αM avec $\alpha\in K^\times$. Dans ce cas, la géodésique Υ_j est orientée dans le sens trigonométrique selon nos conventions. En posant $t_j = -i(z_j-\tau_j')/(z_j-\tau_j)$ on obtient une paramétrisation $t_j \in \mathbb{R}_+^\times$ de cette géodésique qui permet d'identifier le quotient $\Gamma_\tau \backslash R_\tau$ avec le tore T^r en respectant les orientations.

Le changement de variable qui correspond à cette paramétrisation transforme l'identité (41) en l'expression

$$\int_{\Delta_\tau} \omega_{\mathrm{Eis}}^r(s) = \left(\frac{s}{2}\right)^r \left(\frac{N(M)}{d_F}\right)^s \int_{T^r} \sideset{}{'}\sum_{\beta \in M/\mathbb{O}_F^\times} \left|N_{K/\mathbb{Q}}(\beta)\right|^{-s} g_\beta(\beta \centerdot t) d^\times t, \tag{42}$$

où l'on a posé $\beta = \mu\tau + \nu$, élément de $K^\times$ qui parcourt les classes non-nulles de $M/\mathbb{O}_F^\times$ quand le couple (μ, ν) parcourt les classes non-nulles de $\mathbb{O}_F^2/\mathbb{O}_F^\times$, et où $g_\beta : (\mathbb{R}_+^\times)^r \longrightarrow \mathbb{C}$ désigne la fonction auxiliaire

$$g_\beta(t) = \prod_{j=c+1}^{n} \frac{t_j^s(-it_j + \operatorname{signe}(\beta_j \beta_j'))^2}{(t_j^2+1)^{s+1}}.$$

L'étape cruciale consiste maintenant à utiliser une idée due à Hecke [Siegel 1980, p. 86] : on observe d'abord que l'on obtient un système de représentants des classes non-nulles de $M/\mathbb{O}_F^\times$ en considérant la famille $\{\beta\epsilon\}$, lorsque β parcourt les classes non-nulles de $M/\tilde{V}$ et ϵ parcourt $V_1/\{\pm 1\}$.

Par conséquent, l'identité (42) devient

$$\begin{aligned}
\int_{\Delta_\tau} & \omega_{\mathrm{Eis}}^r(s) \\
&= \left(\frac{s}{2}\right)^r \left(\frac{N(M)}{d_F}\right)^s \sideset{}{'}\sum_{\beta \in M/\tilde{V}} \left|N_{K/\mathbb{Q}}(\beta)\right|^{-s} \int_{V_1 \backslash (\mathbb{R}_+^\times)^r} \sum_{\epsilon \in V_1/\{\pm 1\}} g_{\beta\epsilon}(\beta\epsilon \centerdot t) d^\times t \\
&= \left(\frac{s}{2}\right)^r \left(\frac{N(M)}{d_F}\right)^s \sideset{}{'}\sum_{\beta \in M/\tilde{V}} \left|N_{K/\mathbb{Q}}(\beta)\right|^{-s} \int_{(\mathbb{R}_+^\times)^r} g_\beta(\beta \centerdot t) d^\times t. \qquad (43)
\end{aligned}$$

Le changement de variable $u = \beta \centerdot t$ dans la dernière intégrale permet de scinder cette intégrale multiple en un produit de r intégrales de la forme suivante [Siegel 1980, formule (107)] :

$$\int_0^{+\infty} \frac{u_j^s(-iu_j + \operatorname{signe}(\beta_j \beta_j'))^2}{(u_j^2+1)^{s+1}} \frac{du_j}{u_j} = -i \operatorname{signe}(\beta_j \beta_j') \frac{\Gamma\left(\frac{s+1}{2}\right)^2}{\Gamma(s+1)}.$$

Il s'ensuit que

$$\begin{aligned}
\int_{\Delta_\tau} & \omega_{\mathrm{Eis}}^r(s) \\
&= \left(\frac{s\Gamma(\frac{s+1}{2})^2}{2i\Gamma(s+1)}\right)^r \left(\frac{N(M)}{d_F}\right)^s \sideset{}{'}\sum_{\beta \in M/\tilde{V}} \left|N_{K/\mathbb{Q}}(\beta)\right|^{-s} \prod_{j=c+1}^{n} \operatorname{signe}(\beta_j \beta_j'). \qquad (44)
\end{aligned}$$

La formule (40) s'en déduit immédiatement au vu de la définition (21) de $L(M,s)$. □

Corollaire 7.2. *Soit K une extension quadratique de signature $(2r,c)$ du corps F, avec $r+c=n=[F:\mathbb{Q}]$. Soit M un $\mathcal{O}_I$-module projectif dans K. La fonction $L(M,s)$ possède alors un zéro d'ordre $\geq c$ en $s=0$, et l'on a les formules :*

i. *si $r\geq 2$, alors :*
$$\frac{L^{(c)}(M,0)}{c!}=\frac{(2i)^r}{2^n\pi^r}\int_{\Delta_\tau}\frac{\partial^r\tilde{h}(z)}{\partial z_{c+1}\dots\partial z_n}dz_{c+1}\wedge\cdots\wedge dz_n. \tag{45}$$

ii. *si le corps K n'a que deux places réelles $(r=1)$, alors*
$$\frac{L^{(n-1)}(M,0)}{(n-1)!}=\frac{2i}{2^n\pi}\int_{\Delta_\tau}\left(\frac{\partial\tilde{h}(z)}{\partial z_n}-\frac{2^{2n-2}R_F}{z_n-\bar{z}_n}\right)dz_n. \tag{46}$$

Démonstration. D'après [Asai 1970, théorème 3] (voir aussi le théorème 2.1 de [Charollois 2007]), la fonction $E(z,s)$ se prolonge sur $\mathbb{C}$ en une fonction méromorphe de la variable s qui satisfait l'équation fonctionnelle
$$G(2s)E(z,s)=G(2-2s)E(z,1-s) \tag{47}$$
avec $G(s):=d_F^{s/2}\pi^{-ns/2}\Gamma(s/2)^n$. En outre, elle possède un unique pôle simple en $s=1$, et les premiers termes de son développement de Laurent au voisinage de ce pôle sont fournis par la formule limite de Kronecker généralisée :
$$E(z,\ s)=\frac{(2\pi)^nR_F}{4d_F}\left(\frac{1}{s-1}+\gamma_F-\log\prod_{j=1}^n y_j+h(z)\right)+O(s-1), \tag{48}$$
où γ_F est une constante qui ne dépend que de F, et où les deux fonctions h et $\tilde{h}=4^{n-1}R_Fh$ ont été introduites en (9) et (10). Les deux égalités précédentes permettent d'obtenir le développement de Taylor de $E(z,s)$ au voisinage de $s=0$:
$$E(z,s)=-2^{n-2}R_Fs^{n-1}-2^{n-2}R_Fs^n\big(\log N(y)+\gamma'_F-h(z)\big)+O(s^{n+1}), \tag{49}$$
où γ'_F ne dépend que de F. On trouve par conséquent pour $r=n-c\geq 2$:
$$\frac{\partial^rE(z,s)}{\partial z_{c+1}\dots\partial z_n}=\frac{s^n}{2^n}\frac{\partial^r\tilde{h}(z)}{\partial z_{c+1}\dots\partial z_n}+O(s^{n+1}). \tag{50}$$
Pour $r=1$, on a un terme supplémentaire :
$$\frac{\partial E(z,s)}{\partial z_n}=\frac{s^n}{2^n}\left(\frac{\partial\tilde{h}(z)}{\partial z_n}-\frac{2^{2n-2}R_F}{z_n-\bar{z}_n}\right)+O(s^{n+1}). \tag{51}$$

On conclut alors du théorème 7.1 que $L(M,s)$ se prolonge en une fonction méromorphe sur $\mathbb{C}$, dont le développement de Taylor au voisinage de $s=0$ se déduit des identités (50) et (51) ci-dessus :

$$L(M,s)=\frac{(2i)^r s^{n-r}}{2^n\pi^r}\int_{\Delta_\tau}\frac{\partial^r\tilde{h}(z)}{\partial z_{c+1}\dots\partial z_n}dz_{c+1}\wedge\cdots\wedge dz_n+O(s^{n+1-r})$$

pour $r\geq 2$, tandis que pour $r=1$:

$$L(M,s)=\frac{2is^{n-1}}{2^n\pi}\int_{\Delta_\tau}\left(\frac{\partial\tilde{h}(z)}{\partial z_n}-\frac{2^{2n-2}R_F}{z_n-\bar{z}_n}\right)+O(s^n).$$

Les formules (45) et (46) souhaitées en résultent immédiatement. □

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Rouquier blocks of the cyclotomic Ariki–Koike algebras

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The definition of Rouquier for families of characters of Weyl groups in terms of blocks of the associated Iwahori–Hecke algebra has made possible the generalization of this notion to the complex reflection groups. Here we give an algorithm for the determination of the "Rouquier blocks" of the cyclotomic Ariki–Koike algebras.

Introduction

The work of G. Lusztig [1984] on irreducible characters of reductive groups over finite fields has displayed the important role of the families of characters of the Weyl groups concerned. More recent results of Gyoja [1996] and Rouquier [1999] have made possible the definition of a substitute for families of characters which can be applied to all complex reflection groups. In particular, Rouquier has shown that the families of characters of a Weyl group W are exactly the blocks of the Iwahori–Hecke algebra of W over a suitable coefficient ring; this is now called the Rouquier ring. This definition generalizes without problem to the cyclotomic Hecke algebras of all complex reflection groups. Ever since, we have been interested in the determination of the Rouquier blocks of the cyclotomic Hecke algebras of complex reflection groups.

Broué and Kim [2002] presented an algorithm for the determination of the Rouquier blocks for the cyclotomic Hecke algebras of the groups $G(d, 1, r)$ and $G(d, d, r)$. Later Kim [2005] used the same algorithm to determine the Rouquier blocks for the group $G(de, e, r)$. The Rouquier blocks of the spetsial cyclotomic Hecke algebra of many exceptional complex reflection groups have been determined in [Malle and Rouquier 2003]. Finally, in [Chlouveraki 2007], we determined the Rouquier blocks of the cyclotomic Hecke algebras of all exceptional complex reflection groups.

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However, it was recently realized that the algorithm given in [Broué and Kim 2002] works only in the case where d is the power of a prime number. The aim of this paper is to give a complete description of the Rouquier blocks of the cyclotomic Ariki–Koike algebras of the group $G(d, 1, r)$. In order to achieve that, we use the theory of essential hyperplanes introduced in [Chlouveraki 2007]. According to this theory, the Rouquier blocks of the cyclotomic Hecke algebras of any complex reflection group depend on numerical data determined by the generic Hecke algebra, the essential hyperplanes of the group. Thanks to Theorem 2.15, it suffices to study the blocks of the generic Hecke algebra in a finite number of cases in order to obtain the Rouquier blocks for all cyclotomic Hecke algebras.

An algorithm for the blocks of the Ariki–Koike algebras of $G(d, 1, r)$ over any field has been given in [Lyle and Mathas 2007]. This algorithm can be applied to give us the Rouquier blocks of the cyclotomic Ariki–Koike algebras and we use it to obtain a characterization in the combinatorial terms used in [Broué and Kim 2002]. Our main result is Theorem 3.18, which determines completely the Rouquier blocks of the cyclotomic Ariki–Koike algebras. The most important consequence is that we can obtain the Rouquier blocks of a cyclotomic Ariki–Koike algebra of $G(d, 1, r)$ from the families of characters of the Weyl groups of type B_n, $n \leq r$, already determined by Lusztig. This result can also be deduced from the Morita equivalences established in [Dipper and Mathas 2002]. Moreover, we show that the Rouquier blocks in the important case of the spetsial cyclotomic Hecke algebra are the ones given by the algorithm of [Broué and Kim 2002].

Finally, in the case of the Weyl groups, Lusztig attaches to every irreducible character two integers, denoted by a and A, and shows [1987, 3.3 and 3.4] that they are constant on the families. In an analogue way, we can define integers a and A attached to every irreducible character of a cyclotomic Hecke algebra of a complex reflection group. Proposition 3.21 completes the proof of the result in [Broué and Kim 2002, 3.18] to the effect that the integers a and A are constant on the Rouquier blocks of $G(d, 1, r)$. The same result has been obtained by the author for the exceptional complex reflection groups in [Chlouveraki 2008].

1. Blocks and symmetric algebras

For proofs of results not given in this section, see [Broué and Kim 2002] or [Chlouveraki 2007, Chapter 2].

Generalities. Assume that $\mathcal{O}$ is a commutative integral domain with field of fractions F and A is an $\mathcal{O}$-algebra, free and finitely generated as an $\mathcal{O}$-module. We denote by ZA the center of A.

Definition 1.1. The *block-idempotents*, or simply *blocks*, of A are the primitive idempotents of ZA.

Let K be a field extension of F and suppose the K-algebra $KA := K\otimes_{\mathcal{O}} A$ is semisimple. By assumption, KA is isomorphic to a direct product of simple algebras:

$$KA \simeq \prod_{\chi\in\mathrm{Irr}(KA)} M_\chi,$$

where $\mathrm{Irr}(KA)$ denotes the set of irreducible characters of KA and M_χ is a simple K-algebra.

For all $\chi \in \mathrm{Irr}(KA)$, we denote by $\pi_\chi : KA \twoheadrightarrow M_\chi$ the projection onto the χ-factor and by e_χ the element of KA such that:

$$\pi_{\chi'}(e_\chi) = \begin{cases} 1_{M_\chi} & \text{if } \chi = \chi', \\ 0 & \text{if } \chi \neq \chi'. \end{cases}$$

The blocks of the algebra KA are related to those of A as follows.

Theorem 1.2. (1) *We have* $1 = \sum_{\chi\in\mathrm{Irr}(KA)} e_\chi$ *and the set* $\{e_\chi\}_{\chi\in\mathrm{Irr}(KA)}$ *is the set of all the blocks of the algebra* KA.

(2) *There exists a unique partition* $\mathrm{Bl}(A)$ *of* $\mathrm{Irr}(KA)$ *such that*

(a) *For all* $B \in \mathrm{Bl}(A)$, *the idempotent* $e_B := \sum_{\chi\in B} e_\chi$ *is a block of* A.

(b) *We have* $1 = \sum_{B\in\mathrm{Bl}(A)} e_B$ *and for every central idempotent e of A, there exists a subset* $\mathrm{Bl}(A, e)$ *of* $\mathrm{Bl}(A)$ *such that*

$$e = \sum_{B\in\mathrm{Bl}(A,e)} e_B.$$

In particular the set $\{e_B\}_{B\in\mathrm{Bl}(A)}$ *is the set of all the blocks of* A.

If $\chi \in B$ for some $B \in \mathrm{Bl}(A)$, we say that χ *belongs to the block* e_B.

Remark. For all $B \in \mathrm{Bl}(A)$, we have $KAe_B \simeq \prod_{\chi\in B} M_\chi$.

Assumptions 1.3. From now on, we make the following assumptions:

(int) The ring $\mathcal{O}$ is a Noetherian and integrally closed domain with field of fractions F and A is an $\mathcal{O}$-algebra which is free and finitely generated as an $\mathcal{O}$-module.

(spl) The field K is a finite Galois extension of F and the algebra KA is split (i.e., for every simple KA-module V, $\mathrm{End}_{KA}(V) \simeq K$) semisimple.

We denote by $\mathcal{O}_K$ the integral closure of $\mathcal{O}$ in K.

Blocks and integral closure. The Galois group $\mathrm{Gal}(K/F)$ acts on $KA = K \otimes_{\mathcal{O}} A$ (viewed as an F-algebra) as follows: if $\sigma \in \mathrm{Gal}(K/F)$ and $\lambda \otimes a \in KA$, then $\sigma(\lambda\otimes a) := \sigma(\lambda)\otimes a$.

If V is a K-vector space and $\sigma \in \mathrm{Gal}(K/F)$, we denote by ${}^\sigma V$ the K-vector space defined on the additive group V with multiplication $\lambda.v := \sigma^{-1}(\lambda)v$ for all

$\lambda \in K$ and $v \in V$. If $\rho : KA \to \mathrm{End}_K(V)$ is a representation of the K-algebra KA, its composition with the action of σ^{-1} is also a representation ${}^\sigma\rho : KA \to \mathrm{End}_K({}^\sigma V)$:

$$KA \xrightarrow{\sigma^{-1}} KA \xrightarrow{\rho} \mathrm{End}_K(V).$$

We denote by ${}^\sigma\chi$ the character of ${}^\sigma\rho$ and we define the action of $\mathrm{Gal}(K/F)$ on $\mathrm{Irr}(KA)$ as follows: if $\sigma \in \mathrm{Gal}(K/F)$ and $\chi \in \mathrm{Irr}(KA)$, then

$$\sigma(\chi) := {}^\sigma\chi = \sigma \circ \chi \circ \sigma^{-1}.$$

This operation induces an action of $\mathrm{Gal}(K/F)$ on the set of blocks of KA:

$$\sigma(e_\chi) = e_{{}^\sigma\chi} \quad \text{for all } \sigma \in \mathrm{Gal}(K/F),\ \chi \in \mathrm{Irr}(KA).$$

Hence, the group $\mathrm{Gal}(K/F)$ acts on the set of idempotents of $Z\mathcal{O}_K A$ and thus on the set of blocks of $\mathcal{O}_K A$. Since $F \cap \mathcal{O}_K = \mathcal{O}$, the idempotents of ZA are the idempotents of $Z\mathcal{O}_K A$ which are fixed by the action of $\mathrm{Gal}(K/F)$. As a consequence, the primitive idempotents of ZA are sums of the elements of the orbits of $\mathrm{Gal}(K/F)$ on the set of primitive idempotents of $Z\mathcal{O}_K A$. Thus, the blocks of A are in bijection with the orbits of $\mathrm{Gal}(K/F)$ on the set of blocks of $\mathcal{O}_K A$. The following proposition is just a reformulation of this result.

Proposition 1.4. (1) *Let B be a block of A and B' a block of $\mathcal{O}_K A$ contained in B. If $\mathrm{Gal}(K/F)_{B'}$ denotes the stabilizer of B' in $\mathrm{Gal}(K/F)$, then*

$$B = \bigcup_{\sigma \in \mathrm{Gal}(K/F)/\mathrm{Gal}(K/F)_{B'}} \sigma(B'), \quad \textit{that is,} \quad e_B = \sum_{\sigma \in \mathrm{Gal}(K/F)/\mathrm{Gal}(K/F)_{B'}} \sigma(e_{B'}).$$

(2) *Two characters $\chi, \psi \in \mathrm{Irr}(KA)$ are in the same block of A if and only if there exists $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\chi)$ and ψ belong to the same block of $\mathcal{O}_K A$.*

Remark. For all $\chi \in B'$, we have $\mathrm{Gal}(K/F)_\chi \subseteq \mathrm{Gal}(K/F)_{B'}$.

Part (2) of the proposition allows us to transfer the problem of the classification of the blocks of A to that of the classification of the blocks of $\mathcal{O}_K A$.

Blocks and prime ideals. We denote by $\mathrm{Spec}(\mathcal{O})$ the set of prime ideals of $\mathcal{O}$. Since $\mathcal{O}$ is Noetherian and integrally closed, we have

$$\mathcal{O} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})} \mathcal{O}_\mathfrak{p},$$

where $\mathcal{O}_\mathfrak{p} := \{x \in F \mid (\exists a \in \mathcal{O} - \mathfrak{p})(ax \in \mathcal{O})\}$ is the localization of $\mathcal{O}$ at $\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ and $\mathcal{O}_\mathfrak{p} A := \mathcal{O}_\mathfrak{p} \otimes_\mathcal{O} A$. The blocks of $\mathcal{O}_\mathfrak{p} A$ are the $\mathfrak{p}$*-blocks of* A. If $\chi, \psi \in \mathrm{Irr}(KA)$ belong to the same block of $\mathcal{O}_\mathfrak{p} A$, we write $\chi \sim_\mathfrak{p} \psi$.

Proposition 1.5. *Two characters $\chi, \psi \in \mathrm{Irr}(KA)$ belong to the same block of A if and only if there exist a finite sequence $\chi_0, \chi_1, \dots, \chi_n \in \mathrm{Irr}(KA)$ and a finite sequence $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \mathrm{Spec}(\mathbb{O})$ such that*

$$\chi_0 = \chi, \quad \chi_n = \psi, \quad \textit{and} \quad \chi_{j-1} \sim_{\mathfrak{p}_j} \chi_j \quad \textit{for } 1 \le j \le n.$$

Blocks and residue blocks. Let $\mathfrak{p}$ be a maximal ideal of $\mathbb{O}$ and set $k_\mathfrak{p} := \mathbb{O}/\mathfrak{p}$ its residue field. If $\mathbb{O}_\mathfrak{p}$ is the localization of $\mathbb{O}$ at $\mathfrak{p}$, then $k_\mathfrak{p}$ is also the residue field of $\mathbb{O}_\mathfrak{p}$. The natural surjection $\pi_\mathfrak{p} : \mathbb{O}_\mathfrak{p} \twoheadrightarrow k_\mathfrak{p}$ extends to a morphism $\pi_\mathfrak{p} : \mathbb{O}_\mathfrak{p} A \twoheadrightarrow k_\mathfrak{p} A$, which in turn induces a morphism

$$\pi_\mathfrak{p} : Z\mathbb{O}_\mathfrak{p} A \to Zk_\mathfrak{p} A.$$

The following lemma will serve for the proof of Proposition 1.7.

Lemma 1.6. *Let e be an idempotent of $\mathbb{O}_\mathfrak{p} A$ whose image $\bar{e}$ in $k_\mathfrak{p} A$ is central. Then e is central.*

Proof. Set $R := \mathbb{O}_\mathfrak{p} A$. Since $\bar{e}$ is central, we have $\bar{e}k_\mathfrak{p}A(1-\bar{e}) = (1-\bar{e})k_\mathfrak{p}A\bar{e} = \{0\}$, i.e., $eR(1-e) \subseteq \mathfrak{p}R$ and $(1-e)Re \subseteq \mathfrak{p}R$. Since e and $(1-e)$ are idempotents, we get $eR(1-e) \subseteq \mathfrak{p}eR(1-e)$ and $(1-e)Re \subseteq \mathfrak{p}(1-e)Re$. By Nakayama's lemma, $eR(1-e) = (1-e)Re = \{0\}$. Thus, from

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$$

we deduce that $R = eRe \oplus (1-e)R(1-e)$. Consequently, e is central. □

Proposition 1.7. *If $\mathbb{O}_\mathfrak{p}$ is a discrete valuation ring and $K = F$, then the morphism*

$$\pi_\mathfrak{p} : Z\mathbb{O}_\mathfrak{p} A \to Zk_\mathfrak{p} A$$

induces a bijection between the set of blocks of $\mathbb{O}_\mathfrak{p} A$ and the set of blocks of $k_\mathfrak{p} A$.

Proof. From now on, the symbol $\widehat{\ }$ will stand for $\mathfrak{p}$-adic completion. Clearly $\pi_\mathfrak{p}$ sends a block of $\mathbb{O}_\mathfrak{p} A$ to a sum of blocks of $k_\mathfrak{p} A$. Now let $\bar{e}$ be a block of $k_\mathfrak{p} A$. By the idempotent lifting theorems [Thévenaz 1995, Theorem 3.2] and the preceding lemma, $\bar{e}$ is lifted to a sum of central primitive idempotents in $\hat{\mathbb{O}}_\mathfrak{p} A$. However, since KA is split semisimple, the blocks of $\hat{\mathbb{O}}_\mathfrak{p} A$ belong to KA. But $K \cap \hat{\mathbb{O}}_\mathfrak{p} = \mathbb{O}_\mathfrak{p}$ (see [Nagata 1962, 18.4], for example) and $\mathbb{O}_\mathfrak{p} A \cap Z\hat{\mathbb{O}}_\mathfrak{p} A \subseteq Z\mathbb{O}_\mathfrak{p} A$. Therefore, $\bar{e}$ is lifted to a sum of blocks in $\mathbb{O}_\mathfrak{p} A$ and this provides the block bijection. □

Symmetric algebras. Let $\mathbb{O}$ be a ring and let A be an $\mathbb{O}$-algebra. We still suppose that the assumptions 1.3 are satisfied.

Definition 1.8. A *trace function* on A is an $\mathbb{O}$-linear map $t : A \to \mathbb{O}$ such that $t(ab) = t(ba)$ for all $a, b \in A$.

Definition 1.9. We say that a trace function $t : A \to \mathcal{O}$ is a symmetrizing form on A or that A is a symmetric algebra if the morphism

$$\hat{t} : A \to \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O}),\ a \mapsto (x \mapsto \hat{t}(a)(x) := t(ax))$$

is an isomorphism of A-modules-A.

Example 1.10. In the case where $\mathcal{O} = \mathbb{Z}$ and $A = \mathbb{Z}[G]$ (G a finite group), we can define the following symmetrizing (or *canonical*) form on A

$$t : \mathbb{Z}[G] \to \mathbb{Z},\ \sum_{g \in G} a_g g \mapsto a_1,$$

where $a_g \in \mathbb{Z}$ for all $g \in G$.

If $\tau : A \to \mathcal{O}$ is a linear form, we denote by $\tau^\vee$ its inverse image by the isomorphism $\hat{t}$, i.e., $\tau^\vee$ is the element of A such that

$$t(\tau^\vee a) = \tau(a) \text{ for all } a \in A.$$

Lemma 1.11 (see [Geck and Pfeiffer 2000, §7.1], for example).

(1) *τ is a trace function if and only if $\tau^\vee \in ZA$.*

(2) *Let $(e_i)_{i \in I}$ be a basis of A over $\mathcal{O}$ and $(e'_i)_{i \in I}$ the dual basis with respect to t, so $t(e_i e'_j) = \delta_{ij}$. We have $\tau^\vee = \sum_i \tau(e'_i)e_i = \sum_i \tau(e_i)e'_i$ and more generally, for all $a \in A$, we have $\tau^\vee a = \sum_i \tau(e'_i a)e_i = \sum_i \tau(e_i a)e'_i$.*

Schur elements. If A is a symmetric algebra with a symmetrizing form t, we obtain a symmetrizing form t^K on KA by extension of scalars. Every irreducible character $\chi \in \mathrm{Irr}(KA)$ is a trace function on KA and thus we can define $\chi^\vee \in ZKA$. Since KA is a split semisimple K-algebra, we have that $KA \simeq \prod_{\chi \in \mathrm{Irr}(KA)} M_\chi$, where M_χ is a matrix algebra isomorphic to $\mathrm{Mat}_{\chi(1)}(K)$. The map $\pi_\chi : KA \twoheadrightarrow M_\chi$, restricted to ZKA, defines a map $\omega_\chi : ZKA \twoheadrightarrow K$.

Definition 1.12. For all $\chi \in \mathrm{Irr}(KA)$, the *Schur element* of χ with respect to t, denoted by s_χ, is the element of K defined by

$$s_\chi := \omega_\chi(\chi^\vee).$$

Schur elements lie in the integral closure:

Proposition 1.13 [Geck and Pfeiffer 2000, §7.2]. *For all $\chi \in \mathrm{Irr}(KA)$, $s_\chi \in \mathcal{O}_K{}^*$.*

Example 1.14. Let $\mathcal{O} := \mathbb{Z}$, $A := \mathbb{Z}[G]$ (G a finite group) and t the canonical symmetrizing form. If K is an algebraically closed field of characteristic 0, then KA is a split semisimple algebra and $s_\chi = |G|/\chi(1)$ for all $\chi \in \mathrm{Irr}(KA)$. Because of the integrality of the Schur elements, we must have $|G|/\chi(1) \in \mathbb{Z} = \mathbb{Z}_K \cap \mathbb{Q}$ for all $\chi \in \mathrm{Irr}(KA)$. Thus, we have shown that $\chi(1)$ divides $|G|$.

The following properties of the Schur elements can be derived easily from the above (see also [Broué 1991; Geck 1993; Geck and Pfeiffer 2000; Geck and Rouquier 1997; Broué et al. 1999]).

Proposition 1.15.

(1)
$$t=\sum_{\chi\in\mathrm{Irr}(KA)}\frac{1}{s_\chi}\chi.$$

(2) *For all* $\chi\in\mathrm{Irr}(KA)$, *the central primitive idempotent associated with* χ *is*

$$e_\chi=\frac{1}{s_\chi}\chi^\vee.$$

2. Hecke algebras of complex reflection groups

Generic Hecke algebras. Let μ_∞ be the group of all the roots of unity in $\mathbb{C}$ and K a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K. For every integer $d>1$, we set $\zeta_d:=\exp(2\pi i/d)$ and denote by μ_d the group of all the d-th roots of unity.

Let V be a K-vector space of finite dimension r. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo-)reflections acting irreducibly on V. Let us denote by $\mathcal{A}$ the set of the reflecting hyperplanes of W. We set $\mathcal{M}:=\mathbb{C}\otimes V-\bigcup_{H\in\mathcal{A}}\mathbb{C}\otimes H$. For $x_0\in\mathcal{M}$, let $P:=\Pi_1(\mathcal{M},x_0)$ and $B:=\Pi_1(\mathcal{M}/W,x_0)$. Then there exists a short exact sequence (cf. [Broué et al. 1998], §2B):

$$\{1\}\to P\to B\to W\to\{1\}.$$

We denote by τ the central element of P defined by the loop

$$[0,1]\to\mathcal{M},\qquad t\mapsto\exp(2\pi it)x_0.$$

For every orbit $\mathcal{C}$ of W on $\mathcal{A}$, we denote by $e_{\mathcal{C}}$ the common order of the subgroups W_H, where H is any element of $\mathcal{C}$ and W_H the subgroup formed by id_V and all the reflections fixing the hyperplane H.

We choose a set of indeterminates $\boldsymbol{u}=(u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\le j\le e_{\mathcal{C}}-1)}$ and we denote by $\mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}]$ the Laurent polynomial ring in all the indeterminates $\boldsymbol{u}$. We define the *generic Hecke algebra* $\mathcal{H}$ of W to be the quotient of the group algebra $\mathbb{Z}[\boldsymbol{u},\boldsymbol{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\boldsymbol{s}-u_{\mathcal{C},0})(\boldsymbol{s}-u_{\mathcal{C},1})\dots(\boldsymbol{s}-u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

where $\mathcal{C}$ runs over the set $\mathcal{A}/W$ and $\boldsymbol{s}$ runs over the set of monodromy generators around the images in $\mathcal{M}/W$ of the elements of the hyperplane orbit $\mathcal{C}$.

Example 2.1. Let $W:=G_4=\langle s,t\mid sts=tst,\ s^3=t^3=1\rangle$. Then s and t are conjugate in W and their reflecting hyperplanes belong to the same orbit in $\mathcal{A}/W$.

The generic Hecke algebra of W has the presentation

$$\mathcal{H}(G_4) = \big\langle S, T \mid STS = TST,\\ (S-u_0)(S-u_1)(S-u_2) = 0,\ (T-u_0)(T-u_1)(T-u_2) = 0\big\rangle.$$

We make some assumptions for the algebra $\mathcal{H}$. They have been verified for all but a finite number of irreducible complex reflection groups; see [Broué et al. 1999, remarks before 1.17 and §2] and [Geck et al. 2000].

Assumptions 2.2. The algebra $\mathcal{H}$ is a free $\mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$-module of rank $|W|$. Moreover, there exists a linear form $t : \mathcal{H} \to \mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$ with the following properties:

(1) t is a symmetrizing form for $\mathcal{H}$.

(2) Via the specialization $u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^{j}$, the form t becomes the canonical symmetrizing form on the group algebra $\mathbb{Z}W$.

(3) If we denote by $\alpha \mapsto \alpha^*$ the automorphism of $\mathbb{Z}[\boldsymbol{u}, \boldsymbol{u}^{-1}]$ consisting of the simultaneous inversion of the indeterminates, then for all $b \in B$, we have

$$t(b^{-1})^* = \frac{t(b\tau)}{t(\tau)}.$$

We know from [Broué et al. 1999, 2.1] that the form t is unique. From now on we suppose that the assumptions 2.2 are satisfied.

Theorem 2.3 [Malle 1999, 5.2]. *Let*

$$\boldsymbol{v} = (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\le j\le e_{\mathcal{C}}-1)}$$

be a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$ *indeterminates such that* $v_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_{\mathcal{C}}}^{-j} u_{\mathcal{C},j}$ *for every* $\mathcal{C}$ *and* j. *Then the* $K(\boldsymbol{v})$*-algebra* $K(\boldsymbol{v})\mathcal{H}$ *is split semisimple.*

By the Tits deformation theorem (see [Broué et al. 1999, 7.2], for example), it follows that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\chi \mapsto \chi_{\boldsymbol{v}}$ from the set $\mathrm{Irr}(K(\boldsymbol{v})\mathcal{H})$ of absolutely irreducible characters of $K(\boldsymbol{v})\mathcal{H}$ to the set $\mathrm{Irr}(W)$ of absolutely irreducible characters of W, such that the following diagram is commutative:

$$\begin{array}{rccc} \chi_{\boldsymbol{v}}: & \mathcal{H} & \to & \mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}] \\ & \downarrow & & \downarrow \\ \chi: & \mathbb{Z}_K W & \to & \mathbb{Z}_K. \end{array}$$

The following result concerning the form of the Schur elements associated with the irreducible characters of $K(\boldsymbol{v})\mathcal{H}$ is proved using a case by case analysis.

Theorem 2.4 [Chlouveraki 2007, Theorem 3.2.5]. *The Schur element* $s_\chi(\boldsymbol{v})$ *associated with the character* $\chi_{\boldsymbol{v}}$ *of* $K(\boldsymbol{v})\mathcal{H}$ *is an element of* $\mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ *of the form*

$$s_\chi(\boldsymbol{v}) = \xi_\chi N_\chi \prod_{i\in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}}$$

where

- ξ_χ *is an element of* $\mathbb{Z}_K$,
- $N_\chi = \prod_{\mathscr{C},j} v_{\mathscr{C},j}^{b_{\mathscr{C},j}}$ *is a monomial in* $\mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ *such that* $\sum_{j=0}^{e_{\mathscr{C}}-1} b_{\mathscr{C},j} = 0$ *for all* $\mathscr{C} \in \mathscr{A}/W$,
- I_χ *is an index set*,
- $(\Psi_{\chi,i})_{i\in I_\chi}$ *is a family of* K*-cyclotomic polynomials in one variable* (*i.e.*, *minimal polynomials of the roots of unity over* K),
- $(M_{\chi,i})_{i\in I_\chi}$ *is a family of monomials in* $\mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ *and if* $M_{\chi,i} = \prod_{\mathscr{C},j} v_{\mathscr{C},j}^{a_{\mathscr{C},j}}$, *then* $\gcd(a_{\mathscr{C},j}) = 1$ *and* $\sum_{j=0}^{e_{\mathscr{C}}-1} a_{\mathscr{C},j} = 0$ *for all* $\mathscr{C} \in \mathscr{A}/W$,
- $(n_{\chi,i})_{i\in I_\chi}$ *is a family of positive integers.*

This factorization is unique in $K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$. *Moreover, the monomials* $(M_{\chi,i})_{i\in I_\chi}$ *are unique up to inversion, whereas the coefficient* ξ_χ *is unique up to multiplication by a root of unity.*

Remark. The bijection $\mathrm{Irr}(K(\boldsymbol{v})\mathscr{H}) \leftrightarrow \mathrm{Irr}(W)$, $\chi_{\boldsymbol{v}} \mapsto \chi$ implies that the specialization $v_{\mathscr{C},j} \mapsto 1$ sends $s_{\chi_{\boldsymbol{v}}}$ to $|W|/\chi(1)$ (which is the Schur element of χ in the group algebra with respect to the canonical symmetrizing form).

Let $A := \mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ and $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$.

Definition 2.5. Let $M = \prod_{\mathscr{C},j} v_{\mathscr{C},j}^{a_{\mathscr{C},j}}$ be a monomial in A such that $\gcd(a_{\mathscr{C},j}) = 1$. We say that M is $\mathfrak{p}$-essential for a character $\chi \in \mathrm{Irr}(W)$, if there exists a K-cyclotomic polynomial Ψ such that

$$\Psi(M) \text{ divides } s_\chi(\boldsymbol{v}) \quad \text{and} \quad \Psi(1) \in \mathfrak{p}.$$

We say that M is $\mathfrak{p}$-essential for W, if there exists a character $\chi \in \mathrm{Irr}(W)$ such that M is $\mathfrak{p}$-essential for χ.

The next result gives a characterization of $\mathfrak{p}$-essential monomials, which plays an essential role in the proof of Theorem 2.15.

Proposition 2.6 [Chlouveraki 2007, Proposition 3.2.6]. *Let* $M = \prod_{\mathscr{C},j} v_{\mathscr{C},j}^{a_{\mathscr{C},j}}$ *be a monomial in* A *such that* $\gcd(a_{\mathscr{C},j}) = 1$. *We set* $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. *Then*

(1) *The ideal* $\mathfrak{q}_M$ *is a prime ideal of* A.

(2) M *is* $\mathfrak{p}$*-essential for* $\chi \in \mathrm{Irr}(W)$ *if and only if* $s_\chi(\boldsymbol{v})/\xi_\chi \in \mathfrak{q}_M$.

Cyclotomic Hecke algebras. Let y be an indeterminate. We set $x := y^{|\mu(K)|}$.

Definition 2.7. A cyclotomic specialization of $\mathscr{H}$ is a $\mathbb{Z}_K$-algebra morphism $\phi : \mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ with the following properties:

- $\phi : v_{\mathscr{C},j} \mapsto y^{n_{\mathscr{C},j}}$ where $n_{\mathscr{C},j} \in \mathbb{Z}$ for all $\mathscr{C}$ and j.

- For all $\mathcal{C} \in \mathcal{A}/W$, if z is another indeterminate, the element of $\mathbb{Z}_K[y, y^{-1}, z]$ defined by
$$\Gamma_{\mathcal{C}}(y, z) := \prod_{j=0}^{e_{\mathcal{C}}-1} (z - \zeta_{e_{\mathcal{C}}}^j y^{n_{\mathcal{C},j}})$$
is invariant under the action of $\mathrm{Gal}(K(y)/K(x))$.

If ϕ is a cyclotomic specialization of $\mathcal{H}$, the corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[y, y^{-1}]$-algebra, denoted by $\mathcal{H}_\phi$, which is obtained as the specialization of the $\mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra $\mathcal{H}$ via the morphism ϕ. It also has a symmetrizing form t_ϕ defined as the specialization of the canonical form t.

Remark. Sometimes we describe the morphism ϕ by the formula
$$u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j x^{n_{\mathcal{C},j}}.$$
If we now set $q := \zeta x$ for some root of unity $\zeta \in \mu(K)$, then the cyclotomic specialization ϕ becomes a ζ-cyclotomic specialization and $\mathcal{H}_\phi$ can be also considered over $\mathbb{Z}_K[q, q^{-1}]$.

Example 2.8. The spetsial Hecke algebra $\mathcal{H}_q^s(W)$ is the 1-cyclotomic algebra obtained by the specialization
$$u_{\mathcal{C},0} \mapsto q, \ \ u_{\mathcal{C},j} \mapsto \zeta_{e_{\mathcal{C}}}^j \text{ for } 1 \le j \le e_{\mathcal{C}} - 1, \text{ for all } \mathcal{C} \in \mathcal{A}/W.$$
For example, if $W := G_4$, then
$$\mathcal{H}_q^s(W) = \langle S, T \mid STS = TST, (S-q)(S^2+S+1) = (T-q)(T^2+T+1) = 0 \rangle.$$

Proposition 2.9 [Chlouveraki 2007, remarks following Theorem 3.3.3]. *The algebra $K(y)\mathcal{H}_\phi$ is split semisimple.*

When y specializes to 1, the algebra $K(y)\mathcal{H}_\phi$ specializes to the group algebra KW (the form t_ϕ becoming the canonical form on the group algebra). Thus, by the Tits deformation theorem, the specialization $v_{\mathcal{C},j} \mapsto 1$ defines the bijections
$$\begin{array}{ccccc} \mathrm{Irr}(K(\boldsymbol{v})\mathcal{H}) & \leftrightarrow & \mathrm{Irr}(K(y)\mathcal{H}_\phi) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_{\boldsymbol{v}} & \mapsto & \chi_\phi & \mapsto & \chi. \end{array}$$

The following result is an immediate consequence of Theorem 2.4.

Proposition 2.10. *The Schur element $s_{\chi_\phi}(y)$ associated with the irreducible character χ_ϕ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in y of the form*
$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi}}$$
where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Rouquier blocks of the cyclotomic Hecke algebras.

Definition 2.11. The *Rouquier ring of K*, denoted by $\mathcal{R}_K(y)$, is the $\mathbb{Z}_K$-subalgebra of $K(y)$ given by

$$\mathcal{R}_K(y) := \mathbb{Z}_K\big[y, y^{-1}, (y^n-1)^{-1}_{n\geq 1}\big].$$

Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization and $\mathcal{H}_\phi$ the corresponding cyclotomic Hecke algebra. The *Rouquier blocks* of $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$.

Remark. Rouquier [1999] showed that if W is a Weyl group and $\mathcal{H}_\phi$ is obtained via the spetsial cyclotomic specialization (see Example 2.8), then its Rouquier blocks coincide with the families of characters defined by Lusztig. Thus, the Rouquier blocks play an essential role in the program Spets [Broué et al. 1999], whose ambition is to give to complex reflection groups the role of Weyl groups of as yet mysterious structures.

The Rouquier ring has the following interesting properties.

Proposition 2.12 [Chlouveraki 2007, Proposition 3.4.2].

(1) *The group of units $\mathcal{R}_K(y)^\times$ of the Rouquier ring $\mathcal{R}_K(y)$ consists of the elements of the form*

$$uy^n \prod_{\Phi\in \mathrm{Cycl}(K)} \Phi(y)^{n_\phi},$$

where $u \in \mathbb{Z}_K^\times$, $n, n_\phi \in \mathbb{Z}$, $\mathrm{Cycl}(K)$ is the set of K-cyclotomic polynomials and $n_\phi = 0$ for all but a finite number of Φ.

(2) *The prime ideals of $\mathcal{R}_K(y)$ are*
- *the zero ideal $\{0\}$,*
- *the ideals of the form $\mathfrak{p}\mathcal{R}_K(y)$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$,*
- *the ideals of the form $P(y)\mathcal{R}_K(y)$, where $P(y)$ is an irreducible element of $\mathbb{Z}_K[y]$ of degree at least 1, prime to y and to $\Phi(y)$ for all $\Phi \in \mathrm{Cycl}(K)$.*

(3) *The Rouquier ring $\mathcal{R}_K(y)$ is a Dedekind ring.*

Now recall the form of the Schur elements of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ given in Proposition 2.10. If χ_ϕ is an irreducible character of $K(y)\mathcal{H}_\phi$, its Schur element $s_{\chi_\phi}(y)$ is of the form

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi\in C_K} \Phi(y)^{n_{\chi,\phi}}$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Definition 2.13. A prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ lying over a prime number p is ϕ-bad for W, if there exists $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$ with $\psi_{\chi,\phi} \in \mathfrak{p}$. If $\mathfrak{p}$ is ϕ-bad for W, we say that p is a ϕ-bad prime number for W.

Remark. If W is a Weyl group and ϕ is the spetsial cyclotomic specialization, then the ϕ-bad prime ideals are the ideals generated by the bad prime numbers (in the usual sense) for W (see [Geck and Rouquier 1997, 5.2]).

Note that if $\mathfrak{p}$ is ϕ-bad for W, then p must divide the order of the group (since $s_{\chi_\phi}(1) = |W|/\chi(1)$).

Denote by $\mathcal{O}$ the Rouquier ring. By Proposition 1.5, the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the blocks of $\mathcal{O}_{\mathcal{P}}\mathcal{H}_\phi$ for all prime ideals $\mathcal{P}$ of $\mathcal{O}$. However, in all of the following cases, due to the form of the Schur elements, the blocks of $\mathcal{O}_{\mathcal{P}}\mathcal{H}_\phi$ are singletons (i.e., $e_{\chi_\phi} = \chi_\phi^\vee/s_{\chi_\phi} \in \mathcal{O}_{\mathcal{P}}\mathcal{H}_\phi$ for all $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$):

- $\mathcal{P}$ is the zero ideal $\{0\}$.
- $\mathcal{P}$ is of the form $P(y)\mathcal{O}$, where $P(y)$ is an irreducible element of $\mathbb{Z}_K[y]$ of degree at least 1, prime to y and to $\Phi(y)$ for all $\Phi \in \mathrm{Cycl}(K)$.
- $\mathcal{P}$ is of the form $\mathfrak{p}\mathcal{O}$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$ which is not ϕ-bad for W.

Therefore, applying Proposition 1.5, we obtain:

Proposition 2.14. *Two characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exists a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of ϕ-bad prime ideals for W such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$,*
- *for all j $(1 \le j \le n)$, the characters χ_{j-1} and χ_j belong to the same block of $\mathcal{O}_{\mathfrak{p}_j\mathcal{O}}\mathcal{H}_\phi$.*

The above proposition implies that if we know the blocks of the algebra $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ for every ϕ-bad prime ideal $\mathfrak{p}$ for W, then we know the Rouquier blocks of $\mathcal{H}_\phi$. To determine the former, we can use this result:

Theorem 2.15 [Chlouveraki 2007, Theorem 3.2.17]. *Let $A := \mathbb{Z}_K[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ and $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. Let $M_1, \ldots, M_k$ be all the $\mathfrak{p}$-essential monomials for W such that $\phi(M_j) = 1$ for all $j = 1, \ldots, k$. Set $\mathfrak{q}_0 := \mathfrak{p}A$, $\mathfrak{q}_j := \mathfrak{p}A + (M_j - 1)A$ for $j = 1, \ldots, k$ and $\mathcal{Q} := \{\mathfrak{q}_0, \mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$. Two irreducible characters $\chi, \psi \in \mathrm{Irr}(W)$ are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{q}_{j_1}, \ldots, \mathfrak{q}_{j_n} \in \mathcal{Q}$ such that*

- *$\chi_0 = \chi$ and $\chi_n = \psi$,*
- *for all i $(1 \le i \le n)$, the characters χ_{i-1} and χ_i are in the same block of $A_{\mathfrak{q}_{j_i}}\mathcal{H}$.*

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ and $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ a cyclotomic specialization. If $M = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ is a $\mathfrak{p}$-essential monomial for W, then

$$\phi(M) = 1 \iff \sum_{\mathcal{C},j} a_{\mathcal{C},j} n_{\mathcal{C},j} = 0.$$

Set $m := \sum_{\mathcal{C}\in\mathcal{A}/W} e_{\mathcal{C}}$. The hyperplane defined in $\mathbb{C}^m$ by the relation

$$\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0,$$

where $(t_{\mathcal{C},j})_{\mathcal{C},j}$ is a set of m indeterminates, is called a $\mathfrak{p}$-*essential hyperplane* for W. A hyperplane in $\mathbb{C}^m$ is called *essential* for W if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ (Likewise, a monomial is called *essential* for W if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$).

Let H be an essential hyperplane corresponding to the monomial M and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. We denote by $\mathcal{B}_{\mathfrak{p}}^H$ the partition of $\mathrm{Irr}(W)$ into blocks of $A_{\mathfrak{q}_M}\mathcal{H}$, where $\mathfrak{q}_M := (M-1)A + \mathfrak{p}A$. Moreover, we denote by $\mathcal{B}_{\mathfrak{p}}^{\varnothing}$ the partition of $\mathrm{Irr}(W)$ into blocks of $A_{\mathfrak{p}A}\mathcal{H}$.

Definition 2.16. Let H be an essential hyperplane for W. By *Rouquier blocks associated with* H we understand the partition $\mathcal{B}^H$ of $\mathrm{Irr}(W)$ generated by the partition $\mathcal{B}_{\mathfrak{p}}^H$, where $\mathfrak{p}$ runs over the set of prime ideals of $\mathbb{Z}_K$. By *Rouquier blocks with no essential hyperplane* we understand the partition $\mathcal{B}^{\varnothing}$ generated by $\mathcal{B}_{\mathfrak{p}}^{\varnothing}$.

With the help of Proposition 2.14 and Theorem 2.15, we obtain the following characterization for the Rouquier blocks of a cyclotomic Hecke algebra:

Proposition 2.17. *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization. The Rouquier blocks of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ correspond to the partition of* $\mathrm{Irr}(W)$ *generated by the partitions $\mathcal{B}^H$, where H runs over the set of all essential hyperplanes the integers $n_{\mathcal{C},j}$ belong to. If the $n_{\mathcal{C},j}$ belong to no essential hyperplane, then the Rouquier blocks of $\mathcal{H}_\phi$ coincide with the partition $\mathcal{B}^{\varnothing}$.*

Definition and Corollary 2.18. *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization such that the integers $n_{\mathcal{C},j}$ belong to only one essential hyperplane H (resp. to no essential hyperplane). We say that ϕ is a cyclotomic specialization associated with the essential hyperplane H (resp. with no essential hyperplane). The Rouquier blocks of $\mathcal{H}_\phi$ coincide with the partition $\mathcal{B}^H$ (resp. $\mathcal{B}^{\varnothing}$).*

By taking cyclotomic specializations associated to each (or no) essential hyperplane and calculating the Rouquier blocks of the corresponding cyclotomic Hecke algebras, we determined in [Chlouveraki 2007] the Rouquier blocks for all exceptional complex reflection groups. We will do the same for the group $G(d,1,r)$.

The functions a and A. Following the notations in [Broué et al. 1999, 6B], for every element $P(y) \in \mathbb{C}(y)$, we define

- the *valuation* $\mathrm{val}_y P$ *of* $P(y)$ *at* y as the order of $P(y)$ at 0 (we have $\mathrm{val}_y P < 0$ if 0 is a pole of $P(y)$ and $\mathrm{val}_y P > 0$ if 0 is a zero of $P(y)$), and
- the *degree* $\deg_y P$ *of* $P(y)$ *at* y as the opposite of the valuation of $P(1/y)$.

Moreover, if $x := y^{|\mu(K)|}$, then $\mathrm{val}_x P := \dfrac{\mathrm{val}_y P}{|\mu(K)|}$ and $\deg_x P := \dfrac{\deg_y P}{|\mu(K)|}$. For $\chi \in \mathrm{Irr}(W)$, we define

$$a_{\chi_\phi} := \mathrm{val}_x(s_{\chi_\phi}(y)) \text{ and } A_{\chi_\phi} := \deg_x s_{\chi_\phi}(y).$$

Proposition 2.19 [Broué and Kim 2002, Proposition 2.9]. *Let $\chi, \psi \in \mathrm{Irr}(W)$. If χ_ϕ and ψ_ϕ belong to the same Rouquier block, then*

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

3. Rouquier blocks for the Ariki–Koike algebras

We will start this section by introducing some notations and results in combinatorics [Broué and Kim 2002, §3A] that will be useful for the description of the Rouquier blocks of the cyclotomic Ariki–Koike algebras, i.e., the cyclotomic Hecke algebras associated to the group $G(d, 1, r)$.

Combinatorics. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_h)$ be a partition, i.e., a finite decreasing sequence of positive integers: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h \geq 1$. The integer

$$|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_h$$

is called *the size of* λ. We also say that λ *is a partition of* $|\lambda|$. The integer h is called *the height of* λ and we set $h_\lambda := h$. To each partition λ we associate its *β-number*, $\beta_\lambda = (\beta_1, \beta_2, \dots, \beta_h)$, defined by

$$\beta_1 := h + \lambda_1 - 1,\ \beta_2 := h + \lambda_2 - 2,\ \dots,\ \beta_h := h + \lambda_h - h.$$

Multipartitions. Fix a positive integer d. Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ be a d-partition, i.e., a family of partitions indexed by the set $\{0, 1, \dots, d-1\}$. Write

$$h^{(a)} := h_{\lambda^{(a)}},\ \ \beta^{(a)} := \beta_{\lambda^{(a)}};$$

then

$$\lambda^{(a)} = (\lambda_1^{(a)}, \lambda_2^{(a)}, \dots, \lambda_{h^{(a)}}^{(a)}).$$

The integer

$$|\lambda| := \sum_{a=0}^{d-1} |\lambda^{(a)}|$$

is called *the size of* λ. We also say that λ *is a d-partition of* $|\lambda|$.

Ordinary symbols. If $\beta = (\beta_1, \beta_2, \dots, \beta_h)$ is a sequence of positive integers such that $\beta_1 > \beta_2 > \cdots > \beta_h$ and m is a positive integer, then the m-shift of β is the sequence of numbers defined by

$$\beta[m] = (\beta_1 + m,\ \beta_2 + m,\ \dots,\ \beta_h + m,\ m-1,\ m-2,\ \dots,\ 1,\ 0).$$

Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ be a d-partition. The *d-height of* λ is the family

$(h^{(0)}, h^{(1)}, \dots, h^{(d-1)})$; we define the *height of* λ to be the integer

$$h_\lambda := \max\{h^{(a)} \mid (0 \le a \le d-1)\}.$$

Definition 3.1. The ordinary standard symbol of λ is the family of numbers given by

$$B_\lambda = (B_\lambda^{(0)}, B_\lambda^{(1)}, \dots, B_\lambda^{(d-1)}),$$

where, for all a $(0 \le a \le d-1)$, we have

$$B_\lambda^{(a)} := \beta^{(a)}[h_\lambda - h^{(a)}].$$

An ordinary symbol of λ is a symbol obtained from the ordinary standard symbol by shifting all the rows by the same integer.

The ordinary standard symbol of a d-partition λ is of the form

$$\begin{array}{ccccccc} B_\lambda^{(0)} & = & b_1^{(0)} & b_2^{(0)} & \cdots & b_{h_\lambda}^{(0)} \\ B_\lambda^{(1)} & = & b_1^{(1)} & b_2^{(1)} & \cdots & b_{h_\lambda}^{(1)} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ B_\lambda^{(d-1)} & = & b_1^{(d-1)} & b_2^{(d-1)} & \cdots & b_{h_\lambda}^{(d-1)} \end{array}$$

The *ordinary content* of a d-partition of ordinary standard symbol B is the set with repetition

$$\mathrm{Cont}_\lambda = B_\lambda^{(0)} \cup B_\lambda^{(1)} \cup \cdots \cup B_\lambda^{(d-1)}$$

or (with the notations above) the polynomial defined by

$$\mathrm{Cont}_\lambda(x) := \sum_{a,i} x^{b_i^{(a)}}.$$

Example 3.2. Take $d = 2$ and $\lambda = ((2, 1), (3))$. Then

$$B_\lambda = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}.$$

We have $\mathrm{Cont}_\lambda = \{0, 1, 3, 4\}$ or $\mathrm{Cont}_\lambda(x) = 1 + x + x^3 + x^4$.

Charged symbols. From now on, we fix a weight system, i.e., a family of integers

$$m := (m^{(0)}, m^{(1)}, \dots, m^{(d-1)}).$$

Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ be a d-partition, and set

$$hc^{(0)} := h^{(0)} - m^{(0)},\ hc^{(1)} := h^{(1)} - m^{(1)},\ \dots,\ hc^{(d-1)} := h^{(d-1)} - m^{(d-1)}.$$

We define the (d, m)*-charged height of* λ as $(hc^{(0)}, hc^{(1)}, \dots, hc^{(d-1)})$, and the m*-charged height of* λ as the integer

$$hc_\lambda := \max\{hc^{(a)} \mid (0 \le a \le d-1)\}.$$

Definition 3.3. The m-charged standard symbol of λ is the family of numbers defined by
$$Bc_\lambda = (Bc_\lambda^{(0)}, Bc_\lambda^{(1)}, \ldots, Bc_\lambda^{(d-1)}),$$
where, for all a $(0 \le a \le d-1)$, we have
$$Bc_\lambda^{(a)} := \beta^{(a)}[hc_\lambda - hc^{(a)}].$$

An m-charged symbol of λ is a symbol obtained from the m-charged standard symbol by shifting all the rows by the same integer.

Remark. The ordinary symbols correspond to the weight system
$$m^{(0)} = m^{(1)} = \cdots = m^{(d-1)} = 0.$$

The m-charged standard symbol of λ is a tableau of numbers arranged into d rows indexed by the set $\{0, 1, \ldots, d-1\}$ such that the a-th row has length equal to $hc_\lambda + m^{(a)}$. For all a $(0 \le a \le d-1)$, we set $l^{(a)} := hc_\lambda + m^{(a)}$ and we denote by
$$Bc_\lambda^{(a)} \;=\; bc_1^{(a)} \;\; bc_2^{(a)} \;\; \ldots \;\; bc_{l^{(a)}}^{(a)}$$
the a-th row of the m-charged standard symbol.

The *m-charged content* of a d-partition of m-charged standard symbol Bc is the set with repetition
$$\mathrm{Contc}_\lambda = Bc_\lambda^{(0)} \cup Bc_\lambda^{(1)} \cup \cdots \cup Bc_\lambda^{(d-1)}$$
or (with the above notations) the polynomial defined by
$$\mathrm{Contc}_\lambda(x) := \sum_{a,i} x^{bc_i^{(a)}}.$$

Example 3.4. Take $d = 2$, $\lambda = ((2, 1), (3))$ and $m = (-1, 2)$. Then
$$Bc_\lambda = \begin{pmatrix} 3 & 1 & & & \\ 7 & 3 & 2 & 1 & 0 \end{pmatrix}$$
We have $\mathrm{Contc}_\lambda = \{0, 1, 1, 2, 3, 3, 7\}$ or $\mathrm{Contc}_\lambda(x) = 1 + 2x + x^2 + 2x^3 + x^7$.

Generic Ariki–Koike algebras. The group $G(d, 1, r)$ is the group of all monomial $r \times r$ matrices with entries in μ_d. It is isomorphic to the wreath product $\mu_d \wr \mathfrak{S}_r$ and its field of definition is $\mathbb{Q}(\zeta_d)$.

The *generic Ariki–Koike algebra* of $G(d, 1, r)$ [Ariki and Koike 1994; Broué and Malle 1993] is the algebra $\mathcal{H}_{d,r}$ generated over the Laurent ring of polynomials in $d + 1$ indeterminates
$$\mathbb{O}_d := \mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \ldots, u_{d-1}, u_{d-1}^{-1}, x, x^{-1}]$$
by the elements $\boldsymbol{s}, \boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_{r-1}$ satisfying the relations

- $\boldsymbol{s}\boldsymbol{t}_1\boldsymbol{s}\boldsymbol{t}_1 = \boldsymbol{t}_1\boldsymbol{s}\boldsymbol{t}_1\boldsymbol{s}$, $\boldsymbol{s}\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{s}$ for $j \neq 1$,
- $\boldsymbol{t}_j\boldsymbol{t}_{j+1}\boldsymbol{t}_j = \boldsymbol{t}_{j+1}\boldsymbol{t}_j\boldsymbol{t}_{j+1}$, $\boldsymbol{t}_i\boldsymbol{t}_j = \boldsymbol{t}_j\boldsymbol{t}_i$ for $|i-j|>1$,
- $(\boldsymbol{s}-u_0)(\boldsymbol{s}-u_1)\dots(\boldsymbol{s}-u_{d-1}) = (\boldsymbol{t}_j - x)(\boldsymbol{t}_j+1) = 0$.

For every d-partition $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(d-1)})$ of r, we consider the free $\mathcal{O}_d$-module which has as basis the family of standard tableaux of λ. We can give to this module the structure of a $\mathcal{H}_{d,r}$-module [Ariki and Koike 1994; Ariki 1994; Graham and Lehrer 1996] and thus obtain the *Specht module* $\boldsymbol{Sp}^\lambda$ associated to λ.

Set $\mathcal{K}_d := \mathbb{Q}(u_0, u_1, \dots, u_{d-1}, x)$ the field of fractions of $\mathcal{O}_d$. The $\mathcal{K}_d\mathcal{H}_{d,r}$-module $\mathcal{K}_d\boldsymbol{Sp}^\lambda$, obtained by extension of scalars, is absolutely irreducible and every irreducible $\mathcal{K}_d\mathcal{H}_{d,r}$-module is isomorphic to a module of this type. Thus $\mathcal{K}_d$ is a splitting field for $\mathcal{H}_{d,r}$. We denote by χ_λ the (absolutely) irreducible character of the $\mathcal{K}_d\mathcal{H}_{d,r}$-module $\boldsymbol{Sp}^\lambda$.

Since the algebra $\mathcal{K}_d\mathcal{H}_{d,r}$ is split semisimple, the Schur elements of its irreducible characters belong to $\mathcal{O}_d$. The following result by Mathas gives a description of the Schur elements. The same result has been obtained independently by Geck, Iancu and Malle in [Geck et al. 2000].

Proposition 3.5 [Mathas 2004, Corollary 6.5]. *Let λ be a d-partition of r with ordinary standard symbol $B_\lambda = (B_\lambda^{(0)}, B_\lambda^{(1)}, \dots, B_\lambda^{(d-1)})$. Fix $L \geq h_\lambda$, where h_λ is the height of λ. We set $B_{\lambda,L} := (B_\lambda^{(0)}[L-h_\lambda], B_\lambda^{(1)}[L-h_\lambda], \dots, B_\lambda^{(d-1)}[L-h_\lambda]) = (B_{\lambda,L}^{(0)}, B_{\lambda,L}^{(1)}, \dots, B_{\lambda,L}^{(d-1)})$ and $B_{\lambda,L}^{(s)} = (b_1^{(s)}, b_2^{(s)}, \dots, b_L^{(s)})$. Let*

$$a_L := r(d-1) + \binom{d}{2}\binom{L}{2} \quad \textit{and} \quad b_L := dL(L-1)(2dL-d-3)/12.$$

Then the Schur element of the irreducible character χ_λ is given by the formulae $s_\lambda = (-1)^{a_L} x^{b_L}(x-1)^{-r}(u_0u_1\dots u_{d-1})^{-r}\nu_\lambda/\delta_\lambda$, where

$$\nu_\lambda = \prod_{0\leq s<t<d}(u_s-u_t)^L \prod_{0\leq s,t<d}\ \prod_{b_s\in B_{\lambda,L}^{(s)}}\ \prod_{1\leq k\leq b_s}(x^k u_s - u_t)$$

and

$$\delta_\lambda = \prod_{0\leq s<t<d}\ \prod_{(b_s,b_t)\in B_{\lambda,L}^{(s)}\times B_{\lambda,L}^{(t)}} (x^{b_s}u_s - x^{b_t}u_t) \prod_{0\leq s<d}\ \prod_{1\leq i<j\leq L}(x^{b_i^{(s)}}u_s - x^{b_j^{(s)}}u_s).$$

We have already mentioned that the field of definition of $G(d,1,r)$ is $K := \mathbb{Q}(\zeta_d)$. If we set

$$v_j^{|\mu(K)|} := \zeta_d^{-j}u_j\,(0\leq j\leq d-1) \text{ and } z^{|\mu(K)|} := x,$$

then Theorem 2.3 implies that the algebra $K(v_0, v_1, \dots, v_{d-1}, z)\mathcal{H}_{d,r}$ is split semisimple. Proposition 3.5 implies that the essential monomials for $G(d,1,r)$ are of the form

- $z^k v_s v_t^{-1}$ for $0 \le s < t < d$ and $-r < k < r$,
- z.

Remark. The monomial z can be seen as a monomial of the form $z_0 z_1^{-1}$, if, in the definition of the Ariki–Koike algebra, we replace the relation

$$(\boldsymbol{t}_j - x)(\boldsymbol{t}_j + 1) = 0 \text{ by } (\boldsymbol{t}_j - x_0)(\boldsymbol{t}_j + x_1) = 0$$

and we set

$$z_0^{|\mu(K)|} := x_0 \text{ and } z_1^{|\mu(K)|} := x_1.$$

Cyclotomic Ariki–Koike algebras. Let y be an indeterminate and let ϕ be a cyclotomic specialization defined by

$$\phi(v_j) = y^{m_j} \quad (0 \le j < d), \qquad \phi(z) = y^n.$$

If we set $q := y^{|\mu(K)|}$, then ϕ can be described by

$$\phi(u_j) = \zeta_d^j q^{m_j} \quad (0 \le j < d), \qquad \phi(x) = q^n.$$

The corresponding cyclotomic Hecke algebra $(\mathcal{H}_{d,r})_\phi$ can be considered either over the ring $\mathbb{Z}_K[y, y^{-1}]$ or over the ring $\mathbb{Z}_K[q, q^{-1}]$. We define the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$ to be the blocks of $(\mathcal{H}_{d,r})_\phi$ defined over the Rouquier ring $\mathcal{R}_K(y)$ in $K(y)$. However, in other texts, as, for example, in [Broué and Kim 2002], the Rouquier blocks are determined over the Rouquier ring $\mathcal{R}_K(q)$ in $K(q)$. Since $\mathcal{R}_K(y)$ is the integral closure of $\mathcal{R}_K(q)$ in the splitting field $K(y)$ for $(\mathcal{H}_{d,r})_\phi$, Proposition 1.4 establishes a relation between the blocks of $\mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi$ and those of $\mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi$. Moreover, in our case we can prove:

Proposition 3.6. *The blocks of $\mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi$ and the blocks of $\mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi$ coincide.*

Proof. By Proposition 1.4, we know that the blocks of $\mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi$ are unions of the blocks of $\mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi$. Now let e be a block-idempotent of $\mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi$. Since $\mathcal{K}_d$ is a splitting field for $\mathcal{H}_{d,r}$, proposition 1.15 implies that e belongs to $K(q)(\mathcal{H}_{d,r})_\phi$. Thus

$$e \in \mathcal{R}_K(y)(\mathcal{H}_{d,r})_\phi \cap K(q)(\mathcal{H}_{d,r})_\phi = \mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi,$$

since the ring $\mathcal{R}_K(q)$ is integrally closed and $\mathcal{R}_K(y)$ is integral over it ($y^{|\mu(K)|} - q$ vanishes). Thus, e is a sum of blocks of $\mathcal{R}_K(q)(\mathcal{H}_{d,r})_\phi$. □

Residue equivalence. Let ϕ be a cyclotomic specialization like above and set $\mathcal{O} := \mathcal{R}_K(q)$. Following proposition 2.14, in order to obtain the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$, we need to calculate the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_{d,r})_\phi$ for all ϕ-bad prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p. By proposition 2.12 the ring $\mathbb{O}$ is a Dedekind ring and thus $\mathbb{O}_{\mathfrak{p}\mathbb{O}}$ is a discrete valuation ring. If we denote by $k_{\mathfrak{p}}$ its residue field, the blocks of $\mathbb{O}_{\mathfrak{p}\mathbb{O}}(\mathcal{H}_{d,r})_\phi$ coincide with the blocks of $k_{\mathfrak{p}}(\mathcal{H}_{d,r})_\phi$, by Proposition 1.7. We denote the natural surjective map by

$$\pi_{\mathfrak{p}} : \mathbb{O}_{\mathfrak{p}\mathbb{O}} \twoheadrightarrow k_{\mathfrak{p}}.$$

Definition 3.7. The diagram of a d-partition λ is the set

$$[\lambda] := \big\{(i, j, a) \mid (0 \le a \le d-1)(1 \le i \le h^{(a)})(1 \le j \le \lambda_i^{(a)})\big\}.$$

A node is any ordered triple (i, j, a).

The $\mathfrak{p}$-*residue* of the node $x = (i, j, a)$ with respect to ϕ is

$$\operatorname{res}_{\mathfrak{p},\phi}(x) = \begin{cases} \pi_{\mathfrak{p}}(\zeta_d^a q^{m_a} q^{n(j-i)}) & \text{if } n \neq 0,\\ (\pi_{\mathfrak{p}}(j-i), \zeta_d^a q^{m_a}) & \text{if } n = 0 \text{ and } \pi_{\mathfrak{p}}(\zeta_d^a q^{m_a}) \neq \pi_{\mathfrak{p}}(\zeta_d^b q^{m_b}) \text{ for } a \neq b,\\ \pi_{\mathfrak{p}}(\zeta_d^a q^{m_a}) & \text{otherwise.} \end{cases}$$

Let $\operatorname{Res}_{\mathfrak{p},\phi} := \{\operatorname{res}_{\mathfrak{p},\phi}(x) \mid x \in [\lambda] \text{ for some } d\text{-partition } \lambda \text{ of } r\}$ be the set of all possible residues. For any d-partition λ of r and $f \in \operatorname{Res}_{\mathfrak{p},\phi}$, we define

$$C_f(\lambda) = \#\{x \in [\lambda] \mid \operatorname{res}(x) = f\}.$$

Adapting Definition 2.10 of [Lyle and Mathas 2007], we obtain:

Definition 3.8. Let λ and μ be two d-partitions of r. We say that λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ if $C_f(\lambda) = C_f(\mu)$ for all $f \in \operatorname{Res}_{\mathfrak{p},\phi}$.

Then Theorem 2.13 of the same reference implies:

Theorem 3.9. *Two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same block of $k_{\mathfrak{p}}(\mathcal{H}_{d,r})_\phi$ if and only if λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ.*

The above result, in combination with Proposition 2.14 gives:

Corollary 3.10. *Two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if there exists a finite sequence $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(m)}$ of d-partitions of r and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of ϕ-bad prime ideals for W such that*

- *$\lambda_{(0)} = \lambda$ and $\lambda_{(m)} = \mu$,*
- *for all j $(1 \le j \le m)$, the d-partitions $\lambda_{(j-1)}$ and $\lambda_{(j)}$ are $\mathfrak{p}_j$-residue equivalent with respect to ϕ.*

Rouquier blocks and charged content. Theorem 3.13 in [Broué and Kim 2002] gives a description of the Rouquier blocks of the cyclotomic Ariki–Koike algebras when $n \neq 0$. However, in the proof it is supposed that $1-\zeta_d$ always belongs to a prime ideal of $\mathbb{Z}[\zeta_d]$. This is not correct, unless d is the power of a prime number. Therefore, we will state here the part of the theorem that is correct and only for the case $n = 1$.

Theorem 3.11. *Let ϕ be a cyclotomic specialization such that $\phi(x) = q$. If two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then* $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ *with respect to the weight system $m = (m_0, m_1, \ldots, m_{d-1})$. The converse holds when d is the power of a prime number.*

Determination of the Rouquier blocks. In this section, we are going to determine the Rouquier blocks for all cyclotomic Ariki–Koike algebras by determining the Rouquier blocks associated with no and each essential hyperplane for $G(d, 1, r)$. Due to Corollary 2.18, it suffices to consider a cyclotomic specialization associated with no and each essential hyperplane and calculate the Rouquier blocks of the corresponding cyclotomic Hecke algebra. Following the description of the essential monomials in the section on generic Ariki–Koike algebras (page 706), we obtain that the essential hyperplanes for $G(d, 1, r)$ are of the form

- $kN + M_s - M_t = 0$ for $0 \leq s < t < d$ and $-r < k < r$.
- $N = 0$.

Case 1: No essential hyperplane. If ϕ is a cyclotomic specialization associated with no essential hyperplane, then the description of the Schur elements by Proposition 3.5 implies that there are no ϕ-bad prime ideals for $G(d, 1, r)$. Therefore, every irreducible character is a block by itself.

Proposition 3.12. *The Rouquier blocks associated with no essential hyperplane are trivial.*

Case 2: Essential hyperplane of the form $kN + M_s - M_t = 0$. The following result is an immediate consequence of the description of the Schur elements by Proposition 3.5.

Proposition 3.13. *Let s, t, k be three integers such that $0 \leq s < t < d$ and $-r < k < r$. The hyperplane*

$$H : kN + M_s - M_t = 0$$

is essential for $G(d, 1, r)$ if and only if there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$. Moreover, in this case, H is $\mathfrak{p}$-essential for $G(d, 1, r)$.

Example 3.14. The hyperplane $M_0 = M_1$ is 2-essential for $G(2, 1, r)$, whereas it isn't essential for $G(6, 1, r)$, for all $r > 0$.

From now on, we assume that $kN + M_s - M_t = 0$ is an essential hyperplane for $G(d, 1, r)$, i.e., that there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$. Let ϕ be a cyclotomic specialization associated with this essential hyperplane, defined by

$$\phi(u_j) = \zeta_d^j q^{m_j} \ (0 \leq j < d) \quad \text{and} \quad \phi(x) = q^n.$$

Our aim is the determination of the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$.

For the notations used in the following theorem, see pages 702–704.

Proposition 3.15. *Let λ, μ be two d-partitions of r. The irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if the following conditions are satisfied*:

(1) *We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.*

(2) *If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then* $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ *with respect to the weight system (m_s, m_t).*

Proof. We can assume, without loss of generality, that $n = 1$. We can also assume that $m_s = 0$ and $m_t = k$.

Suppose that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. Due to Theorem 3.11, we have $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ with respect to the weight system $m = (m_0, m_1, \ldots, m_{d-1})$. Since the m_a, $a \notin \{s, t\}$ could take any value (as long as they don't belong to another essential hyperplane), we must have that $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$. Moreover, the equality $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ implies that the corresponding m-charged standard symbols Bc_λ and Bc_μ have the same cardinality and thus $hc_\lambda = hc_\mu$. Therefore, we obtain

$$Bc_\lambda^{(a)} = \beta_\lambda^{(a)}[hc_\lambda - hc_\lambda^{(a)}] = \beta_\mu^{(a)}[hc_\mu - hc_\mu^{(a)}] = Bc_\mu^{(a)} \text{ for all } a \notin \{s, t\}.$$

Consequently, we have the following equality between sets with repetition:

$$Bc_\lambda^{(s)} \cup Bc_\lambda^{(t)} = Bc_\mu^{(s)} \cup Bc_\mu^{(t)}.$$

We can assume that the m_a, $a \notin \{s, t\}$ are sufficiently large so that

$$hc_\lambda \in \{hc_\lambda^{(s)}, hc_\lambda^{(t)}\} \text{ and } hc_\mu \in \{hc_\mu^{(s)}, hc_\mu^{(t)}\}.$$

In this case, if $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then

$$Bc_{\lambda^{st}}^{(0)} = Bc_\lambda^{(s)},\ Bc_{\lambda^{st}}^{(1)} = Bc_\lambda^{(t)},\ Bc_{\mu^{st}}^{(0)} = Bc_\mu^{(s)},\ Bc_{\mu^{st}}^{(1)} = Bc_\mu^{(t)}$$

and we obtain $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system (m_s, m_t).

Now suppose that the conditions 1 and 2 are satisfied. Set $l := |\lambda^{st}|$. Due to the first condition, we have $|\mu^{st}| = l$. Let $\mathcal{H}_{2,l}$ be the generic Ariki–Koike algebra of the group $G(2, 1, l)$ defined over the ring

$$\mathbb{Z}[U_s, U_s^{-1}, U_t, U_t^{-1}, X, X^{-1}].$$

The group $G(2,1,l)$ is isomorphic to the cyclic group of order 2 for $l=1$ and to the Coxeter group B_l for $l \geq 2$. Let us consider the cyclotomic specialization

$$\vartheta : U_s \mapsto q^{m_s}, U_t \mapsto -q^{m_t}, X \mapsto q.$$

Due to Theorem 3.11, condition 2 implies that the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$. We conclude that $kN+M_s-M_t=0$ is a 2-essential hyperplane for $G(2,1,l)$ and that, due to Corollary 3.10, λ^{st} and μ^{st} are 2-residue equivalent with respect to ϑ. In order to check whether λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ, we only need to consider the nodes with third entry s or t (thanks to condition 1). The nodes of λ (resp. of μ) with third entry s or t are the nodes of λ^{st} (resp. μ^{st}). The $\mathfrak{p}$-residues of these nodes with respect to ϕ can be obtained by replacing q^{m_s} by $\zeta_d^s q^{m_s}$ and $-q^{m_t}$ by $\zeta_d^t q^{m_t}$ into the 2-residues with respect to ϑ of the nodes belonging to $[\lambda^{st}]$ and $[\mu^{st}]$. Since λ^{st} and μ^{st} are 2-residue equivalent and $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$ (when before we had $1-(-1) \in (2)$), we obtain that λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ. Thus, by Corollary 3.10, $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. □

The following result is a corollary of the above proposition. However, it can also be obtained independently using the Morita equivalences established by [Dipper and Mathas 2002]:

Proposition 3.16. *Let λ, μ be two d-partitions of r. The irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if the following conditions are satisfied*:

(1) *We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s,t\}$.*

(2) *If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$, $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$ and $l := |\lambda^{st}| = |\mu^{st}|$, then the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of the cyclotomic Ariki–Koike algebra of $G(2,1,l)$ obtained via the specialization*

$$\vartheta : U_s \mapsto q^{m_s}, U_t \mapsto -q^{m_t}, X \mapsto q^n.$$

Proof. Following [Dipper and Mathas 2002, Theorem 1.1], we obtain that the algebra $(\mathcal{H}_{d,r})_\phi$ is Morita equivalent to the algebra

$$A := \bigoplus_{\substack{n_1,\dots,n_{d-1}\geq 0\\ n_1+\cdots+n_{d-1}=r}} (\mathcal{H}_{2,n_1})_{\phi'} \otimes \mathcal{H}(\mathfrak{S}_{n_2})_{\phi''} \otimes \cdots \otimes \mathcal{H}(\mathfrak{S}_{n_{d-1}})_{\phi''},$$

where ϕ' is the restriction of ϕ to $\mathbb{Z}[u_s, u_s^{-1}, u_t, u_t^{-1}, x, x^{-1}]$ and ϕ'' is the restriction of ϕ to $\mathbb{Z}[x, x^{-1}]$. Therefore, $(\mathcal{H}_{d,r})_\phi$ and A have the same blocks.

Since $n \neq 0$, the Rouquier blocks of $\mathcal{H}(\mathfrak{S}_{n_2})_{\phi''}, \ldots, \mathcal{H}(\mathfrak{S}_{n_2})_{\phi''}$ are trivial. Thus we obtain that two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if the following conditions are satisfied:

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.

(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$, $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$ and $l := |\lambda^{st}| = |\mu^{st}|$, then the characters $(\chi_{\lambda^{st}})_{\phi'}$ and $(\chi_{\mu^{st}})_{\phi'}$ belong to the same block of $(\mathcal{H}_{2,l})_{\phi'}$ over the Rouquier ring of $\mathbb{Q}(\zeta_d)$.

Since the hyperplane $kN + M_s - M_t = 0$ is a $\mathfrak{p}$-essential hyperplane for $G(d, 1, r)$, Corollary 3.10 implies that the second condition is equivalent to saying that the 2-partitions λ^{st} and μ^{st} are $\mathfrak{p}$-residue equivalent with respect to ϕ'. By replacing $\zeta_d^s q^{m_s}$ by q^{m_s} and $\zeta_d^t q^{m_t}$ by $-q^{m_t}$ into the $\mathfrak{p}$-residues with respect to ϕ' of the nodes of $[\lambda^{st}]$ and $[\mu^{st}]$, we obtain their 2-residues with respect to ϑ. Therefore, the 2-partitions λ^{st} and μ^{st} are $\mathfrak{p}$-residue equivalent with respect to ϕ' if and only if they are 2-residue equivalent with respect to ϑ, i.e., the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$. □

Case 3: Essential hyperplane $N = 0$. Let ϕ be a cyclotomic specialization associated with the essential hyperplane $N = 0$, defined by

$$\phi(u_j) = \zeta_d^j q^{m_j} \ (0 \leq j < d) \quad \text{and} \quad \phi(x) = 1.$$

Proposition 3.17. *Let λ, μ be two d-partitions of r. The following assertions are equivalent*:

(i) *The characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$.*

(ii) $|\lambda^{(a)}| = |\mu^{(a)}|$ *for all $a = 0, 1, \ldots, d-1$.*

Proof. (i) ⇒ (ii) Thanks to Proposition 2.14, we can assume that there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same block of $k_\mathfrak{p}\mathcal{H}_\phi$ (where $k_\mathfrak{p}$ is the $\mathfrak{p}$-residue field of the Rouquier ring). Therefore, by Theorem 3.9, they must be $\mathfrak{p}$-residue equivalent with respect to ϕ. Due to the form of the $\mathfrak{p}$-residue with respect to ϕ and the fact that the m_a $(0 \leq a < d)$ can take any value, we must have

$$\begin{aligned} |\lambda^{(a)}| &= \#\{(i, j, a) \mid (1 \leq i \leq h_\lambda^{(a)})(1 \leq j \leq \lambda_i^{(a)})\} \\ &= \#\{(i, j, a) \mid (1 \leq i \leq h_\mu^{(a)})(1 \leq j \leq \mu_i^{(a)})\} = |\mu^{(a)}| \end{aligned}$$

for all $a = 0, 1, \ldots, d-1$.

(ii) ⇒ (i) Let $a \in \{0, 1, \ldots, d-1\}$. It is enough to show that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block, whenever λ and μ are two d-partitions of r such that $|\lambda^{(a)}| = |\mu^{(a)}|$ and $\lambda^{(b)} = \mu^{(b)}$ for all $b \neq a$, Set $l := |\lambda^{(a)}| = |\mu^{(a)}|$. The generic Ariki–Koike algebra of the symmetric group $\mathfrak{S}_l$ specializes to the group algebra $\mathbb{Z}[\mathfrak{S}_l]$ when x specializes to 1 . For any finite group, it is well known that 1 is the only block-idempotent of the group algebra over $\mathbb{Z}$ (see also [Rouquier 1999], §3, Remark 1). Thus, all irreducible characters of $\mathfrak{S}_l$ belong to the same Rouquier block of $\mathbb{Z}[\mathfrak{S}_l]$. Corollary 3.10 implies that there exist a finite sequence of partitions of l, $\lambda_{(0)}, \lambda_{(1)}, \dots, \lambda_{(m)}$ and a finite sequence of prime numbers of $\mathbb{Z}$, $p_1, p_2, \dots, p_m$ such that

- $\lambda_{(0)} = \lambda^{(a)}$ and $\lambda_{(m)} = \mu^{(a)}$,
- $\lambda_{(i-1)}$ and $\lambda_{(i)}$ are (p_i)-residue equivalent with respect to the specialization sending x to 1, for all $i = 1, \dots, m$.

We define $\lambda_{d,i}$ to be the d-partition of r with

$$\lambda_{d,i}^{(a)} = \lambda_{(i)} \quad \text{and} \quad \lambda_{d,i}^{(b)} = \lambda^{(b)} \text{ for all } b \neq a.$$

Let $\mathfrak{p}_i$ be a prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_i. Then we have

- $\lambda_{d,0} = \lambda$ and $\lambda_{d,m} = \mu$,
- $\lambda_{d,i-1}$ and $\lambda_{d,i}$ are $\mathfrak{p}_i$-residue equivalent with respect to ϕ, for all $i = 1, \dots, m$.

Corollary 3.10 implies that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. □

Conclusion. Let ϕ be a cyclotomic specialization for $\mathcal{H}_{d,r}$, defined by

$$\phi(u_j) = \zeta_d^j q^{m_j} \ (0 \leq j < d) \quad \text{and} \quad \phi(x) = q^n.$$

Let λ and μ be d-partitions of r. We write $\lambda \sim_{R,\phi} \mu$ if there exist two integers s and t with $0 \leq s < t < d$ such that the following conditions are satisfied:

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.

(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system (m_s, m_t) (or, equivalently, the irreducible characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of the cyclotomic Ariki–Koike algebra of $G(2, 1, l)$ obtained via the specialization $\vartheta : U_s \mapsto q^{m_s}, U_t \mapsto -q^{m_t}, X \mapsto q^n$).

(3) There exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$.

Thanks to Propositions 3.12, 3.15 and 3.17, we have this consequence of Proposition 2.17:

Theorem 3.18. *If $n \neq 0$, then two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if there exists a finite sequence $\lambda_{(0)}, \lambda_{(1)}, \dots, \lambda_{(m)}$ of d-partitions of r such that*

- *$\lambda_{(0)} = \lambda$ and $\lambda_{(m)} = \mu$,*

- *for all* i $(1 \leq i \leq m)$, *we have* $\lambda_{(i-1)} \sim_{R,\phi} \lambda_{(i)}$.

If $n = 0$, *then two irreducible characters* $(\chi_\lambda)_\phi$ *and* $(\chi_\mu)_\phi$ *are in the same Rouquier block of* $(\mathscr{H}_{d,r})_\phi$ *if and only if there exists a finite sequence* $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(m)}$ *of* d*-partitions of* r *such that*

- $\lambda_{(0)} = \lambda$ *and* $\lambda_{(m)} = \mu$,
- *for all* i $(1 \leq i \leq m)$, *we have* $\lambda_{(i-1)} \sim_{R,\phi} \lambda_{(i)}$ *or* $|\lambda^{(a)}_{(i-1)}| = |\lambda^{(a)}_{(i)}|$ *for all* $a = 0, 1, \ldots, d-1$.

The spetsial case. In this section, we will show that the Rouquier blocks calculated by the algorithm of [Broué and Kim 2002] are correct, when ϕ is the spetsial cyclotomic specialization (see Example 2.8). We are mostly interested in this case, because, as we have already mentioned, the Rouquier blocks of the spetsial cyclotomic Hecke algebra of a Weyl group coincide with its families of characters.

Let ϕ be a cyclotomic specialization for $\mathscr{H}_{d,r}$, defined by

$$\phi(u_j) = \zeta_d^j q^{m_j} \ (0 \leq j < d) \quad \text{and} \quad \phi(x) = q.$$

Let λ and μ be two d-partitions of r. We write $\lambda \sim_{C,\phi} \mu$ if there exist two integers s and t with $0 \leq s < t < d$ such that the following conditions are satisfied:

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.

(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system (m_s, m_t).

Proposition 3.19. *Let* λ *and* μ *be two* d*-partitions of* r. *We have that* $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ *with respect to the weight system* $(m_0, m_1, \ldots, m_{d-1})$ *if and only if there exists a finite sequence* $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(m)}$ *of* d*-partitions of* r *such that*

- $\lambda_{(0)} = \lambda$ *and* $\lambda_{(m)} = \mu$,
- *for all* i $(1 \leq i \leq m)$, *we have* $\lambda_{(i-1)} \sim_{C,\phi} \lambda_{(i)}$.

Proof. We first show that if $\lambda \sim_{C,\phi} \mu$, then $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$. Let s, t be as in the definition of the relation $\sim_{C,\phi}$. Since $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system (m_s, m_t), we have that $hc_{\lambda^{st}} = hc_{\mu^{st}}$. Moreover, $hc^{(a)}_\lambda = hc^{(a)}_\mu$ for all $a \neq s, t$. Therefore,

$$hc_\lambda = \max\{hc_{\lambda^{st}}, (hc^{(a)}_\lambda)_{a \neq s,t}\} = \max\{hc_{\mu^{st}}, (hc^{(a)}_\mu)_{a \neq s,t}\} = hc_\mu.$$

Set $h := hc_\lambda - hc_{\lambda^{st}} = hc_\mu - hc_{\mu^{st}}$. We have

$$Bc^{(s)}_\lambda = \beta^{(s)}_\lambda[hc_\lambda - h^{(s)}_\lambda + m_s] = \beta^{(s)}_\lambda[hc_{\lambda^{st}} - h^{(s)}_\lambda + m_s + h] = Bc^{(0)}_{\lambda^{st}}[h].$$

Similarly, we obtain that

$$Bc^{(t)}_\lambda = Bc^{(1)}_{\lambda^{st}}[h], \ \ Bc^{(s)}_\mu = Bc^{(0)}_{\mu^{st}}[h] \ \text{and} \ Bc^{(t)}_\mu = Bc^{(1)}_{\mu^{st}}[h].$$

Since
$$Bc^{(0)}_{\lambda^{st}} \cup Bc^{(1)}_{\lambda^{st}} = Bc^{(0)}_{\mu^{st}} \cup Bc^{(1)}_{\mu^{st}},$$
we have
$$Bc^{(0)}_{\lambda^{st}}[h] \cup Bc^{(1)}_{\lambda^{st}}[h] = Bc^{(0)}_{\mu^{st}}[h] \cup Bc^{(1)}_{\mu^{st}}[h]$$
and thus,
$$Bc^{(s)}_{\lambda} \cup Bc^{(t)}_{\lambda} = Bc^{(s)}_{\mu} \cup Bc^{(t)}_{\mu}.$$
Since $Bc^{(a)}_{\lambda} = Bc^{(a)}_{\mu}$ for all $a \neq s, t$, we deduce that $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$.

Now let λ and μ be two d-partitions of r such that $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$. Let p be a prime number such that $p \geq d$. We consider the cyclotomic specialization $\bar{\phi}$ for $\mathcal{H}_{p,r}$, defined by
$$\bar{\phi}(u_j) = \zeta_p^j q^{m_j}\ (0 \leq j < d), \quad \bar{\phi}(u_i) = \zeta_p^i q^M\ (d \leq i < p), \quad \bar{\phi}(x) = q,$$
where $M > m_j + r$ for all j $(0 \leq j < d)$. We define the p-partition $\bar{\lambda}$ of r by
$$\bar{\lambda}^{(j)} := \lambda^{(j)} \text{ for all } j\ (0 \leq j < d) \text{ and } \bar{\lambda}^{(i)} := \varnothing \text{ for all } i\ (d \leq i < p).$$
Similarly, we define $\bar{\mu}$ by
$$\bar{\mu}^{(j)} := \mu^{(j)} \text{ for all } j\ (0 \leq j < d) \text{ and } \bar{\mu}^{(i)} := \varnothing \text{ for all } i\ (d \leq i < p).$$

We have $hc^{(i)}_{\bar{\lambda}} = hc^{(i)}_{\bar{\mu}} = -M$ for all i $(d \leq i < p)$. Moreover, $hc^{(j)}_{\bar{\lambda}} > -M$ and $hc^{(j)}_{\bar{\mu}} > -M$ for all j $(0 \leq j < d)$. Thus $hc_{\bar{\lambda}} = hc_\lambda = hc_\mu = hc_{\bar{\mu}}$. It is immediate, that $\mathrm{Contc}_{\bar{\lambda}} = \mathrm{Contc}_{\bar{\mu}}$ with respect to the weight system
$$(m_0, m_1, \ldots, m_{d-1}, M, M, \ldots, M).$$

Since p is a prime number, Theorem 3.11 implies that the irreducible characters $\chi_{\bar{\lambda}}$ and $\chi_{\bar{\mu}}$ belong to the same Rouquier block of $(\mathcal{H}_{p,r})_{\bar{\phi}}$. Due to Theorem 3.18, there exists a finite sequence $\bar{\lambda}_{(0)}, \bar{\lambda}_{(1)}, \ldots, \bar{\lambda}_{(m)}$ of p-partitions of r such that

- $\bar{\lambda}_{(0)} = \bar{\lambda}$ and $\bar{\lambda}_{(m)} = \bar{\mu}$,
- for all l $(1 \leq l \leq m)$, we have $\bar{\lambda}_{(l-1)} \sim_{R,\bar{\phi}} \bar{\lambda}_{(l)}$ (and thus $\bar{\lambda}_{(l-1)} \sim_{C,\bar{\phi}} \bar{\lambda}_{(l)}$).

Since $\bar{\lambda} \sim_{R,\bar{\phi}} \bar{\lambda}_{(1)}$ and $\bar{\phi}(x) = q \neq 1$, there exist two integers s and t with $0 \leq s < t < p$ such that $\bar{\lambda}$ and $\bar{\lambda}_{(1)}$ belong to the same Rouquier block associated with an essential hyperplane of the form
$$kN + M_s - M_t = 0, \text{ where } -r < k < r$$
and we have $k + m_s - m_t = 0$. Since $M - m_j > r$ for all j with $0 \leq j < d$, we can't have $s < d \leq t$. If $s \geq d$, then $\bar{\lambda}_{(1)}$ is a p-partition of r if and only if $\bar{\lambda}_{(1)} = \bar{\lambda}$. Thus, we must have $t < d$ and since $\bar{\lambda}^{(i)} = \varnothing$ for all i with $d \leq i < p$, we also have $\bar{\lambda}^{(i)}_{(1)} = \varnothing$ for all such i. Inductively, we obtain $\bar{\lambda}^{(i)}_{(l)} = \varnothing$ for all i such that $d \leq i < p$

and all l such that $1 \leq l \leq m$. (The same result can be obtained from the fact that the charged content of two p-partitions linked by $\sim_{R,\bar{\phi}}$ is the same.)

Let $l \in \{0, 1, \ldots, m\}$. Define $\lambda_{(l)}$ to be the d-partition of r such that $\lambda_{(l)}^{(j)} := \bar{\lambda}_{(l)}^{(j)}$ for all j $(0 \leq j < d)$. Then

- $\lambda_{(0)} = \lambda$ and $\lambda_{(m)} = \mu$, and
- for all l $(1 \leq l \leq m)$, we have $\lambda_{(l-1)} \sim_{C,\phi} \lambda_{(l)}$. □

Now assume that ϕ is the spetsial cyclotomic specialization, i.e.,

$$m_0 = 1 \text{ and } m_1 = \cdots = m_{d-1} = 0.$$

Proposition 3.20. *Let ϕ be the spetsial cyclotomic specialization. Two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\lambda)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if* $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$.

Proof. If $(\chi_\lambda)_\phi$ and $(\chi_\lambda)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then, by Theorem 3.11, we have $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$.

Now let λ and μ be two d-partitions of r such that $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$. Thanks to Proposition 3.19, we can assume that $\lambda \sim_{C,\phi} \mu$. Then there exist two integers s and t with $0 \leq s < t < d$ such that

$$\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}} \text{ and } \lambda^{(a)} = \mu^{(a)} \text{ for all } a \neq s, t.$$

Let us suppose that $d = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$, where p_i are prime numbers such that $p_i \neq p_j$ for $i \neq j$. For $i = 1, \ldots, n$, we set $c_i := d/p_i^{a_i}$. Then $\gcd(c_i) = 1$ and, by Bézout's theorem, there exist integers $(b_i)_{1 \leq i \leq n}$ such that $\sum_{i=1}^n b_i c_i = 1$. We have $s - t = \sum_{i=1}^n (s-t) b_i c_i$. We set $k_i := (s-t) b_i c_i$ and we obtain that $s - t = \sum_{i=1}^n k_i$.

For all $i = 1, \ldots, n$, the element $1 - \zeta_d^{c_i}$ belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_i. So is $1 - \zeta_d^{k_i}$.

Let I be a subset of $\{1, \ldots, n\}$ minimal (with respect to inclusion) for the property

$$s - t \equiv \sum_{i \in I} k_i \pmod d,$$

i.e., , if $J \subseteq I$ and

$$s - t \equiv \sum_{j \in J} k_j \pmod d,$$

then $J = I$. Without loss of generality, we can assume that $I = \{1, \ldots, n\}$. Now, for all $1 \leq m \leq n$, set

$$l_m := \sum_{i=1}^{m} k_i \pmod d \text{ and } l_0 := 0.$$

Due to the minimality of I, we have $t + l_i \not\equiv s \pmod d$ for all $i < n$.

The group $\mathfrak{S}_d$ acts naturally on the set of d-partitions of r: Let

$$\nu = (\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(d-1)})$$

be a d-partition of r. If $\tau \in \mathfrak{S}_d$, then

$$\tau(\nu) = (\nu^{(\tau(0))}, \nu^{(\tau(1))}, \ldots, \nu^{(\tau(d-1))}).$$

For $a, b \in \{0, \ldots, d-1\}$, we denote by (a, b) the corresponding transposition. If $a, b \neq 0$, then $\nu \sim_{C,\phi} (a,b)\nu$ (since the ordinary content is stable under the action of (a, b)).

For $i \in I$, set $\sigma_i := (t + l_{i-1} \,(\mathrm{mod}\, d), t + l_i \,(\mathrm{mod}\, d))$. We have that the element

$$\zeta_d^{t+l_{i-1}} - \zeta_d^{t+l_i} = \zeta_d^{t+l_{i-1}}(1 - \zeta_d^{k_i})$$

belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_i. Therefore, if $t + l_{i-1}, t + l_i \not\equiv 0 \,(\mathrm{mod}\, d)$, then $\nu \sim_{R,\phi} \sigma_i(\nu)$ for any d-partition ν of r.

Assume that $t + l_i \not\equiv 0 \,(\mathrm{mod}\, d)$ for all $i < n$. If $\sigma := (t, t + l_{n-1} \,(\mathrm{mod}\, d))$, then

$$\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n-2} \circ \sigma_{n-1} \circ \sigma_{n-2} \circ \cdots \circ \sigma_2 \circ \sigma_1.$$

Theorem 3.18 implies that $(\chi_\lambda)_\phi$ and $(\chi_{\sigma(\lambda)})_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. The same holds for $(\chi_\mu)_\phi$ and $(\chi_{\sigma(\mu)})_\phi$. Since $\lambda \sim_{C,\phi} \mu$ (with respect to s, t), we have that $\sigma(\lambda) \sim_{C,\phi} \sigma(\mu)$ (with respect to $s, t + l_{n-1} \,(\mathrm{mod}\, d)$). Moreover, the element $\zeta_d^s - \zeta_d^{t+l_{n-1}} = \zeta_d^s(1 - \zeta_d^{-k_n})$ belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_n and thus, $\sigma(\lambda) \sim_{R,\phi} \sigma(\mu)$. Consequently, $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$.

Now assume that there exists $1 \leq m < n$ such that

$$t + l_i \not\equiv 0 \,(\mathrm{mod}\, d) \text{ for all } i < m \text{ and } t + l_m \equiv 0 \,(\mathrm{mod}\, d).$$

We will prove that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ by induction on $n - m$.

Let $m = n-1$. We have to distinguish two cases: If $k_{n-1} \not\equiv k_n \,(\mathrm{mod}\, d)$, then we have that $t + l_{n-2} + k_n \not\equiv 0 \,(\mathrm{mod}\, d)$ and we can rearrange the k_i (exchanging k_{n-1} and k_n) so that $t + l_i \not\equiv 0 \,(\mathrm{mod}\, d)$ for all $i < n$. This case has been covered above.

If $k_{n-1} \equiv k_n \,(\mathrm{mod}\, d)$, we set

$$\sigma := (t, t + l_{n-2} \,(\mathrm{mod}\, d)) = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{n-3} \circ \sigma_{n-2} \circ \sigma_{n-3} \circ \cdots \circ \sigma_2 \circ \sigma_1.$$

As above, $(\chi_\lambda)_\phi$ and $(\chi_{\sigma(\lambda)})_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. So do $(\chi_\mu)_\phi$ and $(\chi_{\sigma(\mu)})_\phi$. Since the element

$$\zeta_d^s - \zeta_d^{t+l_{n-2}} = \zeta_d^s - \zeta_d^{s-k_{n-1}-k_n} = \zeta_d^s(1 - \zeta_d^{-2k_n})$$

belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_n, we obtain that $\sigma(\lambda) \sim_{R,\phi} \sigma(\mu)$ and thus $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$.

Now assume that the result holds for integers greater than m. We will show that it holds for m. Suppose that

$$t + l_i \not\equiv 0 \,(\mathrm{mod}\, d) \text{ for all } i < m \quad \text{and} \quad t + l_m \equiv 0 \,(\mathrm{mod}\, d).$$

We again distinguish two cases: If there exists $i_0 > m$ such that $k_{i_0} \not\equiv k_m \,(\mathrm{mod}\, d)$, then we have that $t+l_{m-1}+k_{i_0} \not\equiv 0 \,(\mathrm{mod}\, d)$ and we can rearrange the k_i (exchanging k_m and k_{i_0}) so that $t+l_i \not\equiv 0 \,(\mathrm{mod}\, d)$ for all $i < m+1$. Now, the induction hypothesis and the case $t + l_i \not\equiv 0 \,(\mathrm{mod}\, d)$ for all $i < n$ cover all possibilities. Thus, the result is true.

If $k_i \equiv k_m \,(\mathrm{mod}\, d)$, for all $i > m$, we set

$$\sigma := (t, t + l_{m-1} \,(\mathrm{mod}\, d)) = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_{m-2} \circ \sigma_{m-1} \circ \sigma_{m-2} \circ \cdots \circ \sigma_2 \circ \sigma_1.$$

Again we have $(\chi_\lambda)_\phi$ and $(\chi_{\sigma(\lambda)})_\phi$ in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, and likewise $(\chi_\mu)_\phi$ and $(\chi_{\sigma(\mu)})_\phi$. Since the element

$$\zeta_d^s - \zeta_d^{t+l_{m-1}} = \zeta_d^{t+l_n} - \zeta_d^{t+l_{m-1}} = \zeta_d^{t+l_{m-1}}(\zeta_d^{(n-m+1)k_m} - 1)$$

belongs to the prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_m, we obtain that $\sigma(\lambda) \sim_{R,\phi} \sigma(\mu)$ and thus $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. □

The functions a and A. Let ϕ be a cyclotomic specialization for $\mathcal{H}_{d,r}$, given by

$$\phi(u_j) = \zeta_d^j q^{m_j} \; (0 \le j < d), \quad \phi(x) = q^n.$$

If $n \neq 0$, it follows from [Broué and Kim 2002, Proposition 3.18] that the functions a and A (page 702) are constant on the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$. We will show that this is also true for $n = 0$. The results in Theorem 3.18 reduce this to proving the following:

Proposition 3.21. *Let λ and μ be two d-partitions of r. Let ϕ be a cyclotomic specialization associated with the essential hyperplane $N = 0$. If $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then*

$$a((\chi_\lambda)_\phi) = a((\chi_\mu)_\phi) \text{ and } A((\chi_\lambda)_\phi) = A((\chi_\mu)_\phi).$$

Proof. Thanks to Proposition 2.19, we have that

$$a((\chi_\lambda)_\phi) + A((\chi_\lambda)_\phi) = a((\chi_\mu)_\phi) + A((\chi_\mu)_\phi).$$

Thus, it is enough to show that $A((\chi_\lambda)_\phi) = A((\chi_\mu)_\phi)$.

Set $L := \max\{h_\lambda, h_\mu\}$. Using the notations of Proposition 3.5, it is straightforward to check that, for $x = 1$, the term δ_λ doesn't depend on the d-partition λ. Consequently, we obtain that $A((\chi_\lambda)_\phi) = A((\chi_\mu)_\phi)$ if and only if

$$\deg_q\Big(\prod_{0\le s,t<d}\prod_{b_s\in B^{(s)}_{\lambda,L}}\prod_{1\le k\le b_s}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)\Big) = \deg_q\Big(\prod_{0\le s,t<d}\prod_{b_s\in B^{(s)}_{\mu,L}}\prod_{1\le k\le b_s}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)\Big).$$

Set

$$f_\lambda(q) := \prod_{0\le s,t<d}\prod_{b_s\in B^{(s)}_{\lambda,L}}\prod_{1\le k\le b_s}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big).$$

We have

$$\begin{aligned} f_\lambda(q) &= \prod_{0\le s,t<d}\prod_{b_s\in B^{(s)}_{\lambda,L}}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)^{b_s} \\ &= \prod_{0\le s,t<d}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)^{\sum b_s} = \prod_{0\le s,t<d}\big(\zeta_d^s q^{m_s}-\zeta_d^t q^{m_t}\big)^{|\lambda^{(s)}|+\binom{L}{2}}. \end{aligned}$$

Since $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, by Proposition 3.17, we have $|\lambda^{(s)}| = |\mu^{(s)}|$ for all $s = 0, 1, \ldots, d-1$. Thus, $f_\lambda(q) = f_\mu(q)$, which implies that $\deg_q f_\lambda(q) = \deg_q f_\mu(q)$. Therefore $A((\chi_\lambda)_\phi) = A((\chi_\mu)_\phi)$. $\square$

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