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In this paper we define Banach spaces of overconvergent half-integral weight p-adic modular forms and Banach modules of families of overconvergent half-integral weight p-adic modular forms over admissible open subsets of weight space. Both spaces are equipped with a continuous Hecke action for which U_{p^2} is moreover compact. The modules of families of forms are used to construct an eigencurve parameterizing all finite-slope systems of eigenvalues of Hecke operators acting on these spaces. We also prove an analog of Coleman's theorem stating that overconvergent eigenforms of suitably low slope are classical.

1. Introduction

In [Ramsey 2006], the author set up a geometric theory of modular forms of weight $k/2$ for odd positive integers k, complete with geometrically defined Hecke operators. This approach naturally led to a theory of overconvergent p-adic modular forms of such weights equipped with a Hecke action for which U_{p^2} is compact.

In this paper we define overconvergent half-integral weight p-adic modular forms of general p-adic weights, as well as rigid-analytic families thereof over

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admissible open subsets of weight space. We use the latter spaces and Buzzard's eigenvariety machine [Buzzard 2007] to construct a half-integral weight eigencurve parameterizing all systems of eigenvalues of Hecke operators occurring on spaces of half-integral weight overconvergent eigenforms of finite slope. In contrast to the integral weight situation, this space *does not* parameterize actual forms because a half-integral weight form that is an eigenform for all Hecke operators is not always characterized by its weight and collection of Hecke eigenvalues. We also prove an analog of Coleman's result that overconvergent eigenforms of suitably low slope are classical.

This paper lays the foundation for a forthcoming one in which the author will construct a map from our half-integral weight eigencurve to its integral weight counterpart (at least after passage to the underlying reduced spaces) that rigid-analytically interpolates the classical Shimura lifting introduced in [Shimura 1973].

2. Preliminaries

General notation. Fix a prime number p. The symbol K will always denote a complete and discretely-valued field extension of $\mathbb{Q}_p$. For such K we denote the ring of integers by $\mathcal{O}_K$ and the maximal ideal therein by m_K. The absolute value on K will always be normalized by $|p| = 1/p$.

2.1. ***Modular curves.*** For positive integers N and n, $X_1(N)$ and $X_1(N, n)$ will denote the usual moduli stacks of generalized elliptic curves with level structure. The former classifies generalized elliptic curves with a point P of order N, while the latter classifies generalized elliptic curves with a pair (P, C) consisting of a point P of order N and a cyclic subgroup C of order n meeting the subgroup generated by P trivially (plus a certain ampleness condition for nonsmooth curves). This level structure will always be taken to be the Drinfeld-style level structure found in [Katz and Mazur 1985], [Conrad 2007], and the appendix to this paper, and in all cases the base ring will be a $\mathbb{Z}_{(p)}$-algebra.

Throughout this paper we will make extensive use of certain admissible opens in rigid spaces associated to some of these modular curves. Traditionally these opens were defined using the Eisenstein series E_{p-1}, but this requires that we pose unfavorable restrictions on p and N. Fortunately, more recent papers of Buzzard [2003] and Goren and Kassaei [2006] define these opens and explore their properties in greater generality using alternative techniques. These authors define a "measure of singularity" $v(E) \in \mathbb{Q}^{\geq 0}$ associated to an elliptic curve over a complete extension of $\mathbb{Q}_p$. In case $v(E) \leq p/(p+1)$, one may associate to E a canonical subgroup $H_p(E)$ of order p in an appropriately functorial manner. Moreover, one understands $v(E/C)$ for finite cyclic subgroups $C \subseteq E$ as well as the canonical subgroup of E/C when it exists. Inductively applying this with $C = H_p(E)$, one can define (upon further restricting $v(E)$) canonical subgroups $H_{p^m}(E)$ of higher p-power order. For details regarding these constructions and facts, see [Buzzard 2003, Section 3] and [Goren and Kassaei 2006, Section 4].

We will denote the Tate elliptic curve over $\mathbb{Z}((q))$ by $\underline{\text{Tate}}(q)$; see [Katz 1973]. Our notations concerning the Tate curve differ from those often found in the literature as follows. In the presence of, for example, level N structure, previous authors (for example [Katz 1973]) have preferred to consider the curve $\underline{\text{Tate}}(q^N)$ over the base $\mathbb{Z}((q))$. Points of order N on this curve are used to characterize the behavior of a modular form at the cusps, and are all defined over the fixed ring $\mathbb{Z}((q))[\zeta_N]$ (where ζ_N is some primitive N-th root of 1). We prefer to fix the curve $\underline{\text{Tate}}(q)$ and instead consider *extensions* of the base. Thus, in the presence of level N structure, we introduce the formal variable q_N, and *define* $q = q_N^N$. Then the curve $\underline{\text{Tate}}(q)$ is defined over the subring $\mathbb{Z}((q))$ of $\mathbb{Z}((q_N))$, and all of its N-torsion is defined over the ring $\mathbb{Z}((q_N))[\zeta_N]$. To be precise, the N-torsion is given by

$$\zeta_N^i q_N^j \quad \text{for } 0 \le i, j \le N-1.$$

Cusps will always be referred to by specifying a level structure on the Tate curve.

Suppose that $N \ge 5$ so that we have a fine moduli scheme $X_1(N)_{\mathbb{Q}_p}$, and let $K/\mathbb{Q}_p$ be a finite extension (which will generally be fixed in applications). If $r \in [0, 1] \cap \mathbb{Q}$, then the region in the rigid space $X_1(N)_K^{\text{an}}$ whose points correspond to pairs (E, P) with $v(E) \le r$ is an admissible affinoid open. We denote by $X_1(N)^{\text{an}}_{\ge p^{-r}}$ the connected component of this region that contains the cusp associated to the datum $(\underline{\text{Tate}}(q), \zeta_N)$ for some (equivalently, any) choice of primitive N-th root of unity ζ_N. Similarly, $X_1(N, n)^{\text{an}}_{\ge p^{-r}}$ will denote the connected component of the region defined by $v(E) \le r$ in $X_1(N, n)_K^{\text{an}}$ containing the cusp associated to $(\underline{\text{Tate}}(q), \zeta_N, \langle q_n \rangle)$ for any such ζ_N. For smaller N one defines these spaces by first adding prime-to-p level structure to rigidify the moduli problem and proceeding as above, and then taking invariants. Similarly, the space $X_0(N)^{\text{an}}_{\ge p^{-r}}$ is defined as the quotient of $X_1(N)^{\text{an}}_{\ge p^{-r}}$ by the action of the diamond operators. See [Buzzard 2007, Section 6] for a more detailed discussion of these quotients.

2.2. ***Norms.*** If $\mathfrak{X}$ is an admissible formal scheme over $\mathcal{O}_K$ (in the sense of [Bosch and Lütkebohmert 1993]), we will denote its (Raynaud) generic fiber by $\mathfrak{X}_{\text{rig}}$ and its special fiber by $\mathfrak{X}_0$. In case $\mathfrak{X} = \text{Spf}(\mathcal{A})$ is a formal affine, we have $\mathfrak{X}_{\text{rig}} = \text{Sp}(\mathcal{A} \otimes_{\mathcal{O}_K} K)$ and $\mathfrak{X}_0 = \text{Spec}(\mathcal{A}/\pi\mathcal{A})$, where $\pi \in \mathcal{O}_K$ is a uniformizer. We recall for later use that the natural specialization map $\text{sp} : \mathfrak{X}_{\text{rig}} \to \mathfrak{X}_0$ is surjective on the level of closed points; see [Bosch and Lütkebohmert 1993, Proposition 3.5].

Assume that $\mathfrak{X}$ is reduced, and let $\mathfrak{L}$ be an invertible sheaf on $\mathfrak{X}$ (that is to say, a sheaf of modules on this ringed space that is Zariski-locally free of rank one). For a point $x \in \mathfrak{X}_{\text{rig}}(L)$, let $\hat{x} : \text{Spf}(\mathcal{O}_L) \to \mathfrak{X}$ denote the unique extension of x to the formal model. Then the canonical identification

$$H^0(\mathrm{Sp}(L), x^*\mathfrak{L}_{\mathrm{rig}}) = H^0(\mathrm{Spf}(\mathcal{O}_L), \hat{x}^*\mathfrak{L}) \otimes_{\mathcal{O}_L} L$$

furnishes a norm $|\cdot|_x$ on this one-dimensional vector space by declaring the formal sections on the right to be the unit ball. Now for any admissible open $\mathcal{U} \subseteq \mathfrak{X}_{\mathrm{rig}}$ and any $f \in H^0(\mathcal{U}, \mathfrak{L}_{\mathrm{rig}})$, we define

$$\|f\|_{\mathcal{U}} = \sup_{x \in \mathcal{U}} |x^* f|_x.$$

Note that, in case $\mathfrak{L} = \mathcal{O}_{\mathfrak{X}}$, this is simply the usual supremum norm on functions.

There is no reason for $\|f\|_{\mathcal{U}}$ to be finite in general, but in case $\mathcal{U}$ is affinoid then this is indeed finite and endows $H^0(\mathcal{U}, \mathfrak{L}_{\mathrm{rig}})$ with the structure of a Banach space over K, as we now demonstrate.

Lemma 2.1. *Suppose $\mathfrak{X}$ is a reduced quasicompact admissible formal scheme over $\mathcal{O}_K$, let $\mathfrak{L}$ be an invertible sheaf on $\mathfrak{X}$, and let $\mathcal{U}$ be an admissible affinoid open in $\mathfrak{X}_{\mathrm{rig}}$. Then $H^0(\mathcal{U}, \mathfrak{L}_{\mathrm{rig}})$ is a K-Banach space with respect to $\|\cdot\|_{\mathcal{U}}$.*

Proof. By Raynaud's theorem there is quasicompact admissible formal blowup $\pi : \mathfrak{X}' \to \mathfrak{X}$ and an admissible formal open $\mathfrak{U}$ in $\mathfrak{X}'$ with generic fiber $\mathcal{U}$. For $x \in \mathcal{U}$, let $\hat{x}'$ denote the unique extension to an $\mathcal{O}_L$-valued point of $\mathfrak{U}$, and let $\hat{x}$ denote its image in $\mathfrak{X}$ (which is the same $\hat{x}$ as above by uniqueness). Then we have

$$H^0(\mathrm{Spf}(\mathcal{O}_L), \hat{x}'^*\pi^*\mathfrak{L}) = H^0(\mathrm{Spf}(\mathcal{O}_L), \hat{x}^*\mathfrak{L})$$

as lattices in $H^0(\mathrm{Sp}(L), \mathfrak{L}_{\mathrm{rig}})$. It follows that $|f|_x = |\pi^* f|_x$, and we may compute $\|f\|_{\mathcal{U}}$ using the models $\mathfrak{X}'$ and $\pi^*\mathfrak{L}$. Hence we may as well assume that $\mathcal{U}$ is the generic fiber of an admissible formal open $\mathfrak{U}$ in $\mathfrak{X}$. Furthermore, we may just as well replace $\mathfrak{X}$ by $\mathfrak{U}$ and assume that $\mathfrak{U}$ is the generic fiber of $\mathfrak{X}$ itself.

Cover $\mathfrak{X}$ by a finite collection of admissible formal affine opens $\mathfrak{U}_i$ trivializing $\mathfrak{L}$. Pick a trivializing section ℓ_i of $\mathfrak{L}$ on $\mathfrak{U}_i$. Let $\mathcal{U}_i = (\mathfrak{U}_i)_{\mathrm{rig}}$, so that the $\mathcal{U}_i$ form an admissible cover of $\mathcal{U}$ by admissible affinoid opens. Then, for any section $f \in H^0(\mathcal{U}, \mathfrak{L}_{\mathrm{rig}})$, we may write $f|_{\mathcal{U}_i} = a_i \ell_i$ for a unique $a_i \in \mathcal{O}(\mathcal{U}_i)$, and one easily checks that $\|f\|_{\mathcal{U}} = \max_i \|a_i\|_{\mathrm{sup}}$. The desired assertion now follows easily from the analogous assertion about the supremum norm on a reduced affinoid. □

The following lemma and its corollary establish a sort of maximum modulus principle for these norms.

Lemma 2.2. *Suppose $\mathfrak{X} = \mathrm{Spf}(\mathcal{A})$ is a reduced admissible affine formal scheme over $\mathcal{O}_K$, and let $U \subseteq \mathfrak{X}_0$ be a Zariski-dense open subset of the special fiber. Suppose that the generic fiber $X = \mathrm{Sp}(\mathcal{A} \otimes_{\mathcal{O}_K} K)$ is equidimensional. Then, for any $a \in \mathcal{A} \otimes_{\mathcal{O}_K} K$, the supremum norm of a over X is achieved on $\mathrm{sp}^{-1}(U)$.*

Proof. Let us first prove the lemma in the case that $\mathcal{A}$ is normal. First note that if $\|a\|_{\sup} = 0$, then the result is obvious. Otherwise, since the supremum norm is power-multiplicative we may assume that $\|a\|_{\sup}$ is a norm from K and scale to reduce to the case $\|a\|_{\sup} = 1$. By [de Jong 1995, Theorem 7.4.1] it follows that $a \in \mathcal{A}$ (this is where normality is used). If the reduction $a_0 \in \mathcal{A}_0 = \mathcal{A}/\pi\mathcal{A}$ vanishes at every closed point of U, then it vanishes everywhere by density, so $a_0^n = 0$ in $\mathcal{A}_0$ for some n, which is to say that $\pi \mid a^n$ in $\mathcal{A}$. But this is impossible because by power-multiplicativity we have $\|a^n\|_{\sup} = 1$ for all $n \geq 1$. Thus a_0 must be nonvanishing at some point of U. By the surjectivity of the specialization map we can find a point x reducing to this point. Clearly then $|a(x)| = 1$, which establishes the normal case.

Suppose that X is equidimensional of dimension d. We claim that it follows that the special fiber $\mathfrak{X}_0$ must be equidimensional of dimension d as well. Indeed, inside each irreducible component of this special fiber we can find a nonempty Zariksi-open subset V that does not meet any of the other irreducible components. The generic fiber V_{rig} is an admissible open in X and therefore has dimension d. It follows that V has dimension d, and the claim follows.

Let $f : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ be the normalization map (meaning Spf applied to the normalization map on algebras), and note that this map is finite by general excellence considerations. By [Conrad 1999, Theorem 2.1.3] the generic fiber of this map coincides with the normalization of X. Thus $\widetilde{\mathfrak{X}}_{\mathrm{rig}}$ is also equidimensional of dimension d, and the argument above shows that $\widetilde{\mathfrak{X}}_0$ is equidimensional of dimension d as well. Now since f is finite it follows that f_0 carries generic points to generic points. In particular we see that $f_0^{-1}(U)$ is Zariski-dense in $\widetilde{\mathfrak{X}}_0$. Thus by the normal case proved above, there exists an $x \in \widetilde{\mathfrak{X}}_{\mathrm{rig}}$ reducing to $f_0^{-1}(U)$ at which a (thought of as an element of $\widetilde{\mathcal{A}} \otimes_{\mathcal{O}_K} K$) attains its supremum norm. But then $f(x)$ is a point in X reducing to U with the same property, since the supremum norm of a is the same thought of on X or on $\widetilde{X}$ (since $\widetilde{X} \to X$ is surjective). □

Remark 2.3. Note that the proof in the normal case did not use the equidimensionality hypothesis. This hypothesis may not be required in the general case, but the above proof breaks down without it since it is not clear how to control the special fiber under normalization in general, especially if $\mathfrak{X}_0$ is nonreduced (as is often the case for us).

Corollary 2.4. *Suppose $\mathfrak{X}$ is a reduced quasicompact admissible formal scheme over $\mathcal{O}_K$, let $U \subseteq \mathfrak{X}_0$ be a Zariski-dense open, and let $\mathfrak{L}$ be an invertible sheaf on $\mathfrak{X}$. Assume that $\mathfrak{X}_{\mathrm{rig}}$ is equidimensional. Then, for any $f \in H^0(\mathfrak{X}_{\mathrm{rig}}, \mathfrak{L}_{\mathrm{rig}})$ we have*

$$\|f\|_{\mathfrak{X}_{\mathrm{rig}}} = \sup_{x \in \mathrm{sp}^{-1}(U)} |x^* f|_x = \max_{x \in \mathrm{sp}^{-1}(U)} |x^* f|_x.$$

Proof. Cover $\mathfrak{X}$ with a finite collection of admissible formal affine opens trivializing $\mathfrak{L}$, and apply Lemma 2.2 on each such affine separately. □

The invertible sheaves whose sections we will be taking norms of in this paper will all be of the form $\mathcal{O}_X(D)$ for some divisor D on $X = X_1(N)_K$ or $X_1(N, n)_K$ supported on the cusps. In the end, the main consequence of Corollary 2.4 (namely, Lemma 2.5) will be that these norms are equal to the supremum norm of the restriction of the section in question to the complement of the residue disks around the cusps (where it is simply an analytic function). We feel it worthwhile to give more natural definitions using the above norm machinery in the cases that it applies to (those where we have nice moduli schemes to work with), in the hope that the techniques used and Corollary 2.4 will be useful in other similar situations. The reader content with this equivalent "ad hoc" definition (that is, the supremum norm on the complement of the residue disks around the cusps) can skip to Section 2.3 and ignore the appendix altogether.

In order to endow spaces of sections of a line bundle as in the previous paragraph with norms using the techniques above, we need formal models of the spaces X and sheaves $\mathcal{O}(D)$. For technical reasons (involving regularity of certain moduli stacks), we are forced to work over $\mathbb{Z}_p$ in going about this. The formal models over $\mathcal{O}_K$ will then be obtained by extension of scalars. The general procedure for obtaining formal models over $\mathbb{Z}_p$ goes as follows. Let X denote one the stacks $X_1(N)$ or $X_1(N, n)$ over $\mathbb{Z}_p$, and assume that the generic fiber $X_{\mathbb{Q}_p}$ is a scheme. Let D be a divisor on $X_{\mathbb{Q}_p}$ that is supported on the cusps. If the closure $\overline{D}$ of D in X lies in the maximal open subscheme X^{sch} of X and this subscheme is regular along $\overline{D}$, then this closure is Cartier and we may associate to it the invertible sheaf $\mathcal{O}(\overline{D})$ on X^{sch}. Let $(X^{\text{sch}})\widehat{}$ and $\mathcal{O}(\overline{D})\widehat{}$ denote the formal completions of these objects along the special fiber.

In case $X = X_1(N)$ or $X_1(N, n)$ with $p \nmid n$, assume that N has a divisor that is prime to p and at least 5. Then $X^{\text{sch}} = X$ by [Conrad 2007, Theorem 4.2.1], and X is regular (at least over $\mathbb{Z}_{(p)}$) by [Conrad 2007, Theorem 4.1.1]. That passage to $\mathbb{Z}_p$ preserves regularity follows by excellence considerations from the fact that $\mathbb{Z}_{(p)} \to \mathbb{Z}_p$ is geometrically regular. Strictly speaking, the results of [Conrad 2007] do not apply to $X_1(N, n)$ as stated, but since $p \nmid n$, the proofs of these results are still valid over $\mathbb{Z}_{(p)}$, as is observed in the appendix. Since X is proper over $\mathbb{Z}_p$, we have $\widehat{X}_{\text{rig}} = X^{\text{an}}_{\mathbb{Q}_p}$ (the analytification of the algebraic generic fiber of X) and hence we have a formal model $(\widehat{X}, \mathcal{O}(\overline{D})\widehat{}\,)$ of $(X^{\text{an}}_{\mathbb{Q}_p}, \mathcal{O}(D))$.

Suppose that $X = X_1(Mp, p^2)$ for an integer $M \geq 5$ prime to p. Let D be any divisor supported on the cusps in the connected component $X_1(Mp, p^2)^{\text{an}}_{\geq 1}$ of the ordinary locus. By Theorem A.11, the closure $\overline{D}$ of D in X lies in X^{sch} and is Cartier. Thus we obtain a formal model $((X^{\text{sch}})\widehat{}, \mathcal{O}(\overline{D})\widehat{}\,)$ of $((X^{\text{sch}})\widehat{}_{\text{rig}}, \mathcal{O}(D))$.

Observe that, by Lemma A.9 and the comments that follow it, X^{sch} is simply the complement of a finite collection of cusps on the characteristic p fiber (namely, the ones with nontrivial automorphisms). It follows that the open immersion

$$(X^{\mathrm{sch}})^{\widehat{}}_{\mathrm{rig}} \hookrightarrow (X^{\mathrm{sch}}_{\mathbb{Q}_p})^{\mathrm{an}} \cong X^{\mathrm{an}}_{\mathbb{Q}_p} \tag{1}$$

identifies the Raynaud generic fiber on the left with the complement of the residue disks around the cusps in the analytification on the right that reduce to the missing points in characteristic p. Thus (1) is an isomorphism when restricted to any connected component of the locus defined by $v(E) \le r$ that contains no such cusps. In particular, by Theorem A.11 it is an isomorphism when restricted to $X_1(Mp, p^2)^{\mathrm{an}}_{\ge p^{-r}}$.

Given a complete discretely valued extension $K/\mathbb{Q}_p$, we may extend scalars on our formal models of $\mathcal{O}(D)$ to arrive at norms on the following spaces:

- sections of $\mathcal{O}(D)$ over any admissible open $\mathcal{U}$ in $X = X_1(N)^{\mathrm{an}}_K$ (respectively $X_1(N, n)^{\mathrm{an}}_K$ with $p \nmid n$), where D is (the scalar extension of) a divisor on $X_1(N)_{\mathbb{Q}_p}$ (respectively $X_1(N, n)_{\mathbb{Q}_p}$) and N is divisible by an integer that is prime to p and at least 5; and
- sections of $\mathcal{O}(D)$ over any admissible open $\mathcal{U}$ in $X = X_1(Mp, p^2)^{\mathrm{an}}_{\ge p^{-r}}$, where D is (the scalar extension of) a divisor that is supported on the cusps in $X_1(Mp, p^2)^{\mathrm{an}}_{\mathbb{Q}_p}$ and M is an integer that is prime to p and at least 5.

Lemma 2.5. *Let X, D, and $\mathcal{U}$ be as in either of the two cases above, and assume that $\mathcal{U}$ contains every component of the ordinary locus that it meets. Let $\mathcal{U}'$ denote the complement of the residue disks around the cusps in $\mathcal{U}$. Then, for any $f \in H^0(\mathcal{U}, \mathcal{O}(D))$, we have $\|f\|_{\mathcal{U}} = \|f|_{\mathcal{U}'}\|_{\mathrm{sup}}$.*

Proof. We will treat the case of $X = X_1(N)^{\mathrm{an}}_K$; the other cases are proved in exactly the same manner. First note that, since points in $\mathcal{U}'$ reduce to points outside of the support of $\overline{D}$, the claim is equivalent to the claim that $\|f\|_{\mathcal{U}} = \|f|_{\mathcal{U}'}\|_{\mathcal{U}'}$. That is, the norm on $\mathcal{U}'$ that we have defined using formal models happens to be equal to the supremum norm on $\mathcal{U}'$.

Note that the supersingular loci of $\mathcal{U}$ and $\mathcal{U}'$ coincide, so the contributions to the above norms over this locus are equal, and it suffices to check the assertion upon restriction to the ordinary locus. By assumption, the ordinary locus in $\mathcal{U}$ is a finite union of connected components of the ordinary locus in $X_1(N)^{\mathrm{an}}_K$. Each such component corresponds via reduction to an irreducible component of the special fiber. Let $\mathfrak{X}$ denote the admissible formal open in $X_1(N)^{\widehat{}}$ given by the union of the components so obtained with the supersingular points removed. Then $\mathfrak{X}_{\mathrm{rig}}$ is precisely the ordinary locus in $\mathcal{U}$, and the result now follows from Corollary 2.4 with U equal to the complement of the cusps in $\mathfrak{X}_0$. ☐

Remark 2.6. There remain some curves on which we will need to have norms for sections of $\mathcal{O}(D)$ but to which the norm machinery as set up here does not apply. Namely, for $p \neq 2$ we have the curves $X_1(4p^m)_K^{\mathrm{an}}$ and $X_1(4p^m, p^2)_K^{\mathrm{an}}$, while for $p = 2$ we have $X_1(2^{m+1}N)_K^{\mathrm{an}}$ and $X_1(2^{m+1}N, 4)_K^{\mathrm{an}}$, where $m \geq 1$ and $N \in \{1, 3\}$. The previous lemma suggests an ad hoc workaround to this problem. In case we are working with sections of $\mathcal{O}(D)$ for a cuspidal divisor on one of these curves, we simply *define* the norm to be the supremum norm of the restriction of our section to the complement of the residue disks about the cusps. A more natural definition would likely result from considerations of "formal stacks", but this norm would surely turn out to be equal to ours by an analogue of Lemma 2.5.

2.3. ***Weight space.*** Throughout most of this paper, $\mathcal{W}$ will denote p-adic weight space (everywhere except for the beginning of Section 7, where it is allowed to be a general reduced rigid space for the purpose of reviewing a general construction). That is, $\mathcal{W}$ is a rigid space over $\mathbb{Q}_p$ whose points with values in an extension $K/\mathbb{Q}_p$ are $\mathcal{W}(K) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, K^\times)$. Define $\mathbf{q} = p$ if $p \neq 2$ and $\mathbf{q} = 4$ if $p = 2$. Let $\tau : \mathbb{Z}_p^\times \to (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times \to \mathbb{Q}_p^\times$ denote reduction composed with the Teichmüller lifting, and let $\langle x \rangle = x/\tau(x) \in 1 + \mathbf{q}\mathbb{Z}_p$. For a weight κ we have

$$\kappa(x) = \kappa(\langle x \rangle)\kappa(\tau(x)) = \kappa(\langle x \rangle)\tau(x)^i$$

for a unique integer i with $0 \leq i < \varphi(\mathbf{q})$ (where φ denotes Euler's function). Moreover, this breaks up the space $\mathcal{W}$ as the admissible disjoint union of $\varphi(\mathbf{q})$ admissible opens $\mathcal{W}^i$, each of which is isomorphic to a one-dimensional open ball.

For each positive integer n, let $\mathcal{W}_n$ denote the admissible open subspace of $\mathcal{W}$ whose points are those κ with

$$|\kappa(1+\mathbf{q})^{p^{n-1}} - 1| \leq |\mathbf{q}|.$$

Then $\mathcal{W}_n^i := \mathcal{W}^i \cap \mathcal{W}_n$ is an affinoid disk in $\mathcal{W}^i$, and the $\{\mathcal{W}_n^i\}_n$ form a nested admissible cover of $\mathcal{W}^i$.

To each integer λ we may associate the weight $x \mapsto x^\lambda$. This weight, which by abuse of notation we simply refer to as λ, lies in $\mathcal{W}^i$ for the unique $i \equiv \lambda \pmod{\varphi(\mathbf{q})}$. Also, if λ is an integer and $\psi : (\mathbb{Z}/\mathbf{q}p^{n-1}\mathbb{Z})^\times \to \mathbb{C}_p^\times$ is a character, then $x \mapsto x^\lambda\psi(x)$ is a point in $\mathcal{W}$ (with values in $\mathbb{Q}_p(\mu_{p^{n-1}})$) that lies in $\mathcal{W}_n$, as standard estimates for $|\zeta - 1|$ for roots of unity ζ demonstrate.

3. Some modular functions

Our definition of the spaces of half-integral weight modular forms will follow the general approach of [Coleman and Mazur 1998] (in the integral weight p-adic situation) and [Ramsey 2006] (in the half-integral weight situation). The motivating idea behind this approach is to reduce to weight zero by dividing by a well-understood form of the same weight. For example, if f is a half-integral weight p-adic modular form of weight $k/2$, θ is the usual Jacobi theta function of weight $1/2$, and E_λ is the weight $\lambda = (k-1)/2$ Eisenstein series introduced below, then $f/(E_\lambda\theta)$ should certainly be a meromorphic modular function of weight zero. As we have no working notion of "half-integral weight p-adic modular form", we simply use the weight zero forms so obtained as the *definition* of this notion. One must of course work out issues such as exactly what kind of poles are introduced, how dividing by θE_λ affects the nebentypus character, and how to translate the classical Hecke action into an action on these new forms. The precise definition will be given in Section 4.

This was carried out in [Ramsey 2006] by dividing by θ^k instead of θE_λ. That approach had the disadvantage of limiting us to *classical* weights $k/2$, whereas the current approach will work for more general p-adic weights (and indeed, for families of modular forms) since E_λ interpolates nicely in the variable λ.

This technique of division to reduce to weight zero in order to define modular forms forces us to modify the usual construction of the Hecke operators using the Hecke correspondences on the curve $X_1(N)$ by multiplying by certain functions on the source spaces of these correspondences. Our first task is to define these functions and to establish their overconvergence properties. Since we are dividing by $E_\lambda\theta$ to reduce to weight zero, we will require, for each prime number ℓ, a modular function whose q-expansion (at the appropriate cusp and on the appropriate space, which depends on whether or not $\ell = p$) is

$$\frac{E_\lambda(q_{\ell^2})\theta(q_{\ell^2})}{E_\lambda(q)\theta(q)}.$$

Factoring this into its Eisenstein part and theta part, we split the problem into two problems, the first of which is nearly done in the integral-weight literature (see [Buzzard 2007; Coleman and Mazur 1998]), and the second of which is done in [Ramsey 2006]. We briefly review both problems here, but see these references for details. Note that all analytic spaces in this section are taken over $\mathbb{Q}_p$.

Let $\mathbf{c}$ denote the cusp on $X_1(4)_\mathbb{Q}$ corresponding to the point $\zeta_4 q_2$ of order 4 on the Tate curve. Define a $\mathbb{Q}$-divisor Σ_{4N} on the curve $X_1(4N)_\mathbb{Q}$ by

$$\Sigma_4 := \tfrac{1}{4}\pi^*[\mathbf{c}], \quad \text{where } \pi : X_1(4N)_\mathbb{Q} \to X_1(4)_\mathbb{Q}$$

is the obvious degeneracy map. This divisor is set up to look like the divisor of zeros of the pullback of the Jacobi theta function θ to $X_1(4N)_\mathbb{Q}$ and will later be used to control poles introduced in dividing by $E_\lambda\theta$.

In [Ramsey 2006], we defined a rational function Θ_{ℓ^2} on $X_1(4, \ell^2)_{\mathbb{Q}}$ with divisor

$$\operatorname{div}(\Theta_{\ell^2}) = \pi_2^*\Sigma_4 - \pi_1^*\Sigma_4$$

such that

$$\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle q_{\ell^2}\rangle) = \sum_{n\in\mathbb{Z}} q_{\ell^2}^{n^2} \Big/ \sum_{n\in\mathbb{Z}} q^{n^2} = \theta(q_{\ell^2})/\theta(q).$$

Here π_1 and π_2 are the maps comprising the ℓ^2 Hecke correspondence on $X_1(4)$ and are defined in Section 5.1. Strictly speaking, we had assumed $\ell \neq 2$ in the arguments in [Ramsey 2006], but if one is only interested in the result above, then one can easily check that the arguments work for $\ell = 2$ verbatim.

Let us now turn to the Eisenstein part of the above functions. For further details and proofs of the claims in this paragraph, see [Buzzard 2007, Sections 6 and 7]. Let

$$E(q) := 1 + \frac{2}{\zeta_p(\kappa)} \sum_n \Big(\sum_{d\,|\,n,\, p\nmid d} \kappa(d) d^{-1} \Big) q^n \in \mathcal{O}(\mathcal{W}^0)[\![q]\!]$$

be the q-expansion of the p-deprived Eisenstein family over $\mathcal{W}^0$. Note that there are no problems with zeros of ζ_p since we are restricting our attention to $\mathcal{W}^0$. For a particular choice of $\kappa \in \mathcal{W}^0$, we denote by $E_\kappa(q)$ the expansion obtained by evaluating all of the coefficients at κ. In particular, for a positive integer λ no less than 2 and divisible by $\varphi(\mathbf{q})$, $E_\lambda(q)$ is the q-expansion of the usual p-deprived classical Eisenstein series of weight λ and level p.

Let ℓ be a prime number. If $\ell \neq p$, then there exists a rigid analytic function $\mathbf{E}_\ell$ on $X_0(p\ell)^{\mathrm{an}}_{\geq 1} \times \mathcal{W}^0$ whose q-expansion at $(\underline{\mathrm{Tate}}(q), \mu_{p\ell})$ is $E(q)/E(q^\ell)$. If $\ell = p$, then the same holds with $X_0(p\ell)^{\mathrm{an}}_{\geq 1}$ replaced by $X_0(p)^{\mathrm{an}}_{\geq 1}$ and $\mu_{p\ell}$ replaced by μ_p. Buzzard [2007] shows that there exists a sequence of rational numbers

$$1/(p+1) > r_1 \geq r_2 \geq \cdots \geq r_n \geq \cdots > 0$$

with $r_i < p^{2-i}/\mathbf{q}(1+p)$ such that, when restricted to $X_0(p\ell)^{\mathrm{an}}_{\geq 1} \times \mathcal{W}^0_n$ (respectively, $X_0(p)^{\mathrm{an}}_{\geq 1} \times \mathcal{W}^0_n$ if $\ell = p$), $\mathbf{E}_\ell$ analytically continues to an invertible function on $X_0(p\ell)^{\mathrm{an}}_{\geq p^{-r_n}} \times \mathcal{W}^0_n$ (respectively, $X_0(p)^{\mathrm{an}}_{\geq p^{-r_n}} \times \mathcal{W}^0_n$ if $\ell = p$). Fix such a sequence once and for all. Let us first extend these results to square level.

Lemma 3.1. *Let $\ell \neq p$ be a prime number. There exists an invertible function $\mathbf{E}_{\ell^2}$ on $X_0(p\ell^2)^{\mathrm{an}}_{\geq 1} \times \mathcal{W}^0$ whose q-expansion at $(\underline{\mathrm{Tate}}(q), \mu_{p\ell^2})$ is $E(q)/E(q^{\ell^2})$. Moreover, the function $\mathbf{E}_{\ell^2}$, when restricted to $\mathcal{W}^0_n$, analytically continues to an invertible function on $X_0(p\ell^2)^{\mathrm{an}}_{\geq p^{-r_n}} \times \mathcal{W}^0_n$.*

There exists an invertible function $\mathbf{E}_{p^2}$ on $X_0(p)^{\mathrm{an}}_{\geq 1} \times \mathcal{W}^0$ whose q-expansion at $(\underline{\mathrm{Tate}}(q), \mu_p)$ is $E(q)/E(q^{p^2})$. Moreover, the function $\mathbf{E}_{p^2}$, when restricted to $\mathcal{W}^0_n$, analytically continues to an invertible function on $X_0(p)^{\mathrm{an}}_{\geq p^{-r_n/p}} \times \mathcal{W}^0_n$.

Proof. Let ℓ be a prime different from p. There are two natural maps

$$X_0(p\ell^2)^{\mathrm{an}}_{\mathbb{Q}_p} \to X_0(p\ell)^{\mathrm{an}}_{\mathbb{Q}_p},$$

namely those given on noncuspidal points by

$$(E, C) \xmapsto{d_{\ell,1}} (E, \ell C) \quad \text{and} \quad (E, C) \xmapsto{d_{\ell,2}} (E/p\ell C, C/p\ell C).$$

Both of these restrict to maps

$$d_{\ell,1}, d_{\ell,2} : X_0(p\ell^2)^{\mathrm{an}}_{\geq p^{-r_n}} \to X_0(p\ell)^{\mathrm{an}}_{\geq p^{-r_n}}.$$

We define $\mathbf{E}_{\ell^2}$ to be the invertible function

$$\mathbf{E}_{\ell^2} := d^*_{\ell,1}\mathbf{E}_\ell \cdot d^*_{\ell,2}\mathbf{E}_\ell \in \mathcal{O}(X_0(p\ell^2)^{\mathrm{an}}_{\geq p^{-r_n}} \times \mathcal{W}^0_n)^\times. \tag{2}$$

The q-expansion of $\mathbf{E}_{\ell^2}$ at $(\underline{\mathrm{Tate}}(q), \mu_{p\ell^2})$ is

$$\begin{aligned}\mathbf{E}_\ell(d_{\ell,1}(\underline{\mathrm{Tate}}(q), \mu_{p\ell^2}))&\mathbf{E}_\ell(d_{\ell,2}(\underline{\mathrm{Tate}}(q), \mu_{p\ell^2}))\\ &= \mathbf{E}_\ell(\underline{\mathrm{Tate}}(q), \mu_{p\ell})\mathbf{E}_\ell(\underline{\mathrm{Tate}}(q)/\mu_\ell, \mu_{p\ell^2}/\mu_\ell)\\ &= \mathbf{E}_\ell(\underline{\mathrm{Tate}}(q), \mu_{p\ell})\mathbf{E}_\ell(\underline{\mathrm{Tate}}(q^\ell), \mu_{p\ell})\\ &= \frac{E(q)}{E(q^\ell)}\frac{E(q^\ell)}{E(q^{\ell^2})} = \frac{E(q)}{E(q^{\ell^2})}.\end{aligned}$$

One must take additional care if $\ell = p$. Then there is a well-defined map

$$d : X_0(p)^{\mathrm{an}}_{\geq p^{-r_n/p}} \to X_0(p)^{\mathrm{an}}_{\geq p^{-r_n}}, \quad (E, C) \mapsto (E/C, H_{p^2}/C),$$

where H_{p^2} is the canonical subgroup of E of order p^2. This follows from the fact that $X_0(p)^{\mathrm{an}}_{\geq p^{-r_n/p}}$ consists of pairs (E, C) with C equal to the canonical subgroup of E of order p, and standard facts about quotienting by such subgroups; see for example [Buzzard 2003, Theorem 3.3]. We define an invertible function by

$$\mathbf{E}_{p^2} := \mathbf{E}_p \cdot d^*\mathbf{E}_p \in \mathcal{O}(X_0(p)^{\mathrm{an}}_{\geq p^{-r_n/p}} \times \mathcal{W}^0_n)^\times,$$

where we have implicitly restricted $\mathbf{E}_p$ to

$$X_0(p)^{\mathrm{an}}_{\geq p^{-r_n/p}} \times \mathcal{W}^0_n \subseteq X_0(p)^{\mathrm{an}}_{\geq p^{-r_n}} \times \mathcal{W}^0_n.$$

The q-expansion of $\mathbf{E}_{p^2}$ at $(\underline{\mathrm{Tate}}(q), \mu_p)$ is

$$\begin{aligned}\mathbf{E}_p(\underline{\mathrm{Tate}}(q), \mu_p)&\mathbf{E}_p(d(\underline{\mathrm{Tate}}(q), \mu_p))\\ &= \mathbf{E}_p(\underline{\mathrm{Tate}}(q), \mu_p)\mathbf{E}_p(\underline{\mathrm{Tate}}(q)/\mu_p, \mu_{p^2}/\mu_p)\\ &= \mathbf{E}_p(\underline{\mathrm{Tate}}(q), \mu_p)\mathbf{E}_p(\underline{\mathrm{Tate}}(q^p), \mu_p)\\ &= \frac{E(q)}{E(q^p)}\frac{E(q^p)}{E(q^{p^2})} = \frac{E(q)}{E(q^{p^2})}.\end{aligned}$$

□

Let

$$\pi : X_1(p,\ell^2)^{\mathrm{an}}_{\mathbb{Q}_p} \to \begin{cases} X_0(p\ell^2)^{\mathrm{an}}_{\mathbb{Q}_p} & \text{if } \ell \neq p, \\ X_0(p)^{\mathrm{an}}_{\mathbb{Q}_p} & \text{if } \ell = p \end{cases}$$

denote the map given on noncuspidal points by

$$(E,P,C) \longmapsto \begin{cases} (E/C, (\langle P\rangle + E[\ell^2])/C) & \text{if } \ell \neq p, \\ (E/C, \langle P\rangle/C) & \text{if } \ell = p. \end{cases}$$

Note that we have

$$\pi(\underline{\mathrm{Tate}}(q), \zeta_p, \langle q_{\ell^2}\rangle) = \begin{cases} (\underline{\mathrm{Tate}}(q_{\ell^2}), \mu_{p\ell^2}) & \text{if } \ell \neq p, \\ (\underline{\mathrm{Tate}}(q_{p^2}), \mu_p) & \text{if } \ell = p. \end{cases} \tag{3}$$

This observation suggests that perhaps the components $X_1(p,\ell^2)^{\mathrm{an}}_{\geq p^{-r}}$ should be related (via π) to the components $X_0(p\ell^2)^{\mathrm{an}}_{\geq p^{-r}}$.

Lemma 3.2. *If $\ell \neq p$, then the map π restricts to*

$$\pi : X_1(p,\ell^2)^{\mathrm{an}}_{\geq p^{-r}} \to X_0(p\ell^2)^{\mathrm{an}}_{\geq p^{-r}} \quad \textit{for all } r < p/(1+p).$$

In case $\ell = p$, the map π restricts to

$$X_1(p,p^2)^{\mathrm{an}}_{\geq p^{-p^2 r}} \to X_0(p)^{\mathrm{an}}_{\geq p^{-r}} \quad \textit{for all } r < 1/p(1+p).$$

Proof. First suppose $\ell \neq p$. Let $\mathcal{U}$ denote the entirety of the locus in $X_0(p\ell^2)^{\mathrm{an}}_{\mathbb{Q}_p}$ defined by $v(E) \leq r$. First note that, since quotienting by a subgroup of order prime to p does not change its measure of singularity, the map π restricts to a map

$$X_1(p,\ell^2)^{\mathrm{an}}_{\geq p^{-r}} \to \mathcal{U}.$$

The inverse images of the two connected components of $\mathcal{U}$ under this map are disjoint admissible opens that admissibly cover a connected space, and, by (3), $\pi^{-1}(X_0(p\ell^2)^{\mathrm{an}}_{\geq p^{-r}})$ is nonempty, so this must be all of $X_1(p,\ell^2)^{\mathrm{an}}_{\geq p^{-r}}$. The result follows.

Now suppose that $\ell = p$. Let $\mathcal{U}$ denote the entirety of the locus in $X_0(p)^{\mathrm{an}}_{\mathbb{Q}_p}$ defined by $v(E) \leq r$. Once we verify that π restricts to

$$X_1(p,p^2)^{\mathrm{an}}_{\geq p^{-p^2 r}} \to \mathcal{U},$$

the argument may proceed exactly as above. We claim, moreover, that if (E,P,C) is a point in $X_0(p,p^2)^{\mathrm{an}}_{\geq p^{-p^2 r}}$, then $v(E/C) = v(E)/p^2$. This would follow if we knew that C met the canonical subgroup of E trivially (again by standard facts about quotienting by canonical and noncanonical subgroups of order p, as in [Buzzard 2003, Section 3]), so it suffices to prove that $\langle P\rangle$ *is* the canonical subgroup of E.

The natural map

$$X_1(p,p^2)\to X_0(p),\quad (E,P,C)\mapsto (E,\langle P\rangle)$$

restricts to $X_1(p,p^2)^{\mathrm{an}}_{\geq p^{-r}}\to X_0(p)^{\mathrm{an}}_{\geq p^{-r}}$ by the same connectivity argument used in the $\ell\neq p$ case (since this map clearly doesn't change $v(E)$). But it is well known that the locus $X_0(p)^{\mathrm{an}}_{\geq p^{-r}}$ consists of pairs (E,C) with C equal to the canonical subgroup of E. □

We may pull back the Eisenstein family of Lemma 3.1 for $\ell\neq p$ through the map π to arrive at an invertible function on $X_1(p,\ell^2)^{\mathrm{an}}_{\geq p^{-r_n}}\times\mathcal{W}^0_n$. By the previous lemma, we may also pull back the family for $\ell=p$ through π to arrive at an invertible function on $X_0(p,p^2)^{\mathrm{an}}_{\geq p^{-pr_n}}\times\mathcal{W}^0_n$. For any ℓ, it follows from (3) that the function $\pi^*\mathbf{E}_{\ell^2}$ satisfies

$$\pi^*\mathbf{E}_{\ell^2}(\underline{\mathrm{Tate}}(q),\zeta_p,\langle q_{\ell^2}\rangle)=\frac{E(q_{\ell^2})}{E((q_{\ell^2})^{\ell^2})}=\frac{E(q_{\ell^2})}{E(q)}.$$

To arrive at the functions we need, we simply multiply $\pi^*\mathbf{E}_{\ell^2}$ and Θ_{ℓ^2} (which is constant in the weight). Of course, to do so we must first pull these functions back so that they lie on a common curve. The natural ("smallest") curve to use depends on whether or not $p=2$, since 2 already lies in the Γ_1 part of the level of Θ_{ℓ^2}. The next proposition summarizes the properties of the resulting functions.

Proposition 3.3. *Let p be and ℓ be primes. There exists an element $\mathbf{H}_{\ell^2}$ of*

$$\begin{cases} H^0(X_1(4p,\ell^2)^{\mathrm{an}}_{\geq 1}\times\mathcal{W}^0,\mathcal{O}(\pi_1^*\Sigma_{4p}-\pi_2^*\Sigma_{4p})) & \text{if } p\neq 2,\\ H^0(X_1(4,\ell^2)^{\mathrm{an}}_{\geq 1}\times\mathcal{W}^0,\mathcal{O}(\pi_1^*\Sigma_{4}-\pi_2^*\Sigma_{4})) & \text{if } p=2\end{cases}$$

whose q-expansion at

$$\begin{cases}(\underline{\mathrm{Tate}}(q),\mu_{4p},\langle q_{\ell^2}\rangle)) & \text{if } p\neq 2,\\ (\underline{\mathrm{Tate}}(q),\mu_{4},\langle q_{\ell^2}\rangle)) & \text{if } p=2\end{cases}\qquad \text{is equal to}\quad \frac{E(q_{\ell^2})\theta(q_{\ell^2})}{E(q)\theta(q)}.$$

Moreover, there exists a sequence of rational numbers r_n such that

$$1/(1+p)>r_1\geq r_2\geq\cdots>0$$

with $r_i<p^{2-i}/\mathbf{q}(1+p)$ such that $\mathbf{H}_{\ell^2}$, when restricted to $\mathcal{W}^0_n$, analytically continues to the region

$$\begin{cases} X_1(4p,\ell^2)^{\mathrm{an}}_{\geq p^{-r_n}}\times\mathcal{W}^0_n & \text{if } p\neq 2,\ \ell\neq p,\\ X_1(4p,p^2)^{\mathrm{an}}_{\geq p^{-pr_n}}\times\mathcal{W}^0_n & \text{if } p\neq 2,\ \ell=p,\\ X_1(4,\ell^2)^{\mathrm{an}}_{\geq 2^{-r_n}}\times\mathcal{W}^0_n & \text{if } p=2,\ \ell\neq 2,\\ X_1(4,4)^{\mathrm{an}}_{\geq 2^{-2r_n}}\times\mathcal{W}^0_n & \text{if } p=\ell=2.\end{cases}$$

Finally, we wish to extend $\mathbf{H}_{\ell^2}$ and $E(q)$ to all of $\mathcal{W}$. To do this, we simply pull back through the natural map

$$\mathcal{W} \to \mathcal{W}^0, \quad \kappa \mapsto \kappa \circ \langle \cdot \rangle. \tag{4}$$

When restricted to $\mathcal{W}^i$, this map is simply the isomorphism $\kappa \mapsto \kappa/\tau^i$.

Remark 3.4. We have chosen in the end to use Γ_1-structure on the curves on which the $\mathbf{H}_{\ell^2}$ lie both to rigidify the associated moduli problems over $\mathbb{Q}_p$, as well as because these are the curves that will actually turn up in the sequel. We note, however, that the $\mathbf{H}_{\ell^2}$ are invariant under all diamond automorphisms.

4. The spaces of forms

In this section we define spaces of overconvergent p-adic modular forms as well as families thereof over admissible open subsets of $\mathcal{W}$. Again, the motivating idea behind these definitions is that we have reduced to weight 0 via division by the well-understood forms $E_\lambda\theta$. By "well-understood" we essentially mean two things. The first is that we understand their zeros once we eliminate part of the supersingular locus (and thereby *remove* the zeros of the Eisenstein part). The second is that, by the previous section, we know that there are modular functions with q-expansions

$$\frac{E_\lambda(q_{\ell^2})\theta(q_{\ell^2})}{E_\lambda(q)\theta(q)}$$

that interpolate rigid-analytically in λ, a fact that we will need to define Hecke operators on families in the next section.

Before defining the spaces of forms, we need to make a couple of remarks about diamond automorphisms. For a positive integer N and an element $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, let $\langle d\rangle$ denote the usual diamond automorphism of $X_1(N)$ given on (noncuspidal) points by $(E, P) \mapsto (E, dP)$. Now suppose we are given a factorization $N = N_1N_2$ into relatively prime factors, so the natural reduction map

$$(\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\sim} (\mathbb{Z}/N_1\mathbb{Z})^\times \times (\mathbb{Z}/N_2\mathbb{Z})^\times$$

is an isomorphism. For $a \in (\mathbb{Z}/N_1\mathbb{Z})^\times$ and $b \in (\mathbb{Z}/N_2\mathbb{Z})^\times$ we let $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^\times$ denote the inverse image of the pair (a, b) under the this map. For $a \in (\mathbb{Z}/N_1\mathbb{Z})^\times$, we define $\langle a\rangle_{N_1} := \langle (a, 1)\rangle$, and we refer to these automorphisms as *the diamond automorphisms at* N_1. The diamond automorphisms at N_2 are defined similarly, and we have a factorization $\langle d\rangle = \langle d\rangle_{N_1} \circ \langle d\rangle_{N_2}$. Finally, we observe that the diamond operators on $X_1(4N)^{\mathrm{an}}_K$ preserve the subspaces $X_1(4N)^{\mathrm{an}}_{\geq p^{-r}}$ and the divisor Σ_{4N} in the sense that $\langle d\rangle^{-1}(X_1(4N)^{\mathrm{an}}_{\geq p^{-r}}) = X_1(4N)^{\mathrm{an}}_{\geq p^{-r}}$ and $\langle d\rangle^*\Sigma_{4N} = \Sigma_{4N}$, respectively.

Convention 4.1. By the symbol $\mathcal{O}(\Sigma)$ for a $\mathbb{Q}$-divisor Σ we shall always mean $\mathcal{O}(\lfloor\Sigma\rfloor)$, where $\lfloor\Sigma\rfloor$ is the divisor obtained by taking the floor of each coefficient occurring in Σ.

First we define the spaces of forms of fixed weight. Let N be a positive integer and suppose that either $p \nmid 4N$ or that $p = 2$ and $p \nmid N$.

Definition 4.2. Let $\kappa \in \mathcal{W}^i(K)$ and pick n such that $\kappa \in \mathcal{W}^i_n$. Then, for any rational number r with $0 \le r \le r_n$, we define the space of p-adic half-integral weight modular forms of weight κ, tame level $4N$ (or rather N if $p = 2$), and growth condition p^{-r} over K to be

$$\widetilde{M}_\kappa(4N, K, p^{-r}) := \begin{cases} H^0(X_1(4Np)^{\mathrm{an}}_{\ge p^{-r}}, \mathcal{O}(\Sigma_{4Np}))^{\tau^i} \times \{\kappa\} & \text{if } p \ne 2, \\ H^0(X_1(4N)^{\mathrm{an}}_{\ge 2^{-r}}, \mathcal{O}(\Sigma_{4N}))^{(-1/\,\cdot\,)^i\tau^i} \times \{\kappa\} & \text{if } p = 2, \end{cases}$$

where $(\,\cdot\,)^{\tau^i}$ denotes the τ^i eigenspace for the action of the diamond automorphisms at p, and similarly for $(-1/\,\cdot\,)^i\tau^i$ if $p = 2$.

Remarks 4.3.

- For $p \ne 2$, we have chosen to remove p from the level and only indicate the tame level in the notation because, as we will see, these spaces contain forms of all p-power level. However, for $p = 2$, we have left the 4 in as a reminder that the forms have at least a 4 in the level, as well as for some uniformity in notation.
- Note that this space has been "tagged" with the weight κ because the actual space has only a rather trivial dependence on κ (κ serves only to restrict the admissible K and r and to determine i). The point is that, as we will see, the Hecke action on this space is very sensitive to κ. The tag will generally be ignored in what follows as the weight will be clear from the context.
- This space is endowed with a norm which is defined as in Section 2.2 and is a Banach space over K with respect to this norm.
- We call the forms belonging to spaces with $r > 0$ *overconvergent*. The space of all overconvergent forms (of this weight and level) is the inductive limit
$$\widetilde{M}^\dagger_\kappa(4N, K) = \lim_{r\to 0} \widetilde{M}_\kappa(4N, K, p^{-r}).$$
- In case κ is the character associated to an integer $\lambda \ge 0$, the space of forms defined above would classically be thought of as having weight $\lambda + 1/2$. Our choice of p-adic weight character bookkeeping seems to be the most natural one (the Shimura lifting has the effect of squaring the weight character, for example).
- In case κ is the weight associated to an integer $\lambda \ge 0$, then the definition here is somewhat less general than the definition of the space of forms of weight $\lambda + 1/2$ contained in [Ramsey 2006], due to the need to eliminate enough of

the supersingular locus to get rid of the Eisenstein zeros. The two definitions are (Hecke-equivariantly) isomorphic whenever they are both defined, as we will see in Proposition 6.2.

- The tilde is an homage to the metaplectic literature and will be used henceforth on all half-integral weight objects in order to distinguish them from their integral weight counterparts.

We now turn to the spaces of families of modular forms.

Definition 4.4. Let X be a connected affinoid subdomain of $\mathcal{W}$. Then $X \subseteq \mathcal{W}^i$ for some i since X is connected, and $X \subseteq \mathcal{W}^i_n$ for some n since X is affinoid. For any rational number r with $0 \leq r \leq r_n$, we define the space of families of half-integral weight modular forms of tame level $4N$ and growth condition p^{-r} on X to be

$$\widetilde{M}_X(4N, K, p^{-r}) := \begin{cases} H^0(X_1(4Np)^{\mathrm{an}}_{\geq p^{-r}}, \mathcal{O}(\Sigma_{4Np}))^{\tau^i} \widehat{\otimes}_K \mathcal{O}(X) & \text{if } p \neq 2, \\ H^0(X_1(4N)^{\mathrm{an}}_{\geq 2^{-r}}, \mathcal{O}(\Sigma_{4N}))^{(-1/\cdot)^i \tau^i} \widehat{\otimes}_K \mathcal{O}(X) & \text{if } p = 2. \end{cases}$$

Remarks 4.5.

- We endow $\widetilde{M}_X(4N, K, p^{-r})$ with the completed tensor product norm obtained from the norms defined in Section 2.2 and from the supremum norm on $\mathcal{O}(X)$. The space $\widetilde{M}_X(4N, K, p^{-r})$ with this norm is a Banach module over the Banach algebra $\mathcal{O}(X)$.

- As in the case of fixed weight, the definition depends rather trivially on X, but the Hecke action will be very sensitive to X.

- In general, if X is an affinoid subdomain of $\mathcal{W}$, we define $\widetilde{M}_X$ to be the direct sum of the spaces corresponding to the connected components of X. Also, just as for particular weights, we can talk about the space of all overconvergent families of forms on X, namely $\widetilde{M}^\dagger_X(4N, K) = \lim_{r \to 0} \widetilde{M}_X(4N, K, p^{-r})$.

- Using a simple projector argument, one sees easily that we have a canonical identification

$$H^0(X_1(4Np)^{\mathrm{an}}_{\geq p^{-r}}, \mathcal{O}(\Sigma_{4Np}))^{\tau^i} \widehat{\otimes}_K \mathcal{O}(X) \cong (H^0(X_1(4Np)^{\mathrm{an}}_{\geq p^{-r}}, \mathcal{O}(\Sigma_{4Np})) \widehat{\otimes}_K \mathcal{O}(X))^{\tau^i},$$

and similarly at level $4N$ if $p = 2$. This will prove useful in the next section.

For each X as above and each L-valued point $\kappa \in X$, evaluation at x induces a specialization map $\widetilde{M}_X(4N, K, p^{-r}) \to \widetilde{M}_\kappa(4N, L, p^{-r})$. In the next section we will define a Hecke action on both of these spaces for which such specialization maps are equivariant and which recover the usual Hecke operators on the right side above (in the sense that they are given by the usual formulas on q-expansions).

Each of the spaces of forms that we have defined has a cuspidal subspace consisting of forms that "vanish at the cusps." This notion is a little subtle in half-integral weight because there are often cusps at which *all* forms are *forced* to vanish. To explain this comment and motivate the subsequent definition of the space of cusp forms, let us go back to the motivation behind our definitions of the spaces of forms. If F is a form of half-integral weight in our setting, then $F\theta E$ (where E is an appropriate Eisenstein series) is what we would "classically" like to think of as a half-integral weight form. Indeed, if F is classical (this notion is defined in Section 6), then $F\theta E$ can literally be identified with a classical holomorphic modular form of half-integral weight over $\mathbb{C}$. The condition $\operatorname{div}(F) \geq -\Sigma_{4Np}$ (we are assuming $p \neq 2$ for the sake of this motivation) in our definition is exactly the condition that $F\theta E$ be holomorphic at all cusps. Likewise, the condition that this inequality be strict at all cusps is the condition that $F\theta E$ be cuspidal. But since $\operatorname{div}(F)$ has integral coefficients, the nonstrict inequality *implies* the strict inequality at all cusps where Σ_{4Np} has nonintegral coefficients.

With this in mind, we are led to the following definition of cusp forms. For an integer M, let C_{4M} be the divisor on $X_1(4M)^{\mathrm{an}}_{\mathbb{Q}_p}$ given by the sum of the cusps at which Σ_{4M} has integral coefficients. To define the cuspidal subspace of any of the above spaces of forms, we replace the divisor Σ_{4Np} (respectively Σ_{4N} if $p=2$) by the divisor $\Sigma_{4Np} - C_{4Np}$ (respectively $\Sigma_{4N} - C_{4N}$ if $p = 2$). We will denote the cuspidal subspaces by the letter S instead of M. Thus, for example, if $\kappa \in \mathcal{W}^i_n(K)$ and $0 \leq r \leq r_n$, we define

$$\widetilde{S}_\kappa(4N, K, p^{-r}) = \begin{cases} H^0(X_1(4Np)^{\mathrm{an}}_{\geq p^{-r}}, \mathcal{O}(\Sigma_{4Np} - C_{4Np}))^{\tau^i} \times \{\kappa\} & \\ & \text{if } p \neq 2, \\ H^0(X_1(4N)^{\mathrm{an}}_{\geq 2^{-r}}, \mathcal{O}(\Sigma_{4N} - C_{4N}))^{(-1/\cdot)^i\tau^i} \times \{\kappa\} & \\ & \text{if } p = 2. \end{cases}$$

Remarks 4.3 and 4.5 apply equally well to the corresponding spaces of cusp forms.

5. Hecke operators

Before we construct Hecke operators, we need to make some remarks on diamond operators and nebentypus. Since the p-part of the nebentypus character is encoded as part of the p-adic weight character, we need to separate out the tame part of the diamond action. Fix a weight $\kappa \in \mathcal{W}^i(K)$. To define the tame diamond operators compatibly with the classical definitions and those in [Ramsey 2006], we must twist (at least in the case $p \neq 2$) those obtained via pullback from the automorphism

$\langle\cdot\rangle_{4N}$ by $(-1/\cdot)^i$. That is, for $F \in \widetilde{M}_\kappa(4N, K, p^{-r})$, we define

$$\langle d\rangle_{4N,\kappa}F = \left(\frac{-1}{d}\right)^i \langle d\rangle^*_{4N}F \quad \text{if } p \neq 2,$$
$$\langle d\rangle_{N,\kappa}F = \langle d\rangle^*_N F \quad \text{if } p = 2.$$

Without this twist in the $p \neq 2$ case, the definition would not agree with the classical one because of the particular nature of the automorphy factor of the form θ used in the identification of our forms with classical forms. The same formulas define operators $\langle\cdot\rangle_{4N,X}$ and $\langle\cdot\rangle_{N,X}$ on the space of families of modular forms over $X \subseteq \mathcal{W}^i$. For a more general $X \subseteq \mathcal{W}$, we break into the components in $\mathcal{W}^i$ for each i and define $\langle\cdot\rangle_{4N,X}$ and $\langle\cdot\rangle_{N,X}$ component by component. For a character χ modulo $4N$ (respectively modulo N if $p = 2$), we define the space of forms of tame nebentypus χ to be the χ-eigenspace of $\widetilde{M}_\kappa(4N, K, p^{-r})$ for the operators $\langle\cdot\rangle_{4N,\kappa}$ (respectively $\langle\cdot\rangle_{N,\kappa}$ if $p = 2$). The same definition applies to families of forms. These subspaces are denoted by appending a χ to the list of arguments (for example, $\widetilde{M}_\kappa(4N, K, p^{-r}, \chi)$).

Let $\mathcal{X}$ and $\mathcal{Y}$ be rigid spaces equipped with a pair of maps $\pi_1, \pi_2 : \mathcal{X} \to \mathcal{Y}$, and let D be a $\mathbb{Q}$-divisor on $\mathcal{Y}$ such that $\pi_1^*D - \pi_2^*D$ has integral coefficients. Let $\mathcal{Z} \subseteq \mathcal{X}$ be an admissible affinoid open, and let $H \in H^0(\mathcal{Z}, \mathcal{O}(\pi_1^*D - \pi_2^*D))$. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{Y}$ be admissible affinoid opens such that $\pi_1^{-1}(\mathcal{V}) \cap \mathcal{Z} \subseteq \pi_2^{-1}(\mathcal{U}) \cap \mathcal{Z}$, and suppose that $\pi_1 : \pi_1^{-1}(\mathcal{V}) \cap \mathcal{Z} \to \mathcal{V}$ is finite and flat. Then there is a well-defined map $H^0(\mathcal{U}, \mathcal{O}(D)) \to H^0(\mathcal{V}, \mathcal{O}(D))$ given by the composition

$$H^0(\mathcal{U}, \mathcal{O}(D)) \xrightarrow{\pi_2^*} H^0(\pi_2^{-1}(\mathcal{U}) \cap \mathcal{Z}, \mathcal{O}(\pi_2^*D)) \xrightarrow{\text{res}} H^0(\pi_1^{-1}(\mathcal{V}) \cap \mathcal{Z}, \mathcal{O}(\pi_2^*D))$$
$$\xrightarrow{\cdot H} H^0(\pi_1^{-1}(\mathcal{V}) \cap \mathcal{Z}, \mathcal{O}(\pi_1^*D)) \xrightarrow{\pi_{1*}} H^0(\mathcal{V}, \mathcal{O}(D))$$

in which π_{1*} is the trace map corresponding to the finite and flat map π_1.

5.1. ***Hecke operators for a fixed weight.*** Let N be as above, let ℓ be any prime number, and let

$$\pi_1, \pi_2 : \begin{cases} X_1(4Np, \ell^2)^{\text{an}}_K \to X_1(4Np)^{\text{an}}_K & \text{if } p \neq 2, \\ X_1(4N, \ell^2)^{\text{an}}_K \to X_1(4N)^{\text{an}}_K & \text{if } p = 2 \end{cases}$$

be the maps defined on noncuspidal points of the underlying moduli problem by

$$\pi_1 : (E, P, C) \mapsto (E, P) \quad \text{and} \quad \pi_2 : (E, P, C) \mapsto (E/C, P/C).$$

Suppose that $\ell \neq p$. Then

$$\begin{cases} \pi_1^{-1}(X_1(4Np)^{\text{an}}_{\geq p^{-r}}) = \pi_2^{-1}(X_1(4Np)^{\text{an}}_{\geq p^{-r}}) & \text{if } p \neq 2, \\ \pi_1^{-1}(X_1(4N)^{\text{an}}_{\geq 2^{-r}}) = \pi_2^{-1}(X_1(4N)^{\text{an}}_{\geq 2^{-r}}) & \text{if } p = 2 \end{cases}$$

for any $r < p/(1+p)$, since quotienting an elliptic curve by a subgroup of order prime to p does not change its measure of singularity. Fix a weight $\kappa \in \mathcal{W}^i(K)$, and let $\mathbf{H}_{\ell^2}(\kappa)$ denote the specialization of $\mathbf{H}_{\ell^2}$ to $\kappa \in \mathcal{W}$ (which, recall, is defined to be the specialization of $\mathbf{H}_{\ell^2}$ to $\kappa/\tau^i \in \mathcal{W}^0$). Pick n such that $\kappa \in \mathcal{W}^i_n$, and suppose $0 \le r \le r_n$. Apply the general construction above with the following table:

	$p \neq 2$	$p = 2$
$\mathcal{X}$	$X_1(4Np, \ell^2)^{\text{an}}_K$	$X_1(4N, \ell^2)^{\text{an}}_K$
$\mathcal{Y}$	$X_1(4Np)^{\text{an}}_K$	$X_1(4N)^{\text{an}}_K$
$\mathcal{Z}$	$X_1(4Np, \ell^2)^{\text{an}}_{\ge p^{-r}}$	$X_1(4N, \ell^2)^{\text{an}}_{\ge 2^{-r}}$
D	Σ_{4Np}	Σ_{4N}
H	$\mathbf{H}_{\ell^2}(\kappa)$	$\mathbf{H}_{\ell^2}(\kappa)$
$\mathcal{U} = \mathcal{V}$	$X_1(4Np)^{\text{an}}_{\ge p^{-r}}$	$X_1(4N)^{\text{an}}_{\ge 2^{-r}}$

Then we arrive at an endomorphism of the K-vector space

$$\begin{cases} H^0(X_1(4Np)^{\text{an}}_{\ge p^{-r}}, \mathcal{O}(\Sigma_{4Np})) & \text{if } p \neq 2, \\ H^0(X_1(4N)^{\text{an}}_{\ge 2^{-r}}, \mathcal{O}(\Sigma_{4N})) & \text{if } p = 2. \end{cases}$$

We may easily check that, since the diamond operators act trivially on $\mathbf{H}_{\ell^2}$ (see Remark 3.4), this endomorphism commutes with the action of the diamond operators, and therefore induces an endomorphism of $\widetilde{M}_\kappa(4N, K, p^{-r})$. We define T_{ℓ^2} (or U_{ℓ^2} if $\ell \mid 4N$) to be the quotient of this endomorphism by ℓ^2.

Now suppose that $\ell = p$. Note that

$$\begin{cases} \pi_1^{-1}(X_1(4Np)^{\text{an}}_{\ge p^{-p^2 r}}) \subseteq \pi_2^{-1}(X_1(4Np)^{\text{an}}_{\ge p^{-r}}) & \text{if } p \neq 2, \\ \pi_1^{-1}(X_1(4N)^{\text{an}}_{\ge 2^{-2^2 r}}) \subseteq \pi_2^{-1}(X_1(4N)^{\text{an}}_{\ge 2^{-r}}) & \text{if } p = 2 \end{cases}$$

for any $r < 1/p(1+p)$. This follows from repeated application of the observation (made, for example, in [Buzzard 2003, Theorem 3.3(v)]) that if $v(E) < p/(1+p)$ and C is a subgroup of order p other than the canonical subgroup, then $v(E/C) = v(E)/p$ and the canonical subgroup of E/C is $E[p]/C$.

If $\kappa \in \mathcal{W}^i_n$ and r is chosen so that $0 \le r \le r_n$, then we may apply the construction above with the table

	$p \neq 2$	$p = 2$
$\mathcal{X}$	$X_1(4Np, p^2)^{\text{an}}_K$	$X_1(4N, 4)^{\text{an}}_K$
$\mathcal{Y}$	$X_1(4Np)^{\text{an}}_K$	$X_1(4N)^{\text{an}}_K$
$\mathcal{Z}$	$X_1(4Np, p^2)^{\text{an}}_{\ge p^{-pr}}$	$X_1(4N, 4)^{\text{an}}_{\ge 2^{-2r}}$
D	Σ_{4Np}	Σ_{4N}
H	$\mathbf{H}_{p^2}(\kappa)$	$\mathbf{H}_4(\kappa)$
$\mathcal{U}$	$X_1(4Np)^{\text{an}}_{\ge p^{-r}}$	$X_1(4N)^{\text{an}}_{\ge 2^{-r}}$
$\mathcal{V}$	$X_1(4Np)^{\text{an}}_{\ge p^{-pr}}$	$X_1(4N)^{\text{an}}_{\ge 2^{-2r}}$

to arrive at a linear map

$$\begin{cases} H^0(X_1(4Np)^{\mathrm{an}}_{\geq p^{-r}}, \mathcal{O}(\Sigma_{4Np})) \to H^0(X_1(4Np)^{\mathrm{an}}_{\geq p^{-pr}}, \mathcal{O}(\Sigma_{4Np})) & \text{if } p \neq 2, \\ H^0(X_1(4N)^{\mathrm{an}}_{\geq 2^{-r}}, \mathcal{O}(\Sigma_{4N})) \to H^0(X_1(4N)^{\mathrm{an}}_{\geq 2^{-2r}}, \mathcal{O}(\Sigma_{4N})) & \text{if } p = 2. \end{cases}$$

This map commutes with the diamond operators and restricts to a map

$$\widetilde{M}_\kappa(4N, K, p^{-r}) \to \widetilde{M}_\kappa(4N, K, p^{-pr}).$$

When composed with the natural restriction map

$$\widetilde{M}_\kappa(4N, K, p^{-pr}) \to \widetilde{M}_\kappa(4N, K, p^{-r}) \tag{5}$$

and divided by p^2, we arrive at an endomorphism of $\widetilde{M}_\kappa(4N, K, p^{-r})$, which we denote by U_{p^2}.

Proposition 5.1. *The Hecke operators defined above are continuous.*

Proof. Each of the spaces arising in the construction is a Banach space over K, so it suffices to show that each of the constituent maps of which our Hecke operators are the composition has finite norm. By Lemma 2.5 we may ignore the residue disks around the cusps when computing norms, thereby reducing ourselves to the supremum norm on functions. It follows easily that the pullback, restriction, and trace maps have norm not exceeding 1 and that multiplication by H has norm not exceeding the supremum norm of H on the complement of the residue disks around the cusps. The latter is finite since this complement is affinoid. □

Remarks 5.2.

- In the overconvergent case, that is, when we have $r > 0$, the restriction map (5) is compact; see [Coleman 1997, Proposition A5.2]. It follows that U_{p^2} is compact since it is the composition of a continuous map with a compact map.
- The Hecke operators T_{ℓ^2} and U_{ℓ^2} preserve the space of cusp forms, as can be seen by simply constructing them directly on this space in the same manner as above. The operator U_{p^2} is compact on a space of overconvergent cusp forms.

5.2. *Hecke operators in families.* Let $X \subseteq \mathcal{W}$ be a connected admissible affinoid open. We wish to define endomorphisms of $\widetilde{M}_X(4N, K, p^{-r})$ that interpolate the endomorphisms T_{ℓ^2} and U_{ℓ^2} constructed above for fixed weights $\kappa \in X$.

Suppose that $\ell \neq p$, and adopt the table

	$p \neq 2$	$p = 2$
$\mathcal{U} = \mathcal{V}$	$X_1(4Np)^{\mathrm{an}}_{\geq p^{-r}}$	$X_1(4N)^{\mathrm{an}}_{\geq 2^{-r}}$
$\mathcal{Z}$	$X_1(4Np, \ell^2)^{\mathrm{an}}_{\geq p^{-r}}$	$X_1(4N, \ell^2)^{\mathrm{an}}_{\geq 2^{-r}}$
Σ	Σ_{4Np}	Σ_{4N}

For more compact notation, let us for the rest of this section define

$$M = H^0(\mathcal{U}, \mathcal{O}(\Sigma)), \qquad P = H^0(\pi_1^{-1}(\mathcal{V}) \cap \mathcal{Z}, \mathcal{O}(\pi_1^*\Sigma - \pi_2^*\Sigma)),$$
$$N = H^0(\pi_2^{-1}(\mathcal{U}) \cap \mathcal{Z}, \mathcal{O}(\pi_2^*\Sigma)), \qquad Q = H^0(\pi_1^{-1}(\mathcal{V}) \cap \mathcal{Z}, \mathcal{O}(\pi_1^*\Sigma)).$$
$$L = H^0(\pi_1^{-1}(\mathcal{V}) \cap \mathcal{Z}, \mathcal{O}(\pi_2^*\Sigma)),$$

The Hecke operator T_{ℓ^2} (or U_{ℓ^2} if $\ell \mid 4N$) at a fixed weight was constructed in the previous section by first taking the composition of the following continuous maps: a pullback $M \to N$, a restriction $N \to L$, multiplication by an element of $H \in P$ to arrive at an element of Q, and a trace $Q \to M$. The construction was completed by restricting to an eigenspace of the diamond operators at p and dividing by ℓ^2.

The module of families of forms on X is an eigenspace of $M\widehat{\otimes}_K\mathcal{O}(X)$ (by the final remark in Remarks 4.5). To define T_{ℓ^2} (or U_{ℓ^2}) we begin as in the fixed weight case by defining an endomorphism of $M\widehat{\otimes}_K\mathcal{O}(X)$ and then observing that it commutes with the diamond automorphisms and therefore restricts to an operator on families of modular forms. To define this endomorphism, we modify the above sequence of maps by first applying $\widehat{\otimes}_K\mathcal{O}(X)$ to all of the spaces and taking the unique continuous $\mathcal{O}(X)$-linear extension of each map, with the exception of the multiplication step, where we opt instead to multiply by $\mathbf{H}_{\ell^2}|_X \in P\widehat{\otimes}_K\mathcal{O}(X)$. In so doing, we arrive at an $\mathcal{O}(X)$-linear endomorphism of $M\widehat{\otimes}_K\mathcal{O}(X)$ that is easily seen to commute with the diamond automorphisms, thereby inducing an endomorphism of the module $\widetilde{M}_X(4N, K, p^{-r})$.

Lemma 5.3. *The Hecke operators defined above for families are continuous.*

Proof. By definition, each map arising in the construction is continuous except perhaps for the multiplication map. The proof of the continuity of this map requires several simple facts about completed tensor products, all of which can be found in [Bosch et al. 1984, Section 2.1.7].

It follows trivially from Lemma 2.5 that the multiplication map $L \times P \to Q$ is a bounded K-bilinear map and therefore extends uniquely to a bounded K-linear map $L\widehat{\otimes}_K P \to Q$. Extending scalars to $\mathcal{O}(X)$ and completing, we arrive at a bounded $\mathcal{O}(X)$-linear map $(L\widehat{\otimes}_K P)\widehat{\otimes}_K\mathcal{O}(X) \to Q\widehat{\otimes}_K\mathcal{O}(X)$. There is an isometric isomorphism $(L\widehat{\otimes}_K P)\widehat{\otimes}_K\mathcal{O}(X) \cong (L\widehat{\otimes}_K\mathcal{O}(X))\widehat{\otimes}_{\mathcal{O}(X)}(P\widehat{\otimes}_K\mathcal{O}(X))$, so we conclude that the $\mathcal{O}(X)$-bilinear multiplication map

$$(L\widehat{\otimes}_K\mathcal{O}(X))\widehat{\otimes}_{\mathcal{O}(X)}(P\widehat{\otimes}_K\mathcal{O}(X)) \to Q\widehat{\otimes}_K\mathcal{O}(X)$$

is bounded. In particular, multiplication by $H \in P\widehat{\otimes}_K\mathcal{O}(X)$ is a bounded (and hence continuous) map $\cdot H : L\widehat{\otimes}_K\mathcal{O}(X) \to Q\widehat{\otimes}_K\mathcal{O}(X)$, as desired. □

Remarks 5.4.

- The construction of a continuous endomorphism U_{p^2} is entirely analogous, and once again we find that U_{p^2} is compact in the overconvergent case, that is, whenever $r > 0$.

- The endomorphisms T_{ℓ^2} and U_{ℓ^2} can be extended to $\widetilde{M}_X(4N, K, p^{-r})$ for general admissible affinoid opens X in the usual manner, working component by component.
- All of the Hecke operators defined on families preserve the cuspidal subspaces, as a direct construction on these spaces demonstrates. Again, the operator U_{p^2} is compact on a module of overconvergent cusp forms.

Effect on q-expansions. In this section we will work out the effect of the Hecke operators that we have defined on q-expansions. As in [Ramsey 2006], we must adjust the naive q-expansions obtained by literally evaluating our forms on Tate curves with level structure to get at the classical q-expansions. In particular, by the q-expansion of a form $F \in \widetilde{M}_\kappa(4N, K, p^{-r})$ at the cusp associated to $(\underline{\mathrm{Tate}}(q), \zeta)$, where ζ is a primitive $4Np$-th root of unity if $p \neq 2$ and a primitive $4N$-th root of unity if $p = 2$, we mean $F(\underline{\mathrm{Tate}}(q), \zeta)\theta(q)E_\kappa(q)$. Similarly, for a family $F \in M_X(4N, K, p^{-r})$ the corresponding q-expansion is $F(\underline{\mathrm{Tate}}(q), \zeta)\theta(q)E(q)|_X$ and has coefficients in the ring of analytic functions on X.

Proposition 5.5. *Let F be an element of $\widetilde{M}_\kappa(4N, K, p^{-r})$ or $\widetilde{M}_X(4N, K, p^{-r})$, and let $\sum a_n q^n$ be the q-expansion of F at $(\underline{\mathrm{Tate}}(q), \zeta)$. Then the corresponding q-expansion of $U_{p^2} F$ is $\sum a_{p^2 n} q^n$.*

Proof. We prove the theorem for U_{p^2} acting on $\widetilde{M}_\kappa(4N, K, p^{-r})$. To obtain the result for families, one could either proceed in the same manner or deduce the result for families over X from the result for fixed weight by specializing to weights in X. Let $F \in \widetilde{M}_\kappa(4N, K, p^{-r})$, and suppose that $F(\underline{\mathrm{Tate}}(q), \zeta)\theta(q)E_\kappa(q) = \sum a_n q^n$. The expansion we seek is $(1/p^2)\pi_{1*}(\pi_2^* F \cdot \mathbf{H}_{p^2}(\kappa))(\underline{\mathrm{Tate}}(q), \zeta) \cdot \theta(q)E_\kappa(q)$. The cyclic subgroups of order p^2 that intersect the subgroup generated by ζ trivially are exactly those of the form $\langle \zeta_{p^2}^i q_{p^2} \rangle$ for $0 \leq i \leq p^2 - 1$. Thus we have

$$\begin{aligned}
&\pi_{1*}(\pi_2^* F \cdot \mathbf{H}_{p^2}(\kappa))(\underline{\mathrm{Tate}}(q), \zeta) \\
&\quad = \sum_{i=0}^{p^2-1} (\pi_2^* F \cdot \mathbf{H}_{p^2}(\kappa))(\underline{\mathrm{Tate}}(q), \zeta, \langle \zeta_{p^2}^i q_{p^2} \rangle) \\
&\quad = \sum_{i=0}^{p^2-1} F(\underline{\mathrm{Tate}}(q)/\langle \zeta_{p^2}^i q_{p^2} \rangle, \zeta/\langle \zeta_{p^2}^i q_{p^2} \rangle)\mathbf{H}_{p^2}(\kappa)(\underline{\mathrm{Tate}}(q), \zeta, \langle \zeta_{p^2}^i q_{p^2} \rangle) \\
&\quad = \sum_{i=0}^{p^2-1} F(\underline{\mathrm{Tate}}(\zeta_{p^2}^i q_{p^2}), \zeta)\mathbf{H}_{p^2}(\kappa)(\underline{\mathrm{Tate}}(q), \zeta, \langle \zeta_{p^2}^i q_{p^2} \rangle) \\
&\quad = \sum_{i=0}^{p^2-1} \frac{\sum a_n (\zeta_{p^2}^i q_{p^2})^n}{\theta(\zeta_{p^2}^i q_{p^2})E_\kappa(\zeta_{p^2}^i q_{p^2})} \frac{\theta(\zeta_{p^2}^i q_{p^2})E_\kappa(\zeta_{p^2}^i q_{p^2})}{\theta(q)E_\kappa(q)} = p^2 \frac{\sum a_{p^2 n} q^n}{\theta(q)E_\kappa(q)}. \qquad \square
\end{aligned}$$

The same analysis also proves the following.

Proposition 5.6. *Suppose $\ell \mid 4N$. Let F be an element of $\widetilde{M}_\kappa(4N, K, p^{-r})$ or $\widetilde{M}_X(4N, K, p^{-r})$, and let $\sum a_n q^n$ be the q-expansion of F at $(\underline{\mathrm{Tate}}(q), \zeta)$. Then the corresponding q-expansion of $U_{\ell^2}F$ is then $\sum a_{\ell^2 n} q^n$.*

To work out the effect of T_{ℓ^2} for $\ell \nmid 4Np$ on q-expansions, we will need several more q-expansions of Θ_{ℓ^2} and $\mathbf{E}_{\ell^2}$. For the former, see [Ramsey 2006]. The latter will follow from the following lemma. For $x \in \mathbb{Z}_p^\times$, we denote by $[x]$ the analytic function on $\mathcal{W}$ defined by $[x](\kappa) = \kappa(x)$.

Lemma 5.7. *For $\ell \neq p$, we have*

$$\mathbf{E}_\ell(\underline{\mathrm{Tate}}(q), \mu_p + \langle q_\ell \rangle) = [\langle \ell \rangle] \frac{E(q)}{E(q_\ell)} \quad \textit{and} \quad \mathbf{E}_\ell(\underline{\mathrm{Tate}}(q), \mu_{p\ell}) = \frac{E(q)}{E(q^\ell)}.$$

Proof. The second equality is how we chose to characterize $\mathbf{E}_\ell$ in the first place. We will use it to give an alternative characterization, which we will in turn use to prove the first equality.

By definition, $\mathbf{E}_\ell$ and the coefficients of $E(q)$ are pulled back from their restrictions to $\mathcal{W}^0$ through the map (4). Clearly $[\langle \ell \rangle]$ is the pullback of $[\ell]$ through this map, so it suffices to prove that $\mathbf{E}_\ell(\underline{\mathrm{Tate}}(q), \mu_p + \langle q_\ell \rangle) = [\ell](E(q)/E(q_\ell))$, where the coefficients are now thought of as functions only on $\mathcal{W}^0$. Moreover, it suffices to prove the equality after specialization to integers $\lambda \geq 2$ divisible by $\varphi(\mathbf{q})$, as such integers are Zariski-dense in $\mathcal{W}^0$. Let $E_\lambda(\tau)$ denote the classical analytic p-deprived Eisenstein series of weight λ and level p (normalized to have q-expansion $E_\lambda(q)$). Then

$$\mathbf{E}_\ell^{\mathrm{an}}(\lambda) := E_\lambda(\tau)/E_\lambda(\ell\tau)$$

is a meromorphic function on $X_0(p\ell)_{\mathbb{C}}^{\mathrm{an}}$ with rational q-expansion coefficients, and by GAGA and the q-expansion principle, it yields a rational function on the algebraic curve $X_0(p\ell)_{\mathbb{Q}_p}$. By comparing q-expansions it is evident that the restriction of this function to the region $X_0(p\ell)_{\geq 1}^{\mathrm{an}}$ is equal to the specialization, $\mathbf{E}_\ell(\lambda)$, of $\mathbf{E}_\ell$ to $\lambda \in \mathcal{W}^0$.

It follows that $\mathbf{E}_\ell(\lambda)(\underline{\mathrm{Tate}}(q), \mu_p + \langle q_\ell \rangle) = \mathbf{E}_\ell^{\mathrm{an}}(\lambda)(\underline{\mathrm{Tate}}(q), \mu_p + \langle q_\ell \rangle)$. The right side can be computed using the usual yoga where one pretends to specialize q to $e^{2\pi i \tau}$ and then computes with analytic transformation formulas (see [Ramsey 2006, Section 5] for a rigorous explanation of this yoga). So specializing, we get

$$\mathbf{E}_\ell^{\mathrm{an}}(\lambda)(\underline{\mathrm{Tate}}(q), \mu_p + \langle q_\ell \rangle)(\tau) = \mathbf{E}_\ell^{\mathrm{an}}(\lambda)(\mathbb{C}/\langle 1, \tau \rangle, \langle 1/p \rangle + \langle \tau/\ell \rangle).$$

Choosing a matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \qquad \text{such that } p \mid c \text{ and } \ell \mid d,$$

we arrive at an isomorphism

$$(\mathbb{C}/\langle 1,\tau\rangle, \langle 1/p\rangle + \langle \tau/\ell\rangle) \xrightarrow{\sim} (\mathbb{C}/\langle 1,\gamma\tau\rangle, \langle 1/p\ell\rangle), \quad z \mapsto \frac{z}{c\tau+d}.$$

Thus

$$\mathbf{E}_\ell^{\mathrm{an}}(\lambda)(\mathbb{C}/\langle 1,\tau\rangle, \langle 1/p\rangle + \langle \tau/\ell\rangle) = \mathbf{E}_\ell^{\mathrm{an}}(\lambda)(\mathbb{C}/\langle 1,\gamma\tau\rangle, \langle 1/p\ell\rangle) = \frac{E_\lambda(\gamma\tau)}{E_\lambda(\ell\gamma\tau)}.$$

Now $\ell\gamma\tau = ((a\ell)(\tau/\ell)+b)/(c(\tau/\ell)+d/\ell)$, so we have

$$\frac{E_\lambda(\gamma\tau)}{E_\lambda(\ell\gamma\tau)} = \frac{(c\tau+d)^\lambda E_\lambda(\tau)}{((c\tau+d)/\ell)^\lambda E_\lambda(\tau/\ell)} = \ell^\lambda \frac{E_\lambda(\tau)}{E_\lambda(\tau/\ell)}.$$

The result follows. □

Proposition 5.8. *Let $F \in \widetilde{M}_\kappa(4N, K, p^{-r}, \chi)$ with $\kappa \in \mathcal{W}^i$, and let $\sum a_n q^n$ be the q-expansion of F at $(\underline{\mathrm{Tate}}(q), \zeta)$. Then the corresponding q-expansion of $T_{\ell^2}F$ is $\sum b_n q^n$, where*

$$b_n = a_{\ell^2 n} + \kappa(\ell)\chi(\ell)\ell^{-1}\Big(\frac{(-1)^i n}{\ell}\Big)a_n + \kappa(\ell)^2\chi(\ell)^2\ell^{-1}a_{n/\ell^2}.$$

Let $F \in \widetilde{M}_X(4N, K, p^{-r}, \chi)$ with X a connected affinoid in $\mathcal{W}^i$, and let the q-expansion of F be $\sum a_n q^n$ as above. Then the corresponding q-expansion of $T_{\ell^2}F$ is $\sum b_n q^n$, where

$$b_n = a_{\ell^2 n} + [\ell]\chi(\ell)\ell^{-1}\Big(\frac{(-1)^i n}{\ell}\Big)a_n + [\ell]^2\chi(\ell)^2\ell^{-1}a_{n/\ell^2}.$$

Proof. We prove the first assertion. The second may either be proved directly in the same manner or simply deduced from the first via specialization to individual weights in X. Let $\kappa \in \mathcal{W}(K)$, let $F \in \widetilde{M}_\kappa(4N, K, p^{-r}, \chi)$, and let

$$F(\underline{\mathrm{Tate}}(q), \zeta)\theta(q)E_\kappa(q) = \sum a_n q^n$$

be the q-expansion of F at $(\underline{\mathrm{Tate}}(q), \zeta)$. The corresponding q-expansion of $T_{\ell^2}F$ is

$$\frac{1}{\ell^2}\pi_{1*}(\pi_2^* F \cdot \mathbf{H}_{\ell^2}(\kappa)) \cdot \theta(q)E_\kappa(q). \tag{6}$$

The cyclic subgroups of $\underline{\mathrm{Tate}}(q)$ of order ℓ^2 are the subgroups

$$\mu_{\ell^2}, \quad \langle \zeta_{\ell^2}^i q_{\ell^2}\rangle_{0\le i\le \ell^2-1}, \quad \text{and} \quad \langle \zeta_{\ell^2}^j q_\ell\rangle_{1\le j\le \ell-1}.$$

We examine separately the contribution of each of these types of subgroups to $\pi_{1*}(\pi_2^* F \cdot \mathbf{H}_{\ell^2}(\kappa))$.

First, we have

$$
\begin{aligned}
&F(\underline{\mathrm{Tate}}(q)/\mu_{\ell^2}, \zeta/\mu_{\ell^2})\mathbf{H}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q), \zeta, \mu_{\ell^2}) \\
&\qquad = F(\underline{\mathrm{Tate}}(q^{\ell^2}), \zeta^{\ell^2})\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \mu_{\ell^2})\pi^*\mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q), \zeta_p, \mu_{\ell^2}) \\
&\qquad = F(\underline{\mathrm{Tate}}(q^{\ell^2}), \zeta^{\ell^2})\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \mu_{\ell^2}) \\
&\qquad\qquad\qquad\qquad \times \mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q)/\mu_{\ell^2}, (\mu_p + \underline{\mathrm{Tate}}(q)[\ell^2])/\mu_{\ell^2}) \\
&\qquad = F(\underline{\mathrm{Tate}}(q^{\ell^2}), \zeta^{\ell^2})\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \mu_{\ell^2})\mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q^{\ell^2}), \mu_p + \langle q\rangle).
\end{aligned}
$$

From the definition (2) and Lemma 5.7, we have

$$
\begin{aligned}
&\mathbf{E}_{\ell^2}(\underline{\mathrm{Tate}}(q^{\ell^2}), \mu_p + \langle q\rangle) \\
&\qquad = \mathbf{E}_{\ell}(\underline{\mathrm{Tate}}(q^{\ell^2}), \mu_p + \langle q^{\ell}\rangle)\mathbf{E}_{\ell}(\underline{\mathrm{Tate}}(q^{\ell^2})/\langle q^{\ell}\rangle, (\mu_p + \langle q\rangle)/\langle q^{\ell}\rangle) \\
&\qquad = \mathbf{E}_{\ell}(\underline{\mathrm{Tate}}(q^{\ell^2}), \mu_p + \langle q^{\ell}\rangle)\mathbf{E}_{\ell}(\underline{\mathrm{Tate}}(q^{\ell}), \mu_p + \langle q\rangle) \\
&\qquad = [\langle \ell\rangle]\frac{E(q^{\ell^2})}{E(q^{\ell})}\cdot[\langle \ell\rangle]\frac{E(q^{\ell})}{(q)} = [\langle \ell\rangle]^2\frac{E(q^{\ell^2})}{E(q)}.
\end{aligned}
$$

When specialized to κ, this becomes $\kappa(\langle \ell\rangle)^2 E_\kappa(q^{\ell^2})/E_\kappa(q)$. Referring to [Ramsey 2006], we find

$$\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \mu_{\ell^2}) = \ell\theta(q^{\ell^2})/\theta(q).$$

Thus the contribution of this first subgroup is

$$\frac{\chi(\ell^2)\tau(\ell^2)^i\sum a_n q^{\ell^2 n}}{\theta(q^{\ell^2})E_\kappa(q^{\ell^2})}\,\ell\,\frac{\theta(q^{\ell^2})}{\theta(q)}\kappa(\langle \ell\rangle)^2\frac{E_\kappa(q^{\ell^2})}{E_\kappa(q)} = (\kappa(\langle \ell\rangle)\chi(\ell)\tau(\ell)^i)^2\frac{\ell\sum a_n q^{\ell^2 n}}{\theta(q)E_\kappa(q)}.$$

The subgroups $\langle \zeta_{\ell^2}^a q_{\ell^2}\rangle$ contribute

$$
\begin{aligned}
&\sum_{a=0}^{\ell^2-1} F(\underline{\mathrm{Tate}}(q)/\langle \zeta_{\ell^2}^a q_{\ell^2}\rangle, \zeta/\langle \zeta_{\ell^2}^a q_{\ell^2}\rangle)\mathbf{H}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q), \zeta, \langle \zeta_{\ell^2}^a q_{\ell^2}\rangle) \\
&\qquad = \sum_{a=0}^{\ell^2-1} F(\underline{\mathrm{Tate}}(\zeta_{\ell^2}^a q_{\ell^2}), \zeta)\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle \zeta_{\ell^2}^a q_{\ell^2}\rangle) \\
&\qquad\qquad\qquad\qquad \times \pi^*\mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q), \zeta_p, \langle \zeta_{\ell^2}^a q_{\ell^2}\rangle) \\
&\qquad = \sum_{a=0}^{\ell^2-1} F(\underline{\mathrm{Tate}}(\zeta_{\ell^2}^a q_{\ell^2}), \zeta)\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle \zeta_{\ell^2}^a q_{\ell^2}\rangle) \\
&\qquad\qquad \times \mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q)/\langle \zeta_{\ell^2}^a q_{\ell^2}\rangle, (\mu_p + \underline{\mathrm{Tate}}(q)[\ell^2])/\langle \zeta_{\ell^2}^a q_{\ell^2}\rangle) \\
&\qquad = \sum_{a=0}^{\ell^2-1} F(\underline{\mathrm{Tate}}(\zeta_{\ell^2}^a q_{\ell^2}), \zeta)\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle \zeta_{\ell^2}^a q_{\ell^2}\rangle) \\
&\qquad\qquad\qquad\qquad\qquad \times \mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(\zeta_{\ell^2}^a q_{\ell^2}), \mu_{p\ell^2}).
\end{aligned}
$$

By (2) we have

$$\begin{aligned}\mathbf{E}_{\ell^2}(\underline{\mathrm{Tate}}(\zeta_{\ell^2}^a q_{\ell^2}), \mu_{p\ell^2}) &= \mathbf{E}_\ell(\underline{\mathrm{Tate}}(\zeta_{\ell^2}^a q_{\ell^2}), \mu_{p\ell})\mathbf{E}_\ell(\underline{\mathrm{Tate}}(\zeta_{\ell^2}^a q_{\ell^2})/\mu_\ell, \mu_{p\ell^2}/\mu_\ell)\\ &= \mathbf{E}_\ell(\underline{\mathrm{Tate}}(\zeta_{\ell^2}^a q_{\ell^2}), \mu_{p\ell})\mathbf{E}_\ell(\underline{\mathrm{Tate}}(\zeta_\ell^a q_\ell), \mu_{p\ell})\\ &= \frac{E(\zeta_{\ell^2}^a q_{\ell^2})}{E(\zeta_\ell^a q_\ell)}\frac{E(\zeta_\ell^a q_\ell)}{E(q)} = \frac{E(\zeta_{\ell^2}^a q_{\ell^2})}{E(q)}.\end{aligned}$$

Referring to [Ramsey 2006], we find $\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle\zeta_{\ell^2}^a q_{\ell^2}\rangle) = \theta(\zeta_{\ell^2}^a q_{\ell^2})/\theta(q)$. Thus the total contribution of this collection of subgroups is

$$\sum_{a=0}^{\ell^2-1} \frac{\sum a_n(\zeta_{\ell^2}^a q_{\ell^2})^n}{\theta(\zeta_{\ell^2}^a q_{\ell^2})E_\kappa(\zeta_{\ell^2}^a q_{\ell^2})}\frac{\theta(\zeta_{\ell^2}^a q_{\ell^2})}{\theta(q)}\frac{E_\kappa(\zeta_{\ell^2}^a q_{\ell^2})}{E_\kappa(q)} = \ell^2\frac{\sum a_{\ell^2 n}q^n}{\theta(q)E_\kappa(q)}.$$

The subgroups $\langle\zeta_{\ell^2}^b q_\ell\rangle$ contribute

$$\begin{aligned}\sum_{b=1}^{\ell-1} &F(\underline{\mathrm{Tate}}(q)/\langle\zeta_{\ell^2}^b q_\ell\rangle, \zeta/\langle\zeta_{\ell^2}^b q_\ell\rangle)\mathbf{H}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q), \zeta, \langle\zeta_{\ell^2}^b q_\ell\rangle)\\ &= \sum_{b=1}^{\ell-1} F(\underline{\mathrm{Tate}}(\zeta_\ell^b q), \zeta^\ell)\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle\zeta_{\ell^2}^b q_\ell\rangle)\\ &\qquad\qquad\times \pi^*\mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q), \zeta_p, \langle\zeta_{\ell^2}^b q_\ell\rangle)\\ &= \sum_{b=1}^{\ell-1} F(\underline{\mathrm{Tate}}(\zeta_\ell^b q), \zeta^\ell)\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle\zeta_{\ell^2}^b q_\ell\rangle)\\ &\qquad\times \mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(q)/\langle\zeta_{\ell^2}^b q_\ell\rangle, (\mu_p + \underline{\mathrm{Tate}}(q)[\ell^2])/\langle\zeta_{\ell^2}^b q_\ell\rangle)\\ &= \sum_{b=1}^{\ell-1} F(\underline{\mathrm{Tate}}(\zeta_\ell^b q), \zeta^\ell)\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle\zeta_{\ell^2}^b q_\ell\rangle)\\ &\qquad\qquad\times \mathbf{E}_{\ell^2}(\kappa)(\underline{\mathrm{Tate}}(\zeta_\ell^b q), \mu_p + \langle q_\ell\rangle).\end{aligned}$$

By (2) and Lemma 5.7 we have

$$\begin{aligned}\mathbf{E}_{\ell^2}&(\underline{\mathrm{Tate}}(\zeta_\ell^b q), \mu_p + \langle q_\ell\rangle)\\ &= \mathbf{E}_\ell(\underline{\mathrm{Tate}}(\zeta_\ell^b q), \mu_p + \langle q\rangle)\mathbf{E}_\ell(\underline{\mathrm{Tate}}(\zeta_\ell^b q)/\mu_\ell, (\mu_p + \langle q_\ell\rangle)/\mu_\ell)\\ &= \mathbf{E}_\ell(\underline{\mathrm{Tate}}(\zeta_\ell^b q), \mu_{p\ell})\mathbf{E}_\ell(\underline{\mathrm{Tate}}(q^\ell), \mu_p + \langle q\rangle)\\ &= \frac{E(\zeta_\ell^b q)}{E(q^\ell)}\cdot[\langle\ell\rangle]\frac{E(q^\ell)}{E(q)} = [\langle\ell\rangle]\frac{E(\zeta_\ell^b q)}{E(q)}.\end{aligned}$$

When specialized to κ, this becomes $\kappa(\langle\ell\rangle)E_\kappa(\zeta_\ell^b q)/E_\kappa(q)$. Referring to [Ramsey 2006] we find

$$\Theta_{\ell^2}(\underline{\mathrm{Tate}}(q), \zeta_4, \langle\zeta_\ell^b q\rangle) = \left(\frac{-1}{\ell}\right)\mathfrak{g}_\ell(\zeta_\ell^b)\frac{\theta(\zeta_\ell^b q)}{\theta(q)}, \quad \text{where } \mathfrak{g}_\ell(\zeta) = \sum_{m=1}^{\ell-1}\left(\frac{m}{\ell}\right)\zeta^m$$

is the Gauss sum associated to the ℓ-th root of unity ζ. Thus the total contribution of this third collection of subgroups is

$$\sum_{b=1}^{\ell-1} \frac{\chi(\ell)(-1/\ell)^i \tau(\ell)^i \sum a_n (\zeta_\ell^b q)^n}{\theta(\zeta_\ell^b q) E_\kappa(\zeta_\ell^b q)} \Big(\frac{-1}{\ell}\Big) \mathfrak{g}_\ell(\zeta_\ell^b) \frac{\theta(\zeta_\ell^b q)}{\theta(q)} \kappa(\langle \ell \rangle) \frac{E_\kappa(\zeta_\ell^b q)}{E_\kappa(q)}$$
$$= \kappa(\langle \ell \rangle)\chi(\ell)\Big(\frac{-1}{\ell}\Big)^{i+1} \tau(\ell)^i \frac{\mathfrak{g}_\ell(\zeta_\ell)}{\theta(q)E_\kappa(q)} \sum_n a_n \Big(\sum_{b=1}^{\ell-1} \zeta_\ell^{bn} \Big(\frac{b}{\ell}\Big)\Big) q^n$$
$$= \kappa(\langle \ell \rangle)\chi(\ell)\Big(\frac{-1}{\ell}\Big)^{i+1} \tau(\ell)^i \frac{\mathfrak{g}_\ell(\zeta_\ell)}{\theta(q)E_\kappa(q)} \sum_n a_n \Big(\frac{n}{\ell}\Big) \mathfrak{g}_\ell(\zeta_\ell) q^n$$
$$= \kappa(\langle \ell \rangle)\chi(\ell)\Big(\frac{-1}{\ell}\Big)^{i} \tau(\ell)^i \frac{\ell \sum \big(\frac{n}{\ell}\big) a_n q^n}{\theta(q)E_\kappa(q)}.$$

Adding all this up and plugging into (6), we see that the q-expansion of $T_{\ell^2} F$ is $\sum b_n q^n$, where

$$b_n = a_{\ell^2 n} + \kappa(\langle \ell \rangle)\ell^{-1}\chi(\ell)\Big(\frac{-1}{\ell}\Big)^i \tau(\ell)^i \Big(\frac{n}{\ell}\Big) a_n + \kappa(\langle \ell \rangle)^2 \ell^{-1} \chi(\ell)^2 \tau(\ell)^{2i} a_{n/\ell^2}$$
$$= a_{\ell^2 n} + \kappa(\ell)\ell^{-1}\chi(\ell)\Big(\frac{(-1)^i n}{\ell}\Big) a_n + \kappa(\ell)^2 \ell^{-1}\chi(\ell)^2 a_{n/\ell^2}. \qquad \square$$

6. Classical weights and classical forms

In this section we define classical subspaces of our spaces of modular forms and prove the following analog of Coleman's theorem on overconvergent forms of low slope. Throughout this section k will denote an odd positive integer and we set $\lambda = (k-1)/2$.

Theorem 6.1. *Let m be a positive integer, let $\psi : (\mathbb{Z}/\mathbf{q}p^{m-1}\mathbb{Z})^\times \to K^\times$ be a character, and define $\kappa(x) = x^\lambda \psi(x)$. If $F \in \widetilde{M}_\kappa^\dagger(4N, K)$ satisfies $U_{p^2} F = \alpha F$ with $v(\alpha) < 2\lambda - 1$, then F is classical.*

Our proof follows the approach of Kassaei [2006], which is modular in nature and builds the classical form by analytic continuation and gluing. The term "analytic continuation" has little meaning here since we have only defined our modular forms over restricted regions on the modular curve, owing to the need to avoid Eisenstein zeros. To get around this difficulty, we must invoke the formalism of [Ramsey 2006] for p-adic modular forms of *classical* half-integral weight.

Let N be a positive integer. In [Ramsey 2006] we defined the space of modular forms of weight $k/2$ and level $4N$ over a $\mathbb{Z}[1/4N]$-algebra R to be the R-module

$$\widetilde{M}'_{k/2}(4N, R) := H^0(X_1(4N)_R, \mathcal{O}(k\Sigma_{4N})).$$

Note that this space was denoted $M_{k/2}(4N, R)$ and $k\Sigma_{4N}$ was denoted $\Sigma_{4N,k}$. Roughly speaking, in this space of forms we have divided by θ^k to reduce to weight zero instead of $E_\lambda\theta$. Let $r \in [0, 1] \cap \mathbb{Q}$, and define

$$\widetilde{M}'_{k/2}(4Np^m, K, p^{-r}) = H^0(X_1(4Np^m)^{\mathrm{an}}_{\geq p^{-r}}, \mathcal{O}(k\Sigma_{4Np^m})).$$

It is an easy matter to check that the construction of the Hecke operators T_{ℓ^2} and U_{p^2} in Section 5 (using $H = \Theta^k_{\ell^2}$) adapts to this space of forms and furnishes us with Hecke operators having the expected effect on q-expansions. We will briefly review the construction of U_{p^2} in this context later in this section.

The next proposition relates these spaces of p-adic modular forms to the ones defined in this paper, and will ensure that the latter spaces (and consequently the eigencurve defined later in this paper) see the classical half-integral weight modular forms of arbitrary p-power level. Note that this identification requires knowledge of the action of the diamond operators at p because this data is part of the p-adic weight character.

Proposition 6.2. *Let m be a positive integer, let $\psi : (\mathbb{Z}/\mathbf{q}p^{m-1}\mathbb{Z})^\times \to K^\times$ be a character, and define $\kappa(x) = x^\lambda\psi(x)$. Then, for $0 \leq r \leq r_m$, the space*

$$\widetilde{M}'(4Np^{m+1}/\mathbf{q}, K, p^{-r})^{\langle\cdot\rangle^*_{\mathbf{q}p^{m-1}}=\psi} = \begin{cases} \widetilde{M}'_{k/2}(4Np^m, K, p^{-r})^{\langle\cdot\rangle^*_{p^m}=\psi} & \text{if } p \neq 2 \\ \widetilde{M}'_{k/2}(2^{m+1}N, K, p^{-r})^{\langle\cdot\rangle^*_{2^{m+1}}=\psi} & \text{if } p = 2 \end{cases}$$

is isomorphic to $\widetilde{M}_\kappa(4N, K, p^{-r})$ in a manner compatible with the action of the Hecke operators and tame diamond operators.

Proof. Let i be such that $\kappa \in \mathcal{W}^i$. The complex-analytic modular forms θ^{k-1} and $E_{\kappa\tau^{-i}}$ are each of weight λ. If $p \neq 2$, then θ^{k-1} is invariant under the $\langle d\rangle^*_{\mathbf{q}p^{m-1}}$ while if $p = 2$ it has eigencharacter $(-1/\cdot)^i$. In both cases, $E_{\kappa\tau^{-i}}$ has eigencharacter $\psi\tau^{-i}$ for this action. Standard arguments using GAGA and the q-expansion principle show that the ratio $\theta^{k-1}/E_{\kappa\tau^{-i}}$ furnishes an algebraic rational function on $X_1(4Np^{m+1}/\mathbf{q})_K$. Passing to the p-adic analytification and then restricting to $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\geq p^{-r}}$ shows this function has divisor $(k-1)\Sigma_{4Np^{m+1}/\mathbf{q}}$, since $E_{\kappa\tau^{-i}}$ is invertible in this region for r as in the statement of the proposition (because $\kappa \in \mathcal{W}_m$).

Suppose $F' \in \widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r})$ is a form with eigencharacter ψ for $\langle\cdot\rangle^*_{\mathbf{q}p^{m-1}}$, and let

$$F = F' \cdot \frac{\theta^{k-1}}{E_{\kappa\tau^{-i}}}.$$

Then, for $d \in (\mathbb{Z}/\mathbf{q}p^{m-1}\mathbb{Z})^\times$ we have $\langle d\rangle^*_{\mathbf{q}p^{m-1}} F = \tau(d)^i(-1/\cdot)^i F$. In particular, F is fixed by $\langle d\rangle^*_{p^m}$ with $d \equiv 1 \pmod{\mathbf{q}}$. Consider now the map

$$X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\geq p^{-r}}/\{\langle d\rangle_{\mathbf{q}p^{m-1}} \mid d \equiv 1 (\operatorname{mod} \mathbf{q})\} \longrightarrow \begin{cases} X_1(4Np)^{\mathrm{an}}_{\geq p^{-r}} & \text{if } p \neq 2, \\ X_1(4N)^{\mathrm{an}}_{\geq 2^{-r}} & \text{if } p = 2 \end{cases} \tag{7}$$

induced by $(E, P) \mapsto (E, aP)$, where the integer a is chosen so that

$$\begin{aligned} a &\equiv p^{m-1} (\operatorname{mod} p^m) \quad \text{and } a \equiv 1 (\operatorname{mod} 4N) \quad \text{if } p \neq 2, \\ a &\equiv 2^{m-1} (\operatorname{mod} 2^{m+1}) \text{ and } a \equiv 1 (\operatorname{mod} N) \quad \text{if } p = 2. \end{aligned}$$

The construction of the canonical subgroup of order $\mathbf{q}p^{m-1}$ (defined because $r \leq r_m < p^{2-m}/\mathbf{q}(1+p)$) ensures that this map is an isomorphism. For $p \neq 2$, this map pulls the divisor Σ_{4Np} back to Σ_{4Np^m}, so we conclude that F descends to a section of $\mathcal{O}(\Sigma_{4Np})$ on $X_1(4Np)^{\mathrm{an}}_{\geq p^{-r}}$ and that this section satisfies $\langle d\rangle^*_p F = \tau(d)^i F$ for all $d \in (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times$. For $p = 2$, this map pulls the divisor Σ_{4N} back to $\Sigma_{2^{m+1}N}$, so F descends to a section of $\mathcal{O}(\Sigma_{4N})$ on $X_1(4N)^{\mathrm{an}}_{\geq 2^{-r}}$, and this section satisfies $\langle d\rangle^*_4 F = \tau(d)^i(-1/d)^i F$ for all $d \in (\mathbb{Z}/\mathbf{q}\mathbb{Z})^\times$. Thus we may regard F as an element of $\widetilde{M}_\kappa(4N, K, p^{-r})$. Conversely, for $F \in \widetilde{M}_\kappa(4N, K, p^{-r})$, it is easy to see that

$$F \cdot \frac{E_{\kappa\tau^{-i}}}{\theta^{k-1}} \in \widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r})^{\langle\cdot\rangle_{\mathbf{q}p^{m-1}} = \psi}$$

(where F is implicitly pulled back via the above map (7)) and that this furnishes an inverse to the above map $F' \mapsto F$. That these maps are equivariant with respect to the Hecke action is a formal manipulation with the setup in Section 5 used to define the action on both sides. That it is equivariant with respect to tame diamond operators is trivial, but relies essentially on the "twisted" convention for this action on $\widetilde{M}_\kappa(4N, K, p^{-r})$ (for $p \neq 2$). □

In general, if $\mathcal{U}$ is a connected admissible open in $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K$ containing $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\geq p^{-r}}$ and if $F \in \widetilde{M}_\kappa(4N, K, p^{-r})$ (with κ as in the previous proposition), we will say that F analytically continues to $\mathcal{U}$ if the corresponding form $F' \in \widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r})$ analytically continues to an element of

$$H^0(\mathcal{U}, \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}})). \tag{8}$$

Note that, in case $\mathcal{U}$ is preserved by the diamond operators at p, this analytic continuation automatically lies in the ψ-eigenspace of (8) since $G - \langle d\rangle^*_{\mathbf{q}p^{m-1}} G$ vanishes on the nonempty admissible open $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\geq p^{-r}}$ for all d, and hence must vanish on all of $\mathcal{U}$. In particular, in case $\mathcal{U} = X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K$ we make the following definition.

Definition 6.3. Let $\kappa(x) = x^\lambda \psi(x)$ be as in Proposition 6.2. We say an element $F \in \widetilde{M}_\kappa(4N, K)^\dagger$ is *classical* if it analytically continues in the sense described above to all of $X_1(4Np^{m+1}/\mathbf{q})_K^{\mathrm{an}}$, that is, if it is in the image of the (injective) map

$$\begin{aligned}H^0(X_1(4Np^{m+1}/\mathbf{q})_K^{\mathrm{an}}, \mathcal{O}(k\Sigma_{4Np}))^{\langle\cdot\rangle_{p^m}=\psi} \\ \to \widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r_m})^{\langle\cdot\rangle_{p^m}=\psi} \cong \widetilde{M}_\kappa(4N, K, p^{-r_m}) \\ \hookrightarrow \widetilde{M}_\kappa(4N, K)^\dagger.\end{aligned}$$

The analytic continuation used to prove Theorem 6.1 will proceed in three steps. All of them involve the construction of the operator U_{p^2} on

$$\widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r}),$$

which goes as follows. Let

$$\pi_1, \pi_2 : X_1(4Np^{m+1}/\mathbf{q}, p^2)_K^{\mathrm{an}} \to X_1(4Np^{m+1}/\mathbf{q})_K^{\mathrm{an}}$$

be the usual pair of maps, and let Θ_{p^2} denote the rational function on $X_1(4, p^2)_\mathbb{Q}$ from Section 3. For any pair of admissible open $\mathcal{U}$ and $\mathcal{V}$ in $X_1(4Np^{m+1}/\mathbf{q})_K^{\mathrm{an}}$ with $\pi_1^{-1}\mathcal{V} \subseteq \pi_2^{-1}\mathcal{U}$, we have the map

$$\begin{aligned}H^0(\mathcal{U}, \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}})) &\to H^0(\mathcal{V}, \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}})) \\ F &\mapsto \frac{1}{p^2}\pi_{1*}(\pi_2^* F \cdot \Theta_{p^2}^k).\end{aligned}$$

Note that there is no need to introduce the space $\mathcal{Z}$ as in Section 5 since our "twisting" section $\Theta_{p^2}^k$ is defined on all of $X_1(4Np^{m+1}/\mathbf{q}, p^2)_K^{\mathrm{an}}$. Also, recall from Section 5 that if $0 \le r < 1/p(1+p)$, we have

$$\pi_1^{-1}(X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\ge p^{-p^2 r}}) \subseteq \pi_2^{-1}(X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\ge p^{-r}}).$$

Thus if $F \in \widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r})$ with $r < 1/p(1+p)$ then $U_{p^2}F$ analytically continues to $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\ge p^{-p^2 r}}$. From this simple observation we get the first and easiest analytic continuation result.

Proposition 6.4. *Let $r > 0$, and let $F \in \widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r})$. Suppose that there exists a polynomial $P(T) \in K[T]$ with $P(0) \neq 0$ such that $P(U_{p^2})F$ analytically continues to $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\ge p^{-1/(1+p)}}$. Then F analytically continues to this region as well.*

Proof. Write $P(T) = P_0(T) + a$ with $P_0(0) = 0$ and $a \neq 0$. Then

$$F = \frac{1}{a}(P(U_{p^2})F - P_0(U_{p^2})F).$$

If we have $0 < r < 1/p(1+p)$, then the right side analytically continues to $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\ge p^{-p^2 r}}$ and hence so does F. Since $r > 0$, we may repeat this process until we have analytically continued F to $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\geq p^{-s}}$ for some $s \geq 1/p(1+p)$. Now restrict F to $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\geq p^{-1/p^2(1+p)}}$ and apply the process once more to get the desired result. $\square$

The second analytic continuation step requires that we introduce some admissible opens in $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\mathbb{Q}_p}$ defined in [Buzzard 2003]. Use of the letter $\mathcal{W}$ in this part of the argument is intended to keep the notation parallel to that of [Buzzard 2003] and should not be confused with weight space. If $p \neq 2$, we let $\mathcal{W}_0 \subseteq X_1(4N, p)^{\mathrm{an}}_{\mathbb{Q}_p}$ denote the admissible open subspace whose points reduce to the irreducible component on the special fiber of $X_1(4N, p)$ in characteristic p that contains the cusp associated to the datum $(\underline{\mathrm{Tate}}(q), P, \mu_p)$ for some (equivalently, any) point of order $4N$ on $\underline{\mathrm{Tate}}(q)$. Alternatively, $\mathcal{W}_0$ can be characterized as the complement of the connected component of the ordinary locus in $X_1(4N, p)^{\mathrm{an}}_{\mathbb{Q}_p}$ containing the cusp associated to $(\underline{\mathrm{Tate}}(q), P, \langle q_p \rangle)$ for some (equivalently, any) choice of P. If $p = 2$, we let $\mathcal{W}_0 \subseteq X_1(N, 2)^{\mathrm{an}}_{\mathbb{Q}_p}$ denote the admissible open subspace whose points reduce to the irreducible component on the special fiber of $X_1(N, 2)$ in characteristic 2 that contains the cusp associated to the datum $(\underline{\mathrm{Tate}}(q), P, \mu_2)$ for some (equivalently, any) point of order N on $\underline{\mathrm{Tate}}(q)$. Alternatively, $\mathcal{W}_0$ can be characterized as the complement of the connected component of the ordinary locus in $X_1(N, 2)^{\mathrm{an}}_{\mathbb{Q}_p}$ containing the cusp associated to $(\underline{\mathrm{Tate}}(q), P, \langle q_2 \rangle)$ for some (equivalently, any) choice of P. In particular, $\mathcal{W}_0$ always contains the entire supersingular locus. The reader concerned about problems with small N in these descriptions should focus on the "alternative" versions and the remarks in Section 2.1 about adding level structure and taking invariants.

Buzzard [2003] introduces a map $v' : \mathcal{W}_0 \to \mathbb{Q}$ defined as follows. If $x \in \mathcal{W}_0$ is a cusp, then set $v'(x) = 0$. Otherwise, $x \in \mathcal{W}_0$ corresponds to a triple $(E/L, P, C)$ with E/L an elliptic curve, P a point of order $4N$ (N if $p = 2$) on E, and $C \subset E$ a cyclic subgroup of order p. If E has bad or ordinary reduction, then set $v'(x) = 0$. Otherwise, if $0 < v(E) < p/(1+p)$, then E has a canonical subgroup H of order p, and we define

$$v'(x) = \begin{cases} v(E) & \text{if } H = C, \\ 1 - v(E/C) & \text{if } H \neq C. \end{cases}$$

Finally, if $v(E) \geq p/(1+p)$ we define $v'(x) = p/(1+p)$. Note that v' does not depend on the point P. For a nonnegative integer n, we let V_n denote the region in $\mathcal{W}_0$ defined by the inequality $v' \leq 1 - 1/p^{n-1}(1+p)$. Buzzard proves that V_n is an admissible affinoid open in $\mathcal{W}_0$ for each n, and that $\mathcal{W}_0$ is admissibly covered by the V_n.

Let

$$f : X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\mathbb{Q}_p} \to \begin{cases} X_1(4N, p)^{\mathrm{an}}_{\mathbb{Q}_p} & \text{if } p \neq 2, \\ X_1(N, 2)^{\mathrm{an}}_{\mathbb{Q}_p} & \text{if } p = 2 \end{cases}$$

denote the map characterized by

$$(E, P) \mapsto \begin{cases} (E/\langle 4NpP\rangle, p^m P/\langle 4NpP\rangle, \langle 4NP/\langle 4NpP\rangle\rangle & \text{if } p \neq 2, \\ (E/\langle 2NP\rangle, 2^{m+1}P/\langle 2NP\rangle, \langle NP/\langle 2NP\rangle\rangle) & \text{if } p = 2 \end{cases}$$

on noncuspidal points. Define $\mathcal{W}_1 = f^{-1}(\mathcal{W}_0)$ and $Z_n = f^{-1}(V_n)$ for $n \geq 0$. It follows from the above that $\mathcal{W}_1$ is an admissible open in $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K$ and that $\mathcal{W}_1$ is admissibly covered by the admissible opens Z_n. The latter are affinoid since f is finite.

Lemma 6.5. *The inclusion $\pi_1^{-1}(Z_{n+2}) \subseteq \pi_2^{-1}(Z_n)$ holds for all $n \geq 0$.*

Proof. Since the maps π_1 and π_2 are finite, the stated inclusion is between affinoids and can be checked on noncuspidal points. Then the assertion follows immediately from two applications of [Buzzard 2003, Lemma 4.2(2)]. □

We can now state and prove the second analytic continuation result.

Proposition 6.6. *Let $r > 0$, and let $F \in \widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r})$. Suppose that there exists a polynomial $P(T) \in K[T]$ with $P(0) \neq 0$ such that $P(U_{p^2})F$ extends to $\mathcal{W}_1$. Then F extends to this region as well.*

Proof. Note that

$$X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\geq p^{-1/(1+p)}} = Z_0 \subseteq \mathcal{W}_1$$

so that by Proposition 6.4, F extends to Z_0. Now we proceed inductively to extend F to each Z_n. Let $P(T) = P_0(T) + a$ with $P_0(0) = 0$ and $a \neq 0$. Then

$$F = \frac{1}{a}(P(U_{p^2})F - P_0(U_{p^2})F).$$

Suppose F extends to Z_n for some $n \geq 0$. By hypothesis, $P(U_{p^2})F$ extends to all of $\mathcal{W}_1$, and by the construction of U_{p^2} and Lemma 6.5, $P_0(U_{p^2})F$ extends to Z_{n+2}, and hence so does F. Thus by induction F extends to Z_n for all n, and since $\mathcal{W}_1$ is admissibly covered by the Z_n, F extends to $\mathcal{W}_1$. □

If $p \neq 2$ and $m = 1$ (that is, if there is only one p in the level), then this is the end of the second analytic continuation step. In all other cases, techniques in [Buzzard 2003] allow us to analytically continue to more connected components of the ordinary locus. Define

$$\mathbf{m} = \mathrm{ord}_p(\mathbf{q}p^{m-1}) = \begin{cases} m & \text{if } p \neq 2, \\ m+1 & \text{if } p = 2. \end{cases}$$

We now follow Buzzard: For $0 \leq r \leq \mathbf{m}$ let $\mathcal{U}_r$ denote the admissible open in $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K$ whose noncuspidal points parameterize pairs (E, P) that are either supersingular or satisfy

$$H_{p^{\mathbf{m}-r}}(E) = \begin{cases} H_{p^{m-r}}(E) = \langle 4Np^r P\rangle & \text{if } p \neq 2, \\ H_{2^{m+1-r}}(E) = \langle N2^r P\rangle & \text{if } p = 2. \end{cases}$$

We have

$$\mathcal{W}_1 = \mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \cdots \subseteq \mathcal{U}_{\mathbf{m}} = X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K.$$

The last goal of the second step is to analytically continue eigenforms to $\mathcal{U}_{\mathbf{m}-1}$.

Lemma 6.7. *For $0 \le r \le \mathbf{m}-2$, we have $\pi_1^{-1}(\mathcal{U}_{r+1}) \subseteq \pi_2^{-1}(\mathcal{U}_r)$.*

Proof. As usual, it suffices to check this on noncuspidal points. Moreover, it suffices to check it on ordinary points, since the entire supersingular locus is contained in each $\mathcal{U}_r$. For brevity we will assume $p \neq 2$. The case $p = 2$ is proved in exactly the same manner. Let $(E, P, C) \in \pi_1^{-1}(\mathcal{U}_{r+1})$ be such a point. Then $H_{p^{m-r-1}}(E) = \langle 4Np^{r+1}P\rangle$, and since $r+1 < m$, we conclude that $H_{p^{m-r-1}}(E) \cap C = 0$. Now [Buzzard 2003, Proposition 3.5] implies that $H_{p^r}(E/C)$ is indeed generated by the image of $4Np^r P$ in E/C, so $(E, P, C) \in \pi_2^{-1}(\mathcal{U}_r)$. □

Proposition 6.8. *Let $r > 0$, and let $F \in \widetilde{M}'_{k/2}(4Np^{m+1}/\mathbf{q}, K, p^{-r})$. Suppose that there exists a polynomial $P(T) \in K[T]$ with $P(0) \neq 0$ such that $P(U_{p^2})F$ extends to $\mathcal{U}_{\mathbf{m}-1}$. Then F extends to this region as well.*

Proof. Since $\mathcal{U}_0 = \mathcal{W}_1$, Proposition 6.6 ensures that F analytically continues to $\mathcal{U}_0$. Now we proceed inductively to extend F to each $\mathcal{U}_r$ for $0 \le r \le \mathbf{m}-1$. Let $P(T) = P_0(T) + a$ with $P_0(0) = 0$ and $a \neq 0$. Then

$$F = \frac{1}{a}(P(U_{p^2})F - P_0(U_{p^2})F).$$

Suppose F extends to $\mathcal{U}_r$ for some $0 \le r \le \mathbf{m}-2$. By hypothesis, $P(U_{p^2})F$ extends to all of $\mathcal{U}_{\mathbf{m}-1}$, and by the construction of U_{p^2} and Lemma 6.7, $P_0(U_{p^2})F$ extends to $\mathcal{U}_{r+1}$, and hence so does F. Proceeding inductively, we see that F can be extended all the way to $\mathcal{U}_{\mathbf{m}-1}$. □

The third and most difficult analytic continuation step is to continue to the rest of the curve $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K$. If $p \neq 2$, we let $\mathcal{V}_0$ denote the admissible open in $X_1(4N, p)^{\mathrm{an}}_K$ whose points reduce to the irreducible component on the special fiber in characteristic p that contains the cusp associated to $(\underline{\mathrm{Tate}}(q), P, \langle q_p\rangle)$ for some (equivalently, any) choice of P. On the other hand, if $p = 2$, we let $\mathcal{V}_0$ denote the admissible open in $X_1(N, 2)^{\mathrm{an}}_K$ whose points reduce to the irreducible component on the special fiber in characteristic 2 that contains the cusp associated to $(\underline{\mathrm{Tate}}(q), P, \langle q_2\rangle)$ for some (equivalently, any) choice of P. Let $\mathcal{V}$ denote the preimage of $\mathcal{V}_0$ under the finite map

$$g : X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\mathbb{Q}_p} \to \begin{cases} X_1(4N, p)^{\mathrm{an}}_{\mathbb{Q}_p} & \text{if } p \neq 2, \\ X_1(N, 2)^{\mathrm{an}}_{\mathbb{Q}_p} \text{ if} & p = 2, \end{cases}$$

$$(E, P) \mapsto \begin{cases} (E, p^m P, \langle 4Np^{m-1}P\rangle) & \text{if } p \neq 2, \\ (E, 2^{m+1}P, \langle 2^m NP\rangle) & \text{if } p = 2. \end{cases}$$

Note that the preimage under g of the locus that reduces to the other component of $X_1(4N, p)_{\mathbb{F}_p}$ (or $X_1(N, 2)_{\mathbb{F}_2}$ if $p = 2$) is $\mathcal{U}_{\mathbf{m}-1}$, so in particular $\{\mathcal{U}_{\mathbf{m}-1}, \mathcal{V}\}$ is an admissible cover of $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_{\mathbb{Q}_p}$ and $\mathcal{U}_{\mathbf{m}-1} \cap \mathcal{V}$ is the supersingular locus.

For any subinterval $I \subseteq (p^{-p/(1+p)}, 1]$, let $\mathcal{V}I$ (respectively $\mathcal{U}_{\mathbf{m}-1}I$) denote the admissible open in $\mathcal{V}$ (respectively $\mathcal{U}_{\mathbf{m}-1}$) defined by the condition $p^{-v(E)} \in I$. Note that the complement of $\mathcal{U}_{\mathbf{m}-1}$ in $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K$ is $\mathcal{V}[1, 1]$. Given a U_{p^2}-eigenform of suitably low slope, we will define a function on $\mathcal{V}[1, 1]$ and use the gluing techniques of [Kassaei 2006] to glue it to the analytic continuation of our eigenform to $\mathcal{U}_{\mathbf{m}-1}$ guaranteed by Proposition 6.6. These techniques rely heavily on the norms introduced in Section 2.2. The use of Lemma 2.5 to reduce these norms to the supremum norm on the complement of the residue disks around the cusps will be implicit in many of the estimates that follow.

Over $\mathcal{V}(p^{-1/p(1+p)}, 1]$ we have a section h to π_1 given on noncuspidal points by

$$\begin{gathered} h : \mathcal{V}(p^{-1/p(1+p)}, 1] \to X_1(4Np^{m+1}/\mathbf{q}, p^2)^{\mathrm{an}}_K \\ (E, P) \mapsto (E, P, H_{p^2}). \end{gathered}$$

By standard results on quotienting by the canonical subgroup [Buzzard 2003, Theorem 3.3], the composition $\pi_2 \circ h$ restricts to a map

$$Q : \mathcal{V}(p^{-r}, 1] \to \mathcal{V}(p^{-p^2 r}, 1] \tag{9}$$

for any $0 \leq r \leq 1/p(1 + p)$. Note that since Q preserves the property of having ordinary or supersingular reduction, Q restricts to a map $\mathcal{V}(p^{-r}, 1) \to \mathcal{V}(p^{-p^2 r}, 1)$. Define a meromorphic function ϑ on $\mathcal{V}(p^{-1/p(1+p)}, 1]$ by $\vartheta = h^*\Theta_{p^2}$, and note that

$$\begin{aligned} \operatorname{div}(\vartheta) &= h^*(\pi_2^*\Sigma_{4Np^{m+1}/\mathbf{q}} - \pi_1^*\Sigma_{4Np^{m+1}/\mathbf{q}}) \\ &= Q^*\Sigma_{4Np^{m+1}/\mathbf{q}} - \Sigma_{4Np^{m+1}/\mathbf{q}}. \end{aligned} \tag{10}$$

Let $F \in H^0(\mathcal{U}_{\mathbf{m}-1}, \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}}))$ and suppose that

$$U_{p^2}F = \alpha F + H$$

on $\mathcal{U}_{\mathbf{m}-1}$ for some *classical* form H and some $\alpha \neq 0$. Note that this condition makes sense because $\pi_1^{-1}(\mathcal{U}_{\mathbf{m}-1}) \subseteq \pi_2^{-1}(\mathcal{U}_{\mathbf{m}-1})$ by Lemma 6.7. For a pair $(E, P) \in \mathcal{U}_{\mathbf{m}-1}$ corresponding to a noncuspidal point, we have

$$F(E, P) = \frac{1}{\alpha p^2}\sum_{C} F(E/C, P/C)\Theta^k_{p^2}(E, P, C) - \frac{1}{\alpha}H(E, P), \tag{11}$$

where the sum is over the cyclic subgroups of order p^2 having trivial intersection with the group generated by P. Suppose that (E, P) corresponds to a point in $\mathcal{V}(p^{-1/p(1+p)}, 1)$. Then the subgroup generated by P has trivial intersection with the canonical subgroup H_{p^2}, and thus the canonical subgroup is among the subgroups occurring in the sum above. One can check using [Buzzard 2003, Theorem 3.3] that $(E/H_{p^2}, P/H_{p^2})$ corresponds to a point of $\mathcal{V}(p^{-p/(1+p)}, 1)$, while if $C \neq H_{p^2}$ is a cyclic subgroup of order p^2 with trivial intersection with $\langle P \rangle$, then $(E/C, P/C)$ corresponds to a point of $\mathcal{U}_{\mathbf{m}-1}(p^{-1/p(1+p)}, 1]$. Define F_1 on $\mathcal{V}(p^{-1/p(1+p)}, 1)$ by

$$F_1 = F - \frac{1}{\alpha p^2} \vartheta^k Q^*(F|_{\mathcal{V}(p^{-p/(1+p)},1)}).$$

Lemma 6.9. *The function F_1 on $\mathcal{V}(p^{-1/p(1+p)}, 1)$ extends to an element of*

$$H^0(\mathcal{V}(p^{-1/p(1+p)}, 1], \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}})).$$

Proof. Equation (11) and the comments that follow it show how to define the extension $\widetilde{F}_1$ of F_1, at least on noncuspidal points. For a pair (E, P) corresponding to a noncuspidal point of $\mathcal{V}(p^{-1/p(1+p)}, 1]$, we would like

$$\widetilde{F}_1(E, P) = \frac{1}{\alpha p^2} \sum_C F(E/C, P/C)\Theta^k_{p^2}(E, P, C) - \frac{1}{\alpha} H(E, P),$$

where the sum is over the cyclic subgroups of order p^2 of E not meeting $\langle P \rangle$ and not equal to $H_{p^2}(E)$. We can formalize this as follows.

The canonical subgroup of order p^2 furnishes a section to the finite map

$$\pi_1^{-1}(\mathcal{V}(p^{-1/p(1+p)}, 1]) \xrightarrow{\pi_1} \mathcal{V}(p^{-1/p(1+p)}, 1],$$

and section is an isomorphism onto a connected component of

$$\pi_1^{-1}(\mathcal{V}(p^{-1/p(1+p)}, 1]).$$

Let $\mathcal{Z}$ denote the complement of this connected component. Then π_1 restricts to a finite and flat map $\mathcal{Z} \to \mathcal{V}(p^{-1/p(1+p)}, 1]$. Note that

$$\mathcal{Z} = \pi_1^{-1}(\mathcal{V}(p^{-1/p(1+p)}, 1]) \cap \mathcal{Z} \subseteq \pi_2^{-1}(\mathcal{U}_{\mathbf{m}-1}(p^{-1/p(1+p)}, 1]) \cap \mathcal{Z},$$

as can be checked on noncuspidal points (see the comments following Equation (11)). Now we may apply the general construction of Section 5 with this $\mathcal{Z}$ and define

$$\widetilde{F}_1 = \frac{1}{\alpha p^2} \pi_{1*}(\pi_2^* F \cdot \Theta^k_{p^2}) - \frac{1}{\alpha} H.$$

Then $\widetilde{F}_1 \in H^0(\mathcal{V}(p^{-1/p(1+p)}, 1], \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}}))$, and Equation (11) shows that $\widetilde{F}_1$ extends F_1. □

For $n \geq 1$, we define inductively an element F_n of

$$H^0(\mathcal{V}(p^{-1/p^{2n-1}(1+p)}, 1], \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}})),$$

where F_1 is as above and for $n \geq 1$, we set

$$F_{n+1} = F_1 + \frac{1}{\alpha p^2} \vartheta^k Q^*(F_n|_{\mathcal{V}(p^{-1/p^{2n+1}(1+p)},1]}).$$

Note that (9) and (10) show that the F_n do indeed lie in the spaces indicated. Our goal is to show that the sequence $\{F_n\}$, when restricted to $\mathcal{V}[1, 1]$, converges to an element of G of $H^0(\mathcal{V}[1, 1], \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}}))$ that glues to F in the sense that there exists a global section of $\mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}})$ that restricts to F and G on $\mathcal{U}_{\mathbf{m}-1}$ and $\mathcal{V}[1, 1]$, respectively. To do this we will use Kassaei's gluing lemma [2006]. The following lemmas furnish some necessary norm estimates.

Lemma 6.10. *The function Θ_{p^2} on $Y_1(4, p^2)_{\mathbb{Q}_p}$ is integral. That is, it extends to a regular function on the fine moduli scheme $Y_1(4, p^2)_{\mathbb{Z}_p}$.*

Proof. Each $\Gamma_1(4) \cap \Gamma^0(p^2)$ structure on the elliptic curve $\underline{\text{Tate}}(q)/\mathbb{Q}_p(\!(q)\!)$ lifts trivially to one over the Tate curve thought of as over $\mathbb{Z}_p(\!(q)\!)$. Since the Tate curve is ordinary, such a structure specializes to a unique component of the special fiber $Y_1(4, p^2)_{\mathbb{F}_p}$. Since $Y_1(4, p^2)_{\mathbb{Z}_p}$ is Cohen–Macaulay, the usual argument used to prove the q-expansion principal (as in the proof of [Katz 1973, Corollary 1.6.2]) shows that Θ_{p^2} is integral as long as it has integral q-expansion associated to a level structure specializing to each component of the special fiber. In fact, all q-expansions of Θ_{p^2} are computed explicitly in [Ramsey 2006, Section 5], and all are integral. □

Lemma 6.11. *Let R be an $\mathbb{F}_p$-algebra, let E be an elliptic curve over R, and let $E^{(p)}$ denote the base change of E via the absolute Frobenius morphism on* $\operatorname{Spec}(R)$. *Let* $\operatorname{Fr} : E \to E^{(p)}$ *denote the relative Frobenius morphism. Then for any point P of order* 4 *on E, we have* $\Theta_{p^2}(E, P, \ker(\operatorname{Fr}^2)) = 0$.

Proof. In characteristic p, the forgetful map $Y_1(4, p^2)_{\mathbb{F}_p} \to Y_1(4)_{\mathbb{F}_p}$ has a section given on noncuspidal points by $s : (E, P) \mapsto (E, P, \ker(\operatorname{Fr}^2))$. By Lemma 6.10, we may pull back (the reduction of) Θ_{p^2} through this section to arrive at a regular function on the smooth curve $Y_1(4)_{\mathbb{F}_p}$.

The q-expansion of $s^*\Theta_{p^2}$ at the cusp associated to $(\underline{\text{Tate}}(q), \zeta_4)$ is

$$s^*\Theta_{p^2}(\underline{\text{Tate}}(q), \zeta_4) = \Theta_{p^2}(\underline{\text{Tate}}(q), \zeta_4, (\ker(\operatorname{Fr}^2))).$$

Recall that the map $\underline{\text{Tate}}(q) \to \underline{\text{Tate}}(q^p)$ given by quotienting by μ_p is a lifting of Fr to characteristic zero (more specifically, to the ring $\mathbb{Z}(\!(q)\!)$). Thus the q-expansion we seek is the reduction of

$$\Theta_{p^2}(\underline{\text{Tate}}(q), \zeta_4, \mu_{p^2}) = p \frac{\sum_{n\in\mathbb{Z}} q^{p^2 n^2}}{\sum_{n\in\mathbb{Z}} q^{n^2}}$$

modulo p, which is clearly zero. See [Ramsey 2006, Section 5] for the computation of the above q-expansion in characteristic zero. It follows from the q-expansion principle that $s^*\Theta_{p^2} = 0$, which implies our claim. □

Lemma 6.12. *Let* $0 \le r < 1/p(1+p)$. *Then the section* ϑ *of*

$$\mathcal{O}(\Sigma_{4Np^{m+1}/\mathbf{q}} - Q^*\Sigma_{4Np^{m+1}/\mathbf{q}})$$

satisfies $\|\vartheta\|_{\mathcal{V}[p^{-r},1]} \le p^{pr-1}$.

Proof. By Lemma 2.5, we may, in computing the norm, ignore points reducing to cusps. Let $x \in \mathcal{V}[p^{-r}, 1]$ be outside of this collection of points, so x corresponds to a pair (E, P) with good reduction. Let H_{p^i} denote the canonical subgroup of E of order p^i (for whichever i this is defined). Let $\mathbf{E}$ be a smooth model of E over $\mathcal{O}_L$, and let $\mathbf{P}$ and $\mathbf{H}_{p^2}$ be the extensions of P and H_{p^2} to $\mathbf{E}$, respectively (these $\mathbf{E}$ and $\mathbf{H}$ should not be confused with the functions by the same name from Section 3).

By [Goren and Kassaei 2006, Theorem 3.10], $\mathbf{H}_p$ reduces modulo $p/p^{v(E)}$ to $\ker(\mathrm{Fr})$. Applying this to $\mathbf{E}/\mathbf{H}_p$, we see that $\mathbf{H}_{p^2}/\mathbf{H}_p$ reduces modulo $p/p^{v(E/H_p)}$ to $\ker(\mathrm{Fr})$ on the corresponding reduction of E/H_p. Then from [Buzzard 2003, Theorem 3.3], we know that $v(E/H_p) = pv(E)$, so $p^{1-v(E/H_p)} \mid p^{1-v(E)}$ and we may combine these statements to conclude that $\mathbf{H}_{p^2}$ reduces modulo $p^{1-pv(E)}$ to $\ker(\mathrm{Fr}^2)$ on the reduction of E.

Combining this with the integrality of Θ_{p^2} (from Lemma 6.10), we have

$$h(x) = \Theta_{p^2}(E, P, H_{p^2}) \equiv \Theta_{p^2}(E, P, \ker(\mathrm{Fr}^2)) \pmod{p^{1-pv(E)}}.$$

This is zero by Lemma 6.11, so $|h(x)| \le |p^{1-pv(E)}| = p^{pv(E)-1} \le p^{pr-1}$. □

Proposition 6.13. *Let* $F \in H^0(\mathcal{U}_{\mathbf{m}-1}, \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}}))$. *Suppose* $U_{p^2}F - \alpha F$ *is classical for some* $\alpha \in K$ *with* $v(\alpha) < 2\lambda - 1$. *Then* F *is classical as well.*

Proof. Define F_n as above. We first show that the sequence $F_n|_{\mathcal{V}[1,1]}$ converges. Note that over $\mathcal{V}[1, 1]$ we have

$$\begin{aligned} F_{n+2} - F_{n+1} &= \Big(F_1 + \frac{1}{\alpha p^2}\vartheta^k Q^* F_{n+1}\Big) - \Big(F_1 + \frac{1}{\alpha p^2}\vartheta^k Q^* F_n\Big) \\ &= \frac{1}{\alpha p^2}\vartheta^k Q^*(F_{n+1} - F_n). \end{aligned}$$

By Lemma 6.12 (with $r = 0$), we have

$$\|F_{n+2} - F_{n+1}\|_{\mathcal{V}[1,1]} \le \frac{p^{2-k}}{|\alpha|}\|F_{n+1} - F_n\|_{\mathcal{V}[1,1]}.$$

The hypothesis on α ensures that $(p^{2-k}/|\alpha|)^n \to 0$ as $n \to \infty$ and hence that the sequence has successive differences that tend to zero. Since, by Lemma 2.1,

$H^0(\mathcal{V}[1,1], \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}}))$ is a Banach algebra with respect to $\|\cdot\|_{\mathcal{V}[1,1]}$, it follows that the sequence converges. Set

$$G = \lim_{n\to\infty} F_n|_{\mathcal{V}[1,1]}.$$

Next we apply Kassaei's gluing lemma [Kassaei 2006, Lemma 2.3] to glue G to F as sections of the line bundle $\mathcal{O}(\lfloor k\Sigma_{4Np^{m+1}/\mathbf{q}}\rfloor)$. So that we are gluing over an affinoid as required in the hypotheses of the gluing lemma, we first restrict F to $\mathcal{V}[p^{-1/p(1+p)}, 1)$ and glue G to this restriction to get a section over the smooth affinoid $\mathcal{V}[p^{-1/p(1+p)}, 1]$. Since the pair $\{\mathcal{V}[p^{-1/p(1+p)}, 1], \mathcal{U}_{\mathbf{m}-1}\}$ is an admissible cover of $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K$, this section glues to F to give a global section.

The "auxiliary" approximating sections that are required in the hypotheses of this lemma (denoted F_n in [Kassaei 2006]) are the F_n introduced above. So that the F_n live on affinoids (as in the hypotheses of the gluing lemma), we simply restrict F_n to $\mathcal{V}[p^{-1/p^{2n}(1+p)}, 1]$. The two conditions to be verified are

$$\|F_n - F\|_{\mathcal{V}[p^{-1/p^{2n}(1+p)},1)} \to 0 \quad \text{and} \quad \|F_n - G\|_{\mathcal{V}[1,1]} \to 0.$$

The second of these is simply the definition of G. As for the first, it is not even clear that the indicated norms are finite (since the norms are over non-affinoids). To see that these norms are finite and that the ensuing estimates make sense, we must show that F has finite norm over $\mathcal{V}[p^{-1/p^2(1+p)}, 1)$. It suffices to show that the norms of F over the affinoids $\mathcal{V}_n = \mathcal{V}[p^{-1/p^{2n}(1+p)}, p^{-1/p^{2n+2}(1+p)}]$ are uniformly bounded for $n \geq 1$. The key is that the map Q restricts to a map $Q : \mathcal{V}_n \to \mathcal{V}_{n+1}$ for each $n \geq 1$. Since F_1 extends to the affinoid $\mathcal{V}[p^{-1/p^2(1+p)}, 1]$, its norms over the $\mathcal{V}_n$ are certainly uniformly bounded, say, by M. We have

$$\begin{aligned}
\|F\|_{\mathcal{V}_n} &\leq \max\Big(\|F_1\|_{\mathcal{V}_n}, \Big\|\frac{1}{\alpha p^2}\vartheta^k Q^* F\Big\|_{\mathcal{V}_n}\Big)\\
&\leq \max\Big(M, \frac{p^2}{|\alpha|}\|\vartheta^k\|_{\mathcal{V}_n}\|Q^*F\|_{\mathcal{V}_n}\Big)\\
&\leq \max\Big(M, \frac{p^2}{|\alpha|}\big(p^{1/(p^{2n-1}(1+p))-1}\big)^k\|Q^*F\|_{\mathcal{V}_n}\Big)\\
&\leq \max\Big(M, \frac{p^{2-k}}{|\alpha|}p^{k/(p^{2n-1}(1+p))}\|F\|_{\mathcal{V}_{n-1}}\Big).
\end{aligned}$$

Iterating this, we see that $\|F\|_{\mathcal{V}_n}$ does not exceed the maximum of

$$\max_{0\leq m\leq n-2}\bigg(M\Big(\frac{p^{2-k}}{|\alpha|}\Big)^m p^{\frac{k}{1+p}\,(1/p^{2n-1}+\cdots+1/p^{2(n-m)+1})}\bigg)$$

and

$$\Big(\frac{p^{2-k}}{|\alpha|}\Big)^{n-1} p^{\frac{k}{1+p}\,(1/p^{2n-1}+\cdots+1/p^3)}\|F\|_{\mathcal{V}_1}.$$

The sums in the exponents of are geometric and do not exceed $1/(p^3-p)$. Moreover, the hypothesis on α ensures that $p^{2-k}/|\alpha|<1$. Thus we have

$$\|F\|_{\mathcal{V}_n} \le \max\Big(Mp^{\frac{k}{1+p}\frac{1}{p^3-p}},\, p^{\frac{k}{1+p}\frac{1}{p^3-p}}\|F\|_{\mathcal{V}_1}\Big),$$

which is independent of n, as desired. This ensures that all of the norms encountered below are indeed finite.

From the definition of the F_n, we have

$$\begin{aligned} F_{n+1}-F &= F_1+\frac{1}{\alpha p^2}\vartheta^k Q^* F_n - F \\ &= F-\frac{1}{\alpha p^2}\vartheta^k Q^*F+\frac{1}{\alpha p^2}\vartheta^k Q^*F_n - F = \frac{1}{\alpha p^2}\vartheta^k Q^*(F_n-F). \end{aligned}$$

Taking supremum norms over the appropriate admissible opens, we see

$$\begin{aligned} &\|F_{n+1}-F\|_{\mathcal{V}[p^{-1/(p^{2n+2}(1+p))},1)} \\ &\quad\le \frac{p^2}{|\alpha|}\|\vartheta\|^k_{\mathcal{V}[p^{-1/(p^{2n+2}(1+p))},1)}\|Q^*(F_n-F)\|_{\mathcal{V}[p^{-1/(p^{2n+2}(1+p))},1)} \\ &\quad\le \frac{p^2}{|\alpha|}\big(p^{1/(p^{2n+1}(1+p))-1}\big)^k\|F_n-F\|_{\mathcal{V}[p^{-1/(p^{2n}(1+p)},1))} \\ &\quad= \frac{p^{2-k}}{|\alpha|}p^{k/(p^{2n+1}(1+p))}\|F_n-F\|_{\mathcal{V}[p^{-1/(p^{2n}(1+p))},1)}. \end{aligned}$$

Iterating this we find that

$$\begin{aligned} &\|F_n-F\|_{\mathcal{V}[p^{-1/(p^{2n}(1+p))},1)} \\ &\quad\le \Big(\frac{p^{2-k}}{|\alpha|}\Big)^{n-1}p^{\frac{k}{1+p}\,(1/p^3+1/p^5+\cdots+1/p^{2n-1})}\|F_1-F\|_{\mathcal{V}[p^{-1/p^2(1+p)},1)}. \end{aligned}$$

Again the sum in the exponent is less than $1/(p^3-p)$ for all n, so the hypothesis on α ensures that the above norm tends to zero as $n\to\infty$, as desired □

We are now ready to prove the main result of this section, which is a mild generalization of Theorem 6.1.

Theorem 6.14. *Let m be a positive integer, let $\psi:(\mathbb{Z}/\mathbf{q}p^{m-1}\mathbb{Z})^\times\to K^\times$ be a character, and define $\kappa(x)=x^\lambda\psi(x)$. Let $P(T)\in K[T]$ be a monic polynomial all roots of which have valuation less than $2\lambda-1$. If $F\in\widetilde{M}^\dagger_\kappa(4N,K)$ and $P(U_{p^2})F$ is classical, then F is classical as well.*

Proof. Pick r in $0<r<r_m$ such that $F\in\widetilde{M}_\kappa(4N,K,p^{-r})$, and let $F'\in\widetilde{M}_{k/2}(4Np^{m+1}/\mathbf{q},K,p^{-r})$ be the form corresponding to F under the isomorphism of Proposition 6.2. We must show that F' is classical in the sense that it analytically continues to all of $X_1(4Np^{m+1}/\mathbf{q})^{\mathrm{an}}_K$. Note that $P(0)\neq 0$ for such a

polynomial, so by Proposition 6.8, F' analytically continues to an element of $H^0(\mathcal{U}_{\mathbf{m}-1}, \mathcal{O}(k\Sigma_{4Np^{m+1}/\mathbf{q}}))$. Now we proceed by induction on the degree d of P. The case $d = 1$ is Proposition 6.13. Suppose the result holds for some degree $d \geq 1$, and let $P(T)$ be a polynomial of degree $d+1$ as above. We may pass to a finite extension and write $P(T) = (T-\alpha_1)\cdots(T-\alpha_{d+1})$. The condition that $P(U_{p^2})F'$ is classical implies by the inductive hypothesis that $(U_{p^2} - \alpha_{d+1})F'$ is classical. This implies that F' is classical by the case $d = 1$. □

Remark 6.15. The results of this section likely also follow from the very general classicality machinery developed in [Kassaei 2005], though we have not checked the details.

7. The half-integral weight eigencurve

To construct our eigencurve, we will use the axiomatic version of Coleman and Mazur's Hecke algebra construction, as set up in [Buzzard 2007]. We briefly recall some relevant details.

Let us for the moment allow $\mathcal{W}$ to be any reduced rigid space over K. Let $\mathbf{T}$ be a set with a distinguished element ϕ. Suppose that, for each admissible affinoid open $X \subseteq \mathcal{W}$, we are given a Banach module M_X over $\mathcal{O}(X)$ satisfying a certain technical hypothesis (called *(Pr)* in [Buzzard 2007]), and we are also given a map

$$\mathbf{T} \to \mathrm{End}_{\mathcal{O}(X)}(M_X), \quad t \mapsto t_X$$

whose image consists of commuting endomorphisms and such that ϕ_X is compact for each X. Assume that, for admissible affinoids $X_1 \subseteq X_2 \subseteq \mathcal{W}$, we are given a continuous injective $\mathcal{O}(X_1)$-linear map

$$\alpha_{12} : M_{X_1} \to M_{X_2} \widehat{\otimes}_{\mathcal{O}(X_2)} \mathcal{O}(X_1)$$

that is a "link" in the sense of [Buzzard 2007] and such that $(t_{X_2} \widehat{\otimes} 1) \circ \alpha_{12} = \alpha_{12} \circ t_{X_1}$. Assume moreover that, if $X_1 \subseteq X_2 \subseteq X_3 \subseteq \mathcal{W}$ are admissible affinoids, then $\alpha_{13} = \alpha_{23} \circ \alpha_{12}$ with the obvious notation. Note that the link condition ensures that the characteristic power series $P_X(T)$ of ϕ_X acting on M_X is independent of X in the sense that the image of $P_{X_2}(T)$ under the natural map $\mathcal{O}(X_2)[\![T]\!] \to \mathcal{O}(X_1)[\![T]\!]$ is $P_{X_1}(T)$; see [Buzzard 2007].

Out of this data, Buzzard constructs rigid analytic spaces D and Z, called the *eigenvariety* and *spectral variety*, respectively, equipped with canonical maps

$$D \to Z \to \mathcal{W}. \tag{12}$$

The points of D parameterize systems of eigenvalues of $\mathbf{T}$ acting on the $\{M_X\}$ for which the eigenvalue of ϕ is nonzero, in a sense that will be made precise in Lemma 7.3, while the image of such a point in Z simply records the inverse of the

ϕ eigenvalue and a point of $\mathcal{W}$. If $\mathcal{W}$ is equidimensional of dimension d, then the same is true of both of the spaces D and Z.

As the details of this construction will be required in the next section, we recall them here. The following is the deepest part of the construction.

Theorem 7.1 [Buzzard 2007, Theorem 4.6]. *Let R be a reduced affinoid algebra over K, let $P(T)$ be a Fredholm series over R, and let $Z \subset \mathrm{Sp}(R) \times \mathbb{A}^1$ denote the hypersurface cut out by $P(T)$ equipped with the projection $\pi : Z \to \mathrm{Sp}(R)$. Define $\mathcal{C}(Z)$ to be the collection of admissible affinoid opens Y in Z such that*

- *$Y' = \pi(Y)$ is an admissible affinoid open in $\mathrm{Sp}(R)$,*
- *$\pi : Y \to Y'$ is finite, and*
- *there exists $e \in \mathcal{O}(\pi^{-1}(Y'))$ such that $e^2 = e$ and Y is the zero locus of e.*

Then $\mathcal{C}(Z)$ is an admissible cover of Z.

We will generally take Y' to be connected in what follows. This is not a serious restriction, since Y is the disjoint union of the parts lying over the various connected components of Y'. We also remark that the third of the above conditions follows from the first two (this is observed in [Buzzard 2007], where references to the proof are supplied).

To construct D, first fix an admissible affinoid open $X \subseteq \mathcal{W}$. Let Z_X denote the zero locus of $P_X(T) = \det(1 - \phi_X T \mid M_X)$ in $X \times \mathbb{A}^1$, and let $\pi : Z_X \to X$ denote the projection onto the first factor. Let $Y \in \mathcal{C}(Z_X)$, let $Y' = \pi(Y)$ as above, and assume that Y' is connected. We wish to associate to Y a polynomial factor of $P_{Y'}(T) = \det(1 - (\phi_X \widehat{\otimes} 1)T \mid M_X \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(Y'))$. Since the algebra $\mathcal{O}(Y)$ is a finite and locally free module over $\mathcal{O}(Y')$, we may consider the characteristic polynomial Q' of $T \in \mathcal{O}(Y)$. Since T is a root of its characteristic polynomial, we have a map

$$\mathcal{O}(Y')[T]/(Q'(T)) \to \mathcal{O}(Y). \tag{13}$$

It is shown in [Buzzard 2007, Section 5] that this map is surjective and therefore an isomorphism since both sides are locally free of the same rank.

Now since the natural map $\mathcal{O}(Y')[T]/(Q'(T)) \to \mathcal{O}(Y')\{\{T\}\}/(Q'(T))$ is an isomorphism, it follows that $Q'(T)$ divides $P_{Y'}(T)$ in $\mathcal{O}(Y')\{\{T\}\}$. If a_0 is the constant term of $Q'(T)$, then this divisibility implies that a_0 is a unit. We set $Q(T) = a_0^{-1} Q'(T)$. The spectral theory of compact operators on Banach modules (see [Buzzard 2007, Theorem 3.3]) furnishes a unique decomposition

$$M_X \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(Y') \cong N \oplus F$$

into closed ϕ-invariant $\mathcal{O}(Y')$-submodules such that $Q^*(\phi)$ is zero on N and invertible on F. Moreover, N is projective of rank equal to the degree of Q, and the characteristic power series of ϕ on N is $Q(T)$. The projector $M_X \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{O}(Y') \to N$

is in the closure of $\mathcal{O}(Y')[\phi]$, so N is stable under all of the endomorphisms associated to elements of $\mathbf{T}$. Let $\mathbf{T}(Y)$ denote the $\mathcal{O}(Y')$-subalgebra of $\mathrm{End}_{\mathcal{O}(Y')}(N)$ generated by these endomorphisms. Then $\mathbf{T}(Y)$ is finite over $\mathcal{O}(Y')$ and hence affinoid, so we may set $D_Y = \mathrm{Sp}(\mathbf{T}(Y))$. Because the leading coefficient of Q (that is, the constant term of Q^*) is a unit, there is an isomorphism

$$\mathcal{O}(Y')[T]/(Q(T)) \to \mathcal{O}(Y')[S]/(Q^*(S)), \quad T \mapsto S^{-1}.$$

Thus we obtain a canonical map $D_Y \to Y$, namely, the one corresponding to the map

$$\mathcal{O}(Y) \cong \mathcal{O}(Y')[T]/(Q(T)) \cong \mathcal{O}(Y')[S]/(Q^*(S)) \xrightarrow{S \mapsto \phi} \mathbf{T}(Y)$$

of affinoid algebras.

For general $Y \in \mathcal{C}(Z_X)$, we define D_Y to be the disjoint union of the affinoids defined above from the various connected components of Y'. We then glue the affinoids D_Y for $Y \in \mathcal{C}(Z_X)$ to obtain a rigid space D_X equipped with maps

$$D_X \to Z_X \to X.$$

Finally, we vary X and glue the desired spaces and maps above to obtain the spaces and maps in (12). This final step is where the links α_{ij} above come into play. See [Buzzard 2007] for details.

Definition 7.2. Let L be a complete extension of K. An L-valued system of eigenvalues of $\mathbf{T}$ acting on $\{M_X\}_X$ is a pair (κ, γ) consisting of a map of sets $\gamma : \mathbf{T} \to L$ and a point $\kappa \in \mathcal{W}(L)$ such that there exists an affinoid $X \subseteq \mathcal{W}$ containing κ and a nonzero element $m \in M_X \widehat{\otimes}_{\mathcal{O}(X),\kappa} L$ such that $(t_X \widehat{\otimes} 1)m = \gamma(t)m$ for all $t \in \mathbf{T}$. Such a system of eigenvalues is called ϕ-finite if $\gamma(\phi) \neq 0$.

Let x be an L-valued point of D. Then x lies over a point in $\kappa_x \in \mathcal{W}(L)$ that lies in X for some affinoid X, and x moreover lies in $D_Y(L)$ for some $Y \in \mathcal{C}(Z_X)$. Thus to x and the choice of X and Y corresponds a map $\mathbf{T}(Y) \to L$, and in particular a map of sets $\lambda_x : \mathbf{T} \to L$. Buzzard [2007] proves the following characterization of the points of D.

Lemma 7.3. *The correspondence $x \mapsto (\kappa_x, \lambda_x)$ is a well-defined bijective correspondence between L-valued points of D and ϕ-finite L-valued systems of eigenvalues of* $\mathbf{T}$ *acting on the $\{M_X\}$.*

In our case, we let $\mathcal{W}$ be weight space over $\mathbb{Q}_p$ as in Section 2.3, and let $\mathbf{T}$ be the set of symbols

$$\begin{cases} \{T_{\ell^2}\}_{\ell\nmid 4Np} \cup \{U_{\ell^2}\}_{\ell\mid 4Np} \cup \{\langle d\rangle_{4N}\}_{d\in(\mathbb{Z}/4N\mathbb{Z})^\times} & \text{if } p \neq 2, \\ \{T_{\ell^2}\}_{\ell\nmid 4N} \cup \{U_{\ell^2}\}_{\ell\mid 4N} \cup \{\langle d\rangle_{N}\}_{d\in(\mathbb{Z}/N\mathbb{Z})^\times} & \text{if } p = 2. \end{cases}$$

For an admissible affinoid open $X \subseteq \mathcal{W}$, we let $M_X = \widetilde{M}_X(4N, \mathbb{Q}_p, p^{-r_n})$, where n is the smallest positive integer such that $X \subseteq \mathcal{W}_n$. This module is a direct summand of the $\mathbb{Q}_p$-Banach space

$$\begin{cases} H^0(X_1(4Np)^{\mathrm{an}}_{\geq p^{-r_n}}, \mathcal{O}(\Sigma_{4Np}))\widehat{\otimes}_{\mathbb{Q}_p}\mathcal{O}(X) & \text{if } p \neq 2, \\ H^0(X_1(4N)^{\mathrm{an}}_{\geq 2^{-r_n}}, \mathcal{O}(\Sigma_{4N}))\widehat{\otimes}_{\mathbb{Q}_p}\mathcal{O}(X) & \text{if } p = 2 \end{cases}$$

and therefore satisfies property (Pr) since this latter space is potentially orthonormalizable in the terminology of [Buzzard 2007] by the discussion in [Serre 1962, Section 1]. We take the map $\mathbf{T} \to \mathrm{End}_{\mathcal{O}(X)}(M_X)$ to be the one sending each symbol to the endomorphism by that name defined in Section 5.

Let $X_1 \subseteq X_2 \subseteq \mathcal{W}$ be admissible affinoids, and let n_i be the smallest positive integer with $X_i \subseteq \mathcal{W}_{n_i}$. Then $n_1 \leq n_2$ so that $r_{n_2} \leq r_{n_1}$, and we have an inclusion

$$\widetilde{M}_{X_1}(4N, \mathbb{Q}_p, p^{-r_{n_1}}) \to \widetilde{M}_{X_1}(4N, \mathbb{Q}_p, p^{-r_{n_2}})$$

given by restriction. We define the required continuous injection α_{12} via the diagram

$$\begin{array}{ccc} \widetilde{M}_{X_1}(4N, \mathbb{Q}_p, p^{-r_{n_1}}) & \longrightarrow & \widetilde{M}_{X_1}(4N, \mathbb{Q}_p, p^{-r_{n_2}}) \\ & \searrow^{\alpha_{12}} & \uparrow \wr \\ & & \widetilde{M}_{X_2}(4N, \mathbb{Q}_p, p^{-r_{n_2}})\widehat{\otimes}_{\mathcal{O}(X_2)}\mathcal{O}(X_1) \end{array}$$

and note that the required compatibility condition is satisfied. To see that these maps are links, choose numbers $r_{n_1} = s_0 \geq s_1 > s_2 > \cdots > s_{k-1} \geq s_k = r_{n_2}$ with the property that $p^2 s_{i+1} > s_i$ for all i. Then the map α_{12} factors as the composition of the maps

$$\widetilde{M}_{X_1}(4N, \mathbb{Q}_p, p^{-s_i}) \to \widetilde{M}_{X_1}(4N, \mathbb{Q}_p, p^{-s_{i+1}})$$

for $0 \leq i \leq k-2$ and the map

$$\widetilde{M}_{X_1}(4N, \mathbb{Q}_p, p^{-s_{k-1}}) \to \widetilde{M}_{X_2}(4N, \mathbb{Q}_p, p^{-s_k})\widehat{\otimes}_{\mathcal{O}(X_2)}\mathcal{O}(X_1).$$

The construction of U_{p^2} shows easily that each of these maps is a primitive link.

The result is that we obtain rigid analytic spaces $\widetilde{D}$ and $\widetilde{Z}$, which we call *the half-integral weight eigencurve* and *the half-integral weight spectral curve*, respectively. We also obtain canonical maps $\widetilde{D} \to \widetilde{Z} \to \mathcal{W}$. As usual, the tilde distinguishes these spaces from their integral weight counterparts first constructed in level 1 by Coleman and Mazur and later constructed for general level by Buzzard [2007].

If instead of using the full spaces of forms we use only the cuspidal subspaces everywhere, then we obtain cuspidal versions of all of the above spaces, which we will delineate with a superscript 0. Thus we have $\widetilde{D}^0$ and $\widetilde{Z}^0$ with the usual maps, and the points of these spaces parameterize systems of eigenvalues of the Hecke operators acting on the spaces of cusp forms by Lemma 7.3. We remark that there is a commutative diagram

$$\begin{array}{ccc} \widetilde{D}^0 & \longrightarrow & \widetilde{D} \\ \downarrow & & \downarrow \\ \widetilde{Z}^0 & \longrightarrow & \widetilde{Z} \\ & \searrow \quad \swarrow & \\ & \mathcal{W} & \end{array}$$

where the horizontal maps are injections that identify the cuspidal spaces on the left with unions of irreducible components of the spaces on the right. Proving this is an exercise in the linear algebra that goes into the construction of these eigenvarieties and basic facts about irreducible components of rigid spaces found in [Conrad 1999], and is left to the reader.

For $\kappa \in \mathcal{W}(K)$, let $\widetilde{D}_\kappa$ and $\widetilde{D}^0_\kappa$ denote the fibers $\widetilde{D}$ and $\widetilde{D}^0$ over κ. The following theorem summarizes the basic properties of these eigencurves.

Theorem 7.4. *Let $\kappa \in \mathcal{W}(K)$. For a complete extension L/K, the correspondence $x \mapsto \lambda_x$ is a bijection between the L-valued points of the fiber $\widetilde{D}_\kappa(L)$ and the set of finite-slope systems of eigenvalues of the Hecke operators and tame diamond operators occurring on the space $\widetilde{M}^\dagger_\kappa(4N, L)$ of overconvergent forms of weight κ defined over L. The same statement holds with $\widetilde{D}$ replaced by $\widetilde{D}^0$ and $\widetilde{M}^\dagger_\kappa(4N, L)$ replaced by $\widetilde{S}^\dagger_\kappa(4N, L)$.*

Proof. We prove the statement for the full space of forms. The proof for cuspidal forms is identical. Fix $\kappa \in \mathcal{W}(K)$. Once we establish that the L-valued systems of eigenvalues of the form (κ, γ) occurring on the $\{M_X\}_X$ as defined above are exactly the systems of eigenvalues of the Hecke and tame diamond operators that occur on $\widetilde{M}^\dagger_\kappa(4N, L)$, the result is simply Lemma 7.3 "collated by weight." To see this one simply notes that, for any $f \in \widetilde{M}^\dagger_\kappa(4N, L)$, we have both $f \in \widetilde{M}_\kappa(4N, L, p^{-r_n})$ and $\kappa \in \mathcal{W}_n$ for n sufficiently large. In particular, if f is a nonzero eigenform for the Hecke and tame diamond operators, then the system of eigenvalues associated to f occurs in the module $M_{\mathcal{W}_n}$ for n sufficiently large. □

We remark that the classicality result of Section 6 has the expected consequence that the collection of points of $\widetilde{D}$ corresponding to systems of eigenvalues occurring on classical forms is Zariski-dense in $\widetilde{D}$. This result is contained in [Ramsey 2007].

Appendix: Properties of the stack $X_1(Mp, p^2)$ over $\mathbb{Z}_{(p)}$

by Brian Conrad

In this appendix, we establish some geometric properties concerning the cuspidal locus in compactified moduli spaces for level structures on elliptic curves. We are especially interested in the case of nonétale p-level structures in characteristic p, so it is not sufficient to cite the work in [Deligne and Rapoport 1973] (which requires étale level structures in the treatment of moduli problems for generalized elliptic curves) or [Katz and Mazur 1985] (which works with Drinfeld structures over arbitrary base schemes but avoids nonsmooth generalized elliptic curves). The viewpoints of these works were synthesized in the study of moduli stacks for Drinfeld structures on generalized elliptic curves in [Conrad 2007], and we will use that reference — abbreviated as [C] — as our foundation in what follows.

Motivated by needs in the main text, for a prime p and an integer $M \geq 4$ not divisible by p, we consider the moduli stack $X_1(Mp^r, p^e)$ over $\mathbb{Z}_{(p)}$ that classifies triples (E, P, C), where E is a generalized elliptic curve over a $\mathbb{Z}_{(p)}$-scheme S, $P \in E^{\mathrm{sm}}(S)$ is a Drinfeld $\mathbb{Z}/Mp^r\mathbb{Z}$-structure on E^{sm}, and $C \subseteq E^{\mathrm{sm}}$ is a cyclic subgroup with order p^e such that some reasonable ampleness and compatibility properties for P and C are satisfied. (See Definition A.1 for a precise formulation of these additional properties.) The relevant case for applications to p-adic modular forms with half-integer weight is $e = 2$, but unfortunately such moduli stacks were only considered in [C] when either $r \geq e$ or $r = 0$. (This is sufficient for applications to Hecke operators, and avoids some complications.) We now need to allow $1 \leq r < e$, and the purpose of this appendix is to explain how to include such r and to record some consequences concerning the cusps in these cases. The consequence relevant in the main text is Theorem A.11. To carry out the proofs in this appendix we simply have to adapt some proofs in [C] rather than develop any essentially new ideas. For the convenience of the reader we will usually use [C] as a reference, though it must be stressed that many of the key notions were first introduced in the earlier works [Deligne and Rapoport 1973] and [Katz and Mazur 1985]. In the context of subgroups of the smooth locus on a generalized elliptic curve, we will refer to a Drinfeld $\mathbb{Z}/N\mathbb{Z}$-structure (respectively a Drinfeld $\mathbb{Z}/N\mathbb{Z}$-basis) as a $\mathbb{Z}/N\mathbb{Z}$-structure (respectively $\mathbb{Z}/N\mathbb{Z}$-basis) unless some confusion is possible.

A.1. ***Definitions.*** See [C, Section 2.1] for the definitions of a generalized elliptic curve $f : E \to S$ over a scheme S and of the closed subscheme $S^\infty \subseteq S$ that is the "locus of degenerate fibers" for such an object. (It would be more accurate to write $S^{\infty, f}$, but the abuse of notation should not cause confusion.) Roughly speaking, $E \to S$ is a proper flat family of geometrically connected and semistable curves of arithmetic genus 1 that are either smooth or are so-called Néron polygons, and the relative smooth locus E^{sm} is endowed with a commutative S-group structure that extends (necessarily uniquely) to an action on E such that whenever E_s is a polygon, the action of E_s^{sm} on E_s is via rotations of the polygon. Also, S^∞ is a scheme structure on the set of $s \in S$ such that E_s is not smooth. The definition of the degeneracy locus S^∞ (given in [C, 2.1.8]) makes sense for any proper flat and finitely presented map $C \to S$ with fibers of pure dimension 1. If S' is any S-scheme, then there is an inclusion $S' \times_S S^\infty \subseteq S'^\infty$ as closed subschemes of S' (with S'^∞ corresponding to the S'-curve $C \times_S S'$), but this inclusion can fail to be an equality even when each geometric fiber C_s is smooth of genus 1 or a Néron polygon [C, Example 2.1.11]. Fortunately, if C admits a structure of generalized elliptic curve over S, then this inclusion is always an equality [C, 2.1.12], so the degeneracy locus makes sense on moduli stacks for generalized elliptic curves (where it defines the cusps).

We wish to study moduli spaces for generalized elliptic curves $E_{/S}$ equipped with certain ample level structures defined by subgroups of E^{sm}. Of particular interest are those subgroup schemes $G \subseteq E^{\text{sm}}$ that are not only finite locally free over the base with some constant order n but are even *cyclic* in the sense that *fppf*-locally on the base we can write $G = \langle P \rangle := \sum_{j \in \mathbb{Z}/n\mathbb{Z}} [jP]$ in E^{sm} as Cartier divisors for some n-torsion point P of E^{sm}. By [C, 2.3.5], if P and P' are two such points for the same G, then for any $d \,|\, n$ the points $(n/d)P$ and $(n/d)P'$ are $\mathbb{Z}/(n/d)\mathbb{Z}$-generators of the same S-subgroup of G, so by descent this naturally defines a cyclic S-subgroup $G_d \subseteq G$ of order d even if P does not exist over the given base scheme S. We call G_d the *standard* cyclic subgroup of G with order d. For example, if $d = d'd''$ with $d', d'' \geq 1$ and $\gcd(d', d'') = 1$, then $G_{d'} \times G_{d''} \simeq G_d$ via the group law on G.

Definition A.1. Let $N, n \geq 1$ be integers.

A $\Gamma_1(N)$*-structure* on a generalized elliptic curve $E_{/S}$ is an S-ample $\mathbb{Z}/N\mathbb{Z}$-structure on E^{sm}, which is to say an N-torsion point $P \in E^{\text{sm}}(S)$ such that the relative effective Cartier divisor $D = \sum_{j \in \mathbb{Z}/N\mathbb{Z}} [jP]$ on E^{sm} is an S-subgroup and D_s is ample on E_s for all $s \in S$.

A $\Gamma_1(N, n)$*-structure* on $E_{/S}$ is a pair (P, C), where P is a $\mathbb{Z}/N\mathbb{Z}$-structure on E^{sm} and $C \subseteq E^{\text{sm}}$ is a cyclic S-subgroup with order n such that the relative effective Cartier divisor $D = \sum_{j \in \mathbb{Z}/N\mathbb{Z}} (jP + C)$ on E is S-ample and there is an equality of closed subschemes

$$\sum_{j \in \mathbb{Z}/p^{e_p}\mathbb{Z}} (j(N/p^{e_p})P + C_{p^{e_p}}) = E^{\text{sm}}[p^{e_p}] \tag{1}$$

for all primes $p \,|\, \gcd(N, n)$, with $e_p = \text{ord}_p(\gcd(N, n)) \geq 1$.

Example A.2. Obviously a $\Gamma_1(N, 1)$-structure is the same as a $\Gamma_1(N)$-structure. If $N = 1$, then we refer to $\Gamma_1(1)$-structures as $\Gamma(1)$-structures, and such a structure on a generalized elliptic curve $E_{/S}$ must be the identity section. Thus, by the ampleness requirement, the geometric fibers E_s must be irreducible. Hence, the moduli stack $\mathcal{M}_{\Gamma(1)}$ of $\Gamma(1)$-structures on generalized elliptic curves classifies generalized elliptic curves with geometrically irreducible fibers.

In [C, 2.4.3], the notion of $\Gamma_1(N, n)$-structure is defined as above, but with the additional requirement that $\mathrm{ord}_p(n) \leq \mathrm{ord}_p(N)$ for all primes p such that $p \mid \gcd(N, n)$. This requirement always holds when $n = 1$, and whenever it holds, the standard subgroup $C_{p^{e_p}}$ in (1) is the p-part of C, but it turns out to be unnecessary for the proofs of the basic properties of $\Gamma_1(N, n)$-structures and their moduli, as we shall explain in Section A.2. For example, the proof of [C, 2.4.4] carries over to show that we can replace (1) with the requirement that $\sum_{j \in \mathbb{Z}/d\mathbb{Z}}(j(N/d)P + C_d) = E^{\mathrm{sm}}[d]$ in E for $d = \gcd(N, n)$. Another basic property that carries over to the general case is that if (P, C) is a $\Gamma_1(N, n)$-structure on E, then the relative effective Cartier divisor $\sum_{j \in \mathbb{Z}/N\mathbb{Z}}(jP + C)$ on E^{sm} is an S-subgroup; the proof is given in [C, 2.4.5] under the assumption $\mathrm{ord}_p(n) \leq \mathrm{ord}_p(N)$ for every prime $p \mid \gcd(N, n)$, but the argument works in general once it is observed that after making an *fppf* base change to acquire a $\mathbb{Z}/n\mathbb{Z}$-generator Q of C we can use symmetry in P and Q in the rest of the argument so as to reduce to the case considered in [Conrad 2007].

A.2. ***Moduli stacks.*** As in [C, 2.4.6], for $N, n \geq 1$ we define the moduli stack $\mathcal{M}_{\Gamma_1(N,n)}$ in order to classify $\Gamma_1(N, n)$-structures on generalized elliptic curves over arbitrary schemes, and we let $\mathcal{M}^{\infty}_{\Gamma_1(N,n)} \hookrightarrow \mathcal{M}_{\Gamma_1(N,n)}$ denote the closed substack given by the degeneracy locus for the universal generalized elliptic curve. The arguments in [C, Sections 3.1 and 3.2] carry over verbatim (that is, without using the condition $\mathrm{ord}_p(n) \leq \mathrm{ord}_p(N)$ for all primes $p \mid \gcd(N, n)$) to prove the following result.

Theorem A.3. *The stack $\mathcal{M}_{\Gamma_1(N,n)}$ is an Artin stack that is proper over $\mathbb{Z}$. It is smooth over $\mathbb{Z}[1/Nn]$, and it is Deligne–Mumford away from the open and closed substack in $\mathcal{M}^{\infty}_{\Gamma_1(N,n)}$ classifying degenerate triples (E, P, C) in positive characteristics p such that the p-part of each geometric fiber of C is nonétale and disconnected.*

The proof of [C, 3.3.4] does not use the condition $\mathrm{ord}_p(n) \leq \mathrm{ord}_p(N)$ for all primes $p \mid \gcd(N, n)$ (although this condition is mentioned in the proof), so that argument gives this:

Lemma A.4. *The open substack $\mathcal{M}^{0}_{\Gamma_1(N,n)} = \mathcal{M}_{\Gamma_1(N,n)} - \mathcal{M}^{\infty}_{\Gamma_1(N,n)}$ classifying elliptic curves endowed with a $\Gamma_1(N, n)$-structure is regular and $\mathbb{Z}$-flat with pure relative dimension* 1.

We are interested in the structure of $\mathcal{M}_{\Gamma_1(N,n)}$ around its cuspidal substack, and especially in determining whether it is regular or a scheme near such points. Our analysis of $\mathcal{M}^{\infty}_{\Gamma_1(N,n)}$ rests on the following theorem.

Theorem A.5. *The map $\mathcal{M}_{\Gamma_1(N,n)} \to \mathrm{Spec}(\mathbb{Z})$ is flat and Cohen–Macaulay with pure relative dimension* 1.

Proof. By Lemma A.4, we just have to work along the cusps. Also, it suffices to check the result after localization at each prime p, and if either $p \nmid \gcd(N, n)$ or $1 \le \operatorname{ord}_p(n) \le \operatorname{ord}_p(N)$ then [C, 3.3.1] gives the result over $\mathbb{Z}_{(p)}$. It thus remains to study the cusps in positive characteristic p when $1 \le \operatorname{ord}_p(N) < \operatorname{ord}_p(n)$. As in the cases treated in [Conrad 2007], the key is to study the deformation theory of a related level structure on generalized elliptic curves, the so-called $\widetilde{\Gamma}_1(N, n)$-*structure*: this is a pair (P, Q), where P is a $\mathbb{Z}/N\mathbb{Z}$-structure on the smooth locus and Q is a $\mathbb{Z}/n\mathbb{Z}$-structure on the smooth locus such that $(P, \langle Q\rangle)$ is a $\Gamma_1(N, n)$-structure. The same definition is given in [C, 3.3.2] with the unnecessary restriction $\operatorname{ord}_p(n) \le \operatorname{ord}_p(N)$ for all primes $p \mid \gcd(N, n)$, and the argument that immediately follows that definition works without such a restriction to show that the moduli stack $\mathcal{M}_{\widetilde{\Gamma}_1(N,n)}$ of $\widetilde{\Gamma}_1(N, n)$-structures is a Deligne–Mumford stack over $\mathbb{Z}$ that is a finite flat cover of the proper Artin stack $\mathcal{M}_{\Gamma_1(N,n)}$.

By the Deligne–Mumford property, any $\widetilde{\Gamma}_1(N, n)$-structure $x_0 = (E_0, P_0, Q_0)$ over an algebraically closed field k admits a universal deformation ring. Since $\mathcal{M}_{\widetilde{\Gamma}_1(N,n)}$ is a finite flat cover of $\mathcal{M}_{\Gamma_1(N,n)}$, as in the proof of [C, 3.3.1], it suffices to assume $\operatorname{char}(k) = p > 0$ and to exhibit the deformation ring at x_0 as a finite flat extension of $W(k)[\![x]\!]$ when E_0 is a standard polygon, $n = p^e$, and $N = Mp^r$ with $p \nmid M$ and $e, r \ge 1$. The case $e \le r$ is settled in [Conrad 2007], and we will adapt that argument to handle the case $1 \le r < e$. By the ampleness condition, at least one of MP_0 or Q_0 generates the p-part of the component group of E_0^{sm}, and moreover $\{MP_0, p^{e-r}Q_0\}$ is a Drinfeld $\mathbb{Z}/p^r\mathbb{Z}$-basis of $E_0^{\mathrm{sm}}[p^r]$. We shall break up the problem into three cases, and it is only in Case 3 that we will meet a situation essentially different from that encountered in Conrad's proof for $1 \le e \le r$.

CASE 1: We first assume that MP_0 generates the p-part of the component group, so by the Drinfeld $\mathbb{Z}/p^r\mathbb{Z}$-basis hypothesis, this point is a basis of $E_0^{\mathrm{sm}}(k)[p^\infty]$ over $\mathbb{Z}/p^r\mathbb{Z}$ (as we are in characteristic p and E_0 is a polygon). Hence, $Q_0 = jMP_0$ for a unique $j \in \mathbb{Z}/p^r\mathbb{Z}$ (so $p^{e-r}Q_0 = p^{e-r}jMP_0$). Since n is a p-power, it also follows that $\langle P_0\rangle$ is ample. In particular, (E_0, P_0) is a $\Gamma_1(N)$-structure. Thus, the formation of an infinitesimal deformation (E, P, Q) of (E_0, P_0, Q_0) can be given in three steps: first give an infinitesimal deformation (E, P) of (E_0, P_0) as a $\Gamma_1(N)$-structure, then give a Drinfeld $\mathbb{Z}/p^r\mathbb{Z}$-basis (MP, Q') of $E^{\mathrm{sm}}[p^r]$ with Q' deforming $p^{e-r}Q_0$, and finally specify a p^{e-r}-th root Q of Q' lifting $Q_0 = jMP_0$. The one aspect of this description that merits some explanation is to justify that such a p^{e-r}-th root Q of Q' must be a $\mathbb{Z}/p^e\mathbb{Z}$-structure on E^{sm}. The point Q is clearly killed by p^e, so the Cartier divisor $D = \sum_{j \in \mathbb{Z}/p^e\mathbb{Z}}[jQ]$ in E^{sm} makes sense, and we have to check that it is automatically a subgroup scheme.

For any $t \ge 0$, the identification $(E_0^{\mathrm{sm}})^0[p^t] = \mu_{p^t}$ uniquely lifts to an isomorphism $(E^{\mathrm{sm}})^0[p^t] \simeq \mu_{p^t}$. In particular, if p^ν is the order of the p-part of the cyclic component group of $E_0^{\rm sm}$ (with $\nu \geq r$), then $E^{\rm sm}[p^e]$ is an extension of $\mathbb{Z}/p^j\mathbb{Z}$ by μ_{p^e}, where $j = \min(\nu, e)$. The image of $\langle Q_0 \rangle$ in the component group can be uniquely identified with $\mathbb{Z}/p^i\mathbb{Z}$ (for some $i \leq j$) such that $Q_0 \mapsto 1$, and this $\mathbb{Z}/p^i\mathbb{Z}$ has preimage G in $E^{\rm sm}[p^e]$ that is a p^e-torsion commutative extension of $\mathbb{Z}/p^i\mathbb{Z}$ by μ_{p^e} with $0 \leq i \leq e$. Since Q is a point of G over the (artinian local) base, it follows from [C, 2.3.3] that Q is a $\mathbb{Z}/p^e\mathbb{Z}$-structure on $E^{\rm sm}$ if and only if the point $p^i Q$ in $\mu_{p^{e-i}}$ is a $\mathbb{Z}/p^{e-i}\mathbb{Z}$-generator of $\mu_{p^{e-i}}$. The case $i = e$ is therefore settled, so we can assume $i < e$ (that is, $\langle Q_0 \rangle$ is not étale, or equivalently $p^{e-1}Q_0 = 0$). By hypothesis, $p^{e-r}Q = Q'$ is a $\mathbb{Z}/p^r\mathbb{Z}$-structure on $E^{\rm sm}$ with $1 \leq r < e$, so $p^{e-1}Q = p^{r-1}Q'$ is a $\mathbb{Z}/p\mathbb{Z}$-structure on $E^{\rm sm}$. This $\mathbb{Z}/p\mathbb{Z}$-structure must generate the subgroup $\mu_p \subseteq E^{\rm sm}[p^e]$ since $p^{e-1}Q$ lies in $(E^{\rm sm})^0$ (as $p^{e-1}Q_0 = 0$). Hence, $Q'' = p^i Q$ is a point of $\mu_{p^{e-i}}$ such that $p^{e-i-1}Q''$ is a $\mathbb{Z}/p\mathbb{Z}$-generator of μ_p. Since $\mathbb{Z}/m\mathbb{Z}$-generators of μ_m are simply roots of the cyclotomic polynomial Φ_m [C, 1.12.9], our problem is reduced to the assertion that if s is a positive integer (such as $e - i$), then an element ζ in a ring is a root of the cyclotomic polynomial Φ_{p^s} if $\zeta^{p^{s-1}}$ is a root of Φ_p. This assertion is obvious since $\Phi_{p^s}(T) = \Phi_p(T^{p^{s-1}})$, and so our description of the infinitesimal deformation theory of (E_0, P_0, Q_0) is justified.

The torsion subgroup $E^{\rm sm}[p^r]$ is uniquely an extension of $\mathbb{Z}/p^r\mathbb{Z}$ by μ_{p^r} deforming the canonical such description for $E_0^{\rm sm}[p^r]$, so the condition on Q' is that it has the form $\zeta + p^{e-r}jMP$ for a point ζ of the scheme of generators $\mu_{p^r}^\times$ of $\mu_{p^r} = (E^{\rm sm})^0[p^r]$. Thus, to give Q is to specify a p^{e-r}-th root of ζ in $E^{\rm sm}$ deforming the identity, which is to say a point of $\mu_{p^e}^\times$. It is shown in the proof of [C, 3.3.1] that the universal deformation ring A for (E_0, P_0) is finite flat over $W(k)[\![x]\!]$, and the specification of ζ amounts to giving a root of the cyclotomic polynomial Φ_{p^e}, so the case when MP_0 generates the p-part of the component group of $E_0^{\rm sm}$ is settled (with deformation ring $A[T]/(\Phi_{p^e}(T))$).

CASE 2: Next assume that Q_0 generates the p-part of the component group and that $\langle Q_0 \rangle$ is étale (that is, $Q_0 \in E_0^{\rm sm}(k)$ has order p^e). The point Q_0 must generate $E_0^{\rm sm}(k)[p^\infty]$ over $\mathbb{Z}/p^e\mathbb{Z}$, and the étale hypothesis ensures that Q_0 is a $\mathbb{Z}/p^e\mathbb{Z}$-basis of $E_0^{\rm sm}(k)[p^\infty]$. Thus, $MP_0 = p^{e-r}jQ_0$ for some (unique) $j \in \mathbb{Z}/p^r\mathbb{Z}$. By replacing P with $P - M^{-1}p^{e-r}jQ$ for any infinitesimal deformation (E, P, Q) of (E_0, P_0, Q_0), we can assume that the p-part of P_0 vanishes. The p-part of P must therefore be a point of $\mu_{p^r}^\times$. The $\mathbb{Z}/M\mathbb{Z}$-part of P together with Q constitutes a $\Gamma_1(Mp^e)$-structure on E (in particular, the ampleness condition holds), and this is an étale level structure since the cyclic subgroup $\langle Q_0 \rangle$ in $E_0^{\rm sm}$ is étale. Hence, the infinitesimal deformation functor of (E_0, P_0, Q_0) is pro-represented by $\mu_{p^r}^\times$ over the deformation ring of an étale $\Gamma_1(Mp^e)$-structure. For any $R \geq 1$, deformation rings for étale $\Gamma_1(R)$-structures on polygons over k have the form $W(k)[\![x]\!]$ (as is explained near the end of the proof of [C, 3.3.1], using [C, II, 1.17]), so not only are we done but in this case the deformation ring for (E_0, P_0, Q_0) is the ring $W(k)[\![x]\!][T]/(\Phi_{p^r}(T))$ that is visibly regular.

CASE 3: Finally, assume Q_0 generates the p-part of the component group but that $\langle Q_0\rangle$ is not étale (that is, $Q_0 \in E_0^{\mathrm{sm}}(k)$ has order strictly less than p^e), and so $p^{e-r}Q_0 \in E_0^{\mathrm{sm}}(k)$ has order strictly dividing p^r. Since $\{MP_0, p^{e-r}Q_0\}$ is a Drinfeld $\mathbb{Z}/p^r\mathbb{Z}$-basis of $E_0^{\mathrm{sm}}[p^r]$, the point MP_0 must be a $\mathbb{Z}/p^r\mathbb{Z}$-basis for $E_0^{\mathrm{sm}}(k)[p^r]$. Hence, if we write $P_0 = P_0' + P_0''$ corresponding to the decomposition $\mathbb{Z}/N\mathbb{Z} = (\mathbb{Z}/M\mathbb{Z}) \times (\mathbb{Z}/p^r\mathbb{Z})$, then P_0'' has order exactly p^r in $E_0^{\mathrm{sm}}(k)$. We use P_0'' to identify $E_0^{\mathrm{sm}}(k)[p^r]$ with $\mathbb{Z}/p^r\mathbb{Z}$. It follows that if we make the analogous canonical decomposition $P = P' + P''$ for an infinitesimal deformation (E, P, Q) of (E_0, P_0, Q_0), then the p-part P'' deforms P_0'' and generates an étale subgroup of E^{sm} with order p^r. Thus, P' and Q together constitute a (nonétale) $\Gamma_1(Mp^e)$-structure on E (in particular, the ampleness condition holds), and the data of P'' amounts to a section over $1 \in \mathbb{Z}/p^r\mathbb{Z}$ with respect to the unique quotient map $E^{\mathrm{sm}}[p^r] \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z}$ lifting the quotient map $E_0^{\mathrm{sm}}[p^r] \twoheadrightarrow \mathbb{Z}/p^r\mathbb{Z}$ defined by P_0''. Since the specification of a $\mathbb{Z}/N\mathbb{Z}$-structure on E^{sm} is the "same" as the specification of a pair consisting of $\mathbb{Z}/M\mathbb{Z}$-structure and a $\mathbb{Z}/p^r\mathbb{Z}$-structure [C, 1.7.3], we conclude that the universal deformation ring of (E_0, P_0, Q_0) classifies the fiber over $1 \in \mathbb{Z}/p^r\mathbb{Z}$ in the connected-étale sequence for the p^r-torsion in infinitesimal deformations of the underlying $\Gamma_1(Mp^e)$-structure (E_0, P_0', Q_0). Universal deformation rings for $\Gamma_1(Mp^e)$-structures over k are finite flat over $W(k)[\![x]\!]$ (by the proof of [C, 3.3.1]), so we are therefore done. □

Corollary A.6. *The closed substack $\mathcal{M}^{\infty}_{\Gamma_1(N,n)} \hookrightarrow \mathcal{M}_{\Gamma_1(N,n)}$ is a relative effective Cartier divisor over $\mathbb{Z}$, and it has a reduced generic fiber over $\mathbb{Q}$.*

Proof. The reducedness over $\mathbb{Q}$ is shown in [C, 4.3.2], and the proof works without restriction on $\gcd(N, n)$. Likewise, the proof that $\mathcal{M}^{\infty}_{\Gamma_1(N,n)}$ is a $\mathbb{Z}$-flat Cartier divisor is part of [C, 4.1.1(1)] in case $\mathrm{ord}_p(n) \le \mathrm{ord}_p(N)$ for all primes $p \mid \gcd(N, n)$, but by using the above proof of Theorem A.5, we see that the method of proof works in general. □

Using Lemma A.4, Theorem A.5, and Corollary A.6, Serre's normality criterion can be used to prove normality for $\mathcal{M}_{\Gamma_1(N,n)}$ in general. (This is proved in [C, 4.1.4] subject to the restrictions on $\gcd(N, n)$ in the definition therein of $\Gamma_1(N, n)$-structures, but the argument works in general by using the results that are stated above without any such restriction on $\gcd(N, n)$.) However, the proof of regularity encounters complications at points of a certain locus of cusps in bad characteristics. This problematic locus is defined as follows.

Definition A.7. Let $\mathscr{Z}_{\Gamma_1(N,n)} \hookrightarrow \mathscr{M}^{\infty}_{\Gamma_1(N,n)}$ be the 0-dimensional closed substack with reduced structure that consists of geometric points (E_0, P_0, C_0) in characteristics $p \mid \gcd(N, n)$ such that $1 \leq \mathrm{ord}_p(N) < \mathrm{ord}_p(n)$, C_0 is not étale, and $(N/p^{\mathrm{ord}_p(N)})P_0$ does not generate the p-part of the component group of E_0^{sm}.

Note that if $\mathrm{ord}_p(n) \leq \mathrm{ord}_p(N)$ for all primes $p \mid \gcd(N, n)$ (the situation considered in [Conrad 2007]), then $\mathscr{Z}_{\Gamma_1(N,n)}$ is empty; this includes the case of $\Gamma_1(N)$-structures for any N (take $n = 1$). In all other cases, it is nonempty. The geometric points of $\mathscr{Z}_{\Gamma_1(N,n)}$ correspond to precisely the points in Case 3 in the proof of Theorem A.5. The method in [Conrad 2007] for analyzing regularity along the cusps assumes $\mathscr{Z}_{\Gamma_1(N,n)}$ is empty, and by combining it with the modified arguments in the proof of Theorem A.5 (especially the regularity observation in Case 2) we obtain the following consequence.

Theorem A.8. *Outside the closed substack $\mathscr{Z}_{\Gamma_1(N,n)} \subseteq \mathscr{M}^{\infty}_{\Gamma_1(N,n)}$, the stack $\mathscr{M}_{\Gamma_1(N,n)}$ is regular.*

A.3. ***Applications.*** Before we apply our results, we record a useful lemma.

Lemma A.9. *Let S be a scheme, and let $\mathscr{X}$ be an Artin stack over S. Assume $\mathscr{X}$ is S-separated. The locus of geometric points of $\mathscr{X}$ with trivial automorphism group scheme is an open substack $\mathscr{U} \subseteq \mathscr{X}$ that is an algebraic space. This algebraic space is a scheme if $\mathscr{X}$ is quasifinite over a separated S-scheme.*

Proof. The first part is [C, 2.2.5(2)], and the second part follows from the general fact that an algebraic space that is quasifinite and separated over a scheme is a scheme [C, Theorem A.2]. □

In the setting of Lemma A.9, if $\mathscr{X}$ is quasifinite over a separated S-scheme, then we call $\mathscr{U}$ the *maximal open subscheme* of $\mathscr{X}$. The case of interest to us is $\mathscr{X} = \mathscr{M}_{\Gamma_1(N,n)/S}$ over any scheme S. This is quasifinite over the S-proper stack $\mathscr{M}_{\Gamma(1)/S}$ via fibral contraction away from the identity component, and $\mathscr{M}_{\Gamma(1)/S}$ is quasifinite over $\mathbf{P}^1_S$ via the j-invariant, so $\mathscr{X}$ is quasifinite over the separated S-scheme $\mathbf{P}^1_S$.

We wish to prove results concerning when certain components of $\mathscr{M}^{\infty}_{\Gamma_1(N,n)}$ lie in the maximal open subscheme of $\mathscr{M}_{\Gamma_1(N,n)}$. So we first record a general lemma.

Lemma A.10. *Let $\mathscr{Y}$ be an irreducible Artin stack over $\mathbb{F}_p$, and let $\mathscr{C}$ be a finite locally free commutative $\mathscr{Y}$-group that is cyclic with order p^e. If $\mathscr{C}$ has a multiplicative geometric fiber over $\mathscr{Y}$, then all of its geometric fibers are connected.*

The abstract notion of cyclicity (with no ambient smooth curve group) is developed in [C, 1.5, 1.9, 1.10] over arbitrary base schemes, and the theory carries over when the base is an Artin stack. We will only need the lemma for situations that arise within torsion on generalized elliptic curves (over Artin stacks).

Proof. We can assume $e \geq 1$, and we may replace $\mathcal{C}$ with its standard subgroup $\mathcal{C}_p$ of order p because it is obvious by group theory that a *cyclic* group scheme C of p-power order over an algebraically closed field of characteristic p is étale if and only if its standard subgroup of order p is étale. Hence, we can assume that $\mathcal{C}$ has order p. Our problem is therefore to rule out the existence of étale fibers. By openness of the locus of étale fibers and irreducibility of $\mathcal{Y}$, if there is an étale fiber, then there is a Zariski-dense open $\mathcal{U} \subseteq \mathcal{Y}$ over which $\mathcal{C}$ has étale fibers. In particular, there is some geometric point u of $\mathcal{U}$ that specializes to the geometric point $y \in \mathcal{Y}$ where we assume the fiber is multiplicative, so after pullback to a suitable valuation ring, we get an étale group of order p in characteristic p specializing to a multiplicative one. Passing to Cartier duals gives a multiplicative group of order p having an étale specialization, and this is impossible since multiplicative groups of order p in characteristic p are not étale. □

Theorem A.11. *Let p be a prime, and choose a positive integer M not divisible by p such that $M > 2$. Also fix integers $e, r \geq 0$. If $e = 0$ or $r = 0$, then assume $M \neq 4$. Let $x_0 = (E_0, P_0, C_0)$ be a geometric point on the special fiber of the cuspidal substack in the proper Artin stack $\mathcal{X} = \mathcal{M}_{\Gamma_1(Mp^r, p^e)/\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$, and assume that C_0 is étale.*

Let $\mathcal{Y}$ be the irreducible component of x_0 in $\mathcal{X}_{\mathbb{F}_p}$. For every geometric cusp $x_1 = (E_1, P_1, C_1)$ on $\mathcal{Y}$, the group C_1 is étale and x_1 lies in the maximal open subscheme of $\mathcal{X}$. Moreover, if $x \in \mathcal{X}_{\mathbb{Q}}$ is a cusp specializing into $\mathcal{Y}$, then the Zariski closure D of x in $\mathcal{X}$ lies in the maximal open subscheme and D is Cartier in $\mathcal{X}$.

The case $e = 2$ is required in the main text. It is necessary to avoid the cases $M \leq 2$ and $(M, r) = (4, 0)$ because in these cases there are cusps x_0 in characteristic p as in the theorem such that x_0 admits nontrivial automorphisms (and so x_0 cannot lie in the maximal open subscheme of $\mathcal{X}$).

Proof. We first check that the étale assumption at x_0 is inherited by all geometric cusps $x_1 \in \mathcal{Y}$. Let $(\mathcal{E}, \mathcal{P}, \mathcal{C})$ be the pullback to $\mathcal{Y}$ of the universal family over $\mathcal{X}$. The group $\mathcal{C}$ is cyclic of order p^e with $e \geq 0$, so applying Lemma A.10 to its Cartier dual gives the result (since at a cusp a connected subgroup of p-power order must be multiplicative).

Now we can rename x_1 as x_0 without loss of generality, so we have to check that x_0 lies in the maximal open subscheme of $\mathcal{X}$ and that if $x \in \mathcal{X}_{\mathbb{Q}}$ is a geometric cusp specializing to x_0, then the Zariski closure of x in $\mathcal{X}$ is Cartier. But the étale hypothesis on C_0 ensures that x_0 is not in the closed substack $\mathcal{Z}_{\Gamma_1(Mp^r, p^e)/\mathbb{Z}_{(p)}}$, so by Theorem A.8 the stack $\mathcal{X}$ is regular at x_0. Hence, since $\mathcal{X}$ is $\mathbb{Z}_{(p)}$-flat with pure relative dimension 1 (by Theorem A.5), the desired properties of D at the end of the theorem hold once we know that x_0 is in the maximal open subscheme of $\mathcal{X}$, which is to say that its automorphism group scheme G is trivial. To verify this triviality we will make essential use of the property that C_0 is étale. Let k be the algebraically closed field over which x_0 lives. Since E_0 is d-gon over k for some $d \geq 1$, G is a closed subgroup of the automorphism group $\mu_d \rtimes \langle \mathrm{inv} \rangle$ of the d-gon. Since C_0 is étale with order p^e in characteristic p, it follows that C_0 maps isomorphically into the p-part of the component group of $E_0^{\mathrm{sm}} = \mathbf{G}_m \times (\mathbb{Z}/d\mathbb{Z})$. (In particular, $p^e \mid d$.) If R is an artinian local k-algebra with residue field k, any choice of generator Q_0 of C_0 must be carried to another generator of C_0 by any $g \in G(R)$ since $C_0(R) \to C_0(k)$ is a bijection. But $\mu_d(R)$ acts on $(E_0)_R$ in a manner that preserves the components of the smooth locus, and C_0 meets each component of E_0^{sm} in at most one point. Hence, $G \cap \mu_d$ acts as automorphisms of the $\Gamma_1(Mp^e)$-structure on E_0 defined by $p^r P_0$ and Q_0. Since $Mp^e > 2$ and $Mp^e \neq 4$ (due to the cases we are avoiding), such an ample level structure on a d-gon has trivial automorphism group scheme. This shows that $G \cap \mu_d$ is trivial, so G injects into the group $\mathbb{Z}/2\mathbb{Z}$ of automorphisms of the identity component $\mathbf{G}_m$ of E_0^{sm}. Hence, the contraction operation on E_0 away from $\langle P_0 \rangle$ is faithful on G since contraction does not affect the identity component. It follows that G is a subgroup of the automorphism group of the $\Gamma_1(Mp^r)$-structure obtained by contraction away from $\langle P_0 \rangle$. But $Mp^r \notin \{1, 2, 4\}$ since we assume $M > 2$ and $(M, r) \neq (4, 0)$, so $\Gamma_1(Mp^r)$-structures on polygons have trivial automorphism functor. Thus, $G = \{1\}$ as desired. □

Over the base $\mathbb{Z}_{(p)}$, the results of [C, Sections 3 and 4] concerning the properties of the stack $X_1(N, n)$ carry over if $p \nmid n$. In effect, the hypothesis on $\mathrm{ord}_p(n)$ imposed in [Conrad 2007] only intervenes in the proofs when n is not invertible on the base.

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