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On the additive dilogarithm

Sinan Ünver

Let k be a field of characteristic zero, and let $k[\varepsilon]_n := k[\varepsilon]/(\varepsilon^n)$. We construct an additive dilogarithm $\text{Li}_{2,n} : B_2(k[\varepsilon]_n) \rightarrow k^{\oplus(n-1)}$, where B_2 is the Bloch group which is crucial in studying weight two motivic cohomology. We use this construction to show that the Bloch complex of $k[\varepsilon]_n$ has cohomology groups expressed in terms of the K-groups $K_{(\cdot)}(k[\varepsilon]_n)$ as expected. Finally we compare this construction to the construction of the additive dilogarithm by Bloch and Esnault defined on the complex $T_n\mathbb{Q}(2)(k)$.

1. Introduction

1.1. For any scheme S one expects a category \mathcal{M}_S of motivic (perverse) sheaves over S , which should be an abelian tensor category that satisfies all the formalism of mixed sheaf theory [Beilinson 1987, 5.10]. The Tate sheaves $\mathbb{Z}_{\mathcal{M}}(n)$ should play a special role. Namely, letting

$$H^i(S, \mathbb{Z}_{\mathcal{M}}(n)) := \text{Ext}_{\mathcal{M}_S}^i(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n)),$$

the Chern character map

$$K_{2n-i}(S)_{\mathbb{Q}}^{(n)} \rightarrow H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \quad (1.1.1)$$

from the n -th graded piece of Quillen's K-theory tensored with \mathbb{Q} , defined as the k^n -eigenspace for the k -th Adams operator (Remark 3.1.2), to motivic cohomology of weight n should be an isomorphism when S is regular (loc. cit.). Since \mathcal{M}_S is to have realizations corresponding to various cohomology theories, the regulator map

$$K_{2n-i}(S)_{\mathbb{Q}}^{(n)} \rightarrow H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \rightarrow H_*^i(S, \mathbb{Q}_*(n)),$$

where $*$ is the relevant realization, gives arithmetically important information.

The complexes $\underline{\text{RHom}}_{\text{Zar}}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))$ of sheaves on the Zariski site should have the property that $H^i(S_{\text{Zar}}, \underline{\text{RHom}}_{\text{Zar}}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))) = H^i(S, \mathbb{Z}_{\mathcal{M}}(n))$. Hence the motivic cohomology of S of weight n could be computed as the hypercohomology of a complex of sheaves on S_{Zar} .

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Recently Voevodsky and others have made progress in motivic cohomology [Mazza et al. 2006]. If $S = \text{Spec}(k)$, where k is a field of characteristic zero, Voevodsky constructs a triangulated category $DM_{\text{Nis}}^{\text{eff},-}(k)$ [loc. cit., Chapter 14] and a complex of sheaves $\underline{\mathbb{Z}}(n)$ on the big Zariski site over k , which should be isomorphic to the hypothetical $\underline{\text{RHom}}_{\text{Zar}}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))$ above, such that for any smooth scheme X over k ,

$$H^i(X_{\text{Zar}}, \underline{\mathbb{Z}}(n)) \simeq \text{Ext}_{DM_{\text{Nis}}^{\text{eff},-}}^i(M(X), \underline{\mathbb{Z}}(n))$$

(see [loc. cit., 14.16]), where $M(X)$ is the motive of X [loc. cit., Definition 14.1]. Since $\underline{\mathbb{Z}}(n)$ and Bloch's complex of algebraic cycles of codimension n are isomorphic [loc. cit., Chapter 19], the Bloch–Grothendieck–Riemann–Roch theorem [Bloch 1986] implies that the hypercohomology of $\underline{\mathbb{Q}}(n)$ on X_{Zar} is expressed in terms of the K-groups of X as above:

$$K_{2n-i}(X)_{\mathbb{Q}}^{(n)} \simeq H^i(X_{\text{Zar}}, \underline{\mathbb{Q}}(n)). \quad (1.1.2)$$

In order to study the motivic cohomology of S , it would be sufficient to restrict to a subcategory of \mathcal{M}_S . Let \mathcal{MTM}_S denote the smallest full subcategory of \mathcal{M}_S that contains the Tate motives and is closed under extensions. Then $H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \simeq \text{Ext}_{\mathcal{M}_S}^i(\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n)) = \text{Ext}_{\mathcal{MTM}_S}^i((\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n)))$. The category \mathcal{MTM}_S would be simpler than \mathcal{M}_S . In fact for $S = \text{Spec}(k)$, where k is a number field, Deligne and Goncharov [2005] have constructed a candidate for \mathcal{MTM}_S as a tannakian category, using $DM_{\text{Nis}}^{\text{eff},-}$.

It is natural to expect that \mathcal{MTM}_S can be constructed by using only the relative cohomologies of hyperplane arrangements and in turn that motivic cohomology can be computed using complexes of *linear* algebraic objects [Beilinson et al. 1990], rather than all algebraic cycles. Special degenerate configurations of hyperplanes, called the polylogarithmic configurations [Beilinson et al. 1990; Goncharov 1995], act as building blocks for all configurations and thus play a special role in describing motivic cohomology.

Using the relations satisfied by the polylogarithmic configurations, Goncharov defines a complex $\Gamma_k(n)_{\mathbb{Q}}$ by

$$\mathcal{B}_n(k) \rightarrow \mathcal{B}_{n-1}(k) \otimes k_{\mathbb{Q}}^{\times} \rightarrow \mathcal{B}_{n-2}(k) \otimes \wedge^2 k_{\mathbb{Q}}^{\times} \rightarrow \cdots \rightarrow \mathcal{B}_2(k) \otimes \wedge^{n-2} k_{\mathbb{Q}}^{\times} \rightarrow \wedge^n k_{\mathbb{Q}}^{\times},$$

which he conjectures can be used to compute the motivic cohomology of weight n [Goncharov 1995, Conjectures A and 1.17].

If $k = \mathbb{C}$, integration over the polylogarithmic configurations can be used to define a map $\mathbb{Q}[\mathbb{P}^1(\mathbb{C})] \rightarrow \mathbb{R}$, the single-valued real analytic version of the n -th polylogarithmic function [Goncharov 1995, 1.0], which factors through the projection $\mathbb{Q}[\mathbb{P}^1(\mathbb{C})] \rightarrow \mathcal{B}_n(\mathbb{C})$ (loc. cit.) to give $\mathcal{L}_n : \mathcal{B}_n(\mathbb{C}) \rightarrow \mathbb{R}$, the n -th polylogarithm

that is expected to induce the regulator $K_{2n-1}(\mathbb{C})_{\mathbb{Q}}^{(n)} \simeq H^1(\text{Spec}(\mathbb{C}), \mathbb{Q}_{\mathcal{M}}(n)) \rightarrow \mathcal{B}_n(\mathbb{C}) \rightarrow \mathbb{R}$ [Goncharov 1995, page 224].

For a general field k , one cannot expect a polylogarithm on $\mathcal{B}_n(k)$. However, through his interpretation of hyperbolic scissor congruence groups in terms of mixed Tate motives, Goncharov expected that there should be an infinitesimal polylogarithmic function that acts like a regulator map on $K_{2n-1}(k[\varepsilon]_2, (\varepsilon))^{(n)}$, for any field k of characteristic 0 [Goncharov 1999, pages 616–617; 2004], where $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$. In our notation, assuming the existence of mixed Tate motives and the complex Γ_n over the dual numbers, this translates to the existence of a map

$$\mathcal{B}_n(k[\varepsilon]_2)/\mathcal{B}_n(k) \rightarrow k \tag{1.1.3}$$

that, when composed with $K_{2n-1}(k[\varepsilon]_2, (\varepsilon))^{(n)} \rightarrow \mathcal{B}_n(k[\varepsilon]_2)/\mathcal{B}_n(k)$, gives an isomorphism. The map (1.1.3) is to be an analogue of both the volume map for euclidean scissor congruence groups and of polylogarithms.

In this paper we are interested in this question for weight two. Next we give details about this case.

1.2. Let A be an artinian local ring and I an ideal of A . In the rest of the paper, when we refer to weight two (rational) motivic cohomology of A relative to I , what we mean are the groups $K_3(A, I)_{\mathbb{Q}}^{(2)}$ and $K_2(A, I)_{\mathbb{Q}}^{(2)}$ and not to the Voevodsky motivic cohomology groups in Section 1.1, which were there only to motivate the main results of this paper. This common abuse of notions is partly justified by the expected Chern character isomorphism (1.1.1), which is known to be true when A is a field (1.1.2).

Let k be an algebraically closed field of characteristic 0, let S the semilocal ring of rational functions on \mathbb{A}_k^1 that are regular on $\{0, 1\}$, and let J the Jacobson radical of S .

The first complex computing the weight two motivic cohomology is constructed by Bloch as follows. Localizing \mathbb{A}_k^1 away from 0 and 1 gives an exact sequence

$$0 \rightarrow K_3(k)^{(2)} \rightarrow K_2(S, J) \xrightarrow{\varphi} \bigoplus_{x \in k^\times \setminus \{1\}} k^\times \rightarrow K_2(k) \rightarrow 0$$

(see [Lichtenbaum 1987, proof of 7.1; Bloch 1977]), where φ is the tame symbol map. Let

$$B(k) := K_2(S, J)/\text{im}((1 + J) \otimes k^\times),$$

the part of $K_2(S, J)$ that does not come from the products of weight 1 terms. Then $(\bigoplus_{x \in k^\times \setminus \{1\}} k^\times)/\varphi((1 + J) \otimes k^\times) = k^\times \otimes k^\times$, and the sequence

$$0 \rightarrow K_3(k)_{\mathbb{Q}}^{(2)} \rightarrow B(k)_{\mathbb{Q}} \rightarrow (k^\times \otimes k^\times)_{\mathbb{Q}} \rightarrow K_2(k)_{\mathbb{Q}} \rightarrow 0,$$

remains exact (same references).

In complete analogy, Bloch and Esnault [2003] define a complex that computes the motivic cohomology of $k[t]_2$ relative to the ideal (t) as follows. Let R be the local ring of \mathbb{A}_k^1 at 0. Then localizing away from 0 on \mathbb{A}^1 gives the sequence

$$K_2(k[t], (t^2)) \rightarrow K_2(R, (t^2)) \xrightarrow{\varphi} \bigoplus_{x \in k^\times} k^\times \rightarrow K_1(k[t], (t^2)).$$

Let \mathcal{C} denote the subgroup generated by the symbols $\langle a, b \rangle \in K_2(R, (t^2))$ with $a \in (t^2)$ and $b \in k$, and put $TB(k) := K_2(R, (t^2))/\mathcal{C}$. Then we have $k^\times \otimes k = (\bigoplus_{x \in k^\times} k^\times)/\varphi(\mathcal{C})$ and an exact sequence

$$0 \rightarrow K_2(k[t], (t^2))^{(2)} \rightarrow TB(k) \rightarrow k^\times \otimes k \rightarrow K_1(k[t], (t^2)) \rightarrow 0$$

[Bloch and Esnault 2003, Proposition 2.1 and Corollary 2.5]. Then we have

$$K_2(k[t], (t^2))_{\mathbb{Q}}^{(2)} \simeq K_3(k[t]_2, (t))_{\mathbb{Q}}^{(2)} \quad \text{and} \quad K_1(k[t], (t^2)) \simeq K_2(k[t]_2, (t))$$

(loc. cit.). Therefore the complex $TB(k) \rightarrow k^\times \otimes k$ (tensoring with \mathbb{Q}), really computes the motivic cohomology of $k[t]_2$ relative to (t) . Moreover Bloch and Esnault define a dilogarithm map on $TB(k)$:

Theorem 1.2.1 [Bloch and Esnault 2003, Corollary 2.5]. *Let \mathfrak{m} be the maximal ideal of R . There is a well-defined map $\rho : TB(k) \rightarrow \mathfrak{m}^3/\mathfrak{m}^4$ such that*

$$\rho(\langle a, b \rangle) = -a \cdot db \quad \text{for } \langle a, b \rangle \in K_2(R, (t^2)) \text{ with } a \in \mathfrak{m}^2 \text{ and } b \in R,$$

and ρ induces an isomorphism $K_3(k[t], (t^2))^{(2)} \rightarrow \mathfrak{m}^3/\mathfrak{m}^4$ of abelian groups.

1.3. For k a field of characteristic zero there is another natural complex, which is of more geometric origin and hence easier to relate to various definitions of categories of mixed Tate motives, that computes the weight two motivic cohomology groups of k .

Suppose A is an artinian local ring with residue field k . The Bloch group $B_2(A)$ (denoted by $\mathfrak{p}(A)$ in [Suslin 1990]) is the free abelian group generated by the symbols $[x]$ such that $x(1-x) \in A^\times$, modulo the subgroup generated by elements of the form

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)],$$

for all $x, y \in A^\times$ such that $(1-x)(1-y)(1-x/y) \in A^\times$. The map that sends $[x]$ to $x \wedge (1-x) \in \bigwedge^2_{\mathbb{Z}} A^\times$ induces the two term complex $\gamma_A(2)$ that sits in [1, 2]:

$$\delta_A : B_2(A) \rightarrow \bigwedge^2_{\mathbb{Z}} A^\times. \tag{1.3.1}$$

The complex $\gamma_k(2)$ can be thought of as a more explicit version of $\Gamma_k(2)$. In fact, there is a natural map $\gamma_k(2)_{\mathbb{Q}} \rightarrow \Gamma_k(2)_{\mathbb{Q}}$, which is expected to be an isomorphism

[Goncharov 1995, Conjecture 1.20], and there is an exact sequence [Suslin 1990]

$$0 \rightarrow K_3(k)_{\mathbb{Q}}^{(2)} \rightarrow B_2(k)_{\mathbb{Q}} \rightarrow (\wedge^2 k^\times)_{\mathbb{Q}} \rightarrow K_2(k)_{\mathbb{Q}} \rightarrow 0.$$

For $n \geq 2$, we are interested in the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$, where $\delta_{k[\varepsilon]_n}$ will be denoted by δ_n . We show that it has the expected cohomology:

Theorem 1.3.1. *For k a field of characteristic 0, there is an exact sequence*

$$0 \rightarrow K_3(k[\varepsilon]_n)_{\mathbb{Q}}^{(2)} \longrightarrow B_2(k[\varepsilon]_n)_{\mathbb{Q}} \xrightarrow{\delta_n} (\wedge^2 k[\varepsilon]_n^\times)_{\mathbb{Q}} \longrightarrow K_2(k[\varepsilon]_n)_{\mathbb{Q}} \rightarrow 0.$$

For $n = 2$ this theorem gives a “yes” answer to [Goncharov 2004, Problem 2.3].

While proving the previous theorem we construct an additive dilogarithm map on $B_2(k[\varepsilon]_n)$:

Theorem 1.3.2. *For every $n \geq 2$, there is a natural map*

$$\text{Li}_{2,n} : B_2(k[\varepsilon]_n) \rightarrow k^{\oplus(n-1)}$$

that, when composed with $K_3(k[\varepsilon]_n, (\varepsilon))^{(2)} \hookrightarrow B_2(k[\varepsilon]_n)$, induces an isomorphism $K_3(k[\varepsilon]_n, (\varepsilon))^{(2)} \simeq k^{\oplus(n-1)}$ of abelian groups.

The advantage of defining a dilogarithm map on $B_2(k[\varepsilon]_n)$ is that this group is closely related to the linear algebra-geometric complexes of mixed Tate motives. More precisely, $\text{Li}_{2,n}$ immediately gives an analogue of the volume map for a pair of triangles over $k[\varepsilon]_n$, as in [Beilinson et al. 1990]: All one needs to do is to take the image of the pair of triangles in $B_2(k[\varepsilon]_n)$ under the map in [loc. cit., Proposition 3.7] and then apply $\text{Li}_{2,n}$. In this context Theorems 1.3.1 and 1.3.2 imply that the class of a pair of triangles in $A_2(k[\varepsilon]_n)/A_2(k)$ (loc. cit.) is determined by its image in $\wedge^2 k[\varepsilon]_n^\times / \wedge^2 k^\times$ and its image under $\text{Li}_{2,n}$. This is a precise analogue of Sydler’s theorem on Hilbert’s 3rd problem that the scissors congruence class of a three-dimensional polyhedron is determined by its volume and its Dehn invariant [Goncharov 1999, Section 2.7]. We do not, however, pursue this application here.

1.4. In order to compare $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$ with the complex of Bloch and Esnault, we show that their argument extends to define a complex $T_n\mathbb{Q}(2)(k)$ by

$$T_n B(k) \rightarrow k^\times \otimes (\varepsilon \cdot k[\varepsilon]_n)$$

(for $n = 2$ this is the complex in Section 1.2). Let $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}} = \gamma_k(2)_{\mathbb{Q}} \oplus \gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\circ}$, and note that the cohomology groups of $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\circ}$ and $T_n\mathbb{Q}(2)(k)$ coincide. We define a subcomplex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\prime}$ of $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\circ}$ that has the same cohomology groups, and obtain a direct consequence of Theorems 1.2.1, 1.3.1, and 1.3.2:

Corollary 1.4.1. *For k an algebraically closed field of characteristic 0, the complexes $T_n\mathbb{Q}(2)(k)$ and $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\prime}$ are isomorphic.*

1.5. The paper is organized as follows. In Section 2, we construct the additive dilogarithm, $\text{Li}_{2,n} : B_2(k[\varepsilon]_n) \rightarrow k^{\oplus(n-1)}$. Two results in Section 2 are useful in studying $\text{Li}_{2,n}$. On the one hand, $\text{Li}_{2,n}$ is explicitly described in Proposition 2.2.3 and Definition 2.2.4. On the other hand, $\text{Li}_{2,n}$ has a conceptual description: The image of an element in $B_2(k[\varepsilon]_n)$ under $\text{Li}_{2,n}$ is obtained by lifting that element to an arbitrary element in $B_2(k[\varepsilon]_{2n-1})$ then taking its image in $\bigwedge^2 k[\varepsilon]_{2n-1}^\times$ under the map in (1.3.1), and finally choosing certain algebraic combinations of its coordinates in $\bigwedge^2 k[\varepsilon]_{2n-1}^\times$ as in Propositions 2.1.2, 2.2.1 and 2.2.2. It is this flexibility in the choice of the lifting that is used in the computations in Section 4.

In this paper, rather than working with K-theory we work with cyclic homology most of the time. This is possible since $K_*(k[\varepsilon]_n) = K_*(k[\varepsilon]_n, (\varepsilon)) \oplus K_*(k)$ and by the theorem of Goodwillie [1986], $\text{HC}_{*-1}(k[\varepsilon]_n, (\varepsilon)) \simeq K_*(k[\varepsilon]_n, (\varepsilon))$, where HC denotes cyclic homology with respect to \mathbb{Q} . Note that since we are working with \mathbb{Q} -coefficients, K-theory is nothing other than the primitive part of the rational homology of GL [Loday 1992, Corollary 11.2.12].

In Sections 3.1 through 3.6 we make Goodwillie's theorem explicit, following [Loday 1992], by giving the description of a map from $\text{HC}_2(k[\varepsilon]_n, (\varepsilon))$ to $H_3(\text{GL}(k[\varepsilon]_n), \mathbb{Q})$. Then in Sections 3.7 and 3.8, Suslin and Guin's stability theorem and a construction of Bloch, Suslin and Goncharov is used to construct a map $H_3(\text{GL}(k[\varepsilon]_n), \mathbb{Q}) \rightarrow \ker(\delta_n)$. More details about Section 3 are given in Section 3.1. This explicit description will be needed in Section 4.

In Section 4, we prove Theorem 1.3.2. This is done by first using the description of $\text{HC}_2(k[\varepsilon]_n, (\varepsilon))$ given in [Cathelineau 1990/91] in Section 4.1.1, then constructing certain elements

$$\alpha_w \in \text{HC}_2(k[\varepsilon]_n, \varepsilon)^{(1)} \quad \text{for } n+1 \leq w \leq 2n-1,$$

and chasing the images of these elements under the maps described in Sections 2 and 3. The proof also shows that $\{\alpha_w\}_{n+1 \leq w \leq 2n-1}$ form a basis for $\text{HC}_2(k[\varepsilon]_n, \varepsilon)^{(1)}$.

In Section 5, using [Suslin 1990], [Guin 1989], and Section 4, we show that the infinitesimal part of $\ker(\delta)$ is canonically isomorphic to $\text{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$. From this Theorem 1.3.1 follows.

In Section 6, we first define a subcomplex $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ of $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$. Then we extend the construction of Bloch and Esnault to higher moduli and finally prove Corollary 1.4.1, which compares the two constructions.

Remarks. First, we mention the work of J. Park [2007], which gives an additive Chow theoretic description of the additive dilogarithm of Bloch and Esnault, and the work of K. Rülling [2007], which proves that the complex of additive Chow groups with modulus (not necessarily of 2) has the expected cohomology groups on the level of zero cycles.

Second, there are many problems left unanswered in this note. The most important of these is the construction of additive polylogarithms for higher weights. We have made this construction, but we have yet to prove that the complex has the right cohomology groups. We will address in another paper the question of what happens in characteristic p , and we will also compare our construction to the work of Park and Rülling.

2. Additive dilogarithm

Notation 2.0.1. Let k be a field of characteristic zero. An abelian group A endowed with a group homomorphism $k^\times \rightarrow \text{Aut}_{ab}(A)$ is said to be a k^\times -abelian group; we denote the action of $\lambda \in k^\times$ on $a \in A$ by $\lambda \times a$. If $f : A \rightarrow k$ is an additive map that satisfies $f(\lambda \times a) = \lambda^w \cdot f(a)$ for all $\lambda \in k^\times$ and $a \in A$, then we say that f is of k^\times -weight w .

If V is a k -module with a k^\times -action that is k -linear, that is, defined by a homomorphism $k^\times \rightarrow \text{Aut}_{k\text{-mod}}(V)$, then we let

$$V_{(w)} := \{v \in V \mid \lambda \times v = \lambda^w \cdot v \text{ for all } \lambda \in k^\times\}$$

be the subspace of elements of V of weight w .

Define $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$, $V_m := k[\varepsilon]_m^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and $B_2(k[\varepsilon]_m)$ as in Section 1.3.

For an object A defined over $k[\varepsilon]_m$, we denote by A° its infinitesimal part, for example,

$$B_2(k[\varepsilon]_m) = B_2(k) \oplus B_2(k[\varepsilon]_m)^\circ, \quad k[\varepsilon]_m^\circ = \varepsilon \cdot k[\varepsilon]_m, \quad V_m^\circ = 1 + \varepsilon \cdot k[\varepsilon]_m.$$

When the context requires it we write (say) $K_*(k[\varepsilon]_m)^\circ$ instead of $K_*(k[\varepsilon]_m, (\varepsilon))$. Finally, since in what follows the infinitesimal part A° of an object A is canonically a direct summand of A , we never mention the natural maps $A^\circ \rightarrow A$ and $A \rightarrow A^\circ$, and take other liberties of this kind.

The exponential map gives an isomorphism $k[\varepsilon]_m^\circ \simeq V_m^\circ$, which endows V_m° with a k -space structure. For $\lambda \in k^\times$, the k -algebra map that sends ε to $\lambda \cdot \varepsilon$ defines an action of k^\times on $k[\varepsilon]_m$ and V_m° . Denote the weight i subspace of V_m° under this action by $V_{m,(i)}$, that is,

$$V_{m,(i)} = \{v \in V_m^\circ \mid \lambda \times v = \lambda^i \cdot v \text{ for all } \lambda \in k^\times\} = \{\exp(a \cdot \varepsilon^i) \mid a \in k\}.$$

Then $V_m^\circ = \bigoplus_{1 \leq i \leq m-1} V_{m,(i)}$. To simplify the notation we also put $V_{m,(0)} := k^\times \otimes \mathbb{Q}$.

Let $k[\varepsilon]_m^{\times \times} \subseteq k[\varepsilon]_m^\times$ denote the set of exceptional units, that is, those $a \in k[\varepsilon]_m^\times$ such that $1 - a \in k[\varepsilon]^\times$.

Let $\delta : \mathbb{Q}[k[\varepsilon]_m^{\times \times}] \rightarrow \bigwedge^2 V_m$ be the map that sends $x \in k[\varepsilon]_m^{\times \times}$ to $x \wedge (1 - x)$. If we want to emphasize that we are working over $k[\varepsilon]_m$, we will use the notation δ_m

instead of δ . The map on $B_2(k[\varepsilon]_m)$ induced by δ_m is denoted by the same letter (see (1.3.1)).

2.1. Construction of li_2 . In this section we collect the combinatorial arguments in the construction of the additive dilogarithm over $k[\varepsilon]_n$. The crucial step is the statement that $S_k(m, n)_{(w)}$ is one dimensional in Proposition 2.1.2. This implies that if one thinks that the additive dilogarithm on $k[\varepsilon]_n$ should be constructed by first lifting to $k[\varepsilon]_{2n-1}$ and then using δ , then there is essentially one way to define it. That this is the right definition is justified in the next section.

Definition 2.1.1. Let $n, m \in \mathbb{N}$ such that $2 \leq n \leq m$. Let $\alpha_{m,n} : \mathbb{Q}[k[\varepsilon]_m^{\times \times}] \rightarrow \bigwedge^2 V_m$ denote the map that sends $\gamma \in k[\varepsilon]_m^{\times \times}$ to $\delta(\gamma) - \delta(\gamma | n)$, where $\gamma | n$ is the truncation of γ to the sum of first n powers of ε , that is, if $\gamma = a_0 + a_1 \cdot \varepsilon + \cdots + a_{m-1} \cdot \varepsilon^{m-1}$ then $\gamma | n = a_0 + a_1 \cdot \varepsilon + \cdots + a_{n-1} \cdot \varepsilon^{n-1}$.

Let $V(m, n)$ denote

$$\bigoplus_{\substack{0 < i \leq n-1 \\ n \leq j \leq m-1}} (V_{m, \langle i \rangle} \otimes V_{m, \langle j \rangle}) \subseteq \bigwedge^2 V_m,$$

which we also consider as a quotient of $\bigwedge^2 V_m$ via the direct sum decomposition

$$\bigwedge^2 V_m = \bigoplus_{0 \leq i < j < m} (V_{m, \langle i \rangle} \otimes V_{m, \langle j \rangle}) \oplus \left(\bigoplus_{0 \leq i < m} \bigwedge^2 V_{m, \langle i \rangle} \right). \quad (2.1.1)$$

Finally denote by $V_k(m, n)$ the quotient

$$\bigoplus_{\substack{0 < i \leq n-1 \\ n \leq j \leq m-1}} (V_{m, \langle i \rangle} \otimes_k V_{m, \langle j \rangle})$$

of $V(m, n)$, by $p(m, n) : \bigwedge^2 V_m \rightarrow V_k(m, n)$ the canonical projection, by $S_k(m, n)$ the k^\times -abelian group $V_k(m, n) / \text{im}(p(m, n) \circ \alpha_{m,n})$ and by $S_k(m, n)_{\langle i \rangle}$ the image of $V_k(m, n)_{\langle i \rangle}$ in $S_k(m, n)$. This notation is justified by noting that $S_k(m, n)$ has a natural k -module structure induced from that of $V_k(m, n)$ such that its weight i subspace is equal to $S_k(m, n)_{\langle i \rangle}$ and $S_k(m, n) = \bigoplus_{0 < i} S_k(m, n)_{\langle i \rangle}$.

For $0 < i < j < m$, let $p_{i,j} : \bigwedge^2 V_m \rightarrow V_{m, \langle i \rangle} \otimes V_{m, \langle j \rangle}$ denote the projection determined by the decomposition (2.1.1). Then $l_{i,j} : \bigwedge^2 V_m \rightarrow k$ is defined by letting $(\log \otimes \log)(p_{i,j}(\alpha)) = l_{i,j}(\alpha) \cdot (\varepsilon^i \otimes \varepsilon^j)$ in $k[\varepsilon]_m \otimes_k k[\varepsilon]_m$ for any $\alpha \in \bigwedge^2 V_m$.

Proposition 2.1.2. Let $n, m, w \in \mathbb{N}$ such that $2 \leq n < w \leq \min(2n-1, m)$. Then $S_k(m, n)_{(w)}$ is a one-dimensional k -module. The unique linear functional

$$\text{li}_{2, (m, n), w} : S_k(m, n)_{(w)} \rightarrow k$$

such that $\text{li}_{2,(m,n),w}(\exp(\varepsilon) \otimes \exp(\varepsilon^{w-1})) = 1$ is given by

$$\text{li}_{2,(m,n),w} = \sum_{1 \leq j \leq w-n} j \cdot l_{j,w-j}.$$

Proof. Let $\text{li}_{2,(m,n),w}$ denote the map from $\bigwedge^2 V_m$ to k given by the formula

$$\text{li}_{2,(m,n),w} = \sum_{1 \leq j \leq w-n} j \cdot l_{j,w-j}.$$

We would like to see that $\text{li}_{2,(m,n),w} \circ \alpha_{m,n} = 0$. Fix

$$x := s + s(1-s)a_1\varepsilon + \cdots + s(1-s)a_{m-1}\varepsilon^{m-1} \in k[\varepsilon]_m^{\times \times}.$$

Let $A_m := \{1, \dots, m-1\}$ and let $(A_m)^{\times \alpha}$ denote the cartesian product of A_m with itself α -times. Define $\mathfrak{p} : (A_m)^{\times \alpha} \rightarrow k$ by $\mathfrak{p}(i_1, \dots, i_\alpha) := a_{i_1} \cdot a_{i_2} \cdots a_{i_\alpha}$, and $\mathfrak{w} : (A_m)^{\times \alpha} \rightarrow \mathbb{N}$ by $\mathfrak{w}(i_1, \dots, i_\alpha) := i_1 + i_2 + \cdots + i_\alpha$. Note that even though \mathfrak{p} depends on x , we suppress it from the notation. In order to simplify the notation let $A(\alpha) := (A_m)^{\times \alpha}$ and $B(\alpha) := (A_m)^{\times \alpha} \setminus (A_n)^{\times \alpha}$.

If $1 \leq \alpha, \beta \leq w$, let

$$C(\alpha, \beta) := \{(a, b) \mid a \in A(\alpha), b \in B(\beta), \mathfrak{w}(a) + \mathfrak{w}(b) = w\}.$$

Let the permutation group $S_{\alpha+\beta}$ on $\alpha+\beta$ letters act on $A(\alpha) \times A(\beta)$ by permuting the coordinates. On $C(\alpha, \beta) \subseteq A(\alpha) \times A(\beta)$ consider the following equivalence relation. If $(a, b), (c, d) \in C(\alpha, \beta)$, then (a, b) and (c, d) are equivalent if there exists a permutation $\sigma \in S_{\alpha+\beta}$ such that $(a, b)^\sigma = (c, d)$. Denote the equivalence class of (a, b) by $[(a, b)]$ and the set of all equivalence classes by $\mathcal{S}(\alpha, \beta)$. Let $\mathfrak{p}([(a, b)]) = \mathfrak{p}(a) \cdot \mathfrak{p}(b)$.

Assume from now on that $\alpha + \beta \leq w$. Note that since $w \leq 2n - 1$, any element $(a, b) \in C(\alpha, \beta)$ has a unique coordinate that is greater than or equal to n . Denote this coordinate by $e(a, b)$. Denote by $(a, b)_0$ the element of $C(\alpha, \beta)$ obtained by interchanging the last coordinate of (a, b) with the coordinate containing $e(a, b)$.

Then we define a map $\iota : C(\alpha, \beta) \rightarrow C(\beta, \alpha)$ as follows. Let $(a, b) \in C(\alpha, \beta)$. Then $\iota(a, b) \in C(\beta, \alpha)$ is the element $(a, b)_0$, where we think of both $C(\alpha, \beta)$ and $C(\beta, \alpha)$ as subsets of $A(\alpha) \times A(\beta) \simeq A(\alpha + \beta) \simeq A(\beta) \times A(\alpha)$. This passes to equivalence classes and gives a map $\mathcal{S}(\alpha, \beta) \rightarrow \mathcal{S}(\beta, \alpha)$ that we continue to denote by ι . Note that $\iota^2 = 1$, and if $G \in \mathcal{S}(\alpha, \beta)$, then $\mathfrak{p}(\iota(G)) = \mathfrak{p}(G)$, and

$$\sum_{(a,b) \in G} \mathfrak{w}(a) = \sum_{(c,d) \in \iota(G)} \mathfrak{w}(c).$$

Letting $z = a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_{m-1}\varepsilon^{m-1}$, we have

$$x = s(1 + (1-s)z) \quad \text{and} \quad 1-x = (1-s)(1-sz).$$

Computing in $k[\varepsilon]_m$, this gives

$$\log(x/s) = - \sum_{\ell=1}^{m-1} \frac{(s-1)^\ell z^\ell}{\ell} \quad \text{and} \quad \log((1-x)/(1-s)) = - \sum_{\ell=1}^{m-1} \frac{s^\ell z^\ell}{\ell}.$$

Since $z^\alpha = \sum_{u \in A(\alpha)} \mathfrak{p}(u) \varepsilon^{\mathfrak{w}(u)}$, we have

$$\begin{aligned} \log(x/s) &= - \sum_{\ell=1}^{m-1} \frac{(s-1)^\ell}{\ell} \sum_{u \in A(\ell)} \mathfrak{p}(u) \varepsilon^{\mathfrak{w}(u)}, \\ \log((1-x)/(1-s)) &= - \sum_{\ell=1}^{m-1} \frac{s^\ell}{\ell} \sum_{u \in A(\ell)} \mathfrak{p}(u) \varepsilon^{\mathfrak{w}(u)}. \end{aligned}$$

Then $\text{li}_{2,(m,n),w}(\alpha_{m,n}(x))$ is equal to

$$\begin{aligned} &\sum_{\substack{1 \leq \alpha \leq w \\ 1 \leq \beta \leq w}} \sum_{\substack{a \in A(\alpha) \\ b \in B(\beta) \\ \mathfrak{w}(a) + \mathfrak{w}(b) = w}} \frac{\mathfrak{w}(a) \cdot \mathfrak{p}(a) \cdot \mathfrak{p}(b)}{\alpha \cdot \beta} \cdot ((s-1)^\alpha \cdot s^\beta - s^\alpha \cdot (s-1)^\beta) \\ &= \sum_{\substack{1 \leq \alpha \leq w \\ 1 \leq \beta \leq w}} ((s-1)^\alpha \cdot s^\beta - s^\alpha \cdot (s-1)^\beta) \sum_{G \in \mathcal{F}(\alpha, \beta)} \left(\frac{\mathfrak{p}(G)}{\alpha \cdot \beta} \right) \sum_{(a,b) \in G} \mathfrak{w}(a). \end{aligned}$$

On the other hand

$$\begin{aligned} &\sum_{G \in \mathcal{F}(\alpha, \beta)} \left(\frac{\mathfrak{p}(G)}{\alpha \cdot \beta} \right) \sum_{(a,b) \in G} \mathfrak{w}(a) = \sum_{G \in \mathcal{F}(\alpha, \beta)} \left(\frac{\mathfrak{p}(t(G))}{\alpha \cdot \beta} \right) \sum_{(c,d) \in t(G)} \mathfrak{w}(c) \\ &= \sum_{G \in \mathcal{F}(\beta, \alpha)} \left(\frac{\mathfrak{p}(G)}{\beta \cdot \alpha} \right) \sum_{(a,b) \in G} \mathfrak{w}(a). \end{aligned}$$

Therefore $\text{li}_{2,(m,n),w}(\alpha_{m,n}(x)) = 0$, and we have a linear functional

$$\text{li}_{2,(m,n),w} : S_k(m, n)_{(w)} \rightarrow k.$$

By the definition of $\text{li}_{2,(m,n),w}$ it is clear that $\text{li}_{2,(m,n),w}(\exp(\varepsilon) \otimes \exp(\varepsilon^{w-1})) = 1$.

To finish the proof we only need to show that the space of linear functionals on $S_k(m, n)_{(w)}$ is generated by $\text{li}_{2,(m,n),w}$, or equivalently that if l is a linear combination of $\{l_{2,w-2}, l_{3,w-3}, \dots, l_{w-n,n}\}$ such that $l(\alpha_{m,n}(x)) = 0$ for all $x \in k[\varepsilon]_m^{\times \times}$, then l is zero. So let $l = \sum_{2 \leq i \leq w-n} c_i \cdot l_{i,w-i}$ satisfy $l(\alpha_{m,n}(x)) = 0$ for all $x \in k[\varepsilon]_m^{\times \times}$. Assume that $l \neq 0$ and let i_0 be the smallest integer i such that $c_i \neq 0$. For all $s \in k^{\times \times}$ and $a_1, a_{i_0-1}, a_{w-i_0} \in k$, we have

$$l(\alpha_{m,n}(s + s(1-s) \cdot a_1 \cdot \varepsilon + s(1-s) \cdot a_{i_0-1} \cdot \varepsilon^{i_0-1} + s(1-s) \cdot a_{w-i_0} \cdot \varepsilon^{w-i_0})) = 0.$$

If we denote the left hand side of the above equation by $l(s, a_1, a_{i_0-1}, a_{w-i_0})$, then

$$\begin{aligned} c_{i_0} \cdot \frac{1}{2}((s-1)^2 s - s^2(s-1)) \cdot (a_1 \cdot a_{i_0-1} \cdot a_{w-i_0}) \\ = l(s, a_1, a_{i_0-1}, a_{w-i_0}) - l(s, a_1, 0, a_{w-i_0}) = 0. \end{aligned}$$

Therefore $c_{i_0} = 0$, contradicting the assumption. \square

2.2. Construction of Li. Using the construction in the previous section, we show that $\text{li}_{2,(2n-1,n),w}$ descends to a function on $B_2(k[\varepsilon]_n)$, as defined in Section 1.3.

Proposition 2.2.1. *For $n+1 \leq w \leq 2n-1$, the map*

$$\text{li}_{2,(2n-1,n),w} \circ \delta : \mathbb{Q}[k[\varepsilon]_{2n-1}^{\times \times}] \rightarrow k$$

factors through the canonical projection $\mathbb{Q}[k[\varepsilon]_{2n-1}^{\times \times}] \rightarrow \mathbb{Q}[k[\varepsilon]_n^{\times \times}]$.

We denote the induced map from $\mathbb{Q}[k[\varepsilon]_n^{\times \times}]$ to k by $\text{Li}_{2,n,w}$.

Proof. This follows from the fact that $\text{li}_{2,(2n-1,n),w} \circ \alpha_{2n-1,n} = 0$ by the construction in Proposition 2.1.2. \square

Proposition 2.2.2. *The map $\text{Li}_{2,n,w} : \mathbb{Q}[k[\varepsilon]_n^{\times \times}] \rightarrow k$ factors through the canonical projection $\mathbb{Q}[k[\varepsilon]_n^{\times \times}] \rightarrow B_2(k[\varepsilon]_n)$.*

We continue to denote the induced map by $\text{Li}_{2,n,w}$.

Proof. We need to show that for $x, y \in k[\varepsilon]_n^{\times \times}$ such that $x/y \in k[\varepsilon]_n^{\times \times}$,

$$\text{Li}_{2,n,w}([x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)]) = 0.$$

If \tilde{x} and \tilde{y} are arbitrary liftings of x and y to $k[\varepsilon]_{2n-1}^{\times \times}$, then Proposition 2.2.1 implies that the left side of the last equation is equal to

$$(\text{li}_{2,(2n-1),w} \circ \delta)([\tilde{x}] - [\tilde{y}] + [\tilde{y}/\tilde{x}] - [(1-\tilde{x}^{-1})/(1-\tilde{y}^{-1})] + [(1-\tilde{x})/(1-\tilde{y})]).$$

The proposition then follows from the fact that

$$\delta([\tilde{x}] - [\tilde{y}] + [\tilde{y}/\tilde{x}] - [(1-\tilde{x}^{-1})/(1-\tilde{y}^{-1})] + [(1-\tilde{x})/(1-\tilde{y})]) = 0. \quad \square$$

If $\underline{c} = (c_1, \dots, c_r) \in \mathbb{N}^r$ and $x = s + s(1-s)a_1\varepsilon + \dots + s(1-s)a_{n-1}\varepsilon^{n-1} \in k[\varepsilon]_n^{\times \times}$ then

$$\mathfrak{p}(x; \underline{c}) := a_{c_1} \cdot a_{c_2} \cdots a_{c_r} \quad \text{and} \quad \mathfrak{w}(\underline{c}) := c_1 + \dots + c_r.$$

Let $C(\alpha) := \{1, 2, \dots, n-1\}^{\times \alpha}$.

Proposition 2.2.3. *For $n+1 \leq w \leq 2n-1$, we have*

$$\text{Li}_{2,n,w}([x]) = \sum_{1 \leq \alpha, \beta \leq w} \frac{(s-1)^\alpha \cdot s^\beta - s^\alpha \cdot (s-1)^\beta}{\alpha \cdot \beta} \sum_{\substack{(a,b) \in C(\alpha) \times C(\beta) \\ \mathfrak{w}(a) \leq w-n \\ \mathfrak{w}(a,b) = w}} \mathfrak{w}(a) \cdot \mathfrak{p}(x; (a, b)).$$

Proof. Direct computation. □

Definition 2.2.4. Define the additive dilogarithm $\text{Li}_{2,n} : B_2(k[\varepsilon]_n) \rightarrow k^{\oplus(n-1)}$ by

$$\text{Li}_{2,n} := \bigoplus_{n+1 \leq w \leq 2n-1} \text{Li}_{2,n,w} .$$

3. The map from cyclic homology to the Bloch group

3.1. This section is based on Goodwillie's theorem [1986] and the construction of Bloch [1977], Suslin [1990] and Goncharov [1995] of a map from the K_3 of a field to its Bloch group. Our main reference for cyclic homology and Goodwillie's theorem is [Loday 1992]. Here all cyclic homology groups are relative to \mathbb{Q} .

We will need the following to pass from cyclic homology to the rational homology of GL.

Theorem 3.1.1 [Goodwillie 1986; Loday 1992, Theorem 11.3.1]. *Let A be a \mathbb{Q} -algebra and I a nilpotent ideal in A . Then there is a canonical isomorphism*

$$\text{HC}_{n-1}(A, I) \simeq K_n(A, I)_{\mathbb{Q}} \quad \text{for } n \geq 1.$$

Remark 3.1.2. This isomorphism is compatible with the λ -structures on both sides by [Cathelineau 1990/91, Theorem 1]. Hence, if $\text{HC}_*(A, I)^{(i-1)}$ and $K_*(A, I)_{\mathbb{Q}}^{(i)}$ denote the k^i -eigenspace for the k -th Adams operator (for any k), then the above isomorphism induces an isomorphism $\text{HC}_{*-1}(A, I)^{(i-1)} \simeq K_*(A, I)_{\mathbb{Q}}^{(i)}$ by [loc. cit., corollary in Section 1.3].

For a ring A , the Hurewicz map induces an isomorphism from $\bigoplus_{n>0} K_n(A)_{\mathbb{Q}}$ to the primitive part $\text{Prim } H_*(\text{GL}(A), \mathbb{Q})$ of the homology of GL [Loday 1992, 11.2.12 Corollary]. The map in Theorem 3.1.1 is constructed as the composition of a map from cyclic homology to the primitive part of the homology of GL and then using the inverse of the Hurewicz map. Since we will only need the map from cyclic homology to the homology of GL, we next describe the steps in its construction, following [Loday 1992].

In Section 3.2, cyclic homology of A is computed as the homology of the Connes complex. This section also describes the natural map from the Connes complex to the Chevalley–Eilenberg complex of the Lie algebra \mathfrak{gl} . This map induces an isomorphism from cyclic homology to the primitive homology of \mathfrak{gl} . In Section 3.3, homology of \mathfrak{gl} is replaced with the sum of the homology of its nilpotent parts $\mathfrak{t}_{\sigma}(A, I)$. In Section 3.4, homology of $\mathfrak{t}_{\sigma}(A, I)$ is replaced with that of the completion of its universal enveloping algebra, and in Section 3.5, the latter is replaced with the homology of the group algebra of $T_{\sigma}(A, I)$, via Malčev theory. We reach the group homology of $\text{GL}(A)$ in Section 3.6.

In Section 3.7, this construction in combination with Suslin and Guin's stability theorem [Suslin 1984; Guin 1989] induces a map

$$\mathrm{HC}_{n-1}(A, I) \rightarrow H_n(\mathrm{GL}_n(A), \mathbb{Q}) \quad (3.1.1)$$

when A is an artinian local algebra over \mathbb{Q} and I is a proper ideal of A . We will use this map for $n = 3$.

Finally we use a slight variation of the construction of Suslin and Goncharov in Section 3.8 to get a map $H_3(\mathrm{GL}_3(A), \mathbb{Q}) \rightarrow \ker(\delta)$.

The details can be found in [Loday 1992, Section 11.3] and the references therein. The main result of this section is Proposition 3.8.9.

3.2. Map from cyclic homology to Lie algebra homology.

3.2.1. For any associative \mathbb{Q} -algebra A , the Connes complex $C_*^\lambda(A)$ is defined as follows. Let $\mathbb{Z}/n\mathbb{Z}$ act on $A^{\otimes n}$ by

$$1 \times (a_1 \otimes a_2 \otimes \cdots \otimes a_n) = (-1)^{n-1} a_2 \otimes a_3 \otimes \cdots \otimes a_n \otimes a_1,$$

and let $C_{n-1}^\lambda(A)$ denote the coinvariants of $A^{\otimes n}$ under this action. Define

$$\begin{aligned} b : C_n^\lambda(A) &\rightarrow C_{n-1}^\lambda(A), \\ (a_0, a_1, \dots, a_n) &\mapsto \sum_{0 \leq i \leq n-1} (-1)^i (a_0, \dots, a_i \cdot a_{i+1}, \dots, a_n) \\ &\quad + (-1)^n (a_n \cdot a_0, a_1, \dots, a_{n-1}). \end{aligned}$$

Then $C_*^\lambda(A)$ is the complex

$$\cdots \xrightarrow{b} C_{n+1}^\lambda(A) \xrightarrow{b} C_n^\lambda(A) \xrightarrow{b} \cdots \longrightarrow C_0^\lambda(A) \longrightarrow 0,$$

and $\mathrm{HC}_*(A) = H_*(C_*^\lambda(A))$ [Loday 1992, Theorem 2.1.5]: The cyclic homology relative to \mathbb{Q} can be computed as the homology of the Connes complex.

3.2.2. For \mathfrak{g} a Lie algebra over \mathbb{Q} , the Chevalley–Eilenberg complex $C_*(\mathfrak{g}, \mathbb{Q})$ of \mathfrak{g} with coefficients in \mathbb{Q} is defined by

$$\cdots \xrightarrow{d} \wedge^n \mathfrak{g} \xrightarrow{d} \wedge^{n-1} \mathfrak{g} \longrightarrow \cdots \xrightarrow{d} \mathfrak{g} \xrightarrow{d} \mathbb{Q} \longrightarrow 0,$$

where $d : \wedge^n \mathfrak{g} \rightarrow \wedge^{n-1} \mathfrak{g}$ is given by

$$d(a_1 \wedge a_2 \wedge \cdots \wedge a_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_n.$$

The Lie algebra homology $H_*(\mathfrak{g}, \mathbb{Q})$ of \mathfrak{g} with coefficients in \mathbb{Q} is the homology of the complex $C_*(\mathfrak{g}, \mathbb{Q})$. The diagonal map $\mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ induces a map

$$\Delta : C_*(\mathfrak{g}, \mathbb{Q}) \rightarrow C_*(\mathfrak{g}, \mathbb{Q}) \otimes C_*(\mathfrak{g}, \mathbb{Q}),$$

which makes $(C_*(\mathfrak{g}, \mathbb{Q}), d)$ a DG-coalgebra. Let $\text{Prim } H_*(\mathfrak{g}, \mathbb{Q})$ denote the primitive elements in $H_*(\mathfrak{g}, \mathbb{Q})$, that is, those α such that $\Delta(\alpha) = 1 \otimes \alpha + \alpha \otimes 1$.

Let $\mathfrak{gl}_n(A)$ denote the Lie algebra of $n \times n$ matrices, and let $\mathfrak{gl}(A)$ denote the direct limit $\lim_{n \rightarrow \infty} \mathfrak{gl}_n(A)$ with respect to the natural inclusions $\mathfrak{gl}_n(A) \subseteq \mathfrak{gl}_m(A)$ for $n \leq m$. Then $\mathfrak{gl}(\mathbb{Q})$ acts on $C_*(\mathfrak{gl}(A), \mathbb{Q})$ by

$$[h, g_1 \wedge \cdots \wedge g_n] := \sum_{1 \leq i \leq n} g_1 \wedge \cdots \wedge [h, g_i] \wedge \cdots \wedge g_n.$$

Let $C_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ denote the complex of coinvariants with respect to this action, and let $H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ and $\text{Prim } H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ denote respectively the homology and the primitive part of the homology of the complex $C_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$. Then the theorem of Loday, Quillen, and Tsygan says this:

Theorem 3.2.1 [Loday 1992, Theorem 10.2.4]. *If A is an algebra over \mathbb{Q} , then there is a natural isomorphism*

$$\text{HC}_{*-1}(A) \simeq \text{Prim } H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})} \simeq \text{Prim } H_*(\mathfrak{gl}(A), \mathbb{Q}).$$

Explicitly, the first isomorphism above is induced by the chain map that sends the class of $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $C_{n-1}^\lambda(A)$ to the class of $a_1 e_{12} \wedge a_2 e_{23} \wedge \cdots \wedge a_n e_{n1}$ in $C_n(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$. Here e_{ij} denotes the matrix whose only nonzero entry is the one in the i -th row and the j -th column, which is 1.

3.3. Volodin's construction in the Lie algebra case. Assume that I is a nilpotent ideal of A , and let $\text{HC}_*(A, I)$ denote the cyclic homology of A relative to I , the homology of the complex $C_*^\lambda(A, I)$ that is the kernel of the natural surjection $C_*^\lambda(A) \rightarrow C_*^\lambda(A/I)$.

For any permutation $\sigma \in S_n$, let $\mathfrak{t}_\sigma(A, I)$ denote the Lie subalgebra of $\mathfrak{gl}(A)$ given by $\mathfrak{t}_\sigma(A, I) := \{(a_{ij}) \in \mathfrak{gl}(A) : a_{ij} \in I \text{ if } \sigma(j) \leq \sigma(i)\}$. Let $x(A, I) := \sum_\sigma C_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q})$ denote the sum of the subcomplexes

$$C_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) \subseteq C_*(\mathfrak{gl}(A), \mathbb{Q}),$$

over all n and $\sigma \in S_n$, and let $H_*(\mathfrak{gl}(A, I), \mathbb{Q})$ denote the homology of $x(A, I)$. Then the map in Theorem 3.2.1 induces an isomorphism

$$\text{HC}_{*-1}(A, I) \simeq \text{Prim } H_*(\mathfrak{gl}(A, I), \mathbb{Q}) \simeq \sum_{\sigma} \text{Prim } H_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) \quad (3.3.1)$$

[Loday 1992, Proposition 11.3.12].

3.4. From the Lie algebra to the universal enveloping algebra. For a Lie algebra \mathfrak{g} over \mathbb{Q} , let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra and $\hat{\mathcal{U}}(\mathfrak{g})$ its completion with respect to its augmentation ideal. We will next express the homology of \mathfrak{g} in terms of the homology of $\mathcal{U}(\mathfrak{g})$.

Let B be an associative algebra over \mathbb{Q} endowed with an augmentation map $\varepsilon : B \rightarrow \mathbb{Q}$. Let $C_*(B, \mathbb{Q})$ denote the complex

$$\dots \xrightarrow{b} B^{\otimes n} \xrightarrow{b} B^{\otimes(n-1)} \xrightarrow{b} \dots \xrightarrow{b} \mathbb{Q} \longrightarrow 0,$$

where $b : B^{\otimes n} \rightarrow B^{\otimes(n-1)}$ is the map that sends $b_1 \otimes \dots \otimes b_n$ to

$$\varepsilon(b_1) \cdot b_2 \otimes \dots \otimes b_n + \sum_{1 \leq i \leq n-1} (-1)^i b_1 \otimes \dots \otimes b_i \cdot b_{i+1} \otimes \dots \otimes b_n + (-1)^n \varepsilon(b_n) \cdot b_1 \otimes \dots \otimes b_{n-1}.$$

Let $H_*(B, \mathbb{Q})$ denote the homology of this complex.

Then the natural maps

$$H_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) \simeq H_*(\mathcal{U}(\mathfrak{t}_\sigma(A, I)), \mathbb{Q}) \simeq H_*(\hat{\mathcal{U}}(\mathfrak{t}_\sigma(A, I)), \mathbb{Q}) \quad (3.4.1)$$

are isomorphisms [Loday 1992, Theorem 3.3.2]. Here the first map is induced by the chain map α_{as} , where “as” stands for antisymmetrization, defined by

$$\alpha_{\text{as}}(t_1 \wedge \dots \wedge t_n) = \sum_{\tau \in S_n} \text{sign}(\tau) \cdot t_{\tau(1)} \otimes \dots \otimes t_{\tau(n)}.$$

3.5. Malčev theory. For $\sigma \in S_n$, let $T_\sigma(A, I) \subseteq \text{GL}_n(A)$ denote the group

$$\{1 + (a_{ij}) \in \text{GL}_n(A) \mid a_{ij} \in I \text{ if } \sigma(j) \leq \sigma(i)\}.$$

For a discrete group G , denote by $U(G)$ its group ring over \mathbb{Q} , and by $\hat{U}(G)$ its completion with respect to the augmentation ideal.

Since $T_\sigma(A, I)$ is a unipotent group with Lie algebra $\mathfrak{t}_\sigma(A, I)$, the natural maps

$$H_*(\hat{\mathcal{U}}(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) = H_*(\hat{U}(T_\sigma(A, I)), \mathbb{Q}) \simeq H_*(U(T_\sigma(A, I)), \mathbb{Q}). \quad (3.5.1)$$

are isomorphisms [Loday 1992, Section 11.3.13].

Combining (3.3.1), (3.4.1) and (3.5.1) we get a map

$$\text{HC}_{*-1}(A, I) \rightarrow \sum_{\sigma} H_*(U(T_\sigma(A, I)), \mathbb{Q}) \rightarrow H_*(U(\text{GL}(A)), \mathbb{Q}). \quad (3.5.2)$$

3.6. Group homology. Let G be any (discrete) group and $C_*(G, \mathbb{Q})$ the complex

$$\dots \xrightarrow{d} \mathbb{Q}[G^{n+1}] \xrightarrow{d} \mathbb{Q}[G^n] \xrightarrow{d} \dots \xrightarrow{d} \mathbb{Q}[G] \longrightarrow 0,$$

where $C_n(G, \mathbb{Q}) = \mathbb{Q}[G^{n+1}]$ and d is the map that sends (g_0, g_1, \dots, g_n) to

$$\sum_{0 \leq i \leq n} (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n).$$

Let G act on this complex by multiplication on the left, that is, $g \times (g_0, \dots, g_n) := (g \cdot g_0, \dots, g \cdot g_n)$, and let $H_*(G, \mathbb{Q}) := H_*(C_*(G, \mathbb{Q})_G)$ where the subscript G denotes the space of coinvariants.

The natural map $C_*(U(G), \mathbb{Q}) \rightarrow C_*(G, \mathbb{Q})$ that sends $g_1 \otimes g_2 \otimes \cdots \otimes g_n$ to

$$(1, g_1, g_1 \cdot g_2, \dots, g_1 \cdot g_2 \cdots g_n)$$

induces an isomorphism $H_*(U(G), \mathbb{Q}) \rightarrow H_*(G, \mathbb{Q})$ [Loday 1992, Appendix C.3]. Applying this to $\mathrm{GL}(A)$ and combining with (3.5.2) we obtain the map

$$\mathrm{HC}_{*-1}(A, I) \rightarrow H_*(\mathrm{GL}(A), \mathbb{Q}). \quad (3.6.1)$$

3.7. Suslin's stability theorem. Suslin's stability theorem [1990] was generalized by Guin:

Theorem 3.7.1 [Guin 1989, Section 2]. *For any $1 \leq n$ and any artinian local algebra A over \mathbb{Q} , the map $H_n(\mathrm{GL}_n(A), \mathbb{Q}) \rightarrow H_n(\mathrm{GL}(A), \mathbb{Q})$ induced by the inclusion $\mathrm{GL}_n \hookrightarrow \mathrm{GL}$ is an isomorphism.*

Therefore if A is an artinian local algebra over \mathbb{Q} and I is a proper ideal, we have a map $\rho_1 : \mathrm{HC}_{n-1}(A, I) \rightarrow H_n(\mathrm{GL}_n(A), \mathbb{Q})$.

3.8. Bloch–Suslin map. Let A be an artinian local algebra over \mathbb{Q} with residue field k . We now describe the Bloch–Suslin map [Goncharov 1995, Section 2.6]

$$\rho_2 : H_3(\mathrm{GL}_3(A), \mathbb{Q}) \rightarrow \ker(\delta_A),$$

where $\delta_A : B_2(A)_{\mathbb{Q}} \rightarrow \bigwedge^2 A_{\mathbb{Q}}^{\times}$ is the differential in the Bloch complex.

Definition 3.8.1. Let V be a finite free module over A , and denote $\tilde{C}_m(V)$ by the \mathbb{Q} -vector space with basis consisting of m -tuples (x_0, \dots, x_{m-1}) of elements of V that are in general position, that is, for any $I \subseteq \{0, 1, \dots, m-1\}$ with $|I| \leq \mathrm{rank}(V)$, the set $\{x_i \mid i \in I\}$ can be extended to a basis of V . Let $C_m(V)$ denote the coinvariants of this space under the natural action of $\mathrm{GL}(V)$. Finally, let $\tilde{C}_m(p) := \tilde{C}_m(A^{\oplus p})$ and $C_m(p) := C_m(A^{\oplus p})$.

Remark 3.8.2. Let $\tilde{C}_m(\mathbb{P}(V))$ denote the \mathbb{Q} -space with basis (v_0, \dots, v_{m-1}) of m -tuples of points in $\mathbb{P}(V)$ that are in general position, and define

$$d : \tilde{C}_{m+1}(\mathbb{P}(V)) \rightarrow \tilde{C}_m(\mathbb{P}(V)), \quad (v_0, \dots, v_m) \mapsto \sum_{0 \leq i \leq m} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_m).$$

Let $C_m(\mathbb{P}(V))$ denote the space of coinvariants of $\tilde{C}_m(\mathbb{P}(V))$ under the natural action of $\mathrm{GL}(V)$. Then by identifying $[x]$ with $(0, x, 1, \infty) \in C_4(\mathbb{P}(A^{\oplus 2}))$ and by comparing the dilogarithm relation in the definition of $B_2(A)$ to $d(0, x, y, 1, \infty)$ in $C_4(\mathbb{P}(A^{\oplus 2}))$, we see that

$$B_2(A)_{\mathbb{Q}} = C_4(\mathbb{P}(A^{\oplus 2}))/d(C_5(\mathbb{P}(A^{\oplus 2}))).$$

For (x_1, \dots, x_4) a quadruple of points in \mathbb{P}_A^1 , we denote the corresponding element in $B_2(A)_{\mathbb{Q}}$ by $[x_1, \dots, x_4]$.

Since A is a local ring, a subset of V is in general position if its reduction modulo the maximal ideal is in general position in the k -space $V \otimes_A k$.

Define two maps $d, d' : \tilde{C}_{m+1}(p) \rightarrow \tilde{C}_m(p)$ by

$$\begin{aligned} d(x_0, x_1, \dots, x_m) &= \sum_{0 \leq i \leq m} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_m), \\ d'(x_0, x_1, \dots, x_m) &= \sum_{1 \leq i \leq m} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_m). \end{aligned}$$

Let $\varepsilon : \tilde{C}_1(p) \rightarrow \mathbb{Q}$ be the map that sends each term to the sum of its coefficients.

Lemma 3.8.3. *The following complexes are acyclic.*

$$\begin{aligned} \dots &\xrightarrow{d} \tilde{C}_2(p) \xrightarrow{d} \tilde{C}_1(p) \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0, \\ \dots &\xrightarrow{d'} \tilde{C}_2(p) \xrightarrow{d'} \tilde{C}_1(p) \longrightarrow 0. \end{aligned}$$

Proof. Let $\sum_{j \in J} a_j \cdot (x_0(j), \dots, x_{m-1}(j))$ be an m -cycle in the first or the second complex. Since the reductions modulo the maximal ideal $\{\bar{x}_0(j), \dots, \bar{x}_{m-1}(j)\}$ are in general position in $k^{\oplus P}$ and k is an infinite field, we can choose $\alpha \in A$ such that all $\{\bar{x}_0(j), \dots, \bar{x}_{m-1}(j), \bar{\alpha}\}$ are in general position. Note that if W_i for $1 \leq i \leq r$ are proper subspaces of a vector space W over an infinite field, then $\bigcup_{1 \leq i \leq r} W_i \neq W$. If $m \geq 2$ and $d \sum_{j \in J} a_j \cdot (x_0(j), \dots, x_{m-1}(j)) = 0$, or if $m = 1$ and $\sum_{j \in J} a_j = 0$, we have

$$(-1)^m d \left(\sum_{j \in J} a_j \cdot (x_0(j), \dots, x_{m-1}(j), \alpha) \right) = \sum_{j \in J} a_j \cdot (x_0(j), \dots, x_{m-1}(j)).$$

Similarly, if $m \geq 2$ and $d' \sum_{j \in J} a_j \cdot (x_0(j), \dots, x_{m-1}(j)) = 0$, or if $m = 1$, we have

$$(-1)^m d' \left(\sum_{j \in J} a_j \cdot (x_0(j), \dots, x_{m-1}(j), \alpha) \right) = \sum_{j \in J} a_j \cdot (x_0(j), \dots, x_{m-1}(j)). \quad \square$$

Define maps $\lambda : \tilde{C}_m(p) \rightarrow \tilde{C}_m(p)$ by

$$\lambda(x_0, \dots, x_{m-1}) = \sum_{0 \leq i \leq m-1} \text{sign}(\sigma(m)^i) (x_{\sigma(m)^i(0)}, \dots, x_{\sigma(m)^i(m-1)}),$$

where $\sigma(m) := (0 \ 1 \ \dots \ m-1)$ is the standard m -cyclic permutation.

Then $\lambda \circ d = d' \circ \lambda$, and we have a double complex

$$\begin{array}{ccc} \dots & \xrightarrow{d} & \tilde{C}_3(3) \xrightarrow{d} \tilde{C}_2(3) \\ & & \lambda \downarrow \qquad \lambda \downarrow \\ \dots & \xrightarrow{d'} & \tilde{C}_3(3) \xrightarrow{d'} \tilde{C}_2(3). \end{array}$$

Definition 3.8.4. Let \tilde{D} be the complex associated to the double complex above. That is, $\tilde{D}_0 = \tilde{C}_2(3)$ and $\tilde{D}_i = \tilde{C}_{i+2}(3) \oplus \tilde{C}_{i+1}(3)$ and the maps are given by $(x, y) \rightarrow (d'(x) + \lambda(y), -d(y))$.

Let $\varepsilon : \tilde{D}_0 \rightarrow \mathbb{Q}$ be the map that sends each term to the sum of its coefficients. Then by Lemma 3.8.3 the complex $\cdots \rightarrow \tilde{D}_1 \rightarrow \tilde{D}_0 \xrightarrow{\varepsilon} \mathbb{Q} \rightarrow 0$ is acyclic.

If we endow \tilde{D} with its natural $\mathrm{GL}_3(A)$ action and \mathbb{Q} with the trivial action, then the complex above is an acyclic complex of $\mathrm{GL}_3(A)$ -modules. Therefore we get a canonical map

$$H_3(\mathrm{GL}_3(A), \mathbb{Q}) \rightarrow H_3(D), \quad (3.8.1)$$

where $D := \tilde{D}_{\mathrm{GL}_3(A)}$ is the complex of coinvariants of \tilde{D} .

Next we define a map from $H_3(D)$ to $B_2(A)_{\mathbb{Q}}$. This will be a slight modification of Goncharov's map [1995, Section 2.6].

From the double complex above, we are interested in the part

$$\begin{array}{ccccc} C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) \\ \lambda \downarrow & & \lambda \downarrow & & \lambda \downarrow \\ C_6(3) & \xrightarrow{d'} & C_5(3) & \xrightarrow{d'} & C_4(3). \end{array}$$

We define a map ϕ from this double complex to the double complex

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_2(A)_{\mathbb{Q}} & \longrightarrow & \bigwedge^2 A_{\mathbb{Q}}^{\times}. \end{array}$$

In ϕ , the only nontrivial map

$$\begin{array}{ccc} C_5(3) & \xrightarrow{d'} & C_4(3) \\ \downarrow & & \downarrow \\ B_2(A)_{\mathbb{Q}} & \xrightarrow{\delta} & \bigwedge^2 A_{\mathbb{Q}}^{\times} \end{array} \quad (3.8.2)$$

is a composition of the following two maps:

The first map is

$$\begin{array}{ccc} C_5(3) & \xrightarrow{d'} & C_4(3) \\ -p \downarrow & & p \downarrow \\ C_4(2) & \xrightarrow{d} & C_3(2), \end{array}$$

where $p : C_{m+1}(3) \rightarrow C_m(2)$ sends $(v_0, v_1, \dots, v_{m-1})$ to $(\tilde{v}_1, \dots, \tilde{v}_{m-1})$. Here \tilde{v}_i denotes the image of v_i in $A^{\oplus 3}/\langle v_0 \rangle$, and the term $(\tilde{v}_1, \dots, \tilde{v}_{m-1})$ is identified with an element of $C_m(2)$ by choosing any isomorphism between $A^{\oplus 3}/\langle v_0 \rangle$ and $A^{\oplus 2}$.

The second map is

$$\begin{array}{ccc}
 C_4(2) & \xrightarrow{d} & C_3(2) \\
 \alpha \downarrow & & \beta \downarrow \\
 B_2(A)_{\mathbb{Q}} & \xrightarrow{\delta} & \bigwedge^2 A_{\mathbb{Q}}^{\times}
 \end{array} \tag{3.8.3}$$

where α is the map that sends (v_0, v_1, v_2, v_3) to $[\varrho_0, \varrho_1, \varrho_2, \varrho_3]$. Here ϱ_i denotes the image of v_i in $\mathbb{P}(A^{\oplus 2})$, and $[\varrho_0, \varrho_1, \varrho_2, \varrho_3]$ denotes the image of (v_0, v_1, v_2, v_3) under the map $C_4(\mathbb{P}(A^{\oplus 2})) \rightarrow B_2(A)_{\mathbb{Q}}$, as in Remark 3.8.2. And β is the map that sends (v_0, v_1, v_2) to

$$\left(\frac{v_0 \wedge v_1}{v_1 \wedge v_2} \right) \wedge \left(\frac{v_0 \wedge v_2}{v_1 \wedge v_2} \right).$$

The next three lemmas imply that the maps defined so far can be extended to a map ϕ of the double complexes.

Lemma 3.8.5. *The map $C_6(3) \xrightarrow{d'} C_5(3) \xrightarrow{-p} C_4(2) \xrightarrow{\alpha} B_2(A)_{\mathbb{Q}}$ is zero.*

Proof. This follows from that $-p \circ d'(v_0, v_1, v_2, v_3, v_4, v_5) = d(v_1, v_2, v_3, v_4, v_5)$, and that this maps to zero in $B_2(A)_{\mathbb{Q}}$, by Remark 3.8.2. \square

Lemma 3.8.6. *The map $C_5(3) \xrightarrow{\lambda} C_5(3) \xrightarrow{-p} C_4(2) \xrightarrow{\alpha} B_2(A)_{\mathbb{Q}}$ is zero.*

Proof. (See [Goncharov 1995, Lemma 2.18].) Let $(v_0, \dots, v_4) \in C_5(3)$. Then there is a conic Q passing through the images of the five points v_0, v_1, v_2, v_3, v_4 in the projective plane. Choosing any isomorphism, we identify Q with \mathbb{P}_A^1 . Let the images of v_i be $x_i \in \mathbb{P}_A^1$ under this isomorphism. The composition of the maps in the statement of the lemma then maps (v_0, \dots, v_4) in $C_5(3)$ to

$$- \sum_{0 \leq i \leq 4} [x_i, x_{i+1}, \dots, x_{i+3}]$$

in $B_2(A)_{\mathbb{Q}}$, where the indices are modulo 5.

Claim 3.8.7. *In $B_2(A)_{\mathbb{Q}}$ we have*

$$[x_1, x_2, x_3, x_4] = \text{sign}(\sigma) \cdot [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}] \quad \text{for any } \sigma \in S_4.$$

Proof of the claim. Note that since we are working with \mathbb{Q} -modules we have $[0, x, 1, \infty] = -[0, x/(x-1), 1, \infty]$ by [Suslin 1990, Lemmas 1.2 and 1.5]. Hence

$$[0, x, 1, \infty] = -[x, 0, 1, \infty] \quad \text{and} \quad [0, x, 1, \infty] = -[0, 1/x, 1, \infty]$$

[loc. cit., Lemma 1.2]. Hence $[0, x, 1, \infty] = -[0, 1, x, \infty]$, and using again that $[0, x, 1, \infty] = -[0, x/(x-1), 1, \infty]$, we have $[0, x, 1, \infty] = -[0, x, \infty, 1]$.

Therefore the formula in the statement holds for the transpositions $(1\ 2)$, $(2\ 3)$, and $(3\ 4)$. Since these generate S_4 , the statement follows. \square

Finally by the claim,

$$\sum_{0 \leq i \leq 4} [x_i, x_{i+1}, \dots, x_{i+3}] = \sum_{0 \leq i \leq 4} (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_4]$$

and the right side is zero in $B_2(A)_{\mathbb{Q}}$ by Remark 3.8.2. \square

Lemma 3.8.8. *The map $C_4(3) \xrightarrow{\lambda} C_4(3) \xrightarrow{p} C_3(2) \xrightarrow{\beta} \bigwedge^2 A_{\mathbb{Q}}^{\times}$ is zero.*

Proof. First note that β sends (v_0, v_1, v_2) to

$$\left(\frac{v_0 \wedge v_1}{v_1 \wedge v_2} \right) \wedge \left(\frac{v_0 \wedge v_2}{v_1 \wedge v_2} \right) = \left(\frac{\ell(v_0 \wedge v_1)}{\ell(v_1 \wedge v_2)} \right) \wedge \left(\frac{\ell(v_0 \wedge v_2)}{\ell(v_1 \wedge v_2)} \right),$$

where $\ell : \det_A(A^{\oplus 2}) \rightarrow A$ is any surjective A -linear map. Therefore since we are looking at configurations in general position, the composition $\beta \circ p$ maps $(y_0, y_1, y_2, y_3) \in C_4(3)$ to

$$\left(\frac{y_0 \wedge y_1 \wedge y_2}{y_0 \wedge y_2 \wedge y_3} \right) \wedge \left(\frac{y_0 \wedge y_1 \wedge y_3}{y_0 \wedge y_2 \wedge y_3} \right).$$

This implies the statement by direct computation. \square

Therefore ϕ is a map of double complexes that induces a map $H_3(D) \rightarrow \ker(\delta)$ of the homology of the associated complexes. Combining this with the map

$$H_3(\mathrm{GL}_3(A), \mathbb{Q}) \rightarrow H_3(D)$$

in (3.8.1), we obtain a map $\rho_2 : H_3(\mathrm{GL}_3(A), \mathbb{Q}) \rightarrow \ker(\delta)$.

Therefore applying Sections 3.1–3.7 to $(A, I) = (k[\varepsilon]_n, (\varepsilon))$ proves this:

Proposition 3.8.9. *The composition $T := \rho_2 \circ \rho_1$ defines a natural map*

$$T : \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)} \hookrightarrow \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon)) \rightarrow B_2(k[\varepsilon]_n)_{\mathbb{Q}},$$

whose image lies in $\ker(\delta_n)$.

4. Nonvanishing of $\mathrm{Li}_{2,n}$ on $\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$

4.1. This section shows that $\mathrm{Li}_{2,n}$ is the correct map, as we show that it does not vanish on $\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$. First we describe $\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$ and define some elements α_w in it on which we will evaluate the additive dilogarithm.

4.1.1. Note that $\mathrm{HC}_*(k[\varepsilon]_n, (\varepsilon))$ is a k^{\times} -abelian group, where $\lambda \in k^{\times}$ acts as the map induced by the k -algebra automorphism of $k[\varepsilon]_n$ that sends ε to $\lambda \cdot \varepsilon$. This action is compatible with the decomposition (Remark 3.1.2) of

$$\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon)) = \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)} \oplus \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(2)}$$

[Cathelineau 1990/91, pages 593–594];

$$\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)} = \bigoplus_{n+1 \leq w \leq 2n-1} \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))_{(w)}^{(1)},$$

where each summand is isomorphic to k (loc. cit.); and

$$\mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(2)} = \bigoplus_{1 \leq w \leq n-1} \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))_{(w)}^{(2)},$$

where each summand is isomorphic to $\Omega_{k/\mathbb{Q}}^2$ (loc. cit.).

4.1.2. $\chi(n) = 0$ if n is even and $\chi(n) = 1$ if n is odd. For $n+1 \leq w \leq 2n-1$, let

$$\alpha_w := \sum_{0 \leq j < (2n-1-w)/2} (\varepsilon^{n-1-j}, \varepsilon^{w-n+j}, \varepsilon) + \frac{1}{2} \cdot \chi(w) \cdot (\varepsilon^{(w-1)/2}, \varepsilon^{(w-1)/2}, \varepsilon)$$

in $C_2^\lambda(k[\varepsilon]_n)$. Since α_w is a cycle, as can be checked by direct computation, with k^\times -weight w , it defines an element $\alpha_w \in \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))_{(w)}^{(1)}$ by Section 4.1.1.

4.2. Computation of $\mathrm{Li}_{2,n}$ on HC_2 . In this section, we compute $\mathrm{Li}_{2,n}(T(\alpha_w))$ (which is the same as $\mathrm{Li}_{2,n,w}(T(\alpha_w))$). This we will do in several steps.

4.2.1. From $\mathfrak{gl}_3(k[\varepsilon]_n)$ to $\mathfrak{gl}_2(k[\varepsilon]_n)$. Consider the 2-chain $(\varepsilon^a, \varepsilon^b, \varepsilon) \in C_2^\lambda(k[\varepsilon]_n)$ in the Connes complex, where $a+b \geq n$. By the map in Section 3.2, at the chain complex level, $(\varepsilon^a, \varepsilon^b, \varepsilon)$ goes to $\beta_{a,b} := \varepsilon^a e_{12} \wedge \varepsilon^b e_{23} \wedge \varepsilon e_{31} \in C_3(\mathfrak{gl}_3(k[\varepsilon]_n))_{\mathfrak{gl}_3(\mathbb{Q})}$. Therefore we need to compute the image of

$$\beta_w := \sum_{0 \leq j < (2n-1-w)/2} \beta_{n-1-j, w-n+j} + \frac{1}{2} \chi(w) \beta_{(w-1)/2, (w-1)/2}$$

in k . Let $\gamma_{a,b} := \varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{11}$, and

$$\gamma_w := \sum_{0 \leq j < (2n-1-w)/2} \gamma_{n-1-j, w-n+j} + \frac{1}{2} \chi(w) \gamma_{(w-1)/2, (w-1)/2}.$$

We defined T as the composition

$$\begin{aligned} \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))^{(1)} &\rightarrow \mathrm{Prim} H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})} \simeq \mathrm{Prim} H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q}) \\ &\rightarrow H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q}) \rightarrow H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q}) \rightarrow \ker(\delta). \end{aligned}$$

Let $T' : \mathrm{Prim} H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})} \rightarrow \ker(\delta)$ and $T'' : H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q}) \rightarrow \ker(\delta)$ be the obvious compositions.

The following lemma enables us to work in the homology of $\mathfrak{gl}_2(k[\varepsilon]_n)$ rather than that of $\mathfrak{gl}_3(k[\varepsilon]_n)$.

Lemma 4.2.1. *We have $(\mathrm{Li}_{2,n,w} \circ T')(\beta_w) = (\mathrm{Li}_{2,n,w} \circ T'')(\gamma_w)$.*

Proof. First note that

$$\begin{aligned} d(e_{13} \wedge \varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{31}) &= -\beta_{a,b} + \gamma_{a,b} \\ &\quad - \varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33} - e_{13} \wedge \varepsilon^{a+1} e_{32} \wedge \varepsilon^b e_{21}, \end{aligned}$$

that $\varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33}$ is a cycle; and that $e_{13} \wedge \varepsilon^{a+1} e_{32} \wedge \varepsilon^b e_{21}$ is a boundary in $C_*(\mathfrak{gl}(k[\varepsilon]_n))_{\mathfrak{gl}(\mathbb{Q})}$, since this element corresponds to the element $(1, \varepsilon^{a+1}, \varepsilon^b)$ in the Connes complex and $d(1, \varepsilon^{a+1}, \varepsilon^b, 1) = (1, \varepsilon^{a+1}, \varepsilon^b)$.

Therefore since β_w is a cycle, so is γ_w , and to prove the lemma it suffices to show that $(\text{Li}_{2,n,w} \circ T')(e^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33}) = 0$ for $a + b \geq n$.

Note that since

$$\begin{aligned} d(e_{12} \wedge \varepsilon^a e_{11} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33}) &= -\varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33} \\ &\quad + \varepsilon^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33} - \varepsilon^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}, \end{aligned}$$

it is sufficient to show the vanishing of both

$$\begin{aligned} (\text{Li}_{2,n,w} \circ T')(e^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33}) &= (\text{Li}_{2,n,w} \circ T'')(e^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33}), \\ (\text{Li}_{2,n,w} \circ T')(e^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}) &= (\text{Li}_{2,n,w} \circ T'')(e^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}). \end{aligned}$$

The equalities above follow immediately from the fact that $\varepsilon^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33}$ and $\varepsilon^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}$ are cycles not only in $C_*(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})}$ but also in $C_*(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})$.

Propositions 2.2.1 and 2.2.2 give that $\text{Li}_{2,n,w}(x) = (\text{li}_{2,(2n-1),w} \circ \delta_{2n-1})(\tilde{x})$ for $x \in B_2(k[\varepsilon]_n)$, where $\tilde{x} \in B_2(k[\varepsilon]_{2n-1})$ is any lift of x .

Let $\tilde{\alpha} \in \{\varepsilon^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33}, \varepsilon^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}\} \subseteq C_3(\mathfrak{gl}(k[\varepsilon]_{2n-1}), \mathbb{Q})$, and let α the reduction of $\tilde{\alpha}$ to $C_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})$. Then

$$\text{Li}_{2,n,w}(T''(\alpha)) = (\text{li}_{2,(2n-1),w} \circ \delta_{2n-1})(T''(\tilde{\alpha})).$$

Here T'' denotes the chain map, mapping $C_3(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to $B_2(k[\varepsilon]_{2n-1})_{\mathbb{Q}}$ and $C_2(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to $\wedge^2 V_{2n-1}$, that induces T'' . The map T'' depends on certain choices (see the next paragraph).

Recall how $T''(\tilde{\alpha})$ is defined in Section 3: Through the antisymmetrization map α_{as} (Section 3.4) and the exponential map [Loday 1992, Sections 3.5 and 11.3.13], we get a chain map $C_*(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})^\circ \rightarrow C_*(\hat{U}(\text{GL}_3(k[\varepsilon]_{2n-1})), \mathbb{Q})$. In fact, it is immediately seen that the image of $\tilde{\alpha}$ under these maps lies inside the image of $C_*(U(\text{GL}_3(k[\varepsilon]_{2n-1})), \mathbb{Q})$ in $C_*(\hat{U}(\text{GL}_3(k[\varepsilon]_{2n-1})), \mathbb{Q})$. To get from $C_*(U(\text{GL}_3(k[\varepsilon]_{2n-1})), \mathbb{Q})$ to $C_*(\text{GL}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$ we pass via the map described in Section 3.6. Bypassing the need for stabilization since we are already in GL_3 , and using that \tilde{D} is an acyclic complex of $\text{GL}_3(k[\varepsilon]_{2n-1})$ modules, we get a (non-canonical) map from $C_*(\text{GL}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to \tilde{D} . Finally taking $\text{GL}_3(k[\varepsilon]_{2n-1})$

coinvariants, we end up in the complex D , and using the map of double complexes (induced by (3.8.2)), we pass from D to the complex

$$\gamma_{k[\varepsilon]_{2n-1}}(2)_{\mathbb{Q}} : B_2(k[\varepsilon]_{2n-1})_{\mathbb{Q}} \xrightarrow{\delta_{2n-1}} \bigwedge^2 V_{2n-1}.$$

Since \underline{T}'' is a map of complexes, $\delta_{2n-1}(\underline{T}''(\tilde{\alpha})) = \underline{T}''(d(\tilde{\alpha})) = 0$, as $d(\tilde{\alpha}) = 0$ in $C_*(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$. This implies that

$$\mathrm{Li}_{2,n,w}(T''(\alpha)) = \mathrm{li}_{2,(2n-1,n),w}(\delta_{2n-1}(T''(\tilde{\alpha}))) = 0. \quad \square$$

The next lemma will help us to reduce the computation to \mathfrak{gl}_2 :

Lemma 4.2.2. *The chain γ_w as defined above is a cycle in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})}$ and hence defines an element in $H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$.*

Proof. We already know that γ_w defines a cycle in $C_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})}$. Since $C_i(\mathfrak{gl}_m(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_m(\mathbb{Q})} = (\bigwedge^i \mathfrak{gl}_m(k[\varepsilon]_n))_{\mathfrak{gl}_m(\mathbb{Q})}$ (see Section 3.2.2) and

$$(\bigwedge^i \mathfrak{gl}_m(k[\varepsilon]_n))_{\mathfrak{gl}_m(\mathbb{Q})} = (\bigwedge^i \mathfrak{gl}_i(k[\varepsilon]_n))_{\mathfrak{gl}_i(\mathbb{Q})} \quad \text{for } m \geq i$$

[Loday 1992, Corollary 9.2.8 and (10.2.10.1)], we have

$$d(\gamma_w) = 0 \in C_2(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})} = C_2(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}. \quad \square$$

4.2.2. *From $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ to $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$. In order to continue with the computation of $\mathrm{Li}_{2,n,w}(T'(\gamma_w))$, we need to compute the image of γ_w in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})$. This would be a very long computation, but in fact we will see in this section that we can get away with much less. The following proposition will be crucial.*

Proposition 4.2.3. *For any \mathbb{Q} -algebra A , let $\mathfrak{gl}_n(\mathbb{Q})$ act on $\mathfrak{gl}_n(A)$ by its adjoint action. Let $C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})}$ be the subcomplex of $C_*(\mathfrak{gl}_n(A), \mathbb{Q})$ on which $\mathfrak{gl}_n(\mathbb{Q})$ acts trivially. Then the canonical map*

$$C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})} \rightarrow C_*(\mathfrak{gl}_n(A), \mathbb{Q}) \rightarrow C_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})}$$

is an isomorphism and there is a canonical direct sum of complexes

$$C_*(\mathfrak{gl}_n(A), \mathbb{Q}) = C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})} \oplus L_*, \quad (4.2.1)$$

with $\mathfrak{gl}_n(\mathbb{Q})$ -action, such that L_ is acyclic.*

Proof. This is [Loday 1992, Proposition 10.1.8], taking $\mathfrak{g} = \mathfrak{gl}_n(A)$ and $\mathfrak{h} = \mathfrak{gl}_n(\mathbb{Q})$, and noting the reductivity of $\mathfrak{gl}_n(\mathbb{Q})$ [loc. cit., 10.2.9]. \square

To continue, we need to compute the image γ'_w of γ_w in $H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})$. Then we should lift γ'_w to a chain $\tilde{\gamma}'_w$ in $C'_3(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ and continue

just as in the last part of the proof of Lemma 4.2.1. Namely,

$$\begin{aligned} \text{Li}_{2,n,w}(T'(\gamma_w)) &= \text{Li}_{2,n,w}(T''(\tilde{\gamma}'_w)) = \text{li}_{2,(2n-1,n),w}(\delta_{2n-1}(T''(\tilde{\gamma}'_w))) \\ &= \text{li}_{2,(2n-1,n),w}(T''(d(\tilde{\gamma}'_w))). \end{aligned}$$

Let $\tilde{\gamma}_w^*$ be any chain in $C_3(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$ that has a cycle for its image in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ (under the canonical maps) and that lifts γ_w . Then, by Proposition 4.2.3, the first component $\tilde{\gamma}_w^{*(1)}$ of $\tilde{\gamma}_w^*$ under the decomposition in (4.2.1) is a lift of the element γ'_w . Therefore we can choose $\tilde{\gamma}'_w := \tilde{\gamma}_w^{*(1)}$, and to continue we need to compute $d(\tilde{\gamma}_w^{*(1)}) = d(\tilde{\gamma}_w^*)^{(1)}$.

For the rest of the computation, we will let $\tilde{\gamma}_w^* := \tilde{\gamma}_w$, where

$$\begin{aligned} \tilde{\gamma}_w &:= \sum_{0 \leq j < (2n-1-w)/2} \tilde{\gamma}_{n-1-j, w-n+j} + \frac{1}{2} \chi(w) \tilde{\gamma}_{(w-1)/2, (w-1)/2}, \\ \tilde{\gamma}_{a,b} &:= \varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{11}. \end{aligned}$$

Combining the above we have

$$\text{Li}_{2,n,w}(T'(\gamma_w)) = \text{li}_{2,(2n-1,n),w}(T''(d(\tilde{\gamma}_w)^{(1)})). \quad (4.2.2)$$

Next we will compute $d(\tilde{\gamma}_{a,b})^{(1)}$. For any \mathbb{Q} -algebra A there is a canonical isomorphism for $i \geq n$ given by

$$C_n(\mathfrak{gl}_i(A), \mathbb{Q})_{\mathfrak{gl}_i(\mathbb{Q})} = (\bigwedge^n \mathfrak{gl}_i(A))_{\mathfrak{gl}_i(\mathbb{Q})} \rightarrow (\mathbb{Q}[S_n] \otimes A^{\otimes n})_{S_n}, \quad (4.2.3)$$

where S_n acts on $\mathbb{Q}[S_n]$ by conjugation and on $A^{\otimes n}$ by permuting the factors and multiplying with sign [Loday 1992, 10.2.10.1].

Letting

$$\Gamma_{x,y} := x e_{12} \wedge y e_{21} + x e_{21} \wedge y e_{12} + \frac{1}{2} x (e_{22} - e_{11}) \wedge y (e_{22} - e_{11}) \quad \text{for } x, y \in A,$$

we see by direct computation that $\Gamma_{x,y} \in C'_2(\mathfrak{gl}_2(A), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$.

Under the map (4.2.3),

$$\begin{aligned} \Gamma_{x,y} &\mapsto (3 \cdot \tau) \otimes (x \otimes y), & x(e_{11} - e_{22}) \wedge y e_{11} &\mapsto (1 \cdot \tau) \otimes (x \otimes y), \\ & & x e_{21} \wedge y e_{12} &\mapsto (1 \cdot \tau) \otimes (x \otimes y), \\ & & x e_{12} \wedge y e_{21} &\mapsto (1 \cdot \tau) \otimes (x \otimes y), \end{aligned}$$

where $S_2 = \{\text{id}, \tau\}$. Therefore, using Proposition 4.2.3, we have

$$(x(e_{11} - e_{22}) \wedge y e_{11})^{(1)} = (x e_{21} \wedge y e_{12})^{(1)} = (x e_{12} \wedge y e_{21})^{(1)} = \frac{1}{3} \Gamma_{x,y}.$$

Since $d(\tilde{\gamma}_{a,b}) = \varepsilon^{a+b} (e_{11} - e_{22}) \wedge \varepsilon e_{11} - \varepsilon^b e_{21} \wedge \varepsilon^{a+1} e_{12} - \varepsilon^a e_{12} \wedge \varepsilon^{b+1} e_{21}$, we have

$$d(\tilde{\gamma}_{a,b})^{(1)} = \frac{1}{3} (\Gamma_{\varepsilon^{a+b}, \varepsilon} - \Gamma_{\varepsilon^b, \varepsilon^{a+1}} - \Gamma_{\varepsilon^a, \varepsilon^{b+1}}). \quad (4.2.4)$$

4.2.3. *Fixing a choice for \underline{T}'' .* We need to fix a choice for the restriction of \underline{T}'' to $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$. So, recalling the last part of the proof of Lemma 4.2.1, we need to fix the map from $C_*(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q}) \rightarrow \tilde{D} \rightarrow D \rightarrow \gamma_{k[\varepsilon]_{2n-1}}(2)_{\mathbb{Q}}$, in degree 2.

Fixing v_1, v_2, v_3 any three vectors in $k[\varepsilon]_{2n-1}^{\oplus 2}$ in general position, we define a map that sends $(g_1, g_2, g_3) \in C_2(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to

$$(w, g_1v_1, g_2v_2, g_3v_3) - (w, g_1v_1, g_2v_2, g_2v_3) \\ - (w, g_1v_1, g_1v_2, g_3v_3) + (w, g_1v_1, g_1v_2, g_2v_3)$$

in $\tilde{C}_4(k[\varepsilon]_{2n-1}^{\oplus 3}) = \tilde{C}_4(3) \subseteq \tilde{C}_4(3) \oplus \tilde{C}_3(3)$, where we view

$$k[\varepsilon]_{2n-1}^{\oplus 2} = \{(a_1, a_2, a_3) \in k[\varepsilon]_{2n-1}^{\oplus 3} \mid a_3 = 0\},$$

and we let $w = (0, 0, 1)$. It is seen without difficulty that this map can be extended to a map of complexes $C_*(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q}) \rightarrow \tilde{D}$.

Composing with the remaining map given in (3.8.2) this gives a map that sends (g_1, g_2, g_3) to

$$\beta((g_1v_1, g_2v_2, g_3v_3) - (g_1v_1, g_2v_2, g_2v_3) - (g_1v_1, g_1v_2, g_3v_3) + (g_1v_1, g_1v_2, g_2v_3))$$

in $\bigwedge^2 V_{2n-1}$, where β is the map in (3.8.3). From now on we fix $v_1 := (1, 1)$, $v_2 := (0, 1)$ and $v_3 := (1, 0)$ and denote the resulting map by \underline{T}'' .

4.2.4. *Computing $\mathrm{li}_{2, (2n-1, n), w}(\underline{T}''(\Gamma_{\varepsilon^p, \varepsilon^q}))$.* From (4.2.2) and (4.2.4) we realize that we need to compute $\mathrm{li}_{2, (2n-1, n), w}(\underline{T}''(\Gamma_{\varepsilon^p, \varepsilon^q}))$ for $p + q = w$. We will do this in a few steps.

Lemma 4.2.4. *For $i = 1, 2$ and $p + q = w$, with $p, q \geq 1$, we have*

$$\mathrm{li}_{2, (2n-1, n), w}(\underline{T}''(\varepsilon^p e_{ii} \wedge \varepsilon^q e_{ii})) = 0.$$

Proof. The element $\varepsilon^p e_{ii} \wedge \varepsilon^q e_{ii}$ maps to

$$\varepsilon^p e_{ii} \otimes \varepsilon^q e_{ii} - \varepsilon^q e_{ii} \otimes \varepsilon^p e_{ii} \in C_2(\mathcal{U}(\mathfrak{gl}_2(k[\varepsilon]_{2n-1})), \mathbb{Q}).$$

Since $\varepsilon^x e_{ii} = \log(1 - (1 - \exp(\varepsilon^x e_{ii}))) = -\sum_{1 \leq k} (1 - \exp(\varepsilon^x e_{ii}))^k / k$ for $x \geq 1$, we see that $\varepsilon^p e_{ii} \otimes \varepsilon^q e_{ii}$ is a \mathbb{Q} -linear combination of terms of the form

$$\exp(\varepsilon^s e_{ii})^u \otimes \exp(\varepsilon^t e_{ii})^v.$$

Let $g_1 := \exp(\varepsilon^s e_{ii})^u$ and $g_2 := \exp(\varepsilon^t e_{ii})^v$. Then $g_1 \otimes g_2$ maps to $(1, g_1, g_1g_2)$, which maps to

$$(v_1, g_1v_2, g_1g_2v_3) - (v_1, g_1v_2, g_1v_3) - (v_1, v_2, g_1g_2v_3) + (v_1, v_2, g_1v_3). \quad (4.2.5)$$

Since, depending on i , $g_1(v_2) = v_2$ or $g_1(v_3) = g_1g_2(v_3) = v_3$, we see that the last expression is 0. \square

Lemma 4.2.5. *The value of $\text{li}_{2,(2n-1,n),w}$ on the image of the element $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ in $\bigwedge^2 V_{2n-1}$, under the chain map that we fixed in Section 4.2.3, is 0 if $p+q \neq w$ and $p, q \geq 1$.*

Proof. By Proposition 2.1.2 to compute the value of $\text{li}_{2,(2n-1,n),w}$ on the image of $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ in $\bigwedge^2 V_{2n-1}$, we first need to project that image to $S_k(2n-1, n)_{(w)}$. But for $\lambda \in \mathbb{Q}$, replacing ε with $\lambda\varepsilon$ multiplies $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ by λ^{p+q} , whereas the projection of its image to $S_k(2n-1, n)_{(w)}$ gets multiplied by λ^w . Therefore this projection is 0. Hence the statement in the lemma. \square

Lemma 4.2.6. *For $p+q = w$ with $p, q \geq 1$, we have*

$$\text{li}_{2,(2n-1,n),w}(\underline{T}''(\varepsilon^p e_{22} \wedge \varepsilon^q e_{11})) = \text{li}_{2,(2n-1,n),w}((1 + \varepsilon^q) \wedge (1 + \varepsilon^p)).$$

Proof. The expression $\varepsilon^p e_{22} \wedge \varepsilon^q e_{11}$ maps to

$$\varepsilon^p e_{22} \otimes \varepsilon^q e_{11} - \varepsilon^q e_{11} \otimes \varepsilon^p e_{22}. \quad (4.2.6)$$

Both $\varepsilon^p e_{ii} \otimes \varepsilon^q e_{jj}$ and $\exp(\varepsilon^p e_{ii}) \otimes \exp(\varepsilon^q e_{jj}) - \exp(\varepsilon^p e_{ii}) \otimes 1 - 1 \otimes \exp(\varepsilon^q e_{jj})$ have the same k^\times -weight w component, and therefore by Lemma 4.2.5 have the same image. Note that terms of the form $1 \otimes g$ and $g \otimes 1$ map to 0, because of the computation in (4.2.5). Hence the left side of the expression in the lemma is equal to the image of $\exp(\varepsilon^p e_{22}) \otimes \exp(\varepsilon^q e_{11}) - \exp(\varepsilon^q e_{11}) \otimes \exp(\varepsilon^p e_{22})$. Since $\exp(\varepsilon^q e_{11})v_2 = v_2$, using the expression (4.2.5) we see that $\exp(\varepsilon^q e_{11}) \otimes \exp(\varepsilon^p e_{22})$ maps to 0. Again using (4.2.5) and the definition of β and $\text{li}_{2,(2n-1,n),w}$, we see that $\exp(\varepsilon^p e_{22}) \otimes \exp(\varepsilon^q e_{11})$ maps to $\text{li}_{2,(2n-1,n),w}((1 + \varepsilon^q) \wedge (1 + \varepsilon^p))$. \square

Lemma 4.2.7. *For $p+q = w$ with $p, q \geq 1$, we have*

$$\text{li}_{2,(2n-1,n),w}(\underline{T}''(\varepsilon^p e_{12} \wedge \varepsilon^q e_{21})) = \text{li}_{2,(2n-1,n),w}((1 - \varepsilon^p) \wedge (1 - \varepsilon^q)).$$

Proof. Exactly as in the proof of Lemma 4.2.6, we see that the left side of the expression above is equal to the image of

$$\exp(\varepsilon^p e_{12}) \otimes \exp(\varepsilon^q e_{21}) - \exp(\varepsilon^q e_{21}) \otimes \exp(\varepsilon^p e_{12}).$$

As $\exp(\varepsilon^q e_{21})(v_2) = v_2$, we see, using (4.2.5), that $\exp(\varepsilon^q e_{21}) \otimes \exp(\varepsilon^p e_{12})$ maps to 0. Finally using (4.2.5), and the definition of β and $\text{li}_{2,(2n-1,n),w}$ we see that $\exp(\varepsilon^p e_{12}) \otimes \exp(\varepsilon^q e_{21})$ maps to $\text{li}_{2,(2n-1,n),w}((1 - \varepsilon^p) \wedge (1 - \varepsilon^q))$. \square

Lemma 4.2.8. *For $p+q = w$ with $p, q \geq 1$,*

$$\text{li}_{2,(2n-1,n),w}(\underline{T}''(\Gamma_{\varepsilon^p, \varepsilon^q})) = 3 \text{li}_{2,(2n-1,n),w}((1 - \varepsilon^p) \wedge (1 - \varepsilon^q)).$$

Proof. This follows from Lemmas 4.2.4, 4.2.6, and 4.2.7, together with the fact, which is immediate from the definition of $\text{li}_{2,(2n-1,n),w}$, that

$$\text{li}_{2,(2n-1,n),w}((1 - \varepsilon^p) \wedge (1 - \varepsilon^q)) = \text{li}_{2,(2n-1,n),w}((1 + \varepsilon^p) \wedge (1 + \varepsilon^q)). \quad \square$$

Let $[\cdot]$ denote the greatest integer function.

Theorem 4.2.9. *With the notation as in Section 4.1.2,*

$$\mathrm{Li}_{2,n}(T(\alpha_w)) = \begin{cases} -([\frac{1}{2}(2n-1-w)] + w - n + 1 + \frac{1}{2}\chi(w)) & \text{if } w \neq 2n-1, \\ -\frac{1}{2}(2n-1) & \text{if } w = 2n-1. \end{cases}$$

Proof. Since $\mathrm{Li}_{2,n}(T(\alpha_w)) = \mathrm{Li}_{2,n,w}(T'(\beta_w))$, using Lemma 4.2.1, (4.2.2), (4.2.4) we see that $\mathrm{Li}_{2,n}(T(\alpha_w)) = \frac{1}{2} \mathrm{li}_{2,(2n-1,n),w} \circ \underline{T}''$ evaluated on

$$\sum_{0 \leq j < (2n-1-w)/2} (\Gamma_{\varepsilon^{w-1}, \varepsilon} - \Gamma_{\varepsilon^{w-n+j}, \varepsilon^{n-j}} - \Gamma_{\varepsilon^{n-1-j}, \varepsilon^{w-n+j+1}}) + \frac{1}{2}\chi(w)(\Gamma_{\varepsilon^{w-1}, \varepsilon} - 2\Gamma_{\varepsilon^{(w-1)/2}, \varepsilon^{(w+1)/2}}).$$

Using Lemma 4.2.8 and the definition of $\mathrm{li}_{2,(2n-1,n),w}$ we see that if $w \neq 2n-1$, then the contribution from $j=0$ is $-(w-n+1)$; the contribution from each of the terms where $0 < j < n-1$ is -1 ; the last term contributes $-\frac{1}{2}\chi(w)$.

In the case $w = 2n-1$, there is only one contribution, coming from the last term, and this is $\frac{1}{2}\chi(2n-1)(-1-2(n-1)) = -\frac{1}{2}(2n-1)$. \square

4.3. Proof of Theorem 1.3.2. In order to prove this, by Goodwillie's theorem (Theorem 3.1.1), Remark 3.1.2 and Sections 4.1.1 and 4.1.2, we need only show

$$\mathrm{Li}_{2,n,w} : (k \simeq) \mathrm{HC}_2(k[\varepsilon]_n, (\varepsilon))_{\langle w \rangle}^{(1)} \rightarrow k$$

is an isomorphism. We know that this map is nonzero by Theorem 4.2.9, and replacing ε by $\lambda\varepsilon$ has the effect of multiplication by λ^w , using the vector space structures on both sides [Hesselholt 2005, Proposition 8.1]. This immediately implies the theorem when k is algebraically closed. In the general case, we just need to use Theorem 1.3.2 for \bar{k} , and the equivariance of $\mathrm{Li}_{2,n,w}$ with respect to $\mathrm{Gal}(\bar{k}/k)$ and take galois invariants on both sides.

5. The complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$

5.1. To compute the kernel of δ_n in Theorem 1.3.1, we will need the following proposition. Following Suslin's notation, let $T_m(A) \subseteq \mathrm{GL}_m(A)$ denote the subgroup of diagonal matrices.

Proposition 5.1.1. *The map $\rho_2 : H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \rightarrow \ker(\delta_n)$ from Section 3.8 has the property that*

$$\rho_2(H_3(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q})) = \ker(\delta_n) \quad \text{and} \quad H_3(T_3(k[\varepsilon]_n), \mathbb{Q}) \subseteq \ker(\rho_2).$$

Proof. The first statement is proved in the case of fields in [Suslin 1990, Section 2]. The same proof works for $k[\varepsilon]_n$, if in the first line of [Suslin 1990, page 222], we use [Guin 1989, Theorem 2.2.2] to show that

$$H_*(T_2(k[\varepsilon]_n), \mathbb{Q}) = H_*(UT_2(k[\varepsilon]_n), \mathbb{Q}),$$

where $UT_2(A)$ denotes upper triangular matrices in $GL_2(A)$ (this is denoted by $B_2(A)$ in [Suslin 1990]). We note that there is a slight difference between the construction of our map ρ_2 and the corresponding map of Suslin. Namely, Suslin uses configurations in the projective space rather than the affine space, but the resulting maps $H_3(GL_3(k[\varepsilon]_n), \mathbb{Q}) \rightarrow \ker(\delta_n)$ are the same.

The proof of [Suslin 1990, Proposition 3.1] works for $k[\varepsilon]_n$ as well, proving the second statement. \square

Proposition 5.1.2. *The map $T : HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)} \rightarrow \ker(\delta_n)^\circ$ (see Notation 2.0.1) defined in Proposition 3.8.9 is surjective.*

Proof. Because of Proposition 5.1.1, Theorem 3.1.1 and Remark 3.1.2, it suffices to show that the image of $K_3(k[\varepsilon]_n)_{\mathbb{Q}}^{(2)}$ in $H_3(GL_3(k[\varepsilon]_n), \mathbb{Q})^\circ / H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^\circ$, under the composition of the maps

$$\begin{aligned} K_3(k[\varepsilon]_n)_{\mathbb{Q}}^{(2)} &\rightarrow K_3(k[\varepsilon]_n)_{\mathbb{Q}} \rightarrow H_3(GL(k[\varepsilon]_n), \mathbb{Q}) \simeq H_3(GL_3(k[\varepsilon]_n), \mathbb{Q}) \\ &\rightarrow H_3(GL_3(k[\varepsilon]_n), \mathbb{Q})^\circ / H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^\circ, \end{aligned}$$

contains that of $H_3(GL_2(k[\varepsilon]_n), \mathbb{Q})$ in $H_3(GL_3(k[\varepsilon]_n), \mathbb{Q})^\circ / H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^\circ$.

Let $\bigwedge V$ denote the graded symmetric algebra over a graded vector space V . By the Milnor–Moore theorem, $H_*(GL(A), \mathbb{Q}) \simeq \bigwedge((K_*(A)_{\mathbb{Q}})_{>0})$ [Loday 1992, Corollary 11.2.12]; by the stability theorem,

$$H_3(GL_3(k[\varepsilon]_n), \mathbb{Q}) = H_3(GL(k[\varepsilon]_n), \mathbb{Q})$$

[Guin 1989, Section 2]. Combining these, we obtain

$$H_3(GL_3(k[\varepsilon]_n), \mathbb{Q}) = \bigwedge^3 K_1(k[\varepsilon]_n)_{\mathbb{Q}} \oplus (K_1(k[\varepsilon]_n)_{\mathbb{Q}} \otimes K_2(k[\varepsilon]_n)_{\mathbb{Q}}) \oplus K_3(k[\varepsilon]_n)_{\mathbb{Q}}.$$

The first two components of the decomposition lie inside

$$H_1(GL_1(k[\varepsilon]_n), \mathbb{Q}) \otimes H_2(GL_2(k[\varepsilon]_n), \mathbb{Q}) \subseteq H_3(T_3(k[\varepsilon]_n), \mathbb{Q}),$$

(by the proof of [Suslin 1990, Lemma 4.2]; [Guin 1989]). Therefore it suffices to prove that the image of $K_3(k[\varepsilon]_n)_{\mathbb{Q}}^{(2)}$ under the canonical projection

$$H_3(GL_3(k[\varepsilon]_n), \mathbb{Q}) \rightarrow \text{Prim } H_3(GL_3(k[\varepsilon]_n), \mathbb{Q}) \rightarrow (\text{Prim } H_3(GL_3(k[\varepsilon]_n), \mathbb{Q}))^\circ$$

contains the image of $H_3(GL_2(k[\varepsilon]_n), \mathbb{Q})$.

By the construction of ρ_1 in Sections 3.2–3.7 and Remark 3.1.2, the last translates to showing that the image $\text{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q}))$ of $H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})$ in

$$(\text{Prim } H_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q}))^\circ = \text{HC}_2(k[\varepsilon]_n)^\circ = \text{HC}_2(k[\varepsilon]_n)^{\circ(1)} \oplus \text{HC}_2(k[\varepsilon]_n)^{\circ(2)}$$

is contained in $\text{HC}_2(k[\varepsilon]_n)^{\circ(1)}$.

First note that α_w for $n+1 \leq w \leq 2n-1$ form a basis for $\text{HC}_2(k[\varepsilon]_n)^{\circ(1)}$ by Theorem 4.2.9 and Section 4.1.1. By Lemmas 4.2.1 and 4.2.2 and Proposition 4.2.3 and the discussion following it, the image of α_w in $H_3(\mathfrak{gl}_3(k[\varepsilon]_n))^\circ$ is equal to that of an element $\gamma'_w \in H_3(\mathfrak{gl}_2(k[\varepsilon]_n))^\circ$. This implies immediately that $\text{HC}_2(k[\varepsilon]_n)^{\circ(1)} \subseteq \text{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q}))$.

On the other hand [Loday 1992, Theorems 10.3.4 and 4.6.8] and [Loday and Quillen 1984, Remark 6.10] imply that there is a natural map

$$(\text{Prim } H_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q}))^\circ / \text{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})) \rightarrow \text{HC}_2(k[\varepsilon]_n)^{\circ(2)}$$

which induces an automorphism of $\text{HC}_2(k[\varepsilon]_n)^{\circ(2)}$ when precomposed with

$$\text{HC}_2(k[\varepsilon]_n)^{\circ(2)} \rightarrow (\text{Prim } H_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q}))^\circ / \text{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})).$$

These imply that $\text{im}(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})) = \text{HC}_2(k[\varepsilon]_n)^{\circ(1)}$. \square

The corollary below computes the infinitesimal part of the first cohomology of the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$. Note that $H^1(\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}})^\circ = \ker(\delta_n)^\circ$.

Corollary 5.1.3. *The maps*

$$T : \text{HC}_2(k[\varepsilon]_n, (\varepsilon))^{\circ(1)} \rightarrow \ker(\delta_n)^\circ \quad \text{and} \quad \text{Li}_{2,n} : \ker(\delta_n)^\circ \rightarrow k^{\oplus n-1}$$

are isomorphisms.

Proof. This follows from the fact that T is surjective (Proposition 5.1.2) and that $\text{Li}_{2,n} \circ T$ is an isomorphism (Theorem 1.3.2). \square

Proposition 5.1.4. *There are natural isomorphisms*

$$H^2(\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}})^\circ \simeq \text{HC}_1(k[\varepsilon]_n)^\circ = \text{HC}_1(k[\varepsilon]_n)^{\circ(1)} \simeq \bigoplus_{1 \leq i \leq n-1} \Omega_k^1.$$

Proof. Note that by the definition of Milnor K-theory [Loday 1992, 11.1.16]

$$H^2(\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}) = K_2^M(k[\varepsilon]_n). \quad (5.1.1)$$

Since

$$K_2^M(k[\varepsilon]_n) = K_2(k[\varepsilon]_n) \quad (5.1.2)$$

[Guin 1989, Section 4.2], we have, by [Loday 1992, Proposition 2.1.14],

$$K_2(k[\varepsilon]_n)^\circ = \text{HC}_1(k[\varepsilon]_n)^\circ = \Omega_{k[\varepsilon]_n}^1 / (\Omega_k^1 + d(k[\varepsilon]_n)) \simeq \bigoplus_{1 \leq i \leq n-1} \Omega_k^1.$$

Finally $\mathrm{HC}_1(k[\varepsilon]_n) = \mathrm{HC}_1(k[\varepsilon]_n)^{(1)}$ follows from [loc. cit., Theorem 4.6.7]. \square

5.2. Proof of Theorem 1.3.1. Over k this is the main theorem in [Suslin 1990]. However, note that there the indecomposable quotient $K_3(k)_{\mathrm{ind}, \mathbb{Q}}$ of $K_3(k)_{\mathbb{Q}}$ appears instead of $K_3(k)_{\mathbb{Q}}^{(2)}$. To see that these two groups are canonically isomorphic, see [Lichtenbaum 1987, page 207]. Therefore we only need to compute the cohomology of the infinitesimal part of the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$. And this is done in Corollary 5.1.3 and (5.1.1) and (5.1.2).

6. Comparison with the additive dilogarithm of Bloch and Esnault

In this section we compare the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\circ}$ to the complex $T_n \mathbb{Q}(2)(k)$ of Bloch and Esnault.

6.1. The reduced complex. To make the comparison we first define a subcomplex of $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\circ}$: Define $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\prime}$ to be the subcomplex of $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\circ}$ whose degree 2 term is

$$k^{\times} \otimes V_n^{\circ} \subseteq (\bigwedge^2 V_n)^{\circ}$$

and whose degree 1 term is the inverse image $\delta_n^{-1}(k^{\times} \otimes V_n^{\circ}) \subseteq B_2(k[\varepsilon]_n)_{\mathbb{Q}}^{\circ}$. Denote this last group by $B_2(k[\varepsilon]_n)_{\mathbb{Q}}^{\prime}$. Then we have

$$\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\prime} : B_2(k[\varepsilon]_n)_{\mathbb{Q}}^{\prime} \rightarrow k^{\times} \otimes V_n^{\circ}.$$

We need a lemma to compute the cohomology of this reduced complex.

Lemma 6.1.1. *The natural map $(k^{\times})^{\otimes(i-1)} \otimes k[\varepsilon]_n^{\times} \rightarrow K_i^M(k[\varepsilon]_n)$ is a surjection.*

Proof. By the definition of Milnor K-theory, it is clear that it suffices to prove the lemma for $i = 2$. In this case the lemma follows from the isomorphism

$$K_2(k[\varepsilon]_n) \simeq K_2(k) \oplus \frac{\Omega_{k[\varepsilon]_n}^1}{\Omega_k^1 + d(k[\varepsilon]_n)},$$

[Graham 1973, Theorem 3] and the observation that $k^{\times} \otimes k[\varepsilon]_n^{\times}$ surjects onto the expression on the right, under this isomorphism. Note that $K_2^M(k[\varepsilon]_n) = K_2(k[\varepsilon]_n)$ [Guin 1989]. \square

Proposition 6.1.2. *The inclusion $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\prime} \rightarrow \gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}^{\circ}$ is a quasiisomorphism.*

Proof. The only thing that needs justification is the surjectivity of the induced map on the degree 2 cohomology groups or equivalently the surjectivity of the composition

$$k_{\mathbb{Q}}^{\times} \otimes_{\mathbb{Q}} V_n^{\circ} \rightarrow (\bigwedge^2 V_n)^{\circ} \rightarrow \Omega_{k[\varepsilon]_n}^1 / (\Omega_k^1 + d(k[\varepsilon]_n)),$$

where the last map is the one in the proof of Proposition 5.1.4. But this is exactly Lemma 6.1.1. \square

6.2. The construction of Bloch and Esnault with higher modulus. For the rest of the section we assume that k is algebraically closed. In [2003], Bloch and Esnault construct the additive weight 2 complex with modulus 2; their proof goes through to give a construction for all moduli $n \geq 2$. We describe the properties of this complex below. The proofs and the details of the construction can be found in [Bloch and Esnault 2003, Section 2].

Following their notation, we let R be the local ring of 0 in \mathbb{A}_k^1 . The localization (away from 0) sequence for the pair $(k[t], (t^n))$ splits into the exact sequences

$$K_2(k[t], (t^n)) \rightarrow K_2(R, (t^n)) \xrightarrow{\partial} \bigoplus_{x \in k^\times} K_1(k) \rightarrow K_1(k[t], (t^n)) \rightarrow 0$$

and

$$0 \rightarrow K_1(R, (t^n)) \xrightarrow{\partial} \bigoplus_{x \in k^\times} K_0(k) \rightarrow K_0(k[t], (t^n)) \rightarrow 0,$$

since $K_0(R, (t^n)) = 0$ and the map $K_1(R, (t^n)) \rightarrow \bigoplus_{x \in k^\times} K_0(k)$ is injective, as $K_1(R, (t^n)) = 1 + (t^n)$ and the map is given by the divisor of the function [Lichtenbaum 1987, Appendix]. This description also gives a canonical identification

$$K_0(k[t], (t^n)) = (k[t]_n^\times)^\circ.$$

Using the product structure on K-theory, let

$$T_n B_2(k) := (K_2(R, (t^n)) / \text{im}(K_1(k) \cdot K_1(R, (t^n))))_{\mathbb{Q}},$$

and let $T_n H_M^1(k, 2)$ be the image of $K_2(k[t], (t^n))_{\mathbb{Q}}$ in $T_n B_2(k)$. Then the above exact sequences give the exact sequence

$$0 \rightarrow T_n H_M^1(k, 2) \rightarrow T_n B_2(k) \rightarrow k^\times \otimes V_n^\circ \rightarrow K_1(k[t], (t^n))_{\mathbb{Q}} \rightarrow 0. \quad (6.2.1)$$

We let $T_n \mathbb{Q}(2)(k) : T_n B_2(k) \rightarrow k^\times \otimes V_n^\circ$ denote the middle part of this sequence. This is the exact generalization to higher moduli of the complex considered by Bloch and Esnault [2003] (the complex described in Section 1.2).

We will try to express the cohomology groups of $T_n \mathbb{Q}(2)(k)$ in terms of the groups $K_*(k[t]_n, (t))_{\mathbb{Q}}$.

First note that the long exact sequence for the pair $(k[t], (t^n))$, together with the homotopy invariance of K-theory, gives canonical isomorphisms

$$K_{*+1}(k[t]_n, (t)) \simeq K_*(k[t], (t^n)),$$

and therefore there is a surjection

$$\left(\frac{K_3(k[t]_n, (t))}{K_1(k) \cdot K_2(k[t]_n, (t))} \right)_{\mathbb{Q}} \simeq \left(\frac{K_2(k[t], (t))}{K_1(k) \cdot K_1(k[t], (t))} \right)_{\mathbb{Q}} \rightarrow T_n H_M^1(k, 2). \quad (6.2.2)$$

Lemma 6.2.1. *There is a canonical surjection $K_3(k[t]_n, (t))_{\mathbb{Q}}^{(2)} \rightarrow T_n H_M^1(k, 2)$.*

Proof. By [Lichtenbaum 1987, page 191],

$$K_3(k[t]_n)_\mathbb{Q} = K_3(k[t]_n)_\mathbb{Q}^{(2)} \oplus K_3^M(k[t]_n)_\mathbb{Q},$$

and by Lemma 6.1.1, the image of $K_1(k) \otimes K_2(k[t]_n)$ in $K_3(k[t]_n)$ is $K_3^M(k[t]_n)$. Hence that (6.2.2) is a surjection proves the lemma. \square

Let

$$\rho : T_n B_2(k) = \left(\frac{K_2(R, (t^n))}{K_1(k) \cdot K_1(R, (t^n))} \right)_\mathbb{Q} \rightarrow \left(\frac{K_2(k[t]_{2n-1}, (t^n))}{K_1(k) \cdot K_1(k[t]_{2n-1}, (t^n))} \right)_\mathbb{Q} =: N$$

denote the map induced by reduction modulo (t^{2n-1}) . We will prove that ρ behaves like an additive dilogarithm in this setting.

Proposition 6.2.2. *The composition $K_3(k[t]_n, (t))_\mathbb{Q}^{(2)} \rightarrow T_n H_M^1(k, 2) \rightarrow N$ induced by the inclusion*

$$K_3(k[t]_n, (t))_\mathbb{Q}^{(2)} \rightarrow K_3(k[t]_n, (t))_\mathbb{Q},$$

(6.2.2), and ρ is an isomorphism.

Proof. This map is induced by the long exact sequence of the pair $(k[t]_{2n-1}, (t^n))$:

$$\cdots \rightarrow K_3(k[t]_n, (t)) \rightarrow K_2(k[t]_{2n-1}, (t^n)) \rightarrow K_2(k[t]_{2n-1}, (t)) \rightarrow \cdots$$

By Goodwillie's theorem, Remark 3.1.2 and Section 4.1.1, the map

$$K_3(k[t]_{2n-1}, (t))_\mathbb{Q}^{(2)} \rightarrow K_3(k[t]_n, (t))_\mathbb{Q}^{(2)}$$

is equivalent to a map $k^{\oplus(2n-2)} \rightarrow k^{\oplus(n-1)}$, where the k^\times -weights in the source range in $[2n, 4n-3]$, whereas in the target they range in $[n, 2n-1]$. Therefore this last map is zero and hence $K_3(k[t]_n, (t))_\mathbb{Q}^{(2)} \rightarrow K_2(k[t]_{2n-1}, (t^n))_\mathbb{Q}$ is injective.

By [Stienstra 1981, Theorem 1.11], $K_2(k[t]_{2n-1}, (t^n))_\mathbb{Q} \simeq k^{\oplus(n-1)} \oplus (\Omega_k^1)^{\oplus(n-1)}$, and $K_1(k) \otimes K_1(k[t]_{2n-1}, (t^n)) \rightarrow K_2(k[t]_{2n-1}, (t^n))_\mathbb{Q}$ has image $(\Omega_k^1)^{\oplus(n-1)}$. \square

Corollary 6.2.3. *There are canonical isomorphisms*

$$H^1(T_n \mathbb{Q}(2)(k)) \simeq K_3(k[t]_n, (t))_\mathbb{Q}^{(2)} \simeq \mathrm{HC}_2(k[t]_n, (t))^{(1)},$$

$$H^2(T_n \mathbb{Q}(2)(k)) \simeq K_2(k[t]_n, (t))_\mathbb{Q} \simeq \mathrm{HC}_1(k[t]_n, (t)).$$

Proof. The first isomorphism is an immediate consequence of Lemma 6.2.1 and Proposition 6.2.2, and the second is a consequence of the isomorphism

$$K_2(k[t]_n, (t)) \simeq K_1(k[t], (t^n)),$$

which follows from the long exact sequence for $(k[t], (t^n))$ and the homotopy invariance of K -theory. \square

Proof of Corollary 1.4.1. First we note that the degree 2 terms of $T_n\mathbb{Q}(2)(k)$ and $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ are both equal to $k^\times \otimes V_n^\circ$ and that the cohomology groups of the two complexes are canonically isomorphic (Theorem 1.3.1, Proposition 6.2.2, and Corollary 6.2.3). In both cases the projection from $k^\times \otimes V_n^\circ$ to the degree 2 cohomology is induced by the composition

$$k^\times \otimes V_n^\circ \rightarrow K_2^M(k[\varepsilon]_n) \rightarrow \Omega_{k[\varepsilon]_n}^1 / (\Omega_k^1 + d(k[\varepsilon]_n))$$

(see the proof of Lemma 6.1.1). Therefore the images of $T_n B_2(k)$ and of $B_2(k[\varepsilon]_n)'_{\mathbb{Q}}$ in $k^\times \otimes V_n^\circ$ are the same. The exact sequence (6.2.1) and Proposition 6.2.2 give a splitting of $T_n B_2(k)$; and Theorems 1.3.1 and 1.3.2 give a splitting of $B_2(k[\varepsilon]_n)'_{\mathbb{Q}}$. This proves the corollary. \square

We would like to emphasize that the isomorphism given in the statement of the corollary uses the additive dilogarithm in both constructions and thus should not be considered as natural.

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Nichols algebras with standard braiding

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The class of standard braided vector spaces, introduced by Andruskiewitsch and the author in 2007 to understand the proof of a theorem of Heckenberger, is slightly more general than the class of braided vector spaces of Cartan type. In the present paper, we classify standard braided vector spaces with finite-dimensional Nichols algebra. For any such braided vector space, we give a PBW basis, a closed formula of the dimension and a presentation by generators and relations of the associated Nichols algebra.

Introduction	35
1. PBW bases	38
2. Transformations of braided graded Hopf algebras	44
3. Standard braidings	52
4. Nichols algebras of standard braided vector spaces	60
5. Presentation by generators and relations of Nichols algebras of standard braided vector spaces	71
Acknowledgments	104
References	104

Introduction

A breakthrough in the development of the theory of Hopf algebras occurred with the discovery of quantized enveloping algebras [Drinfel'd 1987; Jimbo 1985]. This special class of Hopf algebras has been intensively studied by many authors and from many points of view. In particular, finite-dimensional analogues of quantized enveloping algebras were introduced and investigated by Lusztig [1990a; 1990b].

About ten year ago, a classification program of pointed Hopf algebras was launched by Andruskiewitsch and Schneider [1998] (see also [Andruskiewitsch and Schneider 2002b]). The success of this program depends on finding solutions to several questions, among them:

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Question 1 [Andruskiewitsch 2002, Question 5.9]. Given a braided vector space of diagonal type V , such that the entries of its matrix are roots of unity, compute the dimension of the associated Nichols algebra $\mathfrak{B}(V)$. If it is finite, give a nice presentation of $\mathfrak{B}(V)$.

Partial answers to this question were given in [Andruskiewitsch and Schneider 2000; Heckenberger 2006b] for the class of braided vector spaces of Cartan type. These answers were already crucial to proving a classification theorem for finite-dimensional Hopf algebras whose group is abelian with prime divisors of the order great than 7 [Andruskiewitsch and Schneider 2005]. Later, a complete answer to the first part of Question 1 was given in [Heckenberger 2006a].

The notion of a standard braided vector space, a special kind of diagonal braided vector space, was introduced in [Andruskiewitsch and Angiono 2008], and is reviewed in Definition 3.5 below. This class includes properly the class of braided vector spaces of Cartan type.

The purpose of this paper is to develop from scratch the theory of standard braided vector spaces. Here are our main contributions:

- We give a complete classification of standard braided vector spaces with finite-dimensional Nichols algebras. As usual, we may assume the connectedness of the corresponding braiding. It turns out that standard braided vector spaces are of Cartan type when the associated Cartan matrix is of type C , D , E or F , see Proposition 3.8. For types A , B , G there are standard braided vector spaces not of Cartan type; these are listed in Propositions 3.9, 3.10 and 3.11. Those of type A_2 and B_2 appeared already in [Graña 2000]. Our classification does not rely on [Heckenberger 2006a], but we can identify our examples in the tables of that reference.
- We describe a concrete PBW (Poincaré–Birkhoff–Witt) basis of the Nichols algebra of a standard braided vector space as in the previous point; this follows from the general theory of Kharchenko [1999] together with [Heckenberger 2006b, Theorem 1]. As an application, we give closed formulas for the dimension of these Nichols algebras.
- We present a concrete set of defining relations of the Nichols algebras of standard braided vector spaces as in the previous points. This is an answer to the second part of Question 1 in the standard case. We note that this seems to be new even for Cartan type, for some values of the roots of unity appearing in the picture. Essentially, these relations are either quantum Serre relations or powers of root vectors; but in some cases, there are some substitutes of the quantum Serre relations due to the smallness of the intervening root vectors. Some of these substitutes can be recognized already in the relations in [Andruskiewitsch and Dăscălescu 2005].

Here is the plan of this article. We start by collecting necessary tools. Namely, we recall the definition of Lyndon words and give some properties about them, such as the Shirshov decomposition, in Section 1A. Next, in Section 1B, we discuss the notions of hyperletter and hyperword, following [Kharchenko 1999] (where they are called superletter and superword); these are certain iterations of braided commutators applied to Lyndon words. In Section 1C, a PBW basis is given for any quotient of the tensor algebra of a diagonal braided vector space V by a Hopf ideal using these hyperwords. This applies in particular to Nichols algebras.

In Section 2, after some technical preparations, we present a transformation of a braided graded Hopf algebra into another, with different space of degree one. This generalizes an analogous transformation for Nichols algebras given in [Heckenberger 2006b, Proposition 1]; see Section 2C.

In Section 3 we classify standard braided vector spaces with finite-dimensional Nichols algebra. In Section 3A, we prove that if the set of PBW generators is finite, the associated generalized Cartan matrix is of finite type. So in Section 3B we obtain all the standard braidings associated to Nichols algebras of finite dimension.

Section 4 is devoted to PBW bases of Nichols algebras of standard braided vector spaces with finite Cartan matrix. In Section 4A we prove that there is exactly one PBW generator whose degree corresponds to each positive root associated to the finite Cartan matrix. We give a set of PBW generators in Section 4B, following a nice presentation from [Lalonde and Ram 1995]. As a consequence, we compute the dimension in Section 4C.

The main result of this paper is the explicit presentation by generators and relations of Nichols algebras of standard braided vector spaces with finite Cartan matrix, given in Section 5. It relies on the explicit PBW basis and transformation described in Section 2C. Section 5A states some relations for Nichols algebras of standard braidings and proves facts about the coproduct. Sections 5B–5D contain the explicit presentation for types A_θ , B_θ and G_2 , respectively. For this, we establish relations among the elements of the PBW basis, inspired in [Andruskiewitsch and Dăscălescu 2005] and [Graña 2000]. We finally prove the presentation in the case of Cartan type in Section 5E. To our knowledge, this is the first self-contained exposition of Nichols algebras of braided vector spaces of Cartan type.

Notation. We fix an algebraically closed field k of characteristic 0; all vector spaces, Hopf algebras and tensor products are considered over k .

For each $N > 0$, \mathbb{G}_N denotes the set of primitive N -th roots of unity in k .

Given $n \in \mathbb{N}$ and $q \in k$, $q \notin \bigcup_{0 \leq j \leq n} \mathbb{G}_j$, we define

$$\binom{n}{j}_q = \frac{(n)_q!}{(k)_q! (n-k)_q!}, \quad \text{where } (n)_q! = \prod_{j=1}^n (k)_q, \quad \text{and } (k)_q = \sum_{j=0}^{k-1} q^j.$$

We define

$$q_h(t) := \frac{t^h - 1}{t - 1} \in \mathbb{k}[t], \quad h \in \mathbb{N}; \quad q_\infty(t) := \frac{1}{1 - t} = \sum_{s=0}^{\infty} t^s \in \mathbb{k}[[t]].$$

For each $\theta \in \mathbb{N}$ and each $n = (n_1, \dots, n_\theta) \in \mathbb{Z}^\theta$, we set $x^n = x_1^{n_1} \dots x_\theta^{n_\theta} \in \mathbb{k}[[x_1^{\pm 1}, \dots, x_\theta^{\pm 1}]]$. For each \mathbb{Z}^θ -graded vector spaces \mathfrak{B} , we denote by $\mathcal{H}_{\mathfrak{B}} = \sum_{n \in \mathbb{Z}^\theta} \dim \mathfrak{B}^n x^n$ the Hilbert series associated to \mathfrak{B} .

Let $C = \bigoplus_{n \in \mathbb{N}_0} C_{i+j}$ be a \mathbb{N}_0 -graded coalgebra, with projections $\pi_n : C \rightarrow C_n$. Given $i, j \geq 0$, we denote by

$$\Delta_{i,j} := (\pi_i \otimes \pi_j) \circ \Delta : C_{i+j} \rightarrow C_i \otimes C_j,$$

the (i, j) -th component of the comultiplication.

1. PBW bases

Let A be an algebra, $P, S \subset A$ and $h : S \mapsto \mathbb{N} \cup \{\infty\}$. Let also $<$ be a linear order on S . Let us denote by $B(P, S, <, h)$ the set

$$\{p s_1^{e_1} \dots s_r^{e_r} : r \in \mathbb{N}_0, s_1 > \dots > s_r, s_i \in S, 0 < e_i < h(s_i), p \in P\}.$$

If $B(P, S, <, h)$ is a linear basis of A , then we say that $(P, S, <, h)$ is a set of *PBW generators* with height h , and that $B(P, S, <, h)$ is a *PBW basis* of A . Occasionally, we shall simply say that S is a PBW basis of A .

In this section, following [Kharchenko 1999], we describe an appropriate PBW basis of a braided graded Hopf algebra $\mathfrak{B} = \bigoplus_{n \in \mathbb{N}} \mathfrak{B}^n$ such that $\mathfrak{B}^1 \cong V$, where V is a braided vector space of diagonal type. This applies in particular, to the Nichols algebra $\mathfrak{B}(V)$. In Section 1A we recall the classical construction of Lyndon words. Let V be a vector space together with a fixed basis. Then there is a basis of the tensor algebra $T(V)$ by certain words satisfying a special condition, called Lyndon words. Each Lyndon word has a canonical decomposition as a product of a pair of smaller Lyndon words, called the Shirshov decomposition.

We briefly recall the notions of a braided vector space (V, c) of diagonal type and of a Nichols algebra in Section 1B. In Section 1C we recall the definition of the hyperletter $[l]_c$, for any Lyndon word l ; this is the braided commutator of the hyperletters corresponding to the words in the Shirshov decomposition. Hyperletters are a set of generators for a PBW basis of $T(V)$ and their classes form a PBW basis of \mathfrak{B} .

1A. Lyndon words. Let $\theta \in \mathbb{N}$. Let X be a set with θ elements and fix an enumeration x_1, \dots, x_θ of X ; this induces a total order on X . Let \mathbb{X} be the corresponding vocabulary (the set of words with letters in X) and consider the lexicographical order on \mathbb{X} .

Definition 1.1. An element $u \in \mathbb{X}$, $u \neq 1$, is called a *Lyndon word* if u is smaller than any of its proper ends; that is, if $u = vw$, $v, w \in \mathbb{X} - \{1\}$, then $u < w$. The set of Lyndon words is denoted by L .

We shall need the following properties of Lyndon words.

- (1) Let $u \in \mathbb{X} - X$. Then u is Lyndon if and only if for any representation $u = u_1u_2$, with $u_1, u_2 \in \mathbb{X}$ not empty, one has $u_1u_2 = u < u_2u_1$.
- (2) Any Lyndon word begins by its smallest letter.
- (3) If $u_1, u_2 \in L$, $u_1 < u_2$, then $u_1u_2 \in L$.

The basic Theorem about Lyndon words, due to Lyndon, says that any word $u \in \mathbb{X}$ has a unique decomposition

$$u = l_1l_2 \dots l_r, \quad (1-1)$$

with $l_i \in L$, $l_r \leq \dots \leq l_1$, as a product of nonincreasing Lyndon words. This is called the *Lyndon decomposition* of $u \in \mathbb{X}$; the $l_i \in L$ appearing in the decomposition (1-1) are called the *Lyndon letters* of u .

The lexicographical order of \mathbb{X} turns out to be the same as the lexicographical order in the Lyndon letters. Namely, if $v = l_1 \dots l_r$ is the Lyndon decomposition of v , then $u < v$ if and only if

- (i) the Lyndon decomposition of u is $u = l_1 \dots l_i$, for some $1 \leq i < r$, or
- (ii) the Lyndon decomposition of u is $u = l_1 \dots l_{i-1}ll'_{i+1} \dots l'_s$, for some $1 \leq i < r$, $s \in \mathbb{N}$ and l, l'_{i+1}, \dots, l'_s in L , with $l < l_i$.

Here is another useful characterization of Lyndon words.

Lemma 1.2 [Kharchenko 1999, p. 6]. *Let $u \in \mathbb{X} - X$. Then $u \in L$ if and only if there exist $u_1, u_2 \in L$ with $u_1 < u_2$ such that $u = u_1u_2$.*

Definition 1.3. Let $u \in L - X$. A decomposition $u = u_1u_2$, with $u_1, u_2 \in L$ such that u_2 is the smallest end among those proper nonempty ends of u is called the *Shirshov decomposition* of u .

Let $u, v, w \in L$ be such that $u = vw$. Then $u = vw$ is the Shirshov decomposition of u if and only if either $v \in X$, or else if $v = v_1v_2$ is the Shirshov decomposition of v , then $w \leq v_2$.

1B. Braided vector spaces of diagonal type and Nichols algebras. A braided vector space is a pair (V, c) , where V is a vector space and $c \in \text{Aut}(V \otimes V)$ is a solution of the braid equation

$$(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).$$

We extend the braiding to $c : T(V) \otimes T(V) \rightarrow T(V) \otimes T(V)$ in the usual way. If $x, y \in T(V)$, the braided commutator is

$$[x, y]_c := \text{multiplication} \circ (\text{id} - c)(x \otimes y). \quad (1-2)$$

Assume that $\dim V < \infty$ and pick a basis $X = \{x_1, \dots, x_\theta\}$ of V ; we may then identify $k\mathbb{X}$ with $T(V)$. We consider the following gradings of the algebra $T(V)$:

- (i) The usual \mathbb{N}_0 -grading $T(V) = \bigoplus_{n \geq 0} T^n(V)$. If ℓ denotes the length of a word in \mathbb{X} , then $T^n(V) = \bigoplus_{x \in \mathbb{X}, \ell(x)=n} kx$.
- (ii) Let $\mathbf{e}_1, \dots, \mathbf{e}_\theta$ be the canonical basis of \mathbb{Z}^θ . Then $T(V)$ is also \mathbb{Z}^θ -graded, where the degree is determined by $\deg x_i = \mathbf{e}_i$, $1 \leq i \leq \theta$.

A braided vector space (V, c) is of *diagonal type* with respect to the basis x_1, \dots, x_θ if there exist $q_{ij} \in k^\times$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $1 \leq i, j \leq \theta$. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow k^\times$ be the bilinear form determined by $\chi(\mathbf{e}_i, \mathbf{e}_j) = q_{ij}$, $1 \leq i, j \leq \theta$. Then

$$c(u \otimes v) = \chi(\deg u, \deg v) v \otimes u \quad (1-3)$$

for any $u, v \in \mathbb{X}$, where $q_{u,v} = \chi(\deg u, \deg v) \in k^\times$. In this case, the braided commutator satisfies a “braided” Jacobi identity as well as braided derivation properties, namely

$$[[u, v]_c, w]_c = [u, [v, w]_c]_c - \chi(\alpha, \beta) v [u, w]_c + \chi(\beta, \gamma) [u, w]_c v, \quad (1-4)$$

$$[u, v w]_c = [u, v]_c w + \chi(\alpha, \beta) v [u, w]_c, \quad (1-5)$$

$$[u v, w]_c = \chi(\beta, \gamma) [u, w]_c v + u [v, w]_c, \quad (1-6)$$

for any homogeneous $u, v, w \in T(V)$, of degrees $\alpha, \beta, \gamma \in \mathbb{N}^\theta$, respectively.

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter–Drinfeld module over H , where H is a Hopf algebra with bijective antipode. Any $V \in {}^H_H\mathcal{YD}$ becomes a braided vector space [Montgomery 1993]. If H is the group algebra of a finite abelian group, then any $V \in {}^H_H\mathcal{YD}$ is a braided vector space of diagonal type. Indeed, $V = \bigoplus_{g \in \Gamma, \chi \in \widehat{\Gamma}} V_g^\chi$, where $V_g^\chi = V^\chi \cap V_g$ with $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$ and $V^\chi = \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in \Gamma\}$. The braiding is given by $c(x \otimes y) = \chi(g)y \otimes x$, for all $x \in V_g, g \in \Gamma, y \in V^\chi, \chi \in \widehat{\Gamma}$.

Reciprocally, any braided vector space of diagonal type can be realized as a Yetter–Drinfeld module over the group algebra of an abelian group.

If $V \in {}^H_H\mathcal{YD}$, the tensor algebra $T(V)$ admits a unique structure of graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ such that $V \subseteq \mathcal{P}(V)$. Following [Andruskiewitsch and Schneider 2002b], we consider the class \mathfrak{S} of all the homogeneous two-sided ideals $I \subseteq T(V)$ such that

- I is generated by homogeneous elements of degree ≥ 2 ,

- I is a Yetter–Drinfeld submodule of $T(V)$, and
- I is a Hopf ideal: $\Delta(I) \subset I \otimes T(V) + T(V) \otimes I$.

The Nichols algebra $\mathfrak{B}(V)$ associated to V is the quotient of $T(V)$ by the maximal element $I(V)$ of \mathfrak{S} .

Let (V, c) be a braided vector space of diagonal type, and assume that $q_{ij} = q_{ji}$ for all i, j . Let Γ be the free abelian group of rank θ , with basis g_1, \dots, g_θ , and define the characters $\chi_1, \dots, \chi_\theta$ of Γ by

$$\chi_j(g_i) = q_{ij}, \quad 1 \leq i, j \leq \theta.$$

Consider V as a Yetter–Drinfeld module over $k\Gamma$ by defining $x_i \in V_{g_i}^{\chi_i}$.

Proposition 1.4 [Lusztig 1993, Proposition 1.2.3; Andruskiewitsch and Schneider 2002b, Proposition 2.10]. *Let $a_1, \dots, a_\theta \in k^\times$. There is a unique bilinear form $(\mid) : T(V) \times T(V) \rightarrow k$ such that $(1\mid 1) = 1$,*

$$(x_i \mid x_j) = \delta_{ij} a_i \text{ for all } i, j, \quad (1-7)$$

$$(x \mid yy') = (x_{(1)} \mid y)(x_{(2)} \mid y') \text{ for all } x, y, y' \in T(V) \quad (1-8)$$

$$(xx' \mid y) = (x \mid y_{(1)})(x' \mid y_{(2)}) \text{ for all } x, x', y \in T(V). \quad (1-9)$$

This form is symmetric and also satisfies

$$(x \mid y) = 0 \text{ for all } x \in T(V)_g, y \in T(V)_h, g, h \in \Gamma, g \neq h. \quad (1-10)$$

The quotient $T(V)/I(V)$, where

$$I(V) := \{x \in T(V) : (x \mid y) = 0 \text{ for all } y \in T(V)\}$$

is the radical of the form, is canonically isomorphic to the Nichols algebra of V . Thus, (\mid) induces a nondegenerate bilinear form on $\mathfrak{B}(V)$ denoted by the same name. \square

If (V, c) is of diagonal type, the ideal $I(V)$ is \mathbb{Z}^θ -homogeneous, since it is the radical of a bilinear form in which the different \mathbb{Z}^θ -homogeneous components are orthogonal; see [Andruskiewitsch and Schneider 2004, Proposition 2.10]. Hence $\mathfrak{B}(V)$ is \mathbb{Z}^θ -graded. The following statement, that we include for later reference, is well-known.

Lemma 1.5. *Let V be a braided vector space of diagonal type, and consider its Nichols algebra $\mathfrak{B}(V)$.*

- If q_{ii} is a root of unity of order $N > 1$, then $x_i^N = 0$.*
- If $i \neq j$, then $(\text{ad}_c x_i)^r(x_j) = 0$ if and only if*

$${}^{(r)}_{q_{ii}}! \prod_{0 \leq k \leq r-1} (1 - q_{ii}^k q_{ij} q_{ji}) = 0.$$

(c) If $i \neq j$ and $q_{ij}q_{ji} = q_{ii}^r$, for some $r \leq 0$, then $(\text{ad}_c x_i)^{1-r}(x_j) = 0$. \square

1C. PBW basis of a quotient of the tensor algebra by a Hopf ideal. Let (V, c) be a braided vector space with a basis $X = \{x_1, \dots, x_\theta\}$; identify $T(V)$ with $k\mathbb{X}$. There is an important graded endomorphism $[\]_c$ of $k\mathbb{X}$ given by

$$[u]_c := \begin{cases} u & \text{if } u = 1 \text{ or } u \in X; \\ [[v]_c, [w]_c]_c & \text{if } u \in L, \ell(u) > 1 \\ & \text{and } u = vw \text{ is the Shirshov decomposition;} \\ [u_1]_c \dots [u_t]_c & \text{if } u \in \mathbb{X} - L \text{ with Lyndon decomposition } u = u_1 \dots u_t. \end{cases}$$

Now assume that (V, c) is of diagonal type with respect to the basis x_1, \dots, x_θ , with matrix (q_{ij}) .

Definition 1.6. The *hyperletter* corresponding to $l \in L$ is the element $[l]_c$. A *hyperword* is a word in hyperletters, and a *monotone hyperword* is a hyperword of the form $W = [u_1]_c^{k_1} \dots [u_m]_c^{k_m}$, where $u_1 > \dots > u_m$.

Remark 1.7. If $u \in L$, then $[u]_c$ is a homogeneous polynomial with coefficients in $\mathbb{Z}[q_{ij}]$ and $[u]_c \in u + k\mathbb{X}_{>u}^{\ell(u)}$.

The hyperletters inherit the order from the Lyndon words; this induces in turn an ordering in the hyperwords (the lexicographical order on the hyperletters). Now, given monotone hyperwords W, V , it can be shown that

$$W = [w_1]_c \dots [w_m]_c > V = [v_1]_c \dots [v_t]_c,$$

where $w_1 \geq \dots \geq w_r$, $v_1 \geq \dots \geq v_s$, if and only if

$$w = w_1 \dots w_m > v = v_1 \dots v_t.$$

Furthermore, the principal word of the polynomial W , when decomposed as sum of monomials, is w with coefficient 1.

Theorem 1.8 [Rosso 1999]. *Let $m, n \in L$, with $m < n$. Then the braided commutator $[[m]_c, [n]_c]_c$ is a $\mathbb{Z}[q_{ij}]$ -linear combination of monotone hyperwords $[l_1]_c, \dots, [l_r]_c, l_i \in L$, such that*

- the hyperletters of those hyperwords satisfy $n > l_i \geq mn$,
- $[mn]_c$ appears in the expansion with a nonzero coefficient, and
- any hyperword appearing in this decomposition satisfies

$$\deg(l_1 \dots l_r) = \deg(mn). \quad \square$$

A crucial result of Rosso describes the behavior of the coproduct of $T(V)$ in the basis of hyperwords.

Lemma 1.9 [Rosso 1999]. *Let $u \in \mathbb{X}$, and $u = u_1 \dots u_r v^m$, $v, u_i \in L$, $v < u_r \leq \dots \leq u_1$ the Lyndon decomposition of u . Then*

$$\Delta([u]_c) = 1 \otimes [u]_c + \sum_{i=0}^m \binom{m}{i}_{q_{v,v}} [u_1]_c \dots [u_r]_c [v]_c^i \otimes [v]_c^{m-i} \\ + \sum_{\substack{l_1 \geq \dots \geq l_p > v, l_i \in L \\ 0 \leq j \leq m}} x_{l_1, \dots, l_p}^{(j)} \otimes [l_1]_c \dots [l_p]_c [v]_c^j,$$

where each $x_{l_1, \dots, l_p}^{(j)}$ is \mathbb{Z}^θ -homogeneous and

$$\deg(x_{l_1, \dots, l_p}^{(j)}) + \deg(l_1 \dots l_p v^j) = \deg(u). \quad \square$$

As in [Ufer 2004], we consider another order in \mathbb{X} ; it is implicit in [Kharchenko 1999].

Definition 1.10. Let $u, v \in \mathbb{X}$. We say that $u > v$ if and only if either $\ell(u) < \ell(v)$, or else $\ell(u) = \ell(v)$ and $u > v$ (lexicographical order). This $>$ is a total order, called the *deg-lex order*.

Note that the empty word 1 is the maximal element for $>$. Also, this order is invariant by right and left multiplication.

Let now I be a proper ideal of $T(V)$, and set $R = T(V)/I$. Let $\pi : T(V) \rightarrow R$ be the canonical projection. Consider the subset of \mathbb{X} given by

$$G_I := \{u \in \mathbb{X} : u \notin k\mathbb{X}_{>u} + I\}.$$

- (a) If $u \in G_I$ and $u = vw$, then $v, w \in G_I$.
- (b) Any word $u \in G_I$ factorizes uniquely as a nonincreasing product of Lyndon words in G_I .

Proposition 1.11 ([Kharchenko 1999]; see also [Rosso 1999]). *The set $\pi(G_I)$ is a basis of R .* □

In what follows, I is a Hopf ideal. We seek to find a PBW basis by hyperwords of the quotient R of $T(V)$. For this, we look at the set

$$S_I := G_I \cap L. \quad (1-11)$$

We then define the function $h_I : S_I \rightarrow \{2, 3, \dots\} \cup \{\infty\}$ by

$$h_I(u) := \min \{t \in \mathbb{N} : u^t \in k\mathbb{X}_{>u^t} + I\}. \quad (1-12)$$

The next result plays a fundamental role in this paper.

Theorem 1.12 [Kharchenko 1999]. *Keep the notation above. Then*

$$B'_I := B(\{1 + I\}, [S_I]_c + I, \langle, h_I)$$

is a PBW basis of $H = T(V)/I$. \square

The next three results are consequences of Theorem 1.12; see [Kharchenko 1999] for their proofs.

Corollary 1.13. *A word u belongs to G_I if and only if the corresponding hyperletter $[u]_c$ is not a linear combination, modulo I , of hyperwords $[w]_c$, $w \succ u$, where all the hyperwords have their hyperletters in S_I .* \square

Proposition 1.14. *In the conditions of the Theorem 1.12, if $v \in S_I$ is such that $h_I(v) < \infty$, then $q_{v,v}$ is a root of unity. In this case, if t is the order of $q_{v,v}$, then $h_I(v) = t$.* \square

Corollary 1.15. *If $h_I(v) := h < \infty$, then $[v]^h$ is a linear combination of hyperwords $[w]_c$, $w \succ v^h$.* \square

2. Transformations of braided graded Hopf algebras

In Section 2C, we shall introduce a transformation over certain graded braided Hopf algebras, generalizing [Heckenberger 2006b, Proposition 1]. It is an instrumental step in the proof of Theorem 5.25, one of the main results of this article.

2A. Preliminaries on braided graded Hopf algebras. Let H be the group algebra of an abelian group Γ . Let $V \in {}^H_H\mathcal{YD}$ with a basis $X = \{x_1, \dots, x_\theta\}$ such that $x_i \in V_{g_i}^{x_i}$, $1 \leq i \leq \theta$. Let $q_{ij} = \chi_j(g_i)$, so that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $1 \leq i, j \leq \theta$.

We fix an ideal I in the class \mathfrak{S} ; we assume that I is \mathbb{Z}^θ -homogeneous. Let $\mathfrak{B} := T(V)/I$: this is a braided graded Hopf algebra, $\mathfrak{B}^0 = \mathbb{k}1$ and $\mathfrak{B}^1 = V$. By definition of $I(V)$, there exists a canonical epimorphism of braided graded Hopf algebras $\pi : \mathfrak{B} \rightarrow \mathfrak{B}(V)$. Let $\sigma_i : \mathfrak{B} \rightarrow \mathfrak{B}$ be the algebra automorphism given by the action of g_i .

For the proof of the next result, see [Andruskiewitsch and Schneider 2002b, 2.8], for example.

Proposition 2.1. (1) *For each $1 \leq i \leq \theta$, there exists a uniquely determined (id, σ_i) -derivation $D_i : \mathfrak{B} \rightarrow \mathfrak{B}$ with $D_i(x_j) = \delta_{i,j}$ for all j .*

(2) *$I = I(V)$ if and only if $\bigcap_{i=1}^\theta \ker D_i = \mathbb{k}1$.* \square

These operators are defined for each $x \in \mathfrak{B}^k$, $k \geq 1$ by the formula

$$\Delta_{n-1,1}(x) = \sum_{i=1}^{\theta} D_i(x) \otimes x_i.$$

Analogously, we can define operators $F_i : \mathfrak{B} \rightarrow \mathfrak{B}$ by $F_i(1) = 0$ and

$$\Delta_{1,n-1}(x) = \sum_{i=1}^{\theta} x_i \otimes F_i(x) \quad \text{for all } x \in \bigoplus_{k>0} \mathfrak{B}^k.$$

Let χ be as in Section 1B. Consider the action \triangleright of $k\mathbb{Z}^\theta$ on \mathfrak{B} given by

$$\mathbf{e}_i \triangleright b = \chi(\mathbf{u}, \mathbf{e}_i)b, \quad b \text{ homogeneous of degree } \mathbf{u} \in \mathbb{Z}^\theta. \quad (2-1)$$

Such operators F_i satisfy $F_i(x_j) = \delta_{i,j}$ for all j , and

$$F_i(b_1 b_2) = F_i(b_1)b_2 + (\mathbf{e}_i \triangleright b_1)F_i(b_2), \quad b_1, b_2 \in \mathfrak{B}.$$

Let $z_r^{(ij)} := (\text{ad}_c x_i)^r(x_j)$, $i, j \in \{1, \dots, \theta\}$, $i \neq j$ and $r \in \mathbb{N}_0$.

Remark 2.2. The operators D_i, F_i satisfy

$$D_i(x_i^n) = (n)_{q_{ii}} x_i^{n-1}, \quad (2-2)$$

$$D_i((\text{ad}_c x_i)^r(x_{j_1} \dots x_{j_s})) = 0 \text{ for } r, s \geq 0, \quad j_k \neq i, \quad (2-3)$$

$$D_j(z_r^{(ij)}) = \prod_{k=0}^{r-1} (1 - q_{ii}^k q_{ij} q_{ji}) x_i^r \text{ for } r \geq 0, \quad (2-4)$$

$$F_i(z_m^{(ij)}) = (m)_{q_{ii}} (1 - q_{ii}^{m-1} q_{ij} q_{ji}) z_{m-1}^{(ij)}, \quad (2-5)$$

$$F_j(z_m^{(ij)}) = 0, \quad m \geq 1. \quad (2-6)$$

The proof of the first three identities is as in [Andruskiewitsch and Schneider 2004, Lemma 3.7]; the proof of the last two is by induction on m .

For each pair $1 \leq i, j \leq \theta$, $i \neq j$, we define

$$M_{i,j}(\mathfrak{B}) := \{(\text{ad}_c x_i)^m(x_j) : m \in \mathbb{N}\}; \quad (2-7)$$

$$m_{ij} := \min \{m \in \mathbb{N}_0 : (m+1)_{q_{ii}} (1 - q_{ii}^m q_{ij} q_{ji}) = 0\}. \quad (2-8)$$

Then either $q_{ii}^{m_{ij}} q_{ij} q_{ji} = 1$, or $q_{ii}^{m_{ij}+1} = 1$, if $q_{ii}^m q_{ij} q_{ji} \neq 1$ for all $m = 0, 1, \dots, m_{ij}$, or such m_{ij} does not exist, in which case we consider $m_{ij} = \infty$.

If $\mathfrak{B} = \mathfrak{B}(V)$, we write simply $M_{i,j} = M_{i,j}(\mathfrak{B}(V))$. Note that $(\text{ad}_c x_i)^{m_{ij}+1} x_j = 0$ and $(\text{ad}_c x_i)^{m_{ij}} x_j \neq 0$, by Lemma 1.5, so

$$|M_{i,j}| = m_{ij} + 1.$$

By Theorem 1.12, the braided graded Hopf algebra \mathfrak{B} has a PBW basis consisting of homogeneous elements (with respect to the \mathbb{Z}^θ -grading). As in [Heckenberger 2006b], we can even assume that

- ⊗ The height of a PBW generator $[u]$, $\deg(u) = d$, is finite if and only if $2 \leq \text{ord}(q_{u,u}) < \infty$, and in such case, $h_{I(V)}(u) = \text{ord}(q_{u,u})$.

This is possible because if the height of $[u]$, $\deg(u) = d$, is finite, then $2 \leq \text{ord}(q_{u,u}) = m < \infty$, by Proposition 1.14. And if $2 \leq \text{ord}(q_{u,u}) = m < \infty$, but $h_{I(V)}(u)$ is infinite, we can add $[u]^m$ to the PBW basis: in this case, $h_{I(V)}(u) = \text{ord}(q_{u,u})$, and $q_{u^m, u^m} = q_{u,u}^{m^2} = 1$.

Let $\Delta^+(\mathfrak{B}) \subseteq \mathbb{N}^n$ be the set of degrees of the generators of the PBW basis, counted with their multiplicities and let also $\Delta(\mathfrak{B}) = \Delta^+(\mathfrak{B}) \cup (-\Delta^+(\mathfrak{B}))$: $\Delta^+(\mathfrak{B})$ is independent of the choice of the PBW basis with the property \circledast (see [Andruskiewitsch and Angiono 2008, Lemma 2.18] for a proof of this statement).

In what follows, we write

$$q_\alpha := \chi(\alpha, \alpha), \quad N_\alpha := \text{ord } q_\alpha, \quad \alpha \in \Delta^+(\mathfrak{B}).$$

2B. Auxiliary results. Let I be \mathbb{Z}^θ -homogeneous ideal in \mathfrak{S} and $\mathfrak{B} = T(V)/I$ as in Section 2A. We shall use repeatedly the following fact.

In what follows, we use the convention $\text{ord } 1 = 1$.

Remark 2.3. If $x_i^N = 0$ in \mathfrak{B} with N minimal (this is called the order of nilpotency of x_i), then q_{ii} is a root of 1 of order N . Hence $(\text{ad}_c x_i)^N x_j = 0$.

The following result extends (18) in the proof of [Heckenberger 2006b, Proposition 1].

Lemma 2.4. For $i \in \{1, \dots, \theta\}$, let \mathcal{K}_i be the subalgebra generated by $\bigcup_{j \neq i} M_{i,j}(\mathfrak{B})$ and denote by n_i the order of q_{ii} . Then there are isomorphisms of graded vector spaces

- $\ker(D_i) \cong \mathcal{K}_i \otimes \mathbb{k}[x_i^{n_i}]$, if $1 < \text{ord } q_{ii} < \infty$ but x_i is not nilpotent, or
- $\ker(D_i) \cong \mathcal{K}_i$, if $\text{ord } q_{ii}$ is the order of nilpotency of x_i or $q_{ii} = 1$.

Moreover,

$$\mathfrak{B} \cong \mathcal{K}_i \otimes \mathbb{k}[x_i]. \quad (2-9)$$

Proof. We assume for simplicity $i = 1$ and consider the PBW basis obtained in the Theorem 1.12. Now $x_1 \in S_I$, and it is the least element of S_I , so each element of B'_I is of the form $[u_1]^{s_1} \dots [u_k]^{s_k} x_1^s$, with $u_k < \dots < u_1, u_i \in S_I \setminus \{x_1\}, 0 < s_i < h_I(u_i), 0 \leq s < h_I(x_1)$. Call $S' = S_I \setminus \{x_1\}$, and

$$B_2 := B(1 + I, [S']_c + I, \langle, h_I|_{S'}),$$

that is, the PBW set generated by $[S']_c + I$, whose height is the restriction of the height of the PBW basis corresponding to S' . We have

$$\mathfrak{B} \cong \mathbb{k}B_2 \otimes \mathbb{k}[x_1].$$

By (2-3), any $(\text{ad}_c x_1)^r(x_j) \in \ker(D_1)$; as D_1 is a skew-derivation, we have $\mathcal{K}_1 \subseteq \ker(D_1)$.

Also, $\text{ad}_c x_1$ is a (σ_1, id) -derivation of \mathfrak{B} . This derivation restricts to an endomorphism of the algebra \mathcal{H}_1 , because if we apply $\text{ad}_c x_1$ to the generators of \mathcal{H}_1 , we obtain another generators (or 0).

We shall prove by induction on the length of u that $[u]_c \in \mathcal{H}_1$ for each $u \in L \setminus \{x_1\}$. If $u = x_j$, $j > 1$, then $[u]_c = x_j \in \mathcal{H}_1$. Now let $u \in L \setminus \{x_1\}$ be of length greater than 1, and (v, w) its Lyndon decomposition. Then:

- If $v \neq x_1$, then $[v]_c, [w]_c \in \mathcal{H}_1$ by induction hypothesis, so

$$[u]_c = [v]_c[w]_c - \chi(\deg v, \deg w)[w]_c[v]_c \in \mathcal{H}_1,$$

because \mathcal{H}_1 is a subalgebra.

- If $v = x_1$, then $[u]_c = \text{ad}_c x_1([w]_c) \subset \text{ad}_c x_1(\mathcal{H}_1) \subseteq \mathcal{H}_1$, because by induction hypothesis $[w]_c \in \mathcal{H}_1$.

Then we prove that $[L]_c \setminus \{x_1\} \subseteq \mathcal{H}_1$, and B_2 is generated by $[L]_c \setminus \{x_1\}$; that is, $\mathbb{k}B_2 \subseteq \mathcal{H}_1$, and $D_1(B_2) = 0$.

If $u \in \ker(D_1)$, we can write $[u]_c = \sum_{w \in B'_1} \alpha_w [w]_c$. If w does not end with x_1 , then $w \in B_2$, and $D_1([w]_c) = 0$. But if $w = u_w x_1^{t_w}$, $[u_w]_c \in B_2$, $0 < t_w < h_I(x_1)$, we have

$$D_1([u]_c) = (t_w)_{q_{11}^{-1}} [u_w]_c x_1^{t_w-1},$$

where $(t_w)_{q_{11}^{-1}} \neq 0$ if n_i does not divide t_w . Then

$$0 = D_1([u]_c) = \sum_{w \in B'_1/t_w > 0} \alpha_w (t_w)_{q_{11}^{-1}} [u_w]_c x_1^{t_w-1},$$

But $[u_w]_c x_1^{t_w-1} \in B_2$, and B_2 is a basis, so $\alpha_w = 0$ for each w such that n_i does not divide t_w . Then $\ker(D_1) = \mathcal{H}_1 \mathbb{k}[x_i^{n_i}]$, so $\ker(D_1) \simeq \mathcal{H}_1 \otimes \mathbb{k}[x_i^{n_i}]$ as \mathbb{k} -vector spaces. This fact and the first part conclude the proof. \square

2C. Transformations of certain braided graded Hopf algebras. Let I be \mathbb{Z}^θ -homogeneous ideal in \mathfrak{S} and $\mathfrak{B} = T(V)/I$ as in the previous subsections. We fix $i \in \{1, \dots, \theta\}$.

Remark 2.5. $\text{ord } q_{ii} = \min\{k \in \mathbb{N} : F_i^k = 0\}$, if $q_{ii} \neq 1$.

Proof. If $k \in \mathbb{N}$, then $F_i(x_i^k) = (k)_{q_{ii}} x_i^{k-1}$, and for all $k \in \mathbb{N}$,

$$F_i^k(x_i^k) = (k)_{q_{ii}^{-1}}!$$

That is, if $F_i^k = 0$, then $(k)_{q_{ii}^{-1}}! = 0$. Hence $\text{ord } q_{ii} \leq \min\{k \in \mathbb{N} : F_i^k = 0\}$.

Reciprocally, if q_{ii} is a root of unity of order k , then $F_i^k(x_i^t) = 0$ for all $t \geq k$ by the previous claim, and $F_i^k(x_i^t) = 0$ for all $t < k$ by degree arguments. Since $F_i(x_j) = 0$ for $j \neq i$, $F_i^k = 0$. \square

We now extend some considerations in [Heckenberger 2006b, p. 180]. We consider the Hopf algebra defined by

$$H_i := \begin{cases} \mathbb{k}\langle y, e_i, e_i^{-1} | e_i y - q_{ii}^{-1} y e_i, y^{N_i} \rangle & \text{where } N_i \text{ is the order of nilpotency} \\ & \text{of } x_i \text{ in } \mathfrak{B}, \text{ if } x_i \text{ is nilpotent,} \\ \mathbb{k}\langle y, e_i, e_i^{-1} | e_i y - q_{ii}^{-1} y e_i \rangle & \text{if } x_i \text{ is not nilpotent,} \end{cases}$$

together with $\Delta(e_i) = e_i \otimes e_i$, $\Delta(y) = e_i \otimes y + y \otimes 1$.

Notice that Δ is well-defined by Remark 2.3. We also consider the action \triangleright of H_i on \mathfrak{B} given by

$$e_i \triangleright b = \chi(\mathbf{u}, \mathbf{e}_i) b, \quad y \triangleright b = F_i(b),$$

if b is homogeneous of degree $\mathbf{u} \in \mathbb{N}^\theta$, extending the previous one defined in (2-1). The action is well-defined by Remark 2.3 and because

$$(e_i y) \triangleright b = e_i \triangleright (F_i(b)) = q_{ii}^{-1} F_i(e_i \triangleright b) = (q_{ii}^{-1} y e_i) \triangleright b \quad \text{for } b \in \mathfrak{B}.$$

It is easy to see that \mathfrak{B} is an H_i -module algebra; hence we can form

$$\mathcal{A}_i := \mathfrak{B} \# H_i.$$

Also, if we denote explicitly by \cdot the multiplication in \mathcal{A}_i , we have

$$(1 \# y) \cdot (b \# 1) = (e_i \triangleright b \# 1) \cdot (1 \# y) + F_i(b) \# 1 \quad \text{for all } b \in \mathfrak{B}. \quad (2-10)$$

As in [Heckenberger 2006b], \mathcal{A}_i is a left Yetter–Drinfeld module over $\mathbb{k}\Gamma$, where the action and the coaction are given by

$$\begin{aligned} g_k \cdot x_j \# 1 &= q_{kj} x_j \# 1, & g_k \cdot 1 \# y &= q_{ki}^{-1} 1 \# y, & g_k \cdot 1 \# e_i &= 1 \# e_i, \\ \delta(x_j \# 1) &= g_j \otimes x_j \# 1, & \delta(1 \# y) &= g_i^{-1} \otimes 1 \# y, & \delta(1 \# e_i) &= 1 \otimes 1 \# e_i, \end{aligned}$$

for each pair $k, j \in \{1, \dots, \theta\}$. Also, \mathcal{A}_i is a $\mathbb{k}\Gamma$ -module algebra.

We now prove a generalization of [Heckenberger 2006b, Proposition 1] in the more general context of our braided Hopf algebras \mathfrak{B} . Although the general strategy of the proof is similar as in *loc. cit.*, many points need slightly different argumentations here.

Theorem 2.6. *Keep the notation above. Assume that $M_{i,j}(\mathfrak{B})$ is finite and*

$$|M_{i,j}(\mathfrak{B})| = m_{ij} + 1, \quad j \in \{1, \dots, \theta\}, j \neq i. \quad (2-11)$$

(1) *Let V_i be the vector subspace of \mathcal{A}_i generated by*

$$\{(\text{ad}_c x_i)^{m_{ij}}(x_j) \# 1 : j \neq i\} \cup \{1 \# y\}.$$

The subalgebra $s_i(\mathfrak{B})$ of \mathcal{A}_i generated by V_i is a graded algebra such that $s_i(\mathfrak{B})^1 \cong V_i$. There exist skew derivations $Y_i : s_i(\mathfrak{B}) \rightarrow s_i(\mathfrak{B})$ such that, for all $b_1, b_2 \in s_i(\mathfrak{B})$, and $l, j \in \{1, \dots, \theta\}$, $j \neq i$,

$$Y_j(b_1 b_2) = b_1 Y_j(b_2) + Y_j(b_2)(g_i^{-m_{ij}} g_j^{-1} \cdot b_2), \quad (2-12)$$

$$Y_i(b_1 b_2) = b_1 Y_i(b_2) + Y_i(b_1)(g_i^{-1} \cdot b_2), \quad (2-13)$$

$$Y_l((\text{ad}_c x_i)^{m_{ij}}(x_j)\#1) = \delta_{lj}, \quad Y_l(1\#y) = \delta_{li}. \quad (2-14)$$

(2) Set $N_i := \{n \in \mathbb{N} : n\mathbf{e}_i \in \Delta(\mathfrak{B})\}$ (by the previous remarks, $N_i = \{1\}$ or $N_i = \{1, h_i\}$). The Hilbert series of $s_i(\mathfrak{B})$ satisfies

$$\mathfrak{H}_{s_i(\mathfrak{B})} = \left(\prod_{\alpha \in \Delta^+(\mathfrak{B}) \setminus N_i \mathbf{e}_i} q_{h_\alpha}(X^{s_i(\alpha)}) \right) \left(\prod_{s \in N_i} q_{h_{s\mathbf{e}_i}}(x_i^s) \right). \quad (2-15)$$

Therefore, if $s_i(\mathfrak{B})$ is a graded braided Hopf algebra,

$$\Delta^+(s_i(\mathfrak{B})) = \{s_i(\Delta^+(\mathfrak{B})) \setminus -N_i \mathbf{e}_i\} \cup N_i \mathbf{e}_i.$$

(3) If $\mathfrak{B} = \mathfrak{B}(V)$, the algebra $s_i(\mathfrak{B})$ is isomorphic to the Nichols algebra $\mathfrak{B}(V_i)$.

Proof. (i) Note that V_i is a Yetter–Drinfeld submodule over $k\Gamma$ of \mathcal{A}_i . Now, $\mathcal{A}_i \cong \mathfrak{B} \otimes H_i$ as graded vector spaces. Let \mathfrak{H}_i be the subalgebra generated by $\bigcup_{j \neq i} M_{i,j}(\mathfrak{B})$, as in Lemma 2.4. Then $s_i(\mathfrak{B}) \subseteq \mathfrak{H}_i \otimes k[y]$, since F_i is a skew-derivation and $F_i(z_k^{(ij)}) = (k)_{q_{ii}}(1 - q_{ii}^{k-1} q_{ij} q_{ji}) z_{k-1}^{(ij)}$, by (2-5). From (2-10),

$$(1\#y) \cdot (z_{m_{ij}}^{(ij)}\#1) = (z_{m_{ij}}^{(ij)}\#1) \cdot (1\#y) + F_i(z_{m_{ij}}^{(ij)})\#1.$$

Also, since $m_{ij} + 1 = |M_{i,j}(\mathfrak{B})|$, we have $(m_{ij})_{q_{ii}}(1 - q_{ii}^{m_{ij}-1} q_{ij} q_{ji}) \neq 0$, so $z_{m_{ij}-1}^{(ij)}\#1$ lies in $s_i(\mathfrak{B})$, and by induction each $z_k^{(ij)}\#1$, for $k = 0, \dots, m_{ij} - 1$, is an element of $s_i(\mathfrak{B})$. Then $\mathfrak{H}_i \otimes k[y] \subseteq s_i(\mathfrak{B})$, and therefore

$$s_i(\mathfrak{B}) = \mathfrak{H}_i \otimes k[y]. \quad (2-16)$$

Thus, $s_i(\mathfrak{B})$ is a graded algebra in ${}_{k\Gamma}^{k\Gamma} \mathcal{YD}$ with $s_i(\mathfrak{B})^1 = V_i$. We have to find the skew derivations $Y_l \in \text{End}(s_i(\mathfrak{B}))$, $l = 1, \dots, \theta$. Set $Y_i := g_i^{-1} \circ \text{ad}(x_i\#1)|_{s_i(\mathfrak{B})}$. Then, for each $b \in \mathfrak{H}_i$ and each $j \neq i$,

$$\text{ad}(x_i\#1)(b\#1) = (\text{ad}_c x_i)(b)\#1,$$

$$\text{ad}(x_i\#1)((\text{ad}_c x_i)^{m_{ij}}(x_j)\#1) = (\text{ad}_c x_i)^{m_{ij}+1}(x_j)\#1 = 0.$$

Also,

$$\begin{aligned} Y_i(1\#y) &= g_i^{-1} \cdot ((x_i\#1) \cdot (1\#y) - (g_i \cdot (1\#y)) \cdot (x_i\#1)) \\ &= g_i^{-1} \cdot (x_i\#y + 1 - q_{ii}(q_{ii}^{-1} x_i\#y)) = 1. \end{aligned}$$

Thus $Y_i \in \text{End}(s_i(\mathfrak{B}))$ satisfies (2-14).

Therefore, $\text{ad}(x_i \# 1)(b_1 b_2) = \text{ad}(x_i \# 1)(b_1) b_2 + (g_i \cdot b_1) \text{ad}(x_i \# 1)(b_2)$, for each pair $b_1, b_2 \in s_i(\mathfrak{B})$, so we conclude that $\text{ad}(x_i \# 1)(s_i(\mathfrak{B})) \subseteq s_i(\mathfrak{B})$, and $Y_i \in \text{End}(s_i(\mathfrak{B}))$ satisfies (2-13).

Before proving that Y_i satisfies (2-12), we need to establish some preliminary facts. Let us fix $j \neq i$, and let $z_k^{(ij)} = (\text{ad}_c x_i)^k(x_j)$ as before. We define inductively

$$\hat{z}_0^{(ij)} := D_j, \quad \hat{z}_{k+1}^{(ij)} := D_i \hat{z}_k^{(ij)} - q_{ii}^k q_{ij} \hat{z}_{k+1}^{(ij)} D_i \in \text{End}(\mathfrak{B}).$$

We calculate

$$\begin{aligned} \lambda_{ij} &:= \hat{z}_{m_{ij}}^{(ij)}(z_{m_{ij}}^{(ij)}) = \sum_{s=0}^{m_{ij}} a_s D_i^{m_{ij}-s} D_j D_i^s(z_{m_{ij}}^{(ij)}) \\ &= (D_i)^{m_{ij}}(D_j)(z_{m_{ij}}^{(ij)}) = \alpha_{m_{ij}}(m_{ij})_{q_{ii}}! \in \mathbb{k}^\times, \end{aligned}$$

where $a_s = (-1)^k \binom{m}{k}_{q_{ii}} q_{ii}^{k(k-1)/2} q_{ij}^k$.

Note that $(D_i)^{m_{ij}+1} D_j(b) = 0$ for all $b \in M_{i,k}$, $k \neq i, j$, and that

$$(D_i)^{m_{ij}+1} D_j(z_r^{(ij)}) = (D_i)^{m_{ij}+1} (q_{ji}^{-r} \alpha_r x_i^r) = 0 \quad \text{for all } r \leq m_{ij},$$

so $(D_i)^{m_{ij}+1} D_j(\mathcal{H}_i) = 0$. This implies that $\hat{z}_{m_{ij}}^{(ij)}(b) \in \mathcal{H}_i$, for each $b \in \mathcal{H}_i$. Now define $Y_j \in \text{End}(s_i(\mathfrak{B}))$ by

$$Y_j(b \# y^m) := q_{ii}^{m m_{ij}} q_{ji}^m \lambda_{ij}^{-1} \hat{z}_{m_{ij}}^{(ij)}(b) \# y^m \quad \text{for } b \in \mathcal{H}_i, m \in \mathbb{N}.$$

We have $Y_j(1 \# y) = 0$, and moreover $Y_j((\text{ad}_c x_i)^{m_{il}}(x_l) \# 1) = 0$ if $l \neq i, j$. By the choice of λ_{ij} , $Y_j((\text{ad}_c x_i)^{m_{ij}}(x_j) \# 1) = 1$.

Now, using that $D_k(g_l \cdot b) = q_{kl} g_l \cdot (D_k(b))$ for each $b \in \mathfrak{B}$ and $k, l \in \{1, \dots, \theta\}$, we prove inductively that for $b_1, b_2 \in \mathcal{H}_i$,

$$\hat{z}_k^{(ij)}(b_1 b_2) = b_1 \hat{z}_k^{(ij)}(b_2) + \hat{z}_k^{(ij)}(b_1)(g_i^k g_j \cdot b_2).$$

Hence,

$$\begin{aligned} Y_j(b_1 \# 1 \cdot b_2 \# 1) &= Y_j(b_1 b_2 \# 1) = \lambda_{ij}^{-1} \hat{z}_{m_{ij}}^{(ij)}(b_1 b_2) \# 1 \\ &= b_2 \# 1 \cdot Y_j(b_2 \# 1) + Y_j(b_1 \# 1) \cdot (g_i^{m_{ij}} g_j \cdot (b_2 \# 1)). \end{aligned}$$

By induction on the degree we prove that F_i commutes with D_i, D_j , so

$$\hat{z}_{m_{ij}}^{(ij)}(F_i(b)) = F_i(\hat{z}_{m_{ij}}^{(ij)}(b)) \quad \text{for all } b \in \mathfrak{B}.$$

Consider $b \in \mathcal{H}_i \subseteq \ker(D_i)$,

$$\begin{aligned} Y_j(b \# 1 \cdot 1 \# y) &= Y_j(b \# y) = q_{ii}^{m_{ij}} q_{ji} \hat{z}_{m_{ij}}^{(ij)}(b) \# y \\ &= b \# 1 \cdot Y_j(1 \# y) + Y_j(b \# 1) \cdot (g_i^{m_{ij}} g_j \cdot (1 \# y)), \end{aligned}$$

where we use that $Y_j(1\#y) = 0$. Since,

$$b_1\#1 \cdot b_2\#y^t = b_1\#1 \cdot b_2\#1 \cdot (1\#y)^t,$$

(2-12) is valid for products of this form. To prove it in the general case, note that

$$(b_1\#y^t) \cdot (b_2\#y^s) = (b_1\#1) \cdot (1\#y)^t \cdot (b_2\#y^s).$$

At this point, we have to prove (2-12) for $b \in \mathcal{K}_i \ker(D_i)$, $s \in \mathbb{N}$:

$$\begin{aligned} Y_j(1\#y \cdot b\#y^s) &= Y_j(F_i(b)\#y^s + (e_i \triangleright b\#y) \cdot 1\#y) \\ &= q_{ii}^{m_{ij}s} q_{ji}^s \lambda_{ij}^{-1} \hat{z}_{m_{ij}}^{(ij)}(F_i(b))\#y^s + q_{ii}^{m_{ij}(s+1)} q_{ji}^{s+1} \lambda_{ij}^{-1} \cdot \hat{z}_{m_{ij}}^{(ij)}(e_i \triangleright b)\#y^{s+1} \\ &= F_i(q_{ii}^{m_{ij}(s+1)} q_{ji}^{s+1} \lambda_{ij}^{-1} \hat{z}_{m_{ij}}^{(ij)}(b))\#y^s + q_{ii}^{m_{ij}} q_{ji} (e_i \triangleright (q_{ii}^{m_{ij}s} q_{ji}^s \lambda_{ij}^{-1} \hat{z}_{m_{ij}}^{(ij)}(b))\#y^s) \\ &= (1\#y) \cdot Y_j(b\#y^s) \\ &= 1\#y \cdot Y_j(b\#y^s) + Y_j(1\#y) \cdot (g_i^{m_{ij}} g_j \cdot b\#y^s), \end{aligned}$$

where we use that $\hat{z}_{m_{ij}}^{(ij)}(e_i \triangleright b) = q_{ii}^{m_{ij}} q_{ji} e_i \triangleright (\hat{z}_{m_{ij}}^{(ij)}(b))$.

(ii) The algebra H_i is \mathbb{Z}^θ -graded, with

$$\deg y = -\mathbf{e}_i, \quad \deg e_i^{\pm 1} = 0.$$

Since \mathfrak{B} and H_i are graded and (2-10) holds, the algebra \mathcal{A}_i is \mathbb{Z}^θ -graded.

Consider the abstract basis $\{u_j\}_{j \in \{1, \dots, \theta\}}$ of V_i . With the grading $\deg u_j = \mathbf{e}_j$, the algebra $\mathfrak{B}(V_i)$ is \mathbb{Z}^θ -graded. Consider also the algebra homomorphism $\Omega : T(V_i) \rightarrow s_i(\mathfrak{B})$ given by

$$\Omega(u_j) := \begin{cases} (\text{ad}_c x_i)^{m_{ij}}(x_j) & \text{if } j \neq i, \\ y & \text{if } j = i. \end{cases}$$

By part (i) of the theorem, Ω is an epimorphism, so it induces an isomorphism between $s_i(\mathfrak{B})' := T(V_i)/\ker \Omega$ and $s_i(\mathfrak{B})$, which we also denote by Ω . We have

$$\begin{aligned} \deg \Omega(u_j) &= \deg((\text{ad}_c x_i)^{m_{ij}}(x_j)) = \mathbf{e}_j + m_{ij}\mathbf{e}_i = s_i(\deg \mathbf{u}_j) \text{ if } j \neq i, \\ \deg \Omega(u_i) &= \deg(y) = -\mathbf{e}_i = s_i(\deg \mathbf{u}_i). \end{aligned}$$

Since Ω is an algebra homomorphism, we have $\deg(\Omega(\mathbf{u})) = s_i(\deg(\mathbf{u}))$ for all $\mathbf{u} \in s_i(\mathfrak{B})'$. Since $s_i^2 = \text{id}$, $s_i(\deg(\Omega(\mathbf{u}))) = \deg(\mathbf{u})$ for all $\mathbf{u} \in s_i(\mathfrak{B})'$, and $\mathfrak{H}_{s_i}(\mathfrak{B})' = s_i(\mathfrak{H}_{s_i}(\mathfrak{B}))$.

From this point on, the proof goes exactly as in [Andruskiewitsch and Angiono 2008, Theorem 3.2].

(iii) This is Proposition 1 in [Heckenberger 2006b]. \square

By Theorem 2.6, the initial braided vector space with matrix $(q_{kj})_{1 \leq k, j \leq \theta}$ is transformed into another braided vector space of diagonal type V_i , with matrix $(\tilde{q}_{kj})_{1 \leq k, j \leq \theta}$, where $\tilde{q}_{jk} = q_{ii}^{m_{ij}m_{ik}} q_{ik}^{m_{ij}} q_{ji}^{m_{ik}} q_{jk}$ for $j, k \in \{1, \dots, \theta\}$.

If $j \neq i$, then $\tilde{m}_{ij} = \min \{m \in \mathbb{N} : (m+1)_{\tilde{q}_{ii}} (\tilde{q}_{ii}^m \tilde{q}_{ij} \tilde{q}_{ji} = 0)\} = m_{ij}$.

For later use in Section 5, we recall a result from [Andruskiewitsch et al. 2008], adapted to diagonal braided vector spaces.

Lemma 2.7 [Andruskiewitsch et al. 2008, Lemma 2.8(ii)]. *Let V be a diagonal braided vector space and I a \mathbb{Z}^θ -homogeneous ideal of $T(V)$. Set $\mathfrak{B} := T(V)/I$ and assume that for all $i \in \{1, \dots, \theta\}$ there exist (id, σ_i) -derivations $D_i : \mathfrak{B} \rightarrow \mathfrak{B}$ with $D_i(x_j) = \delta_{i,j}$ for all j . Then $I \subseteq I(V)$. \square*

That is, the canonical surjective algebra morphisms from $T(V)$ onto \mathfrak{B} and $\mathfrak{B}(V)$ induce a surjective algebra morphism $\mathfrak{B} \rightarrow \mathfrak{B}(V)$.

3. Standard braidings

Heckenberger [2006a] has classified diagonal braidings whose set of PBW generators is finite. Standard braidings form an special subclass, which includes properly braidings of Cartan type.

We first recall the definition of a standard braiding from [Andruskiewitsch and Angiono 2008], and the notion of a Weyl groupoid, introduced in [Heckenberger 2006b]. Then we present the classification of standard braidings, and compare them with [Heckenberger 2006a].

Like Heckenberger, we use the *generalized Dynkin diagram associated to a braided vector space of diagonal type*, with matrix $(q_{ij})_{1 \leq i, j \leq \theta}$: this is a graph with θ vertices, each labeled with the corresponding q_{ii} , and an edge between two vertices i, j labeled with $q_{ij}q_{ji}$ if this scalar is different from 1. So two braided vector spaces of diagonal type have the same generalized Dynkin diagram if and only if they are twist equivalent. We shall assume that the generalized Dynkin diagram is connected, by [Andruskiewitsch and Schneider 2000, Lemma 4.2].

Summarizing, the main result of this section says:

Theorem 3.1. *Any standard braiding is twist equivalent with one or more of*

- a braiding of Cartan type,
- a braiding of type A_θ listed in Proposition 3.9,
- a braiding of type B_θ listed in Proposition 3.10, or
- a braiding of type G_2 listed in Proposition 3.11.

The generalized Dynkin diagrams appearing in Propositions 3.9 and 3.10 correspond to rows 1, 2, 3, 4, 5, 6 in [Heckenberger 2006a, Table C]. The generalized Dynkin diagrams in Proposition 3.11 are (T8) in [Heckenberger 2008, Section 3]. However, our classification does not rely on Heckenberger's papers.

3A. The Weyl groupoid and standard braidings. Let $E = (\mathbf{e}_1, \dots, \mathbf{e}_\theta)$ be the canonical basis of \mathbb{Z}^θ . Consider an arbitrary matrix $(q_{ij})_{1 \leq i, j \leq \theta} \in (\mathbb{k}^\times)^{\theta \times \theta}$, and fix once and for all the bilinear form $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow \mathbb{k}^\times$ determined by

$$\chi(\mathbf{e}_i, \mathbf{e}_j) = q_{ij}, \quad 1 \leq i, j \leq \theta. \quad (3-1)$$

If $F = (\mathbf{f}_1, \dots, \mathbf{f}_\theta)$ is another ordered basis of \mathbb{Z}^θ , then we set $\tilde{q}_{ij} = \chi(\mathbf{f}_i, \mathbf{f}_j)$, $1 \leq i, j \leq \theta$. We call (\tilde{q}_{ij}) the *braiding matrix with respect to the basis F* . Fix $i \in \{1, \dots, \theta\}$. If $1 \leq i, j \leq \theta$, we consider the set

$$\tilde{M}_{ij} := \{m \in \mathbb{N}_0 : (m+1)_{\tilde{q}_{ii}} (\tilde{q}_{ii}^m \tilde{q}_{ij} \tilde{q}_{ji} - 1) = 0\}.$$

If this set is nonempty, its minimal element is denoted \tilde{m}_{ij} (which of course depends on the basis F). Define also $\tilde{m}_{ii} = 2$. Let $s_{i,F} \in \text{GL}(\mathbb{Z}^\theta)$ be the pseudo-reflection given by $s_{i,F}(\mathbf{f}_j) := \mathbf{f}_j + \tilde{m}_{ij}\mathbf{f}_i$, for $j \in \{1, \dots, \theta\}$.

Let G be a group acting on a set X . We define the *transformation groupoid* as $G \times X$ with the operation given by $(g, x)(h, y) = (gh, y)$ if $x = h(y)$, but undefined otherwise.

Definition 3.2. Consider the set \mathfrak{X} of all ordered bases of \mathbb{Z}^θ , and the canonical action of $\text{GL}(\mathbb{Z}^\theta)$ over \mathfrak{X} . The *Weyl groupoid* $W(\chi)$ of the bilinear form χ is the smallest subgroupoid of the transformation groupoid $\text{GL}(\mathbb{Z}^\theta) \times \mathfrak{X}$ that satisfies following properties:

- $(\text{id}, E) \in W(\chi)$,
- if $(\text{id}, F) \in W(\chi)$ and $s_{i,F}$ is defined, then $(s_{i,F}, F) \in W(\chi)$.

Let $\mathfrak{P}(\chi) = \{F : (\text{id}, F) \in W(\chi)\}$ be the set of points of the groupoid $W(\chi)$. The set

$$\Delta(\chi) = \bigcup_{F \in \mathfrak{P}(\chi)} F. \quad (3-2)$$

is called the *generalized root system*¹ associated to χ .

We record for later use the following evident facts.

Remark 3.3. Take $i \in \{1, \dots, \theta\}$ such that $s_{i,E}$ is defined. Set $F = s_{i,E}(E)$ and let (\tilde{q}_{ij}) be the braiding matrix with respect to the basis F . Assume that

¹Following the traditional notation in the theory of Lie algebras, we should speak about systems of real roots, since in the case of braidings of symmetrizable Cartan type one would get just the real roots. But we prefer to follow the denomination in [Andruskiewitsch and Angiono 2008]

- $q_{ii} = -1$ (so $m_{ik} = 0$ if $q_{ik}q_{ki} = 1$ or $m_{ik} = 1$, for each $k \neq i$);
- there exists $j \neq i$ such that $q_{jj}q_{ji}q_{ij} = 1$ (that is, $m_{ij} = m_{ji} = 1$).

Then $\tilde{q}_{jj} = -1$.

Proof. Simply, $\tilde{q}_{jj} = q_{ii}q_{ij}q_{ji}q_{jj} = q_{ii} = -1$. \square

Remark 3.4. If the m_{ij} satisfy $q_{ii}^{m_{ij}} q_{ij}q_{ji} = 1$ for all $j \neq i$, the braiding of V_i is twist equivalent with the corresponding to V .

Define $\alpha : W(\chi) \rightarrow \text{GL}(\theta, \mathbb{Z})$ by $\alpha(s, F) = s$ if $(s, F) \in W(\chi)$, and denote by $W_0(\chi)$ the subgroup generated by the image of α .

Definition 3.5. [Andruskiewitsch and Angiono 2008] We say that χ is *standard* if for any $F \in \mathfrak{P}(\chi)$, the integers m_{rj} are defined, for all $1 \leq r, j \leq \theta$, and the integers m_{rj} for the bases $s_{i,F}(F)$ coincide with those for F for all i, r, j . Clearly it is enough to assume this for the canonical basis E .

We now assume that χ is standard. We set $C := (a_{ij}) \in \mathbb{Z}^{\theta \times \theta}$, where $a_{ij} = -m_{ij}$; this is a generalized Cartan matrix.

Proposition 3.6 [Andruskiewitsch and Angiono 2008]. $W_0(\chi) = \langle s_{i,E} : 1 \leq i \leq \theta \rangle$. Furthermore $W_0(\chi)$ acts freely and transitively on $\mathfrak{P}(\chi)$. \square

Hence, $W_0(\chi)$ is a Coxeter group, and $W_0(\chi)$ and $\mathfrak{P}(\chi)$ have the same cardinality.

Lemma 3.7 [Andruskiewitsch and Angiono 2008]. *The following are equivalent:*

- (1) *The groupoid $W(\chi)$ is finite.*
- (2) *The set $\mathfrak{P}(\chi)$ is finite.*
- (3) *The generalized root system $\Delta(\chi)$ is finite.*
- (4) *The group $W_0(\chi)$ is finite.*
- (5) *The Cartan matrix C is symmetrizable and of finite type.* \square

We shall prove in Theorem 4.1, that if $\Delta(\chi)$ is finite, the matrix C is symmetrizable, hence of finite type. Thus $\mathfrak{B}(V)$ is of finite dimension if and only if the Cartan matrix C is of finite type.

3B. Classification of standard braidings. We now classify standard braidings such that the Cartan matrix is of finite type. We begin with types C_θ, D_θ, E_l ($l = 6, 7, 8$) and F_4 : these standard braidings are necessarily of Cartan type.

Proposition 3.8. *Let V be a braided vector space of standard type, set $\theta = \dim V$, and let $C = (a_{ij})_{i,j \in \{1, \dots, \theta\}}$ be the corresponding Cartan matrix, of type C_θ, D_θ, E_l ($l = 6, 7, 8$) or F_4 . Then V is of Cartan type (associated to the corresponding matrix of finite type).*

Proof. Let V be standard of type C_θ , $\theta \geq 3$.

$$\circ^1 \text{ --- } \circ^2 \text{ --- } \circ^3 \dots \quad \circ^{\theta-2} \text{ --- } \circ^{\theta-1} \longleftarrow \circ^\theta \quad (3-3)$$

Note that $q_{\theta-1,\theta-1} \neq -1$ by Remark 3.3 and the assumption $m_{\theta-1,\theta} = 2$. Since $m_{\theta-1,\theta-2} = 1$, $q_{\theta-1,\theta-1}q_{\theta-1,\theta-2}q_{\theta-2,\theta-1} = 1$. Using Remark 3.3 when $i = \theta - 2$, $j = \theta - 1$, since $\tilde{q}_{\theta-1,\theta-1} \neq -1$ when we transform by $s_{\theta-2}$ (since the new braided vector space is also standard), we have $q_{\theta-2,\theta-2} \neq -1$, so

$$q_{\theta-2,\theta-2}q_{\theta-2,\theta-1}q_{\theta-1,\theta-2} = q_{\theta-2,\theta-2}q_{\theta-2,\theta-3}q_{\theta-3,\theta-2} = 1,$$

and $q_{\theta-1,\theta-1} = q_{\theta-2,\theta-2}$. Inductively,

$$q_{kk}q_{k,k-1}q_{k-1,k} = q_{kk}q_{k,k+1}q_{k+1,k} = q_{11}q_{12}q_{21} = 1, \quad k = 2, \dots, \theta - 1$$

and $q_{11} = q_{22} = \dots = q_{\theta-1,\theta-1}$. So we look at $q_{\theta\theta}$: since $m_{\theta,\theta-1} = 1$, we have $q_{\theta\theta} = -1$ or $q_{\theta\theta}q_{\theta,\theta-1}q_{\theta-1,\theta} = 1$. If $q_{\theta\theta} = -1$, transforming by s_θ , we have

$$\tilde{q}_{\theta-1,\theta-1} = -q^{-1}, \quad \tilde{q}_{\theta-1,\theta}\tilde{q}_{\theta,\theta-1} = q^2,$$

and $q^2 = -1$ since $m_{\theta-1,\theta-2} = 1$. Then

$$q_{\theta\theta}q_{\theta,\theta-1}q_{\theta-1,\theta} = 1, \quad q_{\theta\theta} = q^2,$$

and the braiding is of Cartan type in both cases.

Let V be standard of type D_θ , $\theta \geq 4$.

We prove the statement by induction on θ . Let V be standard of type D_4 , and suppose that $q_{22} = -1$. Let (\tilde{q}_{ij}) the braiding matrix with respect to $F = s_{2,E}(E)$. We calculate for each pair $j \neq k \in \{1, 3, 4\}$:

$$\tilde{q}_{jk}\tilde{q}_{kj} = ((-1)q_{2k}q_{j2}q_{jk})((-1)q_{2j}q_{k2}q_{kj}) = (q_{2k}q_{k2})(q_{2j}q_{j2}),$$

where we use that $q_{jk}q_{kj} = 1$. Since also $\tilde{q}_{jk}\tilde{q}_{kj} = 1$, we have $q_{2k}q_{k2} = (q_{2j}q_{j2})^{-1}$ for $j \neq k$, so $q_{2k}q_{k2} = -1$, $k = 1, 3, 4$, since $q_{2k}q_{k2} \neq 1$. In this case, the braiding is of Cartan type, with $q = -1$. Suppose then $q_{22} \neq -1$. From the fact that $m_{2j} = 1$, we have

$$q_{22}q_{2j}q_{j2} = 1, \quad j = 1, 3, 4.$$

For each j , applying Remark 3.3, we see that $q_{jj} \neq -1$ (since $\tilde{q}_{22} \neq -1$), so $q_{jj}q_{2j}q_{j2} = 1$, for $j = 1, 3, 4$, and the braiding is of Cartan type.

$$\begin{array}{ccc} \circ^1 \text{ --- } \circ^2 \text{ --- } \circ^3 \dots & & \circ^{\theta-2} \text{ --- } \circ^\theta \\ & & \downarrow \\ & & \circ_{\theta-1} \end{array} \quad (3-4)$$

We now suppose the statement valid for θ . Let V be a standard braided vector space of type $D_{\theta+1}$. The subspace generated by $x_2, \dots, x_{\theta+1}$ is a standard braided vector space associated to the matrix $(q_{ij})_{i,j=2,\dots,\theta+1}$, of type D_θ , so it is of Cartan type. To finish, apply Remark 3.3 with $i = 1, j = 2$, to conclude that V is of Cartan type with $q = -1$, or, if $q_{22} \neq -1$, we have $q_{11} \neq -1$ and $q_{11}q_{12}q_{21} = 1$, and in this case it is of Cartan type too (because also $q_{1k}q_{k1} = 1$ when $k > 2$).

Let V be standard of type E_6 . Note that 1, 2, 3, 4, 5 determine a braided vector subspace, which is standard of type D_5 , hence of Cartan type. To prove that $q_{66}q_{65}q_{56} = 1$, we use Remark 3.3 as above.

$$\begin{array}{cccccc} \circ^1 & \text{---} & \circ^2 & \text{---} & \circ^3 & \text{---} & \circ^5 & \text{---} & \circ^6 \\ & & & & | & & & & \\ & & & & \circ^4 & & & & \end{array} \quad (3-5)$$

If V is standard of type E_7 or E_8 , we proceed similarly by reduction to E_6 or E_7 , respectively.

$$\begin{array}{ccccccc} \circ^1 & \text{---} & \circ^2 & \text{---} & \circ^3 & \text{---} & \circ^4 & \text{---} & \circ^6 & \text{---} & \circ^7 \\ & & & & & & | & & & & \\ & & & & & & \circ^5 & & & & \end{array} \quad (3-6)$$

$$\begin{array}{cccccccc} \circ^1 & \text{---} & \circ^2 & \text{---} & \circ^3 & \text{---} & \circ^4 & \text{---} & \circ^5 & \text{---} & \circ^7 & \text{---} & \circ^8 \\ & & & & & & | & & & & & & \\ & & & & & & \circ^6 & & & & & & \end{array} \quad (3-7)$$

Let V be standard of type F_4 . Vertices 2, 3, 4 determine a braided subspace, which is standard of type C_3 , so the q_{ij} satisfy the corresponding relations. Let (\tilde{q}_{ij}) the braiding matrix with respect to $F = s_{2,E}(E)$. Since $\tilde{q}_{13}\tilde{q}_{31} = 1$ and $q_{22}q_{23}q_{32} = 1$, we have $q_{22}q_{12}q_{21} = 1$.

$$\circ^1 \text{ --- } \circ^2 \implies \circ^3 \text{ --- } \circ^4 \quad (3-8)$$

Now, if we suppose $q_{11} = -1$, applying Remark 3.3 we have $q_{22} = -1 = q_{21}q_{12}$, and the corresponding vector space is of Cartan type F_4 , associated to $q \in \mathbb{G}_4$. If $q_{11} \neq -1$, then $q_{11}q_{12}q_{21} = 1$, and the space it again is of Cartan type. \square

To finish the classification of standard braidings, we describe the standard braidings that are not of Cartan type. They are associated to Cartan matrices of type A_θ, B_θ or G_2 .

We use a notation similar to the one in [Heckenberger 2006a] for a special kind of braiding of type A_θ (here we emphasize the positions where $q_{ii} = -1$, which we use to compute the dimension of the corresponding Nichols algebra); $\mathcal{C}(\theta, q; i_1, \dots, i_j)$ corresponds to the generalized Dynkin diagram

$$\circ^1 \text{ --- } \circ^2 \text{ --- } \circ^3 \dots \circ^{\theta-1} \text{ --- } \circ^\theta \quad (3-9)$$

where the following equations hold:

- $q = q_{\theta-1, \theta} q_{\theta, \theta-1} q_{\theta\theta}^2$,
- $(q_{\theta\theta} + 1)(q_{\theta\theta} q_{\theta-1, \theta} q_{\theta, \theta-1} - 1) = (q_{11} + 1)(q_{11} q_{12} q_{21} - 1) = 0$;
- $-q_{ii} = q_{i-1, i} q_{i, i-1} q_{i+1, i} q_{i, i+1} = 1$ if $i \in \{i_1, \dots, i_j\}$.
- $q_{ii} q_{i-1, i} q_{i, i-1} = q_{ii} q_{i+1, i} q_{i, i+1} = 1$, otherwise.

Then $q_{ii} = -1$ if and only if $q_{i-1, i} q_{i, i-1} = (q_{i+1, i} q_{i, i+1})^{-1}$.

Proposition 3.9. *Let V be a braided vector space of diagonal type. Then V is standard of type A_θ if and only if its generalized Dynkin diagram is of the form*

$$\mathcal{C}(\theta, q; i_1, \dots, i_j). \quad (3-10)$$

This braiding is of Cartan type if and only if $j = 0$, or $j = n$ with $q = -1$.

Proof. Let V be a braided vector space of standard A_θ type. For each vertex i , with $1 < i < \theta$, we have $q_{ii} = -1$ or $q_{ii} q_{i, i-1} q_{i-1, i} = q_{ii} q_{i, i+1} q_{i+1, i} = 1$, and similar formulas hold for $i = 1, \theta$. So suppose that $1 < i < \theta$ and $q_{ii} = -1$. We transform by s_i and obtain

$$\tilde{q}_{i-1, i+1} = -q_{i, i+1} q_{i-1, i} q_{i-1, i+1}, \quad \tilde{q}_{i+1, i-1} = -q_{i, i-1} q_{i+1, i} q_{i+1, i-1},$$

and using that $m_{i-1, i+1} = \tilde{m}_{i-1, i+1} = 0$, we have

$$q_{i-1, i+1} q_{i+1, i-1} = 1, \quad \tilde{q}_{i-1, i+1} \tilde{q}_{i+1, i-1} = 1,$$

so we deduce that $q_{i, i+1} q_{i+1, i} = (q_{i, i-1} q_{i-1, i})^{-1}$. Then the corresponding matrix (q_{ij}) is of the form (3-10).

Now consider V of the form (3-10). Assume $q_{ii} = q^{\pm 1}$; if we transform by s_i , the braided vector space V_i is twist equivalent with V by Remark 3.4. Thus, $\tilde{m}_{ij} = m_{ij}$.

Assume $q_{ii} = -1$. We transform by s_i and calculate

$$\tilde{q}_{jj} = (-1)^{m_{ij}^2} (q_{ij} q_{ji})^{m_{ij}} q_{jj} = \begin{cases} q_{jj} & \text{if } |j-i| > 1, \\ (-1) q^{\mp 1} q^{\pm 1} = -1 & \text{if } j = i \pm 1, q_{jj} = q^{\pm 1}, \\ (-1) q^{\pm 1} (-1) = q^{\pm 1} & \text{if } j = i \pm 1, q_{jj} = -1. \end{cases}$$

Also, $\tilde{q}_{ij}\tilde{q}_{ji} = q_{ij}q_{ji}$ if $|j - i| > 1$ and $\tilde{q}_{ij}\tilde{q}_{ji} = q_{ij}^{-1}q_{ji}^{-1}$ if $|j - i| = 1$; moreover

$$\tilde{q}_{kj}\tilde{q}_{jk} = (q_{ik}q_{ki})^{m_{ij}}(q_{ij}q_{ji})^{m_{ik}}q_{kj}q_{jk} = \begin{cases} q_{kj}q_{jk} & \text{if } |j - i| \text{ or } |k - i| > 1, \\ 1 & \text{if } j = i - 1, k = i + 1. \end{cases}$$

Then V_i has a braiding of the above form too, and $(-m_{ij})$ corresponds to the finite Cartan matrix of type A_θ , so it is a standard braiding of type A_θ . Thus this is the complete family of standard braidings of type A_θ . \square

Proposition 3.10. *Let V a diagonal braided vector space. Then V is standard of type B_θ if and only if its generalized Dynkin diagram is of one of these forms:*

- (a) $\overset{\zeta}{\circ} \xrightarrow{q^{-1} q} \circ$ with $\zeta \in \mathbb{G}_3$, $q \neq \zeta$ ($\theta = 2$);
- (b) $\left(\overset{\zeta}{\circ}(\theta-1, q^2; i_1, \dots, i_j)\right) \xrightarrow{q^{-2} q} \circ$ with $q \neq 0, -1$, $0 \leq j \leq \theta - 1$;
- (c) $\left(\overset{\zeta}{\circ}(\theta-1, -\zeta^{-1}; i_1, \dots, i_j)\right) \xrightarrow{-\zeta} \overset{\zeta}{\circ}$ with $\zeta \in \mathbb{G}_3$, $0 \leq j \leq \theta - 1$.

This braiding is of Cartan type if and only if it is as in (b) and $j = 0$.

Proof. First we analyze the case $\theta = 2$. Let V a standard braided vector space of type B_2 . There are several possibilities:

- $q_{11}^2 q_{12} q_{21} = q_{22} q_{21} q_{12} = 1$: this braiding is of Cartan type, with $q = q_{11}$. Note that $q \neq -1$. This braiding has the form (b) with $\theta = 2$, $j = 0$.
- $q_{11}^2 q_{12} q_{21} = 1$, $q_{22} = -1$. We transform by s_2 , obtaining

$$\tilde{q}_{11} = -q_{11}^{-1}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{12}^{-1}q_{21}^{-1}.$$

Thus $\tilde{q}_{11}^2 \tilde{q}_{12}\tilde{q}_{21} = 1$. It has the form (b) with $j = 1$.

- $q_{11} \in \mathbb{G}_3$, $q_{22} q_{21} q_{12} = 1$. We transform by s_1 , obtaining

$$\tilde{q}_{22} = q_{11} q_{12} q_{21}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{11}^2 q_{12}^{-1} q_{21}^{-1}.$$

So $\tilde{q}_{22}\tilde{q}_{21}\tilde{q}_{12} = 1$, which is the case (a).

- $q_{11} \in \mathbb{G}_3$, $q_{22} = -1$: we transform by s_1 , obtaining

$$\tilde{q}_{22} = -q_{12}^2 q_{21}^2 q_{11}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{11}^2 q_{12}^{-1} q_{21}^{-1}.$$

If we transform by s_2 ,

$$\tilde{q}_{11} = -q_{12} q_{21} q_{11}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{12}^{-1} q_{21}^{-1}.$$

So $q_{12} q_{21} = \pm q_{11}$, and we discard the case $q_{12} q_{21} = q_{11}$ because it has been considered before. The braiding has the form (c) with $j = 0$, and is standard.

Conversely, all braidings (a), (b) and (c) are standard of type B_2 .

Now let V be of type B_θ , with $\theta \geq 3$. The first $\theta - 1$ vertices determine a braiding of standard type $A_{\theta-1}$, and the last two determine a braiding of standard type B_2 ; so we have to glue the possible such braidings. The possible cases are the two presented in Proposition 3.10, plus

$$\left(\mathbb{C}(\theta-2, q; i_1, \dots, i_j) \right) \begin{array}{c} q^{-1} \\ \circ \\ q^{-1} \\ \circ \\ \zeta \end{array} .$$

But if we transform by s_θ , we obtain

$$\tilde{q}_{\theta-1, \theta-1} = \zeta q^{-1}, \quad \tilde{q}_{\theta-1, \theta-2} \tilde{q}_{\theta-2, \theta-1} = q^{-1},$$

so $1 = \tilde{q}_{\theta-1, \theta-1} \tilde{q}_{\theta-1, \theta-2} \tilde{q}_{\theta-2, \theta-1}$ and we obtain $q = \pm \zeta^{-1}$, or $\tilde{q}_{\theta-1, \theta-1} = -1$. Then $q = -\zeta^{-1}$ or $q = -1$, so it is of some of the above forms.

To prove that (b) and (c) are standard braidings, we use the following fact: if $m_{ij} = 0$ (that is, $q_{ij}q_{ji} = 1$) and we transform by s_i , then

$$\tilde{q}_{jj} = q_{jj} \quad \text{and} \quad \tilde{q}_{jk} \tilde{q}_{jk} = q_{jk} q_{kj} \quad \text{for } k \neq i.$$

In this case, $m_{ij} = 0$ if $|i - j| > 1$; if, on the contrary, $j = i \pm 1$, we use the fact that the subdiagram determined by these two vertices is standard of type B_2 or type A_2 . So this is the complete family of all twist equivalence classes of standard braidings of type B_θ . \square

Proposition 3.11. *Let V a braided vector space of diagonal type. Then V is standard of type G_2 if and only if its generalized Dynkin diagram is one of the following:*

- (a) $\begin{array}{c} q \quad q^{-3} \quad q^3 \\ \circ \text{---} \circ \end{array}$ with $\text{ord } q \geq 4$;
- (b) $\begin{array}{c} \zeta^2 \quad \zeta \quad \zeta^{-1} \\ \circ \text{---} \circ \end{array}$ or $\begin{array}{c} \zeta^2 \quad \zeta^3 \quad -1 \\ \circ \text{---} \circ \end{array}$ or $\begin{array}{c} \zeta \quad \zeta^5 \quad -1 \\ \circ \text{---} \circ \end{array}$ with $\zeta \in \mathbb{G}_8$.

This braiding is of Cartan type if and only if it is as in (a).

Proof. Let V be a standard braiding of type G_2 . There are four possible cases:

- $q_{11}^3 q_{12} q_{21} = 1$, $q_{22} q_{21} q_{12} = 1$: this braiding is of Cartan type, as in (a), with $q = q_{11}$. If q is a root of unity, then $\text{ord } q \geq 4$ because $m_{12} = 3$.
- $q_{11}^3 q_{12} q_{21} = 1$, $q_{22} = -1$: we transform by s_2 , obtaining

$$\tilde{q}_{11} = -q_{11}^{-2}, \quad \tilde{q}_{12} \tilde{q}_{21} = q_{12}^{-1} q_{21}^{-1}.$$

If $1 = \tilde{q}_{11}^3 \tilde{q}_{12} \tilde{q}_{21} = -q_{11}^{-3}$, then $q_{12} q_{21} = -1$, and the braiding is of Cartan type with $q_{11} \in \mathbb{G}_6$. If not, $1 = \tilde{q}_{11}^4 = q_{11}^{-8}$ and $\text{ord } \tilde{q}_{11} = 4$, so $\text{ord } q_{11} = 8$. Then we can express the braiding in the form of the third diagram in (b).

- $q_{11} \in \mathbb{G}_4$, $q_{22}q_{21}q_{12} = 1$: we transform by s_1 , obtaining

$$\tilde{q}_{22} = q_{11}q_{12}^2q_{21}^2, \quad \tilde{q}_{12}\tilde{q}_{21} = -q_{12}^{-1}q_{21}^{-1}.$$

If $1 = \tilde{q}_{22}\tilde{q}_{21}\tilde{q}_{12} = -q_{11}q_{12}q_{21}$, we have $q_{11}^3q_{12}q_{21} = 1$ because $q_{11}^2 = -1$, and this is a braiding of Cartan type. So consider now the case $-1 = \tilde{q}_{22} = q_{11}q_{12}^2q_{21}^2$, from which $q_{22}^2 = q_{11}^{-1}$ and $q_{22} \in \mathbb{G}_8$. Then we obtain a braiding of the form of the first diagram in (b).

- $q_{11} \in \mathbb{G}_4$, $q_{22} = -1$: we transform by s_2 , obtaining

$$\tilde{q}_{11} = -q_{12}q_{21}q_{11}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{12}^{-1}q_{21}^{-1}.$$

If $\tilde{q}_{11} \in \mathbb{G}_4$, then $(q_{12}q_{21})^4 = 1$. Moreover $q_{12}q_{21} \neq 1$ and $q_{12}q_{21} \neq q_{11}^{-1}$ because $m_{12} = 3$. So $q_{12}q_{21} = -1$ or $q_{12}q_{21} = q_{11} = q_{11}^{-3}$; but these cases have been considered already. There remains to analyze the case

$$1 = \tilde{q}_{11}^3\tilde{q}_{12}\tilde{q}_{21} = q_{11}q_{12}^2q_{21}^2,$$

which we can express in the form of the second diagram in (b), for some $\zeta \in \mathbb{G}_8$.

A simple calculation proves that these braidings are of standard type, so they are all the standard braidings of type G_2 . \square

4. Nichols algebras of standard braided vector spaces

In this section we study Nichols algebras associated to standard braidings. We assume that the Dynkin diagram is connected, as in Section 3. In Section 4A we prove that the set $\Delta^+(\mathfrak{B}(V))$ is in bijection with Δ_C^+ , the set of positive roots associated with the finite Cartan matrix C .

We describe an explicit set of generators in Section 4B, following [Lalonde and Ram 1995]. We adapt their proof since they work on enveloping algebras of simple Lie algebras. In Section 4C, we calculate the dimension of Nichols algebra associated to a standard braided vector space, type by type.

4A. PBW bases of Nichols algebras. We start with a result analogous to [Heckenberger 2006b, Theorem 1], but for braidings of standard type.

Theorem 4.1. *Let V be a braided vector space of standard type with Cartan matrix C . Then the set $\Delta(\mathfrak{B}(V))$ is finite if and only if the Cartan matrix C is symmetrizable and of finite type.*

Proof. Since we assume V of standard type, $\Delta(\mathfrak{B}(V))$ coincides with the set of real roots corresponding to the matrix C by [Heckenberger 2006b, Proposition 1], where we identify corresponding simple roots. Hence, if C is not symmetrizable or not of finite type, the set of real roots is infinite by the classification of finite Coxeter groups, and hence $\Delta(\mathfrak{B}(V))$ is infinite.

Conversely, let C be symmetrizable and of finite type. Then the set of real roots is finite. Take $\alpha \in \Delta(\mathfrak{B}(V))$ and let $k \in \mathbb{N}$, $i_1, \dots, i_k \in \{1, \dots, \theta\}$ be a sequence of integers such that $s_{i_1} \cdots s_{i_k}$ is a longest element in $W_0(\chi)$. Since all roots are positive or negative, there exists $l \in \{1, \dots, k\}$ such that $\beta = s_{i_{l+1}} \cdots s_{i_k}(\alpha)$ is positive and $s_{i_l}(\beta)$ is negative. But then $\beta = \alpha_{i_l}$, and $\alpha = s_{i_k} \cdots s_{i_{l+1}}(\alpha_{i_l})$ is a real root. Thus $\Delta(\mathfrak{B}(V))$ is finite. \square

Corollary 4.2. *Let V be a braided vector space of standard type, set $\theta = \dim V$, and let $C = (a_{ij})_{i,j \in \{1, \dots, \theta\}}$ be the corresponding generalized Cartan matrix of finite type.*

- (a) $\phi(\Delta_C) = \Delta(\mathfrak{B}(V))$, where as before $\phi : \mathbb{Z}\pi \rightarrow \mathbb{Z}^\theta$ is the \mathbb{Z} -linear map determined by $\phi(\alpha_i) := \mathbf{e}_i$.
- (b) *The multiplicity of each root in $\Delta(\mathfrak{B}(V))$ is one.*

Proof. Statement (a) follows from the proof of Theorem 4.1.

Using this condition, since each root is of the form $\beta = w(\alpha_i)$ for some $w \in W$ and $i \in \{1, \dots, \theta\}$, we conclude by applying a certain sequence of transformations s_i that this is the degree corresponding to a generator of the corresponding Nichols algebra, so the multiplicity (which is invariant under these transformations) is 1. \square

4B. Explicit generators for a PBW basis. In view of Corollary 4.2, we restrict our attention to finding one Lyndon word for each positive root of the root system associated with the corresponding finite Cartan matrix.

Proposition 4.3 [Lalonde and Ram 1995, Proposition 2.9]. *Let l be an element of S_I . Then l is of the form $l = l_1 \dots l_k a$, for some $k \in \mathbb{N}_0$, where*

- $l_i \in S_I$ for each $i = 1, \dots, k$;
- l_i is a beginning of l_{i-1} for each $i > 1$; and
- a is a letter.

Also, if $l = uv$ is the Shirshov decomposition, then $u, v \in S_I$. \square

In what follows, we describe a set of Lyndon words for each Cartan matrix of finite type C .

Consider $\alpha = \sum_{j=1}^{\theta} a_j \alpha_j \in \Delta^+$ and let $l_\alpha \in S_I$ be such that $\deg l_\alpha = \alpha$. Let $l_\alpha = l_{\beta_1} \dots l_{\beta_k} x_s$ be a decomposition as above, where $s \in \{1, \dots, \theta\}$ and $\deg l_{\beta_j} = \beta_j$. Since each l_{β_j} is a beginning of $l_{\beta_{j-1}}$, all the words begin with the same letter x' , which satisfies $x' < x_s$ because l is a Lyndon word. Therefore x' is the least letter of l , so

$$x' = x_i, \quad i = \min\{j : a_j \neq 0\} \quad \implies \quad \alpha = \sum_{j=i}^{\theta} a_j \alpha_j.$$

Then $k \leq a_i \leq 3$, for the order given in (3-9), (3-4), (3-5), (3-6), (3-7), (3-8) (the value $a_i = 3$ appears only when C is of type G_2).

Now, each l_{β_j} lies in S_I , so $\beta_j \in \Delta^+$; i.e., it corresponds to a term of the PBW basis. Also $\sum_{j=1}^k \beta_j + \alpha_s = \alpha$. If $k = 2$, we have $\beta_1 - \beta_2 = \sum_{j=1}^{\theta} b_j \alpha_j$ and $b_j \geq 0$, because β_2 is a beginning of β_1 (an analogous claim is valid when the matrix is of type G_2 and $k = 3$). With these rules we define inductively Lyndon words for a PBW basis corresponding with a standard braiding for a fixed order on the letters. This is done as in [Lalonde and Ram 1995], but taking care that in that reference Serre relations are used; here we have quantum Serre relations, and some quantum binomial coefficients may be zero.

Type A_θ : In this case, the roots are of the form

$$\mathbf{u}_{i,j} := \sum_{k=i}^j \alpha_k, \quad 1 \leq i \leq j \leq \theta.$$

By induction on $s = j - i$, we have

$$l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j.$$

This is because when $s = 0$ we have $i = j$, and the unique possibility is $l_{\mathbf{u}_{i,i}} = x_i$. If we remove the last letter (when $j - i > 0$), we must obtain a Lyndon word, so the last letter must be x_j .

Type B_θ : For convenience, we use the following vertex numbering:

$$\circ^1 \longleftarrow \circ^2 \text{ --- } \circ^3 \dots \quad \circ^{\theta-1} \text{ --- } \circ^\theta. \quad (4-1)$$

The roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^j \alpha_k$, or

$$\mathbf{v}_{i,j} := 2 \sum_{k=1}^i \alpha_k + \sum_{k=i+1}^j \alpha_k.$$

In the first case we have $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$, as above. In the second case, if $j = i + 1$, we must have x_{i+1} as the last letter to obtain a decomposition in two words $x_1 \dots x_i$; if $j > i + 1$, the last letter must be x_j , so we obtain

$$l_{\mathbf{v}_{i,j}} = x_1 x_2 \dots x_i x_1 x_2 \dots x_j.$$

Type C_θ : The roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^j \alpha_k$, or

$$\mathbf{w}_{i,j} := \sum_{k=i}^{j-1} \alpha_k + 2 \sum_{k=j}^{\theta-1} \alpha_k + \alpha_\theta, \quad i \leq j < \theta.$$

As before, $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$. Now, if $i < j$, the least letter x_i has degree 1, so if we remove the last letter, we obtain a Lyndon word; that is, $\mathbf{w}_{i,j} - x_s$ is a root, and then $x_s = x_j$, so

$$l_{\mathbf{w}_{i,j}} = x_i x_{i+1} \dots x_{\theta-1} x_{\theta} x_{\theta-1} \dots x_j.$$

When $i = j$, $a_i = 2$, so there are one or two Lyndon words β_j as before. Since $\mathbf{w} - x_s$ is not a root, for $s = i + 1, \dots, \theta$, and $i < s$, there are two Lyndon words $\beta_1 \geq \beta_2$, and $\beta_1 + \beta_2 = 2 \sum_{k=i}^{\theta-1} \alpha_k$. The only possibility is $\beta_1 = \beta_2 = x_i x_{i+1} \dots x_{\theta-1}$; that is,

$$l_{\mathbf{w}_{i,i}} = x_i x_{i+1} \dots x_{\theta-1} x_i x_{i+1} \dots x_{\theta-1} x_{\theta}.$$

Type D_{θ} : the roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^j \alpha_k$, $1 \leq i \leq j \leq \theta$, or

$$\begin{aligned} \mathbf{z}_{i,j} &:= \sum_{k=i}^{j-1} \alpha_k + 2 \sum_{k=j}^{\theta-2} \alpha_k + \alpha_{\theta-1} + \alpha_{\theta}, \quad i < j \leq \theta - 2, \\ \bar{\mathbf{z}}_i &:= \sum_{k=i}^{\theta-2} \alpha_k + \alpha_{\theta}, \quad 1 \leq i \leq \theta - 2. \end{aligned}$$

As above, $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$ if $j \leq n - 1$. When the roots are of type $\bar{\mathbf{z}}_i$, we have $s = \theta$, since $\bar{\mathbf{z}}_i - x_s$ must be a root (if x_s is the last letter); thus $l_{\bar{\mathbf{z}}_i} = x_i x_{i+1} \dots x_{\theta-2} x_{\theta}$ is the unique possibility.

Now, when $\alpha = \mathbf{u}_{i,\theta}$, the last letter is $x_{\theta-1}$ or x_{θ} : if it is x_{θ} , we have $l_{\mathbf{u}_{i,\theta}} = x_i x_{i+1} \dots x_{\theta-1} x_{\theta}$. Since $m_{\theta-1,\theta} = 0$, we have $x_{\theta-1} x_{\theta} = q_{\theta-1,\theta} x_{\theta} x_{\theta-1}$, so

$$x_i x_{i+1} \dots x_{\theta-1} x_{\theta} \equiv x_i x_{i+1} \dots x_{\theta-2} x_{\theta} x_{\theta-1} \pmod{I},$$

and then $x_i x_{i+1} \dots x_{\theta-1} x_{\theta} \notin S_I$. So, $l_{\mathbf{u}_{i,\theta}} = x_i \dots x_{\theta-2} x_{\theta} x_{\theta-1}$.

In the last case, note that if $j = n - 2$, the unique possibility is β_i as before, because the least letter x_i has degree 1 and $x_s = x_{\theta-2}$ (since $\alpha - \alpha_s$ is a root). Hence $l_{\mathbf{z}_{i,\theta-2}} = x_i \dots x_{\theta-2} x_{\theta} x_{\theta-1} x_{\theta-2}$, and inductively,

$$l_{\mathbf{z}_{i,j}} = x_i \dots x_{\theta-2} x_{\theta} x_{\theta-1} x_{\theta-2} \dots x_j.$$

Type E_6 : Let $\alpha = \sum_{j=1}^6 a_j \alpha_j$. If $a_6 = 0$, α corresponds to the Dynkin subdiagram of type D_5 determined by 1, 2, 3, 4, 5, and we obtain l_{α} as above. If $a_1 = 0$ then α corresponds to the Dynkin subdiagram of type D_5 determined by 2, 3, 4, 5, 6; the numbering is different from the one given in (3-4). Anyway, the roots are defined in a similar way, and we obtain the same list as in [Lalonde and Ram 1995, Fig.1]. If $a_4 = 0$, then α corresponds to the Dynkin subdiagram of type A_5 determined by 1, 2, 3, 5, 6.

So we restrict our attention to the case $a_i \neq 0$, $i = 1, 2, 3, 4, 5, 6$. We consider each case in turn:

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$: since $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root, where α_s is the last letter. Then $s = 2$ or $s = 6$. In the second case, $l_{\beta_1} = x_1x_2x_3x_4x_5$, but using that $x_2x_3 = q_{23}x_3x_2$, we have $x_1x_2x_3x_4x_5 \notin S_I$. So $s = 2$, and $l_\alpha = x_1x_3x_4x_5x_6x_2$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$: from $a_1 = 1$, we note that $\alpha - \alpha_s = \beta_1$ is a root. Then $s = 4$, and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$: since $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root. So $s = 3$, and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4x_3$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$: since $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root. The only possibility is $s = 5$, and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4x_5$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$: as above $a_1 = 1$, and $\alpha - \alpha_s = \beta_1$ is a root. So $s = 3$, and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4x_5x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$: since $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root. Then $s = 4$ and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$: since $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root. So $s = 2$, and $l_\alpha = x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4$.

Type E_7 : If $\alpha = \sum_{j=1}^7 a_j \alpha_j$ and $a_7 = 0$, the root corresponds to the subdiagram of type D_6 determined by 1, 2, 3, 4, 5, 6, and we obtain l_α as above. If $a_1 = 0$, it corresponds to the subdiagram of type E_6 determined by 2, 3, 4, 5, 6, 7. If $a_5 = 0$, then α corresponds to the subdiagram of type A_6 determined by 1, 2, 3, 4, 6, 7.

As above, consider each case where $a_i \neq 0$, $i = 1, 2, 3, 4, 5, 6, 7$:

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$: since $a_1 = 1$, $\alpha - \alpha_s = \beta_1$ is a root, if α_s is the last letter. Then $s = 2$ or $s = 7$. In the second case, $l_{\beta_1} = x_1x_2x_3x_4x_5x_6$, but from $x_2x_3 = q_{23}x_3x_2$, we have $x_1x_2x_3x_4x_5x_6x_7 \notin S_I$. So $s = 2$, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$: now $s = 4, 7$. We discard the case $s = 7$ since $m_{47} = 0$; for the case $s = 4$ we have $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$: as above, $s = 3, 7$, but we discard $s = 7$ since $m_{37} = 0$, so $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_3$.
- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$: now $s = 5, 7$, and we discard the case $s = 7$ because $m_{57} = 0$, so $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$: now $s = 3, 7$, and as above we discard the case $s = 7$, so $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$: now $s = 4$, and therefore we have $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$: now $s = 2$, as above, and $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3x_4x_2$.

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: as above, the unique possibility is $s = 6$, so $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 3$, $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 4$, $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 2$, and in this case we obtain $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_2$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 5$, and in this case we obtain $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: as above, $s = 2$, and we get $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 4$, and in this case we obtain $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 3$, and in this case we obtain $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2x_4x_3$.
- $\alpha = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: now there are one or two words β_j . Since $\alpha - \alpha_s \in \Delta^+$ if and only if $s = 1$ and x_1 is not the last letter (because it is the least letter), there are two words β_j . So looking at the roots we obtain $s = 7$, and $l_\alpha = (x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4x_2)(x_1x_3x_4x_5x_6)x_7$

Type E_8 : Consider $\alpha = \sum_{j=1}^8 a_j \alpha_j$; if $a_8 = 0$, the root corresponds to the subdiagram of type D_7 determined by 1, 2, 3, 4, 5, 6, 7, and we obtain l_α as in that case. If $a_1 = 0$, it corresponds to the subdiagram of type E_7 determined by 2, 3, 4, 5, 6, 7, 8. If $a_6 = 0$, then α corresponds to a subdiagram of type A_7 determined by 1, 2, 3, 4, 5, 7, 8.

So, we consider the case $a_i \neq 0$, $i = 1, 2, 3, 4, 5, 6, 7, 8$, and solve it case by case in a similar way as for E_7 , by induction on the height.

Type F_4 : Now $\alpha = \sum_{j=1}^4 a_j \alpha_j$. If $a_4 = 0$, then it corresponds to the subdiagram of type B_3 determined by 1, 2, 3, so we obtain l_α as before. If $a_1 = 0$, α corresponds to the subdiagram of type C_3 determined by 2, 3, 4.

So consider the case $a_i \neq 0$, $i = 1, 2, 3, 4$:

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$: $a_1 = 1$, so $\alpha - \alpha_s = \beta_1$ is a root, where α_s is the last letter. Then $s = 4$, and $l_\alpha = x_1x_2x_3x_4$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$: $a_1 = 1$, so $\alpha - \alpha_s = \beta_1$ is a root. Now $s = 3$ or $s = 4$. If $s = 4$, then $l_\alpha = x_1x_2x_3^2x_4$. But $m_{34} = 2$, so

$$x_3^2x_4 \equiv q_{34}(1 + q_{33})x_3x_4x_3 - q_{33}q_{34}x_4x_3^2 \pmod{I},$$

and $x_1x_2x_3^2x_4 \notin S_I$, a contradiction. So $s = 3$, and we have $l_\alpha = x_1x_2x_3x_4x_3$.

- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$: $a_1 = 1$, and as above, $s = 2$ or $s = 4$: if $s = 4$, then $l_\alpha = x_1x_2x_3^2x_2x_4$, but it is not an element of S_I , because $x_2x_4 \equiv q_{24}x_2x_4 \pmod{I}$. Then $s = 2$, and $l_\alpha = x_1x_2x_3x_4x_3x_2$.
- $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$: $a_1 = 1$, so $s = 3$, and we have $l_\alpha = x_1x_2x_3x_4x_3x_2x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 4$, and $l_\alpha = x_1x_2x_3x_4x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 2$ or $s = 4$, but we discard the case $s = 4$ since $x_2x_4 \equiv q_{24}x_2x_4 \pmod{I}$. So, $l_\alpha = x_1x_2x_3x_4x_3x_4x_2$.
- $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 3$, and $l_\alpha = x_1x_2x_3x_4x_3x_4x_2x_3$.
- $\alpha = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 3$, and $l_\alpha = x_1x_2x_3x_4x_3x_4x_2x_3^2$.
- $\alpha = \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$: $a_1 = 1$, so $s = 2$, and $l_\alpha = x_1x_2x_3x_4x_3x_4x_2x_3^2x_2$.
- $\alpha = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$: $a_1 = 2$, and there are one or two Lyndon words β_j . If there is only one, $\beta_1 = \alpha - \alpha_s \in \Delta^+$. The only possibility is $s = 1$, but it contradicts that l_α is a Lyndon word. Hence there exist $\beta_1, \beta_2 \in \Delta^+$ such that $\beta_1 + \beta_2 = \alpha - \alpha_s$, and β_2 is a beginning of β_1 . So $s = 2$ and $\beta_1 = \beta_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$, i.e., $l_\alpha = x_1x_2x_3x_4x_3x_1x_2x_3x_4x_3x_2$.

Type G_2 : the roots are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2$:

$$l_{\alpha_1} = x_1, \quad l_{\alpha_2} = x_2, \quad l_{m\alpha_1 + \alpha_2} = x_1^m x_2, \quad m = 1, 2, 3.$$

If $\alpha = 3\alpha_1 + 2\alpha_2$, the last letter is x_2 . If we suppose $\beta_1 = 3\alpha_1 + \alpha_2$, then $l_\alpha = x_1^3 x_2^2$, but

$$(\text{ad } x_2)^2 x_1 = x_2^2 x_1 - q_{21}(1 + q_{22})x_2 x_1 x_2 + q_{22} q_{21} x_1 x_2^2 \equiv 0 \pmod{I},$$

so we have

$$x_1^3 x_2^2 \equiv (q_{22}^{-1} + 1)x_1^2 x_2 x_1 x_2 - q_{22}^{-1} q_{21}^{-1} x_1^2 x_2^2 x_1 \pmod{I},$$

and then $l_\alpha = x_1^3 x_2^2 \notin S_I$ because $q_{22}^{-1} q_{21}^{-1} \neq 0$, so there are at least two words β_j . Analogously, if we suppose that there are three words β_j , we obtain $l_{\beta_1} = l_{\beta_2} = x_1 > l_{\beta_3} = x_1 x_2$ since $\beta_1 \geq \beta_2 \geq \beta_3$ and $\beta_1 + \beta_2 + \beta_3 = 3\alpha_1 + \alpha_2$; moreover $l_\alpha = x_1^3 x_2^2 \notin S_I$. So there are two Lyndon words of degree $\beta_1 \geq \beta_2$, and the unique possibility is $\beta_1 = 2\alpha_1 + \alpha_2$, $\beta_2 = \alpha_1$; that is, $l_\alpha = x_1^2 x_2 x_1 x_2$.

4C. Dimensions of Nichols algebras of standard braidings. We begin with the standard braidings of types $C_\theta, D_\theta, E_6, E_7, E_8, F_4$, which are of Cartan type.

Proposition 4.4. *Let V a braided vector space of Cartan type, with $q_{44} \in \mathbb{G}_N$ if V is of type F_4 and $q_{11} \in \mathbb{G}_N$ otherwise, in each case for some $N \in \mathbb{N}$. The dimension of the associated Nichols algebra $\mathfrak{B}(V)$ is as follows:*

$$\text{Type } C_\theta: \dim \mathfrak{B}(V) = \begin{cases} N^{\theta^2} & \text{for } N \text{ odd,} \\ N^{\theta^2}/2^\theta & \text{for } N \text{ even;} \end{cases}$$

$$\underline{\text{Type } F_4}: \dim \mathfrak{B}(V) = \begin{cases} N^{24} & \text{for } N \text{ odd,} \\ N^{24}/2^{12} & \text{for } N \text{ even;} \end{cases}$$

$$\underline{\text{Types } D_\theta, E_6, E_7, E_8}: \dim \mathfrak{B}(V) = N^{|\Delta^+|}.$$

The last case corresponds to simply laced Dynkin diagrams.

Proof. If N is odd, then $\text{ord } q^2 = \text{ord } q = N$, but if N is even, we have $\text{ord } q^2 = N/2$. Since the braiding is of Cartan type,

$$q_{s_i(\alpha)} = \chi(s_i(\alpha), s_i(\alpha)) = \tilde{\chi}(\alpha, \alpha) = \chi(\alpha, \alpha) = q_\alpha.$$

Using this, we just have to determine how many roots there are in the orbit of each simply root.

When V is of type C_θ , we have $q_{ii} = q$, except for $q_{\theta\theta} = q^2$. The roots in the orbit of α_θ by the action of the Weyl group are $q_{w_{ii}}$ for $1 \leq i < \theta$, and the others are in the orbit of α_j , for some $j < \theta$. Hence there are θ roots such that $q_\alpha = q^2$, and $q_\alpha = q$ for the rest.

When V is of type F_4 , we have $q_{11} = q_{22} = q^2$ and $q_{33} = q_{44} = q$. There are exactly 12 roots in the union of orbits corresponding to α_1 and α_2 , and the other 12 are in the union of orbits corresponding to α_3 and α_4 . So

$$|\{\alpha \in \Delta^+ : q_\alpha = q\}| = |\{\alpha \in \Delta^+ : q_\alpha = q^2\}| = 12.$$

When V is of type D or E , all the q_α equal q because $q_{ii} = q$, for all $1 \leq i \leq \theta$. The formula for the dimension follows from Theorem 2.6(ii) and Corollary 4.2. \square

Now we treat the types A_θ , B_θ and G_2 .

Proposition 4.5. *Let V be a standard braided vector space of type A_θ as in Proposition 3.9. The associated Nichols algebra $\mathfrak{B}(V)$ is of finite dimension if and only if q is a root of unity of order $N \geq 2$. In this case,*

$$\dim \mathfrak{B}(V) = 2^{\binom{\theta+1}{2} - \binom{t}{2} - \binom{\theta+1-t}{2}} N^{\binom{t}{2} + \binom{\theta+1-t}{2}}, \quad (4-2)$$

where $t = \theta + 1 - \sum_{k=1}^j (-1)^{j-k} i_k$.

Proof. First, q is a root of unity of order $N \geq 2$ because the height of each PBW generator is finite. To calculate the dimension, recall that from Corollary 4.2, we have to determine q_α for $\alpha \in \Delta_C$. As before, $\mathbf{u}_{ij} = \sum_{k=i}^j \mathbf{e}_k$, $i \leq j$, and we have

$$\Delta(\mathfrak{B}(V)) = \{\mathbf{u}_{ij} : 1 \leq i \leq j \leq \theta\}.$$

If $1 \leq i \leq j \leq \theta$, we define

$$\kappa_{ij} := \text{card}\{k \in \{i, \dots, j\} : q_{kk} = -1\}.$$

We prove by induction on $j - i$ that

- if κ_{ij} is odd, then $q_{\mathbf{u}_j} = -1$;
- if κ_{ij} is even, then $q_{\mathbf{u}_j} = q_{j,j+1}^{-1} q_{j+1,j}^{-1}$.

Here $q_{\theta,\theta+1} q_{\theta+1,\theta} = q_{\theta\theta}^{-2} q_{\theta,\theta-1}^{-1} q_{\theta-1,\theta}^{-1}$.

If $j - i = 0$, then $q_{\mathbf{u}_i} = q_{ii}$; in this case, $\kappa_{ii} = 1$ if $q_{ii} = -1$ or $\kappa_{ii} = 0$ if $q_{ii} = (q_{i,i+1} q_{i+1,i})^{-1} \neq -1$. Now assume this is valid for a certain j , and calculate it for $j + 1$:

$$\begin{aligned} q_{\mathbf{u}_{j+1}} &= \chi(\mathbf{u}_{ij} + \mathbf{e}_{j+1}, \mathbf{u}_j + \mathbf{e}_{j+1}) = q_{\mathbf{u}_j} \chi(\mathbf{u}_{ij}, \mathbf{e}_{j+1}) \chi(\mathbf{e}_{j+1}, \mathbf{u}_j) q_{j+1,j+1} \\ &= q_{\mathbf{u}_j} q_{j,j+1} q_{j+1,j} q_{j+1,j+1} \\ &= \begin{cases} q_{\mathbf{u}_j} & \text{if } q_{j+1,j+1} \neq -1 \ (\kappa_{i,j+1} = \kappa_{ij}), \\ (-1) q q^{-1} = -1 & \text{if } q_{j+1,j+1} = -1, \ \kappa_{ij} \text{ even}, \\ (-1) q (-1) = q & \text{if } q_{j+1,j+1} = -1, \ \kappa_{ij} \text{ odd}. \end{cases} \end{aligned}$$

This proves the inductive step; to calculate the dimension of $\mathfrak{B}(V)$ we have to calculate the number of \mathbf{u}_{ij} such that

$$q_{\mathbf{u}_j} = q_{i,i+1}^{-1} q_{i+1,i}^{-1} = q^{\pm 1},$$

that is, $\text{card}\{\kappa_{ij} : i \leq j, \kappa_{ij} \text{ even}\}$.

We consider an $1 \times (\theta + 1)$ board, numbered from 1 to $\theta + 1$, and recursively paint its squares white or black: square $\theta + 1$ is white, and square i has the same color as square $i + 1$ if and only if $q_{ii} \neq -1$. The possible colorings of this board are in bijective correspondence with the choices of $1 \leq i_1 < \dots < i_j \leq \theta$ for all j (the positions where we put a -1 in the corresponding q_{ii} of the braiding), and the number of white squares is

$$t = 1 + (\theta - i_j) + (i_{j-1} - i_{j-2}) + \dots = \theta + 1 - \sum_{k=1}^j (-1)^{j-k} i_k$$

Thus $\text{card}\{\kappa_{ij} : i \leq j, \kappa_{ij} \text{ even}\}$ is the number of pairs (a, b) (where $a = i$ and $b = j + 1$) such that $1 \leq a < b \leq \theta + 1$ and the squares in positions a and b are of the same color; this number is $\binom{t}{2} + \binom{\theta+1-t}{2}$. This yields (4-2). \square

Proposition 4.6. *Let V be a standard braided vector space of type B_θ as in Proposition 3.10. If the braiding is as in cases (a) or (b) of that proposition, the associated Nichols algebra $\mathfrak{B}(V)$ has finite dimension if and only if q is a root of unity of order $N \geq 2$ in case (a), or $N > 2$ in case (b).*

When finite, the dimension of $\mathfrak{B}(V)$ is as follows, where $t = \theta - \sum_{k=1}^j (-1)^{j-k} i_k$:

- If the braiding is as in (a) of Proposition 3.10,

$$\dim \mathfrak{B}(V) = \begin{cases} 3^3 N^2 & \text{if } 3 \text{ does not divide } N, \\ 3^2 N^2 & \text{if } 3 \text{ divides both } N \text{ and } \text{ord}(\zeta^{-1}q), \\ 3N^2 & \text{if } 3 \text{ divides } N \text{ but not } \text{ord}(\zeta^{-1}q). \end{cases}$$

- If the braiding is as in (b), then $0 \leq j \leq d-1$ and

$$\dim \mathfrak{B}(V) = \begin{cases} 2^{2t(\theta-t)+\theta} k^{\theta^2-2t\theta+2t^2} & \text{if } N = 2k, \\ 2^{(2t+1)(\theta-t)+1} N^{\theta^2-2t\theta+2t^2} & \text{if } N \text{ is odd.} \end{cases}$$

- If the braiding is as in (c), then

$$\dim \mathfrak{B}(V) = 2^{\theta(\theta-1)} 3^{\theta^2-2t\theta+2t^2}.$$

Proof. It is clear that q should be a root of unity if $\dim \mathfrak{B}(V)$ is finite.

We now calculate $\dim \mathfrak{B}(V)$. From Corollary 4.2, we have to determine the q_α for $\alpha \in \Delta_C$, and multiply their orders. As before, $\mathbf{u}_{ij} = \sum_{k=i}^j \mathbf{e}_k$ for $1 \leq i \leq j \leq \theta$ and $\mathbf{v}_{ij} = 2 \sum_{k=1}^i \mathbf{e}_k + \sum_{k=i+1}^j \mathbf{e}_k = 2\mathbf{e}_{1,i} + \mathbf{e}_{i+1,j}$ for $1 \leq i < j \leq \theta$;

$$\Delta(\mathfrak{B}(V)) = \{\mathbf{u}_{ij} : 1 \leq i \leq j \leq \theta\} \cup \{\mathbf{v}_{ij} : 1 \leq i < j \leq \theta\}.$$

We calculate $q_{\mathbf{u}_{ij}}$, $1 < i \leq j \leq \theta$ as above, because they correspond to a braiding of standard $A_{\theta-1}$ type. We also calculate

$$\begin{aligned} q_{\mathbf{v}_{ij}} &= \chi(\mathbf{v}_{ij}, \mathbf{v}_{ij}) = \chi(\mathbf{u}_{1i}, \mathbf{u}_{1i})^4 \chi(\mathbf{u}_{1i}, \mathbf{u}_{i+1,j})^2 \chi(\mathbf{u}_{i+1,j}, \mathbf{u}_{1i})^2 q_{\mathbf{u}_{i+1,j}} \\ &= q_{11}^4 q_{12}^2 q_{21}^2 \left(\prod_{k=2}^i q_{kk}^2 q_{k-1,k} q_{k-1,k} q_{k+1,k} q_{k+1,k} \right)^2 q_{\mathbf{u}_{i+1,j}} = q_{\mathbf{u}_{i+1,j}}, \end{aligned}$$

where we have used the equalities $q_{ij}q_{ji} = 1$ if $|i-j| > 1$, $q_{11}^4 q_{12}^2 q_{21}^2 = 1$, and $q_{kk}^2 q_{k-1,k} q_{k-1,k} q_{k+1,k} q_{k+1,k} = 1$ if $2 \leq k \leq \theta-1$. To calculate the other q_α 's, we analyze each case:

(a) Note that $q_{\mathbf{e}_1} = \zeta$, $q_{\mathbf{e}_1+\mathbf{e}_2} = \zeta$, $q_{2\mathbf{e}_1+\mathbf{e}_2} = \zeta q^{-1}$, $q_{\mathbf{e}_2} = q$. Setting $N' = \text{ord}(\zeta^{-1}q)$, we have $N' = 3N$ if 3 does not divide N ; $N' = N$ if 3 divides both N and N' ; and $N' = N/3$ if 3 divides N but not N' (since $q = \zeta\rho$, with $\rho \in \mathbb{G}_{N'}$).

(b) We have $q_{\mathbf{u}_{1k}} = q^{-1} q_{\mathbf{u}_{2k}}$. This equals $q^2 q^{-1} = q$ if κ_{2k} is even, and $-q^{-1}$ if κ_{2k} is odd; moreover $q_{11} = q$. Also, κ_{2k} is even if and only if $j \in \{i+1, \theta\}$, or $i \in \{i_{j-2}+1, i_{j-1}\}$, and so on. Then, with

$$t = (\theta - i_j) + (i_{j-1} - i_{j-2}) + \cdots = \theta - \sum_{k=1}^j (-1)^{j-k} i_k$$

as in the statement of the proposition, there are t numbers such that $\kappa_{i, \theta-1}$ is even. There are $2\binom{t}{2} + \binom{\theta-t}{2}$ roots such that $q_\alpha = q^2$, $2\left(\binom{\theta}{2} - \binom{t}{2} - \binom{\theta-t}{2}\right)$ roots such

that $q_\alpha = -1$, $t + 1$ roots such that $q_\alpha = q$ and $\theta - 1 - t$ roots such that $q_\alpha = -q^{-1}$. If $N = 2k$, then $\text{ord}(-q^{-1}) = 2k$ and $\text{ord}(q^2) = k$, so

$$\begin{aligned} \dim \mathfrak{B}(V) &= 2^{(\theta-1)\theta-t(t-1)-(\theta-t)(\theta-t-1)} k^{t(t-1)+(\theta-t)(\theta-t-1)} (2k)^\theta \\ &= 2^{2t(\theta-t)+\theta} k^{\theta^2-2t\theta+2t^2}. \end{aligned}$$

If N is odd, then $\text{ord}(-q^{-1}) = 2N$ and $\text{ord}(q^2) = N$, so

$$\begin{aligned} \dim \mathfrak{B}(V) &= 2^{\theta(\theta-1)-t(t-1)-(\theta-t)(\theta-1-t)} N^{t(t-1)+(\theta-t)(\theta-1-t)+t+1} \\ &= (2N)^{\theta-1-t} = 2^{(2t+1)(\theta-t)+1} N^{\theta^2-2t\theta+2t^2}. \end{aligned}$$

(c) In a similar way, we have $q_{\mathbf{u}_i} = (-\zeta^2)q_{\mathbf{u}_{2i}}$, which equals $(-\zeta^2)^2 = \zeta$ if κ_{2i} is even, and $(-1)(-\zeta^2) = \zeta^2$ if κ_{2i} is odd; moreover $q_{11} = \zeta$. There are $2\binom{t}{2} + \binom{\theta-t}{2}$ roots such that $q_\alpha = -\zeta^2$, $2\binom{\theta}{2} - \binom{t}{2} - \binom{\theta-t}{2}$ roots such that $q_\alpha = -1$, $t + 1$ roots such that $q_\alpha = \zeta$ and $\theta - 1 - t$ roots such that $q_\alpha = \zeta^2$. Since $\text{ord} \zeta = \text{ord} \zeta^2 = 3$ and $\text{ord}(-\zeta^2) = 6$, we have

$$\begin{aligned} \dim \mathfrak{B}(V) &= 2^{\theta(\theta-1)-t(t-1)-(\theta-t)(\theta-1-t)} 6^{t(t-1)+(\theta-t)(\theta-1-t)} 3^\theta \\ &= 2^{\theta(\theta-1)} 3^{\theta^2-2t\theta+2t^2}. \end{aligned} \quad \square$$

Proposition 4.7. *Let V be a standard braided vector space of type G_2 as in Proposition 3.11. If the braiding is as in case (a) of that proposition, the associated Nichols algebra $\mathfrak{B}(V)$ is of finite dimension if and only if q is a root of unity of order $N \geq 4$.*

When finite, the dimension of $\mathfrak{B}(V)$ is as follows:

- In case (a) of Proposition 3.11,

$$\dim \mathfrak{B}(V) = \begin{cases} N^6 & \text{if } 3 \text{ does not divide } N, \\ N^6/27 & \text{if } 3 \text{ divides } N. \end{cases}$$

- In case (b), $\dim \mathfrak{B}(V) = 2^{12}$.

Proof. For (a) note that q is a root of unity because x_1 has finite height, and $q_\alpha = q$ if $\alpha \in \{e_1, e_1 + e_2, 2e_1 + e_2\}$, while $q_\alpha = q^3$ if $\alpha \in \{e_2, 3e_1 + e_2, 3e_1 + 2e_2\}$.

Case (b) can be checked as follows:

type	q_{x_2}	$q_{x_1x_2}$	$q_{x_1^3x_2^2}$	$q_{x_1^2x_2}$	$q_{x_1^3x_2}$	q_{x_1}	$\dim \mathfrak{B}(V)$
$\begin{array}{c} \zeta^2 \quad \zeta \quad \zeta^{-1} \\ \circ \text{---} \circ \end{array}$	8	4	2	8	2	4	2^{12}
$\begin{array}{c} \zeta^2 \quad \zeta^3 \quad -1 \\ \circ \text{---} \circ \end{array}$	2	8	2	4	8	4	2^{12}
$\begin{array}{c} \zeta \quad \zeta^5 \quad -1 \\ \circ \text{---} \circ \end{array}$	2	4	8	4	2	8	2^{12}

This completes the proof. □

5. Presentation by generators and relations of Nichols algebras of standard braided vector spaces

In this section we give presentations for the Nichols algebras of standard braided vector spaces. We start with some technical results about relations and PBW bases in Section 5A; also we calculate the coproduct of some hyperwords in $T(V)$. In Sections 5B, 5C and 5D we express the braided commutator of two PBW generators as a combination of elements of the PBW basis under some assumptions. Then we obtain the desired presentation with a proof similar to the ones in [Andruskiewitsch and Dăscălescu 2005] and [Andruskiewitsch and Schneider 2002b]. In Section 5E we solve the problem when the braiding is of Cartan type using the transformation in Section 2C.

For rank two, a set of (not necessarily minimal) relations is given in Theorem 4 of [Heckenberger 2007].

5A. Some general relations. Let V be a standard braided vector space with connected Dynkin diagram and let C be the corresponding Cartan matrix. In what follows we assume that C is symmetrizable and of finite type. Let x_1, \dots, x_θ be an ordered basis of V and $\{x_\alpha : \alpha \in \Delta^+(\mathfrak{B}(V))\}$ a set of PBW generators as in the previous section. Here $x_\alpha \in \mathfrak{B}(V)$ is, by abuse of notation, the image by the canonical projection of $x_\alpha \in T(V)$, the hyperword corresponding to a Lyndon word l_α . As before, we write

$$q_\alpha := \chi(\alpha, \alpha) \quad \text{and} \quad N_\alpha := \text{ord } q_\alpha \quad \text{for } \alpha \in \Delta^+(\mathfrak{B}(V)).$$

Each x_α is homogeneous and has the same degree as l_α . Also,

$$x_\alpha \in T(V)_{g_\alpha}^{\chi_\alpha}, \tag{5-1}$$

where $g_\alpha = g_1^{b_1} \dots g_\theta^{b_\theta}$ and $\chi_\alpha = \chi_1^{b_1} \dots \chi_\theta^{b_\theta}$ if $\alpha = b_1 \mathbf{e}_1 + \dots + b_\theta \mathbf{e}_\theta$.

Proposition 5.1. *If the matrix of the braiding is symmetric, the PBW basis is orthogonal with respect to the bilinear form in Proposition 1.4.*

Proof. We must prove that $(u|v) = 0$, where $u \neq v$ are ordered products of PBW generators (we also allow powers greater than the corresponding heights). We argue by induction on $k := \max\{\ell(u), \ell(v)\}$. If $k = 1$, then v is some x_j and u is either 1 or x_i ; since $(x_i|x_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, \theta\}$, the proposition holds in this case.

Suppose the statement is valid when the length of both words is less than k , and let $u, v \in B_{I(V)}$ be distinct hyperwords such that one (or both) has length k . If both are hyperletters, they have different degrees $\alpha \neq \beta \in \mathbb{Z}^\theta$, so $u = x_\alpha, v = x_\beta$, and $(x_\alpha|x_\beta) = 0$, since the homogeneous components are orthogonal for $(|)$.

Suppose that $u = x_\alpha$ and $v = x_{\beta_1}^{h_1} \dots x_{\beta_m}^{h_m}$, for some $x_{\beta_1} > \dots > x_{\beta_m}$. If u and v have different \mathbb{Z}^θ -degrees, they are orthogonal. Hence we assume that $\alpha = \sum_{j=1}^m h_j \beta_j$. By [Bourbaki 1968, VI, Proposition 19], we can reorder the β_i 's, using h_i copies of β_i , in such a way that each partial sum is a root. By [Rosso 1999, Proposition 21], the order induced by the Lyndon words l_α is convex (the order on Lyndon words used there is the same as ours). Therefore $\beta_m < \alpha$. Using Lemma 1.9 and (1-8),

$$(u|v) = (x_\alpha|w)(1|x_{\beta_m}) + (1|w)(x_{\alpha_n}|x_{\beta_m}) + \sum_{\substack{l_1 \geq \dots \geq l_p > \alpha \\ l_i \in L}} (x_{l_1, \dots, l_p}|w)([l_1]_c \dots [l_p]_c|x_{\beta_m}),$$

where $v = wx_{\beta_m}$. Note that $(1|x_{\beta_m}) = (1|w) = 0$. Also, $[l_1]_c \dots [l_p]_c$ is a linear combination of greater hyperwords of the same degree and an element of $I(V)$. From the inductive hypothesis and the fact that $I(V)$ is the radical of the bilinear form, we see that $([l_1]_c \dots [l_p]_c|x_{\beta_m}) = 0$.

Now consider

$$\begin{aligned} u &= x_{\alpha_1}^{j_1} \dots x_{\alpha_n}^{j_n} \text{ with } x_{\alpha_1} > \dots > x_{\alpha_n}, \\ v &= x_{\beta_1}^{h_1} \dots x_{\beta_m}^{h_m} \text{ with } x_{\beta_1} > \dots > x_{\beta_m}. \end{aligned}$$

Since the bilinear form is symmetric, we may as well assume that $x_{\alpha_n} \leq x_{\beta_m}$. Using Lemma 1.9 and (1-8), we obtain

$$\begin{aligned} (u|v) &= (w|1)(x_{\alpha_n}|v) + \sum_{i=0}^{h_m} \binom{h_m}{i}_{q_{\beta_m}} (w|x_{\beta_1}^{h_1} \dots x_{\beta_{m-1}}^{h_{m-1}} x_{\beta_m}^i)(x_{\alpha_n}|x_{\beta_m}^{h_m-i}) \\ &\quad + \sum_{\substack{l_1 \geq \dots \geq l_p > l, l_i \in L \\ 0 \leq j \leq m}} (w|x_{l_1, \dots, l_p}^{(j)})(x_{\alpha_n}|[l_1]_c \dots [l_p]_c [x_{\beta_m}]_c^j), \end{aligned}$$

where $w = x_{\alpha_1}^{h_1} \dots x_{\alpha_m}^{h_m-1}$. Note that in the first summand, $(w|1) = 0$. In the last sum, $(x_{\alpha_n}|[l_1]_c \dots [l_p]_c [x_{\beta_m}]_c^j)$ vanishes, because by earlier results $[l_1]_c \dots [l_p]_c [x_{\beta_m}]_c^j$ is a combination of hyperwords of the PBW basis greater or equal than it and an element of $I(V)$, then we use induction hypothesis and the fact that $I(V)$ is the radical of this bilinear form. Since $x_{\alpha_n}, x_{\beta_m}^{h_m-i}$ are different elements of the PBW basis for $h_m - i \neq 1$, we have

$$(u|v) = (h_m)_{q_{\beta_m}} (w|x_{\beta_1}^{h_1} \dots x_{\beta_{m-1}}^{h_{m-1}} x_{\beta_m}^{h_m-1})(x_{\alpha_n}|x_{\beta_m}).$$

This is clearly zero if $\alpha_n \neq \beta_m$. To see that it is zero also if $\alpha_n = \beta_m$, note that in that case w and $x_{\beta_1}^{h_1} \dots x_{\beta_{m-1}}^{h_{m-1}} x_{\beta_m}^{h_m-1}$ are different products of PBW generators, and use the induction hypothesis. \square

Corollary 5.2. *If $\alpha \in \Delta^+(\mathfrak{B}(V))$, then $x_\alpha^{N_\alpha} = 0$.*

Proof. Let (q_{ij}) be symmetric. If $u = x_{\alpha_1}^{j_1} \dots x_{\alpha_n}^{j_n}$, $x_{\alpha_1} > \dots > x_{\alpha_n}$, then

$$(u|u) = \prod_{i=1}^n (j_i)_{q_{\alpha_i}}! (x_{\alpha_i}|x_{\alpha_i})^{j_i}, \quad (5-2)$$

where $(x_\alpha|x_\alpha) \neq 0$ for all $\alpha \in \Delta^+(\mathfrak{B}(V))$.

If we consider $u = x_\alpha^{N_\alpha}$, we have $(u|v) = 0$ for each v in the PBW basis (because v is an ordered product of x_β 's different from u), and $(u|u) = 0$ since $q_\alpha \in \mathbb{G}_{N_\alpha}$. Also, $(I(V)|x_\alpha^{N_\alpha}) = 0$, because it is the radical of this bilinear form, so $(T(V)|x_\alpha^{N_\alpha}) = 0$, and then $x_\alpha^{N_\alpha} \in I(V)$. That is, we have $x_\alpha^{N_\alpha} = 0$ in $\mathfrak{B}(V)$.

For the general case, recall that a diagonal braiding is twist equivalent to a braiding with a symmetric matrix [Andruskiewitsch and Schneider 2002a, Theorem 4.5]. Also, there exists a linear isomorphism between the corresponding Nichols algebras. The corresponding x_α are related by a nonzero scalar, because they are an iteration of braided commutators between the hyperwords. \square

In what follows, \mathfrak{J} denotes the family of \mathbb{Z}^θ -graded (hence \mathbb{N} -graded) ideals of $T(V)$ that are generated by their components of degree > 1 . For each $I \in \mathfrak{J}$, $\mathfrak{B} = T(V)/I$ is a \mathbb{Z}^θ -graded algebra such that $\mathfrak{B}^0 = \mathfrak{k}1$ and $\mathfrak{B}^1 \simeq V$.

We shall need some technical results about graded algebras between $T(V)$ and $\mathfrak{B}(V)$. We start with three lemmas dealing with the presence of some important roots in $\Delta(\mathfrak{B})$. Remember that a Lyndon word is a PBW generator in $\mathfrak{B} = T(V)/I$ if it is not a linear combination of greater words modulo I in $T(V)$. We shall relate the absence of some roots in $\Delta(\mathfrak{B})$ (meaning that the Lyndon words of such degrees are linear combinations of greater words modulo I) with the validity of certain relations in \mathfrak{B} .

Lemma 5.3. *Let $i, j \in \{1, \dots, \theta\}$ be distinct, and consider $I \in \mathfrak{J}$, $\mathfrak{B} = T(V)/I$. Let $D_k, k = 1, \dots, \theta$, be skew derivations of \mathfrak{B} as in Proposition 2.1, and assume that $x_i^N = 0$ if $q_{i_1}^{n_1} q_{i_2}^{n_2} \dots q_{i_n}^{n_n} \neq 1$ for all $n \in \mathbb{N}_0$ (where $N = \text{ord } q_{ii}$).*

There exists $m \in \mathbb{N}$ such that $x_i^m x_j$ is a linear combination of greater hyperwords (for a fixed order such that $x_i < x_j$) modulo I if and only if, in \mathfrak{B} .

$$(\text{ad}_c x_i)^{m_{ij}+1} x_j = 0. \quad (5-3)$$

Proof. If $(\text{ad}_c x_i)^m x_j = 0$, there exist $a_k \in \mathfrak{k}$ such that

$$0 = [x_i^m x_j]_c = (\text{ad}_c x_i)^m x_j = x_i^m x_j + \sum_{k=0}^{m-1} a_k x_i^k x_j x_i^{m-k}.$$

Conversely, suppose there exists $m \in \mathbb{N}$ such that $x_i^m x_j$ is a linear combination of greater hyperwords modulo I . Let

$$n = \min \{m \in \mathbb{N} : x_i^m x_j \text{ is a linear combination of greater hyperwords}\}.$$

If $x_i^n = 0$, then q_{ii} is a root of unity. In this case, if N is the order of q_{ii} , then $x_i^N = 0$ and $x_i^{N-1} \neq 0$. Also,

$$(\text{ad}_c x_i)^N x_j = x_i^N x_j + \sum_{s=1}^{N-1} \binom{N}{s}_{q_{ii}} + x_j x_i^N = 0,$$

because $\binom{N}{s}_{q_{ii}} = 0$ for $0 < s < N$. Hence, we can assume $x_i^n \neq 0$ and $(n)_{q_{ii}}! \neq 0$.

Note that $[x_i^{n-k} x_j x_i^k]_c = [x_i^{n-k} x_j]_c x_i^k$. Since \mathfrak{B} is graded, $x_i^n x_j$ is a linear combination of terms $x_i^{n-k} x_j x_i^k$, $0 \leq k < n$. Hence there exist $\alpha_k \in \mathfrak{k}$ such that

$$[x_i^n x_j]_c = \sum_{k=1}^n \alpha_k [x_i^{n-k} x_j]_c x_i^k.$$

Applying D_i we obtain

$$0 = D_i([x_i^n x_j]_c) = \sum_{k=1}^n \alpha_k D_i([x_i^{n-k} x_j]_c x_i^k) = \sum_{k=1}^n \alpha_k (k)_{q_{ii}} [x_i^{n-k} x_j]_c x_i^k.$$

By the hypothesis about n , $\alpha_1 = 0$. Since $(n)_{q_{ii}}! \neq 0$, applying D_i several times we conclude that $\alpha_k = 0$ for $k = 2, \dots, n$. Hence $[x_i^n x_j]_c = 0$. \square

Recall that (5-3) holds in $\mathfrak{B}(V)$, for $1 \leq i \neq j \leq \theta$.

The second lemma is related to Dynkin diagrams of a standard braiding which have two consecutive simple edges.

Lemma 5.4. *Let $I \in \mathfrak{J}$ and $\mathfrak{B} = T(V)/I$. Assume that*

- *there exist skew derivations D_k in \mathfrak{B} as in Proposition 2.1;*
- *there exist different $j, k, l \in \{1, \dots, \theta\}$ such that $m_{kj} = m_{kl} = 1$, $m_{jl} = 0$;*
- *$(\text{ad } x_k)^2 x_j = (\text{ad } x_k)^2 x_l = (\text{ad } x_j) x_l = 0$ hold in \mathfrak{B} ;*
- *$x_k^2 = 0$ if $q_{kk} q_{kj} q_{jk} \neq 1$ or $q_{kk} q_{kl} q_{lk} \neq 1$.*

- (1) *If we order the letters x_1, \dots, x_θ such that $x_j < x_k < x_l$, then $x_j x_k x_l x_k$ is a linear combination of greater words modulo I if and only if, in \mathfrak{B} ,*

$$[(\text{ad } x_j)(\text{ad } x_k) x_l, x_k]_c = 0. \quad (5-4)$$

- (2) *If V is standard and $q_{kk} \neq -1$, then (5-4) holds in \mathfrak{B} .*

- (3) *If V is standard and $\dim \mathfrak{B}(V) < \infty$, then (5-4) holds in $\mathfrak{B} = \mathfrak{B}(V)$.*

Proof. (1) (\Leftarrow) If (5-4) holds, then $x_j x_k x_l x_k$ is a linear combination of greater words, by Remark 1.7, and

$$[x_j x_k x_l x_k]_c = [[x_j x_k x_l]_c, x_k]_c = [(\text{ad } x_j)(\text{ad } x_k) x_l, x_k]_c.$$

(\Rightarrow) If $x_j x_k x_l x_k$ is a linear combination of greater words, then the hyperword $[x_j x_k x_l x_k]_c$ is a linear combination of hyperwords corresponding to words greater than $x_j x_k x_l x_k$ (of the same degree, because \mathfrak{B} is homogeneous); this follows from Remark 1.7. Since $(\text{ad } x_k)^2 x_j = (\text{ad } x_k)^2 x_l = (\text{ad } x_j) x_l = 0$, we do not consider hyperwords with $x_j x_k^2$, $x_k^2 x_l$ and $x_j x_l$ as factors of the corresponding words. Then $[x_j x_k x_l x_k]_c$ is a linear combination of

$$\begin{aligned} [x_k x_l x_k x_j]_c &= [x_k x_l]_c x_k x_j, & [x_l x_k x_j x_k]_c &= x_l x_k [x_j x_k]_c, \\ [x_k x_j x_k x_l]_c &= x_k [x_j x_k x_l]_c, & [x_l x_k^2 x_j]_c &= x_l x_k^2 x_j. \end{aligned}$$

Since $D_j([x_j x_k x_l x_k]_c) = D_j(x_k [x_j x_k x_l]_c) = D_j(x_l x_k [x_j x_k]_c) = 0$, in that linear combination there are no hyperwords ending in x_j ; indeed,

$$D_j([x_k x_l]_c x_k x_j) = [x_k x_l]_c x_k, \quad D_j(x_l x_k^2 x_j) = x_l x_k^2,$$

and $[x_k x_l]_c x_k$, $x_l x_k^2$ are linearly independent. Therefore, there exist $\alpha, \beta \in \mathfrak{k}$ such that

$$[x_j x_k x_l x_k]_c = \alpha x_l x_k [x_j x_k]_c + \beta x_k [x_j x_k x_l]_c.$$

Applying D_l , we have

$$0 = \alpha q_{kj} q_{kk} x_l [x_j x_k]_c + \alpha (1 - q_{kj} q_{jk}) x_l x_k x_j + \beta q_{kk} q_{kj} q_{kl} [x_j x_k x_l]_c.$$

Now $x_l [x_j x_k]_c$, $x_l x_k x_j$ and $[x_j x_k x_l]_c$ are linearly independent by Lemma 2.7, so $\alpha = \beta = 0$.

(2) We assume that some quantum Serre relations hold in \mathfrak{B} ; using them we get

$$\begin{aligned} x_j x_k x_l x_k &= q_{kl}^{-1} (1 + q_{kk})^{-1} x_j x_k^2 x_l + q_{kk} q_{kj} (1 + q_{kk})^{-1} x_j x_l x_k^2 \\ &= q_{kk}^{-1} q_{kj}^{-1} q_{kl}^{-1} x_k x_j x_k x_l + q_{kk}^{-1} q_{kj}^{-1} q_{kl}^{-1} (1 + q_{kk})^{-1} x_k^2 x_j x_l \\ &\quad + q_{kk} q_{kl} q_{jk} (1 + q_{kk})^{-1} x_l x_j x_k^2. \end{aligned}$$

It follows that $x_k x_j x_k x_l \notin G_I$ for an order such that $x_j < x_k < x_l$. Also, $x_j x_l x_k^2 \notin G_I$, since $(\text{ad}_c x_j) x_l = 0$ and (5-4) is valid by part (1).

(3) If V is a standard braided vector space satisfying the conditions of the lemma, consider V_k as the braided vector space obtained transforming by s_k . Then $\tilde{m}_{jl} = 0$. Therefore $\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B}(V_k))$, so $s_k(\mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B}(V))$. It follows that $x_j x_k x_l x_k$ is a linear combination of greater words, since it is a Lyndon word when we consider an order such that $x_j < x_k < x_l$. \square

We now prove two relations involving the double edge in the Dynkin diagram of a standard braiding of type B_θ .

Lemma 5.5. *Let $I \in \mathfrak{J}$ and $\mathfrak{B} = T(V)/I$. Assume that*

- *there exist $j \neq k \in \{1, \dots, \theta\}$ such that $m_{kj} = 2$, $m_{jk} = 1$;*

- there exist skew derivations as in Proposition 2.1;
- the following relations hold in \mathfrak{B} :

$$\begin{aligned} (\operatorname{ad} x_k)^3 x_j &= (\operatorname{ad} x_j)^2 x_k = 0; \\ x_k^3 &= x_j^2 = 0 \quad \text{if } q_{kk}^3 = q_{jj}^2 = 1. \end{aligned} \tag{5-5}$$

- (1) If we order the letters x_1, \dots, x_θ such that $x_k < x_j$, then $x_k^2 x_j x_k x_j$ is a linear combination of greater words modulo I if and only if, in \mathfrak{B} ,

$$[(\operatorname{ad} x_k)^2 x_j, (\operatorname{ad} x_k) x_j]_c = 0. \tag{5-6}$$

- (2) If V is standard, $q_{jj} \neq -1$ and $q_{kk}^2 q_{kj} q_{jk} = 1$, then (5-6) holds in \mathfrak{B} .

- (3) If V is standard and $\dim \mathfrak{B}(V) < \infty$, then (5-6) holds in $\mathfrak{B} = \mathfrak{B}(V)$.

Proof. (1) (\Leftarrow) If (5-6) holds in \mathfrak{B} , then $x_k^2 x_j x_k x_j$ is a linear combination of greater words. This follows from Remark 1.7, and

$$[x_k^2 x_j x_k x_j]_c = [[x_k^2 x_j]_c, [x_k x_j]_c] = [(\operatorname{ad} x_k)^2 x_j, (\operatorname{ad} x_k) x_j]_c.$$

(\Rightarrow) If $x_k^2 x_j x_k x_j$ is a linear combination of greater words, then $[x_k^2 x_j x_k x_j]_c$ is a linear combination of hyperwords corresponding to words greater than $x_k^2 x_j x_k x_j$ (of the same degree, because \mathfrak{B} is homogeneous).

First, there are no hyperwords whose corresponding words have factors $x_k^3 x_j$ or $x_k x_j^2$, by (5-5). Since $[x_k^2 x_j x_k x_j]_c \in \ker D_k$ and

$$\begin{aligned} D_k(x_j [x_k^2 x_j]_c x_k) &= x_j [x_k^2 x_j]_c, \\ D_k([x_k x_j]_c^2 x_k) &= [x_k x_j]_c^2, \\ D_k(x_j [x_k x_j]_c x_k^2) &= (1 + q_{kk}) x_j [x_k x_j]_c x_k, \end{aligned}$$

in that linear combination there are no hyperwords ending in x_k , except $x_j^2 x_k^3$ if $q_{kk} \in \mathbb{G}_3$. We consider $q_{jj} \neq -1$ if $q_{kk} \in \mathbb{G}_3$, since otherwise $x_j^2 x_k^3 = 0$ by assumption. Then there exist $\alpha, \alpha' \in \mathfrak{k}$ such that

$$[x_k^2 x_j x_k x_j]_c = \alpha [x_k x_j x_k^2 x_j]_c + \alpha' x_j^2 x_k^3 = \alpha [x_k x_j]_c [x_k^2 x_j]_c + \alpha' x_j^2 x_k^3.$$

We prove by direct calculation that $D_j([x_k^2 x_j x_k x_j]_c) = 0$. Applying D_j to the previous equality,

$$\begin{aligned} 0 &= \alpha' (1 + q_{jj}) x_j x_k^3 + \chi(\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k + \mathbf{e}_j) \alpha (\operatorname{ad} x_k)^2(x_j) x_k \\ &\quad + (1 - q_{kj} q_{jk}) (1 - q_{kk} q_{kj} q_{jk}) \alpha (\operatorname{ad} x_k)(x_j) x_k^2, \end{aligned}$$

where we use that $(\operatorname{ad} x_k)^3(x_j) = 0$ and

$$x_k (\operatorname{ad} x_k)^m(x_j) = (\operatorname{ad} x_k)^{m+1}(x_j) + q_{kk}^m q_{kj} (\operatorname{ad} x_k)^m(x_j) x_k.$$

Since $(1 - q_{kj}q_{jk})(1 - q_{kk}q_{kj}q_{jk}) \neq 0$ and $(\text{ad } x_k)^2(x_j)x_k$, $(\text{ad } x_k)(x_j)x_k^2$, $x_jx_k^3$ are linearly independent, it follows that $\alpha = \alpha' = 0$.

(2) Using $(\text{ad } x_j)^2x_k = 0$ in the first equality and $(\text{ad } x_k)^3x_j = 0$ in the last expression,

$$\begin{aligned} x_k^2x_jx_kx_j &= (1 + q_{jj})^{-1}q_{jk}^{-1}x_k^2x_j^2x_k + (1 + q_{jj})^{-1}q_{jk}q_{jj}x_k^3x_j^2 \\ &\in (3)_{q_{kk}}(1 + q_{jj})^{-1}q_{kj}q_{jk}q_{jj}x_k^2x_jx_kx_j + \mathbb{k}^{\times}_{>x_k^2x_jx_kx_j}. \end{aligned}$$

Suppose that $(3)_{q_{kk}}(1 + q_{jj})^{-1}q_{kj}q_{jk}q_{jj} = 1$; that is, $(3)_{q_{kk}} = (1 + q_{jj})$. Then $q_{jj} = q_{kk} + q_{kk}^2$, so

$$1 = q_{jj}q_{kj}q_{jk} = q_{kk}q_{kj}q_{jk} + q_{kk}^2q_{kj}q_{jk} = q_{kk}q_{kj}q_{jk} + 1,$$

which is a contradiction since $q_{kk}q_{kj}q_{jk} \in \mathbb{k}^{\times}$. It follows that $x_k^2x_jx_kx_j$ is a linear combination of greater words, so (5-6) follows by previous item.

(3) If V is a standard braided vector space, and we consider V_j as the braided vector space obtained transforming by s_j , then $\tilde{m}_{kj} = 2$. Therefore, $3\mathbf{e}_k + \mathbf{e}_j \notin \Delta^+(\mathfrak{B}(V_k))$, so $s_j(3\mathbf{e}_k + \mathbf{e}_j) = 3\mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B}(V))$. Since $x_k^2x_jx_kx_j$ is a Lyndon word of degree $3\mathbf{e}_k + 2\mathbf{e}_j$ if $x_k < x_j$, then it is a linear combination of greater words. \square

Lemma 5.6. *Let $I \in \mathfrak{J}$ and $\mathfrak{B} = T(V)/I$. Assume that*

- *there exist different $j, k, l \in \{1, \dots, \theta\}$ such that $m_{kj} = 2$, $m_{jk} = m_{jl} = m_{lj} = 1$, $m_{kl} = 0$;*
- *there exist skew derivations in \mathfrak{B} as in Proposition 2.1;*
- *the following relations hold in \mathfrak{B} : (5-4), (5-6),*

$$\begin{aligned} (\text{ad } x_k)^3x_j &= (\text{ad } x_j)^2x_k = (\text{ad } x_j)^2x_l = (\text{ad } x_k)x_l = 0, \\ x_k^3 &= x_j^2 = 0 \text{ if } q_{kk}^3 = q_{jj}^2 = 1. \end{aligned} \tag{5-7}$$

- (1) *If we order the letters x_1, \dots, x_θ so that $x_k < x_j < x_l$, then $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words modulo I if and only if, in \mathfrak{B} ,*

$$[(\text{ad } x_k)^2(\text{ad } x_j)x_l, (\text{ad } x_k)x_j]_c = 0. \tag{5-8}$$

- (2) *If V is a standard braided vector space and $q_{kk} \notin \mathbb{G}_3$, $q_{jj} \neq -1$, then (5-8) holds in \mathfrak{B} .*

- (3) *If V is standard and $\dim \mathfrak{B}(V) < \infty$, then (5-8) holds in $\mathfrak{B}(V)$.*

Proof. (1) (\Leftarrow) As in the last two lemmas, if (5-8) is valid, then $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words, by Remark 1.7, and

$$[x_k^2x_jx_lx_kx_j]_c = [(\text{ad } x_k)^2(\text{ad } x_j)x_l, (\text{ad } x_k)x_j]_c.$$

(\Rightarrow) Suppose that $x_k^2 x_j x_l x_k x_j$ is a linear combination of greater words. Then $[x_k^2 x_j x_l x_k x_j]_c$ is a linear combination of hyperwords corresponding to words greater than $x_k^2 x_j x_l x_k x_j$ (of the same degree, because \mathfrak{B} is homogeneous). We discard those words which have $x_k x_l$, $x_k^3 x_j$, $x_k x_j^2$, $x_j^2 x_l$, $x_k x_j x_l x_j$ and $x_k^2 x_j x_k x_j$, in view of our hypotheses about \mathfrak{B} .

Since $D_k([x_k^2 x_j x_l x_k x_j]_c) = 0$, the coefficients of hyperwords corresponding to words ending in x_k are 0, as in Lemma 5.5, except for $[x_j x_l]_c x_j x_k^3$, $x_l x_j^2 x_k^3$, if $q_{kk} \in \mathbb{G}_3$. Thus

$$[x_k^2 x_j x_l x_k x_j]_c = \alpha [x_k x_j]_c [x_k^2 x_j x_l]_c + \beta [x_k x_j x_l]_c [x_k^2 x_j]_c \\ + \gamma x_l [x_k x_j]_c [x_k^2 x_j]_c + \mu [x_j x_l]_c x_j x_k^3 + \nu x_l x_j^2 x_k^3.$$

By direct calculation, $D_j([x_k^2 x_j x_l x_k x_j]_c) = D_j([x_k^2 x_j x_l]_c) = D_j([x_k x_j x_l]_c) = 0$, so applying D_j to the previous equality we get

$$0 = \alpha q_{jk}^2 q_{jj} q_{jl} x_j [x_k^2 x_j x_l]_c + \beta (1 - q_{kk} q_{kj} q_{jk}) (1 - q_{kk}^2 q_{kj} q_{jk}) [x_k x_j x_l]_c x_k^2 \\ + \gamma (1 - q_{kk} q_{kj} q_{jk}) (1 - q_{kk}^2 q_{kj} q_{jk}) x_l [x_k x_j]_c x_k^2 + \gamma q_{jk}^2 q_{jj} x_l x_j [x_k^2 x_j]_c \\ + \mu [x_j x_l]_c x_k^3 + \nu (1 + q_{jj}) x_l x_j x_k^3,$$

Note that $\nu = 0$ if $q_{jj} \neq -1$; otherwise, $x_j^2 = 0$ by hypothesis, so we can discard this last summand. The other hyperwords appearing in this expression are linearly independent, since the corresponding words are linearly independent by Lemma 2.7. Thus $\alpha = \beta = \gamma = \mu = 0$.

(2) If $q_{kk} \notin \mathbb{G}_3$ and $q_{jj} \neq -1$, then $x_k^2 x_j x_l x_k x_j$ is a linear combination of greater words, as can be seen using the quantum Serre relations in a way similar to that in Lemma 5.6. Now apply part (1).

(3) If V is a standard braided vector space, consider V_k as the braided vector space obtained transforming by s_k . Then $\tilde{m}_{kj} = 2$. Therefore, $\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B}(V_k))$ by Lemma 5.5, so $s_k(\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B}(V))$. Since $x_k^2 x_j x_l x_k x_j$ is a Lyndon word, it follows that it is a linear combination of greater words, and we apply (1). \square

We now give explicit formulas for the comultiplication of these hyperwords.

Lemma 5.7. *Consider the structure of graded braided Hopf algebra of $T(V)$ (see Section 2A). For all $k \neq j$,*

$$\Delta((\text{ad } x_k)^{m_{kj}+1} x_j) = (\text{ad } x_k)^{m_{kj}+1} x_j \otimes 1 + 1 \otimes (\text{ad } x_k)^{m_{kj}+1} x_j \\ + \prod_{1 \leq t \leq m_{kj}} (1 - q_{kk}^t q_{kj} q_{jk}) x_k^{m_{kj}+1} \otimes x_j. \quad (5-9)$$

Proof. We have $F_k((\text{ad } x_k)^{m_{kj}+1} x_j) = 0$ by the definition of m_{kj} and (2-5). Also, $F_l((\text{ad } x_k)^{m_{kj}+1} x_j)$ for $l \neq k$ by (2-6) and the properties of F_l , so

$$\Delta_{1,m_{kj}}((\text{ad } x_k)^{m_{kj}+1}x_j) = \sum_{l=1}^{\theta} x_l \otimes F_l((\text{ad } x_k)^{m_{kj}+1}x_j) = 0.$$

Now $D_k([x_k^i x_j]_c x_k^{s-i}) = 0$ from (2-3), and from (2-4)

$$D_j([x_k^i x_j]_c x_k^{s-i}) = \prod_{1 \leq t \leq m_{kj}} (1 - q_{kk}^t q_{kj} q_{jk}) x_k^{m_{ij}+1},$$

so we deduce that

$$\Delta_{m_{kj},1}((\text{ad } x_k)^{m_{kj}+1}x_j) = \prod_{1 \leq t \leq m_{kj}} (1 - q_{kk}^t q_{kj} q_{jk}) x_k^{m_{ij}+1} \otimes x_j.$$

Since hyperwords form a basis of $T(V)$, we can write, for each $1 < s < m_{kj}$,

$$\begin{aligned} \Delta_{m_{kj}+1-s,s}((\text{ad } x_k)^{m_{kj}+1}x_j) \\ = \sum_{t=0}^{m_{kj}+1-s} \varepsilon_{st} [x_k^t x_j]_c x_k^{m_{kj}+1-s-t} \otimes x_k^s + \sum_{p=0}^s \rho_{sp} x_k^{m_{kj}+1-s} \otimes [x_k^{s-p} x_j]_c x_k^p, \end{aligned}$$

for some $\varepsilon_{st}, \rho_{sp} \in \mathbb{k}$. Then, for each $0 \leq t \leq m_{kj} + 1 - s$,

$$\begin{aligned} 0 &= ((\text{ad } x_k)^{m_{kj}+1}x_j \mid [x_k^t x_j]_c x_k^{m_{kj}+1-t} x_k^s) \\ &= (((\text{ad } x_k)^{m_{kj}+1}x_j)_{(1)} \mid [x_k^t x_j]_c x_k^{m_{kj}+1-t-s}) (((\text{ad } x_k)^{m_{kj}+1}x_j)_{(2)} \mid x_k^s) \\ &= \varepsilon_{st} ([x_k^t x_j]_c x_k^{m_{kj}+1-t-s} \mid [x_k^t x_j]_c x_k^{m_{kj}+1-t-s}) (x_k^s \mid x_k^s) \\ &= \varepsilon_{st} (m_{kj} + 1 - s - t)_{q_{kk}}! (s)_{q_{kk}}! ([x_k^t x_j]_c \mid [x_k^t x_j]_c), \end{aligned}$$

where we have used that $(\text{ad } x_k)^{m_{kj}+1}x_j \in I(V)$ for the first equality, (1-8) for the second, (1-10) and the orthogonality between increasing products of hyperwords for the third, and (5-2) for the last. Since

$$(m_{kj} + 1 - s - t)_{q_{kk}}! (s)_{q_{kk}}! ([x_k^t x_j]_c \mid [x_k^t x_j]_c) \neq 0,$$

we conclude that $\varepsilon_{st} = 0$ for all $0 \leq t \leq m_{kj} + 1 - s$. In a similar way, $\rho_{sp} = 0$ for all $0 \leq p \leq s$, so we obtain (5-9). \square

Lemma 5.8. *Let \mathfrak{B} be a braided graded Hopf algebra provided with an inclusion of braided vector spaces $V \hookrightarrow \mathcal{P}(\mathfrak{B})$. Assume that*

- *there exist $1 \leq j \neq k \neq l \leq \theta$ such that $m_{kj} = m_{kl} = 1, m_{jl} = 0$;*
- *$(\text{ad } x_k)^2 x_j = (\text{ad } x_k)^2 x_l = (\text{ad } x_j) x_l = 0$ in \mathfrak{B} ;*
- *$x_k^2 = 0$ if $q_{kk} q_{kj} q_{jk} \neq 1$ or $q_{kk} q_{kl} q_{lk} \neq 1$.*

Then $u := [(\text{ad } x_j)(\text{ad } x_k)x_l, x_k]_c \in \mathcal{P}(\mathfrak{B})$.

Proof. From (2-3), $D_j(u) = 0$. Also, $D_k((\text{ad } x_j)(\text{ad } x_k)x_l) = 0$, so

$$D_k(u) = (1 - q_{kk}^2 q_{jk} q_{kj} q_{kl} q_{lk})(\text{ad } x_j)(\text{ad } x_k)x_l = 0.$$

From (2-4) and the properties of D_l we have

$$\begin{aligned} D_l(u) &= q_{lk}(1 - q_{kl}q_{lk})[x_j x_k]_c x_k - q_{jk} q_{kk} q_{lk}(1 - q_{kl}q_{lk})x_k [x_j x_k]_c \\ &= q_{lk}(1 - q_{lk}q_{kl})[[x_j x_k]_c, x_k]_c = 0. \end{aligned}$$

Then $\Delta_{31}(u) = 0$. From (2-6) and the properties of F_k and F_l , we have $F_k(u) = F_l(u) = 0$. Using (2-5), we have

$$\begin{aligned} F_j(u) &= (1 - q_{jk}q_{kj})[x_k x_l]_c x_k - q_{jk} q_{kk} q_{lk} q_{kj}(1 - q_{jk}q_{kj})x_k [x_k x_l]_c \\ &= (1 - q_{lk}q_{kl})(1 - q_{kj}q_{jk}q_{kk}^2 q_{lk}q_{jk})[x_k x_l]_c x_k = 0. \end{aligned}$$

Thus $\Delta_{13}(u) = 0$ as well.

Also, we have

$$\Delta(u) = \Delta((\text{ad } x_j)(\text{ad } x_k)x_l) \Delta(x_k) - q_{e_k+e_j+e_j, e_j} \Delta(x_k) \Delta((\text{ad } x_j)(\text{ad } x_k)x_l),$$

and looking at the terms in $\mathfrak{B}^2 \otimes \mathfrak{B}^2$,

$$\begin{aligned} \Delta_{2,2}(u) &= (1 - q_{lk}q_{kl})[x_j x_k]_c \otimes (x_l x_k - q_{kj}q_{jk}q_{kk}^2 q_{lk}x_k x_l) \\ &\quad + (1 - q_{kj}q_{jk})q_{lk}q_{kk}(x_j x_k - q_{jk}x_k x_j) \otimes [x_k x_l]_c \\ &= (1 - q_{kj}q_{jk} - (1 - q_{lk}q_{kl})q_{kk}q_{jk}q_{kj})q_{lk}q_{kk}[x_j x_k]_c \otimes [x_k x_l]_c. \end{aligned}$$

Now a calculation shows that $u \in \mathcal{P}(\mathfrak{B})$:

$$\begin{aligned} 1 - q_{kj}q_{jk} - (1 - q_{lk}q_{kl})q_{kk}q_{jk}q_{kj} &= 1 - q_{kj}q_{jk} - q_{kk}q_{jk}q_{kj} + q_{kk}^{-1} \\ &= q_{kk}^{-1}(1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk}) = 0. \quad \square \end{aligned}$$

Lemma 5.9. *Let \mathfrak{B} be a braided graded Hopf algebra provided with an inclusion of braided vector spaces $V \hookrightarrow \mathcal{P}(\mathfrak{B})$. Assume that*

- *there exist $1 \leq k \neq j \leq \theta$ such that $m_{kj} = 2, m_{jk} = 1$;*
- *$(\text{ad } x_s)^{m_{st}+1}x_t = 0$, for all $1 \leq s \neq t \leq \theta$ in \mathfrak{B} ;*
- *$x_s^{m_{st}+1} = 0$ for each s such that $q_{ss}^{m_{st}}q_{st}q_{ts} \neq 1$, for some $t \neq s$.*

(a) *If $v := [(\text{ad } x_k)^2 x_j, (\text{ad } x_k)x_j]_c$, there exists $b \in \mathfrak{k}$ such that*

$$\Delta(v) = v \otimes 1 + 1 \otimes v + b(1 - q_{kk}^2 q_{kj}^2 q_{jk}^2 q_{jj})x_k^3 \otimes x_j^2. \quad (5-10)$$

(b) *Assume there exist $l \neq j, k$ such that $m_{jl} = m_{lj} = 1, m_{kl} = m_{lk} = 0$, and that (5-4) is valid in \mathfrak{B} . Set*

$$w := [(\text{ad } x_k)^2(\text{ad } x_j)x_l, (\text{ad } x_k)x_j]_c.$$

Then there exist constants $b_1, b_2 \in \mathfrak{k}$ such that

$$\Delta(w) = w \otimes 1 + 1 \otimes w + b_1 v \otimes x_l + b_2(1 - q_{kk}^2 q_{kj} q_{jk}) x_k^3 \otimes ((\text{ad } x_j) x_l) x_j. \quad (5-11)$$

Proof. (a) $F_j(v) = 0$ since v is a braided commutator of two elements in $\ker F_j$. Using (1-4) we have $[(\text{ad } x_k)^2 x_j, x_j]_c = q_{kj}(q_{jj} - q_{kk})[x_k x_j]_c^2$, so we calculate

$$\begin{aligned} F_k(v) &= (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})[x_k x_j]_c^2 - q_{kk}^2 q_{kj}^2 q_{jk} q_{jj}(1-q_{kj}q_{jk})x_j[x_k^2 x_j]_c \\ &\quad + q_{kk}^2 q_{jk}(1-q_{kj}q_{jk})[x_k^2 x_j]_c x_j - q_{kk}^3 q_{kj}^2 q_{jk} q_{jj}(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})[x_k x_j]_c^2 \\ &= q_{kk}^2 q_{jk} q_{kj}(1-q_{kj}q_{jk})(q_{jj} - q_{kk}) \\ &\quad + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})(1-q_{kk}^3 q_{kj}^2 q_{jk}^2 q_{jj})[x_k x_j]_c^2, \end{aligned}$$

which vanishes since the coefficient of $[x_k x_j]_c^2$ is zero for each possible braiding. Thus

$$\Delta_{1,4}(v) = x_k \otimes F_k(v) = 0.$$

Also, $D_k(v) = 0$, and a calculation gives

$$\begin{aligned} D_j(v) &= (1 - q_{kj}q_{jk})([x_k^2 x_j] x_k + (1 - q_{kk}q_{kj}q_{jk})q_{jk}q_{jj}x_k^2[x_k x_j]_c \\ &\quad - q_{kk}^2 q_{kj}^2 q_{jk} q_{jj}(1 - q_{kk}q_{kj}q_{jk})[x_k x_j]_c x_k^2 - q_{kk}^2 q_{kj}^2 q_{jk}^3 q_{jj}^2 x_k[x_k^2 x_j]_c) \\ &= (1 + (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2) \\ &\quad (1 - q_{kj}q_{jk})[x_k^2 x_j] x_k, \end{aligned}$$

where we have reordered the hyperwords and used that $(\text{ad } x_k)^3 x_j = 0$; also,

$$1 + (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2 = 0, \quad (5-12)$$

by calculation for each possible braiding. Thus

$$\Delta_{4,1}(v) = D_j(v) \otimes x_j = 0.$$

To finish, we use the fact that $\Delta(v)$ equals

$$\Delta((\text{ad}_c x_k)^2 x_j) \Delta((\text{ad}_c x_k) x_j) - \chi(2e_k + e_j, e_k + e_j) \Delta((\text{ad}_c x_k) x_j) \Delta((\text{ad}_c x_k)^2 x_j).$$

Looking at the terms in $\mathfrak{B}^3 \otimes \mathfrak{B}^2$ and $\mathfrak{B}^2 \otimes \mathfrak{B}^3$, and using the definition of the braided commutator, we obtain

$$\begin{aligned} \Delta_{32}(v) &= (1 - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2)[x_k^2 x_j]_c \otimes [x_k x_j]_c \\ &\quad + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj}(x_k[x_k x_j]_c - q_{kk}q_{kj}[x_k x_j]_c x_k) \otimes [x_k x_j]_c \\ &\quad + (1 - q_{kj}q_{jk})^2(1 - q_{kk}^2 q_{kj}q_{jk})(1 - q_{kk}^2 q_{kj}^2 q_{jk}^2 q_{jj})x_k^3 \otimes x_j^2 \\ &= (1 + (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2)[x_k^2 x_j]_c \otimes [x_k x_j]_c \\ &\quad + b_1(1 - q_{kk}^2 q_{kj}^2 q_{jk}^2 q_{jj})x_k^3 \otimes x_j^2. \end{aligned}$$

Also,

$$\begin{aligned}
\Delta_{23}(v) &= (1 - q_{kk}q_{kj}q_{jk})(1 - q_{kj}q_{jk})x_k^2 \\
&\quad \otimes \left((1 + q_{kk})q_{kk}q_{jk}[x_kx_j]_c x_j - (1 + q_{kk})q_{kk}^2q_{kj}^2q_{jk}^2q_{jj}x_j[x_kx_j]_c \right. \\
&\quad \left. + x_j[x_kx_j]_c - q_{kk}^4q_{kj}^2q_{jk}^3q_{jj}[x_kx_j]_c x_j \right) \\
&= (1 - q_{kk}^4q_{kj}^3q_{jk}^3q_{jj}^2 + (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj}) \\
&\quad (1 - q_{kk}q_{kj}q_{jk})(1 - q_{kj}q_{jk})x_k^2 \otimes x_j[x_kx_j]_c.
\end{aligned}$$

Using (5-12), we obtain (5-10).

(b) We set $y = (\text{ad } x_k)^2(\text{ad } x_j)x_l$ and $z = (\text{ad } x_k)x_j$. Note that $\Delta(w) = \Delta(y)\Delta(z) - \chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)\Delta(z)\Delta(y)$ and that

$$\begin{aligned}
\Delta(y) &= y \otimes 1 + (1 - q_{jl}q_{lj})(\text{ad } x_k)^2x_j \otimes x_l \\
&\quad + (1 - q_{kj}q_{jk})(1 - q_{kk}q_{kj}q_{jk})x_k^2 \otimes (\text{ad } x_j)x_l \\
&\quad + (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})x_k \otimes (\text{ad } x_k)(\text{ad } x_j)x_l + 1 \otimes y, \\
\Delta(z) &= z \otimes 1 + (1 - q_{kj}q_{jk})x_k \otimes x_j + 1 \otimes z.
\end{aligned}$$

From (2-3) we have $D_k(w) = 0$, and from (2-4),

$$\begin{aligned}
D_l(w) &= (1 - q_{lj}q_{jl})q_{lk}q_{lj} \left[(\text{ad } x_k)^2x_j, (\text{ad } x_k)x_j \right]_c, \\
D_j(w) &= -(1 - q_{kj}q_{jk})q_{kk}^{-2}q_{kj}^{-1}q_{kl}^{-1}(\text{ad } x_k)^3(\text{ad } x_j)x_l \\
&\quad = -(1 - q_{kj}q_{jk})q_{kk}^{-2}q_{kj}^{-1}q_{kl}^{-1}[(\text{ad } x_k)^3x_j, x_l]_c = 0,
\end{aligned}$$

where in the last equality we used (1-4) and the vanishing of $[x_k, x_l]_c = 0$. It follows that

$$\Delta_{51}(w) = (1 - q_{lj}q_{jl})q_{lk}q_{lj} \left[(\text{ad } x_k)^2x_j, (\text{ad } x_k)x_j \right]_c \otimes x_l.$$

Also, $F_j(z) = F_j(y) = F_l(z) = F_l(y) = 0$ by (2-6) and the properties of these skew derivations, so $F_j(w) = F_l(w) = 0$. We now calculate

$$\begin{aligned}
F_k(w) &= (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})[x_kx_jx_l]_c[x_kx_j]_c + q_{kk}^2q_{jk}q_{lk}(1 - q_{kj}q_{jk})[x_k^2x_jx_l]_c x_j \\
&\quad - \chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j) \\
&\quad \left((1 - q_{kj}q_{jk})x_j[x_k^2x_jx_l]_c + (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}q_{jk}[x_kx_jx_l]_c[x_kx_j]_c \right) \\
&= q_{kk}^2q_{jk}q_{lk}(1 - q_{kj}q_{jk})\left[[x_k^2x_jx_l]_c, x_j \right]_c \\
&\quad - (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}^3q_{kj}^2q_{jk}^2q_{jj}q_{lj}q_{lk}\left[[x_kx_j]_c, [x_kx_jx_l]_c \right]_c \\
&= q_{kk}^2q_{kj}q_{jk}q_{jj}q_{lj}q_{lk} \\
&\quad (1 - q_{kj}q_{jk} - (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk})\left[[x_kx_j]_c, [x_kx_jx_l]_c \right]_c \\
&= 0,
\end{aligned}$$

where we used (1-4) and (5-4) in the third equality, and we calculate that

$$1 - q_{kj}q_{jk} - (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk} = 0 \quad (5-13)$$

for each possible standard braiding. It follows that $\Delta_{15}(w) = 0$.

We find each of the other terms of $\Delta(w)$ by direct calculation. First,

$$\begin{aligned} \Delta_{42}(w) &= (1 - \chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)\chi(\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l)) y \otimes z \\ &\quad + (1 - q_{kj}q_{jk})(1 - q_{lj}q_{jl}) \\ &\quad \quad (q_{lk}[x_k^2x_j]_c x_k \otimes x_l x_j - \chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)q_{jk}^2 q_{jj} x_k [x_k^2 x_j]_c \otimes x_j x_l) \\ &\quad + (1 - q_{kj}q_{jk})(1 - q_{kk}q_{kj}q_{jk}) \\ &\quad \quad (\chi(\mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)x_k^2 z - \chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)z x_k^2) \otimes [x_j x_l]_c \\ &= (1 - q_{kj}q_{jk})q_{lk} (1 - q_{jk}q_{kj} + (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}) \\ &\quad \quad [x_k^2 x_j]_c x_k \otimes [x_j x_l]_c, \end{aligned}$$

which is seen to equal 0. In a similar way we calculate

$$\begin{aligned} \Delta_{33}(w) &= (1 - q_{lj}q_{jl})[x_k^2 x_j] \otimes (x_l z - \chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)\chi(\mathbf{e}_k + \mathbf{e}_j, \mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l)z x_l) \\ &\quad + (1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk})\chi(\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)(x_k z - q_{kk}q_{kj}z x_k) \otimes [x_k x_j x_l]_c \\ &\quad + (1 - q_{kk}q_{kj}q_{jk})(1 - q_{kj}q_{jk})^2 x_k^3 \\ &\quad \quad \otimes (\chi(\mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k)[x_j x_l]_c x_j - \chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)\chi(\mathbf{e}_j, 2\mathbf{e}_k)x_j [x_j x_l]_c) \\ &= ((1 + q_{kk})(1 - q_{kk}q_{kj}q_{jk}) - q_{kk}q_{kj}q_{jk}q_{jj}(1 - q_{lj}q_{jl})) \\ &\quad \quad \chi(\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)[x_k^2 x_j]_c \otimes [x_k x_j x_l]_c \\ &\quad + (1 - q_{kk}q_{kj}q_{jk})(1 - q_{kj}q_{jk})^2 (1 - q_{kk}^2 q_{kj}q_{jk})x_k^3 \otimes [x_j x_l]_c x_j, \end{aligned}$$

and the coefficient of $[x_k^2 x_j]_c \otimes [x_k x_j x_l]_c$ is zero (we calculate it for each possible standard braiding). Finally,

$$\begin{aligned} \Delta_{24}(w) &= (1 - q_{kk}q_{kj}q_{jk})(1 - q_{kj}q_{jk})x_k^2 \\ &\quad \otimes ((1 + q_{kk})\chi(\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k)[x_k x_j x_l]_c x_j \\ &\quad \quad - (1 + q_{kk})\chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)q_{jk}x_j [x_k x_j x_l]_c \\ &\quad \quad - \chi(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)\chi(\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k)[[x_k x_j]_c, [x_j x_l]_c]) \\ &= (1 - q_{kk}q_{kj}q_{jk})(1 - q_{kj}q_{jk})\chi(\mathbf{e}_j + \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j)q_{kj} \\ &\quad \quad (q_{kk}(1 - q_{kk}q_{kj}q_{jk}) - q_{jj}(1 - q_{jl}q_{lj}))x_k^2 \otimes x_j [x_k x_j x_l]_c \\ &= 0. \end{aligned}$$

From these calculations, we obtain (5-11). \square

5B. Presentation when the type is A_θ . We now assume V is a standard braided vector space of type A_θ and \mathfrak{B} a \mathbb{Z}^θ -graded algebra, provided with an inclusion of vector spaces $V \hookrightarrow \mathfrak{B}^1 = \bigoplus_{1 \leq j \leq \theta} \mathfrak{B}^{e_j}$. We can extend the braiding to \mathfrak{B} by setting

$$c(u \otimes v) = \chi(\alpha, \beta)v \otimes u, \quad u \in \mathfrak{B}^\alpha, v \in \mathfrak{B}^\beta, \alpha, \beta \in \mathbb{N}^\theta.$$

We assume that on \mathfrak{B} we have

$$\begin{aligned} x_i^2 &= 0 && \text{if } q_{ii} = -1, \\ \text{ad}_c x_i(x_j) &= 0 && \text{if } |j - i| > 1, \\ (\text{ad}_c x_i)^2(x_j)_c &= 0 && \text{if } |j - i| = 1, \\ [(\text{ad}_c x_i)(\text{ad}_c x_{i+1})x_{i+2}, x_{i+1}]_c &= 0 && 2 \leq i \leq \theta - 1. \end{aligned}$$

Using the same notation as in Section 4B,

$$x_{e_i} = x_i, \quad x_{\mathbf{u}_{i,j}} := [x_i, x_{\mathbf{u}_{i+1,j}}]_c \quad (i < j).$$

Lemma 5.10. *Let $1 \leq i \leq j < p \leq r \leq \theta$. The following relations hold in \mathfrak{B} :*

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pr}}]_c = 0, \quad p - j \geq 2; \tag{5-14}$$

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{j+1,r}}]_c = x_{\mathbf{u}_{ir}}. \tag{5-15}$$

Proof. Note that $x_{\mathbf{u}_{pr}}$ belongs to the subalgebra generated by x_p, \dots, x_r , and $[x_{\mathbf{u}_{ij}}, x_s]_c = 0$, for each $p \leq s \leq r$. Equation (5-14) follows from this.

We prove (5-15) by induction on $j - i$: if $i = j$, it is exactly the definition of $x_{\mathbf{u}_{ir}}$. To prove the inductive step, we use the inductive hypothesis, (5-14) and (1-4) (the braided Jacobi identity) to obtain

$$\begin{aligned} [x_{\mathbf{u}_{i,j+1}}, x_{\mathbf{u}_{j+2,r}}]_c &= [[x_{\mathbf{u}_{ij}}, x_{i+1}]_c, x_{\mathbf{u}_{j+2,r}}]_c = [x_{\mathbf{u}_{ij}}, [x_{i+1}, x_{\mathbf{u}_{j+2,r}}]_c]_c \\ &= [x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{j+1,r}}]_c = x_{\mathbf{u}_{ir}}, \end{aligned}$$

and (5-15) is also proved. \square

Lemma 5.11. *If $i < p \leq r < j$, the following relation holds in \mathfrak{B} :*

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pr}}]_c = 0. \tag{5-16}$$

Proof. When $p = r = j - 1$ and $i = j - 2$, note that this is exactly

$$[(\text{ad}_c x_i)(\text{ad}_c x_{i+1})x_{i+2}, x_{i+1}]_c = 0.$$

Then, by (1-4),

$$[x_{\mathbf{u}_{i-1,j}}, x_{j-1}]_c = [[x_{i-1}, x_{\mathbf{u}_{i,j}}]_c, x_{j-1}]_c = [x_{i-1}, [x_{\mathbf{u}_{i,j}}, x_{j-1}]_c]_c.$$

We assume that $j - i > 2$, so $[x_{i-1}, x_{j-1}]_c = 0$ by the hypothesis on \mathfrak{B} . Now we prove the case $p = r = j - 1$ by induction on $p - i$.

Using (1-4) and (5-15), we also have

$$\begin{aligned} [x_{\mathbf{u}_{i,j+1}}, x_p]_c &= [[x_{\mathbf{u}_{i,j}}, x_{j+1}]_c, x_p]_c = [x_{\mathbf{u}_{i,j}}, [x_{j+1}, x_p]_c]_c \\ &\quad + q_{j+1,p} [x_{\mathbf{u}_{i,j}}, x_{j-1}]_c x_{j+1} - \chi(\mathbf{u}_{i,j}, \mathbf{e}_{j+1}) x_{j+1} [x_{\mathbf{u}_{i,j}}, x_{j-1}]_c, \end{aligned}$$

so using that $[x_{j+1}, x_p]_c = 0$ if $j > p$, by induction on $j - p$ we prove (5-16) for the case $p = r$.

For the general case, we use (1-4) one more time as follows

$$\begin{aligned} [x_{\mathbf{u}_{i,j}}, x_{\mathbf{u}_{p,r+1}}]_c &= [x_{\mathbf{u}_{i,j}}, [x_{\mathbf{u}_{pr}}, x_{r+1}]_c]_c = [[x_{\mathbf{u}_{i,j}}, x_{\mathbf{u}_{pr}}]_c, x_{r+1}]_c \\ &\quad - \chi(\mathbf{u}_{pr}, \mathbf{e}_{r+1}) [x_{\mathbf{u}_{ij}}, x_{r+1}]_c x_{\mathbf{u}_{pr}} + \chi(\mathbf{u}_{ij}, \mathbf{u}_{pr}) x_{\mathbf{u}_{pr}} [x_{\mathbf{u}_{ij}}, x_{r+1}]_c, \end{aligned}$$

and we prove (5-16) by induction on $r - p$. \square

Lemma 5.12. *The following relations hold in \mathfrak{B} :*

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{jp}}]_c = 0 \quad \text{if } i \leq j < p, \quad (5-17)$$

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pj}}]_c = 0 \quad \text{if } i < p \leq j. \quad (5-18)$$

Proof. To prove (5-17), note that if $i = j = p - 1$, we have

$$[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{i,i+1}}]_c = [x_i, [x_i, x_{i+1}]_c]_c = (\text{ad } x_i)^2 x_{i+1} = 0.$$

Since $[x_i, x_{\mathbf{u}_{i+2,p}}]_c = 0$ for each $p > i + 1$ by (5-14), we use (1-4), the previous case and (5-15) to obtain

$$[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{ip}}]_c = [x_{\mathbf{u}_{ii}}, [x_{\mathbf{u}_{i,i+1}}, x_{\mathbf{u}_{i+2,p}}]_c]_c = 0.$$

Now, if $i < j < p$, from (5-14) and the relations between the q_{st} we obtain

$$[x_{\mathbf{u}_{i+1,j}}, x_{\mathbf{u}_{ip}}]_c = -\chi(\mathbf{u}_{ip}, \mathbf{u}_{i+1,j}) [x_{\mathbf{u}_{ip}}, x_{\mathbf{u}_{i+1,j}}]_c = 0.$$

Using (1-4) and the previous case we conclude

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{ip}}]_c = [[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{i+1,j}}]_c, x_{\mathbf{u}_{ip}}]_c = 0.$$

The proof of (5-18) is analogous. \square

Lemma 5.13. *If $i < p \leq r < j$, the following relation holds in \mathfrak{B} :*

$$[x_{\mathbf{u}_{ir}}, x_{\mathbf{u}_{pj}}]_c = \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) (1 - q_{r,r+1} q_{r+1,r}) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}}. \quad (5-19)$$

Proof. We calculate

$$\begin{aligned} [x_{\mathbf{u}_{ir}}, x_{\mathbf{u}_{pj}}]_c &= [x_{\mathbf{u}_{ir}}, [x_{\mathbf{u}_{pr}}, x_{\mathbf{u}_{r+1,j}}]_c]_c \\ &= \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}} - \chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j}) x_{\mathbf{u}_{ij}} x_{\mathbf{u}_{pr}} \\ &= (\chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) - \chi(\mathbf{u}_{ij}, \mathbf{u}_{pr}) \chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j})) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}} \\ &= \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) (1 - \chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j}) \chi(\mathbf{u}_{r+1,j}, \mathbf{u}_{pr})) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}}, \end{aligned}$$

where we have used (5-15) in the first equality, (1-4) in the second, (5-18) in the third and the relation between the q_{ij} in the last. \square

We now prove the main theorem of this subsection, namely, the presentation by generators and relations of the Nichols algebra associated to V .

Theorem 5.14. *Let V be a standard braided vector space of type A_θ , where $\theta = \dim V$, and let $C = (a_{ij})_{i,j \in \{1, \dots, \theta\}}$ be the corresponding Cartan matrix of type A_θ .*

The Nichols algebra $\mathfrak{B}(V)$ is presented by the generators x_i , $1 \leq i \leq \theta$, and the relations

$$\begin{aligned} x_\alpha^{N_\alpha} &= 0, & \alpha \in \Delta^+; \\ \text{ad}_c(x_i)^{1-a_{ij}}(x_j) &= 0, & i \neq j; \\ [(\text{ad}_{x_{j-1}})(\text{ad}_{x_j})x_{j+1}, x_j]_c &= 0, & 1 < j < \theta, q_{jj} = -1. \end{aligned}$$

The following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \quad \text{where } 0 \leq h_j < N_{\beta_j} \text{ where } \beta_j \in S_I, \text{ for } 1 \leq j \leq P. \quad (5-20)$$

Proof. From Corollary 4.2 and the definitions of the x_α , we know that the last statement about the PBW basis is true.

Let \mathfrak{B} be the algebra presented by generators x_1, \dots, x_θ and the relations in the statement of the theorem. From Lemmas 5.3, 5.4 and Corollary 5.2 we have a canonical epimorphism $\phi : \mathfrak{B} \rightarrow \mathfrak{B}(V)$. The last relation also holds in \mathfrak{B} for $q_{jj} \neq 1$, by Lemma 5.4(2).

The rest is similar to the proofs of [Andruskiewitsch and Dăscălescu 2005, Lemma 3.7] and [Andruskiewitsch and Schneider 2002b, Lemma 6.12]. Consider the subspace \mathcal{I} of \mathfrak{B} generated by the elements in (5-20). Using Lemmas 5.10, 5.11, 5.12 and 5.13 we prove that \mathcal{I} is an ideal. But $1 \in \mathcal{I}$, so $\mathcal{I} = \mathfrak{B}$.

The images under ϕ of the elements in (5-20) form a basis of $\mathfrak{B}(V)$, so ϕ is an isomorphism. \square

The presentation and dimension of $\mathfrak{B}(V)$ agree with the results presented in [Andruskiewitsch and Dăscălescu 2005] and [Andruskiewitsch and Schneider 2002b].

5C. Presentation when the type is B_θ . We now assume V is a standard braided vector space of type B_θ and \mathfrak{B} is a \mathbb{Z}^θ -graded algebra, provided with an inclusion of vector spaces $V \hookrightarrow \mathfrak{B}^1 = \bigoplus_{1 \leq j \leq \theta} \mathfrak{B}^{e_j}$. Then we can extend the braiding to \mathfrak{B} . We assume the following relations in \mathfrak{B} :

$$\begin{aligned} x_i^2 &= 0 & \text{if } q_{ii} = -1, \\ x_1^3 &= 0 & \text{if } q_{11} \in \mathbb{G}_3, \\ (\text{ad}_c x_i)x_j &= 0 & \text{if } |j - i| > 1, \\ (\text{ad}_c x_i)^2 x_j &= 0 & \text{if } |j - i| = 1 \text{ and } i \neq 1, \end{aligned}$$

$$\begin{aligned}
[(\text{ad}_c x_i)(\text{ad}_c x_{i+1})x_{i+2}, x_{i+1}]_c &= 0 \quad \text{if } 2 \leq i \leq \theta, \\
(\text{ad}_c x_1)^3 x_2 &= 0, \\
[(\text{ad}_c x_1)^2 x_2, (\text{ad}_c x_1)x_2]_c &= 0, \\
[(\text{ad}_c x_1)^2 (\text{ad}_c x_2)x_3, (\text{ad}_c x_1)x_2]_c &= 0.
\end{aligned}$$

Using the same notation as in Section 4B,

$$x_{\mathbf{v}_{ij}} = [x_{\mathbf{u}_i}, x_{\mathbf{u}_j}]_c, \quad 1 \leq i < j \leq \theta.$$

From the proof of the relations corresponding the A_θ case, we know that (5-14), (5-15), (5-16), (5-18) and (5-19) hold for $i \geq 1$, but for relation (5-17) we must assume $i > 1$.

Lemma 5.15. *Suppose $1 \leq s < t$ and $1 < k \leq j$. The following relations hold in \mathfrak{B} :*

$$[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c \begin{cases} = 0 & \text{if } t+1 < k, \\ = x_{\mathbf{v}_{sj}} & \text{if } t+1 = k < j, \\ = 0 & \text{if } s+1 < k \leq j \leq t, \\ = \chi(\mathbf{v}_{st}, \mathbf{u}_{kt})(1 - q_{t,t+1}q_{t+1,t})x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}} & \text{if } s+1 < k \leq t < j, \\ = \chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j})x_{\mathbf{v}_{jt}} & \text{if } s+1 = k \leq j < t, \\ = (\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t}))x_{\mathbf{u}_{1t}}^2 & \text{if } s+1 = k, j = t, \\ \in kx_{\mathbf{v}_{ij}} + kx_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}} + kx_{\mathbf{u}_{s+1,j}}x_{\mathbf{v}_{sj}} & \text{if } s+1 = k \leq t < j, \\ = \gamma_{st}^{kj} x_{\mathbf{u}_{ks}} x_{\mathbf{v}_{jt}} & \text{if } k \leq s < j \leq t, \\ \in kx_{\mathbf{u}_{ks}}x_{\mathbf{v}_{ij}} + kx_{\mathbf{u}_{ks}}x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}} + kx_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}} & \text{if } k \leq s < t < j, \\ = 0 & \text{if } k \leq j \leq s, \end{cases}$$

where $\gamma_{st}^{kj} = \chi(\mathbf{u}_{1t}, \mathbf{u}_{kj})\chi(\mathbf{u}_{1s}, \mathbf{u}_{ks})(1 - q_{s,s+1}q_{s+1,s})$.

Proof. The first, third and last equalities follow from the vanishing of $[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{kj}}]_c$ and $[x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{kj}}]_c = 0$, using (5-14), (5-16), (5-17) or (5-18) as the case maybe, together with (1-4).

For the second case, we use that $[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{t+1,j}}]_c = 0$, (5-15) and (1-4) to obtain

$$x_{\mathbf{v}_{sj}} = [x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1j}}]_c = [x_{\mathbf{u}_{1s}}, [x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{t+1,j}}]_c]_c = [[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1t}}]_c, x_{\mathbf{u}_{t+1,j}}]_c = [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{t+1,j}}]_c.$$

For the fourth, we use (1-4) and the third case to calculate

$$\begin{aligned}
[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c &= [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{kt}}, x_{\mathbf{u}_{t+1,j}}]_c]_c \\
&= \chi(\mathbf{v}_{st}, \mathbf{u}_{kt})x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}} - \chi(\mathbf{u}_{kt}, \mathbf{u}_{t+1,j})x_{\mathbf{v}_{sj}}x_{\mathbf{u}_{kt}} \\
&= \chi(\mathbf{v}_{st}, \mathbf{u}_{kt})(1 - \chi(\mathbf{u}_{kt}, \mathbf{u}_{t+1,j})\chi(\mathbf{u}_{t+1,j}, \mathbf{u}_{kt}))x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}}.
\end{aligned}$$

For the fifth, note that $\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j})^{-1} = \chi(\mathbf{u}_{s+1,j}, \mathbf{u}_{1t})$. Then use (5-15), (5-16) and (1-4) to prove that

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{s+1,j}}]_c &= [[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1t}}]_c, x_{\mathbf{u}_{s+1,j}}]_c \\ &= \chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j})x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}} - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t})x_{\mathbf{u}_{1t}}x_{\mathbf{u}_{1s}} \\ &= \chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j})(x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}} - \chi(\mathbf{u}_{1j}, \mathbf{u}_{1t})x_{\mathbf{u}_{1t}}x_{\mathbf{u}_{1s}}). \end{aligned}$$

The sixth case is similar.

For the seventh case, we use (1-4), (1-5) and the previous case to calculate

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{s+1,j}}]_c &= [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{s+1,t}}, x_{\mathbf{u}_{t+1,j}}]_c]_c \\ &= (\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t}))(x_{\mathbf{u}_{1t}}^2, x_{\mathbf{u}_{t+1,j}}] \\ &\quad + \chi(\mathbf{v}_{st}, \mathbf{u}_{s+1,t})x_{\mathbf{u}_{s+1,t}}x_{\mathbf{v}_{sj}} - \chi(\mathbf{u}_{s+1,t}, \mathbf{u}_{t+1,j})x_{\mathbf{v}_{sj}}x_{\mathbf{u}_{s+1,t}} \\ &= (\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t}))((x_{\mathbf{v}_{tj}} + \chi(\mathbf{u}_{1t}, \mathbf{u}_{1j})x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}}) \\ &\quad + \chi(\mathbf{u}_{1t}, \mathbf{u}_{t+1,j})x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}}) - \chi(\mathbf{u}_{s+1,t}, \mathbf{u}_{t+1,j})x_{\mathbf{v}_{tj}} \\ &\quad + (\chi(\mathbf{v}_{st}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{s+1,t}, \mathbf{u}_{t+1,j}))\chi(\mathbf{v}_{sj}, \mathbf{u}_{s+1,t})x_{\mathbf{u}_{s+1,t}}x_{\mathbf{v}_{sj}}. \end{aligned}$$

We use the previous cases, (5-16) and (5-19) to calculate for the eighth case

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c &= [[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1t}}]_c, x_{\mathbf{u}_{kj}}]_c \\ &= \chi(\mathbf{u}_{1t}, \mathbf{u}_{kj})(\chi(\mathbf{u}_{1s}, \mathbf{u}_{ks})(1 - q_{s,s+1}q_{s+1,s})x_{\mathbf{u}_{ks}}x_{\mathbf{u}_{1j}})x_{\mathbf{u}_{1t}} \\ &\quad - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t})x_{\mathbf{u}_{1t}}(\chi(\mathbf{u}_{1s}, \mathbf{u}_{ks})(1 - q_{s,s+1}q_{s+1,s})x_{\mathbf{u}_{ks}}x_{\mathbf{u}_{1j}}) \\ &= \gamma_{st}^{kj}x_{\mathbf{u}_{ks}}(x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}} - \chi(\mathbf{u}_{1j}, \mathbf{u}_{1t})x_{\mathbf{u}_{1t}}x_{\mathbf{u}_{1j}}). \end{aligned}$$

To conclude, we prove the ninth case in a similar way:

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c &= [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{kt}}, x_{\mathbf{u}_{t+1,j}}]_c]_c \\ &= \gamma_{st}^{kt}(1 - q_{\mathbf{v}_{1t}})[x_{\mathbf{u}_{ks}}x_{\mathbf{u}_{1t}}^2, x_{\mathbf{u}_{t+1,j}}] \\ &\quad + \chi(\mathbf{v}_{st}, \mathbf{u}_{k,t})x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}} - \chi(\mathbf{u}_{kt}, \mathbf{u}_{t+1,j})x_{\mathbf{v}_{sj}}x_{\mathbf{u}_{kt}}. \quad \square \end{aligned}$$

We consider the remaining commutator $[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{jk}}]_c$: when $j = 1$.

Lemma 5.16. *Let $s < t$ in $\{1, \dots, \theta\}$. The following relations hold in \mathfrak{B} :*

$$[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{1k}}]_c = 0 \quad \text{if } s < k \leq t, \quad (5-21)$$

$$[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{st}}]_c = 0. \quad (5-22)$$

Proof. By assumption we have

$$\begin{aligned} [x_{\mathbf{v}_{12}}, x_{\mathbf{u}_{12}}]_c &= [(\text{ad}_c x_1)^2 x_2, (\text{ad}_c x_1)x_2]_c = 0, \\ [x_{\mathbf{v}_{13}}, x_{\mathbf{u}_{12}}]_c &= [(\text{ad}_c x_1)^2 (\text{ad}_c x_2)x_3, (\text{ad}_c x_1)x_2]_c = 0. \end{aligned}$$

For $t \geq 4$, $[x_{\mathbf{u}_{4t}}, x_{\mathbf{u}_{12}}]_c = 0$ by (5-14), and using (1-4),

$$[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{12}}]_c = [[x_{\mathbf{v}_{13}}, x_{\mathbf{u}_{4t}}]_c, x_{\mathbf{u}_{12}}]_c = 0.$$

For each $k \leq t$ we have $[x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{3k}}]_c = [x_1, x_{\mathbf{u}_{3k}}]_c = 0$, so $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{3k}}]_c = 0$. Using (1-4) and (5-15) we have

$$[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1k}}]_c = [x_{\mathbf{v}_{1t}}, [x_{\mathbf{u}_{12}}, x_{\mathbf{u}_{3k}}]_c]_c = 0.$$

Now consider $2 \leq s \leq k$. Since $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1k}}]_c = [x_{\mathbf{u}_{2s}}, x_{\mathbf{u}_{1k}}]_c = 0$ by previous results and (5-16), we conclude from (1-5) and Lemma 5.15 that (5-21) is valid in the general case.

To prove (5-22), we have for $s = 1, t = 2$

$$[x_{\mathbf{u}_{11}}, x_{\mathbf{v}_{12}}]_c = [x_1, x_{\mathbf{v}_{12}}]_c = (\text{ad}_c x_1)^3 x_2 = 0.$$

Using that $[x_1, x_{\mathbf{u}_{3t}}]_c = 0$ if $t \geq 3$ and (1-4), we deduce that

$$[x_{\mathbf{u}_{11}}, x_{\mathbf{v}_{1t}}]_c = [x_1, [x_{\mathbf{v}_{12}}, x_{\mathbf{u}_{3t}}]_c]_c = 0.$$

If $1 < s < t$ we have, by the previous case,

$$[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{1t}}]_c = -\chi(x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{1t}})[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1s}}]_c = 0.$$

By (5-18), $[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{2s}}]_c = 0$. Also, $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{2s}}]_c = \chi(\mathbf{u}_{1t}, \mathbf{u}_{2s})x_{\mathbf{v}_{st}}$, by Lemma 5.15. Equation (5-22) follows by (1-4) and the last three equalities. \square

Lemma 5.17. *Let $s < k < t$. The following relations hold in \mathfrak{B} :*

$$[x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c = \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k})(1 - q_{k,k+1}q_{k+1,k})x_{\mathbf{u}_{1k}}x_{\mathbf{v}_{st}}, \quad (5-23)$$

$$[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{kt}}]_c = \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k})(1 + q_{\mathbf{u}_{1k}})(1 - q_{k,k+1}q_{k+1,k})x_{\mathbf{u}_{1k}}x_{\mathbf{v}_{st}}. \quad (5-24)$$

Proof. The proof follows by (1-4), the second case of Lemma 5.15 and (5-22):

$$\begin{aligned} [x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c &= [x_{\mathbf{v}_{sk}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{k+1,t}}]_c]_c \\ &= \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k})x_{\mathbf{u}_{1k}}x_{\mathbf{v}_{st}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{k+1,t})x_{\mathbf{v}_{st}}x_{\mathbf{u}_{1k}} \\ &= \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k}) \left(1 - \chi(\mathbf{u}_{1k}, \mathbf{u}_{k+1,t})\chi(\mathbf{u}_{k+1,t}, \mathbf{u}_{1k})\right) x_{\mathbf{u}_{1k}}x_{\mathbf{v}_{st}}, \\ [x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{kt}}]_c &= [x_{\mathbf{u}_{1s}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1t}}]_c]_c \\ &= [x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c + \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k})x_{\mathbf{u}_{1k}}x_{\mathbf{v}_{st}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1t})x_{\mathbf{v}_{st}}x_{\mathbf{u}_{1k}} \\ &= \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k})(q_{\mathbf{u}_{1k}}(1 - q_{k,k+1}q_{k+1,k}) + 1 - q_{k,k+1}q_{k+1,k})x_{\mathbf{u}_{1k}}x_{\mathbf{v}_{st}}. \quad \square \end{aligned}$$

We next deal with the expression of the commutator of two words of type $x_{\mathbf{v}_{st}}$.

Lemma 5.18. *Let $s < t$ and $s \leq k < j$, with $k \neq s$ or $j \neq t$. The following relations hold in \mathfrak{B} :*

$$[x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}]_c \begin{cases} = 0 & \text{if } k < j \leq t, \\ = 0 & \text{if } k = s, t < j, \\ = \chi(\mathbf{v}_{st}, \mathbf{v}_{kt})(1 - q_{t,t+1}q_{t+1,t})x_{\mathbf{v}_{kt}}x_{\mathbf{v}_{sj}} & \text{if } k < t < j, \\ = \chi(\mathbf{v}_{st}, \mathbf{u}_{1t})^2(1 - q_{t,t+1}q_{t+1,t}) & \\ \quad (1 - q_{\mathbf{u}_{1t}}q_{t,t+1}q_{t+1,t})x_{\mathbf{u}_{1t}}^2x_{\mathbf{v}_{sj}} & \text{if } k = t < j, \\ \in kx_{\mathbf{v}_{tj}}x_{\mathbf{v}_{sk}} + kx_{\mathbf{v}_{tk}}x_{\mathbf{v}_{sj}} + kx_{\mathbf{u}_{1k}}x_{\mathbf{u}_{1t}}x_{\mathbf{v}_{sj}} & \text{if } t < k < j. \end{cases}$$

Proof. The first and second equalities follow from (1-4) and (5-21), (5-22), respectively. For the third, we use the previous one and (1-4):

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}]_c &= [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1j}}]_c]_c \\ &= \chi(\mathbf{v}_{st}, \mathbf{u}_{1k})x_{\mathbf{u}_{1k}}(\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t})x_{\mathbf{u}_{1t}}x_{\mathbf{v}_{sj}}) \\ &\quad - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j})(\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t})x_{\mathbf{u}_{1t}}x_{\mathbf{v}_{sj}})x_{\mathbf{u}_{1k}} \\ &= (1 - q_{t,t+1}q_{t+1,t})(\chi(\mathbf{v}_{st}, \mathbf{u}_{1k})\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})x_{\mathbf{u}_{1k}}x_{\mathbf{u}_{1t}}x_{\mathbf{v}_{sj}} \\ &\quad - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j})\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})\chi(\mathbf{v}_{sj}, \mathbf{u}_{1k})x_{\mathbf{u}_{1t}}x_{\mathbf{u}_{1k}}x_{\mathbf{v}_{sj}}) \\ &= \chi(\mathbf{v}_{st}, \mathbf{u}_{1k})\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t})(x_{\mathbf{u}_{1k}}x_{\mathbf{u}_{1k}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1t})x_{\mathbf{u}_{1k}}x_{\mathbf{u}_{1k}})x_{\mathbf{v}_{sj}}. \end{aligned}$$

The fourth case is similar to the previous one.

To prove the last case we use (1-4) and Lemma 5.17:

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}]_c &= [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1j}}]_c]_c \\ &= [\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t})x_{\mathbf{u}_{1t}}x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1j}}]_c \\ &\quad + \chi(\mathbf{v}_{st}, \mathbf{u}_{1k})x_{\mathbf{u}_{1k}}(\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t})x_{\mathbf{u}_{1t}}x_{\mathbf{v}_{sj}}) \\ &\quad - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j})(\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t})x_{\mathbf{u}_{1t}}x_{\mathbf{v}_{sj}})x_{\mathbf{u}_{1k}}. \end{aligned}$$

The proof is finished using (1-5) and the previous identities. \square

Theorem 5.19. *Let V be a standard braided vector space of type B_θ , where $\theta = \dim V$, and let $C = (a_{ij})_{i,j \in \{1, \dots, \theta\}}$ be the corresponding Cartan matrix of type B_θ .*

The Nichols algebra $\mathfrak{B}(V)$ is presented by the generators x_i , $1 \leq i \leq \theta$, and the relations

$$\begin{aligned} x_\alpha^{N_\alpha} &= 0, & \alpha \in \Delta^+; \\ \text{ad}_c(x_i)^{1-a_{ij}}(x_j) &= 0, & i \neq j; \\ [(\text{ad } x_{j-1})(\text{ad } x_j)x_{j+1}, x_j]_c &= 0, & 1 < j < \theta, q_{jj} = -1; \\ [(\text{ad } x_1)^2x_2, (\text{ad } x_1)x_2]_c &= 0, & q_{11} \in \mathbb{G}_3 \text{ or } q_{22} = -1; \\ [(\text{ad } x_1)^2(\text{ad } x_2)x_3, (\text{ad } x_1)x_2]_c &= 0, & q_{11} \in \mathbb{G}_3 \text{ or } q_{22} = -1. \end{aligned}$$

The following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \cdots x_{\beta_P}^{h_P}, \quad \text{where } 0 \leq h_j < N_{\beta_j} - 1 \text{ if } \beta_j \in S_I, \text{ for } 1 \leq j \leq P. \quad (5-25)$$

Proof. The proof is analogous to that of Theorem 5.14, since by the previous lemmas we can express the commutator of two generators $x_\alpha < x_\beta$ as a linear combination of monotone hyperwords whose greater hyperletter is great or equal to x_β . \square

5D. Presentation when the type is G_2 . We now consider standard braidings of type G_2 , with $m_{12} = 3, m_{21} = 1$.

Lemma 5.20. *Let $\mathfrak{B} := T(V)/I$, for some $I \in \mathfrak{S}$, and suppose that*

$$x_1^{\text{ord } q_{11}} = 0, \quad x_2^{\text{ord } q_{22}} = 0, \quad (\text{ad } x_1)^4 x_2 = (\text{ad } x_2)^2 x_1 = 0 \quad (5-26)$$

in \mathfrak{B} . Then

$$(a) \quad [x_1^3 x_2 x_1 x_2]_c = 0 \text{ in } \mathfrak{B} \iff 4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B}).$$

Assume further that the equivalent conditions in (a) hold. Then

$$(b) \quad [(\text{ad } x_1)^3 x_2, (\text{ad } x_1)^2 x_2]_c = 0 \text{ in } \mathfrak{B} \iff 5e_1 + 2e_2 \notin \Delta^+(\mathfrak{B}) \text{ and}$$

$$(c) \quad [[x_1^2 x_2 x_1 x_2]_c, [x_1 x_2]_c]_c = 0 \text{ in } \mathfrak{B} \iff 4e_1 + 3e_2 \notin \Delta^+(\mathfrak{B}).$$

Assume also that the equivalent conditions in (b) and those in (c) hold. Then

$$(d) \quad [[x_1^2 x_2]_c, [x_1^2 x_2 x_1 x_2]_c]_c = 0 \text{ in } \mathfrak{B} \iff 5e_1 + 3e_2 \notin \Delta^+(\mathfrak{B}).$$

In particular, all these relations hold when V is a standard braiding and $\mathfrak{B} = \mathfrak{B}(V)$ is finite-dimensional.

Proof. Take the ordering $x_1 < x_2$, and consider a PBW basis as in Theorem 1.12. Define $\gamma_k := \prod_{0 \leq j \leq k-1} (1 - q_{11}^j q_{12} q_{21})$.

(a) If $[x_1^3 x_2 x_1 x_2]_c = 0$, then $4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$ since there are no possible Lyndon words in S_I : $x_1^3 x_2 x_1 x_2$ is the unique Lyndon word such that $x_1^3 x_2$ and $x_1 x_2^2$ are not factors, and it is not in S_I by assumption.

Conversely, if $4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$, then $[x_1^3 x_2 x_1 x_2]_c$ is a linear combination of greater hyperwords, and $[x_1 x_2 x_1^3 x_2]_c$ and $[x_1^2 x_1^2 x_2]_c$ are the only greater hyperwords that are not in S_I and do not end in x_1 (we discard words ending in x_1 since $[x_1^3 x_2 x_1 x_2]_c$ is in $\ker D_1$). Taking their Shirshov decomposition, we see that there exist $\alpha, \beta \in \mathfrak{k}$ such that

$$[x_1^3 x_2 x_1 x_2]_c - \alpha [x_1 x_2]_c [x_1^3 x_2]_c - \beta [x_1^2 x_2]_c^2 = 0. \quad (5-27)$$

Note that $[x_1^3 x_2 x_1 x_2]_c = \text{ad } x_1([x_1^2 x_2 x_1 x_2]_c)$, so by direct calculation,

$$D_2([x_1^2 x_2 x_1 x_2]_c) = 0.$$

Apply D_2 to both sides of equality (5-27) and express the result as a linear combination of $[x_1^3 x_2]_c x_1$, $[x_1^2 x_2]_c x_1^2$ and $[x_1 x_2]_c x_1^3$. The coefficient of $[x_1 x_2]_c x_1^3$ is

$$\alpha(1 - q_{12}q_{21})(1 - q_{11}q_{12}q_{21}),$$

so $\alpha = 0$. Note also that $D_1^2 D_2([x_1^{-1} x_2 x_1 x_2]_c) = 0$; but

$$D_1^2 D_2([x_1^2 x_2]_c^2) = (1 - q_{12}q_{21})(1 - q_{11}q_{12}q_{21})(1 + q_{11})(q_{2e_1+e_2} + 1)[x_1^2 x_2]_c.$$

Looking at the proof of Proposition 4.7, we see that $q_{2e_1+e_2} \neq -1$, so $\beta = 0$.

(b) Assuming (5-26) and the condition in (a), the only possible Lyndon word of degree $5e_1 + 2e_j$ is $x_1^3 x_2 x_1^2 x_2$, and

$$[x_1^2 x_2 x_1 x_2 x_1 x_2]_c = [(\text{ad } x_1)^3 x_2, (\text{ad } x_1)^2 x_2]_c.$$

Then we proceed as before. One implication is clear. For the other, if $5e_1 + 2e_j \notin \Delta^+(\mathfrak{B})$, there exists $\alpha \in \mathfrak{k}$ such that

$$[(\text{ad } x_1)^3 x_2, (\text{ad } x_1)^2 x_2]_c = \alpha(\text{ad } x_1)^2 x_2 (\text{ad } x_1)^3 x_2.$$

Now we apply D_2 and express the equality as a linear combination of $(\text{ad } x_1)^3 x_2 x_1^2$ and $(\text{ad } x_1)^2 x_2 x_1^3$ (using the hypothesis that $(\text{ad } x_1)^4 x_2 = 0$); the coefficient of $(\text{ad } x_1)^2 x_2 x_1^3$ is $\alpha \gamma_3$, so $\alpha = 0$.

(c) The proof is similar. Since we are considering Lyndon words not having $x_1^3 x_2$ or $x_1 x_2^2$ as a factor, the only possible Lyndon word of degree $4e_1 + 3e_j$ is $x_1^2 x_2 x_1 x_2 x_1 x_2$, and

$$[x_1^2 x_2 x_1 x_2 x_1 x_2]_c = [[x_1^2 x_2 x_1 x_2]_c, [x_1 x_2]_c]_c.$$

If $4e_1 + 3e_j \notin \Delta^+(\mathfrak{B})$, there exist $\alpha_i \in \mathfrak{k}$ such that

$$\begin{aligned} & [x_1^2 x_2 (x_1 x_2)^2]_c \\ &= \alpha_1 [x_1 x_2]_c [x_1^2 x_2 x_1 x_2]_c + \alpha_2 [x_1 x_2]_c^2 [x_1^2 x_2]_c + \alpha_3 x_2 [x_1^2 x_2]_c^2 + \alpha_4 x_2 [x_1 x_2]_c [x_1^3 x_2]_c, \end{aligned}$$

since, as above, we are discarding words greater than $x_1^2 x_2 x_1 x_2 x_1 x_2$ ending in x_1 ; we also discard words with factors $x_1^4 x_2$, $x_1 x_2^2$, $x_1^3 x_2 x_1^2 x_2$, by the assumption on \mathfrak{B} . We apply D_2 to this equality. Using the definition of the braided commutator, we express the hyperletter just considered as a linear combination of elements of the PBW basis, having degree $4e_1 + 2e_2$.

The coefficient of $x_2 [x_1 x_2]_c x_1^3$ is $\alpha_4 \gamma_3$ since this PBW generator appears only in the expression of $D_2(x_2 [x_1 x_2]_c [x_1^3 x_2]_c)$. Thus $\alpha_4 = 0$.

Using this fact, we see that the coefficient of $x_2 [x_1^3 x_2]_c x_1$ is

$$\alpha_3 \gamma_2 (1 + q_{11}) q_{11}^2 q_{12} q_{21}^2 q_{22},$$

since this term appears only in the expression of $D_j(x_2 [x_1^2 x_2]_c^2)$. Thus $\alpha_3 = 0$.

Next, the coefficient of $[x_1x_2]_c^2x_1^2$ is $\alpha_2\gamma_2$, so $\alpha_2 = 0$. Now we calculate the coefficient of $[x_1^2x_2]_c^2$:

$$\alpha_1\gamma_2(\chi(\mathbf{e}_1, 5\mathbf{e}_1 + \mathbf{e}_2) - \chi(2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)) = \alpha_1\gamma_2q_{11}q_{12}(q_{11}^3 - q_{22}q_{12}q_{21}).$$

Since $q_{11}^3 \neq q_{22}q_{12}q_{21}$ for each standard braiding, we conclude that $\alpha_1 = 0$.

(d) If the conditions in (b) and (c) hold, the only possible Lyndon word of degree $5e_1 + 3e_2$ not having $x_1^4x_2$ or $x_1x_2^2$ as factors is $x_1^2x_2x_1^2x_2x_1x_2$, and

$$[x_1^2x_2x_1^2x_2x_1x_2]_c = [[x_1^2x_2]_c, [x_1^2x_2x_1x_2]_c].$$

This hyperword is not in S_I if and only if there exist $v_i \in \mathbb{k}$ such that

$$[x_1^2x_2x_1^2x_2x_1x_2]_c = v_1[x_1^2x_2x_1x_2]_c[x_1^2x_2]_c + v_2[x_1x_2]_c[x_1^2x_2]_c^2 + v_3[x_1x_2]_c^2[x_1^3x_2]_c + v_4x_2[x_1^2x_2]_c[x_1^3x_2]_c. \quad (5-28)$$

Apply D_2 and note that $D_2([x_1^2x_2x_1^2x_2x_1x_2]_c) = 0$ under the hypotheses on \mathfrak{B} . Then express the resulting sum as a linear combination of elements of the PBW basis, which have degree $5e_1 + 2e_2$.

The hyperword $x_2[x_1^2x_2]_c[x_1^3]$ appears only for $D_2(x_2[x_1^2x_2]_c[x_1^3x_2]_c)$, and its coefficient is $v_4\gamma_3$, and since $\gamma_3 \neq 0$ we conclude that $v_4 = 0$.

Analogously, $[x_1x_2]_c^2x_1^3$ appears only for $[x_1x_2]_c^2[x_1^3x_2]_c$ (due to $v_4 = 0$). Its coefficient is $v_3\gamma_3$, so $v_3 = 0$.

Note that $D_1^2D_2([x_1^2x_2x_1x_2]_c) = 0$. We apply $D_1^2D_2$ to the expression (5-28), and obtain

$$0 = v_1\gamma_2(1 + q_{11})[x_1^2x_2x_1x_2]_c + v_2\gamma_2(1 + q_{11})(1 + q_{2e_1+e_2})[x_1x_2]_c[x_1^2x_2]_c.$$

The terms $[x_1^2x_2x_1x_2]_c$ and $[x_1x_2]_c[x_1^2x_2]_c$ are linearly independent, since they are linearly independent in $\mathfrak{B}(V)$, and we have a surjection $\mathfrak{B} \rightarrow \mathfrak{B}(V)$. Then

$$v_1\gamma_2(1 + q_{11}) = v_2\gamma_2(1 + q_{11})(1 + q_{2e_1+e_2}) = 0.$$

But for standard braidings of type G_2 we note that $q_{11}, q_{2e_1+e_2} \neq -1$ and $\gamma_2 \neq 0$, so $v_1 = v_2 = 0$.

The last statement is true since

$$\Delta^+(\mathfrak{B}(V)) = \{e_1, e_1 + e_2, 2e_1 + e_2, 3e_1 + e_2, 3e_1 + 2e_2, e_2\},$$

if the braiding is standard of type G_2 . □

Remark 5.21. Let V be a standard braided vector space of type G_2 and let \mathfrak{B} be a braided graded Hopf algebra satisfying the hypotheses of Lemma 5.20. In a similar way to Lemma 5.5, if $q_{11} \notin \mathbb{G}_4$ and $q_{22} \neq -1$, then $5e_1 + 2e_2, 4e_1 + 2e_2, 4e_1 + 3e_2, 5e_1 + 3e_2 \notin \Delta^+(\mathfrak{B})$.

This follows because $x_1^3 x_2 x_1^2 x_2, x_1^2 x_2 x_1 x_2 x_1 x_2, x_1^2 x_2 x_1^2 x_2 x_1 x_2 \notin S_I$, using the quantum Serre relations as in the lemma cited.

Theorem 5.22. *Let V be a standard braided vector space of type G_2 . The Nichols algebra $\mathfrak{B}(V)$ is presented by the generators x_1, x_2 and the relations*

$$\mathrm{ad}_c(x_1)^4(x_2) = \mathrm{ad}_c(x_2)^2(x_1) = 0, \quad x_\alpha^{N_\alpha} = 0, \quad \alpha \in \Delta^+, \quad (5-29)$$

and, if $q_{11} \in \mathbb{G}_4$ or $q_{22} = -1$,

$$[(\mathrm{ad} x_1)^3 x_2, (\mathrm{ad} x_1)^2 x_2]_c = 0, \quad (5-30)$$

$$[x_1, [x_1^2 x_2 x_1 x_2]_c]_c = 0, \quad (5-31)$$

$$[[x_1^2 x_2 x_1 x_2]_c, [x_1 x_2]_c]_c = 0, \quad (5-32)$$

$$[[x_1^2 x_2]_c, [x_1^2 x_2 x_1 x_2]_c]_c = 0. \quad (5-33)$$

The following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_2^{h_{e_2}} [x_1 x_2]_c^{h_{e_1+e_2}} [x_1^2 x_2 x_1 x_2]_c^{h_{3e_1+2e_2}} [x_1^2 x_2]_c^{h_{2e_1+e_2}} [x_1^3 x_2]_c^{h_{3e_1+e_2}} x_1^{h_{e_1}}, \quad 0 \leq h_\alpha \leq N_\alpha - 1. \quad (5-34)$$

Proof. The statement about the PBW basis follows from Corollary 4.2 and the definitions of the x_α .

Let \mathfrak{B} be the algebra presented by the generators x_1, x_2 and the relations (5-29)–(5-33). From Lemma 5.20 and Corollary 5.2, we have a canonical epimorphism of algebras $\phi : \mathfrak{B} \rightarrow \mathfrak{B}(V)$.

Consider the subspace \mathcal{F} of \mathfrak{B} generated by the elements in (5-34). We prove by induction on the sum S of the h_α 's of a such product M that $x_1 M \in \mathcal{F}$; moreover, we prove that it is a linear combination of products whose first hyperletter is less than or equal to the first hyperletter of M . If $S = 0$, we have $M = 1$.

- If $M = x_1^{N_1}$, then $x_1 M = x_1^{N_1+1}$, which is zero if $N_1 = \mathrm{ord} x_1 - 1$.
- If $M = [x_1^3 x_2]_c M'$, then we use that $x_1 [x_1^3 x_2]_c = q_{11}^3 q_{12} [x_1^3 x_2]_c x_1$ to prove that $x_1 M$ lies in \mathcal{F} and either is zero or begins with $[x_1^3 x_2]_c$.
- If $M = [x_1^2 x_2]_c M'$, we have

$$x_1 [x_1^2 x_2]_c = [x_1^3 x_2]_c + q_{11}^2 q_{12} [x_1^2 x_2]_c x_1.$$

We use the inductive step and relation (5-30) to prove that $x_1 M$ lies in \mathcal{F} and is either zero or a linear combination of hyperwords starting with a hyperletter less than or equal to $[x_1^2 x_2]_c$.

- If $M = [x_1^2 x_2 x_1 x_2]_c M'$, we deduce from (5-31) that

$$x_1 [x_1^2 x_2 x_1 x_2]_c = \chi(\mathbf{e}_1, 3\mathbf{e}_1 + 2\mathbf{e}_2) [x_1^2 x_2 x_1 x_2]_c x_1;$$

using also (5-32) and (5-33), we prove that x_1M lies in \mathcal{F} and is either zero or a linear combination of hyperwords that starting with a hyperletter less than or equal to $[x_1^2x_2x_1x_2]_c$.

- If $M = [x_1x_2]_cM'$, observe that

$$x_1[x_1x_2]_c = [x_1^2x_2]_c + q_{11}q_{12}[x_1x_2]_cx_1.$$

Using the inductive step together with (5-31), (5-32), and the equality

$$[x_1^2x_2]_c[x_1x_2]_c = [[x_1^2x_2]_c, [x_1x_2]_c]_c + \chi(2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)[x_1x_2]_c[x_1^2x_2]_c,$$

by the definition of braided commutator, we prove that x_1M lies in \mathcal{F} and is either zero or a linear combination of hyperwords starting with a hyperletter less than or equal to $[x_1x_2]_c$.

- If $M = x_2M'$, we use the equalities $x_1x_2 = [x_1x_2]_c + q_{12}x_2x_1$ and $[[x_1x_2]_c, x_2]_c = 0$ to prove that x_1M lies in \mathcal{F} and is either zero or a linear combination of hyperwords.

Now, x_2M is a product of nonincreasing hyperwords or is zero, for each element in (5-34), so \mathcal{F} is an ideal of \mathfrak{B} containing 1; hence $\mathcal{F} = \mathfrak{B}$. Since the elements in (5-34) are a basis of $\mathfrak{B}(V)$, the map ϕ is an isomorphism. \square

5E. Presentation when the braiding is of Cartan type. In this subsection, we present the Nichols algebra of a diagonal braiding vector space of Cartan type with matrix (q_{ij}) , by generators and relations. This was established in [Andruskiewitsch and Schneider 2002a, Theorem 4.5] assuming that q_{ii} has odd order and that order is not divisible by 3 if i belongs to a component of type G_2 . The proof in loc. cit. combines a reduction to symmetric (q_{ij}) by twisting, with results from [Andersen et al. 1994] and [De Concini and Procesi 1993]. We also note that some particular instances were already proved earlier in this section.

Fix a standard braided vector space V with connected Dynkin diagram and an $i \in \{1, \dots, \theta\}$. Suppose that \mathfrak{B} is a quotient by an ideal $I \in \mathfrak{S}$ of $T(V)$. Assume moreover that V is not of type G_2 and that

$$(5-3) \text{ holds in } \mathfrak{B} \text{ if } 1 \leq i \neq j \leq \theta; \quad (5-35)$$

$$(5-4) \text{ holds in } \mathfrak{B} \text{ if } m_{kj} = m_{kl} = 1 \text{ and } m_{jl} = 0; \quad (5-36)$$

$$(5-6) \text{ holds in } \mathfrak{B} \text{ if } m_{kj} = 2 \text{ and } m_{jk} = 1; \quad (5-37)$$

$$(5-8) \text{ holds in } \mathfrak{B} \text{ if } m_{kj} = 2, m_{jk} = m_{jl} = 1 \text{ and } m_{kl} = 0. \quad (5-38)$$

Note that if (5-3) holds in an algebra with derivations D_k , then (2-11) holds also, by Lemma 2.7. By Theorem 2.6, we have an algebra $s_i(\mathfrak{B})$ provided with skew derivations D_i . We set $\tilde{x}_k = (\text{ad}_c x_i)^{m_{ik}}(x_k)\#1 \in s_i(\mathfrak{B})$, for $k \neq i$, and $\tilde{x}_i = 1\#y$. The elements generate $s_i(\mathfrak{B})^1$ as a vector space.

Lemma 5.23. *Conditions (5-35)–(5-38) are satisfied with $s_i(\mathfrak{B})$ in lieu of \mathfrak{B} .*

Proof of (5-35). Each $m\mathbf{e}_k + \mathbf{e}_j$, $0 \leq m \leq m_{kj}$ is an element of $\Delta(\mathfrak{B}(V_i))$, so $s_i(m\mathbf{e}_k + \mathbf{e}_j) \in \Delta(\mathfrak{B}(V))$. Since we have a surjective morphism of braided graded Hopf algebras $\mathfrak{B} \rightarrow \mathfrak{B}(V)$, we have $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$.

From Lemma 5.3, $(\text{ad}_c \tilde{x}_k)^m \tilde{x}_j = 0$ if and only if $\tilde{x}_k^m \tilde{x}_j$ is a linear combination of greater words, for an order in which $\tilde{x}_k < \tilde{x}_j$ (since we are considering the Cartan case, the condition about the ordering of the \tilde{x}_j is satisfied). Note that $\tilde{x}_k^m \tilde{x}_j$ is the unique Lyndon word of degree $m\mathbf{e}_k + \mathbf{e}_j$. Then, by the relation (2-15) between the Hilbert series of \mathfrak{B} and $s_i(\mathfrak{B})$, the validity of (5-3) for $s_i(\mathfrak{B})$ is equivalent to the condition

$$s_i((m_{kj} + 1)\mathbf{e}_k + \mathbf{e}_j) \notin \Delta^+(\mathfrak{B}).$$

(a) When $k = i \neq j$, this says that $-\mathbf{e}_i + \mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, so (5-3) holds.

(b) To prove (5-3) for $s_i(\mathfrak{B})$ when $j = i$, we show case by case that

$$(m_{ki} + 1)\mathbf{e}_k + ((m_{ki} + 1)m_{ik} - 1)\mathbf{e}_i \notin \Delta^+(\mathfrak{B}).$$

- If $m_{ki} = m_{ik} = 0$, we have $\mathbf{e}_k - \mathbf{e}_i \notin \Delta^+(\mathfrak{B})$.
- If $m_{ki} = m_{ik} = 1$, then $2\mathbf{e}_k + \mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, because $(\text{ad } x_k)^2 x_i = 0$.
- If $m_{ki} = 1$ and $m_{ik} = 2$, then $2\mathbf{e}_k + 3\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, since we can apply Lemma 5.5 to \mathfrak{B} , which satisfies (5-6) by assumption.
- If $m_{ki} = 2$ and $m_{ik} = 1$, then $3\mathbf{e}_k + 2\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, as before.

Thus (5-3) holds for each $k \neq i$.

(c) Now consider $\theta \geq 3$ and $k, j \neq i$.

- If $m_{ik} = m_{ij} = 0$, then $s_i(m\mathbf{e}_k + \mathbf{e}_j) = m\mathbf{e}_k + \mathbf{e}_j$, and $(m_{kj} + 1)\mathbf{e}_k + \mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, since the quantum Serre relation holds in \mathfrak{B} .
- If $m_{ik} = 1$ and $m_{ij} = 0$, then $s_i(m\mathbf{e}_k + \mathbf{e}_j) = m\mathbf{e}_i + m\mathbf{e}_k + \mathbf{e}_j$. If we assume $x_j < x_i < x_k$ and look at the possible Lyndon words in S_I , from (5-3), these words have no factors $x_i^2 x_k, x_j x_i$, so the only possibility is $x_j(x_k x_i)^m$.
 - If $m_{kj} = 0$, then $x_j x_k x_i = q_{jk} x_k x_j x_i$, so $x_j x_k x_i \notin S_I$.
 - If $m_{kj} = 1$, then $x_j x_k x_i x_k \notin S_I$ when $m_{ki} = 1$, since (5-4) is valid in \mathfrak{B} ; while if $m_{ki} = 2$ we have $q_{kk} \neq -1$ and

$$\begin{aligned} x_j(x_k x_i)^2 &= (1 + q_{kk})^{-1} q_{ki}^{-1} x_j x_k^2 x_i^2 + (1 + q_{kk})^{-1} q_{ki} q_{kk}^2 x_j x_i x_k^2 x_i \\ &= q_{ki}^{-1} q_{kj}^{-1} q_{kk}^{-2} x_k x_j x_k x_i^2 + (1 + q_{kk})^{-1} q_{ki}^{-1} q_{kj}^{-2} q_{kk}^{-2} x_k^2 x_j x_i^2 \\ &\quad + (1 + q_{kk})^{-1} q_{ki} q_{kk}^2 q_{ji}^{-1} x_i x_j x_k^2 x_i. \end{aligned}$$

In both cases, $x_j(x_k x_i)^2 \notin S_I$.

- If $m_{kj} = 2$, then $m_{ki} = m_{jk} = 1$ and $q_{kk} \neq -1$. The proof is similar to the previous case.
- If $m_{ik} = 2, m_{ij} = 0$, then $s_i(m\mathbf{e}_k + \mathbf{e}_j) = 2m\mathbf{e}_i + m\mathbf{e}_k + \mathbf{e}_j$ and $m_{kj} = 0, 1$. When $m_{kj} = 0$, the proof is clear as above. When $m_{kj} = 1$, for $j < k < i$ and considering only the quantum Serre relations, the only possible Lyndon word is $x_j(x_k x_i^2)^2$. But since $[[x_i^2 x_k]_c, [x_i x_k]_c]_c = 0$, we deduce that such a word is not in S_I .
- If $m_{ik} = 0, m_{ij} = 1$, then $s_i(m\mathbf{e}_k + \mathbf{e}_j) = \mathbf{e}_i + m\mathbf{e}_k + \mathbf{e}_j$. If $k < i < j$, note that from $x_k x_i, x_k^{m_{kj}+1} x_j \notin S_I$, there are no Lyndon words of degree $\mathbf{e}_i + (m_{kj} + 1)\mathbf{e}_k + \mathbf{e}_j$ in S_I .
- If $m_{ik} = 0, m_{ij} = 2$, then $s_i(m\mathbf{e}_k + \mathbf{e}_j) = 2\mathbf{e}_i + m\mathbf{e}_k + \mathbf{e}_j$, and the proof is analogous to the previous case.
- If $m_{ik} = m_{ij} = 1$, then $m_{kj} = 0$, and $s_i(\mathbf{e}_k + \mathbf{e}_j) = 2\mathbf{e}_i + \mathbf{e}_k + \mathbf{e}_j$, which is not in $\Delta^+(\mathfrak{B})$ from Lemma 5.4.
- If $m_{ik} = 2, m_{ij} = 1$ (it is analogous to $m_{ik} = 1, m_{ij} = 2$), then $m_{kj} = 0$ and $s_i(\mathbf{e}_k + \mathbf{e}_j) = 3\mathbf{e}_i + \mathbf{e}_k + \mathbf{e}_j$. In this way we get $q_{ii} \neq -1$, and if $x_k < x_i < x_j$ the unique Lyndon word without $x_i^2 x_j$ or $x_k x_i^3$ as factors is

$$x_k x_i^2 x_j x_i = (1 + q_{ii})^{-1} q_{ij}^{-1} x_k x_i^3 x_j + (1 + q_{ii})^{-1} q_{ii}^2 q_{ij} x_k x_i x_j x_i^2 \\ \in k(x_i x_k x_i^2 x_j) + k(x_i^2 x_k x_i x_j) + k(x_i^3 x_k x_j) + k(x_k x_i x_j x_i^2),$$

using the quantum Serre relations; hence there are no Lyndon words of degree $3\mathbf{e}_i + \mathbf{e}_k + \mathbf{e}_j$ in S_I .

So, (5-3) holds, for each $k, j \neq i, k \neq j$. □

Proof of (5-36). Assume $m_{kj} = m_{kl} = 1$. We prove case by case that

$$s_i(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) \notin \Delta^+(\mathfrak{B}).$$

- If $m_{ij} = m_{ik} = m_{il} = 0$, then $s_i(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l$, so it follows from Lemma 5.4, because $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$.
- If $m_{ij} \neq 0$ (analogously, if $m_{il} \neq 0$), then $m_{ik} = m_{il} = 0$, because there are no cycles in the Dynkin diagram. Then $s_i(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + m_{ij}\mathbf{e}_i$. If we consider $x_k < x_l < x_j < x_i$, using the equalities $x_k x_i = q_{ki} x_i x_k$, $x_j x_l = q_{jl} x_l x_j$ and $x_l x_i = q_{li} x_i x_l$, and also that $x_k^2 x_l, x_k^2 x_j \notin S_I$, we conclude that no possible Lyndon words of degree $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + m_{ij}\mathbf{e}_i$ can be an element of S_I , except $x_k x_l x_k x_j x_i^{m_{ij}}$; but this, too, is not an element of S_I , because $x_k x_l x_k x_j \notin S_I$. Hence $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + m_{ij}\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$.
- If $m_{ik} = 1$, and therefore $m_{ij} = m_{il} = 0$, then $s_i(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + 2m_{ik}\mathbf{e}_i$. If we consider $x_l < x_i < x_k < x_j$, using the equalities $x_j x_i = q_{ji} x_i x_j$,

$x_j x_l = q_{jl} x_l x_j$ and $x_l x_i = q_{li} x_i x_l$, and also that $x_k^2 x_l, x_k^2 x_j \notin S_I$, we discard as before all possible Lyndon words of degree $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + 2m_{ik}\mathbf{e}_i$, except $x_l x_k x_j x_k x_i^{2m_{ij}}$; but this is not an element of S_I , because $x_k x_l x_k x_j \notin S_I$. Thus $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + 2m_{ij}\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$.

- If $i = j$ (analogously, if $i = l$), then $s_j(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$ if $m_{jk} = 1$ by Lemma 5.4, or $s_j(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + 3\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$ if $m_{jk} = 2$ by Lemma 5.5.
- If $i = k$, then $s_k(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, since $m_{jl} = 0$.

Also, if $\mathbf{u} \in \{\mathbf{e}_k + \mathbf{e}_j, \mathbf{e}_k + \mathbf{e}_l, \mathbf{e}_k, \mathbf{e}_j, \mathbf{e}_l\}$, then $\mathbf{u} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{u}) \in \Delta(\mathfrak{B}(V))$. The canonical surjective algebra morphisms from $T(V)$ to \mathfrak{B} and $\mathfrak{B}(V)$ induce a surjective algebra morphism $\mathfrak{B} \rightarrow \mathfrak{B}(V)$, so $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$; in particular, each $s_i(\mathbf{u})$ lies in $\Delta(\mathfrak{B})$.

Consider a basis as in Proposition 1.11 for an order such that $x_j < x_k < x_l$. From Lemma 2.7, $x_j x_k, x_k x_l, x_j x_k x_l$ are elements of this basis, since they are not linear combinations of greater words modulo I_i , the ideal of $T(V_i)$ such that $s_i(\mathfrak{B}) = T(V_i)/I_i$. In the same way, $(x_k x_l)(x_j x_k), x_l x_k(x_j x_k), (x_k x_l)x_k x_j, x_k(x_j x_k x_l), x_l x_k^2 x_j$ (if $x_k^2 \neq 0$) are elements of this basis, where the parenthesis indicates the Lyndon decomposition as nonincreasing products of Lyndon words. Also, $x_j x_l, x_j x_k^2, x_k^2 x_l$ are not in this basis, by (5-3). By the relation (2-15) between Hilbert series and the fact that $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin s_i(\Delta^+(\mathfrak{B}))$, we note that $x_j x_k x_l x_k$ is not an element of the basis. Thus this word is a linear combination of greater words. By Lemma 5.4, this implies that (5-4) holds in $s_i(\mathfrak{B})$. \square

Proof of (5-37). As before, we prove first that $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) \notin \Delta^+(\mathfrak{B})$ case by case:

- If $m_{ik} = m_{ij} = 0$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B})$ by assumption.
- If $m_{ik} = 0, m_{ij} = 1$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 2\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j$. If we consider an order such that $x_k < x_i < x_j$, a Lyndon word of degree $2e_i + 3e_k + 2e_j$ in S_I begins with x_k , and $x_k x_i$ is not a factor, because $x_k x_i = q_{ki} x_i x_k$. Thus the possible Lyndon words with these conditions are $x_k^2 x_j x_i x_k x_j x_i$ and $x_k^2 x_j x_k x_j x_i^2$; the first is not in S_I because from (5-4) for j, k, i we can express $x_j x_i x_k x_j$ as a linear combination of greater words, and the second is not in S_I because $x_k^2 x_j x_k x_j \notin S_I$.
- If $m_{ik} = 1, m_{ij} = 0$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j$. If we consider an order such that $x_j < x_i < x_k$, a Lyndon word of degree $3e_i + 3e_k + 2e_j$ in S_I begins with x_j , and $x_j x_i$ is not a factor. Using that also $x_i^2 x_k, x_j^2 x_k \notin S_I$, the possible Lyndon word under these conditions is $x_j x_k x_i x_j x_k x_i x_k x_i$. But from the condition on the m_{rs} , we are in cases C_θ or F_4 , and we use that $(\text{ad } x_i)^2 x_k = 0$, $q_{ii} \neq -1$ to replace $x_i x_k x_i$ by a linear combination of $x_i^2 x_k$ and $x_k x_i^2$, and also use $x_j x_i = q_{ji} x_i x_j$, so we conclude that $x_j x_k x_i x_j x_k x_i x_k x_i \notin S_I$.
- If $i = j$, then $s_j(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_k + \mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, since $m_{kj} = 2$.

- If $i = k$, then $s_k(3\mathbf{e}_k + 2\mathbf{e}_j) = \mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, since $m_{jk} = 1$.

If $\mathbf{v} \in \{\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k + \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_j\}$, then $\mathbf{v} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{v}) \in \Delta(\mathfrak{B}(V))$. Since $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$; in particular, each \mathbf{v} lies in $s_i(\Delta(\mathfrak{B}))$.

As in (a), consider a basis as in Proposition 1.11 for an order such that $x_k < x_j$. In a similar way, $x_k x_j, x_k^2 x_j$ are elements of this basis, but $x_k^3 x_j$ and $x_k x_j^2$ are not in this basis by (5-3). By Lemma 2.7, $(x_k x_j)(x_k^2 x_j), x_j(x_k^2 x_j)x_k, (x_k x_j)^2 x_k, x_j(x_k x_j)x_k^2, x_j^2 x_k^3$ (the last if $x_j^2, x_k^3 \neq 0$) are not linear combinations of greater words modulo I_i , so they are elements of the chosen basis. By the relation (2-15) between Hilbert series and the fact that $3\mathbf{e}_k + 2\mathbf{e}_j \notin s_i(\Delta^+(\mathfrak{B}))$, the Lyndon word $x_k^2 x_j x_k x_j$ is not an element of the basis. Thus this word is a linear combination of greater words, and by Lemma 5.5, this implies that (5-6) holds in $s_i(\mathfrak{B})$. \square

Proof of (5-38). We prove case by case that

$$s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) \notin \Delta^+(\mathfrak{B}).$$

- If $m_{ik} = m_{ij} = m_{il} = 0$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l$, and this is not in $\Delta^+(\mathfrak{B})$ by Lemma 5.6.
- If $i \neq j, k, l$ and $m_{ik} \neq 0$, the only possibility is $m_{ik} = m_{ki} = 1$, so V is of type F_4 . Thus $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l$. For the order $x_l < x_j < x_k < x_i$, the only possible Lyndon word without the factors $x_l x_j^2, x_l x_k, x_l x_i, x_j^2 x_k, x_j x_i, x_k x_i^2, x_k^2 x_i$ is $x_l x_j x_k x_i x_j x_k x_i x_k x_i$. Using the quantum Serre relations and the fact that $q_{ii} = q_{kk} \neq -1$, we see that this Lyndon word is not in S_l . Thus $3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$.
- $i \neq j, k, l$ and $m_{ij} \neq 0$: there are no standard braided vector spaces with these values.
- If $i \neq j, k, l$ and $m_{il} \neq 0$, the unique possibility is $m_{il} = m_{li} = 1$. In this case $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l + \mathbf{e}_i$. If we consider $x_k < x_j < x_l < x_i$, the only possible Lyndon word of this degree without the factors $x_k x_l, x_k x_i, x_j x_i, x_k^3 x_j, x_k x_j^2$ is $x_k^2 x_j x_l x_i x_k x_i$. But by assumption,

$$[[x_k^2 x_j x_l]_c, [x_k x_j]_c]_c = [x_i, [x_k x_j]_c]_c = 0,$$

so $[x_k^2 x_j x_l x_i x_k x_i]_c = [[x_k^2 x_j x_l x_i]_c, [x_k x_j]_c]_c = 0$, and $x_k^2 x_j x_l x_i x_k x_i \notin S_l$.

- If $i = k$, then $s_i(3\mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_l) = \mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, by Lemma 5.4.
- If $i = j$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_i + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_i + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, by Lemma 5.6.
- If $i = k$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = \mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, as before.

Now, if $\mathbf{w} \in \{\mathbf{e}_k, \mathbf{e}_j, \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j, \mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, 2\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, 2\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l\}$, then $\mathbf{w} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{w}) \in \Delta(\mathfrak{B}(V))$, hence $s_i(\mathbf{w}) \in \Delta(\mathfrak{B})$.

Consider a basis as in Proposition 1.11 for an order such that $x_k < x_j < x_l$. Then $x_j x_k$ and $x_k x_l$ are elements of this basis. We know that $x_k x_l$, $x_k^3 x_j$, $x_k x_j^2$, $x_k x_j x_l x_k$, $x_k^2 x_j x_k x_j$ are not elements of the basis, since (5-3), (5-4) and (5-6) hold in \mathfrak{B} . By Lemma 2.7, the relation (2-15) between Hilbert series and the fact that $3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin s_i(\Delta^+(\mathfrak{B}))$, the Lyndon word $x_k^2 x_j x_l x_k x_j$ is not an element of the basis. Thus this word is a linear combination of greater words. By Lemma 5.6, this implies that (5-8) holds in $s_i(\mathfrak{B})$. \square

This concludes the proof of Lemma 5.23. Note also that $s_i(\mathfrak{B})$ is of the same type as \mathfrak{B} .

Let V be of a type different from G_2 . We define the algebra $\hat{\mathfrak{B}}(V) := T(V)/\mathfrak{I}(V)$, where $\mathfrak{I}(V)$ is the two-sided ideal of $T(V)$ generated by

- $(\text{ad}_c x_k)^{m_{kj}+1} x_j$, $k \neq j$;
- $[(\text{ad}_c x_j)(\text{ad}_c x_k)x_l, x_k]_c$, $l \neq k \neq j$, $q_{kk} = -1$, $m_{kj} = m_{kl} = 1$;
- $[(\text{ad}_c x_k)^2 x_j, (\text{ad}_c x_k)x_j]_c$, $k \neq j$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, $m_{kj} = 2$, $m_{jk} = 1$;
- $[(\text{ad}_c x_k)^2 (\text{ad}_c x_j)x_l, (\text{ad}_c x_k)x_j]_c$, $k \neq j \neq l$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, $m_{kj} = 2$, $m_{jk} = m_{jl} = 1$.

(Compare with the definitions in Section 4 of [Andruskiewitsch and Schneider 2002a].) Since V is of Cartan type, $\mathfrak{I}(V)$ is a Hopf ideal, by Lemmas 5.7–5.9. Since $\mathfrak{I}(V)$ also is \mathbb{Z}^θ -homogeneous, we have $\mathfrak{I}(V) \in \mathfrak{S}$.

By Lemmas 5.4–5.6, the canonical epimorphism $T(V) \rightarrow \mathfrak{B}(V)$ induces an epimorphism of braided graded Hopf algebras

$$\pi_V : \hat{\mathfrak{B}}(V) \rightarrow \mathfrak{B}(V). \quad (5-39)$$

Also, $\hat{\mathfrak{B}}(V)$ satisfies the conditions in Theorem 2.6 for each $i \in \{1, \dots, \theta\}$, so we can transform it.

Lemma 5.24. *With the notation above, $s_i(\hat{\mathfrak{B}}(V)) \cong \hat{\mathfrak{B}}(V_i)$.*

Proof. By Lemma 5.23, the relations defining $\mathfrak{I}(V_i)$ are satisfied in $s_i(\hat{\mathfrak{B}}(V))$. Thus the canonical projections from $T(V_i)$ onto $\hat{\mathfrak{B}}(V_i)$ and $s_i(\hat{\mathfrak{B}}(V))$ induce a surjective algebra map $\hat{\mathfrak{B}}(V_i) \rightarrow s_i(\hat{\mathfrak{B}}(V))$. Conversely, each relation defining $\mathfrak{I}(V)$ is satisfied in $s_i(\hat{\mathfrak{B}}(V_i))$, so we have the following situation:

$$\begin{array}{ccc} \hat{\mathfrak{B}}(V) & \longrightarrow & s_i(\hat{\mathfrak{B}}(V_i)) \\ & \searrow & \nearrow \\ & & s_i(\hat{\mathfrak{B}}(V)) \\ & \swarrow & \searrow \\ \hat{\mathfrak{B}}(V_i) & \longrightarrow & s_i(\hat{\mathfrak{B}}(V)) \end{array}$$

From the relation (2-15) between Hilbert series, we have, for each $\mathbf{u} \in \mathbb{N}^\theta$,

$$\dim s_i(\hat{\mathfrak{B}}(V))^{\mathbf{u}} = \sum_{\substack{k \in \mathbb{N}: \mathbf{u} - k\mathbf{e}_i \in \mathbb{N}^\theta \\ s_i(\mathbf{u} - k\mathbf{e}_i) \in \mathbb{N}^\theta}} \dim \hat{\mathfrak{B}}(V)^{s_i(\mathbf{u} - k\mathbf{e}_i)},$$

and a analogous relation for $\dim s_i(\hat{\mathfrak{B}}(V_i))^{\mathbf{u}}$. But in view of the previous surjections we have

$$\dim s_i(\hat{\mathfrak{B}}(V))^{\mathbf{u}} \leq \dim \hat{\mathfrak{B}}(V_i)^{\mathbf{u}}, \quad \dim s_i(\hat{\mathfrak{B}}(V_i))^{\mathbf{u}} \leq \dim \hat{\mathfrak{B}}(V)^{\mathbf{u}},$$

for each $\mathbf{u} \in \mathbb{N}^\theta$. Since $s_i^2 = \text{id}$, each of these inequalities is in fact an equality, and $s_i(\hat{\mathfrak{B}}(V)) = \hat{\mathfrak{B}}(V_i)$. \square

We are now able to prove one of the main results of this paper.

Theorem 5.25. *Let V be a braided vector space of Cartan type, of dimension θ , and $C = (a_{ij})_{i,j \in \{1, \dots, \theta\}}$ the corresponding finite Cartan matrix, where $a_{ij} := -m_{ij}$.*

The Nichols algebra $\mathfrak{B}(V)$ is presented by the generators x_i , for $1 \leq i \leq \theta$, and the relations

$$x_\alpha^{N_\alpha} = 0, \quad \alpha \in \Delta^+, \quad (5-40)$$

$$\text{ad}_c(x_k)^{1-a_{kj}}(x_j) = 0, \quad k \neq j. \quad (5-41)$$

If there exist $j \neq k \neq l$ such that $m_{kj} = m_{kl} = 1$, $q_{kk} = -1$, then

$$[(\text{ad } x_k)x_j, (\text{ad } x_k)x_l]_c = 0. \quad (5-42)$$

If there exist $k \neq j$ such that $m_{kj} = 2$, $m_{jk} = 1$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, then

$$[(\text{ad } x_k)^2 x_j, (\text{ad } x_k)x_j]_c = 0. \quad (5-43)$$

If there exist $k \neq j \neq l$ such that $m_{kj} = 2$, $m_{jk} = m_{jl} = 1$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, then

$$[(\text{ad } x_k)^2(\text{ad } x_j)x_l, (\text{ad } x_k)x_j]_c = 0. \quad (5-44)$$

If $\theta = 2$, V if of type G_2 , and $q_{11} \in \mathbb{G}_4$ or $q_{22} = -1$, then

$$[(\text{ad } x_1)^3 x_2, (\text{ad } x_1)^2 x_2]_c = 0, \quad (5-45)$$

$$[x_1, [x_1^2 x_2 x_1 x_2]_c]_c = 0, \quad (5-46)$$

$$[[x_1^2 x_2 x_1 x_2]_c, [x_1 x_2]_c]_c = 0, \quad (5-47)$$

$$[[x_1^2 x_2]_c, [x_1^2 x_2 x_1 x_2]_c]_c = 0. \quad (5-48)$$

The following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \quad \text{where } 0 \leq h_j \leq N_{\beta_j} - 1, \text{ if } \beta_j \in S_I, \text{ for } 1 \leq j \leq P.$$

Proof. We may assume that C is connected. For V of type G_2 , the result was proved in Theorem 5.22. So we can assume $m_{kj} \neq 3$, $k \neq j$.

The statement about the PBW basis was proved in Corollary 4.2; see the definition of the x_α in Section 4B.

Consider the images of the x_α in $\hat{\mathfrak{B}}(V)$; they correspond in $\mathfrak{B}(V)$ with the x_α , and are PBW generators for a basis constructed as in Theorem 1.12, considering the same order in the letters. As we observed in (5-39), there exists a surjective morphism of braided Hopf algebras $\hat{\mathfrak{B}}(V) \rightarrow \mathfrak{B}(V)$, so

$$\Delta(\mathfrak{B}(V)) \subseteq \Delta(\hat{\mathfrak{B}}(V)).$$

Also, $\hat{\mathfrak{B}}(V)$ satisfies the conditions in Theorem 2.6 for each $i \in \{1, \dots, \theta\}$, so we can transform it. By Lemma 5.24, the new algebra is $\hat{\mathfrak{B}}(V_i)$, so we can continue. Consider the sets

$$\hat{\Delta} := \bigcup \{ \Delta(s_{i_1} \dots s_{i_k} \hat{\mathfrak{B}}) : k \in \mathbb{N}, 1 \leq i_1, \dots, i_k \leq \theta \}, \quad \hat{\Delta}^+ := \Delta \cap \mathbb{N}^\theta;$$

$\hat{\Delta}$ is invariant by the s_i . Also, $\Delta(\mathfrak{B}(V)) \subseteq \Delta$, and

$$\Delta(s_{i_1} \dots s_{i_k} \hat{\mathfrak{B}}(V)) = s_{i_1} \dots s_{i_k} \Delta(\hat{\mathfrak{B}}(V)).$$

Consider $\alpha \in \hat{\Delta}^+ \setminus \Delta^+(\mathfrak{B}(V))$. Suppose that α is not of the form $m\alpha_i$ for $m \in \mathbb{N}$ and $i \in \{1, \dots, \theta\}$, and that it is of minimal height among such roots. For each s_i , since α is not a multiple of α_i , we have $s_i(\alpha) \in \Delta^+ \setminus \Delta^+(\mathfrak{B}(V))$; hence $\deg s_i(\alpha) - \deg \alpha \geq 0$. But $\alpha = \sum_{i=1}^\theta b_i \mathbf{e}_i$, so $\sum_{i=1}^\theta b_i a_{ij} \leq 0$, and since $b_i \geq 0$, we have $\sum_{i,j=1}^\theta b_i a_{ij} b_j \leq 0$. This contradicts the fact that (a_{ij}) is definite positive, and $(b_i) \geq 0$, $(b_i) \neq 0$.

Also, $m\mathbf{e}_i \in \Delta^+(\hat{\mathfrak{B}}) \iff m = N_{\mathbf{e}_i}$ or $m = 1$, since $x_i^{N_{\mathbf{e}_i}} \neq 0$. Hence

$$\Delta(\hat{\mathfrak{B}}(V)) = \Delta(\mathfrak{B}(V)) \cup \{N_\alpha \alpha : \alpha \in \Delta(\mathfrak{B}(V))\}.$$

This follows since by Corollary 4.2 each $\alpha \in \Delta^+(\mathfrak{B}(V))$ is of the form

$$\alpha = s_{i_1} \dots s_{i_m}(\mathbf{e}_j), \quad i_1, \dots, i_m, j \in \{1, \dots, \theta\}.$$

Now, $N_{\mathbf{e}_j} \mathbf{e}_j \in \Delta(\hat{\mathfrak{B}}(V))$, so

$$N_\alpha \alpha = N_{\mathbf{e}_j} \alpha = s_{i_1} \dots s_{i_m}(N_{\mathbf{e}_j} \mathbf{e}_j) \in \Delta(\hat{\mathfrak{B}}(V)).$$

Also, each degree $N_\alpha \alpha$ has multiplicity one in $\Delta(\hat{\mathfrak{B}}(V))$.

Suppose there exist Lyndon words of degree $N_\alpha \alpha$, and consider one such word u of minimal height. Let $u = vw$ be a Shirshov decomposition thereof, and put

$$\beta := \deg v, \quad \gamma := \deg w \in \Delta^+(\hat{\mathfrak{B}}(V)).$$

By the preceding assumption, $\beta, \gamma \in \Delta^+(\mathfrak{B}(V))$. Write

$$\alpha = \sum_{k=1}^{\theta} a_k \mathbf{e}_k, \quad \beta = \sum_{k=1}^{\theta} b_k \mathbf{e}_k, \quad \gamma = \sum_{k=1}^{\theta} c_k \mathbf{e}_k,$$

so $N_\alpha a_k = b_k + c_k$, for each $k \in \{1, \dots, \theta\}$. We can assume, by taking a subdiagram if necessary, that $a_1, a_\theta \neq 0$.

Now, if V is of type F_4 and $\beta = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_3 + 3\mathbf{e}_4$, then $c_1 = 0, a_1 = 1, N_\alpha = 2$, or $a_1 = c_1 = 1, N_\alpha = 3$, since $\alpha, \gamma \neq \beta$.

- If $N_\alpha = 3$, then $3a_2 = 3 + c_2$. Hence $c_2 = 0$, so $c_3 = c_4 = 0$, or $c_2 = 3$, and $c_3 = 4, c_4 = 2$. But in both cases we have a contradiction to $\alpha \in \mathbb{N}^4$.
- If $N_\alpha = 2, c_1 = 0$, then c_2 and c_4 are odd, and c_3 is even and nonzero. The only possibility is $\gamma = \mathbf{e}_2 + 2\mathbf{e}_3 + \mathbf{e}_4$, so $\alpha = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 + 2\mathbf{e}_4$. But $q_\alpha = q \neq -1$, so $N_\alpha \neq 2$, which is a contradiction.

Thus we can assume $b_1, c_1 \leq 1$ or $b_\theta, c_\theta \leq 1$, so $a_1 = b_1 = c_1 = 1$ or $a_\theta = b_\theta = c_\theta = 1$; in both cases, $N_\alpha = 2$. For each possible β with $b_1 \neq 0$ (by the assumption that $a_1 \neq 0$, we have $b_1 \neq 0$ or $c_1 \neq 0$), we look for γ such that $\beta + \gamma$ has even coordinates. In types A, D and E there are no such pairs of roots. As for the other types:

- B_θ : $\beta = \mathbf{v}_{i\theta}, \gamma = \mathbf{u}_{i+1,\theta}$. Then $\alpha = \mathbf{u}_{1\theta}$, but $q_\alpha = q_{11} \neq -1$, which is a contradiction.
- C_θ : $\beta = \mathbf{w}_{11}, \gamma = \mathbf{e}_\theta$. Then $\alpha = \mathbf{u}_{1\theta}$, but $q_\alpha = q_{\theta\theta} \neq -1$, which is a contradiction.
- F_4 : $\beta = \mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3 + 2\mathbf{e}_4, \gamma = \mathbf{e}_1 + \mathbf{e}_2$, or $\beta = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 + 2\mathbf{e}_4, \gamma = \mathbf{e}_1$. In both cases, $\alpha = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$, but $q_\alpha = q \neq -1$, which is a contradiction.

Thus each root $N_\alpha \alpha$ corresponds to $x_\alpha^{N_\alpha}$, and each x_α has infinite height, as before. The elements

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \quad \text{where } 0 \leq h_j < \infty, \text{ if } \beta_j \in S_I, \text{ for } 1 \leq j \leq P,$$

form a basis of $\hat{\mathfrak{B}}(V)$ as a vector space.

Now let $\bar{I}(V)$ be the ideal of $T(V)$ generated by the relations (5-41)–(5-44) and (5-40). We have $\mathfrak{J}(V) \subseteq \bar{I}(V) \subseteq I(V)$, so the corresponding projections induce a surjective morphism of algebras $\phi : \mathfrak{B} \rightarrow \mathfrak{B}(V)$, where $\mathfrak{B} := T(V)/\bar{I}(V)$:

$$\begin{array}{ccc} T(V) & \twoheadrightarrow & \hat{\mathfrak{B}}(V) \\ \downarrow & \swarrow & \downarrow \\ \mathfrak{B}(V) & \xleftarrow{\phi} & \mathfrak{B} \end{array}$$

Also, the elements

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \cdots x_{\beta_p}^{h_p}, \quad \text{where } 0 \leq h_j < N_{\beta_j}, \text{ if } \beta_j \in S_I, \text{ for } 1 \leq j \leq P,$$

generate \mathfrak{B} as a vector space, because they correspond to images of generators of $\mathfrak{B}(V)$ and are nonzero (as before, each nonincreasing product of hyperwords such that $h_j \geq N_{\beta_j}$ is zero in \mathfrak{B}). But ϕ is surjective, and the corresponding images of these elements form a basis of $\mathfrak{B}(V)$, so ϕ is an isomorphism. \square

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Frobenius splittings of toric varieties

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We discuss a characteristic free version of Frobenius splittings for toric varieties and give a polyhedral criterion for a toric variety to be diagonally split. We apply this criterion to show that section rings of nef line bundles on diagonally split toric varieties are normally presented and Koszul, and that Schubert varieties are not diagonally split in general.

1. Introduction

Fix an integer q greater than one. Let $T = \text{Spec } \mathbb{Z}[M]$ be the torus with character lattice M , and let N be the dual lattice. Let Σ be a complete fan in $N_{\mathbb{R}}$, with $X = X(\Sigma)$ the associated toric variety over \mathbb{Z} . Multiplication by q preserves the fan and maps the lattice N into itself, and therefore gives an endomorphism

$$F : X \rightarrow X.$$

Each T -orbit in X is a torus that is preserved by F , which acts by taking a point t to t^q . For example, if X is projective space, then F is given in homogeneous coordinates by

$$[x_0 : \cdots : x_n] \mapsto [x_0^q : \cdots : x_n^q].$$

If q is prime and k is the field with q elements, then the restriction of F to the variety X_k is the absolute Frobenius morphism. Pulling back functions by F gives a natural inclusion of \mathbb{C}_X -algebras $F^* : \mathbb{C}_X \hookrightarrow F_*\mathbb{C}_X$.

Definition 1.1. A *splitting of X* is an \mathbb{C}_X -module map $\pi : F_*\mathbb{C}_X \rightarrow \mathbb{C}_X$ such that the composition $\pi \circ F^*$ is the identity on \mathbb{C}_X .

Standard results from the theory of Frobenius splittings generalize in a straightforward way to these splittings of toric varieties. See Section 2 for details.

If Y is a subvariety of X cut out by an ideal sheaf I_Y and $\pi(F_*I_Y)$ is contained in I_Y then we say that π is *compatible* with Y . If Y is a toric variety embedded equivariantly in X , the closure of a subtorus of an orbit in X , then a splitting compatible with Y induces a splitting of Y . We say that X is *diagonally split* if

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there is a splitting of $X \times X$ that is compatible with the diagonal, for some q . Such splittings are of particular interest; by classic arguments of Mehta, Ramanan and Ramanathan, if X is diagonally split then every ample line bundle on X is very ample and defines a projectively normal embedding.

Our main result is a polyhedral criterion for a toric variety to be diagonally split. Let v_ρ denote the primitive generator of a ray, or one-dimensional cone, in Σ . Let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, and let the *diagonal splitting polytope* \mathbb{F}_X be defined by

$$\mathbb{F}_X = \{u \in M_{\mathbb{R}} \mid -1 \leq \langle u, v_\rho \rangle \leq 1 \text{ for all } \rho \in \Sigma\}.$$

We write $\frac{1}{q}M$ for the subgroup of $M_{\mathbb{R}}$ consisting of *fractional lattice points* u such that qu is in M .

Theorem 1.2. *The toric variety X is diagonally split if and only if the interior of \mathbb{F}_X contains representatives of every equivalence class in $\frac{1}{q}M/M$.*

While the existence of a compatible splitting of the diagonal in $X \times X$ implies that section rings of ample line bundles on X are generated in degree one, compatible splittings of large semidiagonals in products of multiple copies of X give further information on these section rings. For example, if the union of $\Delta \times X$ and $X \times \Delta$ is compatibly split in $X \times X \times X$, where Δ is the diagonal in $X \times X$, then it follows from standard arguments that the section ring of each ample line bundle on X is *normally presented*, that is, generated in degree one with relations generated in degree two. For fixed n greater than one, let Δ_i be the large semidiagonal

$$\Delta_i = X^{i-1} \times \Delta \times X^{n-i-1},$$

for $1 \leq i < n$.

Theorem 1.3. *Let X be a diagonally split toric variety. Then $\Delta_1 \cup \dots \cup \Delta_{n-1}$ is compatibly split in X^n .*

In particular, if X is diagonally split then the union of $\Delta \times X$ and $X \times \Delta$ is compatibly split in $X \times X \times X$, so the section ring of any ample line bundle on X is normally presented. Analogous results hold for any finite collection of nef line bundles on X , as we now discuss.

For line bundles L_1, \dots, L_r on X , let $R(L_1, \dots, L_r)$ be the section ring

$$R(L_1, \dots, L_r) = \bigoplus_{(\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r} H^0(X, L_1^{\alpha_1} \otimes \dots \otimes L_r^{\alpha_r}).$$

We consider $R(L_1, \dots, L_r)$ as a graded ring, where the degree of

$$H^0(X, L_1^{\alpha_1} \otimes \dots \otimes L_r^{\alpha_r})$$

is $\alpha_1 + \dots + \alpha_r$. In particular, the degree zero part of $R(L_1, \dots, L_r)$ is \mathbb{Z} . Recall that a graded ring R is *Koszul* if the ideal generated by elements of positive degree

has a linear resolution as an R -module. See [Polishchuk and Positselski 2005] for background on Koszul rings and further details.

Theorem 1.4. *Let X be a complete, diagonally split toric variety, and let*

$$L_1, \dots, L_r$$

be nef line bundles on X . Then the section ring $R(L_1, \dots, L_r)$ is normally presented and Koszul.

In particular, if X is diagonally split then the section ring of any ample line bundle on X is normally presented and Koszul. Well-known open problems ask whether every ample line bundle on a smooth projective toric variety gives a projectively normal embedding [Oda 1997] and, if so, whether its section ring is normally presented [Sturmfels 1996, Conjecture 13.19]. When the section ring is normally presented, it is natural to ask whether it is also Koszul. Addressing these questions and their analogues for singular toric varieties in as many cases as possible is one of the main motivations behind this work.

Remark 1.5. The section ring $R(L_1, \dots, L_r)$ associated to a finite collection of line bundles is canonically identified with the section ring of the line bundle $\mathcal{O}(1)$ on the projectivized vector bundle $\mathbb{P}(L_1 \oplus \dots \oplus L_r)$, which is also a toric variety. If L_1, \dots, L_r are nef and correspond to polytopes P_1, \dots, P_r , then the Cayley sum is the polytope associated to $\mathcal{O}(1)$. Cayley sums have also appeared prominently in recent work related to boundedness questions in toric mirror symmetry [Batyrev and Nill 2008; 2007; Haase et al. 2008].

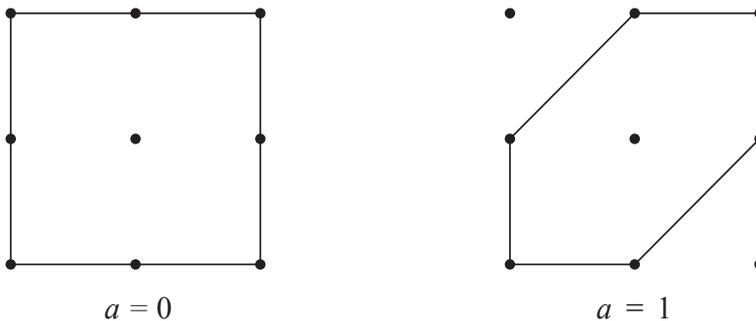
Remark 1.6. Frobenius morphisms and their lifts to characteristic zero have been used powerfully in several other contexts related to the geometry of toric varieties, including by Buch, Lauritzen, Mehta and Thomsen [1997] to prove Bott vanishing and degeneration of the Hodge to de Rham spectral sequence, by Totaro [199?] to give a splitting of the weight filtration on Borel–Moore homology, by Smith [2000] to prove global F -regularity, by Brylinski and Zhang [2003] to prove degeneration of a spectral sequence computing equivariant cohomology with rational coefficients, and by Fujino [2007] to prove vanishing theorems for vector bundles and reflexive sheaves. Frobenius splittings have also played a role in unsuccessful attempts to show that section rings of ample line bundles on smooth toric varieties are normally presented [Bøgvad 1995]. We hope that this work will help revive the insight of Bøgvad and others into the potential usefulness of Frobenius splittings as a tool for understanding ample line bundles on toric varieties.

We conclude the introduction with an example illustrating Theorem 1.2 for Hirzebruch surfaces. As mentioned earlier, the proofs that section rings of ample line bundles on Schubert varieties are normally presented and Koszul via Frobenius

splittings involved compatible splittings of semidiagonals in $(G/B)^n$. It has been an open question for over twenty years whether Schubert varieties themselves are diagonally split (see [Ramanathan 1987, Remark 3.6] and [Brion and Kumar 2005, p. 81]). The following example gives a negative answer: the Hirzebruch surface F_3 is a Schubert variety in the G_2 -flag variety, and F_3 is not diagonally split.

Remark 1.7. To see that F_3 occurs as a Schubert variety in the G_2 -flag variety, first note that for any G , G/B is a \mathbb{P}^1 -bundle over G/P , where P is a minimal parabolic subgroup. If $w = s_1s_2$ is an element of length two in the Weyl group of G , and P is the minimal parabolic corresponding to s_1 , then X_w is a \mathbb{P}^1 -bundle over its image, which is a rational curve in G/P . In particular, X_w is a Hirzebruch surface. Then X_{s_1} is a rational curve in X_w with self-intersection $\langle \alpha_2, \alpha_1^\vee \rangle$, where α_i is the simple root corresponding to s_i , and $\alpha_i^\vee = 2\alpha_i / \langle \alpha_i, \alpha_i \rangle$. See [Kempf 1976, Section 2] for details. For G_2 , we can choose coordinates identifying the root lattice with the sublattice of \mathbb{Z}^3 consisting of those (a_1, a_2, a_3) such that $a_1 + a_2 + a_3 = 0$, with simple roots $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (-1, 2, -1)$. Then X_{s_1} is a curve of self-intersection -3 in X_w , and hence X_w is isomorphic to F_3 . See also [Anderson 2007] for a detailed study of the G_2 -flag variety and its Schubert varieties.

Example 1.8. Let a be a nonnegative integer, and let Σ be the complete fan in \mathbb{R}^2 whose rays are spanned by $(1, 0)$, $(0, 1)$, $(0, -1)$, and $(-1, a)$. Then $X(\Sigma)$ is isomorphic to the Hirzebruch surface F_a , the projectivization of the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ [Fulton 1993, pp. 7–8]. Let $q \geq 2$ be an integer. By Theorem 1.2, X is diagonally split if and only if the fractional lattice points in the interior of \mathbb{F}_X represent every equivalence class in $\frac{1}{q}\mathbb{Z}^2/\mathbb{Z}^2$. The polytopes \mathbb{F}_X for different values of a are shown below.

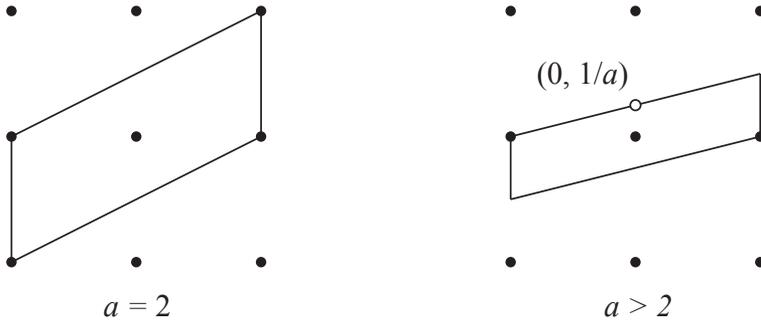


If a is equal to 0 or 1, the interior of \mathbb{F}_X contains the half open unit square $[0, 1) \times [0, 1)$, which contains representatives of every equivalence class in $\frac{1}{q}\mathbb{Z}^2/\mathbb{Z}^2$. Therefore, F_0 and F_1 are diagonally split for all q .

If $a = 2$, then \mathbb{F}_X is the parallelogram with vertices $(\pm 1, 0)$, $(\pm 1, 1)$. For $0 \leq m < q$, the interior of this intersection contains the fractional lattice points

$(m/q, n/q)$ for $(m - q)/2 < n < (m + q)/2$. If q is odd, then these represent every equivalence class in $\frac{1}{q}\mathbb{Z}^2/\mathbb{Z}^2$. In particular, F_2 is diagonally split for q odd.

If a is greater than two, then \mathbb{F}_X is the parallelogram with vertices $(\pm 1, 0)$, $\pm(1, 2/a)$. The only points in the interior of this intersection whose first coordinate is integral, are of the form $(0, y)$ for $-1/a < y < 1/a$. In particular, the equivalence class of $(0, \lfloor q/2 \rfloor/q)$ in $\frac{1}{q}\mathbb{Z}^2/\mathbb{Z}^2$ is not represented by any point in the interior of this intersection. Therefore, F_a is not diagonally split for a greater than two.



2. Preliminaries

Frobenius splittings were introduced and developed by Mehta, Ramanathan, and their collaborators in the 1980s. The original paper of Mehta and Ramanathan is exceedingly well written and remains an excellent first introduction to the subject [1985]. Frobenius splittings were rapidly applied to give elegant unified proofs that all ample line bundles on generalized Schubert varieties of all types are very ample and give projectively normal embeddings whose images are cut out by quadrics [Ramanan and Ramanathan 1985; Ramanathan 1987]. Inamdar and Mehta [1994], and independently Bezrukavnikov [1995], later showed that the homogeneous coordinate rings of these embeddings are Koszul. In characteristic zero, these results are deduced from the positive characteristic case using general semicontinuity theorems. See the recent book of Brion and Kumar [2005] for a unified exposition of these results, along with further details, references, and applications. On toric varieties, the Frobenius endomorphisms lift to endomorphisms over \mathbb{Z} , and it seems easiest and most natural to work independently of the characteristic using these lifted endomorphisms. One feature of this approach is that we can prove results about section rings of toric varieties over \mathbb{Z} , or an arbitrary field, by producing a splitting of the diagonal in $X \times X$ for a single q .

We begin by considering the structure of $F_*\mathcal{O}_X$ as an \mathcal{O}_X -module. As a sheaf of groups, $F_*\mathcal{O}_X$ evaluated on the invariant affine open set U_σ associated to a cone $\sigma \in \Sigma$ is the coordinate ring $\mathbb{Z}[U_\sigma]$, which is usually identified with the semigroup ring $\mathbb{Z}[\sigma^\vee \cap M]$. However, the module structure on $F_*\mathbb{Z}[U_\sigma]$ is different from the

action of $\mathbb{Z}[U_\sigma]$ on itself. For this reason, we identify $F_*\mathbb{Z}[U_\sigma]$ with the semigroup ring of fractional lattice points

$$F_*\mathbb{Z}[U_\sigma] = \mathbb{Z}[\sigma^\vee \cap \frac{1}{q}M],$$

taking a monomial $x^u \in \mathbb{Z}[U_\sigma]$ to $x^{u/q}$. The action of $\mathbb{Z}[U_\sigma]$ on $F_*\mathbb{Z}[U_\sigma]$ is then induced by the natural action of M on $\frac{1}{q}M$, so

$$x^u \cdot x^{u'} = x^{u+u'},$$

for $u \in M$ and $u' \in \frac{1}{q}M$. If Y is a toric variety embedded equivariantly in X , then a splitting π is compatible with Y if and only if the induced map

$$\mathbb{Z}[U_\sigma] \xrightarrow{\sim} \mathbb{Z}[\sigma^\vee \cap \frac{1}{q}M] \xrightarrow{\pi} \mathbb{Z}[U_\sigma]$$

maps $I_Y(U_\sigma)$ into $I_Y(U_\sigma)$ for every $\sigma \in \Sigma$.

We now summarize some basic properties of compatible splittings and their applications to section rings of ample line bundles. Let X be a complete toric variety, and let L be a line bundle on X . A splitting π of X makes \mathbb{O}_X a direct summand of $F_*\mathbb{O}_X$ and hence L a direct summand of $L \otimes F_*\mathbb{O}_X$. By the projection formula, $L \otimes F_*\mathbb{O}_X$ is isomorphic to $F_*(F^*L)$, and we claim that F^*L is isomorphic to L^q . To see this, note that there is a T -invariant Cartier divisor D such that L is isomorphic to $\mathbb{O}(D)$ [Fulton 1993, Section 3.4]. The restriction of D to an invariant affine open U_σ is the divisor of a rational function x^u for some $u \in M$, and hence the restriction of F^*D to U_σ is the divisor of x^{qu} . It follows that F^*L is isomorphic to L^q , as claimed. Now, since cohomology commutes with direct sums, π induces a split injection

$$H^i(X, L) \hookrightarrow H^i(X, L^q),$$

for every i . Iterating this argument gives split injections of $H^i(X, L)$ in $H^i(X, L^{q^r})$ for all positive integers r . In particular, if $H^i(X, L^{q^r})$ vanishes for some r , as is the case when L is ample, then $H^i(X, L)$ vanishes as well.

The proofs of the following five propositions are essentially identical to the standard proofs of the analogous results for Frobenius splittings, and are omitted. See [Brion and Kumar 2005]; Proposition 1.2.1, Theorem 1.2.8, and Exercises 1.5.E.1, 1.5.E.2, and 1.5.E.3, respectively, for the case where the line bundles in question are ample. The extensions to nef bundles can be deduced following the arguments in [Inamdar 1994], using the fact that any nef line bundle on X is the pullback of an ample line bundle on some toric variety X' under a proper birational toric morphism $f : X \rightarrow X'$.

Proposition 2.1. *Let Y and Y' be toric varieties equivariantly embedded in X . If $Y \cup Y'$ is split compatibly in X then Y , Y' , and $Y \cap Y'$ are split compatibly in X .*

Proposition 2.2. *Let Y be a compatibly split subvariety of X . If L is a nef line bundle on X then the restriction map*

$$H^0(X, L) \rightarrow H^0(Y, L)$$

is surjective and $H^1(X, I_Y \otimes L) = 0$.

Proposition 2.3. *Let L_1, \dots, L_r be nef line bundles on X . If the diagonal is compatibly split in $X \times X$ then the section ring $R(L_1, \dots, L_r)$ is normally generated.*

Proposition 2.4. *Let L_1, \dots, L_r be nef line bundles on X . If the union of $\Delta \times X$ and $X \times \Delta$ is compatibly split in $X \times X \times X$ then the section ring $R(L_1, \dots, L_r)$ is normally presented.*

Proposition 2.5. *Let L_1, \dots, L_r be nef line bundles on X . If $\Delta_1 \cup \dots \cup \Delta_{n-1}$ is compatibly split in X^n for every n then the section ring $R(L_1, \dots, L_r)$ is normally presented and Koszul.*

3. Canonical splittings

Every toric variety has a splitting, and among all splittings of X there is a unique one that extends to every toric compactification $X' \supset X$ and lifts to every proper birational toric modification $X'' \rightarrow X'$ of such a compactification. If q is prime and k is the field with q elements, then the restriction of this splitting to X_k is the unique Frobenius splitting that is canonical in the sense of Mathieu [Brion and Kumar 2005, Chapter 4]. We now describe this *canonical splitting*, starting with its restriction to the dense torus T .

Let π_0 be the map of $\mathbb{Z}[T]$ -modules from $F_*\mathbb{Z}[T]$ to $\mathbb{Z}[T]$ given by

$$\pi_0(x^u) = \begin{cases} x^u & \text{if } u \in M, \\ 0 & \text{otherwise.} \end{cases}$$

The pullback map $F^* : \mathbb{Z}[T] \rightarrow F_*\mathbb{Z}[T]$ is induced by the inclusion of M in $\frac{1}{q}M$; if q is prime and k is the field with q elements, then the induced map $k[T] \rightarrow F_*k[T]$ may be identified with the inclusion of $k[T]$ in $k[T]^{1/q}$.

In particular, $\pi_0 \circ F_*$ is the identity, and hence gives a splitting of T .

Proposition 3.1. *For any toric variety X , π_0 extends to a splitting of X .*

Proof. The composition $\pi_0 \circ F^*$ is the identity, and for each affine open U_σ , π_0 maps $\mathbb{Z}[\sigma^\vee \cap \frac{1}{q}M]$ into $\mathbb{Z}[\sigma^\vee \cap M]$. □

Properties of this canonical splitting π_0 are closely related to Smith's proof [2000, Proposition 6.3] that toric varieties are globally F -regular.

Proposition 3.2. *The canonical splitting π_0 is compatible with every T -invariant subvariety.*

Proof. First we claim that π_0 is compatible with the union of the T -invariant divisors. To see this, note that the ideal sheaf I of the union of the invariant divisors is given by

$$I(U_\sigma) = \mathbb{Z}[\text{int}(\sigma^\vee) \cap M],$$

where $\text{int}(\sigma^\vee)$ is the interior of σ^\vee . If u is a fractional lattice point in the interior of σ^\vee then $\pi_0(x^u)$ is either zero or x^u , and so is contained in $I(U_\sigma)$, which proves the claim. The proposition then follows from Proposition 2.1, since every T -invariant subvariety is an intersection of invariant divisors. \square

However, if X is positive dimensional then the canonical splitting π_0 of $X \times X$ is not compatible with the diagonal Δ . To see this, observe that if $u \in \frac{1}{q}M$ is not in M , then $1 - x^u \otimes x^{-u}$ is in F_*I_Δ , but

$$\pi_0(1 - x^u \otimes x^{-u}) = 1,$$

which is not in I_Δ . To apply the standard techniques relating splittings to section rings of ample line bundles discussed in Section 2, we must look for other splittings of $X \times X$, and X^r for r greater than two, that are compatible with the diagonal and the union of the large semidiagonals, respectively.

4. Splittings of diagonals

We now describe the space of all splittings of a toric variety and use this description to characterize diagonally split toric varieties. First, it is helpful to consider the structure of $F_*\mathbb{O}_X$ as an \mathbb{O}_X -module in more detail.

Recall that an equivariant structure, or T -linearization, on a coherent sheaf \mathcal{F} on X is an isomorphism of sheaves on $T \times X$,

$$\varphi : \mu^*\mathcal{F} \rightarrow p^*\mathcal{F},$$

where $\mu : T \times X \rightarrow X$ is the torus action and p is the second projection, that satisfies the usual cocycle condition [Brion and Kumar 2005, Section 2.1]. For example, the natural equivariant structure on \mathbb{O}_X is given by $1 \otimes x^u \mapsto x^{-u} \otimes x^u$. In general, the push forward of an equivariant sheaf under an equivariant morphism does not carry a natural equivariant structure. However, the equivariant endomorphism F has the property that $F_*\mathbb{O}_X$ is *equivariantizable* [Bøgvad 1998; Thomsen 2000]; it is possible to choose an equivariant structure as follows. First, choose representatives u_1, \dots, u_s of the cosets in $\frac{1}{q}M/M$. Let φ be the map from $\mu^*F_*\mathbb{O}_X$ to $p^*F_*\mathbb{O}_X$ that takes $1 \otimes x^u$ to $x^{u_i - u} \otimes x^u$, for u in the coset $u_i + M$. It is straightforward to check that φ is an isomorphism and gives an equivariant structure on $F_*\mathbb{O}_X$, as required.

A splitting of X restricts to a splitting of T , and two splittings of X agree if and only if they agree on T , so we describe the space of all splittings of X in terms of

splittings of T that extend to X , as follows. For fractional lattice points $a \in \frac{1}{q}M$, let

$$\pi_a : F_*\mathbb{Z}[T] \rightarrow \mathbb{Z}[T]$$

be the map given by

$$\pi_a(x^u) = \begin{cases} x^{a+u} & \text{if } a+u \text{ is in } M, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.1. *The set of maps π_a , for a in $\frac{1}{q}M$, is a \mathbb{Z} -basis for*

$$\text{Hom}(F_*\mathbb{Z}[T], \mathbb{Z}[T]).$$

Proof. The maps π_a are independent, and the free generators x^{u_1}, \dots, x^{u_s} for $F_*\mathbb{Z}[T]$ can be sent to an arbitrary s -tuple of elements of $\mathbb{Z}[T]$ by a suitable linear combination of the maps π_a . \square

If we choose an equivariant structure for $\text{Hom}(F_*\mathbb{Z}[T], \mathbb{Z}[T])$, as above, then the maps π_a form a T -eigenbasis. Therefore, a rational section

$$\pi = c_1\pi_{a_1} + \dots + c_r\pi_{a_r}$$

of the sheaf $\mathcal{H}om(F_*\mathbb{O}_X, \mathbb{O}_X)$, with each c_i nonzero, extends to X if and only if each π_{a_i} is regular on X . For a ray, or one-dimensional cone ρ in Σ , we write v_ρ for the primitive generator of ρ .

Proposition 4.2. *Let U_σ be an affine toric variety. Then π_a is regular on U_σ if and only if $\langle a, v_\rho \rangle$ is greater than minus one for each ray ρ in σ .*

Proof. The map π_a is regular on U_σ if and only if it takes

$$\mathbb{Z}[\sigma^\vee \cap \frac{1}{q}M]$$

into $\mathbb{Z}[\sigma^\vee \cap M]$. Suppose $\langle a, v_\rho \rangle$ is greater than minus one for each ray ρ in σ and u is in $\sigma^\vee \cap \frac{1}{q}M$. Either $\pi_a(x^u)$ is zero or $a+u$ is in M and $\langle u, v_\rho \rangle$ is a nonnegative integer for all rays ρ in σ , and hence $a+u$ is in σ^\vee . Therefore π_a extends to U_σ . Conversely, if $\langle a, v_\rho \rangle$ is less than or equal to minus one for some ρ , then it is straightforward to produce points $u \in \sigma^\vee$ such that $a+u$ is in M , but not in σ^\vee . In this case, π_a is not regular on U_σ . \square

We follow the usual toric convention fixing $K = -\sum D_\rho$, the sum of the prime T -invariant divisors each with multiplicity minus one, as a convenient representative of the canonical class. The polytope associated to a divisor $D = \sum d_\rho D_\rho$ is

$$P_D = \{u \in M_\mathbb{R} \mid \langle u, v_\rho \rangle \geq -d_\rho \text{ for all } \rho\}.$$

In particular, the polytope P_{-K} associated to the anticanonical divisor is

$$P_{-K} = \{u \in M_\mathbb{R} \mid \langle u, v_\rho \rangle \geq -1 \text{ for all } \rho\}.$$

The interior of the polytope P_{-K} controls the space of \mathbb{O}_X -module maps from $F_*\mathbb{O}_X$ to \mathbb{O}_X as follows.

Proposition 4.3. *The set of maps π_a for fractional lattice points a in the interior of P_{-K} is a basis for $\text{Hom}(F_*\mathbb{O}_X, \mathbb{O}_X)$.*

Proof. If a is not in the interior of P_{-K} then $\langle a, v_\rho \rangle$ is less than or equal to minus one for some ray $\rho \in \Sigma$ and then π_a is not regular on U_ρ . Conversely, if a is in the interior of P_{-K} , then π_a extends to every invariant affine open subvariety of X , by Proposition 4.2, and therefore is regular on X . \square

Remark 4.4. When X is smooth, Proposition 4.3 corresponds to the natural identification between $\text{Hom}(F_*\mathbb{O}_X, \mathbb{O}_X)$ and $H^0(X, K_X^{1-q})$ given by duality for finite flat morphisms [Brion and Kumar 2005, Section 1.3].

Proposition 4.5. *A map $\sum c_a \pi_a$ in $\text{Hom}(F_*\mathbb{O}_X, \mathbb{O}_X)$ is a splitting if and only if $c_0 = 1$.*

Proof. Zero is the only lattice point in the interior of $P_{-K} \cap M$, so the image of x^u under $\pi = \sum c_a \pi_a$ is equal to $c_0 x^u$ for $u \in M$. In particular, since F^* maps x^u to x^u in $F_*\mathbb{O}_X$, $\pi \circ F^*$ is the identity if and only if c_0 is equal to one. \square

The set of splittings of X is an affine hyperplane in $\text{Hom}(F_*\mathbb{O}_X, \mathbb{O}_X)$ by Proposition 4.5. For any subvariety $Y \subset X$, the condition that $\pi(F_*I_Y)$ is contained in I_Y cuts out a linear subspace of $\text{Hom}(F_*\mathbb{O}_X, \mathbb{O}_X)$. So the set of splittings of X that are compatible with Y is an affine subspace of $\text{Hom}(F_*\mathbb{O}_X, \mathbb{O}_X)$, which may be empty. We now prove Theorem 1.2, which gives a necessary and sufficient condition for the space of splittings of $X \times X$ that are compatible with the diagonal to be nonempty.

Proof of Theorem 1.2. Suppose

$$\pi = \sum c_{a,a'} \pi_{a,a'}$$

is a splitting of $X \times X$ that is compatible with the diagonal. Then the restriction of π to the dense torus is a splitting compatible with the diagonal in $T \times T$. For any $u \in \frac{1}{q}M$, we have

$$1 - x^u \otimes x^{-u}$$

in F_*I_Δ , where I_Δ is the ideal of the diagonal in $T \times T$. Since $\pi(1)$ is equal to one, the restriction of $\pi(x^u \otimes x^{-u})$ to the diagonal must also be equal to one. Now the restriction of $\pi(x^u \otimes x^{-u})$ to the diagonal is a Laurent polynomial in $\mathbb{Z}[T]$ whose constant term is

$$\sum_{a \in [u]} c_{-a,a}$$

where $[u]$ is the coset of u in $\frac{1}{q}M/M$. Since the polytope associated to $-K_{X \times X}$ is $P_{-K_X} \times P_{-K_X}$, there must be a representative a of $[u]$ such that both a and $-a$ are

contained in the interior of P_{-K_X} , which means that a is contained in the interior of the diagonal splitting polytope

$$\mathbb{F}_X = P_{-K_X} \cap -P_{-K_X}.$$

For the converse, suppose that every nonzero equivalence class $[u_i]$ in $\frac{1}{q}M/M$ has a representative a_i in the interior of \mathbb{F}_X . Then

$$\pi_\Delta = \pi_0 + \sum_{i=1}^s \pi_{a_i, -a_i}$$

is a splitting, and we claim that π_Δ is compatible with the diagonal. To see this, note that the ideal of the diagonal in $U_\sigma \times U_\sigma$ is generated by the Laurent polynomials $1 - x^u \otimes x^{-u}$ for u in $\sigma^\vee \cap M$. Then F_*I_Δ is generated as a $\mathbb{Z}[U_\sigma \times U_\sigma]$ -module by the

$$x^b - x^b \cdot (x^u \otimes x^{-u}),$$

as $b = (b_1, b_2)$ ranges over $\frac{1}{q}(M \times M)$ and u ranges over $\frac{1}{q}M$. Now the restriction of $\pi_\Delta(x^b)$ to the diagonal is $x^{b_1+b_2}$ if $b_1 + b_2$ is in M and zero otherwise. In particular, the restriction of $\pi_\Delta(x^b - x^b \cdot (x^u \otimes x^{-u}))$ to the diagonal vanishes, as required. \square

Proof of Theorem 1.3. Suppose the diagonal is compatibly split in $X \times X$. Then every nonzero equivalence class $[u_j]$ in $\frac{1}{q}M/M$ is represented by a fractional lattice point a_j in the interior of \mathbb{F}_X , by Theorem 1.2.

A splitting of X^n is compatible with the union $\Delta_1 \cup \dots \cup \Delta_{n-1}$ if it is compatible with each Δ_i . For $u \in M$, let $u^{(i)}$ denote the lattice point in M^n whose only nonzero coordinate is the i -th one, which is equal to u . The ideal of Δ_i is generated by the functions

$$1 - x^{u^{(i)}} \cdot x^{-u^{(i+1)}}.$$

We claim that the splitting

$$\pi = \pi_0 + \sum_{i=1}^{n-1} \sum_j \pi_{a_j^{(i)} - a_j^{(i+1)}}$$

is compatible with Δ_i for $1 \leq i < n$, and hence with the union $\Delta_1 \cup \dots \cup \Delta_{n-1}$. The proof of the claim is then similar to the proof of Theorem 1.2 above, and the theorem follows. \square

Proof of Theorem 1.4. Suppose the diagonal is compatibly split in $X \times X$. Then $\Delta_1 \cup \dots \cup \Delta_{n-1}$ is compatibly split in X^n for every n , by Theorem 1.3. Therefore, for any nef line bundles L_1, \dots, L_r on X the section ring $R(L_1, \dots, L_r)$ is normal and Koszul, by Proposition 2.5. \square

Remark 4.6. Pairs of opposite lattice points u and $-u$ in the polytopes associated to anticanonical divisors have also appeared in relation to the classification of smooth toric Fano varieties. In particular, Ewald [1988] conjectured twenty years ago that if X is a smooth toric Fano variety then \mathbb{F}_X contains a basis for the character lattice M . Ewald’s conjecture has been verified for smooth toric Fano varieties of dimension less than or equal to seven by Øbro [2007, Section 4.1]. However, it remains unknown in higher dimensions whether there exists a single nonzero lattice point u in \mathbb{F}_X [Kreuzer and Nill 2007, Section 4.6].

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Algebra & Number Theory

Volume 3 No. 1 2009

On the additive dilogarithm	1
SINAN ÜNVER	
Nichols algebras with standard braiding	35
IVAN EZEQUIEL ANGIOLO	
Frobenius splittings of toric varieties	107
SAM PAYNE	



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