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The class of standard braided vector spaces, introduced by Andruskiewitsch and the author in 2007 to understand the proof of a theorem of Heckenberger, is slightly more general than the class of braided vector spaces of Cartan type. In the present paper, we classify standard braided vector spaces with finite-dimensional Nichols algebra. For any such braided vector space, we give a PBW basis, a closed formula of the dimension and a presentation by generators and relations of the associated Nichols algebra.

Introduction

A breakthrough in the development of the theory of Hopf algebras occurred with the discovery of quantized enveloping algebras [Drinfel'd 1987; Jimbo 1985]. This special class of Hopf algebras has been intensively studied by many authors and from many points of view. In particular, finite-dimensional analogues of quantized enveloping algebras were introduced and investigated by Lusztig [1990a; 1990b].

About ten year ago, a classification program of pointed Hopf algebras was launched by Andruskiewitsch and Schneider [1998] (see also [Andruskiewitsch and Schneider 2002b]). The success of this program depends on finding solutions to several questions, among them:

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Question 1 [Andruskiewitsch 2002, Question 5.9]. Given a braided vector space of diagonal type V, such that the entries of its matrix are roots of unity, compute the dimension of the associated Nichols algebra $\mathfrak{B}(V)$. If it is finite, give a nice presentation of $\mathfrak{B}(V)$.

Partial answers to this question were given in [Andruskiewitsch and Schneider 2000; Heckenberger 2006b] for the class of braided vector spaces of Cartan type. These answers were already crucial to proving a classification theorem for finite-dimensional Hopf algebras whose group is abelian with prime divisors of the order great than 7 [Andruskiewitsch and Schneider 2005]. Later, a complete answer to the first part of Question 1 was given in [Heckenberger 2006a].

The notion of a standard braided vector space, a special kind of diagonal braided vector space, was introduced in [Andruskiewitsch and Angiono 2008], and is reviewed in Definition 3.5 below. This class includes properly the class of braided vector spaces of Cartan type.

The purpose of this paper is to develop from scratch the theory of standard braided vector spaces. Here are our main contributions:

- We give a complete classification of standard braided vector spaces with finite-dimensional Nichols algebras. As usual, we may assume the connectedness of the corresponding braiding. It turns out that standard braided vector spaces are of Cartan type when the associated Cartan matrix is of type C, D, E or F, see Proposition 3.8. For types A, B, G there are standard braided vector spaces not of Cartan type; these are listed in Propositions 3.9, 3.10 and 3.11. Those of type A_2 and B_2 appeared already in [Graña 2000]. Our classification does not rely on [Heckenberger 2006a], but we can identify our examples in the tables of that reference.
- We describe a concrete PBW (Poincaré–Birkhoff–Witt) basis of the Nichols algebra of a standard braided vector space as in the previous point; this follows from the general theory of Kharchenko [1999] together with [Heckenberger 2006b, Theorem 1]. As an application, we give closed formulas for the dimension of these Nichols algebras.
- We present a concrete set of defining relations of the Nichols algebras of standard braided vector spaces as in the previous points. This is an answer to the second part of Question 1 in the standard case. We note that this seems to be new even for Cartan type, for some values of the roots of unity appearing in the picture. Essentially, these relations are either quantum Serre relations or powers of root vectors; but in some cases, there are some substitutes of the quantum Serre relations due to the smallness of the intervening root vectors. Some of these substitutes can be recognized already in the relations in [Andruskiewitsch and Dăscălescu 2005].

Here is the plan of this article. We start by collecting necessary tools. Namely, we recall the definition of Lyndon words and give some properties about them, such as the Shirshov decomposition, in Section 1A. Next, in Section 1B, we discuss the notions of hyperletter and hyperword, following [Kharchenko 1999] (where they are called superletter and superword); these are certain iterations of braided commutators applied to Lyndon words. In Section 1C, a PBW basis is given for any quotient of the tensor algebra of a diagonal braided vector space V by a Hopf ideal using these hyperwords. This applies in particular to Nichols algebras.

In Section 2, after some technical preparations, we present a transformation of a braided graded Hopf algebra into another, with different space of degree one. This generalizes an analogous transformation for Nichols algebras given in [Heckenberger 2006b, Proposition 1]; see Section 2C.

In Section 3 we classify standard braided vector spaces with finite-dimensional Nichols algebra. In Section 3A, we prove that if the set of PBW generators is finite, the associated generalized Cartan matrix is of finite type. So in Section 3B we obtain all the standard braidings associated to Nichols algebras of finite dimension.

Section 4 is devoted to PBW bases of Nichols algebras of standard braided vector spaces with finite Cartan matrix. In Section 4A we prove that there is exactly one PBW generator whose degree corresponds to each positive root associated to the finite Cartan matrix. We give a set of PBW generators in Section 4B, following a nice presentation from [Lalonde and Ram 1995]. As a consequence, we compute the dimension in Section 4C.

The main result of this paper is the explicit presentation by generators and relations of Nichols algebras of standard braided vector spaces with finite Cartan matrix, given in Section 5. It relies on the explicit PBW basis and transformation described in Section 2C. Section 5A states some relations for Nichols algebras of standard braidings and proves facts about the coproduct. Sections 5B–5D contain the explicit presentation for types A_θ, B_θ and G_2, respectively. For this, we establish relations among the elements of the PBW basis, inspired in [Andruskiewitsch and Dăscălescu 2005] and [Graña 2000]. We finally prove the presentation in the case of Cartan type in Section 5E. To our knowledge, this is the first self-contained exposition of Nichols algebras of braided vector spaces of Cartan type.

Notation. We fix an algebraically closed field $\Bbbk$ of characteristic 0; all vector spaces, Hopf algebras and tensor products are considered over $\Bbbk$.

For each $N > 0$, $\mathbb{G}_N$ denotes the set of primitive N-th roots of unity in $\Bbbk$.

Given $n \in \mathbb{N}$ and $q \in \Bbbk$, $q \notin \bigcup_{0 \le j \le n} \mathbb{G}_j$, we define

$$\binom{n}{j}_q = \frac{(n)_q!}{(k)_q!\,(n-k)_q!}, \quad \text{where } (n)_q! = \prod_{j=1}^{n} (k)_q, \quad \text{and } (k)_q = \sum_{j=0}^{k-1} q^j.$$

We define

$$\mathfrak{q}_h(t) := \frac{t^h - 1}{t - 1} \in \Bbbk[t], \quad h \in \mathbb{N}; \quad \mathfrak{q}_\infty(t) := \frac{1}{1-t} = \sum_{s=0}^{\infty} t^s \in \Bbbk[[t]].$$

For each $\theta \in \mathbb{N}$ and each $n = (n_1, \dots, n_\theta) \in \mathbb{Z}^\theta$, we set $x^n = x_1^{n_1} \cdots x_\theta^{n_\theta} \in \Bbbk[[x_1^{\pm 1}, \dots, x_\theta^{\pm 1}]]$. For each $\mathbb{Z}^\theta$-graded vector spaces $\mathfrak{B}$, we denote by $\mathcal{H}_{\mathfrak{B}} = \sum_{n \in \mathbb{Z}^\theta} \dim \mathfrak{B}^n x^n$ the Hilbert series associated to $\mathfrak{B}$.

Let $C = \bigoplus_{n \in \mathbb{N}_0} C_{i+j}$ be a $\mathbb{N}_0$-graded coalgebra, with projections $\pi_n : C \to C_n$. Given $i, j \geq 0$, we denote by

$$\Delta_{i,j} := (\pi_i \otimes \pi_j) \circ \Delta : C_{i+j} \to C_i \otimes C_j,$$

the (i, j)-th component of the comultiplication.

1. PBW bases

Let A be an algebra, $P, S \subset A$ and $h : S \mapsto \mathbb{N} \cup \{\infty\}$. Let also $<$ be a linear order on S. Let us denote by $B(P, S, <, h)$ the set

$$\left\{ p\, s_1^{e_1} \dots s_t^{e_t} : t \in \mathbb{N}_0,\ s_1 > \dots > s_t,\ s_i \in S,\ 0 < e_i < h(s_i),\ p \in P \right\}.$$

If $B(P, S, <, h)$ is a linear basis of A, then we say that $(P, S, <, h)$ is a set of *PBW generators* with height h, and that $B(P, S, <, h)$ is a *PBW basis* of A. Occasionally, we shall simply say that S is a PBW basis of A.

In this section, following [Kharchenko 1999], we describe an appropriate PBW basis of a braided graded Hopf algebra $\mathfrak{B} = \bigoplus_{n \in \mathbb{N}} \mathfrak{B}^n$ such that $\mathfrak{B}^1 \cong V$, where V is a braided vector space of diagonal type. This applies in particular, to the Nichols algebra $\mathfrak{B}(V)$. In Section 1A we recall the classical construction of Lyndon words. Let V be a vector space together with a fixed basis. Then there is a basis of the tensor algebra $T(V)$ by certain words satisfying a special condition, called Lyndon words. Each Lyndon word has a canonical decomposition as a product of a pair of smaller Lyndon words, called the Shirshov decomposition.

We briefly recall the notions of a braided vector space (V, c) of diagonal type and of a Nichols algebra in Section 1B. In Section 1C we recall the definition of the hyperletter $[l]_c$, for any Lyndon word l; this is the braided commutator of the hyperletters corresponding to the words in the Shirshov decomposition. Hyperletters are a set of generators for a PBW basis of $T(V)$ and their classes form a PBW basis of $\mathfrak{B}$.

1A. ***Lyndon words.*** Let $\theta \in \mathbb{N}$. Let X be a set with θ elements and fix an enumeration $x_1, \dots, x_\theta$ of X; this induces a total order on X. Let $\mathbb{X}$ be the corresponding vocabulary (the set of words with letters in X) and consider the lexicographical order on $\mathbb{X}$.

Definition 1.1. An element $u \in \mathbb{X}$, $u \neq 1$, is called a *Lyndon word* if u is smaller than any of its proper ends; that is, if $u = vw$, $v, w \in \mathbb{X} - \{1\}$, then $u < w$. The set of Lyndon words is denoted by L.

We shall need the following properties of Lyndon words.

(1) Let $u \in \mathbb{X} - X$. Then u is Lyndon if and only if for any representation $u = u_1u_2$, with $u_1, u_2 \in \mathbb{X}$ not empty, one has $u_1u_2 = u < u_2u_1$.

(2) Any Lyndon word begins by its smallest letter.

(3) If $u_1, u_2 \in L$, $u_1 < u_2$, then $u_1u_2 \in L$.

The basic Theorem about Lyndon words, due to Lyndon, says that any word $u \in \mathbb{X}$ has a unique decomposition

$$u = l_1 l_2 \dots l_r, \tag{1-1}$$

with $l_i \in L$, $l_r \leq \dots \leq l_1$, as a product of nonincreasing Lyndon words. This is called the *Lyndon decomposition* of $u \in \mathbb{X}$; the $l_i \in L$ appearing in the decomposition (1-1) are called the *Lyndon letters* of u.

The lexicographical order of $\mathbb{X}$ turns out to be the same as the lexicographical order in the Lyndon letters. Namely, if $v = l_1 \dots l_r$ is the Lyndon decomposition of v, then $u < v$ if and only if

(i) the Lyndon decomposition of u is $u = l_1 \dots l_i$, for some $1 \leq i < r$, or

(ii) the Lyndon decomposition of u is $u = l_1 \dots l_{i-1} l l'_{i+1} \dots l'_s$, for some $1 \leq i < r$, $s \in \mathbb{N}$ and $l, l'_{i+1}, \dots, l'_s$ in L, with $l < l_i$.

Here is another useful characterization of Lyndon words.

Lemma 1.2 [Kharchenko 1999, p. 6]. *Let $u \in \mathbb{X} - X$. Then $u \in L$ if and only if there exist $u_1, u_2 \in L$ with $u_1 < u_2$ such that $u = u_1u_2$.*

Definition 1.3. Let $u \in L - X$. A decomposition $u = u_1u_2$, with $u_1, u_2 \in L$ such that u_2 is the smallest end among those proper nonempty ends of u is called the *Shirshov decomposition* of u.

Let $u, v, w \in L$ be such that $u = vw$. Then $u = vw$ is the Shirshov decomposition of u if and only if either $v \in X$, or else if $v = v_1v_2$ is the Shirshov decomposition of v, then $w \leq v_2$.

1B. ***Braided vector spaces of diagonal type and Nichols algebras.*** A braided vector space is a pair (V, c), where V is a vector space and $c \in \mathrm{Aut}(V \otimes V)$ is a solution of the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

We extend the braiding to $c : T(V)\otimes T(V) \to T(V)\otimes T(V)$ in the usual way. If $x, y \in T(V)$, the braided commutator is

$$[x, y]_c := \text{multiplication} \circ (\mathrm{id} - c)\,(x\otimes y)\,. \tag{1-2}$$

Assume that $\dim V < \infty$ and pick a basis $X = \{x_1, \dots, x_\theta\}$ of V; we may then identify $\Bbbk\mathbb{X}$ with $T(V)$. We consider the following gradings of the algebra $T(V)$:

(i) The usual $\mathbb{N}_0$-grading $T(V) = \bigoplus_{n\geq 0} T^n(V)$. If ℓ denotes the length of a word in $\mathbb{X}$, then $T^n(V) = \bigoplus_{x\in\mathbb{X},\, \ell(x)=n} \Bbbk x$.

(ii) Let $\mathbf{e}_1, \dots, \mathbf{e}_\theta$ be the canonical basis of $\mathbb{Z}^\theta$. Then $T(V)$ is also $\mathbb{Z}^\theta$-graded, where the degree is determined by $\deg x_i = \mathbf{e}_i$, $1 \leq i \leq \theta$.

A braided vector space (V, c) is of *diagonal type* with respect to the basis $x_1, \dots x_\theta$ if there exist $q_{ij} \in \Bbbk^\times$ such that $c(x_i\otimes x_j) = q_{ij} x_j\otimes x_i$, $1 \leq i, j \leq \theta$. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times$ be the bilinear form determined by $\chi(\mathbf{e}_i, \mathbf{e}_j) = q_{ij}$, $1 \leq i, j \leq \theta$. Then

$$c(u\otimes v) = \chi(\deg u, \deg v) v\otimes u \tag{1-3}$$

for any $u, v \in \mathbb{X}$, where $q_{u,v} = \chi(\deg u, \deg v) \in \Bbbk^\times$. In this case, the braided commutator satisfies a "braided" Jacobi identity as well as braided derivation properties, namely

$$\big[[u, v]_c\,, w\big]_c = \big[u, [v, w]_c\big]_c - \chi(\alpha, \beta) v\ [u, w]_c + \chi(\beta, \gamma)\, [u, w]_c\ v, \tag{1-4}$$

$$[u, v\ w]_c = [u, v]_c w + \chi(\alpha, \beta) v\ [u, w]_c, \tag{1-5}$$

$$[u\ v, w]_c = \chi(\beta, \gamma) [u, w]_c\ v + u\ [v, w]_c, \tag{1-6}$$

for any homogeneous $u, v, w \in T(V)$, of degrees $\alpha, \beta, \gamma \in \mathbb{N}^\theta$, respectively.

We denote by ${}^H_H\mathcal{YD}$ the category of Yetter–Drinfeld module over H, where H is a Hopf algebra with bijective antipode. Any $V \in {}^H_H\mathcal{YD}$ becomes a braided vector space [Montgomery 1993]. If H is the group algebra of a finite abelian group, then any $V \in {}^H_H\mathcal{YD}$ is a braided vector space of diagonal type. Indeed, $V = \bigoplus_{g\in\Gamma,\chi\in\widehat{\Gamma}} V_g^\chi$, where $V_g^\chi = V^\chi \cap V_g$ with $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$ and $V^\chi = \{v \in V \mid g \cdot v = \chi(g) v \text{ for all } g \in \Gamma\}$. The braiding is given by $c(x \otimes y) = \chi(g) y \otimes x$, for all $x \in V_g$, $g \in \Gamma$, $y \in V^\chi$, $\chi \in \widehat{\Gamma}$.

Reciprocally, any braided vector space of diagonal type can be realized as a Yetter–Drinfeld module over the group algebra of an abelian group.

If $V \in {}^H_H\mathcal{YD}$, the tensor algebra $T(V)$ admits a unique structure of graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ such that $V \subseteq \mathcal{P}(V)$. Following [Andruskiewitsch and Schneider 2002b], we consider the class $\mathfrak{S}$ of all the homogeneous two-sided ideals $I \subseteq T(V)$ such that

- I is generated by homogeneous elements of degree ≥ 2,

- I is a Yetter–Drinfeld submodule of $T(V)$, and
- I is a Hopf ideal: $\Delta(I) \subset I \otimes T(V) + T(V) \otimes I$.

The Nichols algebra $\mathfrak{B}(V)$ associated to V is the quotient of $T(V)$ by the maximal element $I(V)$ of $\mathfrak{S}$.

Let (V, c) be a braided vector space of diagonal type, and assume that $q_{ij} = q_{ji}$ for all i, j. Let Γ be the free abelian group of rank θ, with basis $g_1, \dots, g_\theta$, and define the characters $\chi_1, \dots, \chi_\theta$ of Γ by

$$\chi_j(g_i) = q_{ij}, \quad 1 \le i, j \le \theta.$$

Consider V as a Yetter–Drinfeld module over $\mathsf{k}\Gamma$ by defining $x_i \in V_{g_i}^{\chi_i}$.

Proposition 1.4 [Lusztig 1993, Proposition 1.2.3; Andruskiewitsch and Schneider 2002b, Proposition 2.10]. *Let $a_1, \dots, a_\theta \in \mathsf{k}^\times$. There is a unique bilinear form $(\,|\,) : T(V) \times T(V) \to \mathsf{k}$ such that* $(1|1) = 1$,

$$(x_i|x_j) = \delta_{ij} a_i \textit{ for all } i, j, \tag{1-7}$$

$$(x|yy') = (x_{(1)}|y)(x_{(2)}|y') \textit{ for all } x, y, y' \in T(V) \tag{1-8}$$

$$(xx'|y) = (x|y_{(1)})(x'|y_{(2)}) \textit{ for all } x, x', y \in T(V). \tag{1-9}$$

This form is symmetric and also satisfies

$$(x|y) = 0 \quad \textit{for all } x \in T(V)_g,\ y \in T(V)_h,\ g, h \in \Gamma,\ g \ne h. \tag{1-10}$$

The quotient $T(V)/I(V)$, where

$$I(V) := \{x \in T(V) : (x|y) = 0 \textit{ for all } y \in T(V)\}$$

is the radical of the form, is canonically isomorphic to the Nichols algebra of V. Thus, $(\,|\,)$ induces a nondegenerate bilinear form on $\mathfrak{B}(V)$ denoted by the same name. □

If (V, c) is of diagonal type, the ideal $I(V)$ is $\mathbb{Z}^\theta$-homogeneous, since it is the radical of a bilinear form in which the different $\mathbb{Z}^\theta$-homogeneous components are orthogonal; see [Andruskiewitsch and Schneider 2004, Proposition 2.10]. Hence $\mathfrak{B}(V)$ is $\mathbb{Z}^\theta$-graded. The following statement, that we include for later reference, is well-known.

Lemma 1.5. *Let V be a braided vector space of diagonal type, and consider its Nichols algebra $\mathfrak{B}(V)$.*

(a) *If q_{ii} is a root of unity of order $N > 1$, then $x_i^N = 0$.*

(b) *If $i \ne j$, then $(\operatorname{ad}_c x_i)^r(x_j) = 0$ if and only if*

$$(r)_{q_{ii}}! \prod_{0 \le k \le r-1} (1 - q_{ii}^k q_{ij} q_{ji}) = 0.$$

(c) *If* $i \neq j$ *and* $q_{ij}q_{ji} = q_{ii}^r$, *for some* $r \leq 0$, *then* $(\mathrm{ad}_c\, x_i)^{1-r}(x_j) = 0$. □

1C. ***PBW basis of a quotient of the tensor algebra by a Hopf ideal.*** Let (V, c) be a braided vector space with a basis $X = \{x_1, \dots, x_\theta\}$; identify $T(V)$ with $\Bbbk\mathbb{X}$. There is an important graded endomorphism $[]_c$ of $\Bbbk\mathbb{X}$ given by

$$[u]_c := \begin{cases} u & \text{if } u = 1 \text{ or } u \in X; \\ \big[[v]_c, [w]_c\big]_c & \text{if } u \in L,\ \ell(u) > 1 \\ & \qquad \text{and } u = vw \text{ is the Shirshov decomposition;} \\ [u_1]_c \dots [u_t]_c & \text{if } u \in \mathbb{X} - L \text{ with Lyndon decomposition } u = u_1 \dots u_t. \end{cases}$$

Now assume that (V, c) is of diagonal type with respect to the basis $x_1, \dots, x_\theta$, with matrix (q_{ij}).

Definition 1.6. The *hyperletter* corresponding to $l \in L$ is the element $[l]_c$. A *hyperword* is a word in hyperletters, and a *monotone hyperword* is a hyperword of the form $W = [u_1]_c^{k_1} \dots [u_m]_c^{k_m}$, where $u_1 > \dots > u_m$.

Remark 1.7. If $u \in L$, then $[u]_c$ is a homogeneous polynomial with coefficients in $\mathbb{Z}\left[q_{ij}\right]$ and $[u]_c \in u + \Bbbk\mathbb{X}_{>u}^{\ell(u)}$.

The hyperletters inherit the order from the Lyndon words; this induces in turn an ordering in the hyperwords (the lexicographical order on the hyperletters). Now, given monotone hyperwords W, V, it can be shown that

$$W = [w_1]_c \dots [w_m]_c > V = [v_1]_c \dots [v_t]_c,$$

where $w_1 \geq \dots \geq w_r, \quad v_1 \geq \dots \geq v_s$, if and only if

$$w = w_1 \dots w_m > v = v_1 \dots v_t.$$

Furthermore, the principal word of the polynomial W, when decomposed as sum of monomials, is w with coefficient 1.

Theorem 1.8 [Rosso 1999]. *Let* $m, n \in L$, *with* $m < n$. *Then the braided commutator* $[[m]_c, [n]_c]_c$ *is a* $\mathbb{Z}[q_{ij}]$*-linear combination of monotone hyperwords* $[l_1]_c, \dots, [l_r]_c$, $l_i \in L$, *such that*

- *the hyperletters of those hyperwords satisfy* $n > l_i \geq mn$,
- $[mn]_c$ *appears in the expansion with a nonzero coefficient, and*
- *any hyperword appearing in this decomposition satisfies*

$$\deg(l_1 \dots l_r) = \deg(mn).$$ □

A crucial result of Rosso describes the behavior of the coproduct of $T(V)$ in the basis of hyperwords.

Lemma 1.9 [Rosso 1999]. *Let $u \in \mathbb{X}$, and $u = u_1 \dots u_r v^m$, $v, u_i \in L, v < u_r \le \dots \le u_1$ the Lyndon decomposition of u. Then*

$$\Delta([u]_c) = 1 \otimes [u]_c + \sum_{i=0}^{m} \binom{m}{i}_{q_{v,v}} [u_1]_c \dots [u_r]_c [v]_c^i \otimes [v]_c^{m-i} + \sum_{\substack{l_1 \ge \dots \ge l_p > v,\ l_i \in L \\ 0 \le j \le m}} x^{(j)}_{l_1,\dots,l_p} \otimes [l_1]_c \dots [l_p]_c [v]_c^j,$$

where each $x^{(j)}_{l_1,\dots,l_p}$ is $\mathbb{Z}^\theta$-homogeneous and

$$\deg(x^{(j)}_{l_1,\dots,l_p}) + \deg(l_1 \dots l_p v^j) = \deg(u). \qquad \square$$

As in [Ufer 2004], we consider another order in $\mathbb{X}$; it is implicit in [Kharchenko 1999].

Definition 1.10. Let $u, v \in \mathbb{X}$. We say that $u \succ v$ if and only if either $\ell(u) < \ell(v)$, or else $\ell(u) = \ell(v)$ and $u > v$ (lexicographical order). This $\succ$ is a total order, called the *deg-lex order*.

Note that the empty word 1 is the maximal element for $\succ$. Also, this order is invariant by right and left multiplication.

Let now I be a proper ideal of $T(V)$, and set $R = T(V)/I$. Let $\pi : T(V) \to R$ be the canonical projection. Consider the subset of $\mathbb{X}$ given by

$$G_I := \{u \in \mathbb{X} : u \notin \Bbbk\mathbb{X}_{\succ u} + I\}.$$

(a) If $u \in G_I$ and $u = vw$, then $v, w \in G_I$.

(b) Any word $u \in G_I$ factorizes uniquely as a nonincreasing product of Lyndon words in G_I.

Proposition 1.11 ([Kharchenko 1999]; see also [Rosso 1999]). *The set $\pi(G_I)$ is a basis of R.* $\square$

In what follows, I is a Hopf ideal. We seek to find a PBW basis by hyperwords of the quotient R of $T(V)$. For this, we look at the set

$$S_I := G_I \cap L. \tag{1-11}$$

We then define the function $h_I : S_I \to \{2, 3, \dots\} \cup \{\infty\}$ by

$$h_I(u) := \min\left\{t \in \mathbb{N} : u^t \in \Bbbk\mathbb{X}_{\succ u^t} + I\right\}. \tag{1-12}$$

The next result plays a fundamental role in this paper.

Theorem 1.12 [Kharchenko 1999]. *Keep the notation above. Then*

$$B'_I := B\left(\{1+I\}, [S_I]_c + I, <, h_I\right)$$

is a PBW basis of $H = T(V)/I$. □

The next three results are consequences of Theorem 1.12; see [Kharchenko 1999] for their proofs.

Corollary 1.13. *A word u belongs to G_I if and only if the corresponding hyperletter $[u]_c$ is not a linear combination, modulo I, of hyperwords $[w]_c$, $w \succ u$, where all the hyperwords have their hyperletters in S_I.* □

Proposition 1.14. *In the conditions of the Theorem 1.12, if $v \in S_I$ is such that $h_I(v) < \infty$, then $q_{v,v}$ is a root of unity. In this case, if t is the order of $q_{v,v}$, then $h_I(v) = t$.* □

Corollary 1.15. *If $h_I(v) := h < \infty$, then $[v]^h$ is a linear combination of hyperwords $[w]_c$, $w \succ v^h$.* □

2. Transformations of braided graded Hopf algebras

In Section 2C, we shall introduce a transformation over certain graded braided Hopf algebras, generalizing [Heckenberger 2006b, Proposition 1]. It is an instrumental step in the proof of Theorem 5.25, one of the main results of this article.

2A. ***Preliminaries on braided graded Hopf algebras.*** Let H be the group algebra of an abelian group Γ. Let $V \in {}^H_H\mathcal{YD}$ with a basis $X = \{x_1, \dots, x_\theta\}$ such that $x_i \in V^{\chi_i}_{g_i}$, $1 \le i \le \theta$. Let $q_{ij} = \chi_j(g_i)$, so that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, $1 \le i, j \le \theta$.

We fix an ideal I in the class $\mathfrak{S}$; *we assume that I is $\mathbb{Z}^\theta$-homogeneous*. Let $\mathfrak{B} := T(V)/I$: this is a braided graded Hopf algebra, $\mathfrak{B}^0 = \Bbbk 1$ and $\mathfrak{B}^1 = V$. By definition of $I(V)$, there exists a canonical epimorphism of braided graded Hopf algebras $\pi : \mathfrak{B} \to \mathfrak{B}(V)$. Let $\sigma_i : \mathfrak{B} \to \mathfrak{B}$ be the algebra automorphism given by the action of g_i.

For the proof of the next result, see [Andruskiewitsch and Schneider 2002b, 2.8], for example.

Proposition 2.1. (1) *For each $1 \le i \le \theta$, there exists a uniquely determined* (id, σ_i)*-derivation $D_i : \mathfrak{B} \to \mathfrak{B}$ with $D_i(x_j) = \delta_{i,j}$ for all j.*

(2) *$I = I(V)$ if and only if* $\bigcap_{i=1}^{\theta} \ker D_i = \Bbbk 1$. □

These operators are defined for each $x \in \mathfrak{B}^k$, $k \ge 1$ by the formula

$$\Delta_{n-1,1}(x) = \sum_{i=1}^{\theta} D_i(x) \otimes x_i.$$

Analogously, we can define operators $F_i : \mathfrak{B} \to \mathfrak{B}$ by $F_i(1) = 0$ and

$$\Delta_{1,n-1}(x) = \sum_{i=1}^{\theta} x_i \otimes F_i(x) \quad \text{for all } x \in \bigoplus_{k>0} \mathfrak{B}^k.$$

Let χ be as in Section 1B. Consider the action $\triangleright$ of $\Bbbk\mathbb{Z}^\theta$ on $\mathfrak{B}$ given by

$$\mathbf{e}_i \triangleright b = \chi(\mathbf{u}, \mathbf{e}_i)b, \quad b \text{ homogeneous of degree } \mathbf{u} \in \mathbb{Z}^\theta. \tag{2-1}$$

Such operators F_i satisfy $F_i(x_j) = \delta_{i,j}$ for all j, and

$$F_i(b_1 b_2) = F_i(b_1)b_2 + (\mathbf{e}_i \triangleright b_1)F_i(b_2), \quad b_1, b_2 \in \mathfrak{B}.$$

Let $z_r^{(ij)} := (\operatorname{ad}_c x_i)^r(x_j)$, $i, j \in \{1, \dots, \theta\}$, $i \neq j$ and $r \in \mathbb{N}_0$.

Remark 2.2. The operators D_i, F_i satisfy

$$D_i(x_i^n) = (n)_{q_{ii}} x_i^{n-1}, \tag{2-2}$$

$$D_i\left((\operatorname{ad}_c x_i)^r(x_{j_1} \dots x_{j_s})\right) = 0 \text{ for } r, s \geq 0, \ j_k \neq i, \tag{2-3}$$

$$D_j(z_r^{(ij)}) = \prod_{k=0}^{r-1}(1 - q_{ii}^k q_{ij} q_{ji}) x_i^r \text{ for } r \geq 0, \tag{2-4}$$

$$F_i(z_m^{(ij)}) = (m)_{q_{ii}}(1 - q_{ii}^{m-1} q_{ij} q_{ji}) z_{m-1}^{(ij)}, \tag{2-5}$$

$$F_j(z_m^{(ij)}) = 0, \ m \geq 1. \tag{2-6}$$

The proof of the first three identities is as in [Andruskiewitsch and Schneider 2004, Lemma 3.7]; the proof of the last two is by induction on m.

For each pair $1 \leq i, j \leq \theta$, $i \neq j$, we define

$$M_{i,j}(\mathfrak{B}) := \left\{(\operatorname{ad}_c x_i)^m(x_j) : m \in \mathbb{N}\right\}; \tag{2-7}$$

$$m_{ij} := \min\left\{m \in \mathbb{N}_0 : (m+1)_{q_{ii}}(1 - q_{ii}^m q_{ij} q_{ji}) = 0\right\}. \tag{2-8}$$

Then either $q_{ii}^{m_{ij}} q_{ij} q_{ji} = 1$, or $q_{ii}^{m_{ij}+1} = 1$, if $q_{ii}^m q_{ij} q_{ji} \neq 1$ for all $m = 0, 1, \dots, m_{ij}$, or such m_{ij} does not exist, in which case we consider $m_{ij} = \infty$.

If $\mathfrak{B} = \mathfrak{B}(V)$, we write simply $M_{i,j} = M_{i,j}(\mathfrak{B}(V))$. Note that $(\operatorname{ad}_c x_i)^{m_{ij}+1} x_j = 0$ and $(\operatorname{ad}_c x_i)^{m_{ij}} x_j \neq 0$, by Lemma 1.5, so

$$\left|M_{i,j}\right| = m_{ij} + 1.$$

By Theorem 1.12, the braided graded Hopf algebra $\mathfrak{B}$ has a PBW basis consisting of homogeneous elements (with respect to the $\mathbb{Z}^\theta$-grading). As in [Heckenberger 2006b], we can even assume that

⊛ The height of a PBW generator $[u]$, $\deg(u) = d$, is finite if and only if $2 \leq \operatorname{ord}(q_{u,u}) < \infty$, and in such case, $h_{I(V)}(u) = \operatorname{ord}(q_{u,u})$.

This is possible because if the height of $[u]$, $\deg(u) = d$, is finite, then $2 \leq \operatorname{ord}(q_{u,u}) = m < \infty$, by Proposition 1.14. And if $2 \leq \operatorname{ord}(q_{u,u}) = m < \infty$, but $h_{I(V)}(u)$ is infinite, we can add $[u]^m$ to the PBW basis: in this case, $h_{I(V)}(u) = \operatorname{ord}(q_{u,u})$, and $q_{u^m,u^m} = q_{u,u}^{m^2} = 1$.

Let $\Delta^+(\mathfrak{B}) \subseteq \mathbb{N}^n$ be the set of degrees of the generators of the PBW basis, counted with their multiplicities and let also $\Delta(\mathfrak{B}) = \Delta^+(\mathfrak{B}) \cup \big(-\Delta^+(\mathfrak{B})\big)$: $\Delta^+(\mathfrak{B})$ is independent of the choice of the PBW basis with the property $\circledast$ (see [Andruskiewitsch and Angiono 2008, Lemma 2.18] for a proof of this statement).

In what follows, we write

$$q_\alpha := \chi(\alpha,\alpha), \qquad N_\alpha := \operatorname{ord} q_\alpha, \quad \alpha \in \Delta^+(\mathfrak{B}).$$

2B. *Auxiliary results.* Let I be $\mathbb{Z}^\theta$-homogeneous ideal in $\mathfrak{S}$ and $\mathfrak{B} = T(V)/I$ as in Section 2A. We shall use repeatedly the following fact.

In what follows, we use the convention $\operatorname{ord} 1 = 1$.

Remark 2.3. If $x_i^N = 0$ in $\mathfrak{B}$ with N minimal (this is called the order of nilpotency of x_i), then q_{ii} is a root of 1 of order N. Hence $(\operatorname{ad}_c x_i)^N x_j = 0$.

The following result extends (18) in the proof of [Heckenberger 2006b, Proposition 1].

Lemma 2.4. *For $i \in \{1,\dots,\theta\}$, let $\mathcal{K}_i$ be the subalgebra generated by $\bigcup_{j\neq i} M_{i,j}(\mathfrak{B})$ and denote by n_i the order of q_{ii}. Then there are isomorphisms of graded vector spaces*

- $\ker(D_i) \cong \mathcal{K}_i \otimes \mathsf{k}\left[x_i^{n_i}\right]$, *if* $1 < \operatorname{ord} q_{ii} < \infty$ *but x_i is not nilpotent, or*
- $\ker(D_i) \cong \mathcal{K}_i$, *if* $\operatorname{ord} q_{ii}$ *is the order of nilpotency of x_i or $q_{ii} = 1$.*

Moreover,

$$\mathfrak{B} \cong \mathcal{K}_i \otimes \mathsf{k}[x_i]. \tag{2-9}$$

Proof. We assume for simplicity $i = 1$ and consider the PBW basis obtained in the Theorem 1.12. Now $x_1 \in S_I$, and it is the least element of S_I, so each element of B'_I is of the form $[u_1]^{s_1} \dots [u_k]^{s_k} x_1^s$, with $u_k < \dots < u_1$, $u_i \in S_I \setminus \{x_1\}$, $0 < s_i < h_I(u_i)$, $0 \leq s < h_I(x_1)$. Call $S' = S_I \setminus \{x_1\}$, and

$$B_2 := B\left(1 + I, [S']_c + I, <, h_I|_{S'}\right),$$

that is, the PBW set generated by $[S']_c + I$, whose height is the restriction of the height of the PBW basis corresponding to S'. We have

$$\mathfrak{B} \cong \mathsf{k}B_2 \otimes \mathsf{k}\,[x_1]\,.$$

By (2-3), any $(\operatorname{ad}_c x_1)^r(x_j) \in \ker(D_1)$; as D_1 is a skew-derivation, we have $\mathcal{K}_1 \subseteq \ker(D_1)$.

Also, $\mathrm{ad}_c\, x_1$ is a (σ_1, id)-derivation of $\mathfrak{B}$. This derivation restricts to an endomorphism of the algebra $\mathcal{K}_1$, because if we apply $\mathrm{ad}_c\, x_1$ to the generators of $\mathcal{K}_1$, we obtain another generators (or 0).

We shall prove by induction on the length of u that $[u]_c \in \mathcal{K}_1$ for each $u \in L \setminus \{x_1\}$. If $u = x_j$, $j > 1$, then $[u]_c = x_j \in \mathcal{K}_1$. Now let $u \in L \setminus \{x_1\}$ be of length greater than 1, and (v, w) its Lyndon decomposition. Then:

- If $v \neq x_1$, then $[v]_c, [w]_c \in \mathcal{K}_1$ by induction hypothesis, so
$$[u]_c = [v]_c[w]_c - \chi(\deg v, \deg w)[w]_c[v]_c \in \mathcal{K}_1,$$
because $\mathcal{K}_1$ is a subalgebra.
- If $v = x_1$, then $[u]_c = \mathrm{ad}_c\, x_1([w]_c) \subset \mathrm{ad}_c\, x_1(\mathcal{K}_1) \subseteq \mathcal{K}_1$, because by induction hypothesis $[w]_c \in \mathcal{K}_1$.

Then we prove that $[L]_c \setminus \{x_1\} \subseteq \mathcal{K}_1$, and B_2 is generated by $[L]_c \setminus \{x_1\}$; that is, $\Bbbk B_2 \subseteq \mathcal{K}_1$, and $D_1(B_2) = 0$.

If $u \in \ker(D_1)$, we can write $[u]_c = \sum_{w \in B'_I} \alpha_w [w]_c$. If w does not end with x_1, then $w \in B_2$, and $D_1([w]_c) = 0$. But if $w = u_w x_1^{t_w}$, $[u_w]_c \in B_2$, $0 < t_w < h_I(x_1)$, we have
$$D_1\left([w]_c\right) = (t_w)_{q_{11}^{-1}} [u_w]_c x_1^{t_w - 1},$$
where $(t_w)_{q_{11}^{-1}} \neq 0$ if n_i does not divide t_w. Then
$$0 = D_1([u]_c) = \sum_{w \in B'_I / t_w > 0} \alpha_w (t_w)_{q_{11}^{-1}} [u_w]_c x_1^{t_w - 1},$$

But $[u_w]_c x_1^{t_w - 1} \in B_2$, and B_2 is a basis, so $\alpha_w = 0$ for each w such that n_i does not divide t_w. Then $\ker(D_1) = \mathcal{K}_1 \Bbbk[x_i^{n_i}]$, so $\ker(D_1) \simeq \mathcal{K}_1 \otimes \Bbbk[x_i^{n_i}]$ as $\Bbbk$-vector spaces. This fact and the first part conclude the proof. □

2C. *Transformations of certain braided graded Hopf algebras.* Let I be $\mathbb{Z}^\theta$-homogeneous ideal in $\mathfrak{S}$ and $\mathfrak{B} = T(V)/I$ as in the previous subsections. *We fix $i \in \{1, \dots, \theta\}$.*

Remark 2.5. $\operatorname{ord} q_{ii} = \min\{k \in \mathbb{N} : F_i^k = 0\}$, if $q_{ii} \neq 1$.

Proof. If $k \in \mathbb{N}$, then $F_i(x_i^k) = (k)_{q_{ii}} x_i^{k-1}$, and for all $k \in \mathbb{N}$,
$$F_i^k(x_i^k) = (k)_{q_{ii}^{-1}}!.$$
That is, if $F_i^k = 0$, then $(k)_{q_{ii}^{-1}}! = 0$. Hence $\operatorname{ord} q_{ii} \leq \min\{k \in \mathbb{N} : F_i^k = 0\}$. Reciprocally, if q_{ii} is a root of unity of order k, then $F_i^k(x_i^t) = 0$ for all $t \geq k$ by the previous claim, and $F_i^k(x_i^t) = 0$ for all $t < k$ by degree arguments. Since $F_i(x_j) = 0$ for $j \neq i$, $F_i^k = 0$. □

We now extend some considerations in [Heckenberger 2006b, p. 180]. We consider the Hopf algebra defined by

$$H_i := \begin{cases} \Bbbk\langle y, e_i, e_i^{-1} | e_i y - q_{ii}^{-1} y e_i, y^{N_i}\rangle & \text{where } N_i \text{ is the order of nilpotency of } x_i \text{ in } \mathfrak{B}, \text{ if } x_i \text{ is nilpotent,} \\ \Bbbk\langle y, e_i, e_i^{-1} | e_i y - q_{ii}^{-1} y e_i\rangle & \text{if } x_i \text{ is not nilpotent,} \end{cases}$$

together with $\Delta(e_i) = e_i \otimes e_i$, $\Delta(y) = e_i \otimes y + y \otimes 1$.

Notice that Δ is well-defined by Remark 2.3. We also consider the action $\triangleright$ of H_i on $\mathfrak{B}$ given by

$$e_i \triangleright b = \chi(\mathbf{u}, \mathbf{e}_i) b, \qquad y \triangleright b = F_i(b),$$

if b is homogeneous of degree $\mathbf{u} \in \mathbb{N}^\theta$, extending the previous one defined in (2-1). The action is well-defined by Remark 2.3 and because

$$(e_i y) \triangleright b = e_i \triangleright (F_i(b)) = q_{ii}^{-1} F_i(e_i \triangleright b) = (q_{ii}^{-1} y e_i) \triangleright b \quad \text{for } b \in \mathfrak{B}.$$

It is easy to see that $\mathfrak{B}$ is an H_i-module algebra; hence we can form

$$\mathcal{A}_i := \mathfrak{B} \# H_i.$$

Also, if we denote explicitly by $\cdot$ the multiplication in $\mathcal{A}_i$, we have

$$(1\#y) \cdot (b\#1) = (e_i \triangleright b\#1) \cdot (1\#y) + F_i(b)\#1 \quad \text{for all } b \in \mathfrak{B}. \tag{2-10}$$

As in [Heckenberger 2006b], $\mathcal{A}_i$ is a left Yetter–Drinfeld module over $\Bbbk\Gamma$, where the action and the coaction are given by

$$g_k \cdot x_j\#1 = q_{kj} x_j\#1, \quad g_k \cdot 1\#y = q_{ki}^{-1} 1\#y, \quad g_k \cdot 1\#e_i = 1\#e_i,$$
$$\delta(x_j\#1) = g_j \otimes x_j\#1, \quad \delta(1\#y) = g_i^{-1} \otimes 1\#y, \quad \delta(1\#e_i) = 1 \otimes 1\#e_i,$$

for each pair $k, j \in \{1, \dots, \theta\}$. Also, $\mathcal{A}_i$ is a $\Bbbk\Gamma$-module algebra.

We now prove a generalization of [Heckenberger 2006b, Proposition 1] in the more general context of our braided Hopf algebras $\mathfrak{B}$. Although the general strategy of the proof is similar as in *loc. cit.*, many points need slightly different argumentations here.

Theorem 2.6. *Keep the notation above. Assume that $M_{i,j}(\mathfrak{B})$ is finite and*

$$\left|M_{i,j}(\mathfrak{B})\right| = m_{ij} + 1, \quad j \in \{1, \dots, \theta\}, j \neq i. \tag{2-11}$$

(1) *Let V_i be the vector subspace of $\mathcal{A}_i$ generated by*

$$\left\{(\mathrm{ad}_c\, x_i)^{m_{ij}}(x_j)\#1 : j \neq i\right\} \cup \{1\#y\}.$$

The subalgebra $s_i(\mathfrak{B})$ of $\mathcal{A}_i$ generated by V_i is a graded algebra such that $s_i(\mathfrak{B})^1 \cong V_i$. There exist skew derivations $Y_i : s_i(\mathfrak{B}) \to s_i(\mathfrak{B})$ such that, for all $b_1, b_2 \in s_i(\mathfrak{B})$, and $l, j \in \{1, \dots, \theta\}$, $j \neq i$,

$$Y_j(b_1b_2) = b_1Y_j(b_2) + Y_j(b_2)\big(g_i^{-m_{ij}} g_j^{-1} \cdot b_2\big), \tag{2-12}$$

$$Y_i(b_1b_2) = b_1Y_i(b_2) + Y_i(b_1)\big(g_i^{-1} \cdot b_1\big), \tag{2-13}$$

$$Y_l((\operatorname{ad}_c x_i)^{m_{ij}}(x_j)\#1) = \delta_{lj}, \qquad Y_l(1\#y) = \delta_{li}. \tag{2-14}$$

(2) *Set $N_i := \{n \in \mathbb{N} : n\mathbf{e}_i \in \Delta(\mathfrak{B})\}$ (by the previous remarks, $N_i = \{1\}$ or $N_i = \{1, h_i\}$). The Hilbert series of $s_i(\mathfrak{B})$ satisfies*

$$\mathcal{H}_{s_i(\mathfrak{B})} = \Bigg(\prod_{\alpha \in \Delta^+(\mathfrak{B})\setminus N_i\mathbf{e}_i} \mathfrak{q}_{h_\alpha}(X^{s_i(\alpha)})\Bigg)\Bigg(\prod_{s\in N_i} \mathfrak{q}_{h_{s\mathbf{e}_i}}(x_i^s)\Bigg). \tag{2-15}$$

Therefore, if $s_i(\mathfrak{B})$ is a graded braided Hopf algebra,

$$\Delta^+(s_i(\mathfrak{B})) = \big\{s_i\left(\Delta^+(\mathfrak{B})\right) \setminus -N_i\mathbf{e}_i\big\} \cup N_i\mathbf{e}_i.$$

(3) *If $\mathfrak{B} = \mathfrak{B}(V)$, the algebra $s_i(\mathfrak{B})$ is isomorphic to the Nichols algebra $\mathfrak{B}(V_i)$.*

Proof. (i) Note that V_i is a Yetter–Drinfeld submodule over $\Bbbk\Gamma$ of $\mathcal{A}_i$. Now, $\mathcal{A}_i \cong \mathfrak{B} \otimes H_i$ as graded vector spaces. Let $\mathcal{K}_i$ be the subalgebra generated by $\bigcup_{j\neq i} M_{i,j}(\mathfrak{B})$, as in Lemma 2.4. Then $s_i(\mathfrak{B}) \subseteq \mathcal{K}_i \otimes \Bbbk[y]$, since F_i is a skew-derivation and $F_i\big(z_k^{(ij)}\big) = (k)_{q_{ii}}(1 - q_{ii}^{k-1}q_{ij}q_{ji})z_{k-1}^{(ij)}$, by (2-5). From (2-10),

$$(1\#y) \cdot \big(z_{m_{ij}}^{(ij)}\#1\big) = \big(z_{m_{ij}}^{(ij)}\#1\big) \cdot (1\#y) + F_i\big(z_{m_{ij}}^{(ij)}\big)\#1.$$

Also, since $m_{ij}+1 = \big|M_{i,j}(\mathfrak{B})\big|$, we have $(m_{ij})_{q_{ii}}(1-q_{ii}^{m_{ij}-1}q_{ij}q_{ji}) \neq 0$, so $z_{m_{ij}-1}^{(ij)}\#1$ lies in $s_i(\mathfrak{B})$, and by induction each $z_k^{(ij)}\#1$, for $k = 0, \dots, m_{ij} - 1$, is an element of $s_i(\mathfrak{B})$. Then $\mathcal{K}_i \otimes \Bbbk[y] \subseteq s_i(\mathfrak{B})$, and therefore

$$s_i(\mathfrak{B}) = \mathcal{K}_i \otimes \Bbbk[y]. \tag{2-16}$$

Thus, $s_i(\mathfrak{B})$ is a graded algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with $s_i(\mathfrak{B})^1 = V_i$. We have to find the skew derivations $Y_l \in \operatorname{End}(s_i(\mathfrak{B}))$, $l = 1, \dots, \theta$. Set $Y_i := g_i^{-1} \circ \operatorname{ad}(x_i\#1)|_{s_i(\mathfrak{B})}$. Then, for each $b \in \mathcal{K}_i$ and each $j \neq i$,

$$\begin{aligned}
&\operatorname{ad}(x_i\#1)(b\#1) = (\operatorname{ad}_c x_i)(b)\#1,\\
&\operatorname{ad}(x_i\#1)\left((\operatorname{ad}_c x_i)^{m_{ij}}(x_j)\#1\right) = (\operatorname{ad}_c x_i)^{m_{ij}+1}(x_j)\#1 = 0.
\end{aligned}$$

Also,

$$\begin{aligned}
Y_i(1\#y) &= g_i^{-1} \cdot \big((x_i\#1) \cdot (1\#y) - (g_i \cdot (1\#y)) \cdot (x_i\#1)\big)\\
&= g_i^{-1} \cdot \big(x_i\#y + 1 - q_{ii}(q_{ii}^{-1}x_i\#y)\big) = 1.
\end{aligned}$$

Thus $Y_i \in \operatorname{End}(s_i(\mathfrak{B}))$ satisfies (2-14).

Therefore, $\operatorname{ad}(x_i\#1)(b_1b_2) = \operatorname{ad}(x_i\#1)(b_1)b_2 + (g_i \cdot b_1)\operatorname{ad}(x_i\#1)(b_2)$, for each pair $b_1, b_2 \in s_i(\mathfrak{B})$, so we conclude that $\operatorname{ad}(x_i\#1)(s_i(\mathfrak{B})) \subseteq s_i(\mathfrak{B})$, and $Y_i \in \operatorname{End}(s_i(\mathfrak{B}))$ satisfies (2-13).

Before proving that Y_i satisfies (2-12), we need to establish some preliminary facts. Let us fix $j \neq i$, and let $z_k^{(ij)} = (\operatorname{ad}_c x_i)^k(x_j)$ as before. We define inductively

$$\hat{z}_0^{(ij)} := D_j, \quad \hat{z}_{k+1}^{(ij)} := D_i \hat{z}_k^{(ij)} - q_{ii}^k q_{ij} \hat{z}_{k+1}^{(ij)} D_i \in \operatorname{End}(\mathfrak{B}).$$

We calculate

$$\begin{aligned}\lambda_{ij} := \hat{z}_{m_{ij}}^{(ij)}\big(z_{m_{ij}}^{(ij)}\big) &= \sum_{s=0}^{m_{ij}} a_s D_i^{m_{ij}-s} D_j D_i^s \big(z_{m_{ij}}^{(ij)}\big) \\ &= (D_i)^{m_{ij}}(D_j)\big(z_{m_{ij}}^{(ij)}\big) = \alpha_{m_{ij}} (m_{ij})_{q_{ii}}! \in \mathsf{k}^\times,\end{aligned}$$

where $a_s = (-1)^k \binom{m}{k}_{q_{ii}} q_{ii}^{k(k-1)/2} q_{ij}^k$.

Note that $(D_i)^{m_{ij}+1} D_j(b) = 0$ for all $b \in M_{i,k}$, $k \neq i, j$, and that

$$(D_i)^{m_{ij}+1} D_j(z_r^{(ij)}) = (D_i)^{m_{ij}+1}\left(q_{ji}^{-r}\alpha_r x_i^r\right) = 0 \quad \text{for all } r \leq m_{ij},$$

so $(D_i)^{m_{ij}+1} D_j(\mathcal{K}_i) = 0$. This implies that $\hat{z}_{m_{ij}}^{(ij)}(b) \in \mathcal{K}_i$, for each $b \in \mathcal{K}_i$. Now define $Y_j \in \operatorname{End}(s_i(\mathfrak{B}))$ by

$$Y_j(b\#y^m) := q_{ii}^{m m_{ij}} q_{ji}^m \lambda_{ij}^{-1} \hat{z}_{m_{ij}}^{(ij)}(b)\#y^m \quad \text{for } b \in \mathcal{K}_i,\ m \in \mathbb{N}.$$

We have $Y_j(1\#y) = 0$, and moreover $Y_j((\operatorname{ad}_c x_i)^{m_{il}}(x_l)\#1) = 0$ if $l \neq i, j$. By the choice of λ_{ij}, $Y_j((\operatorname{ad}_c x_i)^{m_{ij}}(x_j)\#1) = 1$.

Now, using that $D_k(g_l \cdot b) = q_{kl} g_l \cdot (D_k(b))$ for each $b \in \mathfrak{B}$ and $k, l \in \{1, \dots, \theta\}$, we prove inductively that for $b_1, b_2 \in \mathcal{K}_i$,

$$\hat{z}_k^{(ij)}(b_1b_2) = b_1 \hat{z}_k^{(ij)}(b_2) + \hat{z}_k^{(ij)}(b_1)(g_i^k g_j \cdot b_2).$$

Hence,

$$\begin{aligned}Y_j(b_1\#1 \cdot b_2\#1) = Y_j(b_1b_2\#1) &= \lambda_{ij}^{-1} \hat{z}_{m_{ij}}(b_1b_2)\#1 \\ &= b_2\#1 \cdot Y_j(b_2\#1) + Y_j(b_1\#1) \cdot \big(g_i^{m_{ij}} g_j \cdot (b_2\#1)\big).\end{aligned}$$

By induction on the degree we prove that F_i commutes with D_i, D_j, so

$$\hat{z}_{m_{ij}}^{(ij)}(F_i(b)) = F_i\big(\hat{z}_{m_{ij}}^{(ij)}(b)\big) \quad \text{for all } b \in \mathfrak{B}.$$

Consider $b \in \mathcal{K}_i \subseteq \ker(D_i)$,

$$\begin{aligned}Y_j\ (b\#1 \cdot 1\#y) = Y_j\ (b\#y) &= q_{ii}^{m_{ij}} q_{ji} \hat{z}_{m_{ij}}^{(ij)}(b)\#y \\ &= b\#1 \cdot Y_j(1\#y) + Y_j(b\#1) \cdot \big(g_i^{m_{ij}} g_j \cdot (1\#y)\big),\end{aligned}$$

where we use that $Y_j(1\#y)=0$. Since,

$$b_1\#1\cdot b_2\#y^t=b_1\#1\cdot b_2\#1\cdot(1\#y)^t\,,$$

(2-12) is valid for products of this form. To prove it in the general case, note that

$$(b_1\#y^t)\cdot(b_2\#y^s)=(b_1\#1)\cdot(1\#y)^t\cdot(b_2\#y^s).$$

At this point, we have to prove (2-12) for $b\in\mathcal{K}_i\ker(D_i)$, $s\in\mathbb{N}$:

$$\begin{aligned}
&Y_j(1\#y\cdot b\#y^s)\\
&\quad=Y_j\big(F_i(b)\#y^s+(e_i\triangleright b\#y)\cdot 1\#y\big)\\
&\quad=q_{ii}^{m_{ij}s}q_{ji}^s\lambda_{ij}^{-1}\hat{z}_{m_{ij}}^{(ij)}(F_i(b))\#y^s+q_{ii}^{m_{ij}(s+1)}q_{ji}^{s+1}\lambda_{ij}^{-1}\cdot\hat{z}_{m_{ij}}^{(ij)}(e_i\triangleright b)\#y^{s+1}\\
&\quad=F_i\big(q_{ii}^{m_{ij}(s+1)}q_{ji}^{s+1}\lambda_{ij}^{-1}\hat{z}_{m_{ij}}^{(ij)}(b)\big)\#y^s+q_{ii}^{m_{ij}}q_{ji}\big(e_i\triangleright\big(q_{ii}^{m_{ij}s}q_{ji}^s\lambda_{ij}^{-1}\hat{z}_{m_{ij}}^{(ij)}(b)\big)\#y^s\big)\\
&\quad=(1\#y)\cdot Y_j(b\#y^s)\\
&\quad=1\#y\cdot Y_j\left(b\#y^s\right)+Y_j(1\#y)\cdot\big(g_i^{m_{ij}}g_j\cdot b\#y^s\big),
\end{aligned}$$

where we use that $\hat{z}_{m_{ij}}^{(ij)}(e_i\triangleright b)=q_{ii}^{m_{ij}}q_{ji}e_i\triangleright(\hat{z}_{m_{ij}}^{(ij)}(b))$.

(ii) The algebra H_i is $\mathbb{Z}^\theta$-graded, with

$$\deg y=-\mathbf{e}_i,\quad \deg e_i^{\pm1}=0.$$

Since $\mathfrak{B}$ and H_i are graded and (2-10) holds, the algebra $\mathcal{A}_i$ is $\mathbb{Z}^\theta$-graded.

Consider the abstract basis $\{u_j\}_{j\in\{1,\dots,\theta\}}$ of V_i. With the grading $\deg u_j=\mathbf{e}_j$, the algebra $\mathfrak{B}(V_i)$ is $\mathbb{Z}^\theta$-graded. Consider also the algebra homomorphism $\Omega: T(V_i)\to s_i(\mathfrak{B})$ given by

$$\Omega(u_j):=\begin{cases}(\operatorname{ad}_c x_i)^{m_{ij}}(x_j) & \text{if } j\neq i,\\ y & \text{if } j=i.\end{cases}$$

By part (i) of the theorem, Ω is an epimorphism, so it induces an isomorphism between $s_i(\mathfrak{B})':=T(V_i)/\ker\Omega$ and $s_i(\mathfrak{B})$, which we also denote by Ω. We have

$$\begin{aligned}
&\deg\Omega(u_j)=\deg\big((\operatorname{ad}_c x_i)^{m_{ij}}(x_j)\big)=\mathbf{e}_j+m_{ij}\mathbf{e}_i=s_i(\deg\mathbf{u}_j)\text{ if } j\neq i,\\
&\deg\Omega(u_i)=\deg(y)=-\mathbf{e}_i=s_i(\deg\mathbf{u}_i).
\end{aligned}$$

Since Ω is an algebra homomorphism, we have $\deg(\Omega(\mathbf{u}))=s_i(\deg(\mathbf{u}))$ for all $\mathbf{u}\in s_i(\mathfrak{B})'$. Since $s_i^2=\mathrm{id}$, $s_i(\deg(\Omega(\mathbf{u})))=\deg(\mathbf{u})$ for all $\mathbf{u}\in s_i(\mathfrak{B})'$, and $\mathfrak{H}_{s_i(\mathfrak{B})'}=s_i(\mathfrak{H}_{s_i(\mathfrak{B})})$.

From this point on, the proof goes exactly as in [Andruskiewitsch and Angiono 2008, Theorem 3.2].

(iii) This is Proposition 1 in [Heckenberger 2006b]. □

By Theorem 2.6, the initial braided vector space with matrix $(q_{kj})_{1\le k,j\le\theta}$ is transformed into another braided vector space of diagonal type V_i, with matrix $(\tilde{q}_{kj})_{1\le k,j\le\theta}$, where $\tilde{q}_{jk} = q_{ii}^{m_{ij}m_{ik}} q_{ik}^{m_{ij}} q_{ji}^{m_{ik}} q_{jk}$ for $j,k \in \{1,\dots,\theta\}$.

If $j \neq i$, then $\widetilde{m}_{ij} = \min\left\{m \in \mathbb{N} : (m+1)_{\tilde{q}_{ii}} \left(\tilde{q}_{ii}^m \tilde{q}_{ij}\tilde{q}_{ji} = 0\right)\right\} = m_{ij}$.

For later use in Section 5, we recall a result from [Andruskiewitsch et al. 2008], adapted to diagonal braided vector spaces.

Lemma 2.7 [Andruskiewitsch et al. 2008, Lemma 2.8(ii)]. *Let V be a diagonal braided vector space and I a $\mathbb{Z}^\theta$-homogeneous ideal of $T(V)$. Set $\mathfrak{B} := T(V)/I$ and assume that for all $i \in \{1,\dots,\theta\}$ there exist* (id, σ_i)*-derivations $D_i : \mathfrak{B} \to \mathfrak{B}$ with $D_i(x_j) = \delta_{i,j}$ for all j. Then $I \subseteq I(V)$.* □

That is, the canonical surjective algebra morphisms from $T(V)$ onto $\mathfrak{B}$ and $\mathfrak{B}(V)$ induce a surjective algebra morphism $\mathfrak{B} \to \mathfrak{B}(V)$.

3. Standard braidings

Heckenberger [2006a] has classified diagonal braidings whose set of PBW generators is finite. Standard braidings form an special subclass, which includes properly braidings of Cartan type.

We first recall the definition of a standard braiding from [Andruskiewitsch and Angiono 2008], and the notion of a Weyl groupoid, introduced in [Heckenberger 2006b]. Then we present the classification of standard braidings, and compare them with [Heckenberger 2006a].

Like Heckenberger, we use the *generalized Dynkin diagram associated* to a braided vector space of diagonal type, with matrix $(q_{ij})_{1\le i,j\le\theta}$: this is a graph with θ vertices, each labeled with the corresponding q_{ii}, and an edge between two vertices i, j labeled with $q_{ij}q_{ji}$ if this scalar is different from 1. So two braided vector spaces of diagonal type have the same generalized Dynkin diagram if and only if they are twist equivalent. We shall assume that the generalized Dynkin diagram is connected, by [Andruskiewitsch and Schneider 2000, Lemma 4.2].

Summarizing, the main result of this section says:

Theorem 3.1. *Any standard braiding is twist equivalent with one or more of*

- *a braiding of Cartan type,*
- *a braiding of type A_θ listed in Proposition 3.9,*
- *a braiding of type B_θ listed in Proposition 3.10, or*
- *a braiding of type G_2 listed in Proposition 3.11.*

The generalized Dynkin diagrams appearing in Propositions 3.9 and 3.10 correspond to rows 1, 2, 3, 4, 5, 6 in [Heckenberger 2006a, Table C]. The generalized Dynkin diagrams in Proposition 3.11 are (T8) in [Heckenberger 2008, Section 3]. However, our classification does not rely on Heckenberger's papers.

3A. ***The Weyl groupoid and standard braidings.*** Let $E = (\mathbf{e}_1, \dots, \mathbf{e}_\theta)$ be the canonical basis of $\mathbb{Z}^\theta$. Consider an arbitrary matrix $(q_{ij})_{1 \le i,j \le \theta} \in (\Bbbk^\times)^{\theta \times \theta}$, and fix once and for all the bilinear form $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times$ determined by

$$\chi(\mathbf{e}_i, \mathbf{e}_j) = q_{ij}, \qquad 1 \le i, j \le \theta. \tag{3-1}$$

If $F = (\mathbf{f}_1, \dots, \mathbf{f}_\theta)$ is another ordered basis of $\mathbb{Z}^\theta$, then we set $\tilde{q}_{ij} = \chi(\mathbf{f}_i, \mathbf{f}_j)$, $1 \le i, j \le \theta$. We call $(\tilde{q}_{ij})$ the *braiding matrix with respect to the basis* F. Fix $i \in \{1, \dots, \theta\}$. If $1 \le i, j \le \theta$, we consider the set

$$\widetilde{M}_{ij} := \{m \in \mathbb{N}_0 : (m+1)_{\tilde{q}_{ii}} (\tilde{q}_{ii}^m \tilde{q}_{ij} \tilde{q}_{ji} - 1) = 0\}.$$

If this set is nonempty, its minimal element is denoted $\tilde{m}_{ij}$ (which of course depends on the basis F). Define also $\tilde{m}_{ii} = 2$. Let $s_{i,F} \in \mathrm{GL}(\mathbb{Z}^\theta)$ be the pseudo-reflection given by $s_{i,F}(\mathbf{f}_j) := \mathbf{f}_j + \tilde{m}_{ij} \mathbf{f}_i$, for $j \in \{1, \dots, \theta\}$.

Let G be a group acting on a set X. We define the *transformation groupoid* as $G \times X$ with the operation given by $(g, x)(h, y) = (gh, y)$ if $x = h(y)$, but undefined otherwise.

Definition 3.2. Consider the set $\mathfrak{X}$ of all ordered bases of $\mathbb{Z}^\theta$, and the canonical action of $\mathrm{GL}(\mathbb{Z}^\theta)$ over $\mathfrak{X}$. The *Weyl groupoid* $W(\chi)$ of the bilinear form χ is the smallest subgroupoid of the transformation groupoid $\mathrm{GL}(\mathbb{Z}^\theta) \times \mathfrak{X}$ that satisfies following properties:

- $(\mathrm{id}, E) \in W(\chi)$,
- if $(\mathrm{id}, F) \in W(\chi)$ and $s_{i,F}$ is defined, then $(s_{i,F}, F) \in W(\chi)$.

Let $\mathfrak{P}(\chi) = \{F : (\mathrm{id}, F) \in W(\chi)\}$ be the set of points of the groupoid $W(\chi)$. The set

$$\Delta(\chi) = \bigcup_{F \in \mathfrak{P}(\chi)} F. \tag{3-2}$$

is called the *generalized root system*[1] associated to χ.

We record for later use the following evident facts.

Remark 3.3. Take $i \in \{1, \dots, \theta\}$ such that $s_{i,E}$ is defined. Set $F = s_{i,E}(E)$ and let $(\tilde{q}_{ij})$ be the braiding matrix with respect to the basis F. Assume that

[1]Following the traditional notation in the theory of Lie algebras, we should speak about systems of real roots, since in the case of braidings of symmetrizable Cartan type one would get just the real roots. But we prefer to follow the denomination in [Andruskiewitsch and Angiono 2008]

- $q_{ii} = -1$ (so $m_{ik} = 0$ if $q_{ik}q_{ki} = 1$ or $m_{ik} = 1$, for each $k \neq i$);
- there exists $j \neq i$ such that $q_{jj}q_{ji}q_{ij} = 1$ (that is, $m_{ij} = m_{ji} = 1$).

Then $\tilde{q}_{jj} = -1$.

Proof. Simply, $\tilde{q}_{jj} = q_{ii}q_{ij}q_{ji}q_{jj} = q_{ii} = -1$. □

Remark 3.4. If the m_{ij} satisfy $q_{ii}^{m_{ij}} q_{ij}q_{ji} = 1$ for all $j \neq i$, the braiding of V_i is twist equivalent with the corresponding to V.

Define $\alpha : W(\chi) \to \mathrm{GL}(\theta, \mathbb{Z})$ by $\alpha(s, F) = s$ if $(s, F) \in W(\chi)$, and denote by $W_0(\chi)$ the subgroup generated by the image of α.

Definition 3.5. [Andruskiewitsch and Angiono 2008] We say that χ is *standard* if for any $F \in \mathfrak{P}(\chi)$, the integers m_{rj} are defined, for all $1 \leq r, j \leq \theta$, and the integers m_{rj} for the bases $s_{i,F}(F)$ coincide with those for F for all i, r, j. Clearly it is enough to assume this for the canonical basis E.

We now assume that χ is standard. We set $C := (a_{ij}) \in \mathbb{Z}^{\theta\times\theta}$, where $a_{ij} = -m_{ij}$; this is a generalized Cartan matrix.

Proposition 3.6 [Andruskiewitsch and Angiono 2008]. $W_0(\chi) = \langle s_{i,E} : 1 \leq i \leq \theta\rangle$. *Furthermore $W_0(\chi)$ acts freely and transitively on $\mathfrak{P}(\chi)$.* □

Hence, $W_0(\chi)$ is a Coxeter group, and $W_0(\chi)$ and $\mathfrak{P}(\chi)$ have the same cardinality.

Lemma 3.7 [Andruskiewitsch and Angiono 2008]. *The following are equivalent*:

(1) *The groupoid $W(\chi)$ is finite.*

(2) *The set $\mathfrak{P}(\chi)$ is finite.*

(3) *The generalized root system $\Delta(\chi)$ is finite.*

(4) *The group $W_0(\chi)$ is finite.*

(5) *The Cartan matrix C is symmetrizable and of finite type.* □

We shall prove in Theorem 4.1, that if $\Delta(\chi)$ is finite, the matrix C is symmetrizable, hence of finite type. Thus $\mathfrak{B}(V)$ is of finite dimension if and only if the Cartan matrix C is of finite type.

3B. *Classification of standard braidings.* We now classify standard braidings such that the Cartan matrix is of finite type. We begin with types C_θ, D_θ, E_l ($l = 6, 7, 8$) and F_4: these standard braidings are necessarily of Cartan type.

Proposition 3.8. *Let V be a braided vector space of standard type, set $\theta = \dim V$, and let $C = (a_{ij})_{i,j\in\{1,\dots,\theta\}}$ be the corresponding Cartan matrix, of type C_θ, D_θ, E_l ($l = 6, 7, 8$) or F_4. Then V is of Cartan type (associated to the corresponding matrix of finite type).*

Proof. Let V be standard of type C_θ, $\theta \geq 3$.

$$\circ^1 \;\text{———}\; \circ^2 \;\text{———}\; \circ^3 \;\ldots\; \circ^{\theta-2} \;\text{———}\; \circ^{\theta-1} \Longleftarrow \circ^{\theta} \tag{3-3}$$

Note that $q_{\theta-1,\theta-1} \neq -1$ by Remark 3.3 and the assumption $m_{\theta-1,\theta} = 2$. Since $m_{\theta-1,\theta-2} = 1$, $q_{\theta-1,\theta-1}q_{\theta-1,\theta-2}q_{\theta-2,\theta-1} = 1$. Using Remark 3.3 when $i = \theta - 2$, $j = \theta - 1$, since $\tilde{q}_{\theta-1,\theta-1} \neq -1$ when we transform by $s_{\theta-2}$ (since the new braided vector space is also standard), we have $q_{\theta-2,\theta-2} \neq -1$, so

$$q_{\theta-2,\theta-2}q_{\theta-2,\theta-1}q_{\theta-1,\theta-2} = q_{\theta-2,\theta-2}q_{\theta-2,\theta-3}q_{\theta-3,\theta-2} = 1,$$

and $q_{\theta-1,\theta-1} = q_{\theta-2,\theta-2}$. Inductively,

$$q_{kk}q_{k,k-1}q_{k-1,k} = q_{kk}q_{k,k+1}q_{k+1,k} = q_{11}q_{12}q_{21} = 1, \quad k = 2, \ldots, \theta - 1$$

and $q_{11} = q_{22} = \ldots = q_{\theta-1,\theta-1}$. So we look at $q_{\theta\theta}$: since $m_{\theta,\theta-1} = 1$, we have $q_{\theta\theta} = -1$ or $q_{\theta\theta}q_{\theta,\theta-1}q_{\theta-1,\theta} = 1$. If $q_{\theta\theta} = -1$, transforming by s_θ, we have

$$\tilde{q}_{\theta-1,\theta-1} = -q^{-1}, \quad \tilde{q}_{\theta-1,\theta}\tilde{q}_{\theta,\theta-1} = q^2,$$

and $q^2 = -1$ since $m_{\theta-1,\theta-2} = 1$. Then

$$q_{\theta\theta}q_{\theta,\theta-1}q_{\theta-1,\theta} = 1, \quad q_{\theta\theta} = q^2,$$

and the braiding is of Cartan type in both cases.

Let V be standard of type D_θ, $\theta \geq 4$.

We prove the statement by induction on θ. Let V be standard of type D_4, and suppose that $q_{22} = -1$. Let $(\tilde{q}_{ij})$ the braiding matrix with respect to $F = s_{2,E}(E)$. We calculate for each pair $j \neq k \in \{1, 3, 4\}$:

$$\tilde{q}_{jk}\tilde{q}_{kj} = \left((-1)q_{2k}q_{j2}q_{jk}\right)\left((-1)q_{2j}q_{k2}q_{kj}\right) = (q_{2k}q_{k2})\left(q_{2j}q_{j2}\right),$$

where we use that $q_{jk}q_{kj} = 1$. Since also $\tilde{q}_{jk}\tilde{q}_{kj} = 1$, we have $q_{2k}q_{k2} = (q_{2j}q_{j2})^{-1}$ for $j \neq k$, so $q_{2k}q_{k2} = -1$, $k = 1, 3, 4$, since $q_{2k}q_{k2} \neq 1$. In this case, the braiding is of Cartan type, with $q = -1$. Suppose then $q_{22} \neq -1$. From the fact that $m_{2j} = 1$, we have

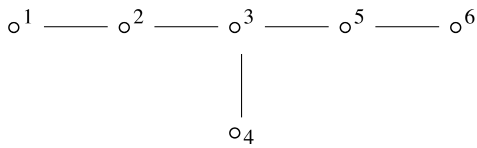
$$q_{22}q_{2j}q_{j2} = 1, \quad j = 1, 3, 4.$$

For each j, applying Remark 3.3, we see that $q_{jj} \neq -1$ (since $\tilde{q}_{22} \neq -1$), so $q_{jj}q_{2j}q_{j2} = 1$, for $j = 1, 3, 4$, and the braiding is of Cartan type.

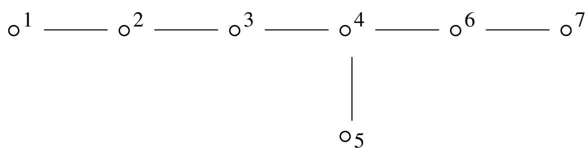
$$\begin{array}{ccccccccc} \circ^1 & \text{———} & \circ^2 & \text{———} & \circ^3 \ldots & & \circ^{\theta-2} & \text{———} & \circ^{\theta} \\ & & & & & & | & & \\ & & & & & & \circ_{\theta-1} & & \end{array} \tag{3-4}$$

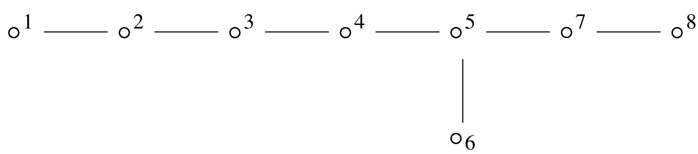
We now suppose the statement valid for θ. Let V be a standard braided vector space of type $D_{\theta+1}$. The subspace generated by $x_2, \dots, x_{\theta+1}$ is a standard braided vector space associated to the matrix $(q_{ij})_{i,j=2,\dots,\theta+1}$, of type D_θ, so it is of Cartan type. To finish, apply Remark 3.3 with $i=1$, $j=2$, to conclude that V is of Cartan type with $q=-1$, or, if $q_{22}\neq -1$, we have $q_{11}\neq -1$ and $q_{11}q_{12}q_{21}=1$, and in this case it is of Cartan type too (because also $q_{1k}q_{k1}=1$ when $k>2$).

Let V be standard of type E_6. Note that $1, 2, 3, 4, 5$ determine a braided vector subspace, which is standard of type D_5, hence of Cartan type. To prove that $q_{66}q_{65}q_{56}=1$, we use Remark 3.3 as above.

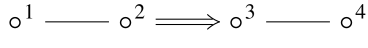

(3-5)

If V is standard of type E_7 or E_8, we proceed similarly by reduction to E_6 or E_7, respectively.


(3-6)


(3-7)

Let V be standard of type F_4. Vertices $2, 3, 4$ determine a braided subspace, which is standard of type C_3, so the q_{ij} satisfy the corresponding relations. Let $(\tilde{q}_{ij})$ the braiding matrix with respect to $F=s_{2,E}(E)$. Since $\tilde{q}_{13}\tilde{q}_{31}=1$ and $q_{22}q_{23}q_{32}=1$, we have $q_{22}q_{12}q_{21}=1$.


(3-8)

Now, if we suppose $q_{11}=-1$, applying Remark 3.3 we have $q_{22}=-1=q_{21}q_{12}$, and the corresponding vector space is of Cartan type F_4, associated to $q\in\mathbb{G}_4$. If $q_{11}\neq -1$, then $q_{11}q_{12}q_{21}=1$, and the space it again is of Cartan type. □

To finish the classification of standard braidings, we describe the standard braidings that are not of Cartan type. They are associated to Cartan matrices of type A_θ, B_θ or G_2.

We use a notation similar to the one in [Heckenberger 2006a] for a special kind of braiding of type A_θ (here we emphasize the positions where $q_{ii}=-1$, which we use to compute the dimension of the corresponding Nichols algebra); $\mathscr{C}(\theta, q; i_1, \dots, i_j)$ corresponds to the generalized Dynkin diagram

$$\circ^1 \text{———} \circ^2 \text{———} \circ^3 \;\dots\qquad \circ^{\theta-1} \text{———} \circ^{\theta} \tag{3-9}$$

where the following equations hold:

- $q = q_{\theta-1,\theta}q_{\theta,\theta-1}q_{\theta\theta}^2$,
- $(q_{\theta\theta}+1)(q_{\theta\theta}q_{\theta-1,\theta}q_{\theta,\theta-1}-1) = (q_{11}+1)(q_{11}q_{12}q_{21}-1) = 0$;
- $-q_{ii} = q_{i-1,i}q_{i,i-1}q_{i+1,i}q_{i,i+1} = 1$ if $i \in \{i_1, \dots, i_j\}$.
- $q_{ii}q_{i-1,i}q_{i,i-1} = q_{ii}q_{i+1,i}q_{i,i+1} = 1$, otherwise.

Then $q_{ii}=-1$ if and only if $q_{i-1,i}q_{i,i-1} = (q_{i+1,i}q_{i,i+1})^{-1}$.

Proposition 3.9. *Let V be a braided vector space of diagonal type. Then V is standard of type A_θ if and only if its generalized Dynkin diagram is of the form*

$$\mathscr{C}(\theta, q; i_1, \dots, i_j). \tag{3-10}$$

This braiding is of Cartan type if and only if $j=0$, or $j=n$ with $q=-1$.

Proof. Let V be a braided vector space of standard A_θ type. For each vertex i, with $1<i<\theta$, we have $q_{ii}=-1$ or $q_{ii}q_{i,i-1}q_{i-1,i} = q_{ii}q_{i,i+1}q_{i+1,i}=1$, and similar formulas hold for $i=1,\theta$. So suppose that $1<i<\theta$ and $q_{ii}=-1$. We transform by s_i and obtain

$$\tilde q_{i-1,i+1} = -q_{i,i+1}q_{i-1,i}q_{i-1,i+1}, \quad \tilde q_{i+1,i-1} = -q_{i,i-1}q_{i+1,i}q_{i+1,i-1},$$

and using that $m_{i-1,i+1} = \widetilde m_{i-1,i+1} = 0$, we have

$$q_{i-1,i+1}q_{i+1,i-1} = 1, \quad \tilde q_{i-1,i+1}\tilde q_{i+1,i-1} = 1,$$

so we deduce that $q_{i,i+1}q_{i+1,i} = (q_{i,i-1}q_{i-1,i})^{-1}$. Then the corresponding matrix (q_{ij}) is of the form (3-10).

Now consider V of the form (3-10). Assume $q_{ii}=q^{\pm1}$; if we transform by s_i, the braided vector space V_i is twist equivalent with V by Remark 3.4. Thus, $\widetilde m_{ij} = m_{ij}$.

Assume $q_{ii}=-1$. We transform by s_i and calculate

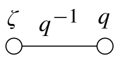
$$\tilde q_{jj} = (-1)^{m_{ij}^2}(q_{ij}q_{ji})^{m_{ij}}q_{jj} = \begin{cases} q_{jj} & \text{if } |j-i|>1,\\ (-1)q^{\mp1}q^{\pm1} = -1 & \text{if } j=i\pm1,\ q_{jj}=q^{\pm1},\\ (-1)q^{\pm1}(-1) = q^{\pm1} & \text{if } j=i\pm1,\ q_{jj}=-1.\end{cases}$$

Also, $\tilde{q}_{ij}\tilde{q}_{ji} = q_{ij}q_{ji}$ if $|j-i|>1$ and $\tilde{q}_{ij}\tilde{q}_{ji} = q_{ij}^{-1}q_{ji}^{-1}$ if $|j-i|=1$; moreover

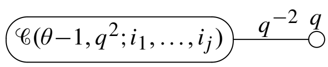
$$\tilde{q}_{kj}\tilde{q}_{jk} = (q_{ik}q_{ki})^{m_{ij}}(q_{ij}q_{ji})^{m_{ik}}q_{kj}q_{jk} = \begin{cases} q_{kj}q_{jk} & \text{if } |j-i| \text{ or } |k-i|>1, \\ 1 & \text{if } j=i-1, k=i+1. \end{cases}$$

Then V_i has a braiding of the above form too, and $(-m_{ij})$ corresponds to the finite Cartan matrix of type A_θ, so it is a standard braiding of type A_θ. Thus this is the complete family of standard braidings of type A_θ. □

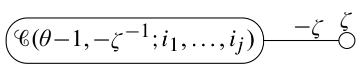
Proposition 3.10. *Let V a diagonal braided vector space. Then V is standard of type B_θ if and only if its generalized Dynkin diagram is of one of these forms*:

(a) 

with $\zeta \in \mathbb{G}_3$, $q \neq \zeta$ $(\theta = 2)$;

(b) 

with $q \neq 0, -1$, $0 \le j \le \theta - 1$;

(c) 

with $\zeta \in \mathbb{G}_3$, $0 \le j \le \theta - 1$.

This braiding is of Cartan type if and only if it is as in (b) and $j = 0$.

Proof. First we analyze the case $\theta = 2$. Let V a standard braided vector space of type B_2. There are several possibilities:

- $q_{11}^2 q_{12}q_{21} = q_{22}q_{21}q_{12} = 1$: this braiding is of Cartan type, with $q = q_{11}$. Note that $q \neq -1$. This braiding has the form (b) with $\theta = 2$, $j = 0$.
- $q_{11}^2 q_{12}q_{21} = 1$, $q_{22} = -1$. We transform by s_2, obtaining

$$\tilde{q}_{11} = -q_{11}^{-1}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{12}^{-1}q_{21}^{-1}.$$

  Thus $\tilde{q}_{11}^2\tilde{q}_{12}\tilde{q}_{21} = 1$. It has the form (b) with $j = 1$.
- $q_{11} \in \mathbb{G}_3$, $q_{22}q_{21}q_{12} = 1$. We transform by s_1, obtaining

$$\tilde{q}_{22} = q_{11}q_{12}q_{21}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{11}^2 q_{12}^{-1}q_{21}^{-1}.$$

  So $\tilde{q}_{22}\tilde{q}_{21}\tilde{q}_{12} = 1$, which is the case (a).
- $q_{11} \in \mathbb{G}_3$, $q_{22} = -1$: we transform by s_1, obtaining

$$\tilde{q}_{22} = -q_{12}^2 q_{21}^2 q_{11}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{11}^2 q_{12}^{-1}q_{21}^{-1}.$$

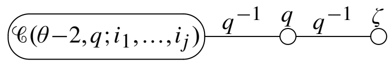
  If we transform by s_2,

$$\tilde{q}_{11} = -q_{12}q_{21}q_{11}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{12}^{-1}q_{21}^{-1}.$$

  So $q_{12}q_{21} = \pm q_{11}$, and we discard the case $q_{12}q_{21} = q_{11}$ because it has been considered before. The braiding has the form (c) with $j = 0$, and is standard.

Conversely, all braidings (a), (b) and (c) are standard of type B_2.

Now let V be of type B_θ, with $\theta \geq 3$. The first $\theta-1$ vertices determine a braiding of standard type $A_{\theta-1}$, and the last two determine a braiding of standard type B_2; so we have to glue the possible such braidings. The possible cases are the two presented in Proposition 3.10, plus



.

But if we transform by s_θ, we obtain

$$\tilde{q}_{\theta-1,\theta-1} = \zeta q^{-1}, \quad \tilde{q}_{\theta-1,\theta-2}\tilde{q}_{\theta-2,\theta-1} = q^{-1},$$

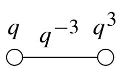
so $1 = \tilde{q}_{\theta-1,\theta-1}\tilde{q}_{\theta-1,\theta-2}\tilde{q}_{\theta-2,\theta-1}$ and we obtain $q = \pm\zeta^{-1}$, or $\tilde{q}_{\theta-1,\theta-1} = -1$. Then $q = -\zeta^{-1}$ or $q = -1$, so it is of some of the above forms.

To prove that (b) and (c) are standard braidings, we use the following fact: if $m_{ij} = 0$ (that is, $q_{ij}q_{ji} = 1$) and we transform by s_i, then

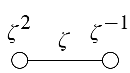
$$\tilde{q}_{jj} = q_{jj} \qquad \text{and} \qquad \tilde{q}_{jk}\tilde{q}_{jk} = q_{jk}q_{kj} \quad \text{for } k \neq i.$$

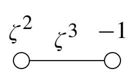
In this case, $m_{ij} = 0$ if $|i-j| > 1$; if, on the contrary, $j = i \pm 1$, we use the fact that the subdiagram determined by these two vertices is standard of type B_2 or type A_2. So this is the complete family of all twist equivalence classes of standard braidings of type B_θ. □

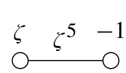
Proposition 3.11. *Let V a braided vector space of diagonal type. Then V is standard of type G_2 if and only if its generalized Dynkin diagram is one of the following*:

(a) 

with $\operatorname{ord} q \geq 4$;

(b) 

or 

or 

with $\zeta \in \mathbb{G}_8$.

This braiding is of Cartan type if and only if it is as in (a).

Proof. Let V be a standard braiding of type G_2. There are four possible cases:

- $q_{11}^3 q_{12}q_{21} = 1$, $q_{22}q_{21}q_{12} = 1$: this braiding is of Cartan type, as in (a), with $q = q_{11}$. If q is a root of unity, then $\operatorname{ord} q \geq 4$ because $m_{12} = 3$.
- $q_{11}^3 q_{12}q_{21} = 1$, $q_{22} = -1$: we transform by s_2, obtaining

$$\tilde{q}_{11} = -q_{11}^{-2}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{12}^{-1}q_{21}^{-1}.$$

If $1 = \tilde{q}_{11}^3\tilde{q}_{12}\tilde{q}_{21} = -q_{11}^{-3}$, then $q_{12}q_{21} = -1$, and the braiding is of Cartan type with $q_{11} \in \mathbb{G}_6$. If not, $1 = \tilde{q}_{11}^4 = q_{11}^{-8}$ and $\operatorname{ord}\tilde{q}_{11} = 4$, so $\operatorname{ord} q_{11} = 8$. Then we can express the braiding in the form of the third diagram in (b).

- $q_{11} \in \mathbb{G}_4,\ q_{22}q_{21}q_{12} = 1$: we transform by s_1, obtaining
$$\tilde{q}_{22} = q_{11}q_{12}^2q_{21}^2, \quad \tilde{q}_{12}\tilde{q}_{21} = -q_{12}^{-1}q_{21}^{-1}.$$
If $1 = \tilde{q}_{22}\tilde{q}_{21}\tilde{q}_{12} = -q_{11}q_{12}q_{21}$, we have $q_{11}^3q_{12}q_{21} = 1$ because $q_{11}^2 = -1$, and this is a braiding of Cartan type. So consider now the case $-1 = \tilde{q}_{22} = q_{11}q_{12}^2q_{21}^2$, from which $q_{22}^2 = q_{11}^{-1}$ and $q_{22} \in \mathbb{G}_8$. Then we obtain a braiding of the form of the first diagram in (b).
- $q_{11} \in \mathbb{G}_4,\ q_{22} = -1$: we transform by s_2, obtaining
$$\tilde{q}_{11} = -q_{12}q_{21}q_{11}, \quad \tilde{q}_{12}\tilde{q}_{21} = q_{12}^{-1}q_{21}^{-1}.$$
If $\tilde{q}_{11} \in \mathbb{G}_4$, then $(q_{12}q_{21})^4 = 1$. Moreover $q_{12}q_{21} \neq 1$ and $q_{12}q_{21} \neq q_{11}^{-1}$ because $m_{12} = 3$. So $q_{12}q_{21} = -1$ or $q_{12}q_{21} = q_{11} = q_{11}^{-3}$; but these cases have been considered already. There remains to analyze the case
$$1 = \tilde{q}_{11}^3\tilde{q}_{12}\tilde{q}_{21} = q_{11}q_{12}^2q_{21}^2,$$
which we can express in the form of the second diagram in (b), for some $\zeta \in \mathbb{G}_8$.

A simple calculation proves that these braidings are of standard type, so they are all the standard braidings of type G_2. □

4. Nichols algebras of standard braided vector spaces

In this section we study Nichols algebras associated to standard braidings. We assume that the Dynkin diagram is connected, as in Section 3. In Section 4A we prove that the set $\Delta^+(\mathfrak{B}(V))$ is in bijection with Δ_C^+, the set of positive roots associated with the finite Cartan matrix C.

We describe an explicit set of generators in Section 4B, following [Lalonde and Ram 1995]. We adapt their proof since they work on enveloping algebras of simple Lie algebras. In Section 4C, we calculate the dimension of Nichols algebra associated to a standard braided vector space, type by type.

4A. ***PBW bases of Nichols algebras.*** We start with a result analogous to [Heckenberger 2006b, Theorem 1], but for braidings of standard type.

Theorem 4.1. *Let V be a braided vector space of standard type with Cartan matrix C. Then the set $\Delta(\mathfrak{B}(V))$ is finite if and only if the Cartan matrix C is symmetrizable and of finite type.*

Proof. Since we assume V of standard type, $\Delta(\mathfrak{B}(V))$ coincides with the set of real roots corresponding to the matrix C by [Heckenberger 2006b, Proposition 1], where we identify corresponding simple roots. Hence, if C is not symmetrizable or not of finite type, the set of real roots is infinite by the classification of finite Coxeter groups, and hence $\Delta(\mathfrak{B}(V))$ is infinite.

Conversely, let C be symmetrizable and of finite type. Then the set of real roots is finite. Take $\alpha \in \Delta(\mathfrak{B}(V))$ and let $k \in \mathbb{N}$, $i_1, \dots, i_k \in \{1, \dots, \theta\}$ be a sequence of integers such that $s_{i_1} \cdots s_{i_k}$ is a longest element in $W_0(\chi)$. Since all roots are positive or negative, there exists $l \in \{1, \dots, k\}$ such that $\beta = s_{i_{l+1}} \cdots s_{i_k}(\alpha)$ is positive and $s_{i_l}(\beta)$ is negative. But then $\beta = \alpha_{i_l}$, and $\alpha = s_{i_k} \cdots s_{i_{l+1}}(\alpha_{i_l})$ is a real root. Thus $\Delta(\mathfrak{B}(V))$ is finite. □

Corollary 4.2. *Let V be a braided vector space of standard type, set $\theta = \dim V$, and let $C = (a_{ij})_{i,j \in \{1,\dots,\theta\}}$ be the corresponding generalized Cartan matrix of finite type.*

(a) $\phi(\Delta_C) = \Delta(\mathfrak{B}(V))$, *where as before $\phi : \mathbb{Z}\pi \to \mathbb{Z}^\theta$ is the $\mathbb{Z}$-linear map determined by $\phi(\alpha_i) := \mathbf{e}_i$.*

(b) *The multiplicity of each root in $\Delta(\mathfrak{B}(V))$ is one.*

Proof. Statement (a) follows from the proof of Theorem 4.1.

Using this condition, since each root is of the form $\beta = w(\alpha_i)$ for some $w \in W$ and $i \in \{1, \dots, \theta\}$, we conclude by applying a certain sequence of transformations s_i that this is the degree corresponding to a generator of the corresponding Nichols algebra, so the multiplicity (which is invariant under these transformations) is 1. □

4B. *Explicit generators for a PBW basis.* In view of Corollary 4.2, we restrict our attention to finding one Lyndon word for each positive root of the root system associated with the corresponding finite Cartan matrix.

Proposition 4.3 [Lalonde and Ram 1995, Proposition 2.9]. *Let l be an element of S_I. Then l is of the form $l = l_1 \dots l_k a$, for some $k \in \mathbb{N}_0$, where*

- *$l_i \in S_I$ for each $i = 1, \dots, k$;*
- *l_i is a beginning of l_{i-1} for each $i > 1$; and*
- *a is a letter.*

Also, if $l = uv$ is the Shirshov decomposition, then $u, v \in S_I$. □

In what follows, we describe a set of Lyndon words for each Cartan matrix of finite type C.

Consider $\alpha = \sum_{j=1}^{\theta} a_j \alpha_j \in \Delta^+$ and let $l_\alpha \in S_I$ be such that $\deg l_\alpha = \alpha$. Let $l_\alpha = l_{\beta_1} \dots l_{\beta_k} x_s$ be a decomposition as above, where $s \in \{1, \dots, \theta\}$ and $\deg l_{\beta_j} = \beta_j$. Since each l_{β_j} is a beginning of $l_{\beta_{j-1}}$, all the words begin with the same letter x', which satisfies $x' < x_s$ because l is a Lyndon word. Therefore x' is the least letter of l, so

$$x' = x_i, \quad i = \min\{j : a_j \neq 0\} \quad \Longrightarrow \quad \alpha = \sum_{j=i}^{\theta} a_j \alpha_j.$$

Then $k \le a_i \le 3$, for the order given in (3-9), (3-4), (3-5), (3-6), (3-7), (3-8) (the value $a_i = 3$ appears only when C is of type G_2).

Now, each l_{β_j} lies in S_I, so $\beta_j \in \Delta^+$; i.e., it corresponds to a term of the PBW basis. Also $\sum_{j=1}^{k} \beta_j + \alpha_s = \alpha$. If $k = 2$, we have $\beta_1 - \beta_2 = \sum_{j=1}^{\theta} b_j \alpha_j$ and $b_j \ge 0$, because β_2 is a beginning of β_1 (an analogous claim is valid when the matrix is of type G_2 and $k = 3$). With these rules we define inductively Lyndon words for a PBW basis corresponding with a standard braiding for a fixed order on the letters. This is done as in [Lalonde and Ram 1995], but taking care that in that reference Serre relations are used; here we have quantum Serre relations, and some quantum binomial coefficients may be zero.

Type A_θ: In this case, the roots are of the form

$$\mathbf{u}_{i,j} := \sum_{k=i}^{j} \alpha_k, \quad 1 \le i \le j \le \theta.$$

By induction on $s = j - i$, we have

$$l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j.$$

This is because when $s = 0$ we have $i = j$, and the unique possibility is $l_{\mathbf{u}_{i,i}} = x_i$. If we remove the last letter (when $j - i > 0$), we must obtain a Lyndon word, so the last letter must be x_j.

Type B_θ: For convenience, we use the following vertex numbering:

$$\circ^{1} \Longleftarrow \circ^{2} \text{———} \circ^{3} \dots \qquad \circ^{\theta-1} \text{———} \circ^{\theta} . \tag{4-1}$$

The roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^{j} \alpha_k$, or

$$\mathbf{v}_{i,j} := 2\sum_{k=1}^{i} \alpha_k + \sum_{k=i+1}^{j} \alpha_k.$$

In the first case we have $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$, as above. In the second case, if $j = i + 1$, we must have x_{i+1} as the last letter to obtain a decomposition in two words $x_1 \cdots x_i$; if $j > i + 1$, the last letter must be x_j, so we obtain

$$l_{\mathbf{v}_{i,j}} = x_1 x_2 \dots x_i x_1 x_2 \dots x_j.$$

Type C_θ: The roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^{j} \alpha_k$, or

$$\mathbf{w}_{i,j} := \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{\theta-1} \alpha_k + \alpha_\theta, \quad i \le j < \theta.$$

As before, $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$. Now, if $i < j$, the least letter x_i has degree 1, so if we remove the last letter, we obtain a Lyndon word; that is, $\mathbf{w}_{i,j} - x_s$ is a root, and then $x_s = x_j$, so

$$l_{\mathbf{w}_{i,j}} = x_i x_{i+1} \dots x_{\theta-1} x_\theta x_{\theta-1} \dots x_j.$$

When $i = j$, $a_i = 2$, so there are one or two Lyndon words β_j as before. Since $\mathbf{w} - x_s$ is not a root, for $s = i+1, \dots, \theta$, and $i < s$, there are two Lyndon words $\beta_1 \geq \beta_2$, and $\beta_1 + \beta_2 = 2\sum_{k=i}^{\theta-1} \alpha_k$. The only possibility is $\beta_1 = \beta_2 = x_i x_{i+1} \dots x_{\theta-1}$; that is,

$$l_{\mathbf{w}_{i,i}} = x_i x_{i+1} \dots x_{\theta-1} x_i x_{i+1} \dots x_{\theta-1} x_\theta.$$

Type D_θ: the roots are of the form $\mathbf{u}_{i,j} := \sum_{k=i}^{j} \alpha_k,\ 1 \leq i \leq j \leq \theta$, or

$$\mathbf{z}_{i,j} := \sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{\theta-2} \alpha_k + \alpha_{\theta-1} + \alpha_\theta,\ i < j \leq \theta - 2,$$

$$\bar{\mathbf{z}}_i := \sum_{k=i}^{\theta-2} \alpha_k + \alpha_\theta,\ 1 \leq i \leq \theta - 2.$$

As above, $l_{\mathbf{u}_{i,j}} = x_i x_{i+1} \dots x_j$ if $j \leq n-1$. When the roots are of type $\bar{\mathbf{z}}_i$, we have $s = \theta$, since $\bar{\mathbf{z}}_i - x_s$ must be a root (if x_s is the last letter); thus $l_{\bar{\mathbf{z}}_i} = x_i x_{i+1} \dots x_{\theta-2} x_\theta$ is the unique possibility.

Now, when $\alpha = \mathbf{u}_{i,\theta}$, the last letter is $x_{\theta-1}$ or x_θ: if it is x_θ, we have $l_{\mathbf{u}_{i,\theta}} = x_i x_{i+1} \dots x_{\theta-1} x_\theta$. Since $m_{\theta-1,\theta} = 0$, we have $x_{\theta-1} x_\theta = q_{\theta-1,\theta} x_\theta x_{\theta-1}$, so

$$x_i x_{i+1} \dots x_{\theta-1} x_\theta \equiv x_i x_{i+1} \dots x_{\theta-2} x_\theta x_{\theta-1} \mod I,$$

and then $x_i x_{i+1} \dots x_{\theta-1} x_\theta \notin S_I$. So, $l_{\mathbf{u}_{i,\theta}} = x_i \dots x_{\theta-2} x_\theta x_{\theta-1}$.

In the last case, note that if $j = n-2$, the unique possibility is β_t as before, because the least letter x_i has degree 1 and $x_s = x_{\theta-2}$ (since $\alpha - \alpha_s$ is a root). Hence $l_{\mathbf{z}_{i,\theta-2}} = x_i \dots x_{\theta-2} x_\theta x_{\theta-1} x_{\theta-2}$, and inductively,

$$l_{\mathbf{z}_{i,j}} = x_i \dots x_{\theta-2} x_\theta x_{\theta-1} x_{\theta-2} \dots x_j.$$

Type E_6: Let $\alpha = \sum_{j=1}^{6} a_j \alpha_j$. If $a_6 = 0$, α corresponds to the Dynkin subdiagram of type D_5 determined by $1, 2, 3, 4, 5$, and we obtain l_α as above. If $a_1 = 0$ then α corresponds to the Dynkin subdiagram of type D_5 determined by $2, 3, 4, 5, 6$; the numbering is different from the one given in (3-4). Anyway, the roots are defined in a similar way, and we obtain the same list as in [Lalonde and Ram 1995, Fig.1]. If $a_4 = 0$, then α corresponds to the Dynkin subdiagram of type A_5 determined by $1, 2, 3, 5, 6$.

So we restrict our attention to the case $a_i \neq 0,\ i = 1, 2, 3, 4, 5, 6$. We consider each case in turn:

- $\alpha=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6$: since $a_1=1$, $\alpha-\alpha_s=\beta_1$ is a root, where α_s is the last letter. Then $s=2$ or $s=6$. In the second case, $l_{\beta_1}=x_1x_2x_3x_4x_5$, but using that $x_2x_3=q_{23}x_3x_2$, we have $x_1x_2x_3x_4x_5\notin S_I$. So $s=2$, and $l_\alpha=x_1x_3x_4x_5x_6x_2$.
- $\alpha=\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6$: from $a_1=1$, we note that $\alpha-\alpha_s=\beta_1$ is a root. Then $s=4$, and $l_\alpha=x_1x_3x_4x_5x_6x_2x_4$.
- $\alpha=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6$: since $a_1=1$, $\alpha-\alpha_s=\beta_1$ is a root. So $s=3$, and $l_\alpha=x_1x_3x_4x_5x_6x_2x_4x_3$.
- $\alpha=\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$: since $a_1=1$, $\alpha-\alpha_s=\beta_1$ is a root. The only possibility is $s=5$, and $l_\alpha=x_1x_3x_4x_5x_6x_2x_4x_5$.
- $\alpha=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$: as above $a_1=1$, and $\alpha-\alpha_s=\beta_1$ is a root. So $s=3$, and $l_\alpha=x_1x_3x_4x_5x_6x_2x_4x_5x_3$.
- $\alpha=\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$: since $a_1=1$, $\alpha-\alpha_s=\beta_1$ is a root. Then $s=4$ and $l_\alpha=x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4$.
- $\alpha=\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$: since $a_1=1$, $\alpha-\alpha_s=\beta_1$ is a root. So $s=2$, and $l_\alpha=x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4$.

Type E_7: If $\alpha=\sum_{j=1}^{7}a_j\alpha_j$ and $a_7=0$, the root corresponds to the subdiagram of type D_6 determined by $1,2,3,4,5,6$, and we obtain l_α as above. If $a_1=0$, it corresponds to the subdiagram of type E_6 determined by $2,3,4,5,6,7$. If $a_5=0$, then α corresponds to the subdiagram of type A_6 determined by $1,2,3,4,6,7$.

As above, consider each case where $a_i\neq 0,\ i=1,2,3,4,5,6,7$:

- $\alpha=\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7$: since $a_1=1$, $\alpha-\alpha_s=\beta_1$ is a root, if α_s is the last letter. Then $s=2$ or $s=7$. In the second case, $l_{\beta_1}=x_1x_2x_3x_4x_5x_6$, but from $x_2x_3=q_{23}x_3x_2$, we have $x_1x_2x_3x_4x_5x_6x_7\notin S_I$. So $s=2$, and $l_\alpha=x_1x_3x_4x_5x_6x_7x_2$.
- $\alpha=\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$: now $s=4,7$. We discard the case $s=7$ since $m_{47}=0$; for the case $s=4$ we have $l_\alpha=x_1x_3x_4x_5x_6x_7x_2x_4$.
- $\alpha=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7$: as above, $s=3,7$, but we discard $s=7$ since $m_{37}=0$, so $l_\alpha=x_1x_3x_4x_5x_6x_7x_2x_4x_3$.
- $\alpha=\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7$: now $s=5,7$, and we discard the case $s=7$ because $m_{57}=0$, so $l_\alpha=x_1x_3x_4x_5x_6x_7x_2x_4x_5$.
- $\alpha=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7$: now $s=3,7$, and as above we discard the case $s=7$, so $l_\alpha=x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3$.
- $\alpha=\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7$: now $s=4$, and therefore we have $l_\alpha=x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3x_4$.
- $\alpha=\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7$: now $s=2$, as above, and $l_\alpha=x_1x_3x_4x_5x_6x_7x_2x_4x_5x_3x_4x_2$.

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: as above, the unique possibility is $s = 6$, so $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 3$, $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 4$, $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 2$, and in this case we obtain $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_2$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 5$, and in this case we obtain $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: as above, $s = 2$, and we get $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2$.
- $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 4$, and in this case we obtain $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2x_4$.
- $\alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: $s = 3$, and in this case we obtain $l_\alpha = x_1x_3x_4x_5x_6x_7x_2x_4x_5x_6x_3x_4x_5x_2x_4x_3$.
- $\alpha = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$: now there are one or two words β_j. Since $\alpha - \alpha_s \in \Delta^+$ if and only if $s = 1$ and x_1 is not the last letter (because it is the least letter), there are two words β_j. So looking at the roots we obtain $s = 7$, and $l_\alpha = (x_1x_3x_4x_5x_6x_2x_4x_5x_3x_4x_2)(x_1x_3x_4x_5x_6)x_7$

Type E_8: Consider $\alpha = \sum_{j=1}^{8} a_j\alpha_j$; if $a_8 = 0$, the root corresponds to the subdiagram of type D_7 determined by $1, 2, 3, 4, 5, 6, 7$, and we obtain l_α as in that case. If $a_1 = 0$, it corresponds to the subdiagram of type E_7 determined by $2, 3, 4, 5, 6, 7, 8$. If $a_6 = 0$, then α corresponds to a subdiagram of type A_7 determined by $1, 2, 3, 4, 5, 7, 8$.

So, we consider the case $a_i \neq 0$, $i = 1, 2, 3, 4, 5, 6, 7, 8$, and solve it case by case in a similar way as for E_7, by induction on the height.

Type F_4: Now $\alpha = \sum_{j=1}^{4} a_j\alpha_j$. If $a_4 = 0$, then it corresponds to the subdiagram of type B_3 determined by $1, 2, 3$, so we obtain l_α as before. If $a_1 = 0$, α corresponds to the subdiagram of type C_3 determined by $2, 3, 4$.

So consider the case $a_i \neq 0$, $i = 1, 2, 3, 4$:

- $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$: $a_1 = 1$, so $\alpha - \alpha_s = \beta_1$ is a root, where α_s is the last letter. Then $s = 4$, and $l_\alpha = x_1x_2x_3x_4$.
- $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$: $a_1 = 1$, so $\alpha - \alpha_s = \beta_1$ is a root. Now $s = 3$ or $s = 4$. If $s = 4$, then $l_\alpha = x_1x_2x_3^2x_4$. But $m_{34} = 2$, so

$$x_3^2x_4 \equiv q_{34}(1 + q_{33})x_3x_4x_3 - q_{33}q_{34}x_4x_3^2 \mod I,$$

and $x_1x_2x_3^2x_4 \notin S_I$, a contradiction. So $s = 3$, and we have $l_\alpha = x_1x_2x_3x_4x_3$.

- $\alpha=\alpha_1+2\alpha_2+2\alpha_3+\alpha_4$: $a_1=1$, and as above, $s=2$ or $s=4$: if $s=4$, then $l_\alpha=x_1x_2x_3^2x_2x_4$, but it is not an element of S_I, because $x_2x_4\equiv q_{24}x_2x_4 \mod I$. Then $s=2$, and $l_\alpha=x_1x_2x_3x_4x_3x_2$.
- $\alpha=\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$: $a_1=1$, so $s=3$, and we have $l_\alpha=x_1x_2x_3x_4x_3x_2x_3$.
- $\alpha=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$: $a_1=1$, so $s=4$, and $l_\alpha=x_1x_2x_3x_4x_3x_4$.
- $\alpha=\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4$: $a_1=1$, so $s=2$ or $s=4$, but we discard the case $s=4$ since $x_2x_4\equiv q_{24}x_2x_4 \mod I$. So, $l_\alpha=x_1x_2x_3x_4x_3x_4x_2$.
- $\alpha=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$: $a_1=1$, so $s=3$, and $l_\alpha=x_1x_2x_3x_4x_3x_4x_2x_3$.
- $\alpha=\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4$: $a_1=1$, so $s=3$, and $l_\alpha=x_1x_2x_3x_4x_3x_4x_2x_3^2$.
- $\alpha=\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$: $a_1=1$, so $s=2$, and $l_\alpha=x_1x_2x_3x_4x_3x_4x_2x_3^2x_2$.
- $\alpha=2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$: $a_1=2$, and there are one or two Lyndon words β_j. If there is only one, $\beta_1=\alpha-\alpha_s\in\Delta^+$. The only possibility is $s=1$, but it contradicts that l_α is a Lyndon word. Hence there exist $\beta_1,\beta_2\in\Delta^+$ such that $\beta_1+\beta_2=\alpha-\alpha_s$, and β_2 is a beginning of β_1. So $s=2$ and $\beta_1=\beta_2=\alpha_1+\alpha_2+2\alpha_3+\alpha_4$, i.e., $l_\alpha=x_1x_2x_3x_4x_3x_1x_2x_3x_4x_3x_2$.

Type G_2: the roots are $\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2$:

$$l_{\alpha_1}=x_1, \qquad l_{\alpha_2}=x_2, \qquad l_{m\alpha_1+\alpha_2}=x_1^mx_2, \quad m=1,2,3.$$

If $\alpha=3\alpha_1+2\alpha_2$, the last letter is x_2. If we suppose $\beta_1=3\alpha_1+\alpha_2$, then $l_\alpha=x_1^3x_2^2$, but

$$(\operatorname{ad} x_2)^2x_1=x_2^2x_1-q_{21}(1+q_{22})x_2x_1x_2+q_{22}q_{21}x_1x_2^2\equiv 0 \mod I,$$

so we have

$$x_1^3x_2^2\equiv(q_{22}^{-1}+1)x_1^2x_2x_1x_2-q_{22}^{-1}q_{21}^{-1}x_1^2x_2^2x_1 \mod I,$$

and then $l_\alpha=x_1^3x_2^2\notin S_I$ because $q_{22}^{-1}q_{21}^{-1}\neq 0$, so there are at least two words β_j. Analogously, if we suppose that there are three words β_j, we obtain $l_{\beta_1}=l_{\beta_2}=x_1>l_{\beta_3}=x_1x_2$ since $\beta_1\geq\beta_2\geq\beta_3$ and $\beta_1+\beta_2+\beta_3=3\alpha_1+\alpha_2$; moreover $l_\alpha=x_1^3x_2^2\notin S_I$. So there are two Lyndon words of degree $\beta_1\geq\beta_2$, and the unique possibility is $\beta_1=2\alpha_1+\alpha_2$, $\beta_2=\alpha_1$; that is, $l_\alpha=x_1^2x_2x_1x_2$.

4C. *Dimensions of Nichols algebras of standard braidings.* We begin with the standard braidings of types C_θ, D_θ, E_6, E_7, E_8, F_4, which are of Cartan type.

Proposition 4.4. *Let V a braided vector space of Cartan type, with $q_{44}\in\mathbb{G}_N$ if V is of type F_4 and $q_{11}\in\mathbb{G}_N$ otherwise, in each case for some $N\in\mathbb{N}$. The dimension of the associated Nichols algebra $\mathfrak{B}(V)$ is as follows*:

Type C_θ: $\dim\mathfrak{B}(V)=\begin{cases}N^{\theta^2} & \textit{for } N \textit{ odd,}\\ N^{\theta^2}/2^\theta & \textit{for } N \textit{ even;}\end{cases}$

Type F_4: $\dim \mathfrak{B}(V) = \begin{cases} N^{24} & \textit{for N odd,} \\ N^{24}/2^{12} & \textit{for N even;} \end{cases}$

Types D_θ, E_6, E_7, E_8: $\dim \mathfrak{B}(V) = N^{|\Delta^+|}$.

The last case corresponds to simply laced Dynkin diagrams.

Proof. If N is odd, then $\operatorname{ord} q^2 = \operatorname{ord} q = N$, but if N is even, we have $\operatorname{ord} q^2 = N/2$. Since the braiding is of Cartan type,

$$q_{s_i(\alpha)} = \chi\,(s_i(\alpha), s_i(\alpha)) = \widetilde{\chi}(\alpha, \alpha) = \chi(\alpha, \alpha) = q_\alpha.$$

Using this, we just have to determine how many roots there are in the orbit of each simply root.

When V is of type C_θ, we have $q_{ii} = q$, except for $q_{\theta\theta} = q^2$. The roots in the orbit of α_θ by the action of the Weyl group are $q_{\mathbf{w}_{ii}}$ for $1 \le i < \theta$, and the others are in the orbit of α_j, for some $j < \theta$. Hence there are θ roots such that $q_\alpha = q^2$, and $q_\alpha = q$ for the rest.

When V is of type F_4, we have $q_{11} = q_{22} = q^2$ and $q_{33} = q_{44} = q$. There are exactly 12 roots in the union of orbits corresponding to α_1 and α_2, and the other 12 are in the union of orbits corresponding to α_3 and α_4. So

$$\big|\{\alpha \in \Delta^+ : q_\alpha = q\}\big| = \big|\{\alpha \in \Delta^+ : q_\alpha = q^2\}\big| = 12.$$

When V is of type D or E, all the q_α equal q because $q_{ii} = q$, for all $1 \le i \le \theta$.

The formula for the dimension follows from Theorem 2.6(ii) and Corollary 4.2. □

Now we treat the types A_θ, B_θ and G_2.

Proposition 4.5. *Let V be a standard braided vector space of type A_θ as in Proposition 3.9. The associated Nichols algebra $\mathfrak{B}(V)$ is of finite dimension if and only if q is a root of unity of order $N \ge 2$. In this case,*

$$\dim \mathfrak{B}(V) = 2^{\binom{\theta+1}{2} - \binom{t}{2} - \binom{\theta+1-t}{2}} N^{\binom{t}{2} + \binom{\theta+1-t}{2}}, \tag{4-2}$$

where $t = \theta + 1 - \sum_{k=1}^{j} (-1)^{j-k} i_k$.

Proof. First, q is a root of unity of order $N \ge 2$ because the height of each PBW generator is finite. To calculate the dimension, recall that from Corollary 4.2, we have to determine q_α for $\alpha \in \Delta_C$. As before, $\mathbf{u}_{ij} = \sum_{k=i}^{j} \mathbf{e}_k$, $i \le j$, and we have

$$\Delta(\mathfrak{B}(V)) = \{\mathbf{u}_{ij} : 1 \le i \le j \le \theta\}.$$

If $1 \le i \le j \le \theta$, we define

$$\kappa_{ij} := \operatorname{card}\big\{k \in \{i, \dots, j\} : q_{kk} = -1\big\}.$$

We prove by induction on $j-i$ that

- if κ_{ij} is odd, then $q_{\mathbf{u}_{ij}}=-1$;
- if κ_{ij} is even, then $q_{\mathbf{u}_{ij}}=q_{j,j+1}^{-1}q_{j+1,j}^{-1}$.

Here $q_{\theta,\theta+1}q_{\theta+1,\theta}=q_{\theta\theta}^{-2}q_{\theta,\theta-1}^{-1}q_{\theta-1,\theta}^{-1}$.

If $j-i=0$, then $q_{\mathbf{u}_{ii}}=q_{ii}$; in this case, $\kappa_{ii}=1$ if $q_{ii}=-1$ or $\kappa_{ii}=0$ if $q_{ii}=(q_{i,i+1}q_{i+1,i})^{-1}\neq -1$. Now assume this is valid for a certain j, and calculate it for $j+1$:

$$\begin{aligned} q_{\mathbf{u}_{i,j+1}} &= \chi(\mathbf{u}_{ij}+\mathbf{e}_{j+1},\mathbf{u}_{ij}+\mathbf{e}_{j+1}) = q_{\mathbf{u}_{ij}}\chi(\mathbf{u}_{ij},\mathbf{e}_{j+1})\chi(\mathbf{e}_{j+1},\mathbf{u}_{ij})q_{j+1,j+1} \\ &= q_{\mathbf{u}_{ij}}q_{j,j+1}q_{j+1,j}q_{j+1,j+1} \\ &= \begin{cases} q_{\mathbf{u}_{ij}} & \text{if } q_{j+1,j+1}\neq -1 \ (\kappa_{i,j+1}=\kappa_{ij}), \\ (-1)qq^{-1}=-1 & \text{if } q_{j+1,j+1}=-1,\ \kappa_{ij} \text{ even}, \\ (-1)q(-1)=q & \text{if } q_{j+1,j+1}=-1,\ \kappa_{ij} \text{ odd}. \end{cases} \end{aligned}$$

This proves the inductive step; to calculate the dimension of $\mathfrak{B}(V)$ we have to calculate the number of $\mathbf{u}_{ij}$ such that

$$q_{\mathbf{u}_{ij}}=q_{i,i+1}^{-1}q_{i+1,i}^{-1}=q^{\pm 1},$$

that is, $\operatorname{card}\{\kappa_{ij}: i\leq j, \kappa_{ij} \text{ even}\}$.

We consider an $1\times(\theta+1)$ board, numbered from 1 to $\theta+1$, and recursively paint its squares white or black: square $\theta+1$ is white, and square i has the same color as square $i+1$ if and only if $q_{ii}\neq -1$. The possible colorings of this board are in bijective correspondence with the choices of $1\leq i_1<\cdots<i_j\leq\theta$ for all j (the positions where we put a -1 in the corresponding q_{ii} of the braiding), and the number of white squares is

$$t=1+(\theta-i_j)+(i_{j-1}-i_{j-2})+\cdots=\theta+1-\sum_{k=1}^{j}(-1)^{j-k}i_k$$

Thus $\operatorname{card}\{\kappa_{ij}: i\leq j, \kappa_{ij} \text{ even}\}$ is the number of pairs (a,b) (where $a=i$ and $b=j+1$) such that $1\leq a<b\leq\theta+1$ and the squares in positions a and b are of the same color; this number is $\binom{t}{2}+\binom{\theta+1-t}{2}$. This yields (4-2). □

Proposition 4.6. *Let V be a standard braided vector space of type B_θ as in Proposition 3.10. If the braiding is as in cases* (a) *or* (b) *of that proposition, the associated Nichols algebra $\mathfrak{B}(V)$ has finite dimension if and only if q is a root of unity of order $N\geq 2$ in case* (a), *or $N>2$ in case* (b).

When finite, the dimension of $\mathfrak{B}(V)$ is as follows, where $t=\theta-\sum_{k=1}^{j}(-1)^{j-k}i_k$:

- *If the braiding is as in* (a) *of Proposition 3.10*,

$$\dim \mathfrak{B}(V) = \begin{cases} 3^3 N^2 & \text{if } 3 \text{ does not divide } N, \\ 3^2 N^2 & \text{if } 3 \text{ divides both } N \text{ and } \operatorname{ord}(\zeta^{-1}q), \\ 3N^2 & \text{if } 3 \text{ divides } N \text{ but not } \operatorname{ord}(\zeta^{-1}q). \end{cases}$$

- *If the braiding is as in* (b), *then* $0 \leq j \leq d-1$ *and*

$$\dim \mathfrak{B}(V) = \begin{cases} 2^{2t(\theta-t)+\theta} k^{\theta^2-2t\theta+2t^2} & \text{if } N = 2k, \\ 2^{(2t+1)(\theta-t)+1} N^{\theta^2-2t\theta+2t^2} & \text{if } N \text{ is odd.} \end{cases}$$

- *If the braiding is as in* (c), *then*

$$\dim \mathfrak{B}(V) = 2^{\theta(\theta-1)} 3^{\theta^2-2t\theta+2t^2}.$$

Proof. It is clear that q should be a root of unity if $\dim \mathfrak{B}(V)$ is finite.

We now calculate $\dim \mathfrak{B}(V)$. From Corollary 4.2, we have to determine the q_α for $\alpha \in \Delta_C$, and multiply their orders. As before, $\mathbf{u}_{ij} = \sum_{k=i}^{j} \mathbf{e}_k$ for $1 \leq i \leq j \leq \theta$ and $\mathbf{v}_{ij} = 2\sum_{k=1}^{i} e_k + \sum_{k=i+1}^{j} e_k = 2e_{1,i} + e_{i+1,j}$ for $1 \leq i < j$; hence

$$\Delta(\mathfrak{B}(V)) = \{\mathbf{u}_{ij} : 1 \leq i \leq j \leq \theta\} \cup \{\mathbf{v}_{ij} : 1 \leq i < j \leq \theta\}.$$

We calculate $q_{\mathbf{u}_{ij}},\ 1 < i \leq j \leq \theta$ as above, because they correspond to a braiding of standard $A_{\theta-1}$ type. We also calculate

$$\begin{aligned} q_{\mathbf{v}_{ij}} &= \chi(\mathbf{v}_{ij}, \mathbf{v}_{ij}) = \chi(\mathbf{u}_{1i}, \mathbf{u}_{1i})^4 \chi(\mathbf{u}_{1i}, \mathbf{u}_{i+1,j})^2 \chi(\mathbf{u}_{i+1,j}, \mathbf{u}_{1i})^2 q_{\mathbf{u}_{i+1,j}} \\ &= q_{11}^4 q_{12}^2 q_{21}^2 \left(\prod_{k=2}^{i} q_{kk}^2 q_{k-1,k} q_{k-1,k} q_{k+1,k} q_{k+1,k} \right)^2 q_{\mathbf{u}_{i+1,j}} = q_{\mathbf{u}_{i+1,j}}, \end{aligned}$$

where we have used the equalities $q_{ij}q_{ji} = 1$ if $|i-j| > 1$, $q_{11}^4 q_{12}^2 q_{21}^2 = 1$, and $q_{kk}^2 q_{k-1,k} q_{k-1,k} q_{k+1,k} q_{k+1,k} = 1$ if $2 \leq k \leq \theta - 1$. To calculate the other q_α's, we analyze each case:

(a) Note that $q_{\mathbf{e}_1} = \zeta,\ q_{\mathbf{e}_1+\mathbf{e}_2} = \zeta,\ q_{2\mathbf{e}_1+\mathbf{e}_2} = \zeta q^{-1},\ q_{\mathbf{e}_2} = q$. Setting $N' = \operatorname{ord}(\zeta^{-1}q)$, we have $N' = 3N$ if 3 does not divide N; $N' = N$ if 3 divides both N and N'; and $N' = N/3$ if 3 divides N but not N' (since $q = \zeta\rho$, with $\rho \in \mathbb{G}_{N'}$).

(b) We have $q_{\mathbf{u}_{1k}} = q^{-1} q_{\mathbf{u}_{2k}}$. This equals $q^2 q^{-1} = q$ if κ_{2k} is even, and $-q^{-1}$ if κ_{2k} is odd; moreover $q_{11} = q$. Also, κ_{2k} is even if and only if $j \in \{i_j + 1, \theta\}$, or $i \in \{i_{j-2} + 1, i_{j-1}\}$, and so on. Then, with

$$t = (\theta - i_j) + (i_{j-1} - i_{j-2}) + \cdots = \theta - \sum_{k=1}^{j} (-1)^{j-k} i_k$$

as in the statement of the proposition, there are t numbers such that $\kappa_{i,\theta-1}$ is even. There are $2\left(\binom{t}{2} + \binom{\theta-t}{2}\right)$ roots such that $q_\alpha = q^2$, $2\left(\binom{\theta}{2} - \binom{t}{2} - \binom{\theta-t}{2}\right)$ roots such that $q_\alpha = -1$, $t+1$ roots such that $q_\alpha = q$ and $\theta - 1 - t$ roots such that $q_\alpha = -q^{-1}$. If $N = 2k$, then $\operatorname{ord}(-q^{-1}) = 2k$ and $\operatorname{ord}(q^2) = k$, so

$$\begin{aligned}\dim \mathfrak{B}(V) &= 2^{(\theta-1)\theta - t(t-1) - (\theta-t)(\theta-t-1)} k^{t(t-1)+(\theta-t)(\theta-t-1)} (2k)^{\theta} \\ &= 2^{2t(\theta-t)+\theta} k^{\theta^2-2t\theta+2t^2}.\end{aligned}$$

If N is odd, then $\operatorname{ord}(-q^{-1}) = 2N$ and $\operatorname{ord}(q^2) = N$, so

$$\begin{aligned}\dim \mathfrak{B}(V) = 2^{\theta(\theta-1)-t(t-1)-(\theta-t)(\theta-1-t)} N^{t(t-1)+(\theta-t)(\theta-1-t)+t+1} \\ (2N)^{\theta-1-t} = 2^{(2t+1)(\theta-t)+1} N^{\theta^2-2t\theta+2t^2}.\end{aligned}$$

(c) In a similar way, we have $q_{\mathbf{u}_{1i}} = (-\zeta^2) q_{\mathbf{u}_{2i}}$, which equals $(-\zeta^2)^2 = \zeta$ if κ_{2i} is even, and $(-1)(-\zeta^2) = \zeta^2$ if κ_{2i} is odd; moreover $q_{11} = \zeta$. There are $2\left(\binom{t}{2} + \binom{\theta-t}{2}\right)$ roots such that $q_\alpha = -\zeta^2$, $2\left(\binom{\theta}{2} - \binom{t}{2} - \binom{\theta-t}{2}\right)$ roots such that $q_\alpha = -1$, $t+1$ roots such that $q_\alpha = \zeta$ and $\theta - 1 - t$ roots such that $q_\alpha = \zeta^2$. Since $\operatorname{ord}\zeta = \operatorname{ord}\zeta^2 = 3$ and $\operatorname{ord}(-\zeta^2) = 6$, we have

$$\begin{aligned}\dim \mathfrak{B}(V) &= 2^{\theta(\theta-1)-t(t-1)-(\theta-t)(\theta-1-t)} 6^{t(t-1)+(\theta-t)(\theta-1-t)} 3^{\theta} \\ &= 2^{\theta(\theta-1)} 3^{\theta^2-2t\theta+2t^2}.\end{aligned}$$ □

Proposition 4.7. *Let V be a standard braided vector space of type G_2 as in Proposition 3.11. If the braiding is as in case* (a) *of that proposition, the associated Nichols algebra $\mathfrak{B}(V)$ is of finite dimension if and only if q is a root of unity of order $N \geq 4$.*

When finite, the dimension of $\mathfrak{B}(V)$ is as follows:

- *In case* (a) *of Proposition 3.11,*

$$\dim \mathfrak{B}(V) = \begin{cases} N^6 & \text{if 3 does not divide } N, \\ N^6/27 & \text{if 3 divides } N. \end{cases}$$

- *In case* (b), $\dim \mathfrak{B}(V) = 2^{12}$.

Proof. For (a) note that q is a root of unity because x_1 has finite height, and $q_\alpha = q$ if $\alpha \in \{e_1, e_1+e_2, 2e_1+e_2\}$, while $q_\alpha = q^3$ if $\alpha \in \{e_2, 3e_1+e_2, 3e_1+2e_2\}$.

Case (b) can be checked as follows:

type	q_{x_2}	$q_{x_1x_2}$	$q_{x_1^3x_2^2}$	$q_{x_1^2x_2}$	$q_{x_1^3x_2}$	q_{x_1}	$\dim \mathfrak{B}(V)$
$\overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}$	8	4	2	8	2	4	2^{12}
$\overset{\zeta^2}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}$	2	8	2	4	8	4	2^{12}
$\overset{\zeta}{\circ}\overset{\zeta^5}{\text{———}}\overset{-1}{\circ}$	2	4	8	4	2	8	2^{12}

This completes the proof. □

5. Presentation by generators and relations of Nichols algebras of standard braided vector spaces

In this section we give presentations for the Nichols algebras of standard braided vector spaces. We start with some technical results about relations and PBW bases in Section 5A; also we calculate the coproduct of some hyperwords in $T(V)$. In Sections 5B, 5C and 5D we express the braided commutator of two PBW generators as a combination of elements of the PBW basis under some assumptions. Then we obtain the desired presentation with a proof similar to the ones in [Andruskiewitsch and Dăscălescu 2005] and [Andruskiewitsch and Schneider 2002b]. In Section 5E we solve the problem when the braiding is of Cartan type using the transformation in Section 2C.

For rank two, a set of (not necessarily minimal) relations is given in Theorem 4 of [Heckenberger 2007].

5A. ***Some general relations.*** Let V be a standard braided vector space with connected Dynkin diagram and let C be the corresponding Cartan matrix. In what follows we assume that C is symmetrizable and of finite type. Let $x_1, \dots, x_\theta$ be an ordered basis of V and $\{x_\alpha : \alpha \in \Delta^+(\mathfrak{B}(V))\}$ a set of PBW generators as in the previous section. Here $x_\alpha \in \mathfrak{B}(V)$ is, by abuse of notation, the image by the canonical projection of $x_\alpha \in T(V)$, the hyperword corresponding to a Lyndon word l_α. As before, we write

$$q_\alpha := \chi(\alpha, \alpha) \quad \text{and} \quad N_\alpha := \operatorname{ord} q_\alpha \quad \text{for } \alpha \in \Delta^+(\mathfrak{B}(V)).$$

Each x_α is homogeneous and has the same degree as l_α. Also,

$$x_\alpha \in T(V)^{\chi_\alpha}_{g_\alpha}, \tag{5-1}$$

where $g_\alpha = g_1^{b_1} \dots g_\theta^{b_\theta}$ and $\chi_\alpha = \chi_1^{b_1} \dots \chi_\theta^{b_\theta}$ if $\alpha = b_1\mathbf{e}_1 + \dots + b_\theta\mathbf{e}_\theta$.

Proposition 5.1. *If the matrix of the braiding is symmetric, the PBW basis is orthogonal with respect to the bilinear form in Proposition 1.4.*

Proof. We must prove that $(u|v) = 0$, where $u \neq v$ are ordered products of PBW generators (we also allow powers greater than the corresponding heights). We argue by induction on $k := \max\{\ell(u), \ell(v)\}$. If $k = 1$, then v is some x_j and u is either 1 or x_i; since $(x_i|x_j) = \delta_{ij}$ for all $i, j \in \{1, \dots, \theta\}$, the proposition holds in this case.

Suppose the statement is valid when the length of both words is less than k, and let $u, v \in B_{I(V)}$ be distinct hyperwords such that one (or both) has length k. If both are hyperletters, they have different degrees $\alpha \neq \beta \in \mathbb{Z}^\theta$, so $u = x_\alpha$, $v = x_\beta$, and $(x_\alpha|x_\beta) = 0$, since the homogeneous components are orthogonal for $(\ |\ )$.

Suppose that $u=x_\alpha$ and $v=x_{\beta_1}^{h_1}\dots x_{\beta_m}^{h_m}$, for some $x_{\beta_1}>\dots>x_{\beta_m}$. If u and v have different $\mathbb{Z}^\theta$-degrees, they are orthogonal. Hence we assume that $\alpha=\sum_{j=1}^m h_m\beta_m$. By [Bourbaki 1968, VI, Proposition 19], we can reorder the β_i's, using h_i copies of β_i, in such a way that each partial sum is a root. By [Rosso 1999, Proposition 21], the order induced by the Lyndon words l_α is convex (the order on Lyndon words used there is the same as ours). Therefore $\beta_m<\alpha$. Using Lemma 1.9 and (1-8),

$$(u|v)=(x_\alpha|w)(1|x_{\beta_m})+(1|w)(x_{\alpha_n}|x_{\beta_m})+\sum_{\substack{l_1\geq\dots\geq l_p>\alpha\\ l_i\in L}}(x_{l_1,\dots,l_p}|w)([l_1]_c\cdots[l_p]_c|x_{\beta_m}),$$

where $v=wx_{\beta_n}$. Note that $(1|x_{\beta_m})=(1|w)=0$. Also, $[l_1]_c\cdots[l_p]_c$ is a linear combination of greater hyperwords of the same degree and an element of $I(V)$. From the inductive hypothesis and the fact that $I(V)$ is the radical of the bilinear form, we see that $([l_1]_c\cdots[l_p]_c|x_{\beta_m})=0$.

Now consider

$$u=x_{\alpha_1}^{j_1}\dots x_{\alpha_n}^{j_n}\text{ with }x_{\alpha_1}>\dots>x_{\alpha_n},$$
$$v=x_{\beta_1}^{h_1}\dots x_{\beta_m}^{h_m}\text{ with }x_{\beta_1}>\dots>x_{\beta_m}.$$

Since the bilinear form is symmetric, we may as well assume that $x_{\alpha_n}\leq x_{\beta_m}$. Using Lemma 1.9 and (1-8), we obtain

$$(u|v)=(w|1)(x_{\alpha_n}|v)+\sum_{i=0}^{h_m}\binom{h_m}{i}_{q_{\beta_m}}(w|x_{\beta_1}^{h_1}\dots x_{\beta_{m-1}}^{h_{m-1}}x_{\beta_m}^{i})(x_{\alpha_n}|x_{\beta_m}^{h_m-i})$$
$$+\sum_{\substack{l_1\geq\dots\geq l_p>l,l_i\in L\\ 0\leq j\leq m}}(w|x_{l_1,\dots,l_p}^{(j)})(x_{\alpha_n}|[l_1]_c\dots[l_p]_c\left[x_{\beta_m}\right]_c^j),$$

where $w=x_{\alpha_1}^{h_1}\dots x_{\alpha_m}^{h_m-1}$. Note that in the first summand, $(w|1)=0$. In the last sum, $(x_{\alpha_n}|[l_1]_c\dots[l_p]_c[x_{\beta_m}]_c^j)$ vanishes, because by earlier results $[l_1]_c\dots[l_p]_c[x_{\beta_m}]_c^j$ is a combination of hyperwords of the PBW basis greater or equal than it and an element of $I(V)$, then we use induction hypothesis and the fact that $I(V)$ is the radical of this bilinear form. Since $x_{\alpha_n},x_{\beta_m}^{h_m-i}$ are different elements of the PBW basis for $h_m-i\neq 1$, we have

$$(u|v)=(h_m)_{q_{\beta_m}}(w|x_{\beta_1}^{h_1}\dots x_{\beta_{m-1}}^{h_{m-1}}x_{\beta_m}^{h_m-1})(x_{\alpha_n}|x_{\beta_m}).$$

This is clearly zero if $\alpha_n\neq\beta_m$. To see that it is zero also if $\alpha_n=\beta_m$, note that in that case w and $x_{\beta_1}^{h_1}\dots x_{\beta_{m-1}}^{h_{m-1}}x_{\beta_m}^{h_m-1}$ are different products of PBW generators, and use the induction hypothesis. □

Corollary 5.2. *If $\alpha\in\Delta^+(\mathfrak{B}(V))$, then $x_\alpha^{N_\alpha}=0$.*

Proof. Let (q_{ij}) be symmetric. If $u = x_{\alpha_1}^{j_1} \dots x_{\alpha_n}^{j_n},\ x_{\alpha_1} > \dots > x_{\alpha_n}$, then

$$(u|u) = \prod_{i=1}^{n} (j_i)_{q_{\alpha_i}}! \, (x_{\alpha_i}|x_{\alpha_i})^{j_i}, \tag{5-2}$$

where $(x_\alpha|x_\alpha) \neq 0$ for all $\alpha \in \Delta^+(\mathfrak{B}(V))$.

If we consider $u = x_\alpha^{N_\alpha}$, we have $(u|v) = 0$ for each v in the PBW basis (because v is an ordered product of x_β's different from u), and $(u|u) = 0$ since $q_\alpha \in \mathbb{G}_{N_\alpha}$. Also, $(I(V)|x_\alpha^{N_\alpha}) = 0$, because it is the radical of this bilinear form, so $(T(V)|x_\alpha^{N_\alpha}) = 0$, and then $x_\alpha^{N_\alpha} \in I(V)$. That is, we have $x_\alpha^{N_\alpha} = 0$ in $\mathfrak{B}(V)$.

For the general case, recall that a diagonal braiding is twist equivalent to a braiding with a symmetric matrix [Andruskiewitsch and Schneider 2002a, Theorem 4.5]. Also, there exists a linear isomorphism between the corresponding Nichols algebras. The corresponding x_α are related by a nonzero scalar, because they are an iteration of braided commutators between the hyperwords. □

In what follows, $\mathfrak{J}$ denotes the family of $\mathbb{Z}^\theta$-graded (hence $\mathbb{N}$-graded) ideals of $T(V)$ that are generated by their components of degree > 1. For each $I \in \mathfrak{J}$, $\mathfrak{B} = T(V)/I$ is a $\mathbb{Z}^\theta$-graded algebra such that $\mathfrak{B}^0 = \mathsf{k}1$ and $\mathfrak{B}^1 \simeq V$.

We shall need some technical results about graded algebras between $T(V)$ and $\mathfrak{B}(V)$. We start with three lemmas dealing with the presence of some important roots in $\Delta(\mathfrak{B})$. Remember that a Lyndon word is a PBW generator in $\mathfrak{B} = T(V)/I$ if it is not a linear combination of greater words modulo I in $T(V)$. We shall relate the absence of some roots in $\Delta(\mathfrak{B})$ (meaning that the Lyndon words of such degrees are linear combinations of greater words modulo I) with the validity of certain relations in $\mathfrak{B}$.

Lemma 5.3. *Let $i, j \in \{1, \dots, \theta\}$ be distinct, and consider $I \in \mathfrak{J}$, $\mathfrak{B} = T(V)/I$. Let D_k, $k = 1, \dots, \theta$, be skew derivations of $\mathfrak{B}$ as in Proposition 2.1, and assume that $x_i^N = 0$ if $q_{ii}^n q_{ij} q_{ji} \neq 1$ for all $n \in \mathbb{N}_0$ (where $N = \operatorname{ord} q_{ii}$).*

There exists $m \in \mathbb{N}$ such that $x_i^m x_j$ is a linear combination of greater hyperwords (for a fixed order such that $x_i < x_j$) modulo I if and only if, in $\mathfrak{B}$.

$$(\operatorname{ad}_c x_i)^{m_{ij}+1} x_j = 0. \tag{5-3}$$

Proof. If $(\operatorname{ad}_c x_i)^m x_j = 0$, there exist $a_k \in \mathsf{k}$ such that

$$0 = [x_i^m x_j]_c = (\operatorname{ad}_c x_i)^m x_j = x_i^m x_j + \sum_{k=0}^{m-1} a_k x_i^k x_j x_i^{m-k}.$$

Conversely, suppose there exists $m \in \mathbb{N}$ such that $x_i^m x_j$ is a linear combination of greater hyperwords modulo I. Let

$$n = \min\{m \in \mathbb{N} : x_i^m x_j \text{ is a linear combination of greater hyperwords}\}.$$

If $x_i^n = 0$, then q_{ii} is a root of unity. In this case, if N is the order of q_{ii}, then $x_i^N = 0$ and $x_i^{N-1} \neq 0$. Also,

$$(\mathrm{ad}_c\, x_i)^N x_j = x_i^N x_j + \sum_{s=1}^{N-1} \binom{N}{s}_{q_{ii}} + x_j x_i^N = 0,$$

because $\binom{N}{s}_{q_{ii}} = 0$ for $0 < s < N$. Hence, we can assume $x_i^n \neq 0$ and $(n)_{q_{ii}}! \neq 0$.

Note that $[x_i^{n-k} x_j x_i^k]_c = [x_i^{n-k} x_j]_c x_i^k$. Since $\mathfrak{B}$ is graded, $x_i^n x_j$ is a linear combination of terms $x_i^{n-k} x_j x_i^k$, $0 \leq k < n$. Hence there exist $\alpha_k \in \Bbbk$ such that

$$[x_i^n x_j]_c = \sum_{k=1}^{n} \alpha_k [x_i^{n-k} x_j]_c x_i^k.$$

Applying D_i we obtain

$$0 = D_i\big([x_i^n x_j]_c\big) = \sum_{k=1}^{n} \alpha_k D_i\big([x_i^{n-k} x_j]_c x_i^k\big) = \sum_{k=1}^{n} \alpha_k (k)_{q_{ii}} [x_i^{n-k} x_j]_c x_i^k.$$

By the hypothesis about n, $\alpha_1 = 0$. Since $(n)_{q_{ii}}! \neq 0$, applying D_i several times we conclude that $\alpha_k = 0$ for $k = 2, \dots, n$. Hence $[x_i^n x_j]_c = 0$. □

Recall that (5-3) holds in $\mathfrak{B}(V)$, for $1 \leq i \neq j \leq \theta$.

The second lemma is related to Dynkin diagrams of a standard braiding which have two consecutive simple edges.

Lemma 5.4. *Let $I \in \mathfrak{J}$ and $\mathfrak{B} = T(V)/I$. Assume that*

- *there exist skew derivations D_k in $\mathfrak{B}$ as in Proposition 2.1*;
- *there exist different $j, k, l \in \{1, \dots, \theta\}$ such that $m_{kj} = m_{kl} = 1$, $m_{jl} = 0$*;
- $(\mathrm{ad}\, x_k)^2 x_j = (\mathrm{ad}\, x_k)^2 x_l = (\mathrm{ad}\, x_j) x_l = 0$ *hold in* $\mathfrak{B}$;
- $x_k^2 = 0$ *if* $q_{kk} q_{kj} q_{jk} \neq 1$ *or* $q_{kk} q_{kl} q_{lk} \neq 1$.

(1) *If we order the letters $x_1, \dots, x_\theta$ such that $x_j < x_k < x_l$, then $x_j x_k x_l x_k$ is a linear combination of greater words modulo I if and only if, in $\mathfrak{B}$,*

$$\big[(\mathrm{ad}\, x_j)(\mathrm{ad}\, x_k) x_l, x_k\big]_c = 0. \tag{5-4}$$

(2) *If V is standard and $q_{kk} \neq -1$, then* (5-4) *holds in* $\mathfrak{B}$.

(3) *If V is standard and* $\dim \mathfrak{B}(V) < \infty$, *then* (5-4) *holds in* $\mathfrak{B} = \mathfrak{B}(V)$.

Proof. (1) ($\Leftarrow$) If (5-4) holds, then $x_j x_k x_l x_k$ is a linear combination of greater words, by Remark 1.7, and

$$[x_j x_k x_l x_k]_c = \big[[x_j x_k x_l]_c, x_k\big]_c = \big[(\mathrm{ad}\, x_j)(\mathrm{ad}\, x_k) x_l, x_k\big]_c.$$

($\Rightarrow$) If $x_jx_kx_lx_k$ is a linear combination of greater words, then the hyperword $\left[x_jx_kx_lx_k\right]_c$ is a linear combination of hyperwords corresponding to words greater than $x_jx_kx_lx_k$ (of the same degree, because $\mathfrak{B}$ is homogeneous); this follows from Remark 1.7. Since $(\operatorname{ad} x_k)^2x_j = (\operatorname{ad} x_k)^2x_l = (\operatorname{ad} x_j)x_l = 0$, we do not consider hyperwords with $x_jx_k^2$, $x_k^2x_l$ and x_jx_l as factors of the corresponding words. Then $[x_jx_kx_lx_k]_c$ is a linear combination of

$$[x_kx_lx_kx_j]_c = [x_kx_l]_cx_kx_j, \qquad [x_lx_kx_jx_k]_c = x_lx_k[x_jx_k]_c,$$
$$[x_kx_jx_kx_l]_c = x_k[x_jx_kx_l]_c, \qquad [x_lx_k^2x_j]_c = x_lx_k^2x_j.$$

Since $D_j([x_jx_kx_lx_k]_c) = D_j(x_k[x_jx_kx_l]_c) = D_j(x_lx_k[x_jx_k]_c) = 0$, in that linear combination there are no hyperwords ending in x_j; indeed,

$$D_j([x_kx_l]_cx_kx_j) = [x_kx_l]_cx_k, \quad D_j(x_lx_k^2x_j) = x_lx_k^2,$$

and $[x_kx_l]_cx_k$, $x_lx_k^2$ are linearly independent. Therefore, there exist $\alpha, \beta \in \mathsf{k}$ such that

$$[x_jx_kx_lx_k]_c = \alpha x_lx_k[x_jx_k]_c + \beta x_k[x_jx_kx_l]_c.$$

Applying D_l, we have

$$0 = \alpha q_{kj}q_{kk}x_l[x_jx_k]_c + \alpha(1 - q_{kj}q_{jk})x_lx_kx_j + \beta q_{kk}q_{kj}q_{kl}[x_jx_kx_l]_c.$$

Now $x_l[x_jx_k]_c$, $x_lx_kx_j$ and $[x_jx_kx_l]_c$ are linearly independent by Lemma 2.7, so $\alpha = \beta = 0$.

(2) We assume that some quantum Serre relations hold in $\mathfrak{B}$; using them we get

$$\begin{aligned} x_jx_kx_lx_k &= q_{kl}^{-1}(1+q_{kk})^{-1}x_jx_k^2x_l + q_{kk}q_{kj}(1+q_{kk})^{-1}x_jx_lx_k^2 \\ &= q_{kk}^{-1}q_{kj}^{-1}q_{kl}^{-1}x_kx_jx_kx_l + q_{kk}^{-1}q_{kj}^{-1}q_{kl}^{-1}(1+q_{kk})^{-1}x_k^2x_jx_l \\ &\qquad + q_{kk}q_{kl}q_{jk}(1+q_{kk})^{-1}x_lx_jx_k^2. \end{aligned}$$

It follows that $x_kx_jx_kx_l \notin G_I$ for an order such that $x_j < x_k < x_l$. Also, $x_jx_lx_k^2 \notin G_I$, since $(\operatorname{ad}_c x_j)x_l = 0$ and (5-4) is valid by part (1).

(3) If V is a standard braided vector space satisfying the conditions of the lemma, consider V_k as the braided vector space obtained transforming by s_k. Then $\widetilde{m}_{jl} = 0$. Therefore $\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B}(V_k))$, so $s_k(\mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B}(V))$. It follows that $x_jx_kx_lx_k$ is a linear combination of greater words, since it is a Lyndon word when we consider an order such that $x_j < x_k < x_l$. □

We now prove two relations involving the double edge in the Dynkin diagram of a standard braiding of type B_θ.

Lemma 5.5. *Let $I \in \mathfrak{I}$ and $\mathfrak{B} = T(V)/I$. Assume that*

- *there exist $j \neq k \in \{1, \dots, \theta\}$ such that $m_{kj} = 2, m_{jk} = 1$;*

- *there exist skew derivations as in Proposition 2.1*;
- *the following relations hold in* $\mathfrak{B}$:

$$(\operatorname{ad} x_k)^3 x_j = (\operatorname{ad} x_j)^2 x_k = 0; \tag{5-5}$$
$$x_k^3 = x_j^2 = 0 \qquad \text{if } q_{kk}^3 = q_{jj}^2 = 1.$$

(1) *If we order the letters* $x_1, \dots, x_\theta$ *such that* $x_k < x_j$, *then* $x_k^2 x_j x_k x_j$ *is a linear combination of greater words modulo* I *if and only if*, *in* $\mathfrak{B}$,

$$\big[(\operatorname{ad} x_k)^2 x_j, (\operatorname{ad} x_k) x_j\big]_c = 0. \tag{5-6}$$

(2) *If* V *is standard*, $q_{jj} \neq -1$ *and* $q_{kk}^2 q_{kj} q_{jk} = 1$, *then* (5-6) *holds in* $\mathfrak{B}$.

(3) *If* V *is standard and* $\dim \mathfrak{B}(V) < \infty$, *then* (5-6) *holds in* $\mathfrak{B} = \mathfrak{B}(V)$.

Proof. (1) ($\Leftarrow$) If (5-6) holds in $\mathfrak{B}$, then $x_k^2 x_j x_k x_j$ is a linear combination of greater words. This follows from Remark 1.7, and

$$[x_k^2 x_j x_k x_j]_c = \big[[x_k^2 x_j]_c, [x_k x_j]_c\big]_c = \big[(\operatorname{ad} x_k)^2 x_j, (\operatorname{ad} x_k) x_j\big]_c .$$

($\Rightarrow$) If $x_k^2 x_j x_k x_j$ is a linear combination of greater words, then $[x_k^2 x_j x_k x_j]_c$ is a linear combination of hyperwords corresponding to words greater than $x_k^2 x_j x_k x_j$ (of the same degree, because $\mathfrak{B}$ is homogeneous).

First, there are no hyperwords whose corresponding words have factors $x_k^3 x_j$ or $x_k x_j^2$, by (5-5). Since $[x_k^2 x_j x_k x_j]_c \in \ker D_k$ and

$$D_k(x_j [x_k^2 x_j]_c x_k) = x_j [x_k^2 x_j]_c,$$
$$D_k([x_k x_j]_c^2 x_k) = [x_k x_j]_c^2,$$
$$D_k(x_j [x_k x_j]_c x_k^2) = (1 + q_{kk}) x_j [x_k x_j]_c x_k,$$

in that linear combination there are no hyperwords ending in x_k, except $x_j^2 x_k^3$ if $q_{kk} \in \mathbb{G}_3$. We consider $q_{jj} \neq -1$ if $q_{kk} \in \mathbb{G}_3$, since otherwise $x_j^2 x_k^3 = 0$ by assumption. Then there exist $\alpha, \alpha' \in \Bbbk$ such that

$$[x_k^2 x_j x_k x_j]_c = \alpha [x_k x_j x_k^2 x_j]_c + \alpha' x_j^2 x_k^3 = \alpha [x_k x_j]_c [x_k^2 x_j]_c + \alpha' x_j^2 x_k^3.$$

We prove by direct calculation that $D_j([x_k^2 x_j x_k x_j]_c) = 0$. Applying D_j to the previous equality,

$$\begin{aligned} 0 =& \alpha'(1 + q_{jj}) x_j x_k^3 + \chi(\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k + \mathbf{e}_j) \alpha (\operatorname{ad} x_k)^2 (x_j) x_k \\ &+ (1 - q_{kj} q_{jk})(1 - q_{kk} q_{kj} q_{jk}) \alpha (\operatorname{ad} x_k)(x_j) x_k^2, \end{aligned}$$

where we use that $(\operatorname{ad} x_k)^3 (x_j) = 0$ and

$$x_k (\operatorname{ad} x_k)^m (x_j) = (\operatorname{ad} x_k)^{m+1} (x_j) + q_{kk}^m q_{kj} (\operatorname{ad} x_k)^m (x_j) x_k.$$

Since $(1-q_{kj}q_{jk})(1-q_{kk}q_{kj}q_{jk}) \neq 0$ and $(\operatorname{ad} x_k)^2(x_j)x_k$, $(\operatorname{ad} x_k)(x_j)x_k^2$, $x_jx_k^3$ are linearly independent, it follows that $\alpha = \alpha' = 0$.

(2) Using $(\operatorname{ad} x_j)^2 x_k = 0$ in the first equality and $(\operatorname{ad} x_k)^3 x_j = 0$ in the last expression,

$$\begin{aligned} x_k^2x_jx_kx_j &= (1+q_{jj})^{-1}q_{jk}^{-1}x_k^2x_j^2x_k + (1+q_{jj})^{-1}q_{jk}q_{jj}x_k^3x_j^2 \\ &\in (3)_{q_{kk}}(1+q_{jj})^{-1}q_{kj}q_{jk}q_{jj}x_k^2x_jx_kx_j + \mathsf{k}\mathbb{X}_{>x_k^2x_jx_kx_j}. \end{aligned}$$

Suppose that $(3)_{q_{kk}}(1+q_{jj})^{-1}q_{kj}q_{jk}q_{jj} = 1$; that is, $(3)_{q_{kk}} = (1+q_{jj})$. Then $q_{jj} = q_{kk} + q_{kk}^2$, so

$$1 = q_{jj}q_{kj}q_{jk} = q_{kk}q_{kj}q_{jk} + q_{kk}^2q_{kj}q_{jk} = q_{kk}q_{kj}q_{jk} + 1,$$

which is a contradiction since $q_{kk}q_{kj}q_{jk} \in \mathsf{k}^\times$. It follows that $x_k^2x_jx_kx_j$ is a linear combination of greater words, so (5-6) follows by previous item.

(3) If V is a standard braided vector space, and we consider V_j as the braided vector space obtained transforming by s_j, then $\widetilde{m}_{kj} = 2$. Therefore, $3\mathbf{e}_k + \mathbf{e}_j \notin \Delta^+(\mathfrak{B}(V_k))$, so $s_j(3\mathbf{e}_k+\mathbf{e}_j) = 3\mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B}(V))$. Since $x_k^2x_jx_kx_j$ is a Lyndon word of degree $3\mathbf{e}_k + 2\mathbf{e}_j$ if $x_k < x_j$, then it is a linear combination of greater words. □

Lemma 5.6. *Let $I \in \mathfrak{I}$ and $\mathfrak{B} = T(V)/I$. Assume that*

- *there exist different $j, k, l \in \{1, \dots, \theta\}$ such that $m_{kj} = 2$, $m_{jk} = m_{jl} = m_{lj} = 1$, $m_{kl} = 0$;*
- *there exist skew derivations in $\mathfrak{B}$ as in Proposition 2.1;*
- *the following relations hold in $\mathfrak{B}$:* (5-4), (5-6),

$$\begin{aligned} (\operatorname{ad} x_k)^3 x_j = (\operatorname{ad} x_j)^2 x_k = (\operatorname{ad} x_j)^2 x_l = (\operatorname{ad} x_k) x_l = 0, \\ x_k^3 = x_j^2 = 0 \ \text{ if } q_{kk}^3 = q_{jj}^2 = 1. \end{aligned} \tag{5-7}$$

(1) *If we order the letters $x_1, \dots, x_\theta$ so that $x_k < x_j < x_l$, then $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words modulo I if and only if, in $\mathfrak{B}$,*

$$\left[(\operatorname{ad} x_k)^2(\operatorname{ad} x_j)x_l, (\operatorname{ad} x_k)x_j\right]_c = 0. \tag{5-8}$$

(2) *If V is a standard braided vector space and $q_{kk} \notin \mathbb{G}_3$, $q_{jj} \neq -1$, then* (5-8) *holds in $\mathfrak{B}$.*

(3) *If V is standard and* $\dim \mathfrak{B}(V) < \infty$, *then* (5-8) *holds in* $\mathfrak{B}(V)$.

Proof. (1) ($\Leftarrow$) As in the last two lemmas, if (5-8) is valid, then $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words, by Remark 1.7, and

$$[x_k^2x_jx_lx_kx_j]_c = \left[(\operatorname{ad} x_k)^2(\operatorname{ad} x_j)x_l, (\operatorname{ad} x_k)x_j\right]_c.$$

(⇒) Suppose that $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words. Then $[x_k^2x_jx_lx_kx_j]_c$ is a linear combination of hyperwords corresponding to words greater than $x_k^2x_jx_lx_kx_j$ (of the same degree, because $\mathfrak{B}$ is homogeneous). We discard those words which have x_kx_l, $x_k^3x_j$, $x_kx_j^2$, $x_j^2x_l$, $x_kx_jx_lx_j$ and $x_k^2x_jx_kx_j$, in view of our hypotheses about $\mathfrak{B}$.

Since $D_k([x_k^2x_jx_lx_kx_j]_c)=0$, the coefficients of hyperwords corresponding to words ending in x_k are 0, as in Lemma 5.5, except for $[x_jx_l]_cx_jx_k^3$, $x_lx_j^2x_k^3$, if $q_{kk}\in\mathbb{G}_3$. Thus

$$[x_k^2x_jx_lx_kx_j]_c=\alpha[x_kx_j]_c[x_k^2x_jx_l]_c+\beta[x_kx_jx_l]_c[x_k^2x_j]_c + \gamma\, x_l[x_kx_j]_c[x_k^2x_j]_c+\mu[x_jx_l]_cx_jx_k^3+\nu x_lx_j^2x_k^3.$$

By direct calculation, $D_j([x_k^2x_jx_lx_kx_j]_c)=D_j([x_k^2x_jx_l]_c)=D_j([x_kx_jx_l]_c)=0$, so applying D_j to the previous equality we get

$$0=\alpha q_{jk}^2q_{jj}q_{jl}x_j[x_k^2x_jx_l]_c+\beta(1-q_{kk}q_{kj}q_{jk})(1-q_{kk}^2q_{kj}q_{jk})[x_kx_jx_l]_cx_k^2 + \gamma\,(1-q_{kk}q_{kj}q_{jk})(1-q_{kk}^2q_{kj}q_{jk})x_l[x_kx_j]_cx_k^2+\gamma\, q_{jk}^2q_{jj}x_lx_j[x_k^2x_j]_c + \mu[x_jx_l]_cx_k^3+\nu(1+q_{jj})x_lx_jx_k^3,$$

Note that $\nu=0$ if $q_{jj}\neq-1$; otherwise, $x_j^2=0$ by hypothesis, so we can discard this last summand. The other hyperwords appearing in this expression are linearly independent, since the corresponding words are linearly independent by Lemma 2.7. Thus $\alpha=\beta=\gamma=\mu=0$.

(2) If $q_{kk}\notin\mathbb{G}_3$ and $q_{jj}\neq-1$, then $x_k^2x_jx_lx_kx_j$ is a linear combination of greater words, as can be seen using the quantum Serre relations in a way similar to that in Lemma 5.6. Now apply part (1).

(3) If V is a standard braided vector space, consider V_k as the braided vector space obtained transforming by s_k. Then $\widetilde{m}_{kj}=2$. Therefore, $\mathbf{e}_k+2\mathbf{e}_j+\mathbf{e}_l\notin\Delta^+(\mathfrak{B}(V_k))$ by Lemma 5.5, so $s_k(\mathbf{e}_k+2\mathbf{e}_j+\mathbf{e}_l)=3\mathbf{e}_k+2\mathbf{e}_j+\mathbf{e}_l\notin\Delta^+(\mathfrak{B}(V))$. Since $x_k^2x_jx_lx_kx_j$ is a Lyndon word, it follows that it is a linear combination of greater words, and we apply (1). □

We now give explicit formulas for the comultiplication of these hyperwords.

Lemma 5.7. *Consider the structure of graded braided Hopf algebra of* $T(V)$ (see Section 2A). *For all* $k\neq j$,

$$\Delta((\operatorname{ad}x_k)^{m_{kj}+1}x_j)=(\operatorname{ad}x_k)^{m_{kj}+1}x_j\otimes 1\,+\,1\otimes(\operatorname{ad}x_k)^{m_{kj}+1}x_j + \prod_{1\leq t\leq m_{kj}}(1-q_{kk}^tq_{kj}q_{jk})x_k^{m_{ij}+1}\otimes x_j. \quad (5\text{-}9)$$

Proof. We have $F_k((\operatorname{ad}x_k)^{m_{kj}+1}x_j)=0$ by the definition of m_{kj} and (2-5). Also, $F_l((\operatorname{ad}x_k)^{m_{kj}+1}x_j)$ for $l\neq k$ by (2-6) and the properties of F_l, so

$$\Delta_{1,m_{kj}}((\operatorname{ad} x_k)^{m_{kj}+1}x_j) = \sum_{l=1}^{\theta} x_l \otimes F_l((\operatorname{ad} x_k)^{m_{kj}+1}x_j) = 0.$$

Now $D_k([x_k^i x_j]_c x_k^{s-i}) = 0$ from (2-3), and from (2-4)

$$D_j([x_k^i x_j]_c x_k^{s-i}) = \prod_{1\le t\le m_{kj}} (1 - q_{kk}^t q_{kj} q_{jk}) x_k^{m_{ij}+1},$$

so we deduce that

$$\Delta_{m_{kj},1}((\operatorname{ad} x_k)^{m_{kj}+1}x_j) = \prod_{1\le t\le m_{kj}} (1 - q_{kk}^t q_{kj} q_{jk}) x_k^{m_{ij}+1} \otimes x_j.$$

Since hyperwords form a basis of $T(V)$, we can write, for each $1 < s < m_{kj}$,

$$\begin{aligned}\Delta_{m_{kj}+1-s,s}&((\operatorname{ad} x_k)^{m_{kj}+1}x_j)\\ &= \sum_{t=0}^{m_{kj}+1-s} \varepsilon_{st}[x_k^t x_j]_c x_k^{m_{kj}+1-s-t} \otimes x_k^s + \sum_{p=0}^{s} \rho_{sp} x_k^{m_{kj}+1-s} \otimes [x_k^{s-p} x_j]_c x_k^p,\end{aligned}$$

for some $\varepsilon_{st}, \rho_{sp} \in \mathsf{k}$. Then, for each $0 \le t \le m_{kj}+1-s$,

$$\begin{aligned} 0 &= \big((\operatorname{ad} x_k)^{m_{kj}+1}x_j \;\big|\; [x_k^t x_j]_c x_k^{m_{kj}+1-t} x_k^s\big)\\ &= \big(((\operatorname{ad} x_k)^{m_{kj}+1}x_j)_{(1)} \;\big|\; [x_k^t x_j]_c x_k^{m_{kj}+1-t-s}\big)\big(((\operatorname{ad} x_k)^{m_{kj}+1}x_j)_{(2)} \;\big|\; x_k^s\big)\\ &= \varepsilon_{st}\big([x_k^t x_j]_c x_k^{m_{kj}+1-t-s} \;\big|\; [x_k^t x_j]_c x_k^{m_{kj}+1-t-s}\big)(x_k^s | x_k^s)\\ &= \varepsilon_{st}(m_{kj}+1-s-t)_{q_{kk}}!\,(s)_{q_{kk}}!\,\big([x_k^t x_j]_c \;\big|\; [x_k^t x_j]_c\big),\end{aligned}$$

where we have used that $(\operatorname{ad} x_k)^{m_{kj}+1}x_j \in I(V)$ for the first equality, (1-8) for the second, (1-10) and the orthogonality between increasing products of hyperwords for the third, and (5-2) for the last. Since

$$(m_{kj}+1-s-t)_{q_{kk}}!\,(s)_{q_{kk}}!\,([x_k^t x_j]_c | [x_k^t x_j]_c) \neq 0,$$

we conclude that $\varepsilon_{st} = 0$ for all $0 \le t \le m_{kj}+1-s$. In a similar way, $\rho_{sp} = 0$ for all $0 \le p \le s$, so we obtain (5-9). □

Lemma 5.8. *Let $\mathfrak{B}$ be a braided graded Hopf algebra provided with an inclusion of braided vector spaces $V \hookrightarrow \mathcal{P}(\mathfrak{B})$. Assume that*

- *there exist $1 \le j \neq k \neq l \le \theta$ such that $m_{kj} = m_{kl} = 1$, $m_{jl} = 0$;*
- *$(\operatorname{ad} x_k)^2 x_j = (\operatorname{ad} x_k)^2 x_l = (\operatorname{ad} x_j) x_l = 0$ in $\mathfrak{B}$;*
- *$x_k^2 = 0$ if $q_{kk}q_{kj}q_{jk} \neq 1$ or $q_{kk}q_{kl}q_{lk} \neq 1$.*

Then $u := [(\operatorname{ad} x_j)(\operatorname{ad} x_k) x_l, x_k]_c \in \mathcal{P}(\mathfrak{B})$.

Proof. From (2-3), $D_j(u)=0$. Also, $D_k((\operatorname{ad} x_j)(\operatorname{ad} x_k)x_l)=0$, so

$$D_k(u)=(1-q_{kk}^2q_{jk}q_{kj}q_{kl}q_{lk})(\operatorname{ad} x_j)(\operatorname{ad} x_k)x_l=0.$$

From (2-4) and the properties of D_l we have

$$\begin{aligned} D_l(u) &= q_{lk}(1-q_{kl}q_{lk})[x_jx_k]_c x_k - q_{jk}q_{kk}q_{lk}(1-q_{kl}q_{lk})x_k[x_jx_k]_c \\ &= q_{lk}(1-q_{lk}q_{kl})\big[[x_jx_k]_c, x_k\big]_c = 0. \end{aligned}$$

Then $\Delta_{31}(u)=0$. From (2-6) and the properties of F_k and F_l, we have $F_k(u)=F_l(u)=0$. Using (2-5), we have

$$\begin{aligned} F_j(u) &= (1-q_{jk}q_{kj})[x_kx_l]_c x_k - q_{jk}q_{kk}q_{lk}q_{kj}(1-q_{jk}q_{kj})x_k[x_kx_l]_c \\ &= (1-q_{lk}q_{kl})(1-q_{kj}q_{jk}q_{kk}^2q_{lk}q_{jk})[x_kx_l]_c x_k = 0. \end{aligned}$$

Thus $\Delta_{13}(u)=0$ as well.

Also, we have

$$\Delta(u)=\Delta\big((\operatorname{ad} x_j)(\operatorname{ad} x_k)x_l\big)\Delta(x_k)-q_{\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_j,\mathbf{e}_j}\Delta(x_k)\Delta\big((\operatorname{ad} x_j)(\operatorname{ad} x_k)x_l\big),$$

and looking at the terms in $\mathfrak{B}^2\otimes\mathfrak{B}^2$,

$$\begin{aligned} \Delta_{2,2}(u) &= (1-q_{lk}q_{kl})[x_jx_k]_c\otimes\big(x_lx_k-q_{kj}q_{jk}q_{kk}^2q_{lk}x_kx_l\big) \\ &\qquad + (1-q_{kj}q_{jk})q_{lk}q_{kk}\big(x_jx_k-q_{jk}x_kx_j\big)\otimes[x_kx_l]_c \\ &= \big(1-q_{kj}q_{jk}-(1-q_{lk}q_{kl})\,q_{kk}q_{jk}q_{kj}\big)\,q_{lk}q_{kk}[x_jx_k]_c\otimes[x_kx_l]_c. \end{aligned}$$

Now a calculation shows that $u\in\mathcal{P}(\mathfrak{B})$:

$$\begin{aligned} 1-q_{kj}q_{jk}-(1-q_{lk}q_{kl})q_{kk}q_{jk}q_{kj} &= 1-q_{kj}q_{jk}-q_{kk}q_{jk}q_{kj}+q_{kk}^{-1} \\ &= q_{kk}^{-1}(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})=0. \qquad \square \end{aligned}$$

Lemma 5.9. *Let $\mathfrak{B}$ be a braided graded Hopf algebra provided with an inclusion of braided vector spaces $V\hookrightarrow\mathcal{P}(\mathfrak{B})$. Assume that*

- *there exist $1\le k\ne j\le\theta$ such that $m_{kj}=2$, $m_{jk}=1$;*
- $(\operatorname{ad} x_s)^{m_{st}+1}x_t=0$, *for all $1\le s\ne t\le\theta$ in $\mathfrak{B}$;*
- $x_s^{m_{st}+1}=0$ *for each s such that $q_{ss}^{m_{st}}q_{st}q_{ts}\ne 1$, for some $t\ne s$.*

(a) *If $v:=\big[(\operatorname{ad} x_k)^2x_j, (\operatorname{ad} x_k)x_j\big]_c$, there exists $b\in\Bbbk$ such that*

$$\Delta(v)=v\otimes 1+1\otimes v+b(1-q_{kk}^2q_{kj}^2q_{jk}^2q_{jj})x_k^3\otimes x_j^2. \tag{5-10}$$

(b) *Assume there exist $l\ne j,k$ such that $m_{jl}=m_{lj}=1$, $m_{kl}=m_{lk}=0$, and that* (5-4) *is valid in $\mathfrak{B}$. Set*

$$w:=\big[(\operatorname{ad} x_k)^2(\operatorname{ad} x_j)x_l, (\operatorname{ad} x_k)x_j\big]_c.$$

Then there exist constants $b_1, b_2 \in \mathsf{k}$ *such that*

$$\Delta(w) = w \otimes 1 + 1 \otimes w + b_1 v \otimes x_l + b_2(1 - q_{kk}^2 q_{kj} q_{jk}) x_k^3 \otimes \big((\operatorname{ad} x_j) x_l\big)\, x_j. \quad (5\text{-}11)$$

Proof. (a) $F_j(v) = 0$ since v is a braided commutator of two elements in $\ker F_j$. Using (1-4) we have $[(\operatorname{ad} x_k)^2 x_j, x_j]_c = q_{kj}(q_{jj} - q_{kk})[x_k x_j]_c^2$, so we calculate

$$\begin{aligned} F_k(v) &= (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})[x_kx_j]_c^2 - q_{kk}^2 q_{kj}^2 q_{jk} q_{jj}(1-q_{kj}q_{jk}) x_j [x_k^2 x_j]_c \\ &\quad + q_{kk}^2 q_{jk}(1-q_{kj}q_{jk})[x_k^2 x_j]_c x_j - q_{kk}^3 q_{kj}^2 q_{jk}^2 q_{jj}(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})[x_kx_j]_c^2 \\ &= q_{kk}^2 q_{jk} q_{kj}(1-q_{kj}q_{jk})(q_{jj}-q_{kk}) \\ &\quad + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})(1-q_{kk}^3 q_{kj}^2 q_{jk}^2 q_{jj})[x_kx_j]_c^2, \end{aligned}$$

which vanishes since the coefficient of $[x_k x_j]_c^2$ is zero for each possible braiding. Thus

$$\Delta_{1,4}(v) = x_k \otimes F_k(v) = 0.$$

Also, $D_k(v) = 0$, and a calculation gives

$$\begin{aligned} D_j(v) &= (1 - q_{kj}q_{jk})\big([x_k^2 x_j] x_k + (1 - q_{kk}q_{kj}q_{jk}) q_{jk} q_{jj} x_k^2 [x_k x_j]_c \\ &\quad - q_{kk}^2 q_{kj}^2 q_{jk} q_{jj}(1 - q_{kk}q_{kj}q_{jk})[x_kx_j]_c x_k^2 - q_{kk}^2 q_{kj}^2 q_{jk}^3 q_{jj}^2 x_k [x_k^2 x_j]\big) \\ &= \big(1 + (1+q_{kk})(1 - q_{kk}q_{kj}q_{jk}) q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2\big) \\ &\quad (1 - q_{kj}q_{jk})[x_k^2 x_j] x_k, \end{aligned}$$

where we have reordered the hyperwords and used that $(\operatorname{ad} x_k)^3 x_j = 0$; also,

$$1 + (1+q_{kk})(1 - q_{kk}q_{kj}q_{jk}) q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2 = 0, \quad (5\text{-}12)$$

by calculation for each possible braiding. Thus

$$\Delta_{4,1}(v) = D_j(v) \otimes x_j = 0.$$

To finish, we use the fact that $\Delta(v)$ equals

$$\Delta((\operatorname{ad}_c x_k)^2 x_j)\,\Delta((\operatorname{ad}_c x_k) x_j) - \chi(2e_k + e_j, e_k + e_j)\,\Delta((\operatorname{ad}_c x_k) x_j)\,\Delta((\operatorname{ad}_c x_k)^2 x_j).$$

Looking at the terms in $\mathfrak{B}^3 \otimes \mathfrak{B}^2$ and $\mathfrak{B}^2 \otimes \mathfrak{B}^3$, and using the definition of the braided commutator, we obtain

$$\begin{aligned} &\Delta_{32}(v) \\ &= (1 - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2)[x_k^2 x_j]_c \otimes [x_k x_j]_c \\ &\quad + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk}) q_{kk}q_{kj}q_{jk}q_{jj}\big(x_k[x_kx_j]_c - q_{kk}q_{kj}[x_kx_j]_c x_k\big) \otimes [x_kx_j]_c \\ &\quad + (1 - q_{kj}q_{jk})^2 (1 - q_{kk}^2 q_{kj} q_{jk})(1 - q_{kk}^2 q_{kj}^2 q_{jk}^2 q_{jj}) x_k^3 \otimes x_j^2 \\ &= \big(1 + (1+q_{kk})(1 - q_{kk}q_{kj}q_{jk}) q_{kk}q_{kj}q_{jk}q_{jj} - q_{kk}^4 q_{kj}^3 q_{jk}^3 q_{jj}^2\big)[x_k^2 x_j]_c \otimes [x_k x_j]_c \\ &\quad + b_1(1 - q_{kk}^2 q_{kj}^2 q_{jk}^2 q_{jj}) x_k^3 \otimes x_j^2. \end{aligned}$$

Also,

$$\begin{aligned}\Delta_{23}(v) &= (1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})x_k^2\\ &\qquad\otimes\big((1+q_{kk})q_{kk}q_{jk}[x_kx_j]_c x_j-(1+q_{kk})q_{kk}^2q_{kj}^2q_{jk}^2q_{jj}x_j[x_kx_j]_c\\ &\qquad\qquad+x_j[x_kx_j]_c-q_{kk}^4q_{kj}^2q_{jk}^3q_{jj}[x_kx_j]_c x_j\big)\\ &=\big(1-q_{kk}^4q_{kj}^3q_{jk}^3q_{jj}^2+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}q_{jj}\big)\\ &\qquad(1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})x_k^2\otimes x_j[x_kx_j]_c.\end{aligned}$$

Using (5-12), we obtain (5-10).

(b) We set $y=(\operatorname{ad}x_k)^2(\operatorname{ad}x_j)x_l$ and $z=(\operatorname{ad}x_k)x_j$. Note that $\Delta(w)=\Delta(y)\Delta(z)-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\Delta(z)\Delta(y)$ and that

$$\begin{aligned}\Delta(y) &= y\otimes 1+(1-q_{jl}q_{lj})(\operatorname{ad}x_k)^2x_j\otimes x_l\\ &\qquad+(1-q_{kj}q_{jk})(1-q_{kk}q_{kj}q_{jk})x_k^2\otimes(\operatorname{ad}x_j)x_l\\ &\qquad\qquad+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})x_k\otimes(\operatorname{ad}x_k)(\operatorname{ad}x_j)x_l+1\otimes y,\\ \Delta(z) &= z\otimes 1+(1-q_{kj}q_{jk})x_k\otimes x_j+1\otimes z.\end{aligned}$$

From (2-3) we have $D_k(w)=0$, and from (2-4),

$$\begin{aligned}D_l(w) &= (1-q_{lj}q_{jl})q_{lk}q_{lj}\left[(\operatorname{ad}x_k)^2x_j,(\operatorname{ad}x_k)x_j\right]_c,\\ D_j(w) &= -(1-q_{kj}q_{jk})q_{kk}^{-2}q_{kj}^{-1}q_{kl}^{-1}(\operatorname{ad}x_k)^3(\operatorname{ad}x_j)x_l\\ &= -(1-q_{kj}q_{jk})q_{kk}^{-2}q_{kj}^{-1}q_{kl}^{-1}[(\operatorname{ad}x_k)^3x_j,x_l]_c=0,\end{aligned}$$

where in the last equality we used (1-4) and the vanishing of $[x_k,x_l]_c=0$. It follows that

$$\Delta_{51}(w)=(1-q_{lj}q_{jl})q_{lk}q_{lj}\left[(\operatorname{ad}x_k)^2x_j,(\operatorname{ad}x_k)x_j\right]_c\otimes x_l.$$

Also, $F_j(z)=F_j(y)=F_l(z)=F_l(y)=0$ by (2-6) and the properties of these skew derivations, so $F_j(w)=F_l(w)=0$. We now calculate

$$\begin{aligned}&F_k(w)\\ &\quad=(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})[x_kx_jx_l]_c[x_kx_j]_c+q_{kk}^2q_{jk}q_{lk}(1-q_{kj}q_{jk})[x_k^2x_jx_l]_c x_j\\ &\qquad-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l,\mathbf{e}_k+\mathbf{e}_j)\\ &\qquad\quad\big((1-q_{kj}q_{jk})x_j[x_k^2x_jx_l]_c+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{jk}[x_kx_jx_l]_c[x_kx_j]_c\big)\\ &\quad=q_{kk}^2q_{jk}q_{lk}(1-q_{kj}q_{jk})\big[[x_k^2x_jx_l]_c,x_j\big]_c\\ &\qquad\qquad-(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}^3q_{kj}^2q_{jk}^2q_{jj}q_{lj}q_{lk}\big[[x_kx_j]_c,[x_kx_jx_l]_c\big]_c\\ &\quad=q_{kk}^2q_{kj}q_{jk}q_{jj}q_{lj}q_{lk}\\ &\qquad\qquad\big(1-q_{kj}q_{jk}-(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}\big)\big[[x_kx_j]_c,[x_kx_jx_l]_c\big]_c\\ &\quad=0,\end{aligned}$$

where we used (1-4) and (5-4) in the third equality, and we calculate that

$$1 - q_{kj}q_{jk} - (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk} = 0 \tag{5-13}$$

for each possible standard braiding. It follows that $\Delta_{15}(w) = 0$.

We find each of the other terms of $\Delta(w)$ by direct calculation. First,

$$\begin{aligned}
&\Delta_{42}(w)\\
&\quad = \big(1-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)\chi(\mathbf{e}_k+\mathbf{e}_j, 2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l)\big)\, y\otimes z\\
&\qquad + (1-q_{kj}q_{jk})(1-q_{lj}q_{jl})\\
&\qquad\qquad \big(q_{lk}[x_k^2x_j]_c x_k\otimes x_lx_j - \chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)q_{jk}^2q_{jj}x_k[x_k^2x_j]_c\otimes x_jx_l\big)\\
&\qquad + (1-q_{kj}q_{jk})(1-q_{kk}q_{kj}q_{jk})\\
&\qquad\qquad \big(\chi(\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)x_k^2z - \chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)zx_k^2\big)\otimes[x_jx_l]_c\\
&\quad = (1-q_{kj}q_{jk})q_{lk}\big(1-q_{jk}q_{kj}+(1+q_{kk})(1-q_{kk}q_{kj}q_{jk})q_{kk}q_{kj}q_{jk}\big)\\
&\qquad\qquad [x_k^2x_j]_cx_k\otimes[x_jx_l]_c,
\end{aligned}$$

which is seen to equal 0. In a similar way we calculate

$$\begin{aligned}
&\Delta_{33}(w)\\
&\quad = (1-q_{lj}q_{jl})[x_k^2x_j]\otimes\big(x_lz-\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)\chi(\mathbf{e}_k+\mathbf{e}_j, \mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l)zx_l\big)\\
&\qquad + (1+q_{kk})(1-q_{kk}q_{kj}q_{jk})\chi(\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)(x_kz-q_{kk}q_{kj}zx_k)\otimes[x_kx_jx_l]_c\\
&\qquad + (1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})^2x_k^3\\
&\qquad\qquad \otimes\big(\chi(\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k)[x_jx_l]_cx_j - \chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)\chi(\mathbf{e}_j, 2\mathbf{e}_k)x_j[x_jx_l]_c\big)\\
&\quad = \big((1+q_{kk})(1-q_{kk}q_{kj}q_{jk})-q_{kk}q_{kj}q_{jk}q_{jj}(1-q_{lj}q_{jl})\big)\\
&\qquad\qquad \chi(\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)[x_k^2x_j]_c\otimes[x_kx_jx_l]_c\\
&\qquad + (1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})^2(1-q_{kk}^2q_{kj}q_{jk})x_k^3\otimes[x_jx_l]_cx_j,
\end{aligned}$$

and the coefficient of $[x_k^2x_j]_c\otimes[x_kx_jx_l]_c$ is zero (we calculate it for each possible standard braiding). Finally,

$$\begin{aligned}
&\Delta_{24}(w)\\
&\quad = (1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})x_k^2\\
&\qquad \otimes\big((1+q_{kk})\chi(\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k)[x_kx_jx_l]_cx_j\\
&\qquad\qquad -(1+q_{kk})\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)q_{jk}x_j[x_kx_jx_l]_c\\
&\qquad\qquad\qquad -\chi(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)\chi(\mathbf{e}_k+\mathbf{e}_j, 2\mathbf{e}_k)\big[[x_kx_j]_c, [x_jx_l]_c\big]_c\big)\\
&\quad = (1-q_{kk}q_{kj}q_{jk})(1-q_{kj}q_{jk})\chi(\mathbf{e}_j+\mathbf{e}_l, \mathbf{e}_k+\mathbf{e}_j)q_{kj}\\
&\qquad\qquad \big(q_{kk}(1-q_{kk}q_{kj}q_{jk})-q_{jj}(1-q_{jl}q_{lj})\big)x_k^2\otimes x_j[x_kx_jx_l]_c\\
&\quad = 0.
\end{aligned}$$

From these calculations, we obtain (5-11). □

5B. ***Presentation when the type is A_θ.*** We now assume V is a standard braided vector space of type A_θ and $\mathfrak{B}$ a $\mathbb{Z}^\theta$-graded algebra, provided with an inclusion of vector spaces $V \hookrightarrow \mathfrak{B}^1 = \bigoplus_{1\le j\le\theta} \mathfrak{B}^{\mathbf{e}_j}$. We can extend the braiding to $\mathfrak{B}$ by setting

$$c(u\otimes v) = \chi(\alpha,\beta) v\otimes u, \quad u\in\mathfrak{B}^\alpha, v\in\mathfrak{B}^\beta,\ \alpha,\beta\in\mathbb{N}^\theta.$$

We assume that on $\mathfrak{B}$ we have

$$\begin{aligned} x_i^2 &= 0 \quad \text{if } q_{ii}=-1,\\ \operatorname{ad}_c x_i(x_j) &= 0 \quad \text{if } |j-i|>1,\\ (\operatorname{ad}_c x_i)^2(x_j)_c &= 0 \quad \text{if } |j-i|=1,\\ \big[(\operatorname{ad}_c x_i)(\operatorname{ad}_c x_{i+1})x_{i+2}, x_{i+1}\big]_c &= 0 \quad 2\le i\le\theta-1. \end{aligned}$$

Using the same notation as in Section 4B,

$$x_{\mathbf{e}_i} = x_i, \qquad x_{\mathbf{u}_{i,j}} := [x_i, x_{\mathbf{u}_{i+1,j}}]_c \quad (i<j).$$

Lemma 5.10. *Let $1\le i\le j<p\le r\le\theta$. The following relations hold in $\mathfrak{B}$:*

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pr}}]_c = 0,\ \ p-j\ge 2; \tag{5-14}$$

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{j+1,r}}]_c = x_{\mathbf{u}_{ir}}. \tag{5-15}$$

Proof. Note that $x_{\mathbf{u}_{pr}}$ belongs to the subalgebra generated by $x_p,\dots,x_r$, and $[x_{\mathbf{u}_{ij}}, x_s]_c = 0$, for each $p\le s\le r$. Equation (5-14) follows from this.

We prove (5-15) by induction on $j-i$: if $i=j$, it is exactly the definition of $x_{\mathbf{u}_{ir}}$. To prove the inductive step, we use the inductive hypothesis, (5-14) and (1-4) (the braided Jacobi identity) to obtain

$$\begin{aligned} [x_{\mathbf{u}_{i,j+1}}, x_{\mathbf{u}_{j+2,r}}]_c &= \big[[x_{\mathbf{u}_{ij}}, x_{i+1}]_c, x_{\mathbf{u}_{j+2,r}}\big]_c = \big[x_{\mathbf{u}_{ij}}, [x_{i+1}, x_{\mathbf{u}_{j+2,r}}]_c\big]_c\\ &= [x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{j+1,r}}]_c = x_{\mathbf{u}_{ir}}, \end{aligned}$$

and (5-15) is also proved. □

Lemma 5.11. *If $i<p\le r<j$, the following relation holds in $\mathfrak{B}$:*

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pr}}]_c = 0. \tag{5-16}$$

Proof. When $p=r=j-1$ and $i=j-2$, note that this is exactly

$$\big[(\operatorname{ad}_c x_i)(\operatorname{ad}_c x_{i+1})x_{i+2}, x_{i+1}\big]_c = 0.$$

Then, by (1-4),

$$[x_{\mathbf{u}_{i-1,j}}, x_{j-1}]_c = [[x_{i-1}, x_{\mathbf{u}_{i,j}}]_c, x_{j-1}]_c = [x_{i-1}, [x_{\mathbf{u}_{i,j}}, x_{j-1}]_c]_c.$$

We assume that $j-i>2$, so $[x_{i-1}, x_{j-1}]_c = 0$ by the hypothesis on $\mathfrak{B}$. Now we prove the case $p=r=j-1$ by induction on $p-i$.

Using (1-4) and (5-15), we also have

$$[x_{\mathbf{u}_{i,j+1}}, x_p]_c = \big[[x_{\mathbf{u}_{i,j}}, x_{j+1}]_c, x_p\big]_c = \big[x_{\mathbf{u}_{i,j}}, [x_{j+1}, x_p]_c\big]_c$$
$$+q_{j+1,p}[x_{\mathbf{u}_{i,j}}, x_{j-1}]_c x_{j+1} - \chi(\mathbf{u}_{i,j}, \mathbf{e}_{j+1}) x_{j+1}[x_{\mathbf{u}_{i,j}}, x_{j-1}]_c,$$

so using that $[x_{j+1}, x_p]_c = 0$ if $j > p$, by induction on $j-p$ we prove (5-16) for the case $p = r$.

For the general case, we use (1-4) one more time as follows

$$[x_{\mathbf{u}_{i,j}}, x_{\mathbf{u}_{p,r+1}}]_c = \big[x_{\mathbf{u}_{i,j}}, [x_{\mathbf{u}_{pr}}, x_{r+1}]_c\big]_c = \big[[x_{\mathbf{u}_{i,j}}, x_{\mathbf{u}_{pr}}]_c, x_{r+1}\big]_c$$
$$-\chi(\mathbf{u}_{pr}, \mathbf{e}_{r+1})[x_{\mathbf{u}_{ij}}, x_{r+1}]_c x_{\mathbf{u}_{pr}} + \chi(\mathbf{u}_{ij}, \mathbf{u}_{pr}) x_{\mathbf{u}_{pr}}[x_{\mathbf{u}_{ij}}, x_{r+1}]_c,$$

and we prove (5-16) by induction on $r-p$. □

Lemma 5.12. *The following relations hold in* $\mathfrak{B}$:

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{ip}}]_c = 0 \quad \text{if } i \le j < p, \tag{5-17}$$
$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{pj}}]_c = 0 \quad \text{if } i < p \le j. \tag{5-18}$$

Proof. To prove (5-17), note that if $i = j = p-1$, we have

$$[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{i,i+1}}]_c = \big[x_i, [x_i, x_{i+1}]_c\big]_c = (\operatorname{ad} x_i)^2 x_{i+1} = 0.$$

Since $[x_i, x_{\mathbf{u}_{i+2,p}}]_c = 0$ for each $p > i+1$ by (5-14), we use (1-4), the previous case and (5-15) to obtain

$$[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{ip}}]_c = \big[x_{\mathbf{u}_{ii}}, [x_{\mathbf{u}_{i,i+1}}, x_{\mathbf{u}_{i+2,p}}]_c\big]_c = 0.$$

Now, if $i < j < p$, from (5-14) and the relations between the q_{st} we obtain

$$[x_{\mathbf{u}_{i+1,j}}, x_{\mathbf{u}_{ip}}]_c = -\chi(\mathbf{u}_{ip}, \mathbf{u}_{i+1,j})[x_{\mathbf{u}_{ip}}, x_{\mathbf{u}_{i+1,j}}]_c = 0.$$

Using (1-4) and the previous case we conclude

$$[x_{\mathbf{u}_{ij}}, x_{\mathbf{u}_{ip}}]_c = \big[[x_{\mathbf{u}_{ii}}, x_{\mathbf{u}_{i+1,j}}]_c, x_{\mathbf{u}_{ip}}\big]_c = 0.$$

The proof of (5-18) is analogous. □

Lemma 5.13. *If* $i < p \le r < j$, *the following relation holds in* $\mathfrak{B}$:

$$[x_{\mathbf{u}_{ir}}, x_{\mathbf{u}_{pj}}]_c = \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr})(1 - q_{r,r+1}q_{r+1,r}) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}}. \tag{5-19}$$

Proof. We calculate

$$\begin{aligned}[x_{\mathbf{u}_{ir}}, x_{\mathbf{u}_{pj}}]_c &= \big[x_{\mathbf{u}_{ir}}, [x_{\mathbf{u}_{pr}}, x_{\mathbf{u}_{r+1,j}}]_c\big]_c \\ &= \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}} - \chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j}) x_{\mathbf{u}_{ij}} x_{\mathbf{u}_{pr}} \\ &= \big(\chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) - \chi(\mathbf{u}_{ij}, \mathbf{u}_{pr})\chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j})\big) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}} \\ &= \chi(\mathbf{u}_{ir}, \mathbf{u}_{pr}) \big(1 - \chi(\mathbf{u}_{pr}, \mathbf{u}_{r+1,j})\chi(\mathbf{u}_{r+1,j}, \mathbf{u}_{pr})\big) x_{\mathbf{u}_{pr}} x_{\mathbf{u}_{ij}},\end{aligned}$$

where we have used (5-15) in the first equality, (1-4) in the second, (5-18) in the third and the relation between the q_{ij} in the last. □

We now prove the main theorem of this subsection, namely, the presentation by generators and relations of the Nichols algebra associated to V.

Theorem 5.14. *Let V be a standard braided vector space of type A_θ, where $\theta = \dim V$, and let $C = (a_{ij})_{i,j\in\{1,\dots,\theta\}}$ be the corresponding Cartan matrix of type A_θ.*

The Nichols algebra $\mathfrak{B}(V)$ is presented by the generators x_i, $1 \le i \le \theta$, and the relations

$$x_\alpha^{N_\alpha} = 0, \quad \alpha \in \Delta^+;$$
$$\mathrm{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0, \quad i \neq j;$$
$$\big[(\mathrm{ad}\, x_{j-1})(\mathrm{ad}\, x_j)x_{j+1}, x_j\big]_c = 0, \quad 1 < j < \theta,\ q_{jj} = -1.$$

The following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \quad \text{where } 0 \le h_j < N_{\beta_j} \text{ where } \beta_j \in S_I, \text{ for } 1 \le j \le P. \tag{5-20}$$

Proof. From Corollary 4.2 and the definitions of the x_α, we know that the last statement about the PBW basis is true.

Let $\mathfrak{B}$ be the algebra presented by generators $x_1, \dots, x_\theta$ and the relations in the statement of the theorem. From Lemmas 5.3, 5.4 and Corollary 5.2 we have a canonical epimorphism $\phi : \mathfrak{B} \to \mathfrak{B}(V)$. The last relation also holds in $\mathfrak{B}$ for $q_{jj} \neq 1$, by Lemma 5.4(2).

The rest is similar to the proofs of [Andruskiewitsch and Dăscălescu 2005, Lemma 3.7] and [Andruskiewitsch and Schneider 2002b, Lemma 6.12]. Consider the subspace $\mathcal{I}$ of $\mathfrak{B}$ generated by the elements in (5-20). Using Lemmas 5.10, 5.11, 5.12 and 5.13 we prove that $\mathcal{I}$ is an ideal. But $1 \in \mathcal{I}$, so $\mathcal{I} = \mathfrak{B}$.

The images under ϕ of the elements in (5-20) form a basis of $\mathfrak{B}(V)$, so ϕ is an isomorphism. □

The presentation and dimension of $\mathfrak{B}(V)$ agree with the results presented in [Andruskiewitsch and Dăscălescu 2005] and [Andruskiewitsch and Schneider 2002b].

5C. ***Presentation when the type is B_θ.*** We now assume V is a standard braided vector space of type B_θ and $\mathfrak{B}$ is a $\mathbb{Z}^\theta$-graded algebra, provided with an inclusion of vector spaces $V \hookrightarrow \mathfrak{B}^1 = \bigoplus_{1\le j\le\theta} \mathfrak{B}^{\mathbf{e}_j}$. Then we can extend the braiding to $\mathfrak{B}$. We assume the following relations in $\mathfrak{B}$:

$$x_i^2 = 0 \quad \text{if } q_{ii} = -1,$$
$$x_1^3 = 0 \quad \text{if } q_{11} \in \mathbb{G}_3,$$
$$(\mathrm{ad}_c\, x_i)x_j = 0 \quad \text{if } |j-i| > 1,$$
$$(\mathrm{ad}_c\, x_i)^2 x_j = 0 \quad \text{if } |j-i| = 1 \text{ and } i \neq 1,$$

$$\big[(\mathrm{ad}_c\, x_i)(\mathrm{ad}_c\, x_{i+1})x_{i+2},\, x_{i+1}\big]_c = 0 \quad \text{if } 2 \le i \le \theta,$$
$$(\mathrm{ad}_c\, x_1)^3 x_2 = 0,$$
$$\big[(\mathrm{ad}_c\, x_1)^2 x_2,\, (\mathrm{ad}_c\, x_1)x_2\big]_c = 0,$$
$$\big[(\mathrm{ad}\, x_1)^2(\mathrm{ad}\, x_2)x_3,\, (\mathrm{ad}\, x_1)x_2\big]_c = 0.$$

Using the same notation as in Section 4B,

$$x_{\mathbf{v}_{ij}} = [x_{\mathbf{u}_{1i}}, x_{\mathbf{u}_{1j}}]_c, \quad 1 \le i < j \le \theta.$$

From the proof of the relations corresponding the A_θ case, we know that (5-14), (5-15), (5-16), (5-18) and (5-19) hold for $i \ge 1$, but for relation (5-17) we must assume $i > 1$.

Lemma 5.15. *Suppose* $1 \le s < t$ *and* $1 < k \le j$. *The following relations hold in* $\mathfrak{B}$:

$$[x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c \begin{cases} = 0 & \text{if } t+1 < k, \\ = x_{\mathbf{v}_{sj}} & \text{if } t+1 = k < j, \\ = 0 & \text{if } s+1 < k \le j \le t, \\ = \chi(\mathbf{v}_{st}, \mathbf{u}_{kt})(1 - q_{t,t+1}q_{t+1,t})x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}} & \text{if } s+1 < k \le t < j, \\ = \chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,j})x_{\mathbf{v}_{jt}} & \text{if } s+1 = k \le j < t, \\ = (\chi(\mathbf{u}_{1t}, \mathbf{u}_{s+1,t}) - \chi(\mathbf{u}_{1s}, \mathbf{u}_{1t}))x^2_{\mathbf{u}_{1t}} & \text{if } s+1 = k,\ j = t, \\ \in \mathsf{k}x_{\mathbf{v}_{tj}} + \mathsf{k}x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}} + \mathsf{k}x_{\mathbf{u}_{s+1,j}}x_{\mathbf{v}_{sj}} & \text{if } s+1 = k \le t < j, \\ = \gamma_{st}^{kj}\, x_{\mathbf{u}_{ks}}x_{\mathbf{v}_{jt}} & \text{if } k \le s < j \le t, \\ \in \mathsf{k}x_{\mathbf{u}_{ks}}x_{\mathbf{v}_{tj}} + \mathsf{k}x_{\mathbf{u}_{ks}}x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}} + \mathsf{k}x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}} & \text{if } k \le s < t < j, \\ = 0 & \text{if } k \le j \le s, \end{cases}$$

where $\gamma_{st}^{kj} = \chi(\mathbf{u}_{1t}, \mathbf{u}_{kj})\chi(\mathbf{u}_{1s}, \mathbf{u}_{ks})(1 - q_{s,s+1}q_{s+1,s})$.

Proof. The first, third and last equalities follow from the vanishing of $[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{kj}}]_c$ and $[x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{kj}}]_c = 0$, using (5-14), (5-16), (5-17) or (5-18) as the case maybe, together with (1-4).

For the second case, we use that $[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{t+1,j}}]_c = 0$, (5-15) and (1-4) to obtain

$$x_{\mathbf{v}_{sj}} = [x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1j}}]_c = \big[x_{\mathbf{u}_{1s}}, [x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{t+1,j}}]_c\big]_c = \big[[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{1t}}]_c, x_{\mathbf{u}_{t+1,j}}\big]_c = [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{t+1,j}}]_c.$$

For the fourth, we use (1-4) and the third case to calculate

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{u}_{kj}}]_c &= [x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{kt}}, x_{\mathbf{u}_{t+1,j}}]_c]_c \\ &= \chi(\mathbf{v}_{st}, \mathbf{u}_{kt})x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}} - \chi(\mathbf{u}_{kt}, \mathbf{u}_{t+1,j})x_{\mathbf{v}_{sj}}x_{\mathbf{u}_{kt}} \\ &= \chi(\mathbf{v}_{st}, \mathbf{u}_{kt})(1 - \chi(\mathbf{u}_{kt}, \mathbf{u}_{t+1,j})\chi(\mathbf{u}_{t+1,j}, \mathbf{u}_{kt}))x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}}. \end{aligned}$$

For the fifth, note that $\chi(\mathbf{u}_{1t},\mathbf{u}_{s+1,j})^{-1}=\chi(\mathbf{u}_{s+1,j},\mathbf{u}_{1t})$. Then use (5-15), (5-16) and (1-4) to prove that

$$\begin{aligned}[x_{\mathbf{v}_{st}},x_{\mathbf{u}_{s+1,j}}]_c &= \big[[x_{\mathbf{u}_{1s}},x_{\mathbf{u}_{1t}}]_c,x_{\mathbf{u}_{s+1,j}}\big]_c\\ &=\chi(\mathbf{u}_{1t},\mathbf{u}_{s+1,j})x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}}-\chi(\mathbf{u}_{1s},\mathbf{u}_{1t})x_{\mathbf{u}_{1t}}x_{\mathbf{u}_{1s}}\\ &=\chi(\mathbf{u}_{1t},\mathbf{u}_{s+1,j})(x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}}-\chi(\mathbf{u}_{1j},\mathbf{u}_{1t})x_{\mathbf{u}_{1t}}x_{\mathbf{u}_{1s}}).\end{aligned}$$

The sixth case is similar.

For the seventh case, we use (1-4), (1-5) and the previous case to calculate

$$\begin{aligned}[x_{\mathbf{v}_{st}},x_{\mathbf{u}_{s+1,j}}]_c &= \big[x_{\mathbf{v}_{st}},[x_{\mathbf{u}_{s+1,t}},x_{\mathbf{u}_{t+1,j}}]_c\big]_c\\ &=(\chi(\mathbf{u}_{1t},\mathbf{u}_{s+1,t})-\chi(\mathbf{u}_{1s},\mathbf{u}_{1t}))[x^2_{\mathbf{u}_{1t}},x_{\mathbf{u}_{t+1,j}}]\\ &\qquad+\chi(\mathbf{v}_{st},\mathbf{u}_{s+1,t})x_{\mathbf{u}_{s+1,t}}x_{\mathbf{v}_{sj}}-\chi(\mathbf{u}_{s+1,t},\mathbf{u}_{t+1,j})x_{\mathbf{v}_{sj}}x_{\mathbf{u}_{s+1,t}}\\ &=(\chi(\mathbf{u}_{1t},\mathbf{u}_{s+1,t})-\chi(\mathbf{u}_{1s},\mathbf{u}_{1t}))((x_{\mathbf{v}_{tj}}+\chi(\mathbf{u}_{1t},\mathbf{u}_{1j})x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}})\\ &\quad+\chi(\mathbf{u}_{1t},\mathbf{u}_{t+1,j})x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}})-\chi(\mathbf{u}_{s+1,t},\mathbf{u}_{t+1,j})x_{\mathbf{v}_{tj}}\\ &\qquad+(\chi(\mathbf{v}_{st},\mathbf{u}_{s+1,t})-\chi(\mathbf{u}_{s+1,t},\mathbf{u}_{t+1,j})\chi(\mathbf{v}_{sj},\mathbf{u}_{s+1,t}))x_{\mathbf{u}_{s+1,t}}x_{\mathbf{v}_{sj}}.\end{aligned}$$

We use the previous cases, (5-16) and (5-19) to calculate for the eighth case

$$\begin{aligned}[x_{\mathbf{v}_{st}},x_{\mathbf{u}_{kj}}]_c &= [[x_{\mathbf{u}_{1s}},x_{\mathbf{u}_{1t}}]_c,x_{\mathbf{u}_{kj}}]_c\\ &=\chi(\mathbf{u}_{1t},\mathbf{u}_{kj})(\chi(\mathbf{u}_{1s},\mathbf{u}_{ks})(1-q_{s,s+1}q_{s+1,s})x_{\mathbf{u}_{ks}}x_{\mathbf{u}_{1j}})x_{\mathbf{u}_{1t}}\\ &\qquad-\chi(\mathbf{u}_{1s},\mathbf{u}_{1t})x_{\mathbf{u}_{1t}}(\chi(\mathbf{u}_{1s},\mathbf{u}_{ks})(1-q_{s,s+1}q_{s+1,s})x_{\mathbf{u}_{ks}}x_{\mathbf{u}_{1j}})\\ &=\gamma_{st}^{kj}x_{\mathbf{u}_{ks}}(x_{\mathbf{u}_{1j}}x_{\mathbf{u}_{1t}}-\chi(\mathbf{u}_{1j},\mathbf{u}_{1t})x_{\mathbf{u}_{1t}}x_{\mathbf{u}_{1j}}).\end{aligned}$$

To conclude, we prove the ninth case in a similar way:

$$\begin{aligned}[x_{\mathbf{v}_{st}},x_{\mathbf{u}_{kj}}]_c &= [x_{\mathbf{v}_{st}},[x_{\mathbf{u}_{kt}},x_{\mathbf{u}_{t+1,j}}]_c]_c\\ &=\gamma_{st}^{kt}(1-q_{\mathbf{v}_{1t}})[x_{\mathbf{u}_{ks}}x^2_{\mathbf{u}_{1t}},x_{\mathbf{u}_{t+1,j}}]\\ &\qquad+\chi(\mathbf{v}_{st},\mathbf{u}_{k,t})x_{\mathbf{u}_{kt}}x_{\mathbf{v}_{sj}}-\chi(\mathbf{u}_{kt},\mathbf{u}_{t+1,j})x_{\mathbf{v}_{sj}}x_{\mathbf{u}_{kt}}.\end{aligned}$$ □

We consider the remaining commutator $\big[x_{\mathbf{v}_{st}},x_{\mathbf{u}_{jk}}\big]_c$: when $j=1$.

Lemma 5.16. *Let $s<t$ in $\{1,\dots,\theta\}$. The following relations hold in $\mathfrak{B}$:*

$$[x_{\mathbf{v}_{st}},x_{\mathbf{u}_{1k}}]_c=0 \quad \textit{if } s<k\le t, \tag{5-21}$$

$$[x_{\mathbf{u}_{1s}},x_{\mathbf{v}_{st}}]_c=0. \tag{5-22}$$

Proof. By assumption we have

$$\begin{aligned}[x_{\mathbf{v}_{12}},x_{\mathbf{u}_{12}}]_c &= \big[(\mathrm{ad}_c\,x_1)^2x_2,(\mathrm{ad}_c\,x_1)x_2\big]_c=0,\\ [x_{\mathbf{v}_{13}},x_{\mathbf{u}_{12}}]_c &= \big[(\mathrm{ad}_c\,x_1)^2(\mathrm{ad}_c\,x_2)x_3,(\mathrm{ad}_c\,x_1)x_2\big]_c=0.\end{aligned}$$

For $t \geq 4$, $[x_{\mathbf{u}_{4t}}, x_{\mathbf{u}_{12}}]_c = 0$ by (5-14), and using (1-4),

$$[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{12}}]_c = \big[[x_{\mathbf{v}_{13}}, x_{\mathbf{u}_{4t}}]_c, x_{\mathbf{u}_{12}}\big]_c = 0.$$

For each $k \leq t$ we have $[x_{\mathbf{u}_{1t}}, x_{\mathbf{u}_{3k}}]_c = [x_1, x_{\mathbf{u}_{3k}}]_c = 0$, so $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{3k}}]_c = 0$. Using (1-4) and (5-15) we have

$$[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1k}}]_c = \big[x_{\mathbf{v}_{1t}}, [x_{\mathbf{u}_{12}}, x_{\mathbf{u}_{3k}}]_c\big]_c = 0.$$

Now consider $2 \leq s \leq k$. Since $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1k}}]_c = [x_{\mathbf{u}_{2s}}, x_{\mathbf{u}_{1k}}]_c = 0$ by previous results and (5-16), we conclude from (1-5) and Lemma 5.15 that (5-21) is valid in the general case.

To prove (5-22), we have for $s = 1, t = 2$

$$[x_{\mathbf{u}_{11}}, x_{\mathbf{v}_{12}}]_c = [x_1, x_{\mathbf{v}_{12}}]_c = (\operatorname{ad}_c x_1)^3 x_2 = 0.$$

Using that $[x_1, x_{\mathbf{u}_{3t}}]_c = 0$ if $t \geq 3$ and (1-4), we deduce that

$$[x_{\mathbf{u}_{11}}, x_{\mathbf{v}_{1t}}]_c = \big[x_1, [x_{\mathbf{v}_{12}}, x_{\mathbf{u}_{3t}}]_c\big]_c = 0.$$

If $1 < s < t$ we have, by the previous case,

$$[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{1t}}]_c = -\chi(x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{1t}})[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{1s}}]_c = 0.$$

By (5-18), $[x_{\mathbf{u}_{1s}}, x_{\mathbf{u}_{2s}}]_c = 0$. Also, $[x_{\mathbf{v}_{1t}}, x_{\mathbf{u}_{2s}}]_c = \chi(\mathbf{u}_{1t}, \mathbf{u}_{2s}) x_{\mathbf{v}_{st}}$, by Lemma 5.15. Equation (5-22) follows by (1-4) and the last three equalities. □

Lemma 5.17. *Let $s < k < t$. The following relations hold in $\mathfrak{B}$:*

$$[x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c = \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k})(1 - q_{k,k+1}q_{k+1,k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}}, \tag{5-23}$$

$$[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{kt}}]_c = \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k})(1 + q_{\mathbf{u}_{1k}})(1 - q_{k,k+1}q_{k+1,k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}}. \tag{5-24}$$

Proof. The proof follows by (1-4), the second case of Lemma 5.15 and (5-22):

$$\begin{aligned}
[x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c &= \big[x_{\mathbf{v}_{sk}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{k+1,t}}]_c\big]_c \\
&= \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{k+1,t}) x_{\mathbf{v}_{st}} x_{\mathbf{u}_{1k}} \\
&= \chi(\mathbf{v}_{sk}, \mathbf{u}_{1k}) \big(1 - \chi(\mathbf{u}_{1k}, \mathbf{u}_{k+1,t})\chi(\mathbf{u}_{k+1,t}, \mathbf{u}_{1k})\big) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}}, \\
[x_{\mathbf{u}_{1s}}, x_{\mathbf{v}_{kt}}]_c &= \big[x_{\mathbf{u}_{1s}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1t}}]_c\big]_c \\
&= [x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1t}}]_c + \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1t}) x_{\mathbf{v}_{st}} x_{\mathbf{u}_{1k}} \\
&= \chi(\mathbf{u}_{1s}, \mathbf{u}_{1k})(q_{\mathbf{u}_{1k}}(1 - q_{k,k+1}q_{k+1,k}) + 1 - q_{k,k+1}q_{k+1,k}) x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{st}}. \quad \square
\end{aligned}$$

We next deal with the expression of the commutator of two words of type $x_{\mathbf{v}_{st}}$.

Lemma 5.18. *Let $s < t$ and $s \le k < j$, with $k \ne s$ or $j \ne t$. The following relations hold in $\mathfrak{B}$:*

$$[x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}]_c \begin{cases} = 0 & \text{if } k < j \le t, \\ = 0 & \text{if } k = s,\ t < j, \\ = \chi(\mathbf{v}_{st}, \mathbf{v}_{kt})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{v}_{kt}} x_{\mathbf{v}_{sj}} & \text{if } k < t < j, \\ = \chi(\mathbf{v}_{st}, \mathbf{u}_{1t})^2 (1 - q_{t,t+1}q_{t+1,t}) & \\ \qquad (1 - q_{\mathbf{u}_{1t}} q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}}^2 x_{\mathbf{v}_{sj}} & \text{if } k = t < j, \\ \in \Bbbk x_{\mathbf{v}_{tj}} x_{\mathbf{v}_{sk}} + \Bbbk x_{\mathbf{v}_{tk}} x_{\mathbf{v}_{sj}} + \Bbbk x_{\mathbf{u}_{1k}} x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}} & \text{if } t < k < j. \end{cases}$$

Proof. The first and second equalities follow from (1-4) and (5-21), (5-22), respectively. For the third, we use the previous one and (1-4):

$$\begin{aligned} &[x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}]_c \\ &\quad = \big[x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1j}}]_c\big]_c \\ &\quad = \chi(\mathbf{v}_{st}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1k}} \big(\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}\big) \\ &\qquad\qquad - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j}) \big(\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}\big) x_{\mathbf{u}_{1k}} \\ &\quad = (1 - q_{t,t+1}q_{t+1,t}) \big(\chi(\mathbf{v}_{st}, \mathbf{u}_{1k}) \chi(\mathbf{v}_{st}, \mathbf{u}_{1t}) x_{\mathbf{u}_{1k}} x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}} \\ &\qquad\qquad - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j}) \chi(\mathbf{v}_{st}, \mathbf{u}_{1t}) \chi(\mathbf{v}_{sj}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1t}} x_{\mathbf{u}_{1k}} x_{\mathbf{v}_{sj}}\big) \\ &\quad = \chi(\mathbf{v}_{st}, \mathbf{u}_{1k}) \chi(\mathbf{v}_{st}, \mathbf{u}_{1t}) (1 - q_{t,t+1}q_{t+1,t}) \big(x_{\mathbf{u}_{1k}} x_{\mathbf{u}_{1k}} - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1t}) x_{\mathbf{u}_{1k}} x_{\mathbf{u}_{1k}}\big) x_{\mathbf{v}_{sj}}. \end{aligned}$$

The fourth case is similar to the previous one.

To prove the last case we use (1-4) and Lemma 5.17:

$$\begin{aligned} [x_{\mathbf{v}_{st}}, x_{\mathbf{v}_{kj}}]_c &= \big[x_{\mathbf{v}_{st}}, [x_{\mathbf{u}_{1k}}, x_{\mathbf{u}_{1j}}]_c\big]_c \\ &= [\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sk}}, x_{\mathbf{u}_{1j}}]_c \\ &\qquad + \chi(\mathbf{v}_{st}, \mathbf{u}_{1k}) x_{\mathbf{u}_{1k}} (\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}) \\ &\qquad - \chi(\mathbf{u}_{1k}, \mathbf{u}_{1j}) (\chi(\mathbf{v}_{st}, \mathbf{u}_{1t})(1 - q_{t,t+1}q_{t+1,t}) x_{\mathbf{u}_{1t}} x_{\mathbf{v}_{sj}}) x_{\mathbf{u}_{1k}}. \end{aligned}$$

The proof is finished using (1-5) and the previous identities. □

Theorem 5.19. *Let V be a standard braided vector space of type B_θ, where $\theta = \dim V$, and let $C = (a_{ij})_{i,j \in \{1,\dots,\theta\}}$ be the corresponding Cartan matrix of type B_θ.*

The Nichols algebra $\mathfrak{B}(V)$ is presented by the generators x_i, $1 \le i \le \theta$, and the relations

$$\begin{aligned} x_\alpha^{N_\alpha} &= 0, & & \alpha \in \Delta^+; \\ \operatorname{ad}_c(x_i)^{1-a_{ij}}(x_j) &= 0, & & i \ne j; \\ \big[(\operatorname{ad} x_{j-1})(\operatorname{ad} x_j) x_{j+1}, x_j\big]_c &= 0, & & 1 < j < \theta,\ q_{jj} = -1; \\ \big[(\operatorname{ad} x_1)^2 x_2, (\operatorname{ad} x_1) x_2\big]_c &= 0, & & q_{11} \in \mathbb{G}_3 \text{ or } q_{22} = -1; \\ \big[(\operatorname{ad} x_1)^2 (\operatorname{ad} x_2) x_3, (\operatorname{ad} x_1) x_2\big]_c &= 0, & & q_{11} \in \mathbb{G}_3 \text{ or } q_{22} = -1. \end{aligned}$$

The following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \quad \textit{where } 0 \le h_j < N_{\beta_j} - 1 \textit{ if } \beta_j \in S_I, \textit{ for } 1 \le j \le P. \tag{5-25}$$

Proof. The proof is analogous to that of Theorem 5.14, since by the previous lemmas we can express the commutator of two generators $x_\alpha < x_\beta$ as a linear combination of monotone hyperwords whose greater hyperletter is great or equal to x_β. □

5D. *Presentation when the type is G_2.* We now consider standard braidings of type G_2, with $m_{12} = 3, m_{21} = 1$.

Lemma 5.20. *Let* $\mathfrak{B} := T(V)/I$, *for some* $I \in \mathfrak{S}$, *and suppose that*

$$x_1^{\operatorname{ord} q_{11}} = 0, \quad x_2^{\operatorname{ord} q_{22}} = 0, \quad (\operatorname{ad} x_1)^4 x_2 = (\operatorname{ad} x_2)^2 x_1 = 0 \tag{5-26}$$

in $\mathfrak{B}$. *Then*

(a) $[x_1^3 x_2 x_1 x_2]_c = 0$ *in* $\mathfrak{B} \iff 4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$.

Assume further that the equivalent conditions in (a) *hold. Then*

(b) $\big[(\operatorname{ad} x_1)^3 x_2, (\operatorname{ad} x_1)^2 x_2\big]_c = 0$ *in* $\mathfrak{B} \iff 5e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$ *and*

(c) $\big[[x_1^2 x_2 x_1 x_2]_c, [x_1 x_2]_c\big]_c = 0$ *in* $\mathfrak{B} \iff 4e_1 + 3e_2 \notin \Delta^+(\mathfrak{B})$.

Assume also that the equivalent conditions in (b) *and those in* (c) *hold. Then*

(d) $\big[[x_1^2 x_2]_c, [x_1^2 x_2 x_1 x_2]_c\big]_c = 0$ *in* $\mathfrak{B} \iff 5e_1 + 3e_2 \notin \Delta^+(\mathfrak{B})$.

In particular, all these relations hold when V *is a standard braiding and* $\mathfrak{B} = \mathfrak{B}(V)$ *is finite-dimensional.*

Proof. Take the ordering $x_1 < x_2$, and consider a PBW basis as in Theorem 1.12. Define $\gamma_k := \prod_{0 \le j \le k-1} (1 - q_{11}^j q_{12} q_{21})$.

(a) If $[x_1^3 x_2 x_1 x_2]_c = 0$, then $4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$ since there are no possible Lyndon words in S_I: $x_1^3 x_2 x_1 x_2$ is the unique Lyndon word such that $x_1^3 x_2$ and $x_1 x_2^2$ are not factors, and it is not in S_I by assumption.

Conversely, if $4e_1 + 2e_2 \notin \Delta^+(\mathfrak{B})$, then $[x_1^3 x_2 x_1 x_2]_c$ is a linear combination of greater hyperwords, and $[x_1 x_2 x_1^3 x_2]_c$ and $[x_1^2 x_1^2 x_2]$ are the only greater hyperwords that are not in S_I and do not end in x_1 (we discard words ending in x_1 since $[x_1^3 x_2 x_1 x_2]_c$ is in $\ker D_1$). Taking their Shirshov decomposition, we see that there exist $\alpha, \beta \in \mathsf{k}$ such that

$$[x_1^3 x_2 x_1 x_2]_c - \alpha [x_1 x_2]_c [x_1^3 x_2]_c - \beta [x_1^2 x_2]_c^2 = 0. \tag{5-27}$$

Note that $[x_1^3 x_2 x_1 x_2]_c = \operatorname{ad} x_1([x_1^2 x_2 x_1 x_2]_c)$, so by direct calculation,

$$D_2([x_1^2 x_2 x_1 x_2]_c) = 0.$$

Apply D_2 to both sides of equality (5-27) and express the result as a linear combination of $[x_1^3x_2]_cx_1$, $[x_1^2x_2]_cx_1^2$ and $[x_1x_2]_cx_1^3$. The coefficient of $[x_1x_2]_cx_1^3$ is

$$\alpha(1-q_{12}q_{21})(1-q_{11}q_{12}q_{21}),$$

so $\alpha=0$. Note also that $D_1^2D_2\big([x\_1^3x_2x_1x_2]_c\big)=0$; but

$$D_1^2D_2\big([x_1^2x_2]_c^2\big)=(1-q_{12}q_{21})(1-q_{11}q_{12}q_{21})(1+q_{11})(q_{2e_1+e_2}+1)[x_1^2x_2]_c.$$

Looking at the proof of Proposition 4.7, we see that $q_{2e_1+e_2}\neq -1$, so $\beta=0$.

(b) Assuming (5-26) and the condition in (a), the only possible Lyndon word of degree $5e_1+2e_j$ is $x_1^3x_2x_1^2x_2$, and

$$[x_1^2x_2x_1x_2x_1x_2]_c=\big[(\operatorname{ad}x_1)^3x_2,\,(\operatorname{ad}x_1)^2x_2\big]_c\,.$$

Then we proceed as before. One implication is clear. For the other, if $5e_1+2e_j\notin\Delta^+(\mathfrak{B})$, there exists $\alpha\in\mathsf{k}$ such that

$$\big[(\operatorname{ad}x_1)^3x_2,\,(\operatorname{ad}x_1)^2x_2\big]_c=\alpha(\operatorname{ad}x_1)^2x_2(\operatorname{ad}x_1)^3x_2.$$

Now we apply D_2 and express the equality as a linear combination of $(\operatorname{ad}x_1)^3x_2x_1^2$ and $(\operatorname{ad}x_1)^2x_2x_1^3$ (using the hypothesis that $(\operatorname{ad}x_1)^4x_2=0$); the coefficient of $(\operatorname{ad}x_1)^2x_2x_1^3$ is $\alpha\gamma_3$, so $\alpha=0$.

(c) The proof is similar. Since we are considering Lyndon words not having $x_1^3x_2$ or $x_1x_2^2$ as a factor, the only possible Lyndon word of degree $4e_1+3e_j$ is $x_1^2x_2x_1x_2x_1x_2$, and

$$[x_1^2x_2x_1x_2x_1x_2]_c=\big[[x_1^2x_2x_1x_2]_c,\,[x_1x_2]_c\big]_c\,.$$

If $4e_1+3e_j\notin\Delta^+(\mathfrak{B})$, there exist $\alpha_i\in\mathsf{k}$ such that

$$\begin{aligned}&\big[x_1^2x_2(x_1x_2)^2\big]_c\\&\quad=\alpha_1[x_1x_2]_c[x_1^2x_2x_1x_2]_c+\alpha_2[x_1x_2]_c^2[x_1^2x_2]_c+\alpha_3x_2[x_1^2x_2]_c^2+\alpha_4x_2[x_1x_2]_c[x_1^3x_2]_c,\end{aligned}$$

since, as above, we are discarding words greater than $x_1^2x_2x_1x_2x_1x_2$ ending in x_1; we also discard words with factors $x_1^4x_2$, $x_1x_2^2$, $x_1^3x_2x_1^2x_2$, by the assumption on $\mathfrak{B}$. We apply D_2 to this equality. Using the definition of the braided commutator, we express the hyperletter just considered as a linear combination of elements of the PBW basis, having degree $4e_1+2e_2$.

The coefficient of $x_2[x_1x_2]_cx_1^3$ is $\alpha_4\gamma_3$ since this PBW generator appears only in the expression of $D_2(x_2[x_1x_2]_c[x_1^3x_2]_c)$. Thus $\alpha_4=0$.

Using this fact, we see that the coefficient of $x_2[x_1^3x_2]_cx_1$ is

$$\alpha_3\gamma_2(1+q_{11})q_{11}^2q_{12}q_{21}^2q_{22},$$

since this term appears only in the expression of $D_j(x_2[x_1^2x_2]_c^2)$. Thus $\alpha_3=0$.

Next, the coefficient of $[x_1x_2]_c^2x_1^2$ is $\alpha_2\gamma_2$, so $\alpha_2 = 0$. Now we calculate the coefficient of $[x_1^2x_2]_c^2$:

$$\alpha_1\gamma_2\big(\chi(\mathbf{e}_1, 5\mathbf{e}_1+\mathbf{e}_2) - \chi(2\mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2)\big) = \alpha_1\gamma_2 q_{11}q_{12}\left(q_{11}^3 - q_{22}q_{12}q_{21}\right).$$

Since $q_{11}^3 \neq q_{22}q_{12}q_{21}$ for each standard braiding, we conclude that $\alpha_1 = 0$.

(d) If the conditions in (b) and (c) hold, the only possible Lyndon word of degree $5e_1+3e_2$ not having $x_1^4x_2$ or $x_1x_2^2$ as factors is $x_1^2x_2x_1^2x_2x_1x_2$, and

$$[x_1^2x_2x_1^2x_2x_1x_2]_c = \big[[x_1^2x_2]_c, [x_1^2x_2x_1x_2]_c\big]_c.$$

This hyperword is not in S_I if and only if there exist $\nu_i \in \Bbbk$ such that

$$[x_1^2x_2x_1^2x_2x_1x_2]_c = \nu_1[x_1^2x_2x_1x_2]_c[x_1^2x_2]_c + \nu_2[x_1x_2]_c[x_1^2x_2]_c^2 \\ + \nu_3[x_1x_2]_c^2[x_1^3x_2]_c + \nu_4x_2[x_1^2x_2]_c[x_1^3x_2]_c. \quad (5\text{-}28)$$

Apply D_2 and note that $D_2([x_1^2x_2x_1^2x_2x_1x_2]_c) = 0$ under the hypotheses on $\mathfrak{B}$. Then express the resulting sum as a linear combination of elements of the PBW basis, which have degree $5e_1+2e_2$.

The hyperword $x_2[x_1^2x_2]x_1^3$ appears only for $D_2(x_2[x_1^2x_2]_c[x_1^3x_2]_c)$, and its coefficient is $\nu_4\gamma_3$, and since $\gamma_3 \neq 0$ we conclude that $\nu_4 = 0$.

Analogously, $[x_1x_2]_c^2x_1^3$ appears only for $[x_1x_2]_c^2[x_1^3x_2]_c$ (due to $\nu_4 = 0$). Its coefficient is $\nu_3\gamma_3$, so $\nu_3 = 0$.

Note that $D_1^2D_2([x_1^2x_2x_1x_2]_c) = 0$. We apply $D_1^2D_2$ to the expression (5-28), and obtain

$$0 = \nu_1\gamma_2(1+q_{11})[x_1^2x_2x_1x_2]_c + \nu_2\gamma_2(1+q_{11})(1+q_{2\mathbf{e}_1+\mathbf{e}_2})[x_1x_2]_c[x_1^2x_2]_c.$$

The terms $[x_1^2x_2x_1x_2]_c$ and $[x_1x_2]_c[x_1^2x_2]_c$ are linearly independent, since they are linearly independent in $\mathfrak{B}(V)$, and we have a surjection $\mathfrak{B} \to \mathfrak{B}(V)$. Then

$$\nu_1\gamma_2(1+q_{11}) = \nu_2\gamma_2(1+q_{11})(1+q_{2\mathbf{e}_1+\mathbf{e}_2}) = 0.$$

But for standard braidings of type G_2 we note that $q_{11}, q_{2\mathbf{e}_1+\mathbf{e}_2} \neq -1$ and $\gamma_2 \neq 0$, so $\nu_1 = \nu_2 = 0$.

The last statement is true since

$$\Delta^+(\mathfrak{B}(V)) = \{e_1, e_1+e_2, 2e_1+e_2, 3e_1+e_2, 3e_1+2e_2, e_2\},$$

if the braiding is standard of type G_2. □

Remark 5.21. Let V be a standard braided vector space of type G_2 and let $\mathfrak{B}$ be a braided graded Hopf algebra satisfying the hypotheses of Lemma 5.20. In a similar way to Lemma 5.5, if $q_{11} \notin \mathbb{G}_4$ and $q_{22} \neq -1$, then $5e_1+2e_2, 4e_1+2e_2 4e_1+3e_2, 5e_1+3e_2 \notin \Delta^+(\mathfrak{B})$.

This follows because $x_1^3x_2x_1^2x_2, x_1^2x_2x_1x_2x_1x_2, x_1^2x_2x_1^2x_2x_1x_2 \notin S_I$, using the quantum Serre relations as in the lemma cited.

Theorem 5.22. *Let V be a standard braided vector space of type G_2. The Nichols algebra $\mathfrak{B}(V)$ is presented by the generators x_1, x_2 and the relations*

$$\operatorname{ad}_c(x_1)^4(x_2) = \operatorname{ad}_c(x_2)^2(x_1) = 0, \qquad x_\alpha^{N_\alpha} = 0,\ \alpha \in \Delta^+, \tag{5-29}$$

and, if $q_{11} \in \mathbb{G}_4$ or $q_{22} = -1$,

$$\big[(\operatorname{ad} x_1)^3 x_2, (\operatorname{ad} x_1)^2 x_2\big]_c = 0, \tag{5-30}$$

$$\big[x_1, [x_1^2x_2x_1x_2]_c\big]_c = 0, \tag{5-31}$$

$$\big[[x_1^2x_2x_1x_2]_c, [x_1x_2]_c\big]_c = 0, \tag{5-32}$$

$$\big[[x_1^2x_2]_c, [x_1^2x_2x_1x_2]_c\big]_c = 0. \tag{5-33}$$

The following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_2^{h_{e_2}} [x_1x_2]_c^{h_{e_1+e_2}} [x_1^2x_2x_1x_2]_c^{h_{3e_1+2e_2}} [x_1^2x_2]_c^{h_{2e_1+e_2}} [x_1^3x_2]_c^{h_{3e_1+e_2}} x_1^{h_{e_1}}, \quad 0 \le h_\alpha \le N_\alpha - 1. \tag{5-34}$$

Proof. The statement about the PBW basis follows from Corollary 4.2 and the definitions of the x_α.

Let $\mathfrak{B}$ be the algebra presented by the generators x_1, x_2 and the relations (5-29)–(5-33). From Lemma 5.20 and Corollary 5.2, we have a canonical epimorphism of algebras $\phi : \mathfrak{B} \to \mathfrak{B}(V)$.

Consider the subspace $\mathcal{I}$ of $\mathfrak{B}$ generated by the elements in (5-34). We prove by induction on the sum S of the h_α's of a such product M that $x_1M \in \mathcal{I}$; moreover, we prove that it is a linear combination of products whose first hyperletter is less than or equal to the first hyperletter of M. If $S = 0$, we have $M = 1$.

- If $M = x_1^{N_1}$, then $x_1M = x_1^{N_1+1}$, which is zero if $N_1 = \operatorname{ord} x_1 - 1$.
- If $M = [x_1^3x_2]_cM'$, then we use that $x_1[x_1^3x_2]_c = q_{11}^3q_{12}[x_1^3x_2]_cx_1$ to prove that x_1M lies in $\mathcal{I}$ and either is zero or begins with $[x_1^3x_2]_c$.
- If $M = [x_1^2x_2]_cM'$, we have

$$x_1[x_1^2x_2]_c = [x_1^3x_2]_c + q_{11}^2q_{12}[x_1^2x_2]_cx_1.$$

  We use the inductive step and relation (5-30) to prove that x_1M lies in $\mathcal{I}$ and is either zero or a linear combination of hyperwords starting with a hyperletter less than or equal to $[x_1^2x_2]_c$.
- If $M = [x_1^2x_2x_1x_2]_cM'$, we deduce from (5-31) that

$$x_1[x_1^2x_2x_1x_2]_c = \chi(\mathbf{e}_1, 3\mathbf{e}_1 + 2\mathbf{e}_2)[x_1^2x_2x_1x_2]_cx_1;$$

using also (5-32) and (5-33), we prove that x_1M lies in $\mathcal{I}$ and is either zero or a linear combination of hyperwords that starting with a hyperletter less than or equal to $[x_1^2x_2x_1x_2]_c$.

- If $M = [x_1x_2]_cM'$, observe that

$$x_1[x_1x_2]_c = [x_1^2x_2]_c + q_{11}q_{12}[x_1x_2]_cx_1.$$

Using the inductive step together with (5-31), (5-32), and the equality

$$[x_1^2x_2]_c[x_1x_2]_c = \big[[x_1^2x_2]_c, [x_1x_2]_c\big]_c + \chi(2\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2)[x_1x_2]_c[x_1^2x_2]_c,$$

by the definition of braided commutator, we prove that x_1M lies in $\mathcal{I}$ and is either zero or a linear combination of hyperwords starting with a hyperletter less than or equal to $[x_1x_2]_c$.

- If $M = x_2M'$, we use the equalities $x_1x_2 = [x_1x_2]_c + q_{12}x_2x_1$ and $[[x_1x_2]_c, x_2]_c = 0$ to prove that x_1M lies in $\mathcal{I}$ and is either zero or a linear combination of hyperwords.

Now, x_2M is a product of nonincreasing hyperwords or is zero, for each element in (5-34), so $\mathcal{I}$ is an ideal of $\mathfrak{B}$ containing 1; hence $\mathcal{I} = \mathfrak{B}$. Since the elements in (5-34) are a basis of $\mathfrak{B}(V)$, the map ϕ is an isomorphism. □

5E. ***Presentation when the braiding is of Cartan type.*** In this subsection, we present the Nichols algebra of a diagonal braiding vector space of Cartan type with matrix (q_{ij}), by generators and relations. This was established in [Andruskiewitsch and Schneider 2002a, Theorem 4.5] assuming that q_{ii} has odd order and that order is not divisible by 3 if i belongs to a component of type G_2. The proof in loc. cit. combines a reduction to symmetric (q_{ij}) by twisting, with results from [Andersen et al. 1994] and [De Concini and Procesi 1993]. We also note that some particular instances were already proved earlier in this section.

Fix a standard braided vector space V with connected Dynkin diagram and an $i \in \{1, \dots, \theta\}$. Suppose that $\mathfrak{B}$ is a quotient by an ideal $I \in \mathfrak{S}$ of $T(V)$. *Assume moreover that V is not of type G_2 and that*

$$\text{(5-3) holds in } \mathfrak{B} \text{ if } 1 \le i \ne j \le \theta; \tag{5-35}$$

$$\text{(5-4) holds in } \mathfrak{B} \text{ if } m_{kj} = m_{kl} = 1 \text{ and } m_{jl} = 0; \tag{5-36}$$

$$\text{(5-6) holds in } \mathfrak{B} \text{ if } m_{kj} = 2 \text{ and } m_{jk} = 1; \tag{5-37}$$

$$\text{(5-8) holds in } \mathfrak{B} \text{ if } m_{kj} = 2,\ m_{jk} = m_{jl} = 1 \text{ and } m_{kl} = 0. \tag{5-38}$$

Note that if (5-3) holds in an algebra with derivations D_k, then (2-11) holds also, by Lemma 2.7. By Theorem 2.6, we have an algebra $s_i(\mathfrak{B})$ provided with skew derivations D_i. We set $\tilde{x}_k = (\mathrm{ad}_c\, x_i)^{m_{ik}}(x_k)\#1 \in s_i(\mathfrak{B})$, for $k \ne i$, and $\tilde{x}_i = 1\#y$. The elements generate $s_i(\mathfrak{B})^1$ as a vector space.

Lemma 5.23. *Conditions* (5-35)–(5-38) *are satisfied with* $s_i(\mathfrak{B})$ *in lieu of* $\mathfrak{B}$.

Proof of (5-35). Each $m\mathbf{e}_k + e_j$, $0 \le m \le m_{kj}$ is an element of $\Delta(\mathfrak{B}(V_i))$, so $s_i(m\mathbf{e}_k + \mathbf{e}_j) \in \Delta(\mathfrak{B}(V))$. Since we have a surjective morphism of braided graded Hopf algebras $\mathfrak{B} \to \mathfrak{B}(V)$, we have $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$.

From Lemma 5.3, $(\operatorname{ad}_c \tilde{x}_k)^m \tilde{x}_j = 0$ if and only if $\tilde{x}_k^m \tilde{x}_j$ is a linear combination of greater words, for an order in which $\tilde{x}_k < \tilde{x}_j$ (since we are considering the Cartan case, the condition about the ordering of the $\tilde{x}_j$ is satisfied). Note that $\tilde{x}_k^m \tilde{x}_j$ is the unique Lyndon word of degree $m\mathbf{e}_k + e_j$. Then, by the relation (2-15) between the Hilbert series of $\mathfrak{B}$ and $s_i(\mathfrak{B})$, the validity of (5-3) for $s_i(\mathfrak{B})$ is equivalent to the condition

$$s_i((m_{kj}+1)\mathbf{e}_k + \mathbf{e}_j) \notin \Delta^+(\mathfrak{B}).$$

(a) When $k = i \neq j$, this says that $-\mathbf{e}_i + \mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, so (5-3) holds.

(b) To prove (5-3) for $s_i(\mathfrak{B})$ when $j = i$, we show case by case that

$$(m_{ki}+1)\mathbf{e}_k + ((m_{ki}+1)m_{ik} - 1)\mathbf{e}_i \notin \Delta^+(\mathfrak{B}).$$

- If $m_{ki} = m_{ik} = 0$, we have $\mathbf{e}_k - \mathbf{e}_i \notin \Delta^+(\mathfrak{B})$.
- If $m_{ki} = m_{ik} = 1$, then $2\mathbf{e}_k + \mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, because $(\operatorname{ad} x_k)^2 x_i = 0$.
- If $m_{ki} = 1$ and $m_{ik} = 2$, then $2\mathbf{e}_k + 3\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, since we can apply Lemma 5.5 to $\mathfrak{B}$, which satisfies (5-6) by assumption.
- If $m_{ki} = 2$ and $m_{ik} = 1$, then $3\mathbf{e}_k + 2\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, as before.

Thus (5-3) holds for each $k \neq i$.

(c) Now consider $\theta \ge 3$ and $k, j \neq i$.

- If $m_{ik} = m_{ij} = 0$, then $s_i(m\mathbf{e}_k + \mathbf{e}_j) = m\mathbf{e}_k + \mathbf{e}_j$, and $(m_{kj}+1)\mathbf{e}_k + \mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, since the quantum Serre relation holds in $\mathfrak{B}$.
- If $m_{ik} = 1$ and $m_{ij} = 0$, then $s_i(m\mathbf{e}_k + \mathbf{e}_j) = m\mathbf{e}_i + m\mathbf{e}_k + \mathbf{e}_j$. If we assume $x_j < x_i < x_k$ and look at the possible Lyndon words in S_I, from (5-3), these words have no factors $x_i^2 x_k$, $x_j x_i$, so the only possibility is $x_j(x_k x_i)^m$.
  - If $m_{kj} = 0$, then $x_j x_k x_i = q_{jk} x_k x_j x_i$, so $x_j x_k x_i \notin S_I$.
  - If $m_{kj} = 1$, then $x_j x_k x_l x_k \notin S_I$ when $m_{ki} = 1$, since (5-4) is valid in $\mathfrak{B}$; while if $m_{ki} = 2$ we have $q_{kk} \neq -1$ and

$$\begin{aligned} x_j(x_k x_i)^2 &= (1+q_{kk})^{-1} q_{ki}^{-1}\, x_j x_k^2 x_i^2 + (1+q_{kk})^{-1} q_{ki} q_{kk}^2\, x_j x_i x_k^2 x_i \\ &= q_{ki}^{-1} q_{kj}^{-1} q_{kk}^{-2}\, x_k x_j x_k x_i^2 + (1+q_{kk})^{-1} q_{ki}^{-1} q_{kj}^{-2} q_{kk}^{-2}\, x_k^2 x_j x_i^2 \\ &\qquad + (1+q_{kk})^{-1} q_{ki} q_{kk}^2 q_{ji}\, x_i x_j x_k^2 x_i. \end{aligned}$$

  In both cases, $x_j(x_k x_i)^2 \notin S_I$.

– If $m_{kj}=2$, then $m_{ki}=m_{jk}=1$ and $q_{kk}\neq -1$. The proof is similar to the previous case.

- If $m_{ik}=2, m_{ij}=0$, then $s_i(m\mathbf{e}_k+\mathbf{e}_j)=2m\mathbf{e}_i+m\mathbf{e}_k+\mathbf{e}_j$ and $m_{kj}=0,1$. When $m_{kj}=0$, the proof is clear as above. When $m_{kj}=1$, for $j<k<i$ and considering only the quantum Serre relations, the only possible Lyndon word is $x_j(x_kx_i^2)^2$. But since $\big[[x_i^2x_k]_c,[x_ix_k]_c\big]_c=0$, we deduce that such a word is not in S_I.
- If $m_{ik}=0, m_{ij}=1$, then $s_i(m\mathbf{e}_k+\mathbf{e}_j)=\mathbf{e}_i+m\mathbf{e}_k+\mathbf{e}_j$. If $k<i<j$, note that from $x_kx_i, x_k^{m_{kj}+1}x_j\notin S_I$, there are no Lyndon words of degree $\mathbf{e}_i+(m_{kj}+1)\mathbf{e}_k+\mathbf{e}_j$ in S_I.
- If $m_{ik}=0, m_{ij}=2$, then $s_i(m\mathbf{e}_k+\mathbf{e}_j)=2\mathbf{e}_i+m\mathbf{e}_k+\mathbf{e}_j$, and the proof is analogous to the previous case.
- If $m_{ik}=m_{ij}=1$, then $m_{kj}=0$, and $s_i(\mathbf{e}_k+\mathbf{e}_j)=2\mathbf{e}_i+\mathbf{e}_k+\mathbf{e}_j$, which is not in $\Delta^+(\mathfrak{B})$ from Lemma 5.4.
- If $m_{ik}=2, m_{ij}=1$ (it is analogous to $m_{ik}=1, m_{ij}=2$), then $m_{kj}=0$ and $s_i(\mathbf{e}_k+\mathbf{e}_j)=3\mathbf{e}_i+\mathbf{e}_k+\mathbf{e}_j$. In this way we get $q_{ii}\neq -1$, and if $x_k<x_i<x_j$ the unique Lyndon word without $x_i^2x_j$ or $x_kx_i^3$ as factors is

$$x_kx_i^2x_jx_i=(1+q_{ii})^{-1}q_{ij}^{-1}x_kx_i^3x_j+(1+q_{ii})^{-1}q_{ii}^2q_{ij}x_kx_ix_jx_i^2$$
$$\in \mathsf{k}(x_ix_kx_i^2x_j)+\mathsf{k}(x_i^2x_kx_ix_j)+\mathsf{k}(x_i^3x_kx_j)+\mathsf{k}(x_kx_ix_jx_i^2),$$

using the quantum Serre relations; hence there are no Lyndon words of degree $3\mathbf{e}_i+\mathbf{e}_k+\mathbf{e}_j$ in S_I.

So, (5-3) holds, for each $k, j\neq i, k\neq j$. □

Proof of (5-36). Assume $m_{kj}=m_{kl}=1$. We prove case by case that

$$s_i(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l)\notin\Delta^+(\mathfrak{B}).$$

- If $m_{ij}=m_{ik}=m_{il}=0$, then $s_i(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l)=2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l$, so it follows from Lemma 5.4, because $2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l\notin\Delta^+(\mathfrak{B})$.
- If $m_{ij}\neq 0$ (analogously, if $m_{il}\neq 0$), then $m_{ik}=m_{il}=0$, because there are no cycles in the Dynkin diagram. Then $s_i(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l)=2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l+m_{ij}\mathbf{e}_i$. If we consider $x_k<x_l<x_j<x_i$, using the equalities $x_kx_i=q_{ki}x_ix_k$, $x_jx_l=q_{jl}x_lx_j$ and $x_lx_i=q_{li}x_ix_l$, and also that $x_k^2x_l, x_k^2x_j\notin S_I$, we conclude that no possible Lyndon words of degree $2e_k+e_j+e_l+m_{ij}e_i$ can be an element of S_I, except $x_kx_lx_kx_jx_i^{m_{ij}}$; but this, too, is not an element of S_I, because $x_kx_lx_kx_j\notin S_I$. Hence $2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l+m_{ij}\mathbf{e}_i\notin\Delta^+(\mathfrak{B})$.
- If $m_{ik}=1$, and therefore $m_{ij}=m_{il}=0$, then $s_i(2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l)=2\mathbf{e}_k+\mathbf{e}_j+\mathbf{e}_l+2m_{ik}\mathbf{e}_i$. If we consider $x_l<x_i<x_k<x_j$, using the equalities $x_jx_i=q_{ji}x_ix_j$,

$x_jx_l = q_{jl}x_lx_j$ and $x_lx_i = q_{li}x_ix_l$, and also that $x_k^2x_l, x_k^2x_j \notin S_I$, we discard as before all possible Lyndon words of degree $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + 2m_{ik}\mathbf{e}_i$, except $x_lx_kx_jx_kx_i^{2m_{ij}}$; but this is not an element of S_I, because $x_kx_lx_kx_j \notin S_I$. Thus $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l + 2m_{ij}\mathbf{e}_i \notin \Delta^+(\mathfrak{B})$.

- If $i = j$ (analogously, if $i = l$), then $s_j(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$ if $m_{jk} = 1$ by Lemma 5.4, or $s_j(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = 2\mathbf{e}_k + 3\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$ if $m_{jk} = 2$ by Lemma 5.5.
- If $i = k$, then $s_k(2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l) = \mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, since $m_{jl} = 0$.

Also, if $\mathbf{u} \in \{\mathbf{e}_k + \mathbf{e}_j, \mathbf{e}_k + \mathbf{e}_l, \mathbf{e}_k, \mathbf{e}_j, \mathbf{e}_l\}$, then $\mathbf{u} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{u}) \in \Delta(\mathfrak{B}(V))$. The canonical surjective algebra morphisms from $T(V)$ to $\mathfrak{B}$ and $\mathfrak{B}(V)$ induce a surjective algebra morphism $\mathfrak{B} \to \mathfrak{B}(V)$, so $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$; in particular, each $s_i(\mathbf{u})$ lies in $\Delta(\mathfrak{B})$.

Consider a basis as in Proposition 1.11 for an order such that $x_j < x_k < x_l$. From Lemma 2.7, x_jx_k, x_kx_l, $x_jx_kx_l$ are elements of this basis, since they are not linear combinations of greater words modulo I_i, the ideal of $T(V_i)$ such that $s_i(\mathfrak{B}) = T(V_i)/I_i$. In the same way, $(x_kx_l)(x_jx_k)$, $x_lx_k(x_jx_k)$, $(x_kx_l)x_kx_j$, $x_k(x_jx_kx_l)$, $x_lx_k^2x_j$ (if $x_k^2 \neq 0$) are elements of this basis, where the parenthesis indicates the Lyndon decomposition as nonincreasing products of Lyndon words. Also, x_jx_l, $x_jx_k^2$, $x_k^2x_l$ are not in this basis, by (5-3). By the relation (2-15) between Hilbert series and the fact that $2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l \notin s_i\left(\Delta^+(\mathfrak{B})\right)$, we note that $x_jx_kx_lx_k$ is not an element of the basis. Thus this word is a linear combination of greater words. By Lemma 5.4, this implies that (5-4) holds in $s_i(\mathfrak{B})$. □

Proof of (5-37). As before, we prove first that $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) \notin \Delta^+(\mathfrak{B})$ case by case:

- If $m_{ik} = m_{ij} = 0$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B})$ by assumption.
- If $m_{ik} = 0, m_{ij} = 1$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 2\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j$. If we consider an order such that $x_k < x_i < x_j$, a Lyndon word of degree $2e_i + 3e_k + 2e_j$ in S_I begins with x_k, and x_kx_i is not a factor, because $x_kx_i = q_{ki}x_ix_j$. Thus the possible Lyndon words with these conditions are $x_k^2x_jx_ix_kx_jx_i$ and $x_k^2x_jx_kx_jx_i^2$; the first is not in S_I because from (5-4) for j, k, i we can express $x_jx_ix_kx_j$ as a linear combination of greater words, and the second is not in S_I because $x_k^2x_jx_kx_j \notin S_I$.
- If $m_{ik} = 1, m_{ij} = 0$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j$. If we consider an order such that $x_j < x_i < x_k$, a Lyndon word of degree $3e_i + 3e_k + 2e_j$ in S_I begins with x_j, and x_jx_i is not a factor. Using that also $x_i^2x_k, x_j^2x_k \notin S_I$, the possible Lyndon word under these conditions is $x_jx_kx_ix_jx_kx_ix_kx_i$. But from the condition on the m_{rs}, we are in cases C_θ or F_4, and we use that $(\operatorname{ad} x_i)^2x_k = 0$, $q_{ii} \neq -1$ to replace $x_ix_kx_i$ by a linear combination of $x_i^2x_k$ and $x_kx_i^2$, and also use $x_jx_i = q_{ji}x_ix_j$, so we conclude that $x_jx_kx_ix_jx_kx_ix_kx_i \notin S_I$.
- If $i = j$, then $s_j(3\mathbf{e}_k + 2\mathbf{e}_j) = 3\mathbf{e}_k + \mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, since $m_{kj} = 2$.

- If $i = k$, then $s_k(3\mathbf{e}_k + 2\mathbf{e}_j) = \mathbf{e}_k + 2\mathbf{e}_j \notin \Delta^+(\mathfrak{B})$, since $m_{jk} = 1$.

If $\mathbf{v} \in \{\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k + \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_j\}$, then $\mathbf{v} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{v}) \in \Delta(\mathfrak{B}(V))$. Since $\Delta(\mathfrak{B}(V)) \subseteq \Delta(\mathfrak{B})$; in particular, each $\mathbf{v}$ lies in $s_i(\Delta(\mathfrak{B}))$.

As in (a), consider a basis as in Proposition 1.11 for an order such that $x_k < x_j$. In a similar way, $x_k x_j$, $x_k^2 x_j$ are elements of this basis, but $x_k^3 x_j$ and $x_k x_j^2$ are not in this basis by (5-3). By Lemma 2.7, $(x_k x_j)(x_k^2 x_j)$, $x_j(x_k^2 x_j)x_k$, $(x_k x_j)^2 x_k$, $x_j(x_k x_j)x_k^2$, $x_j^2 x_k^3$ (the last if $x_j^2, x_k^3 \neq 0$) are not linear combinations of greater words modulo I_i, so they are elements of the chosen basis. By the relation (2-15) between Hilbert series and the fact that $3\mathbf{e}_k + 2\mathbf{e}_j \notin s_i\left(\Delta^+(\mathfrak{B})\right)$, the Lyndon word $x_k^2 x_j x_k x_j$ is not an element of the basis. Thus this word is a linear combination of greater words, and by Lemma 5.5, this implies that (5-6) holds in $s_i(\mathfrak{B})$. □

Proof of (5-38). We prove case by case that

$$s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) \notin \Delta^+(\mathfrak{B}).$$

- If $m_{ik} = m_{ij} = m_{il} = 0$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l$, and this is not in $\Delta^+(\mathfrak{B})$ by Lemma 5.6.
- If $i \neq j, k, l$ and $m_{ik} \neq 0$, the only possibility is $m_{ik} = m_{ki} = 1$, so V is of type F_4. Thus $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l$. For the order $x_l < x_j < x_k < x_i$, the only possible Lyndon word without the factors $x_l x_j^2$, $x_l x_k$, $x_l x_i$, $x_j^2 x_k$, $x_j x_i$, $x_k x_i^2$, $x_k^2 x_i$ is $x_l x_j x_k x_i x_j x_k x_i x_k x_i$. Using the quantum Serre relations and the fact that $q_{ii} = q_{kk} \neq -1$, we see that this Lyndon word is not in S_I. Thus $3\mathbf{e}_i + 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$.
- $i \neq j, k, l$ and $m_{ij} \neq 0$: there are no standard braided vector spaces with these values.
- If $i \neq j, k, l$ and $m_{il} \neq 0$, the unique possibility is $m_{il} = m_{li} = 1$. In this case $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l + \mathbf{e}_i$. If we consider $x_k < x_j < x_l < x_i$, the only possible Lyndon word of this degree without the factors $x_k x_l$, $x_k x_i$, $x_j x_i$, $x_k^3 x_j$, $x_k x_j^2$ is $x_k^2 x_j x_l x_i x_k x_i$. But by assumption,

$$\big[[x_k^2 x_j x_l]_c, [x_k x_j]_c\big]_c = \big[x_i, [x_k x_j]_c\big]_c = 0,$$

so $[x_k^2 x_j x_l x_i x_k x_i]_c = \big[[x_k^2 x_j x_l x_i]_c, [x_k x_j]_c\big]_c = 0$, and $x_k^2 x_j x_l x_i x_k x_i \notin S_I$.
- If $i = k$, then $s_i(3\mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_l) = \mathbf{e}_i + 2\mathbf{e}_j + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, by Lemma 5.4.
- If $i = j$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_i + \mathbf{e}_l) = 3\mathbf{e}_k + 2\mathbf{e}_i + \mathbf{e}_l \notin \Delta^+(\mathfrak{B})$, by Lemma 5.6.
- If $i = k$, then $s_i(3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_i) = \mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_i \notin \Delta^+(\mathfrak{B})$, as before.

Now, if $\mathbf{w} \in \{\mathbf{e}_k, \mathbf{e}_j, \mathbf{e}_l, \mathbf{e}_k + \mathbf{e}_j, \mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, 2\mathbf{e}_k + \mathbf{e}_j, 2\mathbf{e}_k + \mathbf{e}_j + \mathbf{e}_l, 2\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l\}$, then $\mathbf{w} \in \Delta(\mathfrak{B}(V_i))$, so $s_i(\mathbf{w}) \in \Delta(\mathfrak{B}(V))$, hence $s_i(\mathbf{w}) \in \Delta(\mathfrak{B})$.

Consider a basis as in Proposition 1.11 for an order such that $x_k < x_j < x_l$. Then $x_j x_k$ and $x_k x_l$ are elements of this basis. We know that $x_k x_l$, $x_k^3 x_j$, $x_k x_j^2$, $x_k x_j x_l x_k$, $x_k^2 x_j x_k x_j$ are not elements of the basis, since (5-3), (5-4) and (5-6) hold in $\mathfrak{B}$. By Lemma 2.7, the relation (2-15) between Hilbert series and the fact that $3\mathbf{e}_k + 2\mathbf{e}_j + \mathbf{e}_l \notin s_i(\Delta^+(\mathfrak{B}))$, the Lyndon word $x_k^2 x_j x_l x_k x_j$ is not an element of the basis. Thus this word is a linear combination of greater words. By Lemma 5.6, this implies that (5-8) holds in $s_i(\mathfrak{B})$. □

This concludes the proof of Lemma 5.23. Note also that $s_i(\mathfrak{B})$ is of the same type as $\mathfrak{B}$.

Let V be of a type different from G_2. We define the algebra $\hat{\mathfrak{B}}(V) := T(V)/\mathfrak{I}(V)$, where $\mathfrak{I}(V)$ is the two-sided ideal of $T(V)$ generated by

- $(\mathrm{ad}_c\, x_k)^{m_{kj}+1} x_j$, $k \neq j$;
- $\big[(\mathrm{ad}_c\, x_j)(\mathrm{ad}_c\, x_k) x_l, x_k\big]_c$, $l \neq k \neq j$, $q_{kk} = -1$, $m_{kj} = m_{kl} = 1$;
- $\big[(\mathrm{ad}_c\, x_k)^2 x_j, (\mathrm{ad}_c\, x_k) x_j\big]_c$, $k \neq j$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, $m_{kj} = 2, m_{jk} = 1$;
- $\big[(\mathrm{ad}_c\, x_k)^2 (\mathrm{ad}_c\, x_j) x_l, (\mathrm{ad}_c\, x_k) x_j\big]_c$, $k \neq j \neq l$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, $m_{kj} = 2$, $m_{jk} = m_{jl} = 1$.

(Compare with the definitions in Section 4 of [Andruskiewitsch and Schneider 2002a].) Since V is of Cartan type, $\mathfrak{I}(V)$ is a Hopf ideal, by Lemmas 5.7–5.9. Since $\mathfrak{I}(V)$ also is $\mathbb{Z}^\theta$-homogeneous, we have $\mathfrak{I}(V) \in \mathfrak{S}$.

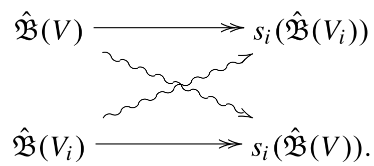
By Lemmas 5.4–5.6, the canonical epimorphism $T(V) \to \mathfrak{B}(V)$ induces an epimorphism of braided graded Hopf algebras

$$\pi_V : \hat{\mathfrak{B}}(V) \to \mathfrak{B}(V). \tag{5-39}$$

Also, $\hat{\mathfrak{B}}(V)$ satisfies the conditions in Theorem 2.6 for each $i \in \{1, \dots, \theta\}$, so we can transform it.

Lemma 5.24. *With the notation above*, $s_i(\hat{\mathfrak{B}}(V)) \cong \hat{\mathfrak{B}}(V_i)$.

Proof. By Lemma 5.23, the relations defining $\mathfrak{I}(V_i)$ are satisfied in $s_i(\hat{\mathfrak{B}}(V))$. Thus the canonical projections from $T(V_i)$ onto $\hat{\mathfrak{B}}(V_i)$ and $s_i(\hat{\mathfrak{B}}(V))$ induce a surjective algebra map $\hat{\mathfrak{B}}(V_i) \to s_i(\hat{\mathfrak{B}}(V))$. Conversely, each relation defining $\mathfrak{I}(V)$ is satisfied in $s_i(\hat{\mathfrak{B}}(V_i))$, so we have the following situation:



From the relation (2-15) between Hilbert series, we have, for each $\mathbf{u} \in \mathbb{N}^\theta$,

$$\dim s_i(\hat{\mathfrak{B}}(V))^{\mathbf{u}} = \sum_{\substack{k\in\mathbb{N}:\, \mathbf{u}-k\mathbf{e}_i\in\mathbb{N}^\theta \\ s_i(\mathbf{u}-k\mathbf{e}_i)\in\mathbb{N}^\theta}} \dim \hat{\mathfrak{B}}(V)^{s_i(\mathbf{u}-k\mathbf{e}_i)},$$

and a analogous relation for $\dim s_i(\hat{\mathfrak{B}}(V_i))^{\mathbf{u}}$. But in view of the previous surjections we have

$$\dim s_i(\hat{\mathfrak{B}}(V))^{\mathbf{u}} \le \dim \hat{\mathfrak{B}}(V_i)^{\mathbf{u}}, \quad \dim s_i(\hat{\mathfrak{B}}(V_i))^{\mathbf{u}} \le \dim \hat{\mathfrak{B}}(V)^{\mathbf{u}},$$

for each $\mathbf{u} \in \mathbb{N}^\theta$. Since $s_i^2 = \mathrm{id}$, each of these inequalities is in fact an equality, and $s_i(\hat{\mathfrak{B}}(V)) = \hat{\mathfrak{B}}(V_i)$. □

We are now able to prove one of the main results of this paper.

Theorem 5.25. *Let V be a braided vector space of Cartan type, of dimension θ, and $C = (a_{ij})_{i,j\in\{1,\dots,\theta\}}$ the corresponding finite Cartan matrix, where $a_{ij} := -m_{ij}$.*

The Nichols algebra $\mathfrak{B}(V)$ is presented by the generators x_i, for $1 \le i \le \theta$, and the relations

$$x_\alpha^{N_\alpha} = 0, \quad \alpha \in \Delta^+, \tag{5-40}$$

$$\mathrm{ad}_c(x_k)^{1-a_{kj}}(x_j) = 0, \quad k \ne j. \tag{5-41}$$

If there exist $j \ne k \ne l$ such that $m_{kj} = m_{kl} = 1$, $q_{kk} = -1$, then

$$\big[(\mathrm{ad}\, x_k)x_j, (\mathrm{ad}\, x_k)x_l\big]_c = 0. \tag{5-42}$$

If there exist $k \ne j$ such that $m_{kj} = 2$, $m_{jk} = 1$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, then

$$\big[(\mathrm{ad}\, x_k)^2 x_j, (\mathrm{ad}\, x_k)x_j\big]_c = 0. \tag{5-43}$$

If there exist $k \ne j \ne l$ such that $m_{kj} = 2$, $m_{jk} = m_{jl} = 1$, $q_{kk} \in \mathbb{G}_3$ or $q_{jj} = -1$, then

$$\big[(\mathrm{ad}\, x_k)^2(\mathrm{ad}\, x_j)x_l, (\mathrm{ad}\, x_k)x_j\big]_c = 0. \tag{5-44}$$

If $\theta = 2$, V if of type G_2, and $q_{11} \in \mathbb{G}_4$ or $q_{22} = -1$, then

$$\big[(\mathrm{ad}\, x_1)^3 x_2, (\mathrm{ad}\, x_1)^2 x_2\big]_c = 0, \tag{5-45}$$

$$\big[x_1, \big[x_1^2 x_2 x_1 x_2\big]_c\big]_c = 0, \tag{5-46}$$

$$\big[\big[x_1^2 x_2 x_1 x_2\big]_c, [x_1 x_2]_c\big]_c = 0, \tag{5-47}$$

$$\big[\big[x_1^2 x_2\big]_c, \big[x_1^2 x_2 x_1 x_2\big]_c\big]_c = 0. \tag{5-48}$$

The following elements constitute a basis of $\mathfrak{B}(V)$:

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \quad \textit{where } 0 \le h_j \le N_{\beta_j} - 1, \textit{ if } \beta_j \in S_I, \textit{ for } 1 \le j \le P.$$

Proof. We may assume that C is connected. For V of type G_2, the result was proved in Theorem 5.22. So we can assume $m_{kj} \neq 3$, $k \neq j$.

The statement about the PBW basis was proved in Corollary 4.2; see the definition of the x_α in Section 4B.

Consider the images of the x_α in $\hat{\mathfrak{B}}(V)$; they correspond in $\mathfrak{B}(V)$ with the x_α, and are PBW generators for a basis constructed as in Theorem 1.12, considering the same order in the letters. As we observed in (5-39), there exists a surjective morphism of braided Hopf algebras $\hat{\mathfrak{B}}(V) \to \mathfrak{B}(V)$, so

$$\Delta(\mathfrak{B}(V)) \subseteq \Delta(\hat{\mathfrak{B}}(V)).$$

Also, $\hat{\mathfrak{B}}(V)$ satisfies the conditions in Theorem 2.6 for each $i \in \{1, \dots, \theta\}$, so we can transform it. By Lemma 5.24, the new algebra is $\hat{\mathfrak{B}}(V_i)$, so we can continue. Consider the sets

$$\hat{\Delta} := \bigcup \{\Delta(s_{i_1} \dots s_{i_k} \hat{\mathfrak{B}}) : k \in \mathbb{N}, 1 \leq i_1, \dots, i_k \leq \theta\}, \quad \hat{\Delta}^+ := \Delta \cap \mathbb{N}^\theta;$$

$\hat{\Delta}$ is invariant by the s_i. Also, $\Delta(\mathfrak{B}(V)) \subseteq \Delta$, and

$$\Delta(s_{i_1} \dots s_{i_k} \hat{\mathfrak{B}}(V)) = s_{i_1} \dots s_{i_k} \Delta(\hat{\mathfrak{B}}(V)).$$

Consider $\alpha \in \hat{\Delta}^+ \setminus \Delta^+(\mathfrak{B}(V))$. Suppose that α is not of the form $m\alpha_i$ for $m \in \mathbb{N}$ and $i \in \{1, \dots, \theta\}$, and that it is of minimal height among such roots. For each s_i, since α is not a multiple of α_i, we have $s_i(\alpha) \in \Delta^+ \setminus \Delta^+ (\mathfrak{B}(V))$; hence $\deg s_i(\alpha) - \deg \alpha \geq 0$. But $\alpha = \sum_{i=1}^{\theta} b_i \mathbf{e}_i$, so $\sum_{i=1}^{\theta} b_i a_{ij} \leq 0$, and since $b_i \geq 0$, we have $\sum_{i,j=1}^{\theta} b_i a_{ij} b_j \leq 0$. This contradicts the fact that (a_{ij}) is definite positive, and $(b_i) \geq 0$, $(b_i) \neq 0$.

Also, $m\mathbf{e}_i \in \Delta^+(\hat{\mathfrak{B}}) \iff m = N_{\mathbf{e}_i}$ or $m = 1$, since $x_i^{N_{\mathbf{e}_i}} \neq 0$. Hence

$$\Delta(\hat{\mathfrak{B}}(V)) = \Delta(\mathfrak{B}(V)) \cup \{N_\alpha \alpha : \alpha \in \Delta(\mathfrak{B}(V))\}.$$

This follows since by Corollary 4.2 each $\alpha \in \Delta^+(\mathfrak{B}(V))$ is of the form

$$\alpha = s_{i_1} \cdots s_{i_m}(\mathbf{e}_j), \quad i_1, \dots, i_m, j \in \{1, \dots, \theta\}.$$

Now, $N_{\mathbf{e}_j} \mathbf{e}_j \in \Delta(\hat{\mathfrak{B}}(V))$, so

$$N_\alpha \alpha = N_{\mathbf{e}_j} \alpha = s_{i_1} \dots s_{i_m}(N_{\mathbf{e}_j} \mathbf{e}_j) \in \Delta(\hat{\mathfrak{B}}(V)).$$

Also, each degree $N_\alpha \alpha$ has multiplicity one in $\Delta(\hat{\mathfrak{B}}(V))$.

Suppose there exist Lyndon words of degree $N_\alpha \alpha$, and consider one such word u of minimal height. Let $u = vw$ be a Shirshov decomposition thereof, and put

$$\beta := \deg v, \ \gamma := \deg w \in \Delta^+(\hat{\mathfrak{B}}(V)).$$

By the preceding assumption, $\beta, \gamma \in \Delta^+(\mathfrak{B}(V))$. Write

$$\alpha = \sum_{k=1}^{\theta} a_k \mathbf{e}_k, \quad \beta = \sum_{k=1}^{\theta} b_k \mathbf{e}_k, \quad \gamma = \sum_{k=1}^{\theta} c_k \mathbf{e}_k,$$

so $N_\alpha a_k = b_k + c_k$, for each $k \in \{1, \dots, \theta\}$. We can assume, by taking a subdiagram if necessary, that $a_1, a_\theta \neq 0$.

Now, if V is of type F_4 and $\beta = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_3 + 3\mathbf{e}_4$, then $c_1 = 0$, $a_1 = 1$, $N_\alpha = 2$, or $a_1 = c_1 = 1$, $N_\alpha = 3$, since $\alpha, \gamma \neq \beta$.

- If $N_\alpha = 3$, then $3a_2 = 3 + c_2$. Hence $c_2 = 0$, so $c_3 = c_4 = 0$, or $c_2 = 3$, and $c_3 = 4$, $c_4 = 2$. But in both cases we have a contradiction to $\alpha \in \mathbb{N}^4$.
- If $N_\alpha = 2$, $c_1 = 0$, then c_2 and c_4 are odd, and c_3 is even and nonzero. The only possibility is $\gamma = \mathbf{e}_2 + 2\mathbf{e}_3 + \mathbf{e}_4$, so $\alpha = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 + 2\mathbf{e}_4$. But $q_\alpha = q \neq -1$, so $N_\alpha \neq 2$, which is a contradiction.

Thus we can assume $b_1, c_1 \leq 1$ or $b_\theta, c_\theta \leq 1$, so $a_1 = b_1 = c_1 = 1$ or $a_\theta = b_\theta = c_\theta = 1$; in both cases, $N_\alpha = 2$. For each possible β with $b_1 \neq 0$ (by the assumption that $a_1 \neq 0$, we have $b_1 \neq 0$ or $c_1 \neq 0$), we look for γ such that $\beta + \gamma$ has even coordinates. In types A, D and E there are no such pairs of roots. As for the other types:

- B_θ: $\beta = \mathbf{v}_{i\theta}$, $\gamma = \mathbf{u}_{i+1,\theta}$. Then $\alpha = \mathbf{u}_{1\theta}$, but $q_\alpha = q_{11} \neq -1$, which is a contradiction.
- C_θ: $\beta = \mathbf{w}_{11}$, $\gamma = \mathbf{e}_\theta$. Then $\alpha = \mathbf{u}_{1\theta}$, but $q_\alpha = q_{\theta\theta} \neq -1$, which is a contradiction.
- F_4: $\beta = \mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3 + 2\mathbf{e}_4$, $\gamma = \mathbf{e}_1 + \mathbf{e}_2$, or $\beta = \mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3 + 2\mathbf{e}_4$, $\gamma = \mathbf{e}_1$. In both cases, $\alpha = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$, but $q_\alpha = q \neq -1$, which is a contradiction.

Thus each root $N_\alpha \alpha$ corresponds to $x_\alpha^{N_\alpha}$, and each x_α has infinite height, as before. The elements

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \quad \text{where } 0 \leq h_j < \infty, \text{ if } \beta_j \in S_I, \text{ for } 1 \leq j \leq P,$$

form a basis of $\hat{\mathfrak{B}}(V)$ as a vector space.

Now let $\bar{I}(V)$ be the ideal of $T(V)$ generated by the relations (5-41)–(5-44) and (5-40). We have $\mathfrak{I}(V) \subseteq \bar{I}(V) \subseteq I(V)$, so the corresponding projections induce a surjective morphism of algebras $\phi : \mathfrak{B} \to \mathfrak{B}(V)$, where $\mathfrak{B} := T(V)/\bar{I}(V)$:

$$\begin{array}{ccc} T(V) & \twoheadrightarrow & \hat{\mathfrak{B}}(V) \\ \downarrow & \swarrow & \downarrow \\ \mathfrak{B}(V) & \overset{\phi}{\twoheadleftarrow} & \mathfrak{B} \end{array}$$

Also, the elements

$$x_{\beta_1}^{h_1} x_{\beta_2}^{h_2} \dots x_{\beta_P}^{h_P}, \quad \text{where } 0 \le h_j < N_{\beta_j}, \text{ if } \beta_j \in S_I, \text{ for } 1 \le j \le P,$$

generate $\mathfrak{B}$ as a vector space, because they correspond to images of generators of $\hat{\mathfrak{B}}(V)$ and are nonzero (as before, each nonincreasing product of hyperwords such that $h_j \ge N_{\beta_j}$ is zero in $\mathfrak{B}$). But ϕ is surjective, and the corresponding images of these elements form a basis of $\mathfrak{B}(V)$, so ϕ is an isomorphism. □

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