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We present a relationship between continued fractions and Weyl groupoids of Cartan schemes of rank two. This allows one to decide easily if a given Cartan scheme of rank two admits a finite root system. We obtain obstructions and sharp bounds for the entries of the Cartan matrices.

1. Introduction

Root systems and crystallographic Coxeter groups are key tools in the study of semisimple Lie algebras [Bourbaki 1968]. In the structure theory of pointed Hopf algebras [Montgomery 1993] a similar role is expected to be played by Weyl groupoids and their root systems. Let us give some hints towards this claim. The most striking results on pointed Hopf algebras rely on the lifting method of Andruskiewitsch and Schneider [1998]. Based on it, many new examples of finite-dimensional pointed Hopf algebras have been detected, and fairly general classification results were achieved [Andruskiewitsch and Schneider 2005; Heckenberger 2009]. The first step in the lifting method is the determination of finite-dimensional Nichols algebras of finite group type. The upper triangular part of a small quantum group, also called Frobenius–Lusztig kernel, is a prominent example. A very natural symmetry object of Nichols algebras of finite group type is the Weyl groupoid. This was observed first in [Heckenberger 2006] for Nichols algebras of diagonal type, and then in [Andruskiewitsch et al. 2008] in a very general setting.

An axiomatic approach to Weyl groupoids and their root systems, without referring to Nichols algebras, was initiated in [Heckenberger and Yamane 2008]. The theory includes and extends the theory of crystallographic Coxeter groups, but contains even such examples which do not seem to be related to Nichols algebras of diagonal type. In this paper we use the language and some structural and classification results achieved in [Cuntz and Heckenberger 2008]; see Section 2 for the most essential definitions and facts.

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For the classification of Nichols algebras of diagonal type it is crucial to be able to decide whether a given Cartan scheme (a categorical generalization of the notion of a generalized Cartan matrix; see Definition 2.1) admits a finite root system. Because of the large variety of examples, this seems to be a difficult task. In our paper, we present a very efficient method for Cartan schemes of rank two. It relies on a relationship between Cartan schemes of rank two and continued fractions [Perron 1929]. Instead of giving a complete list of Cartan schemes of rank two admitting a finite root system (which is then unique by a result in [Cuntz and Heckenberger 2008]), we present an algorithm in Theorem 6.19. It works with very elementary operations on sequences of positive integers and transforms any Cartan scheme into another one, for which the answer is known. The algorithm is based on various observations: on the introduction and study of coverings of Cartan schemes in Section 3, on an old theorem of Stern, Pringsheim, and Tietze, and a variation of a transformation formula for continued fractions (Section 4 and Lemma 5.2) on the characterization of simple connected Cartan schemes admitting a finite root system in terms of certain sequences of positive integers (Proposition 6.5 and Theorem 6.6), and on the description of Cartan schemes with object change diagram a cycle using characteristic sequences (Definition 6.9). As an application, in Section 7 we give obstructions for the entries of the Cartan matrices in a Cartan scheme admitting a finite root system. We present the power of our method on a small example at the end of Section 6.

We are confident that a suitable generalization of our method to Cartan schemes and Weyl groupoids of higher rank would have a deep impact on the classification of Nichols algebras, and consider it as a great challenge for the future.

2. Cartan schemes, root systems, and their Weyl groupoids

If not stated otherwise, we follow the notation in [Cuntz and Heckenberger 2008]. Let us start by recalling the main definitions.

Let I be a nonempty finite set and $\{\alpha_i \mid i \in I\}$ the standard basis of $\mathbb{Z}^I$. By [Kac 1990, §1.1] a generalized Cartan matrix $C = (c_{ij})_{i,j\in I}$ is a matrix in $\mathbb{Z}^{I\times I}$ such that

(M1) $c_{ii} = 2$ and $c_{jk} \le 0$ for all $i, j, k \in I$ with $j \ne k$,

(M2) if $i, j \in I$ and $c_{ij} = 0$, then $c_{ji} = 0$.

Definition 2.1. Let A be a nonempty set, $\rho_i : A \to A$ a map for all $i \in I$, and $C^a = (c^a_{jk})_{j,k\in I}$ a generalized Cartan matrix in $\mathbb{Z}^{I\times I}$ for all $a \in A$. The quadruple

$$\mathscr{C} = \mathscr{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$$

is called a *Cartan scheme* if

(C1) $\rho_i^2 = \mathrm{id}$ for all $i \in I$,

(C2) $c^a_{ij} = c^{\rho_i(a)}_{ij}$ for all $a \in A$ and $i, j \in I$.

Remark 2.2. The preceding definition of a Cartan scheme has the striking advantage to be very simple, but sufficiently powerful to admit the definition of a Weyl groupoid, as we will see below. For some investigations it can be of advantage to consider more general axioms (for example by allowing the maps ρ_i to be partially defined) or to impose additional restrictions (like (C3) below, or other for example to exclude the existence of associated roots which are neither positive nor negative). We will mostly consider Cartan schemes admitting a root system. This restriction still gives many more examples than those coming from contragredient Lie superalgebras and Nichols algebras of diagonal type with finite root system. Nevertheless, up to now no further axioms on Cartan schemes are known which keep this property.

Two Cartan schemes

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A}) \quad \text{and} \quad \mathcal{C}' = \mathcal{C}'(I', A', (\rho'_i)_{i\in I'}, ({C'}^a)_{a\in A'})$$

are termed *equivalent* if there are bijections $\varphi_0 : I \to I'$ and $\varphi_1 : A \to A'$ such that

$$\varphi_1(\rho_i(a)) = \rho'_{\varphi_0(i)}(\varphi_1(a)), \qquad c^{\varphi_1(a)}_{\varphi_0(i)\varphi_0(j)} = c^a_{ij} \tag{2-1}$$

for all $i, j \in I$ and $a \in A$.

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $i \in I$ and $a \in A$ define $\sigma^a_i \in \mathrm{Aut}(\mathbb{Z}^I)$ by

$$\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i \qquad \text{for all } j \in I. \tag{2-2}$$

The *Weyl groupoid of* $\mathcal{C}$ is the category $\mathcal{W}(\mathcal{C})$ such that $\mathrm{Ob}(\mathcal{W}(\mathcal{C})) = A$ and the morphisms are generated by the maps $\sigma^a_i \in \mathrm{Hom}(a, \rho_i(a))$ with $i \in I$, $a \in A$. In this paper, we will always denote the set of all morphisms of $\mathcal{W}(\mathcal{C})$ by $\mathrm{Hom}(\mathcal{W}(\mathcal{C}))$. Formally, for $a, b \in A$ the set $\mathrm{Hom}(a, b)$ consists of the triples (b, f, a), where

$$f = \sigma^{\rho_{i_{n-1}}\cdots\rho_{i_1}(a)}_{i_n} \cdots \sigma^{\rho_{i_1}(a)}_{i_2}\sigma^a_{i_1}$$

and $b = \rho_{i_n}\cdots\rho_{i_2}\rho_{i_1}(a)$ for some $n \in \mathbb{N}_0$ and $i_1, \ldots, i_n \in I$. The composition is induced by the group structure of $\mathrm{Aut}(\mathbb{Z}^I)$:

$$(a_3, f_2, a_2) \circ (a_2, f_1, a_1) = (a_3, f_2 f_1, a_1)$$

for all $(a_3, f_2, a_2), (a_2, f_1, a_1) \in \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$. By abuse of notation we will write $f \in \mathrm{Hom}(a, b)$ instead of $(b, f, a) \in \mathrm{Hom}(a, b)$.

The cardinality of I is termed the *rank of* $\mathcal{W}(\mathcal{C})$. A Cartan scheme is called *connected* if its Weyl groupoid is connected, that is, if for all $a, b \in A$ there exists $w \in \mathrm{Hom}(a, b)$.

In many cases it will be natural to assume that a Cartan scheme satisfies the following additional property.

(C3) If $a, b \in A$ and $(b, \mathrm{id}, a) \in \mathrm{Hom}(a, b)$, then $a = b$.

Definition 2.3. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. For all $a \in A$ let $R^a \subset \mathbb{Z}^I$, and define $m^a_{i,j} = |R^a \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ for all $i, j \in I$ and $a \in A$. We say that

$$\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$$

is a *root system of type* $\mathcal{C}$ if it satisfies the following axioms.

(R1) $R^a = R^a_+ \cup -R^a_+$, where $R^a_+ = R^a \cap \mathbb{N}_0^I$, for all $a \in A$.

(R2) $R^a \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ for all $i \in I$, $a \in A$.

(R3) $\sigma^a_i(R^a) = R^{\rho_i(a)}$ for all $i \in I$, $a \in A$.

(R4) If $i, j \in I$ and $a \in A$ such that $i \neq j$ and $m^a_{i,j}$ is finite, then $(\rho_i\rho_j)^{m^a_{i,j}}(a) = a$.

If $\mathcal{R}$ is a root system of type $\mathcal{C}$, then we say that $\mathcal{W}(\mathcal{R}) = \mathcal{W}(\mathcal{C})$ is the *Weyl groupoid of* $\mathcal{R}$. Further, $\mathcal{R}$ is called *connected* if $\mathcal{C}$ is a connected Cartan scheme. If $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ is a root system of type $\mathcal{C}$ and $\mathcal{R}' = \mathcal{R}'(\mathcal{C}', (R'^a)_{a\in A'})$ is a root system of type $\mathcal{C}'$, then we say that $\mathcal{R}$ and $\mathcal{R}'$ are *equivalent* if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent Cartan schemes given by maps $\varphi_0 : I \to I'$, $\varphi_1 : A \to A'$ as in Definition 2.1, and if the map $\varphi_0^* : \mathbb{Z}^I \to \mathbb{Z}^{I'}$ given by $\varphi_0^*(\alpha_i) = \alpha_{\varphi_0(i)}$ satisfies $\varphi_0^*(R^a) = R'^{\varphi_1(a)}$ for all $a \in A$.

There exist many interesting examples of root systems of type $\mathcal{C}$ related to semisimple Lie algebras, Lie superalgebras and Nichols algebras of diagonal type, respectively. For further details and results we refer to [Heckenberger and Yamane 2008] and [Cuntz and Heckenberger 2008].

Convention 2.4. In connection with Cartan schemes, upper indices usually refer to elements of A. Often, these indices will be omitted if they are uniquely determined by the context.

Remark 2.5. If $\mathcal{C}$ is a Cartan scheme and there exists a root system of type $\mathcal{C}$, then $\mathcal{C}$ satisfies (C3) by [Heckenberger and Yamane 2008, Lemma 8(iii)].

Definition 4.3 of [Cuntz and Heckenberger 2008] introduced the concept of an *irreducible* root system of type $\mathcal{C}$. By Proposition 4.6 of the same paper, if $\mathcal{C}$ is a connected Cartan scheme and $\mathcal{R}$ is a finite root system of type $\mathcal{C}$, then $\mathcal{R}$ is irreducible if and only if the generalized Cartan matrix C^a is indecomposable for one (equivalently, for all) $a \in A$.

Here is a fundamental result about Weyl groupoids.

Theorem 2.6 [Heckenberger and Yamane 2008, Theorem 1]. *Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme and $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ a root system of type $\mathcal{C}$. Let $\mathcal{W}$ be the abstract groupoid with* $\mathrm{Ob}(\mathcal{W}) = A$ *such that* $\mathrm{Hom}(\mathcal{W})$ *is generated by abstract morphisms $s_i^a \in \mathrm{Hom}(a, \rho_i(a))$, where $i \in I$ and $a \in A$, satisfying the relations*

$$s_i s_i 1_a = 1_a, \quad (s_j s_k)^{m^a_{j,k}} 1_a = 1_a, \qquad a \in A,\ i, j, k \in I,\ j \neq k$$

(see Convention 2.4). *Here 1_a is the identity of the object a, and $(s_j s_k)^\infty 1_a$ is understood to be 1_a. The functor $\mathcal{W} \to \mathcal{W}(\mathcal{R})$, which is the identity on the objects, and on the set of morphisms is given by $s_i^a \mapsto \sigma_i^a$ for all $i \in I$, $a \in A$, is an isomorphism of groupoids.*

Definition 2.7. Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a Cartan scheme. Let Γ be a nondirected graph such that the vertices of Γ correspond to the elements of A. Assume that for all $i \in I$ and $a \in A$ with $\rho_i(a) \neq a$ there is precisely one edge between the vertices a and $\rho_i(a)$ with label i, and all edges of Γ are given in this way. The graph Γ is called the *object change diagram* of $\mathcal{C}$. If $\mathcal{R} = \mathcal{R}(\mathcal{C}, (R^a)_{a\in A})$ is a root system of type $\mathcal{C}$, then we also say that Γ is the object change diagram of $\mathcal{R}$.

3. Coverings of Cartan schemes, Weyl groupoids, and root systems

Two Cartan schemes can be related to each other in different ways. In this section we analyze coverings of Cartan schemes. The definition is motivated by the corresponding notion in topology.

Definition 3.1. Let

$$\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A}) \quad \text{and} \quad \mathcal{C}' = \mathcal{C}'(I, A', (\rho'_i)_{i\in I}, (C'^a)_{a\in A'})$$

be connected Cartan schemes. Let $\pi : A' \to A$ be a map such that $C^{\pi(a)} = C'^a$ for all $a \in A'$ and the diagrams

$$\begin{tikzcd} A' \arrow[r, "\rho'_i"] \arrow[d, "\pi"'] & A' \arrow[d, "\pi"] \\ A \arrow[r, "\rho_i"'] & A \end{tikzcd} \tag{3-1}$$

commute for all $i \in I$. We say that $\pi : \mathcal{C}' \to \mathcal{C}$ is a *covering*, and that $\mathcal{C}'$ is a *covering of $\mathcal{C}$*.

The composition of two coverings is again one. For any covering $\pi : \mathcal{C}' \to \mathcal{C}$ of Cartan schemes $\mathcal{C}', \mathcal{C}$, the map $\pi : A' \to A$ is surjective by (3-1), since A' is nonempty and $\mathcal{C}$ is connected.

Remark 3.2. Many of the following results can be formulated without assuming that $\mathcal{C}$ and/or $\mathcal{C}'$ in Definition 3.1 are connected Cartan schemes. In that case one should assume that π is a surjective map. However, in the applications we are interested in, all Cartan schemes are connected, and hence we prefer the above definition in order to simplify the terminology.

Any covering $\pi : \mathcal{C}' \to \mathcal{C}$ of Cartan schemes $\mathcal{C}', \mathcal{C}$ induces a covariant functor $F_\pi : \mathcal{W}(\mathcal{C}') \to \mathcal{W}(\mathcal{C})$ by letting

$$F_\pi(a') = \pi(a'), \quad F_\pi(\sigma_i^{a'}) = \sigma_i^{\pi(a')} \qquad \text{for all } i \in I,\ a' \in A'.$$

In this case the Weyl groupoid $\mathcal{W}(\mathcal{C}')$ is termed a *covering of* $\mathcal{W}(\mathcal{C})$, and the functor F_π a covering of Weyl groupoids.

First we need a technical result.

Lemma 3.3. *Let $\pi : \mathcal{C}' \to \mathcal{C}$ be a covering, and assume that $\mathcal{C}'$ satisfies Axiom* (C3).

(1) *$\mathcal{C}$ satisfies* (C3).

(2) *Let $a \in A$ and $a', a'' \in A'$ such that $\pi(a') = \pi(a'') = a$. If there exists $w' \in$* $\mathrm{Hom}(a', a'')$ *such that $F_\pi(w') \in F_\pi(\mathrm{End}(a'))$, then $a' = a''$.*

Proof. (1) Let $a \in A$. If $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$, then Definition 3.1 gives that $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma_{i_k}^a = \sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma_{i_k}^{a'}$ in $\mathrm{Aut}(\mathbb{Z}^I)$ for all $a' \in A'$ with $\pi(a') = a$. Assume now that $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma_{i_k}^a = \mathrm{id}$. Then $\rho'_{i_1} \cdots \rho'_{i_k}(a') = a'$ for all $a' \in A'$ with $\pi(a') = a$, since $\mathcal{C}'$ satisfies (C3). Hence $\rho_{i_1} \cdots \rho_{i_k}(a) = a$ by (3-1). This yields the claim.

(2) Let $w'' \in \mathrm{End}(a')$ with $F_\pi(w'') = F_\pi(w')$. Then $F_\pi(w' w''^{-1}) = \mathrm{id}_a$, and hence $w' w''^{-1} = \mathrm{id}$ in $\mathrm{Aut}(\mathbb{Z}^I)$. Since $\mathcal{C}'$ satisfies (C3), it follows that $w' w''^{-1} = \mathrm{id}_{a'}$, and hence $a' = a''$. □

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i \in I}, (C^a)_{a \in A})$ be a connected Cartan scheme, $\mathcal{W}(\mathcal{C})$ its Weyl groupoid, and $a \in A$. Coverings of $\mathcal{C}$ can be parametrized by subgroups of $\mathrm{End}(a) \subset \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$ (up to conjugation).

Proposition 3.4. (1) *Let $\mathcal{C}'$ be a connected Cartan scheme and assume that $\pi : \mathcal{C}' \to \mathcal{C}$ is a covering. Let $a' \in A'$ with $\pi(a') = a$.*

(a) *The group homomorphism $F_\pi : \mathrm{End}(a') \to \mathrm{End}(a)$ is injective.*

(b) *For each $b' \in A'$ with $\pi(b') = a$ the subgroup $F_\pi(\mathrm{End}(b'))$ of $\mathrm{End}(a)$ is conjugate to $F_\pi(\mathrm{End}(a'))$.*

(c) *If U' is a subgroup of $\mathrm{End}(a)$ conjugate to $F_\pi(\mathrm{End}(a'))$, then there exists $b' \in A'$ with $\pi(b') = a$ and $F_\pi(\mathrm{End}(b')) = U'$.*

(2) *Suppose that $U \subset \mathrm{End}(a)$ is a subgroup. There exists a covering $\pi : \mathscr{C}' \to \mathscr{C}$ and $b' \in A'$ such that*

$$F_\pi(\mathrm{End}(b')) = U, \tag{3-2}$$

$$|\pi^{-1}(b)| = [\mathrm{End}(a) : U] \quad \textit{for all } b \in A. \tag{3-3}$$

If $\mathscr{C}$ satisfies Axiom (C3), then up to equivalence there is a unique covering $\mathscr{C}'$ satisfying (3-2) and Axiom (C3). For this covering (3-3) holds.

Proof. (1A) Each element $w' \in \mathrm{End}(a')$ is a product of $\sigma_i^{b'}$ for some $i \in I$ and $b' \in A'$. Moreover, w' can be naturally regarded as an element in $\mathrm{Aut}(\mathbb{Z}^I)$. The same is true for $w \in \mathrm{End}(a)$. Since $C'^{b'} = C^{\pi(b')}$ for all $b' \in A'$, $F_\pi(w')$ identifies with the same element of $\mathrm{Aut}(\mathbb{Z}^I)$ as w'.

(1B) Let $b' \in A'$. Since $\mathscr{C}'$ is connected, there exists $w' \in \mathrm{Hom}(a', b')$. Then $\mathrm{End}(b') = w' \,\mathrm{End}(a') w'^{-1}$. Since F_π is a functor,

$$F_\pi(\mathrm{End}(b')) = F_\pi(w') F_\pi(\mathrm{End}(a')) F_\pi(w')^{-1}.$$

(1C) Assume that $w \in \mathrm{End}(a)$ such that $U' = w F_\pi(\mathrm{End}(a'))w^{-1}$. Then $w = \sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma_{i_k}^a$ for some $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$. Let $w' = \sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma_{i_k}^{a'}$ and $b' = \rho'_{i_1} \cdots \rho'_{i_k}(a')$. Then $\mathrm{End}(b') = w' \,\mathrm{End}(a') w'^{-1}$, and hence $F_\pi(\mathrm{End}(b')) = w F_\pi(\mathrm{End}(a'))w^{-1} = U'$.

(2) We construct $\mathscr{C}'$ explicitly. Let

$$A' = \mathrm{Hom}(\mathscr{W}(\mathscr{C}))/U = \big\{ gU \subset \mathrm{Hom}(a, b) \mid b \in A,\, g \in \mathrm{Hom}(a, b) \big\}$$

be the set of left cosets. For all $i \in I$ and $gU \in A'$ with $g \in \mathrm{Hom}(a, b)$, where $b \in A$, define $C'^{gU} = C^b$ and $\rho'_i(gU) = \sigma_i^b gU$. Then $\rho'_i : A' \to A'$ satisfies (C1) since $\sigma_i^{\rho_i(b)} \sigma_i^b = \mathrm{id}$ and $\rho_i^2 = \mathrm{id}$, and $\mathscr{C}'$ fulfills (C3), since $\mathscr{C}$ does. Since $\mathscr{C}$ is connected, $\mathscr{C}' = \mathscr{C}'(I, A', (\rho'_i)_{i \in I}, (C'^{a'})_{a' \in A'})$ is a connected Cartan scheme. Define $\pi : A' \to A$ by $\pi(gU) = b$ for all $b \in A$, $g \in \mathrm{Hom}(a, b)$. Then $F_\pi(\mathrm{End}(1_a U)) = U$ and $|\pi^{-1}(a)| = [\mathrm{End}(a) : U]$. Since $\mathscr{C}'$ is connected, $|\pi^{-1}(b)| = |\pi^{-1}(a)|$ for all $b \in A$.

Assume that $\mathscr{C}$ satisfies (C3). We show that $\mathscr{C}'$ satisfies (C3). For $l \in \{1, 2\}$ let $a_l \in A$ and $g_l \in \mathrm{Hom}(a, a_l)$ such that $(g_1 U, \mathrm{id}, g_2 U) \in \mathrm{Hom}(\mathscr{W}(\mathscr{C}'))$. Then there exist $k \in \mathbb{N}_0$ and $i_1, \dots, i_k \in I$ such that $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma_{i_k}^{a_2} g_2 U = g_1 U$ and that $\sigma_{i_1} \cdots \sigma_{i_{k-1}} \sigma_{i_k}^{a_2} = \mathrm{id}$ in $\mathrm{Aut}(\mathbb{Z}^I)$. Since $\mathscr{C}$ fulfills (C3), we obtain that $a_1 = a_2$, and hence $g_2 U = g_1 U$. Therefore $\mathscr{C}'$ satisfies (C3).

Finally, let $\pi : \mathscr{C}' \to \mathscr{C}$ and $\pi'' : \mathscr{C}'' \to \mathscr{C}$ be coverings of $\mathscr{C}$ satisfying (C3), and assume that there exist $b' \in A'$, $b'' \in A''$ such that $\pi(b') = \pi''(b'') = a$ and $F_\pi(\mathrm{End}(b')) = F_{\pi''}(\mathrm{End}(b'')) = U$. We have to show that $\mathscr{C}'$ and $\mathscr{C}''$ are equivalent Cartan schemes. Define $\phi : A' \to A''$ by

$$\phi(\rho'_{i_1} \cdots \rho'_{i_k}(b')) = \rho''_{i_1} \cdots \rho''_{i_k}(b'') \qquad \text{for all } k \in \mathbb{N}_0,\, i_1, \dots, i_k \in I.$$

Then ϕ is well-defined: Assume that $\rho'_{i_1}\cdots\rho'_{i_k}(b')=b'$. Then $\sigma_{i_1}\cdots\sigma_{i_k}^{b'}\in\mathrm{End}(b')$, and hence an application of π and F_π gives $\rho_{i_1}\cdots\rho_{i_k}(a)=a$, $\sigma_{i_1}\cdots\sigma_{i_k}^{a}\in U$. Thus $F_{\pi''}(\sigma_{i_1}\cdots\sigma_{i_k}^{b''})\in U$, and hence Lemma 3.3(2) gives that $\rho''_{i_1}\cdots\rho''_{i_k}(b'')=b''$. The compatibility of ϕ with ρ', ρ'', $C'^{b'}$, $C''^{b''}$ is fulfilled by Definition 3.1 and by definition of ϕ. Further, $\phi: A'\to A''$ is a bijection, the construction of ϕ^{-1} being analogous. Hence ϕ gives rise to an equivalence of the Cartan schemes $\mathcal{C}'$ and $\mathcal{C}''$. □

Definition 3.5. We say that a Cartan scheme $\mathcal{C}$ is *simply connected* if $\mathrm{End}(a)$ is the trivial group for all $a\in A$.

Corollary 3.6. *Let $\mathcal{C}$ be a connected Cartan scheme satisfying (C3). Then up to equivalence there exists a unique covering $\mathcal{C}'$ of $\mathcal{C}$ which is simply connected and satisfies (C3).*

As usual, this simply connected covering of $\mathcal{C}$ is called the *universal covering*.

Proof. The claim follows from Proposition 3.4(2) by setting $U=\{1\}$. □

Proposition 3.7. *Let $\mathcal{C}$, $\mathcal{C}'$ be connected Cartan schemes and $\pi:\mathcal{C}'\to\mathcal{C}$ a covering.*

(1) *If there exists a root system $\mathcal{R}'$ of type $\mathcal{C}'$, then the equations*

$$R^a=\bigcap_{\substack{a'\in A'\\ \pi(a')=a}} R'^{a'} \qquad \text{for all } a\in A \tag{3-4}$$

define a root system $\mathcal{R}$ of type $\mathcal{C}$.

(2) *If there exists a root system $\mathcal{R}$ of type $\mathcal{C}$, and $\mathcal{C}'$ satisfies (C3), then the equations*

$$R'^{a'}=R^{\pi(a')} \qquad \text{for all } a'\in A' \tag{3-5}$$

define a root system $\mathcal{R}'$ of type $\mathcal{C}'$.

Proof. (1) By Definition 3.1 and Axioms (R1)–(R4) for $\mathcal{R}'$, the Axioms (R1)–(R4) are fulfilled for $\mathcal{R}$.

(2) Since Axioms (R1)–(R3) hold for $\mathcal{R}$, they also hold for $\mathcal{R}'$. Suppose that $i,j\in I$ and $a'\in A'$ such that $i\neq j$ and that $m_{i,j}^{a'}=m_{i,j}^{a}$ is finite, where $a=\pi(a')$. Then $(\sigma_i\sigma_j)^{m_{i,j}^a}1_a=\mathrm{id}_a$ by Theorem 2.6. Hence $(\sigma_i\sigma_j)^{m_{i,j}^a}1_{a'}=\mathrm{id}$, and (C3) for $\mathcal{C}'$ implies that $(\rho'_i\rho'_j)^{m_{i,j}^{a'}}(a')=a'$. Thus (R4) holds for $\mathcal{R}'$ and hence $\mathcal{R}'$ is a root system of type $\mathcal{C}'$. □

4. Continued fractions

Continued fractions are related to Weyl groupoids of Cartan schemes of rank two. We recall some basic facts about continued fractions and formulate the facts we will use in our study.

A *continued fraction* is a sequence of indeterminates $a_1, a_2, a_3, \dots, b_0, b_1, \dots$ written in the form

$$b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} \tag{4-1}$$

(see [Perron 1929] for an introduction). We assume the a_i and b_i are integers. The *convergents* of (4-1) are the numbers

$$\frac{A_\nu}{B_\nu} = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots + \frac{a_\nu|}{|b_\nu},$$

for $\nu \in \mathbb{N}$, also given by the recursion

$$\begin{pmatrix} B_0 & A_0 \\ B_{-1} & A_{-1} \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} b_\nu & a_\nu \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_{\nu-1} & A_{\nu-1} \\ B_{\nu-2} & A_{\nu-2} \end{pmatrix} = \begin{pmatrix} B_\nu & A_\nu \\ B_{\nu-1} & A_{\nu-1} \end{pmatrix}. \tag{4-2}$$

One says that the continued fraction (4-1) is *convergent* if, for some $\nu_0 \in \mathbb{N}$, the sequence $(A_\nu/B_\nu)_{\nu \geq \nu_0}$ is well-defined and converges in $\mathbb{R}$.

The case where all a_ν are 1 is the most important one and well understood. However, we will be interested in a different case: From now on, let $a_\nu = -1$, $b_\nu \in \mathbb{N}$ for all ν and assume that the sequence $b_1, b_2, \dots$ is periodic. For any $i \in \mathbb{Z}$, let

$$\eta(i) = \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{4-3}$$

We will often need the following equations, which hold for all $i, j, k \in \mathbb{Z}$.

$$\eta(i)^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & i \end{pmatrix}, \tag{4-4}$$

$$\eta(i)\eta(j) = \begin{pmatrix} ij-1 & -i \\ j & -1 \end{pmatrix}, \tag{4-5}$$

$$\eta(i)\eta(j)\eta(k) = \begin{pmatrix} (ij-1)k-i & -(ij-1) \\ jk-1 & -j \end{pmatrix}, \tag{4-6}$$

$$\tau\eta(i)\tau = \eta(i)^{-1}, \quad \tau\eta(i)^{-1}\tau = \eta(i), \tag{4-7}$$

where

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4-8}$$

By (4-2),

$$\begin{pmatrix} B_n \\ B_{n-1} \end{pmatrix} = \eta(b_n)\cdots\eta(b_1)\begin{pmatrix} B_0 \\ B_{-1} \end{pmatrix}.$$

The product $\eta(b_n)\cdots\eta(b_1)$ will appear in the study of Weyl groupoids of rank two. In particular, we will need to know for which sequences $b_n,\dots,b_1$ this product has finite order. If it has finite order, then, since $B_{-1}=0$, there exists $\nu\in\mathbb{N}$ such that $B_\nu=0$.

The following fact is well-known. Variations of it were considered for example by Stern, Pringsheim, and Tietze; see respectively Satz 15 (§51), Satz 24 (§53), and Satz 1 (§35) in [Perron 1929].

Theorem 4.1. *If $a_\nu=-1$ and $b_\nu\geq 2$ for all $\nu\in\mathbb{N}$, then the continued fraction $\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\cdots$ is convergent.*

Thus we get:

Corollary 4.2. *Let $n\in\mathbb{N}$ and $b_1,\dots,b_n\in\mathbb{Z}$. If $b_i\geq 2$ for all $i\in\{1,\dots,n\}$, then $\eta(b_1)\cdots\eta(b_n)$ does not have finite order.*

Proof. Assume $b_i\geq 2$ for all $i\in\{1,\dots,n\}$. If $\eta(b_1)\cdots\eta(b_n)$ had finite order, then the periodic continued fraction

$$\frac{-1|}{|b_n}+\frac{-1|}{|b_{n-1}}+\cdots+\frac{-1|}{|b_1}+\frac{-1|}{|b_n}+\frac{-1|}{|b_{n-1}}+\cdots+\frac{-1|}{|b_1}+\frac{-1|}{|b_n}+\cdots$$

would have infinitely many convergents with denominator 0. This is a contradiction to Theorem 4.1. □

One can also prove Corollary 4.2 without Theorem 4.1, using for example [Heckenberger 2008, Lemma 9].

5. Distinguished finite sequences of integers

We now study a special class of finite sequences of positive integers. They correspond to a class of continued fractions which are not convergent. Later we will use these sequences to classify finite root systems of type $\mathscr{C}$ and rank two. Recall the definition of the map $\eta:\mathbb{Z}\to \mathrm{SL}(2,\mathbb{Z})$ from (4-3).

Definition 5.1. Let $\mathscr{A}$ denote the set of finite sequences $(c_1,\dots,c_n)$ of integers such that $n\geq 1$ and $\eta(c_1)\cdots\eta(c_n)=-\mathrm{id}$. Let $\mathscr{A}^+$ be the subset of $\mathscr{A}$ formed by those $(c_1,\dots,c_n)\in\mathscr{A}$, for which $c_i\geq 1$ for all $i\in\{1,\dots,n\}$ and the entries in the first column of $\eta(c_1)\cdots\eta(c_i)$ are nonnegative for all $i<n$.

The following lemma will be crucial for our analysis of $\mathscr{A}^+$. It is related to a well-known transformation formula for continued fractions [Perron 1929, §37, Equations (1), (2)].

Lemma 5.2. *Let $n \geq 3$ and $c = (c_1, 1, c_3, c_4, \dots, c_n)$ such that $c_i \in \mathbb{Z}$ for all $i \in \{1, \dots, n\}$. Let $c' = (c_1 - 1, c_3 - 1, c_4, \dots, c_n)$.*

(1) *$c' \in \mathcal{A}$ if and only if $c \in \mathcal{A}$.*

(2) *$c' \in \mathcal{A}^+$ if and only if $c \in \mathcal{A}^+$, $c_1, c_3 \geq 2$.*

(3) *If $c \in \mathcal{A}^+$, then either $n = 3$, $c_1 = c_3 = 1$ or $n > 3$, $c_1, c_3 \geq 2$.*

Proof. If $i, k \in \mathbb{Z}$, then

$$\eta(i)\eta(1)\eta(k) = \begin{pmatrix} ik - i - k & 1 - i \\ k - 1 & -1 \end{pmatrix} = \eta(i-1)\eta(k-1)$$

by (4-5) and (4-6). This gives (1). By (4-5), the first column of $\eta(c_1)\eta(1)$ contains only nonnegative integers if and only if $c_1 \geq 1$. Thus (2) holds. Let $c \in \mathcal{A}^+$ such that $c_1 = 1$ or $c_3 = 1$. Then (4-6) gives that the upper left entry of $\eta(c_1)\eta(1)\eta(c_3)$ is -1, and hence $n = 3$. Then $c \in \mathcal{A}$ implies that $c_1 = c_3 = 1$. Hence (3) is proven. □

Proposition 5.3. *Let $n \in \mathbb{N}$ and $(c_1, \dots, c_n) \in \mathcal{A}^+$.*

(1) *Let $i, j \in \{1, \dots, n\}$ with $i \leq j$ and $(i, j) \neq (1, n)$. Then*

$$\eta(c_i)\eta(c_{i+1}) \cdots \eta(c_j) \in \mathrm{SL}(2, \mathbb{Z})$$

such that the first column contains only nonnegative and the second only non-positive integers.

(2) *Let $i \in \{1, \dots, n\}$. Then $(c_i, c_{i+1}, \dots, c_n, c_1, \dots, c_{i-1}) \in \mathcal{A}^+$.*

(3) *$(c_n, c_{n-1}, \dots, c_2, c_1) \in \mathcal{A}^+$.*

(4) *If $n \leq 3$ then $(c_1, \dots, c_n) = (1, 1, 1)$.*

Proof. (1) We proceed by induction on the lexicographically ordered pairs (i, j).

If $i = j$ then we are done, since the matrix $\eta(c_i)$ satisfies the claim.

Let $i, j \in \{1, \dots, n\}$ with $i < j$ and $(i, j) \neq (1, n)$. Assume that the claim holds for all pairs $(i', j') \in \{1, \dots, n\}$ such that $i' \leq j'$ and either $i' < i$ or $i' = i$, $j' < j$. Let

$$\eta(c_i) \cdots \eta(c_j) = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{Z}$. Clearly, $-ad + bc = 1$ since $\eta(k) \in \mathrm{SL}(2, \mathbb{Z})$ for all $k \in \mathbb{Z}$. Moreover, (4-4) gives that

$$\eta(c_i) \cdots \eta(c_{j-1}) = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & c_j \end{pmatrix} = \begin{pmatrix} b & -(bc_j - a) \\ d & -(dc_j - c) \end{pmatrix}.$$

Hence $b, d \geq 0$ by induction hypothesis.

If $i = 1$, then $a, c \geq 0$ by definition of $\mathcal{A}^+$ and the assumption $(i, j) \neq (1, n)$, and hence we are done. Otherwise

$$\eta(c_{i-1}) \cdots \eta(c_j) = \begin{pmatrix} c_{i-1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = \begin{pmatrix} c_{i-1}a - c & d - c_{i-1}b \\ a & -b \end{pmatrix},$$

and hence $a > 0$ by induction hypothesis. Since $a, b, d \geq 0$, we get $bc = 1 + ad \geq 1$, and hence $c > 0$, which proves the claim.

(2) It suffices to prove the claim for $i = 2$. If $\eta(c_1) \cdots \eta(c_n) = -\mathrm{id}$, then clearly $\eta(c_2) \cdots \eta(c_n)\eta(c_1) = -\mathrm{id}$. Let $j \in \{2, \ldots, n\}$. Then the entries in the first column of $\eta(c_2) \cdots \eta(c_j)$ are nonnegative by part (1) of the proposition. This gives (2).

(3) Recall the definition of τ in (4-8). Then (4-7) gives that

$$\eta(c_n)\eta(c_{n-1}) \cdots \eta(c_1) = \tau\,\eta(c_n)^{-1}\eta(c_{n-1})^{-1} \cdots \eta(c_1)^{-1}\tau = -\mathrm{id}$$

since $\eta(c_1) \cdots \eta(c_n) = -\mathrm{id}$. Therefore $(c_n, c_{n-1}, \ldots, c_1) \in \mathcal{A}$.

Let $2 \leq i \leq n$ and assume that

$$\eta(c_i)\eta(c_{i+1}) \cdots \eta(c_n) = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{Z}$. Then $a, b, c, d \geq 0$ and $bc - ad = 1$ by part (1) of the proposition. We obtain that

$$\begin{aligned} \eta(c_n) \cdots \eta(c_i) &= \tau\,\eta(c_n)^{-1} \cdots \eta(c_i)^{-1}\tau \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -d & b \\ -c & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix}. \end{aligned}$$

Thus $(c_n, c_{n-1}, \ldots, c_1) \in \mathcal{A}^+$.

(4) Equations $\eta(c_1) = -\mathrm{id}$, $\eta(c_1)\eta(c_2) = -\mathrm{id}$ have no solutions with $c_1, c_2 \in \mathbb{N}$ by (4-3), (4-5). Let now $n = 3$ and $c_1, c_2, c_3 \in \mathbb{N}$. If $c_1, c_2, c_3 \geq 2$, then $(c_1, c_2, c_3) \notin \mathcal{A}$ by Corollary 4.2. Otherwise $c_1 = c_2 = c_3 = 1$ by Lemma 5.2(3) and part (2) of the proposition. Relation $(1, 1, 1) \in \mathcal{A}^+$ holds by (4-5) with $i = j = 1$. □

By Proposition 5.3(2) and (3), the dihedral group $\mathbb{D}_n$ of $2n$ elements, where $n \in \mathbb{N}$, acts on sequences of length n in $\mathcal{A}^+$ by cyclic permutation of the entries and by reflections. This action gives rise to an equivalence relation $\sim$ on $\mathcal{A}^+$ by taking the orbits of the action as equivalence classes. For brevity we will usually not distinguish between elements of $\mathcal{A}^+$ and $\mathcal{A}^+/\!\sim$. By Proposition 5.3(4) there is precisely one element of $\mathcal{A}^+/\!\sim$ of length 3.

Lemma 5.2 suggests to introduce a further equivalence relation $\approx$ on $\mathcal{A}^+$. Let $n, m \in \mathbb{N}$ with $m \geq n$, and let $c = (c_1, \ldots, c_n)$, $d = (d_1, \ldots, d_m) \in \mathcal{A}^+$. We write $c \approx' d$ if and only if $m = n$, $c \sim d$ or $m = n+1$, $d = (c_1+1, 1, c_2+1, c_3, c_4, \ldots, c_n)$.

Figure 1. Left: chain diagram. Right: cycle diagram.

Definition 5.4. Let $c, d \in \mathscr{A}^+$. Write $c \approx d$ if and only if there exists $k \in \mathbb{N}$ and a sequence $c = e_1, e_2, \dots, e_k = d$ of elements of $\mathscr{A}^+$, such that $e_i \approx' e_{i+1}$ or $e_{i+1} \approx' e_i$ for all $i \in \{1, 2, \dots, k-1\}$.

Clearly, $\approx$ is an equivalence relation on $\mathscr{A}^+$. We are interested in the equivalence classes of $\mathscr{A}^+/\approx$.

Theorem 5.5. *The only element of* $\mathscr{A}^+/\approx$ *is* $(1, 1, 1)$.

Proof. Let $n \geq 1$ and $c = (c_1, \dots, c_n) \in \mathscr{A}^+$. By Proposition 5.3(4) it suffices to prove that if $n \geq 4$, then $c \approx c'$ for some $c' = (c'_1, c'_2, \dots, c'_{n-1}) \in \mathscr{A}^+$.

Assume that $n \geq 4$. By Corollary 4.2 there exists $i \in \{1, \dots, n\}$ such that $c_i = 1$. By Proposition 5.3(2) and the definition of $\approx$ we may assume that $c_2 = 1$. Now apply Lemma 5.2(2) and (3) to obtain the desired $c' \in \mathscr{A}^+$. □

Corollary 5.6. *If* $n \in \mathbb{N}$, $(c_1, \dots, c_n) \in \mathscr{A}^+$, *then* $\sum_{i=1}^n c_i = 3(n-2)$.

Proof. The expression $\sum_{i=1}^n c_i - 3(n-2)$ is zero for $c = (1, 1, 1)$ and is an invariant of $\approx$. □

6. Connected root systems of rank two

Throughout this section I will denote a two-element set, A a finite set, and $\mathscr{C} = \mathscr{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ a connected Cartan scheme. Since $\rho_i^2 = \mathrm{id}$ for all $i \in I$, and $\mathscr{C}$ is connected, the object change diagram of $\mathscr{C}$ is either a chain (if ρ_i has a fixed point for some $i \in I$) or a cycle; see Figure 1.

Recall that an element $w \in \mathrm{Hom}(\mathscr{W}(\mathscr{C}))$ is called *even* if $\det(w) = 1$.

Lemma 6.1. *The object change diagram of* $\mathscr{C}$ *is a cycle if and only if* $\mathrm{End}(a)$ *contains only even elements* (*for all* $a \in A$).

Proof. If the object change diagram of $\mathscr{C}$ is a cycle, then for all $a \in A$, $\mathrm{End}(a)$ consists of the elements $(\sigma_i\sigma_j)^{k|A|/2}1_a$, where $k \in \mathbb{Z}$ and $I = \{i, j\}$. These are all even. Otherwise the object change diagram of $\mathscr{C}$ is a chain, and there exists $a \in A$ and $i \in I$ such that $\rho_i(a) = a$. Then $\mathrm{End}(a)$ is generated by σ_i^a and $(\sigma_j\sigma_i)^{|A|-1}\sigma_j^a$ which are odd. □

Assume that $\mathscr{C}$ admits a finite root system. The next proposition explains the relationship between the $m_{i,j}^a$ and the number $|A|$ of objects. For this, we need the following standard lemma.

Lemma 6.2. *Let $M \in \mathrm{GL}(2,\mathbb{Z})$. If the order e of M is finite, then*

$$-2 \le \mathrm{tr}(M) \le 2, \qquad e \in \{1,2,3,4,6\}.$$

Proposition 6.3. *Assume that $I = \{i, j\}$ and that $\mathscr{C}$ admits a finite root system. Then $m^a_{i,j} = m^a_{j,i} = |R^a_+|$ for all objects a. If the object change diagram is a cycle, then $m^a_{i,j} = \frac{1}{2}m\,|A|$ for some $m \in \{1,2,3,4,6\}$. If it is a chain, then $m^a_{i,j} = m\,|A|$ with the same possibilities for m.*

Proof. We have $m^a_{i,j} = m^a_{j,i} = |R^a_+|$ by Definition 2.3 for all objects a. Axiom (R3) from the same definition implies that $m^a_{i,j}$ does not depend on a. Let $d = |A|$ if the object change diagram is a chain and $d = |A|/2$ if it is a cycle. Then $(\sigma_j\sigma_i)^k 1_a \in \mathrm{End}(a)$, $k \in \mathbb{N}_0$, if and only if $k \in \mathbb{N}_0 d$. Theorem 2.6 and Lemma 6.2 give that $md = m^a_{i,j}$ for some $m \in \{1,2,3,4,6\}$. □

We are going to give a characterization of finite connected irreducible root systems of type $\mathscr{C}$. First we analyze root systems with simply connected Cartan schemes.

Lemma 6.4. *Assume that $\mathscr{C}$ is simply connected and that $\mathscr{R}$ is a finite root system of type $\mathscr{C}$. Then the object change diagram of $\mathscr{C}$ is a cycle with $|R^a|$ vertices, where $a \in A$.*

Proof. Since $\mathscr{C}$ is simply connected, $\mathrm{End}(a) = \{1\}$ for all $a \in A$. By Lemma 6.1 the object change diagram of $\mathscr{C}$ is a cycle. Now

$$\left|\mathrm{Hom}(\mathscr{W}(\mathscr{C}))1_a\right| = |A| \cdot \left|\mathrm{End}(a)\right|$$

since $\mathscr{C}$ is connected. Again, $\mathscr{C}$ is simply connected, hence $|A| = \left|\mathrm{Hom}(\mathscr{W}(\mathscr{C}))1_a\right|$. This is equal to $2|R^a_+|$ by Theorem 2.6, since $|I| = 2$. □

Proposition 6.5. *Assume that $I = \{i, j\}$ and that $\mathscr{R}$ is a finite irreducible root system of type $\mathscr{C}$. Let $a \in A$ and $n = |R^a_+|$. Let $a_1, a_2, \ldots, a_{2n} \in A$ and $c_1, c_2, \ldots, c_{2n} \in \mathbb{Z}$ such that*

$$\begin{aligned} a_{2r-1} &= (\rho_j\rho_i)^{r-1}(a), & a_{2r} &= \rho_i(\rho_j\rho_i)^{r-1}(a),\\ c_{2r-1} &= -c^{a_{2r-1}}_{ij}, & c_{2r} &= -c^{a_{2r}}_{ji} \end{aligned} \tag{6-1}$$

for all $r \in \{1,2,\ldots,n\}$. Then $(c_1, c_2, \ldots, c_n) \in \mathscr{A}^+$, $c_{n+r} = c_r$ for all $r \in \{1,2,\ldots,n\}$, and $\rho_j(a_{2n}) = a$.

Proof. For all $r \in \mathbb{Z}$ let $i_r \in I$ such that $i_r = i$ for r odd and $i_r = j$ for r even. Let $\theta_{2r-1} = \sigma_i^{a_{2r-1}}\tau$, $\theta_{2r} = \tau\sigma_j^{a_{2r}} \in \mathrm{SL}(2,\mathbb{Z})$ for all $r \in \{1,\ldots,n\}$. Then $\theta_r = \eta(c_r)$ for all $r \in \{1,\ldots,2n\}$. Since $\mathscr{R}$ is irreducible, $c_r > 0$ for all r. By Lemmas 4 and 7 of [Heckenberger and Yamane 2008], $\ell(\sigma_{i_n}^{a_n}\cdots\sigma_{i_2}^{a_2}\sigma_{i_1}^{a}) = n$. Hence

$$\sigma_{i_n}^{a_n}\cdots\sigma_{i_2}^{a_2}\sigma_{i_1}^{a}(\{\alpha_1,\alpha_2\}) = \{-\alpha_1,-\alpha_2\}$$

by Lemma 8(iii) of the same work. Thus $\theta_n \cdots \theta_2\theta_1(\{\alpha_1, \alpha_2\}) = \{-\alpha_1, -\alpha_2\}$, and since $\det\theta_r = 1$ for all r, we conclude that $\theta_n \cdots \theta_2\theta_1 = -\mathrm{id}$. Hence $(c_n, \dots, c_2, c_1)$ lies in $\mathscr{A}$.

Clearly, if $2 \le r \le n$, then the first column of $\theta_n \cdots \theta_{r+1}\theta_r$ has nonnegative entries if and only if $\sigma_{i_n} \cdots \sigma_{i_{r+1}}\sigma_{i_r}^{a_r}(\alpha_{i_{r-1}})$ is a positive root. The latter is true by [Heckenberger and Yamane 2008, Lemma 4], and hence $(c_n, \dots, c_2, c_1) \in \mathscr{A}^+$. Then $(c_1, c_2, \dots, c_n) \in \mathscr{A}^+$ by Proposition 5.3(3).

Replacing in the construction a by a_2 and i by j, we find that $(c_2, \dots, c_n, c_{n+1})$ lies in $\mathscr{A}^+$. Then $\eta(c_1)^{-1} = -\eta(c_2)\cdots\eta(c_n) = \eta(c_{n+1})^{-1}$, and hence $c_1 = c_{n+1}$. Thus $c_{n+r} = c_r$ for all $r \in \{1, 2, \dots, n\}$ by induction on r. Finally, $\rho_j(a_{2n}) = (\rho_j\rho_i)^n(a) = a$ by (R4). $\square$

The construction in Proposition 6.5 associates to any pair $(i, a) \in I \times A$ a sequence $(c_1, c_2, \dots, c_n) \in \mathscr{A}^+$. This defines a map

$$\Phi : I \times A \to \mathscr{A}^+.$$

Proposition 6.5 gives immediately, that

$$\Phi(j, a) = (c_n, c_{n-1}, \dots, c_1), \quad \Phi(j, \rho_i(a)) = (c_2, c_3, \dots, c_n, c_1). \tag{6-2}$$

Thus, by definition of $\sim$, the induced map $\Phi : I \times A \to \mathscr{A}^+/{\sim}$ is constant. But we can say more.

Theorem 6.6. *Let $n \in \mathbb{N}$ and $c = (c_1, c_2, \dots, c_n) \in \mathscr{A}^+$. Then there is a unique* (*up to equivalence*) *finite connected irreducible root system $\mathscr{R}$ with simply connected Cartan scheme of rank two such that $c \in \operatorname{Im}\Phi$.*

Proof. Assume that $c \in \mathscr{A}^+$, $\mathscr{R}$ is a connected irreducible root system of rank two, $i \in I$, and $a \in A$ such that $\Phi(i, a) = c$. If the Cartan scheme of $\mathscr{R}$ is simply connected, then by Lemma 6.4 and Proposition 6.5 the object change diagram of $\mathscr{R}$ is a cycle and $|A| = 2n$. The Cartan matrices C^a and the sets R^a, where $a \in A$, are then uniquely determined by the construction in Proposition 6.5. Thus $\mathscr{R}$ is uniquely determined. We describe $\mathscr{R}$ explicitly.

Let $I = \{i, j\}$ and let $A = \{a_1, \dots, a_{2n}\}$ be a set with $2n$ elements. Define $\rho_i, \rho_j : A \to A$ such that

$$\begin{aligned} \rho_i(a_{2r-1}) &= a_{2r}, & \rho_i(a_{2r}) &= a_{2r-1}, \\ \rho_j(a_{2r}) &= a_{2r+1}, & \rho_j(a_{2r+1}) &= a_{2r} \end{aligned} \tag{6-3}$$

for all $r \in \{1, 2, \dots, n\}$, where $a_{2n+1} = a_1$. Then $\rho_i^2 = \rho_j^2 = \mathrm{id}$. Let $c_{ln+r} = c_r$ for all $r \in \{1, 2, \dots, n\}$ and $l \in \mathbb{Z}$, and define

$$C^{a_{2r-1}} = \begin{pmatrix} 2 & -c_{2r-1} \\ -c_{2r-2} & 2 \end{pmatrix}, \quad C^{a_{2r}} = \begin{pmatrix} 2 & -c_{2r-1} \\ -c_{2r} & 2 \end{pmatrix} \tag{6-4}$$

for all $r \in \{1, 2, \dots, n\}$. Since $c_r \in \mathbb{N}$ for all $r \in \{1, 2, \dots, 2n\}$, the matrices C^{a_r} satisfy (M1) and (M2). Since also (C1) and (C3) hold, $\mathscr{C} = \mathscr{C}(I, A, (\rho_i, \rho_j), (C^a)_{a\in A})$ is a connected Cartan scheme.

Now define

$$R^{a_{2r-1}} = \left\{\pm\eta(c_{2r-1})\,\eta(c_{2r})\cdots\eta(c_{2r-2+l})\tbinom{1}{0} \;\middle|\; 0 \le l \le n-1\right\},$$
$$R^{a_{2r}} = \left\{\pm\tau\eta(c_{2r})\eta(c_{2r+1})\cdots\eta(c_{2r+l-1})\tbinom{1}{0} \;\middle|\; 0 \le l \le n-1\right\},$$

for all $r \in \{1, 2, \dots, n\}$. Note that $|R^a_+| = n$ for all $a \in A$. Indeed, otherwise $\eta(c_r)\eta(c_{r+1})\cdots\eta(c_{r+l-1})\binom{1}{0} = \binom{1}{0}$ for some $r \in \{1, \dots, 2n\}$ and $l \in \{1, \dots, n-1\}$. Then

$$\eta(c_{r+1})\eta(c_{r+2})\cdots\eta(c_{r+l-1})\tbinom{1}{0} = \eta(c_r)^{-1}\tbinom{1}{0} = \tbinom{0}{-1},$$

a contradiction to Proposition 5.3(1) and (2).

Axiom (R1) is fulfilled by Proposition 5.3(2). Let $r \in \{1, 2, \dots, 2n\}$. Equation $\eta(c_r)\eta(c_{r+1})\cdots\eta(c_{r+n-1}) = -\mathrm{id}$ implies that

$$\eta(c_r)\eta(c_{r+1})\cdots\eta(c_{r+n-2}) = -\eta(c_{r+n-1})^{-1},$$

and hence $\pm\alpha_1, \pm\alpha_2 \in R^{a_r}$. Since $\tau, \eta(l) \in \mathrm{SL}(2, \mathbb{Z})$ for all $l \in \mathbb{Z}$, we get (R2). (R4) holds by (6-3), since $|R^a_+| = n$ for all $a \in A$.

Now we prove (R3). Let $r \in \{1, 2, \dots, 2n\}$. Then $\sigma_i^{a_r} = \eta(-c_{ij}^{a_r})\tau = \tau\eta(-c_{ij}^{a_r})^{-1}$ by (6-4), (4-7). If r is odd, then

$$\sigma_i^{a_r}(R^{a_r}) = \tau\eta(c_r)^{-1}\left(\left\{\pm\eta(c_r)\eta(c_{r+1})\cdots\eta(c_{r+l-1})\tbinom{1}{0} \;\middle|\; 0 \le l \le n-1\right\}\right)$$
$$\subset R^{a_{r+1}} = R^{\rho_i(a_r)},$$

and if r is even, then

$$\sigma_i^{a_r}(R^{a_r}) = \eta(c_{r-1})\tau\left(\left\{\pm\tau\eta(c_r)\eta(c_{r+1})\cdots\eta(c_{r+l-1})\tbinom{1}{0} \;\middle|\; 0 \le l \le n-1\right\}\right)$$
$$\subset R^{a_{r-1}} = R^{\rho_i(a_r)}.$$

Similarly, $\sigma_j^{a_r} = \tau\eta(c_{r-1})$ for odd r and $\sigma_j^{a_r} = \eta(c_r)^{-1}\tau$ for even r. Hence $\sigma_j^{a_r}(R^{a_r}) \subset R^{\rho_j(a_r)}$, (R3) holds, and $\mathscr{R}$ is a finite irreducible root system of type $\mathscr{C}$. The Cartan scheme $\mathscr{C}$ is simply connected, since $\left|\mathrm{Hom}(\mathscr{W}(\mathscr{C}))1_{a_1}\right| = 2n = |A|$ and $(c_1, \dots, c_n) \in \mathscr{A}$. Finally, $\Phi(i, a_1) = (c_1, \dots, c_n)$ because of (6-1), (6-3), and (6-4). □

Corollary 6.7. *Assume that there is a finite root system $\mathscr{R}$ of type $\mathscr{C}$. Then there are $a \in A$ and $i, j \in I$ with $i \neq j$ such that $c^a_{ij} = 0$ or $c^a_{ij} = -1$.*

Proof. If $\mathscr{R}$ is not irreducible, then $C^a_{ij} = 0$ for all $a \in A$ and $i, j \in I$ with $i \neq j$; see the end of Section 2. Otherwise Proposition 6.5 gives that the negatives of the entries of the Cartan matrices of $\mathcal{C}$ give rise to a sequence $(c_1, \dots, c_n) \in \mathcal{A}^+$. By Corollary 4.2, this sequence has an entry 1, and the corollary is proven. $\square$

Remark 6.8. The assumption in Corollary 6.7 can be weakened for example by requiring only that $\mathcal{W}(\mathcal{C})$ is finite. We don't work out the details, since we are mainly interested in Cartan schemes admitting (finite) root systems.

We are going to give a very effective algorithm to decide if our given connected Cartan scheme $\mathcal{C}$ admits a finite irreducible root system. The central notions towards this will be the characteristic sequences and centrally symmetric Cartan schemes. Our algorithm can also be used to get a more precise classification of root systems of rank two, for example in form of explicit lists for a given number of objects.

Definition 6.9. Assume that the object change diagram of $\mathcal{C}$ is a cycle. Let $i \in I$, $a \in A$, and define $a_1, \dots, a_{|A|} \in A$ and $c_1, \dots, c_{|A|} \in \mathbb{N}_0$ by

$$a_{2k-1} = (\rho_j \rho_i)^{k-1}(a), \qquad a_{2k} = (\rho_i \rho_j)^{k-1} \rho_i(a),$$
$$c_{2k-1} = -c_{ij}^{a_{2k-1}}, \qquad c_{2k} = -c_{ji}^{a_{2k}}$$

for all $k \in \{1, 2, \dots, |A|/2\}$, where $I = \{i, j\}$. Then $(c_1, c_2, \dots, c_{|A|})$ is called the *characteristic sequence of* $\mathcal{C}$ with respect to i and a. The Cartan scheme $\mathcal{C}$ is termed *centrally symmetric* if $c_k = c_{k+|A|/2}$ for all $k \in \{1, 2, \dots, |A|/2\}$. In this case we write also $(c_1, c_2, \dots, c_{|A|/2})^2$ for $(c_1, c_2, \dots, c_{|A|})$.

Remark 6.10. Let $(c_1, c_2, \dots, c_{|A|})$ be the characteristic sequence of $\mathcal{C}$ with respect to i and a. Then the characteristic sequences with respect to j and a and i and $\rho_i(a)$, respectively, are $(c_{|A|}, c_{|A|-1}, \dots, c_1)$ and $(c_1, c_{|A|}, c_{|A|-1}, \dots, c_3, c_2)$, respectively. Thus if $\mathcal{C}$ is centrally symmetric with respect to i and a, it is also centrally symmetric with respect to j and a and i and $\rho_i(a)$, respectively. Since $\mathcal{C}$ is connected, this means that $\mathcal{C}$ being centrally symmetric is independent of the choice of $i \in I$ and $a \in A$.

Remark 6.11. Characteristic sequences must not be confused with elements of $\mathcal{A}$ or $\mathcal{A}^+$. Their precise relationship will not be needed in the sequel, so we don't work it out in detail.

Remark 6.12. Let $n \in \mathbb{N}$ and let $c = (c_1, c_2, \dots, c_{2n})$ be a sequence of positive integers. By axioms (M1) and (C3) there is a unique (up to equivalence) connected Cartan scheme $\mathcal{C}$ with object change diagram a cycle, such that the characteristic sequence of $\mathcal{C}$ (with respect to some $i \in I$ and $a \in A$) is c.

Remark 6.13. Assume that $\mathcal{C}$ is simply connected, and that there exists a finite irreducible root system of type $\mathcal{C}$. Then $\mathcal{C}$ is centrally symmetric by Lemma 6.4 and Proposition 6.5.

Remark 6.14. Assume that the object change diagram of $\mathscr{C}$ is a cycle. By Lemma 6.1 and Proposition 3.4 the object change diagram of an n-fold covering $\mathscr{C}'$ of $\mathscr{C}$, where $n \in \mathbb{N}$, is a cycle. The characteristic sequence of $\mathscr{C}'$ is just the n-fold repetition of the characteristic sequence of $\mathscr{C}$. Thus an n-fold covering of $\mathscr{C}$ is centrally symmetric if and only if $\mathscr{C}$ is centrally symmetric or n is even.

Lemma 6.15. *Assume that there exists a finite irreducible root system of type $\mathscr{C}$. Suppose that the object change diagram of $\mathscr{C}$ is a chain. Then there is a unique double covering $\mathscr{C}'$ of $\mathscr{C}$ and a finite irreducible root system of type $\mathscr{C}'$ such that the object change diagram of $\mathscr{C}'$ is a cycle.*

Proof. By assumption there exists $a \in A$ and $i \in I$ such that $\rho_i(a) = a$. Then $\mathrm{End}(a)$ is generated by σ_i^a and $\tau^a = (\sigma_j \sigma_i)^{|A|-1} \sigma_j^a$, where $I = \{i, j\}$. Since σ_i^a, τ^a are reflections, for the subgroup $U = \langle \sigma_i^a \tau^a \rangle \subset \mathrm{End}(a)$ we obtain that $[\mathrm{End}(a) : U] = 2$, and U consists of even elements. By Proposition 3.4(2) there exists a unique double covering $\mathscr{C}'$ of $\mathscr{C}$ satisfying Axiom (C3) such that $\mathrm{End}(a') \simeq U$ for all $a' \in A'$. By Lemma 6.1 the object change diagram of $\mathscr{C}'$ is a cycle. The uniqueness of $\mathscr{C}'$ holds, since U is the unique subgroup of $\mathrm{End}(a)$ consisting of even elements and satisfying $[\mathrm{End}(a) : U] = 2$. The existence of a finite irreducible root system of type $\mathscr{C}'$ follows from Proposition 3.7(2). $\square$

Remark 6.16. If $\mathscr{C}'$ is a Cartan scheme with object change diagram a cycle, then $\mathscr{C}'$ is the double covering of a Cartan scheme with object change diagram a chain if and only if there exist $i \in I'$, $a \in A'$, such that the characteristic sequence of $\mathscr{C}'$ with respect to i and a is of the form $(c_1, \ldots, c_n, c_{n+1}, c_n, c_{n-1}, \ldots, c_2)$ with $n = |A'|/2$ and $c_1, \ldots, c_{n+1} \in \mathbb{N}_0$.

Lemma 6.17. *Assume that there exists a finite irreducible root system of type $\mathscr{C}$. Suppose that the object change diagram of $\mathscr{C}$ is a cycle, and that $\mathscr{C}$ is not centrally symmetric. Then there is a unique double covering $\mathscr{C}'$ of $\mathscr{C}$ which admits a (finite irreducible) root system. The Cartan scheme $\mathscr{C}'$ is centrally symmetric.*

Proof. Since the object change diagram of $\mathscr{C}$ is a cycle, $\mathrm{End}(a)$ is cyclic for all $a \in A$. The universal covering of $\mathscr{C}$ is centrally symmetric by Remark 6.13. Since $\mathscr{C}$ is not centrally symmetric, $|\mathrm{End}(a)|$ is even by Remark 6.14 and Proposition 3.4(2). By Proposition 3.4(2) there is a unique double covering $\mathscr{C}'$ of $\mathscr{C}$ satisfying (C3). It admits a finite irreducible root system of type $\mathscr{C}'$ by Proposition 3.7(2). All coverings of $\mathscr{C}$ admitting a root system fulfill (C3). Hence $\mathscr{C}'$ is the only double covering of $\mathscr{C}$ admitting a root system. This $\mathscr{C}'$ is centrally symmetric by Remark 6.14. $\square$

Remark 6.18. Let $\mathscr{C}'$ be a Cartan scheme with object change diagram a centrally symmetric cycle, and $n = |A'|$. Then $\mathscr{C}'$ is the double covering of a Cartan scheme with object change diagram a not centrally symmetric cycle if and only if $n \in 4\mathbb{N}$, and with respect to one (equivalently, all) pair $(i', a') \in I' \times A'$ the characteristic sequence of $\mathscr{C}'$ is not of the form

$$(c_1, c_2, \dots, c_{n/4}, c_1, c_2, \dots, c_{n/4})^2,$$

where $c_1, \dots, c_{n/4} \in \mathbb{N}_0$.

In order to decide if a given connected Cartan scheme admits a finite root system, Lemmas 6.15 and 6.17 allow us to concentrate on centrally symmetric Cartan schemes. Further, since the classification of finite root systems with at most three objects is known [Cuntz and Heckenberger 2008], we may assume that the Cartan scheme has at least 4 objects.

For any matrix C, let C^{t} denote the transpose of C.

Theorem 6.19. *Let $\mathscr{C} = \mathscr{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected centrally symmetric Cartan scheme with $|A| \geq 4$.*

(1) *Assume that the characteristic sequence of $\mathscr{C}$ contains 0. Then $c^a_{ij} = 0$ for all $a \in A$ and $i, j \in I$ with $i \neq j$. Moreover, $\mathscr{C}$ admits a finite root system if and only if $|A| = 4$.*

(2) *If all entries of the characteristic sequence of $\mathscr{C}$ are at least two, then $\mathscr{C}$ does not admit a finite root system.*

(3) *Assume that the characteristic sequence of $\mathscr{C}$ is of the form*

$$c = (c_1, 1, c_3, c_4, \dots, c_{|A|/2})^2.$$

(Thus $0 \notin \{c_1, \dots, c_{|A|/2}\}$ by (1).) If $c_1 = 1$ or $c_3 = 1$, then there is a finite root system of type $\mathscr{C}$ if and only if $|A| = 6$ and $c_1 = c_3 = 1$. If $c_1 > 1$ and $|A| = 4$, then there is a finite root system of type $\mathscr{C}$ if and only if $c_1 \in \{2, 3\}$. If $c_1 > 1$, $c_3 > 1$, and $|A| \geq 6$, then there is a finite root system of type $\mathscr{C}$ if and only if the Cartan scheme with object change diagram a cycle with $|A|-2$ edges and with characteristic sequence

$$(c_1 - 1, c_3 - 1, c_4, \dots, c_{|A|/2})^2 \tag{6-5}$$

admits a finite root system.

Proof. (1) follows from (M2), (C3), and (R4), and (2) from Corollary 6.7.

(3) If $c_1 = 1$ or $c_3 = 1$, then there exists $a \in A$ such that $c^a_{ij} = c^a_{ji} = -1$, where $I = \{i, j\}$. Then Lemma 4.8 of [Cuntz and Heckenberger 2008] gives that $m^a_{i,j} = 3$ and $c_r = 1$ for all $r \in \{1, 3, 4, \dots, |A|/2\}$. By (R4) we get $|A| = 6$.

Assume next that $c_1 > 1$ and $|A| = 4$. Then $C^a = C^b$ for all $a, b \in A$, and hence $\mathscr{C}$ admits a finite root system if and only if C^a is of finite type and (R4) holds (see Theorem 3.3 of the same reference), that is, $c_1 \in \{2, 3\}$.

Finally, assume that $c_1 > 1$, $c_3 > 1$, $|A| \geq 6$, and $\mathcal{C}$ admits a finite root system. By Proposition 3.7, the universal covering $\mathcal{C}'$ of $\mathcal{C}$ admits a finite root system. Hence A' is finite by (C1) and (R4). Therefore $\mathrm{End}(a) \subset \mathrm{Hom}(\mathcal{W}(\mathcal{C}))$ is finite for all $a \in A$ by (3-3). Let $m = |\mathrm{End}(a)|$. Remark 6.14 and Lemma 6.4 tell that the object change diagram of $\mathcal{C}'$ is a centrally symmetric cycle, and the characteristic sequence of $\mathcal{C}'$ is an m-fold repetition of c. Let

$$\tilde{c} = (c_1, 1, c_3, c_4, \dots, c_{|A|/2}).$$

By Proposition 6.5 the m-fold repetition of $\tilde{c}$ is an element of $\mathcal{A}^+$. Since $|A| \geq 6$, Lemma 5.2(2) gives that the m-fold repetition of

$$\tilde{c}' = (c_1 - 1, c_3 - 1, c_4, \dots, c_{|A|/2})$$

is in $\mathcal{A}^+$. Let $\mathcal{C}''$ be the connected simply connected Cartan scheme which corresponds to the m-fold repetition of $\tilde{c}'$ via Theorem 6.6. It admits a finite root system. Now $\mathcal{C}''$ is the m-fold covering of a Cartan scheme $\mathcal{C}'''$ with characteristic sequence given in (6-5). Hence Proposition 3.7 gives that $\mathcal{C}'''$ admits a finite root system.

We have shown that if $\mathcal{C}$ admits a finite root system, then also $\mathcal{C}'''$. The proof of the converse goes in the same way, and we are done. □

Example 6.20. Consider the connected Cartan scheme $\mathcal{C}$ of rank two with 4 objects, object change diagram a cycle and characteristic sequence $(5, 1, 2, 2)$. To check that $\mathcal{C}$ admits a finite root system, consider the double covering $\mathcal{C}'$ corresponding to the characteristic sequence $(5, 1, 2, 2)^2$. By Proposition 3.7, $\mathcal{C}$ admits a finite root system if and only if $\mathcal{C}'$ does. Theorem 6.19(3) allows one to replace $\mathcal{C}'$ by the Cartan scheme with characteristic sequence $(4, 1, 2)^2$ and then $(3, 1)^2$. Thus $\mathcal{C}$ admits a finite root system.

If we start with the characteristic sequence $(5, 1, 2, 3)$ for $\mathcal{C}$, then the analogous arguments produce the characteristic sequences $(5, 1, 2, 3)^2$, $(4, 1, 3)^2$ and $(3, 2)^2$, and then $\mathcal{C}$ does not admit a finite root system by Theorem 6.19(2).

Example 6.21. Theorem 6.19 also enables us to list all connected centrally symmetric Cartan schemes which admit a finite root system to a fixed number of objects. For example if $|A| = 4$, then there are 3 such schemes and they belong to the characteristic sequences $(0, 0)^2$, $(1, 2)^2$, $(1, 3)^2$. Therefore by Theorem 6.19(2) and (3), the only connected centrally symmetric Cartan schemes (up to equivalence) which have 6 objects and admit a finite root system are those that correspond to the characteristic sequences $(1, 1, 1)^2$, $(2, 1, 3)^2$ and $(2, 1, 4)^2$, and, if $|A| = 8$, then we obtain $(2, 1, 2, 1)^2$, $(3, 1, 2, 3)^2$, $(2, 2, 1, 4)^2$, $(3, 1, 4, 1)^2$, $(3, 1, 2, 4)^2$, $(2, 2, 1, 5)^2$ and $(3, 1, 5, 1)^2$. Similarly, we have 15, 47, 136 connected centrally symmetric Cartan schemes up to equivalence with 10, 12, 14 objects, respectively, admitting a finite root system.

According to Lemma 6.17 and the above lists for $|A| = 4$ and $|A| = 8$, the complete list of all characteristic sequences to irreducible Cartan schemes which admit a finite root system, with object change diagram a cycle and 4 objects is thus: $(1, 2, 1, 2)$, $(1, 3, 1, 3)$, $(3, 1, 2, 3)$, $(2, 2, 1, 4)$, $(3, 1, 4, 1)$, $(3, 1, 2, 4)$, $(2, 2, 1, 5)$, $(3, 1, 5, 1)$.

Remark 6.16 and the list for $|A| = 8$ also supports us with Cartan schemes with 4 objects which admit a finite root system and have a chain as object change diagram. The symmetry property mentioned in Remark 6.16 is fulfilled for the sequences $(2, 1, 2, 1)^2$ (also in the form $(1, 2, 1, 2)^2$), $(3, 1, 4, 1)^2$ (also in the form $(4, 1, 3, 1)^2$), and $(3, 1, 5, 1)^2$ (also in the form $(5, 1, 3, 1)^2$). This yields the following 6 Cartan schemes with 4 objects (the Cartan matrices represent the objects).

$$
\begin{array}{c}
\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \\
\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \\
\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \\
\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \\
\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \\
\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \overset{1}{\text{———}} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \overset{2}{\text{———}} \begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}
\end{array}
$$

To complete the classification of all connected Cartan schemes with finite root system and 4 objects, it remains to calculate all connected Cartan schemes with finite root system and 8 objects with object change diagram a not centrally symmetric cycle, and then to apply Remark 6.16 to them, as indicated above, to get all chains with 4 objects. This is certainly an easy task for a computer but there are too many such Cartan schemes to list them here.

7. Bounds

Let $\mathcal{C} = \mathcal{C}(I, A, (\rho_i)_{i\in I}, (C^a)_{a\in A})$ be a connected Cartan scheme of rank two admitting a finite irreducible root system of type $\mathcal{C}$. Then A is finite by (C1) and (R4). Let $-q = -q(\mathcal{C})$ denote the sum of all nondiagonal entries of the Cartan matrices of $\mathcal{C}$, and $h = |\mathrm{End}(a)|$ for an $a \in A$. Then $|\mathrm{End}(b)| = h$ for all $b \in A$, since $\mathcal{C}$ is connected.

Theorem 7.1. *We have $h(6|A| - q) = 24$ and*

$$|R^a_+| = \frac{h|A|}{2} = \frac{12|A|}{6|A| - q}.$$

Proof. The universal covering $\mathcal{C}'$ of $\mathcal{C}$ has $h|A|$ objects by (3-3), and $q(\mathcal{C}')/4 = 3(h|A|/2 - 2)$ by Proposition 6.5 and Corollary 5.6. Since $q(\mathcal{C}') = hq(\mathcal{C})$, we obtain that $hq = 6(h|A| - 4)$. Hence $h(6|A| - q) = 24$. Lemma 6.4 tells that $|R^a_+| = h|A|/2$. This yields the claim. □

Remark 7.2. Proposition 6.3 and Theorem 7.1 give that $h \in \{1, 2, 3, 4, 6\}$ if the object change diagram of $\mathcal{C}$ is a cycle, and $h/2 \in \{1, 2, 3, 4, 6\}$ if it is a chain. But this result could have been obtained much easier. Nevertheless, Theorem 7.1 gives a restriction for $q = 6|A| - 24/h$ for given number $|A|$ of objects in a finite irreducible root system.

Next we give sharp bounds for the entries of the Cartan matrices.

Proposition 7.3. *Assume that $|A| \geq 2$. Let $c \leq 0$ be an entry of C^a for some $a \in A$. If the object change diagram is a cycle (chain), then $|c| \leq |A| + 1$ ($|c| \leq 2|A| + 1$).*

Proof. Assume first that the object change diagram of $\mathcal{C}$ is a cycle. If $|A| \geq 4$ and $\mathcal{C}$ is centrally symmetric, then Theorem 6.19(2) and (3) yields by induction on $|A|$, that $|c| \leq |A|/2 + 1$. If $\mathcal{C}$ is not centrally symmetric, then by Lemma 6.17 there exists a double covering of $\mathcal{C}$ which is centrally symmetric. Hence $|c| \leq |A| + 1$.

If the object change diagram of $\mathcal{C}$ is a chain, then by Lemma 6.15 there exists a double covering of $\mathcal{C}$ which has a cycle as object change diagram. Hence $|c| \leq 2|A| + 1$. □

Proposition 7.4. *For all $n \geq 1$ there exist finite connected irreducible root systems $\mathcal{R}$ of rank two with $|A| = 2n$ and object change diagram a cycle or $|A| = n$ and object change diagram a chain such that $-(2n + 1)$ is an entry in a Cartan matrix C^a, $a \in A$.*

Proof. For $n = 1$ the claim follows from [Cuntz and Heckenberger 2008, Proposition 5.2].

Theorem 6.19 tells that for all $n \geq 2$ the Cartan scheme $\mathcal{C}_n$ with $4n$ objects, object change diagram a cycle, and characteristic sequence

$$(3, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}}, 1, 2n+1, 1, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}})^2 \tag{7-1}$$

admits a finite irreducible root system with $|A| = 4n$. Indeed, if $n = 2$, then using Theorem 6.19(3) we can transform the sequence $(3, 1, 5, 1)^2$ first to $(2, 4, 1)^2$. By changing the reference object, the latter is equivalent to $(4, 1, 2)^2$, and using the same result we may reduce it to $(3, 1)^2$. If $n > 2$, then using Theorem 6.19(3) we may transform the sequence in (7-1) in two steps, first to

$$(3, \underbrace{2, 2, \dots, 2}_{n-3 \text{ times}}, 1, 2n, 1, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}})^2,$$

and then to

$$(3, \underbrace{2, 2, \dots, 2}_{n-3 \text{ times}}, 1, 2n-1, 1, \underbrace{2, 2, \dots, 2}_{n-3 \text{ times}})^2.$$

By induction on n we obtain that $\mathscr{C}_n$ admits a finite irreducible root system. By Remark 6.18, $\mathscr{C}_n$ is the double covering of a Cartan scheme $\mathscr{C}'_n$ with $2n$ objects, object change diagram a cycle, and characteristic sequence

$$(3, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}}, 1, 2n+1, 1, \underbrace{2, 2, \dots, 2}_{n-2 \text{ times}}).$$

By Proposition 3.7, $\mathscr{C}'_n$ admits a finite irreducible root system $\mathscr{R}'$, and $\mathscr{R}'$ is such a root system we are looking for. By Remark 6.16, $\mathscr{C}'_n$ is the double covering of a Cartan scheme $\mathscr{C}''_n$ with n objects and object change diagram a chain. By Proposition 3.7, $\mathscr{C}''_n$ admits a finite irreducible root system $\mathscr{R}''$, and the proposition is proven. □

Corollary 7.5. *Any $c \in \mathbb{N}$ occurs as the negative of an entry of a Cartan matrix of a finite connected irreducible root system of rank two.*

Proof. For even c use the appropriate intermediate step in the proof of Proposition 7.4. □

Corollary 7.6. *For $r, n \in \mathbb{N}$, there are only finitely many finite root systems $\mathscr{R}$ of rank r with n objects.*

Proof. Let I, A be finite sets with $|I| = r$ and $|A| = n$, and let $\mathscr{R}$ be a finite root system of rank r with object set A. For all $i, j \in I$ with $i \neq j$ the restriction $\mathscr{R}|_{\{i,j\}}$ (see [Cuntz and Heckenberger 2008, Definition 4.1]) is a finite root system of rank two. Hence the entries of the Cartan matrices of $\mathscr{R}$ are bounded by $2|A| + 1$ by Proposition 7.3. Since for all $i \in I$, ρ_i is one of finitely many permutations of A, and since finite root systems are uniquely determined by their Cartan scheme, the claim is proven. □

References

[Andruskiewitsch and Schneider 1998] N. Andruskiewitsch and H.-J. Schneider, "Lifting of quantum linear spaces and pointed Hopf algebras of order p^3", *J. Algebra* **209**:2 (1998), 658–691. MR 99k:16075 Zbl 55.0262.09

[Andruskiewitsch and Schneider 2005] N. Andruskiewitsch and H.-J. Schneider, "On the classification of finite-dimensional pointed Hopf algebras", Preprint, 2005. To appear in *Ann. Math.* arXiv math.QA/0502157

[Andruskiewitsch et al. 2008] N. Andruskiewitsch, I. Heckenberger, and H.-J. Schneider, "The Nichols algebra of a semisimple Yetter–Drinfeld module", Preprint, 2008. arXiv 0803.2430

[Bourbaki 1968] N. Bourbaki, *Groupes et algèbres de Lie. ch. 4, 5 et 6*, Actualités Scientifiques et Industrielles **1337**, Hermann, Paris, 1968. MR 39 #1590 Zbl 0186.33001

[Cuntz and Heckenberger 2008] M. Cuntz and I. Heckenberger, "Weyl groupoids with at most three objects", Preprint, 2008. arXiv 0805.1810

[Heckenberger 2006] I. Heckenberger, "The Weyl groupoid of a Nichols algebra of diagonal type", *Invent. Math.* **164**:1 (2006), 175–188. MR 2007e:16047 Zbl 05027328

[Heckenberger 2008] I. Heckenberger, "Rank 2 Nichols algebras with finite arithmetic root system", *Algebr. Represent. Theory* **11**:2 (2008), 115–132. MR 2009a:16080 Zbl 05250098

[Heckenberger 2009] I. Heckenberger, "Classification of arithmetic root systems", *Adv. Math.* **220**:1 (2009), 59–124. MR 2462836 Zbl 05376870

[Heckenberger and Yamane 2008] I. Heckenberger and H. Yamane, "A generalization of Coxeter groups, root systems, and Matsumoto's theorem", *Math. Z.* **259** (2008), 255–276. MR 2009e:20087 Zbl 05267026

[Kac 1990] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, 1990. MR 92k:17038 Zbl 0716.17022

[Montgomery 1993] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics **82**, Amer. Math. Soc., 1993. MR 94i:16019 Zbl 0793.16029

[Perron 1929] O. Perron, *Die Lehre von den Kettenbrüchen*, Teubner, Leipzig, 1929.

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