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We show that the universal unitary completion of certain locally algebraic representation of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ with $p > 2$ is nonzero, topologically irreducible, admissible and corresponds to a 2-dimensional crystalline representation with nonsemisimple Frobenius via the p-adic Langlands correspondence for G.

1. Introduction

Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and B be the subgroup of upper-triangular matrices in G. Let L be a finite extension of $\mathbb{Q}_p$.

Theorem 1.1. *Assume that $p > 2$, let $k \geq 2$ be an integer, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character with $\chi(p)^2 p^{k-1} \in \mathfrak{o}_L^\times$. Assume there exists a G-invariant norm $\|\cdot\|$ on $(\mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$. Then the completion E is a topologically irreducible, admissible Banach space representation of G. If we let E^0 be the unit ball in E, then*

$$V_{k,2\chi(p)^{-1}} \otimes (\chi|\chi|) \cong L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(E^0/\varpi_L^n E^0),$$

where $\mathbf{V}$ is Colmez's Montreal functor and $V_{k,2\chi(p)^{-1}}$ is a 2-dimensional irreducible crystalline representation of $\mathcal{G}_{\mathbb{Q}_p}$, the absolute Galois group of $\mathbb{Q}_p$, with Hodge–Tate weights $(0, k-1)$ and the trace of crystalline Frobenius equal to $2\chi(p)^{-1}$.

As we explain in Section 5, the existence of such G-invariant norm follows from [Colmez 2008]. Our result addresses [Berger and Breuil 2007, remarque 5.3.5]. In other words, the completion E fits into the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

The idea is to approximate $(\mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$ with representations $(\mathrm{Ind}_B^G \chi\delta_x \otimes \chi\delta_{x^{-1}}|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$, where $\delta_x : \mathbb{Q}_p^\times \to L^\times$ is an unramified character with $\delta_x(p) = x \in 1 + \mathfrak{p}_L$. If $x^2 \neq 1$, then $\chi\delta_x \neq \chi\delta_{x^{-1}}$ and the analogue of Theorem 1.1 is a result of Berger and Breuil [2007]. This allows to deduce admissibility. This approximation process relies on the results of [Vignéras 2008].

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Using Colmez's functor $\mathbf{V}$, we may then transfer the question of irreducibility to the Galois side. Here, we use the fact that for $p > 2$ the representation $V_{k,\pm 2p^{(k-1)/2}}$ sits in the p-adic family studied by Berger, Li and Zhu [2004].

2. Notation

We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. We let val be the valuation on $\overline{\mathbb{Q}}_p$ such that $\mathrm{val}(p) = 1$, and we set $|x| := p^{-\mathrm{val}(x)}$. Let L be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$, let $\mathfrak{o}_L$ be the ring of integers of L, let ϖ_L be a uniformizer, and let $\mathfrak{p}_L$ be the maximal ideal of $\mathfrak{o}_L$. Given a character $\chi : \mathbb{Q}_p^\times \to L^\times$, we consider χ as a character of the absolute Galois group $\mathcal{G}_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ via the local class field theory by sending the geometric Frobenius to p.

Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$, and let B be the subgroup of upper-triangular matrices. Given two characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to L^\times$, we consider $\chi_1 \otimes \chi_2$ as a character of B sending a matrix $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to $\chi_1(a)\chi_2(d)$. Let Z be the centre of G. Define

$$K := \mathrm{GL}_2(\mathbb{Z}_p), \qquad K_m := \begin{pmatrix} 1 + p^m\mathbb{Z}_p & p^m\mathbb{Z}_p \\ p^m\mathbb{Z}_p & 1 + p^m\mathbb{Z}_p \end{pmatrix} \quad \text{for } m \geq 1,$$
$$I := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}, \quad I_m := \begin{pmatrix} 1 + p^m\mathbb{Z}_p & p^{m-1}\mathbb{Z}_p \\ p^m\mathbb{Z}_p & 1 + p^m\mathbb{Z}_p \end{pmatrix} \quad \text{for } m \geq 1.$$

Let $\mathfrak{K}_0$ be the G-normalizer of K, so that $\mathfrak{K}_0 = KZ$, and let $\mathfrak{K}_1$ be the G-normalizer of I, so that $\mathfrak{K}_1$ is generated as a group by I and $\Pi := \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. We note that if $m \geq 1$, then K_m is normal in $\mathfrak{K}_0$ and I_m is normal in $\mathfrak{K}_1$. We denote $s := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

3. Diagrams

Let R be a commutative ring, (typically $R = L$, $\mathfrak{o}_L$ or $\mathfrak{o}_L/\mathfrak{p}_L^n$). By a diagram D of R-modules, we mean the data (D_0, D_1, r), where D_0 is an $R[\mathfrak{K}_0]$-module, D_1 is an $R[\mathfrak{K}_1]$-module and $r : D_1 \to D_0$ is a $\mathfrak{K}_0 \cap \mathfrak{K}_1 = IZ$-equivariant homomorphism of R-modules. A morphism α between two diagrams D and D' is given by (α_0, α_1), where $\alpha_0 : D_0 \to D_0'$ is a morphism of $R[\mathfrak{K}_0]$-modules, $\alpha_1 : D_1 \to D_1'$ is a morphism of $R[\mathfrak{K}_1]$-modules, and the diagram

$$\begin{tikzcd} D_0 \arrow[r, "\alpha_0"] & D_0' \\ D_1 \arrow[u, "r"] \arrow[r, "\alpha_1"] & D_1' \arrow[u, "r'"'] \end{tikzcd} \tag{1}$$

commutes in the category of $R[IZ]$-modules. The condition (1) is important, since one can have two diagrams of R-modules D and D', such that $D_0 \cong D_0'$ as $R[\mathfrak{K}_0]$-modules and $D_1 \cong D_1'$ as $R[\mathfrak{K}_1]$-modules, but $D \not\cong D'$ as diagrams. The diagrams of R-modules with the above morphisms form an abelian category. To a diagram D one may associate a complex

$$\text{c-Ind}_{\mathfrak{K}_1}^{G} D_1 \otimes \delta \xrightarrow{\partial} \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0 \tag{2}$$

of G-representations, where $\delta : \mathfrak{K}_1 \to R^\times$ is the character $\delta(g) := (-1)^{\mathrm{val}(\det g)}$; $\text{c-Ind}_{\mathfrak{K}_i}^{G} D_i$ denotes the space of functions $f : G \to D_i$ such that $f(kg) = kf(g)$ for $k \in \mathfrak{K}_i$ and $g \in G$, and f is supported only on finitely many cosets $\mathfrak{K}_i g$. To describe ∂, we note that Frobenius reciprocity gives

$$\mathrm{Hom}_G(\text{c-Ind}_{\mathfrak{K}_1}^{G} D_1 \otimes \delta, \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0) \cong \mathrm{Hom}_{\mathfrak{K}_1}(D_1 \otimes \delta, \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0);$$

now $\mathrm{Ind}_{IZ}^{\mathfrak{K}_1} D_0$ is a direct summand of the restriction of $\text{c-Ind}_{\mathfrak{K}_0}^{G} D_0$ to $\mathfrak{K}_1$, and

$$\mathrm{Hom}_{\mathfrak{K}_1}(D_1 \otimes \delta, \mathrm{Ind}_{IZ}^{\mathfrak{K}_1} D_0) \cong \mathrm{Hom}_{IZ}(D_1, D_0),$$

since δ is trivial on IZ. Composition of the maps above yields a map

$$\mathrm{Hom}_{IZ}(D_1, D_0) \to \mathrm{Hom}_G(\text{c-Ind}_{\mathfrak{K}_1}^{G} D_1 \otimes \delta, \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0).$$

We let ∂ be the image of r. We define $H_0(D)$ to be the cokernel of ∂ and $H_1(D)$ to be the kernel of ∂. So we have this exact sequence of G-representations:

$$0 \to H_1(D) \to \text{c-Ind}_{\mathfrak{K}_1}^{G} D_1 \otimes \delta \xrightarrow{\partial} \text{c-Ind}_{\mathfrak{K}_0}^{G} D_0 \to H_0(D) \to 0 \tag{3}$$

Further, if r is injective then one may show that $H_1(D) = 0$; see [Vignéras 2008, Proposition 0.1]. To a diagram D one may associate a G-equivariant coefficient system $\mathcal{V}$ of R-modules on the Bruhat–Tits tree; see [Paškūnas 2004, Section 5]. Then $H_0(D)$ and $H_1(D)$ compute the homology of the coefficient system $\mathcal{V}$, and the map ∂ has a natural interpretation. Assume that $R = L$ (or any field of characteristic 0), and let π be a smooth irreducible representation of G on an L-vector space, so that for all $v \in \pi$ the subgroup $\{g \in G : gv = v\}$ is open in G. Since the action of G is smooth, there exists an $m \geq 0$ such that $\pi^{I_m} \neq 0$. To π we may associate a diagram $D := (\pi^{I_m} \hookrightarrow \pi^{K_m})$. As a very special case of a result by Schneider and Stuhler [1997, Theorem V.1; 1993, Section 3], we obtain that $H_0(D) \cong \pi$.

We are going to compute such diagrams D, attached to smooth principal series representations of G on L-vector spaces. Given smooth characters $\theta_1, \theta_2 : \mathbb{Z}_p^\times \to L^\times$ and $\lambda_1, \lambda_2 \in L^\times$, we define a diagram $D(\lambda_1, \lambda_2, \theta_1, \theta_2)$ as follows. Let $c \geq 1$ be an integer such that θ_1 and θ_2 are trivial on $1 + p^c\mathbb{Z}_p$. Set $J_c := (K \cap B)K_c = (I \cap B)K_c$, so that J_c is a subgroup of I. Let $\theta : J_c \to L^\times$ be the character $\theta\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := \theta_1(a)\theta_2(d)$. Let $D_0 := \mathrm{Ind}_{J_c}^{K} \theta$, and let $p \in Z$ act on D_0 by a scalar $\lambda_1\lambda_2$, so that D_0 is a representation of $\mathfrak{K}_0$. Set $D_1 := D_0^{I_c}$, so that D_1 is naturally a representation of IZ.

We are going to put an action of Π on D_1, so that D_1 is a representation of $\mathfrak{K}_1$. Let

$$V_1 := \{f \in D_1 : \operatorname{Supp} f \subseteq I\}, \quad V_s := \{f \in D_1 : \operatorname{Supp} f \subseteq J_c s I\}. \tag{4}$$

Since I contains K_1, we have $J_c s I = (B \cap K)sI = IsI$; hence $D_1 = V_1 \oplus V_s$. For all $f_1 \in V_1$ and $f_s \in V_s$, we define $\Pi \cdot f_1 \in V_s$ and $\Pi \cdot f_s \in V_1$ such that

$$[\Pi \cdot f_1](sg) := \lambda_1 f_1(\Pi^{-1} g \Pi), \quad [\Pi \cdot f_s](g) = \lambda_2 f_s(s \Pi g \Pi^{-1}) \quad \text{for all } g \in I. \tag{5}$$

Every $f \in D_1$ can be written uniquely as $f = f_1 + f_s$, with $f_1 \in V_1$ and $f_s \in V_s$, and we define $\Pi \cdot f := \Pi \cdot f_1 + \Pi \cdot f_s$.

Lemma 3.1. *Equation (5) defines an action of $\mathfrak{K}_1$ on D_1. We denote the diagram $D_1 \hookrightarrow D_0$ by $D(\lambda_1, \lambda_2, \theta_1, \theta_2)$. Let $\pi := \operatorname{Ind}_B^G \chi_1 \otimes \chi_2$ be a smooth principal series representation of G, with*

$$\chi_1(p) = \lambda_1, \quad \chi_2(p) = \lambda_2, \quad \chi_1|_{\mathbb{Z}_p^\times} = \theta_1, \quad \chi_2|_{\mathbb{Z}_p^\times} = \theta_2.$$

There exists an isomorphism of diagrams $D(\lambda_1, \lambda_2, \theta_1, \theta_2) \cong (\pi^{I_c} \hookrightarrow \pi^{K_c})$. In particular, we have a G-equivariant isomorphism $H_0(D(\lambda_1, \lambda_2, \theta_1, \theta_2)) \cong \pi$.

Proof. We note that $p \in Z$ acts on π by a scalar $\lambda_1 \lambda_2$. Since $G = BK$, we have $\pi|_K \cong \operatorname{Ind}_{B \cap K}^K \theta$, and so the map $f \mapsto [g \mapsto f(g)]$ induces an isomorphism $\iota_0 : \pi^{K_c} \cong \operatorname{Ind}_{J_c}^K \theta = D_0$. Let

$$\mathcal{F}_1 := \{f \in \pi : \operatorname{Supp} f \subseteq BI\} \quad \text{and} \quad \mathcal{F}_s := \{f \in \pi : \operatorname{Supp} f \subseteq BsI\}.$$

Iwasawa decomposition gives $G = BI \cup BsI$; hence $\pi = \mathcal{F}_1 \oplus \mathcal{F}_s$. If $f_1 \in \mathcal{F}_1$, then $\operatorname{Supp}(\Pi f_1) = (\operatorname{Supp} f_1)\Pi^{-1} \subseteq BI\Pi^{-1} = BsI$. Moreover,

$$\begin{aligned}[\Pi f_1](sg) = f_1(sg\Pi) &= f_1(s\Pi(\Pi^{-1} g \Pi)) \\ &= \chi_1(p) f_1(\Pi^{-1} g \Pi) \quad \text{for all } g \in I.\end{aligned} \tag{6}$$

Similarly, if $f_s \in \mathcal{F}_s$, then $\operatorname{Supp}(\Pi f_s) = (\operatorname{Supp} f_s)\Pi^{-1} \subseteq BsI\Pi^{-1} = BI$, and

$$\begin{aligned}[\Pi f_s](g) = f_1(g\Pi) &= f_1((\Pi s)s(\Pi^{-1} g \Pi)) \\ &= \chi_2(p) f_s(s(\Pi^{-1} g \Pi)) \quad \text{for all } g \in I.\end{aligned} \tag{7}$$

Now $\pi^{I_c} = \mathcal{F}_1^{I_c} \oplus \mathcal{F}_s^{I_c} \subset \pi^{K_c}$. Let ι_1 be the restriction of ι_0 to π^{I_c}. Then it is immediate that $\iota_1(\mathcal{F}_1^{I_c}) = V_1$ and $\iota_1(\mathcal{F}_s^{I_1}) = V_s$, where V_1 and V_s are as above. Moreover, if $f \in D_1$ and $\Pi \cdot f$ is given by (5), then $\Pi \cdot f = \iota_1(\Pi \iota_1^{-1}(f))$. Since $\mathfrak{K}_1$ acts on π^{I_c}, Equation (5) defines an action of $\mathfrak{K}_1$ on D_1 such that ι_1 is $\mathfrak{K}_1$-equivariant. Hence, (ι_0, ι_1) is an isomorphism of diagrams $(\pi^{I_c} \hookrightarrow \pi^{K_c}) \cong (D_1 \hookrightarrow D_0)$. $\square$

4. The main result

Lemma 4.1. *Let U be a finite dimensional L-vector space with subspaces U_1, U_2 such that $U = U_1 \oplus U_2$. For $x \in L$ define a map $\phi_x : U \to U$ by $\phi_x(v_1+v_2) = xv_1+v_2$ for all $v_1 \in U_1$ and $v_2 \in U_2$. Let M be an $\mathfrak{o}_L$-lattice in V. Then there exists an integer $a \geq 1$ such that $\phi_x(M) = M$ for $x \in 1+\mathfrak{p}_L^a$.*

Proof. Let N denote the image of M in U/U_2. Then N contains $(M \cap U_1) + U_2$, and both are lattices in U/U_2. Define $a \geq 1$ to be the smallest integer such that $\mathfrak{p}_L^{-a}(M \cap U_1) + U_2$ contains N. Suppose that $x \in 1 + \mathfrak{p}_F^a$ and $v \in M$. We may write $v = \lambda v_1 + v_2$, with $v_1 \in M \cap U_1$, $v_2 \in U_2$ and $\lambda \in \mathfrak{p}_L^{-a}$. Now $\phi_x(v) = v + \lambda(x-1)v_1 \in M$. Hence we get $\phi_x(M) \subseteq M$ and $\phi_{x^{-1}}(M) \subseteq M$. Applying $\phi_{x^{-1}}$ to the first inclusion gives $M \subseteq \phi_{x^{-1}}(M)$. ☐

We fix an integer $k \geq 2$ and set $W := \mathrm{Sym}^{k-2} L^2$, an algebraic representation of G. Let $\pi := \pi(\chi_1, \chi_2) := \mathrm{Ind}_B^G \chi_1 \otimes \chi_2$ be a smooth principal series L-representation of G. We say that $\pi \otimes W$ admits a G-invariant norm if there exists a norm $\|\cdot\|$ on $\pi \otimes W$ with respect to which $\pi \otimes W$ is a normed L-vector space such that $\|gv\| = \|v\|$ for all $v \in \pi \otimes W$ and $g \in G$.

Let $c \geq 1$ be an integer such that both χ_1 and χ_2 are trivial on $1 + p^c\mathbb{Z}_p$. Let D be the diagram $\pi^{I_c} \otimes W \hookrightarrow \pi^{K_c} \otimes W$. Since $H_0(\pi^{I_c} \hookrightarrow \pi^{K_c}) \cong \pi$, by tensoring (2) with W we obtain $H_0(D) \cong \pi \otimes W$. Assume that $\pi \otimes W$ admits a G-invariant norm $\|\cdot\|$, and set $(\pi \otimes W)^0 := \{v \in \pi \otimes W : \|v\| \leq 1\}$. Then we may define a diagram $\mathfrak{D} = (\mathfrak{D}_1 \hookrightarrow \mathfrak{D}_0)$ of $\mathfrak{o}_L$-modules by

$$\mathfrak{D} := ((\pi^{I_c} \otimes W) \cap (\pi \otimes W)^0 \hookrightarrow (\pi^{K_c} \otimes W) \cap (\pi \otimes W)^0).$$

In this case Vignéras [2008] has shown that the inclusion $\mathfrak{D} \hookrightarrow D$ induces a G-equivariant injection $H_0(\mathfrak{D}) \hookrightarrow H_0(D)$ such that $H_0(\mathfrak{D}) \otimes_{\mathfrak{o}_L} L = H_0(D)$ and $H_1(\mathfrak{D}) = 0$. Moreover, $H_0(\mathfrak{D})$ does not contain an $\mathfrak{o}_L$-submodule isomorphic to L; see [Vignéras 2008, Proposition 0.1]. Since $H_0(D)$ is an L-vector space of countable dimension, this implies that $H_0(\mathfrak{D})$ is a free $\mathfrak{o}_L$-module. By tensoring (2) with $\mathfrak{o}_L/\mathfrak{p}_L^n$, we obtain

$$H_0(\mathfrak{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^n \cong H_0(\mathfrak{D} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^n). \tag{8}$$

Proposition 4.2. *Let $\pi = \pi(\chi_1, \chi_2)$ be a smooth principal series representation, assume that $\pi \otimes W$ admits a G-invariant norm, and let $\mathfrak{D}$ be as above. Then there exists an integer $a \geq 1$ such that for all $x \in 1 + \mathfrak{p}_F^b$, with $b \geq a$, there exists both a finitely generated $\mathfrak{o}_L[G]$-module M in $\pi(\chi_1\delta_{x^{-1}}, \chi_2\delta_x) \otimes W$ that is free as an $\mathfrak{o}_L$-module, and a G-equivariant isomorphism*

$$M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b \cong H_0(\mathfrak{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b,$$

where $\delta_x : \mathbb{Q}_p^\times \to L^\times$ is an unramified character with $\delta_x(p) = x$.

Proof. Apply Lemma 4.1 to $U = D_1$, $U_1 = V_1 \otimes W$, $U_2 = V_s \otimes W$ and $M = \mathfrak{D}_1$, where V_1 and V_s are given by (4). We get an integer $a \geq 1$ such that $\phi_x(\mathfrak{D}_1) = \mathfrak{D}_1$ for all $x \in 1 + \mathfrak{p}_L^a$. It is immediate that ϕ_x is IZ-equivariant. We define a new action $\star$ of Π on D_1 by $\Pi \star v := \phi_x(\Pi \phi_x^{-1}(v))$. This gives us a new diagram $D(x)$, so that $D(x)_0 = D_0$ as a representation of $\mathfrak{K}_0$, $D(x)_1 = D_1$ as a representation of IZ, the IZ-equivariant injection $D(x)_1 \hookrightarrow D(x)_0$ is equal to the IZ-equivariant injection $D_1 \hookrightarrow D_0$, but the action of Π on D_1 is given by $\star$, (here by $=$ we really mean an equality, not an isomorphism). If $f_1 \in V_1$ and $f_s \in V_s$ then

$$\Pi \star (f_1 \otimes w) = f_s' \otimes (\Pi w), \quad \Pi \star (f_s \otimes w) = f_1' \otimes (\Pi w) \quad \text{for all } w \in W,$$

where $f_s' \in V_s$, $f_1' \in V_1$ and for all $g \in I$ we have

$$f_s'(sg) = x^{-1}[\Pi \cdot f_1](sg) = x^{-1}\lambda_1 f_1(\Pi^{-1} g \Pi), \tag{9}$$

$$f_1'(g) = x[\Pi \cdot f_s](g) = x\lambda_2 f_s(s \Pi g \Pi^{-1}). \tag{10}$$

Hence, we have an isomorphism of diagrams $D(x) \cong D(x^{-1}\lambda_1, x\lambda_2, \theta_1, \theta_2)$, and so Lemma 3.1 gives $H_0(D(x)) \cong \pi(\chi_1 \delta_{x^{-1}}, \chi_2 \delta_x) \otimes W$. Now let $b \geq a$ be an integer and suppose that $x \in 1 + \mathfrak{p}_L^b$. Since $\Pi \cdot \mathfrak{D}_1 = \phi_x(\mathfrak{D}_1) = \phi_x^{-1}(\mathfrak{D}_1) = \mathfrak{D}_1$, we get

$$\Pi \star (\mathfrak{D}_0 \cap D_1) = \Pi \star \mathfrak{D}_1 = \phi_x(\Pi \phi_x^{-1}(\mathfrak{D}_1)) = \mathfrak{D}_1.$$

So if we let $\mathfrak{D}(x)_0 := \mathfrak{D}_0$ and $\mathfrak{D}(x)_1 := \mathfrak{D}(x)_0 \cap D(x)_1$, where Π acts on $\mathfrak{D}(x)_1$ by $\star$, then the diagram $\mathfrak{D}(x) := (\mathfrak{D}(x)_1 \hookrightarrow \mathfrak{D}(x)_0)$ is an integral structure in $D(x)$ in the sense of [Vignéras 2008]. The results of Vignéras cited above imply that $M := H_0(\mathfrak{D}(x))$ is a finitely generated $\mathfrak{o}_L[G]$-submodule of $\pi(\chi_1 \delta_{x^{-1}}, \chi_2 \delta_x) \otimes W$, which is free as an $\mathfrak{o}_L$-module, and $M \otimes_{\mathfrak{o}_L} L \cong \pi(\chi_1 \delta_{x^{-1}}, \chi_2 \delta_x) \otimes W$. Moreover, since ϕ_x is the identity modulo $\mathfrak{p}_L^b$, we have $\Pi \star v \equiv \Pi \cdot v \pmod{\varpi_L^b \mathfrak{D}_1}$ for all $v \in \mathfrak{D}_1$, and so the identity map $\mathfrak{D}(x)_0 \to \mathfrak{D}_0$ induces an isomorphism of diagrams $\mathfrak{D}(x) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b \cong \mathfrak{D} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b$. Now (8) gives $H_0(\mathfrak{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b \cong M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^b$. □

Let $k \geq 2$ be an integer and $a_p \in \mathfrak{p}_L$. Following [Breuil 2003] we define a filtered φ-module D_{k,a_p} as the following data: a 2-dimensional L-vector space D with basis $\{e_1, e_2\}$, an L-linear automorphism $\varphi : D \to D$ given by

$$\varphi(e_1) = p^{k-1} e_2 \quad \text{and} \quad \varphi(e_2) = -e_1 + a_p e_2,$$

and a decreasing filtration $(\mathrm{Fil}^i D)_{i \in \mathbb{Z}}$ by L-subspaces such that if $i \leq 0$ then $\mathrm{Fil}^i D = D$, if $1 \leq i \leq k-1$ then $\mathrm{Fil}^i D = L e_1$, and if $i \geq k$ then $\mathrm{Fil}^i D = 0$. We set $V_{k,a_p} := \mathrm{Hom}_{\varphi,\mathrm{Fil}^{\cdot}}(D_{k,a_p}, B_{cris})$. Then V_{k,a_p} is a 2-dimensional L-linear absolutely irreducible crystalline representation of $\mathcal{G}_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with Hodge–Tate weights 0 and $k-1$. We denote by χ_{k,a_p} the trace character of V_{k,a_p}. Since $\mathcal{G}_{\mathbb{Q}_p}$ is compact and the action is continuous, $\mathcal{G}_{\mathbb{Q}_p}$ stabilizes some $\mathfrak{o}_L$-lattice in V_{k,a_p}, and so χ_{k,a_p} takes values in $\mathfrak{o}_L$.

Proposition 4.3. *Let m be the largest integer such that $m \le (k-2)/(p-1)$. Let $a_p, a'_p \in \mathfrak{p}_L$, and assume that $\mathrm{val}(a_p) > m$ and $\mathrm{val}(a'_p) > m$. Let $n \ge em$ be an integer, where $e := e(L/\mathbb{Q}_p)$ is the ramification index. Suppose $a_p \equiv a'_p \pmod{\mathfrak{p}_L^n}$. Then $\chi_{k,a_p}(g) \equiv \chi_{k,a'_p}(g) \pmod{\mathfrak{p}_L^{n-em}}$ for all $g \in \mathcal{G}_{\mathbb{Q}_p}$.*

Proof. This a consequence of a result of Berger, Li and Zhu [Berger et al. 2004], where the authors construct $\mathcal{G}_{\mathbb{Q}_p}$-invariant lattices T_{k,a_p} in V_{k,a_p}. The assumption $a_p \equiv a'_p \pmod{\mathfrak{p}_L^n}$ implies $T_{k,a_p} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{n-em} \cong T_{k,a'_p} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{n-em}$; see their [Remark 4.1.2(2)]. This implies the congruences of characters. □

Let $k \ge 2$ be an integer and choose $\lambda_1, \lambda_2 \in L$ such that $\lambda_1 + \lambda_2 = a_p$ and $\lambda_1\lambda_2 = p^{k-1}$ (enlarge L if necessary). Assume $\mathrm{val}(\lambda_1) \ge \mathrm{val}(\lambda_2) > 0$. Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to L^\times$ be unramified characters, with $\chi_1(p) = \lambda_1^{-1}$ and $\chi_2(p) = \lambda_2^{-1}$. Let M be a finitely generated $\mathfrak{o}_L[G]$-module in $\pi(\chi_1, \chi_2|\cdot|^{-1}) \otimes W$, where $W := \mathrm{Sym}^{k-2} L^2$. In the case $\lambda_1 \ne \lambda_2$, Berger and Breuil have shown that the unitary L-Banach space representation

$$E_{k,a_p} := L \otimes_{\mathfrak{o}_L} \varprojlim M/\varpi_L^n M$$

of G is nonzero, topologically irreducible, admissible in the sense of [Schneider and Teitelbaum 2002], and contains $\pi(\chi_1, \chi_2|\cdot|^{-1}) \otimes W$ as a dense G-invariant subspace [Berger and Breuil 2007, Section 5.3]. Moreover, the dual of E_{k,a_p} is isomorphic to the representation of Borel subgroup B constructed from the (φ, Γ)-module of V_{k,a_p}.

Let $\mathrm{Rep}_{\mathfrak{o}_L} G$ be the category of finite length $\mathfrak{o}_L[G]$-modules with a central character such that the action of G is smooth (that is, the stabilizer of a vector is an open subgroup of G). Let $\mathrm{Rep}_{\mathfrak{o}_L} \mathcal{G}_{\mathbb{Q}_p}$ be the category of continuous representations of $\mathcal{G}_{\mathbb{Q}_p}$ on $\mathfrak{o}_L$-modules of finite length. Colmez [2008, IV.2.14] has defined an exact covariant functor $\mathbf{V} : \mathrm{Rep}_{\mathfrak{o}_L} G \to \mathrm{Rep}_{\mathfrak{o}_L} \mathcal{G}_{\mathbb{Q}_p}$. The constructions in [Berger and Breuil 2007] and [Colmez 2008] are mutually inverse to one another. This means if we assume $\lambda_1 \ne \lambda_2$ and let M be as above, then

$$V_{k,a_p} \cong L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(M/\varpi_L^n M). \tag{11}$$

That $M/\varpi_L^n M$ is an $\mathfrak{o}_L[G]$-module of finite length follows from [Berger 2005, Theorem A].

Theorem 4.4. *Assume that $p > 2$. Let $\lambda = \pm p^{(k-1)/2}$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character with $\chi(p) = \lambda^{-1}$. Assume there exists a G-invariant norm $\|\cdot\|$ on $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$, where $W := \mathrm{Sym}^{k-2} L^2$. Let E be the completion of $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ with respect to $\|\cdot\|$. Then E is a nonzero, topologically* *irreducible, admissible Banach space representation of G. If we let E^0 be the unit ball in E, then $V_{k,2\lambda} \otimes (\chi|\chi|) \cong L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(E^0/\varpi_L^n E^0)$.*

Proof. Since the character $\chi|\chi|$ is integral, by twisting we may assume that χ is unramified. We denote the diagram

$$\pi(\chi, \chi|\cdot|^{-1})^{I_1} \otimes W \hookrightarrow \pi(\chi, \chi|\cdot|^{-1})^{K_1} \otimes W$$

by $D = (D_1 \hookrightarrow D_0)$. Let $\mathfrak{D} = (\mathfrak{D}_1 \hookrightarrow \mathfrak{D}_0)$ be the diagram of $\mathfrak{o}_L$-modules with $\mathfrak{D}_1 = D_1 \cap E^0$ and $\mathfrak{D}_0 = D_0 \cap E^0$. Let $a \geq 1$ be the integer Proposition 4.2 gives. For each $j \geq 0$, we fix $x_j \in 1 + \mathfrak{p}_L^{a+j}$ with $x_j \neq 1$ and a finitely generated $\mathfrak{o}_L[G]$-submodule M_j in $\pi(\chi\delta_{x_j^{-1}}, \chi\delta_{x_j}|\cdot|^{-1}) \otimes W$ (which is then a free $\mathfrak{o}_L$-module) such that

$$H_0(\mathfrak{D}) \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j} \cong M_j \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j}.$$

This is possible by Proposition 4.2. To ease the notation we set $M := H_0(\mathfrak{D})$. Let $a_p(j) := \lambda x_j^{-1} + \lambda x_j$, let $a_p := 2\lambda$, and let m be the largest integer such that $m \leq (k-2)/(p-1)$. Since $p > 2$, $x_j + x_j^{-1}$ is a unit in $\mathfrak{o}_L$, we have $\mathrm{val}(a_p(j)) = \mathrm{val}(a_p) = (k-1)/2 > m$. (Here we really need $p > 2$.) Moreover, we have $a_p \equiv a_p(j) \pmod{\mathfrak{p}_L^{j+a+em}}$, where $e := e(L/\mathbb{Q}_p)$ is the ramification index. Now since $x_j \neq 1$ we get that $\lambda x_j \neq \lambda x_j^{-1}$, and hence we may apply the results of Berger and Breuil to $\pi(\chi\delta_{x_j^{-1}}, \chi\delta_{x_j}|\cdot|^{-1}) \otimes W$. By (11),

$$T_{k,a_p(j)} := \varprojlim \mathbf{V}(M_j/\varpi_L^n M_j)$$

is a $\mathcal{G}_{\mathbb{Q}_p}$-invariant lattice in $V_{k,a_p(j)}$. Since $M \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j} \cong M_j \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j}$ we get

$$\mathbf{V}(M/\varpi_L^{a+j}M) \cong \mathbf{V}(M_j/\varpi_L^{a+j}M_j) \cong T_{k,a_p(j)} \otimes_{\mathfrak{o}_L} \mathfrak{o}_L/\mathfrak{p}_L^{a+j}. \tag{12}$$

Set $V := L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(M/\varpi_L^n M)$. Then (12) implies that V is a 2-dimensional L-vector space. Let χ_V be the trace character of V. Then it follows from (12) that $\chi_V \equiv \chi_{k,a_p(j)} \pmod{\mathfrak{p}_L^{a+j}}$. Since $a_p \equiv a_p(j) \pmod{\mathfrak{p}_L^{a+j+em}}$, Proposition 4.3 says that $\chi_{k,a_p} \equiv \chi_{k,a_p(j)} \pmod{\mathfrak{p}_L^{a+j}}$. We obtain $\chi_V \equiv \chi_{k,a_p} \pmod{\mathfrak{p}_L^{a+j}}$ for all $j \geq 0$. This gives us $\chi_V = \chi_{k,a_p}$. Since V_{k,a_p} is irreducible, the equality of characters implies $V \cong V_{k,a_p}$.

Set $\widehat{M} := \varprojlim M/\varpi_L^n M$, and $E' := \widehat{M} \otimes_{\mathfrak{o}_L} L$. Since M is a free $\mathfrak{o}_L$-module, we get an injection $M \hookrightarrow \widehat{M}$. In particular, E' contains $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ as a dense G-invariant subspace. We claim that E' is a topologically irreducible and admissible G-representation. Now Theorem 4.1.1 and Proposition 4.1.4 of [Berger et al. 2004] say that the semisimplification of $T_{k,a_p(j)} \otimes_{\mathfrak{o}_L} k_L$ is irreducible if $p+1 \nmid k-1$ and is otherwise isomorphic to

$$\begin{pmatrix} \mu_{\sqrt{-1}} & 0 \\ 0 & \mu_{-\sqrt{-1}} \end{pmatrix} \otimes \omega^{(k-1)/(p+1)},$$

where $\mu_{\pm\sqrt{-1}}$ is the unramified character sending arithmetic Frobenius to $\pm\sqrt{-1}$, and ω is the cyclotomic character. Then [Berger 2005, Theorem A] implies that if $p+1 \nmid k-1$, then $M_j \otimes_{\mathfrak{o}_L} k_L$ is an irreducible supersingular representation of G, and if $p+1 \mid k-1$, then the semisimplification of $M_j \otimes_{\mathfrak{o}_L} k_L$ is a direct sum of two irreducible principal series. The irreducibility of principal series follows from [Barthel and Livné 1994, Theorem 33], since $\sqrt{-1} \neq \pm 1$, as $p > 2$. Since $M \otimes_{\mathfrak{o}_L} k_L \cong M_j \otimes_{\mathfrak{o}_L} k_L$, we get that $M \otimes_{\mathfrak{o}_L} k_L$ is an admissible representation of G (so that for every open subgroup $\mathcal{U}$ of G, the space of $\mathcal{U}$-invariants is finite dimensional). This implies that E' is admissible.

Suppose that E_1 is a closed G-invariant subspace of E' with $E' \neq E_1$. Let $E_1^0 := E_1 \cap \widehat{M}$. We obtain a G-equivariant injection $E_1^0 \otimes_{\mathfrak{o}_L} k_L \hookrightarrow M \otimes_{\mathfrak{o}_L} k_L$. If $E_1^0 \otimes_{\mathfrak{o}_L} k_L = 0$ or $M \otimes_{\mathfrak{o}_L} k_L$, then Nakayama's lemma gives $E_1^0 = 0$ or $E_1^0 = \widehat{M}$, respectively. If $p+1 \nmid k-1$, then $M \otimes_{\mathfrak{o}_L} k_L$ is irreducible and we are done. If $p+1 \mid k-1$, then $E_1^0 \otimes_{\mathfrak{o}_L} k_L$ is an irreducible principal series, and so $\mathbf{V}(E_1^0 \otimes_{\mathfrak{o}_L} k_L)$ is one-dimensional [Colmez 2008, IV.4.17]. But then $V_1 := L \otimes_{\mathfrak{o}_L} \varprojlim \mathbf{V}(E_1^0/\varpi_L^n E_1^0)$ is a 1-dimensional subspace of V_{k,a_p} stable under the action of $\mathcal{G}_{\mathbb{Q}_p}$. Since V_{k,a_p} is irreducible we obtain a contradiction.

Since E' is a completion of $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ with respect to a finitely generated $\mathfrak{o}_L[G]$-submodule, E' is in fact the universal completion; see for example [Emerton 2005, Proposition 1.17]. In particular, we obtain a nonzero G-equivariant map of L-Banach space representations $E' \to E$, but since E' is irreducible and $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ is dense in E, this map is an isomorphism. □

Corollary 4.5. *Assume that $p > 2$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character such that $\chi(p)^2 p^{k-1} = 1$. Assume that there is a G-invariant norm $\|\cdot\|$ on $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$, where $W := \mathrm{Sym}^{k-2} L^2$. Then every bounded G-invariant $\mathfrak{o}_L$-lattice in $\pi(\chi, \chi|\cdot|^{-1}) \otimes W$ is finitely generated as an $\mathfrak{o}_L[G]$-module.*

Proof. The existence of a G-invariant norm implies that the universal completion is nonzero. It follows from Theorem 4.4 that the universal completion is topologically irreducible and admissible. The assertion follows from the proof of [Berger and Breuil 2007, Corollary 5.3.4]. □

For the purposes of [Paškūnas 2008] we record the following corollary to the proof of Theorem 4.4.

Corollary 4.6. *Assume $p > 2$, and let $\chi : \mathbb{Q}_p^\times \to L^\times$ be a smooth character such that $\chi^2(p) p^{k-1}$ is a unit in $\mathfrak{o}_L$. Assume there exists a unitary L-Banach space representation $(E, \|\cdot\|)$ of G containing $(\mathrm{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \mathrm{Sym}^{k-2} L^2$ as a dense G-invariant subspace and satisfying $\|E\| \subseteq |L|$. Then there exists $x \in 1+\mathfrak{p}_L$*

with $x^2 \neq 1$ and a unitary completion E_x of $(\operatorname{Ind}_B^G \chi\delta_x \otimes \chi\delta_{x^{-1}}|\cdot|^{-1}) \otimes \operatorname{Sym}^{k-2} L^2$ such that $E^0 \otimes_{\mathfrak{o}_L} k_L \cong E_x^0 \otimes_{\mathfrak{o}_L} k_L$, where E_x^0 is the unit ball in E_x and E^0 is the unit ball in E.

Proof. Let $\pi := \operatorname{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}$ and $M := (\pi \otimes W) \cap E^0$. Now $M \cap \varpi_L E^0 = (\pi \otimes W) \cap \varpi_L E^0 = \varpi_L M$. So $\iota : M/\varpi_L M \hookrightarrow E^0/\varpi_L E^0$ is a G-equivariant injection. We claim that ι is a surjection. Let $v \in E^0$. Since $\pi \otimes W$ is dense in E, there exists a sequence $\{v_n\}_{n\geq 1}$ in $\pi \otimes W$ such that $\lim v_n = v$. We also have $\lim \|v_n\| = \|v\|$. Since $\|E\| \subseteq |L| \cong \mathbb{Z}$, there exists an $m \geq 0$ such that $v_n \in M$ for all $n \geq m$. This implies the surjectivity of ι. So we get $M \otimes_{\mathfrak{o}_L} k_L \cong E^0 \otimes_{\mathfrak{o}_L} k_L$.

By Corollary 4.5 we may find $u_1, \dots, u_n \in M$ that generate M as an $\mathfrak{o}_L[G]$-module. Further, $u_i = \sum_{j=1}^{m_i} v_{ij} \otimes w_{ij}$ with $v_{ij} \in \pi$ and $w_{ij} \in W$. Since π is a smooth representation of G, there exists an integer $c \geq 1$ such that v_{ij} is fixed by K_c for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$. Set

$$\mathfrak{D} := \big((\pi^{I_c} \otimes W) \cap M \hookrightarrow (\pi^{K_c} \otimes W) \cap M\big), \quad D := \big(\pi^{I_c} \otimes W \hookrightarrow \pi^{K_c} \otimes W\big)$$

and let M' be the image of $H_0(\mathfrak{D}) \hookrightarrow H_0(D) \cong \pi \otimes W$. It follows from (3) that M' is generated by $(\pi^{K_c} \otimes W) \cap M$ as an $\mathfrak{o}_L[G]$-module. Hence, $M' \subseteq M$. By construction $(\pi^{K_c} \otimes W) \cap M$ contains $u_1, \dots u_n$, and so $M \subseteq M'$. In particular, $H_0(\mathfrak{D}) \otimes_{\mathfrak{o}_L} k_L \cong M \otimes_{\mathfrak{o}_L} k_L$. The claim follows from the proof of Theorem 4.4. $\square$

5. Existence

Recent results of Colmez, which appeared after the first version of this note, imply the existence of a G-invariant norm on $(\operatorname{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \operatorname{Sym}^{k-2} L^2$ for $\chi^2(p)p^{k-1} \in \mathfrak{o}_L^\times$, thus making our results unconditional. We briefly explain this.

We continue to assume that $p > 2$, that $k \geq 2$ is an integer and that $a_p = 2p^{(k-1)/2}$. The representation V_{k,a_p} of $\mathcal{G}_{\mathbb{Q}_p}$ sits in the p-adic family of Berger, Li and Zhu, [2004, 3.2.5]. Moreover, all the other points in the family correspond to the crystalline representations with distinct Frobenius eigenvalues, to which the theory of [Berger and Breuil 2007] applies. Hence [Colmez 2008, II.3.1 and IV.4.11] imply that there exists an irreducible unitary L-Banach space representation Π of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that $\mathbf{V}(\Pi) \cong V_{k,a_p}$. If $p \geq 5$ or $p = 3$ and $k \not\equiv 3 \pmod 8$ and $k \not\equiv 7 \pmod 8$, the existence of such Π also follows from [Kisin 2008]. It follows from [Colmez 2008, VI.6.46] that the set of locally algebraic vectors Π^{alg} of Π is isomorphic to

$$(\operatorname{Ind}_B^G \chi \otimes \chi|\cdot|^{-1}) \otimes \operatorname{Sym}^{k-2} L^2,$$

where $\chi : \mathbb{Q}_p^\times \to L^\times$ is an unramified character with $\chi(p) = p^{-(k-1)/2}$. The restriction of the G-invariant norm of Π to Π^{alg} solves the problem. Also, if $\delta : \mathbb{Q}_p^\times \to L^\times$ is a unitary character, then we also obtain a G-invariant norm on $\Pi^{\mathrm{alg}} \otimes \delta \circ \det$.

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