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Frank Himstedt and Peter Symonds

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We consider a finite group acting on a graded module and define an equivariant degree that generalizes the usual nonequivariant degree. The value of this degree is a module for the group, up to a rational multiple. We investigate how this behaves when the module is a ring and apply our results to reprove some results of Kuhn on the cohomology of groups.

1. Introduction

We consider a finitely generated graded module M over a graded ring R that is finitely generated over some base field k and such that R_0 is finite-dimensional over k. We suppose that there is a finite group G that acts on M, preserving the grading and commuting with R.

To this data we associate a formal Laurent series $[M]$ in t in which the coefficient of t^r is the homogeneous part M_r, *considered as a kG-module*. The difficulty of the theory depends on whether we wish to keep track of these modules up to isomorphism (that is, in the Green ring) or only up to composition factors (in the representation ring). We develop both cases.

This series $[M]$ is shown to satisfy a form of the Hilbert–Serre Theorem (in particular it is a rational function, or at least a sum of them in the Green ring case). We define the equivariant degree $\deg_G M$ to be the coefficient of the leading term when we expand $[M]$ as a Laurent series in $1-t$. This is a kG-module up to rational multiple, although there is sometimes a problem of whether it is well defined in the Green ring case. The dimension of this module agrees with the usual definition of the degree in the nonequivariant case.

We investigate various properties of the equivariant degree; Theorem 6.4, in particular, lists several equivalent characterizations.

In Section 7, we go on to consider the case of the homogeneous coordinate ring on a projective variety and show that in this case the degree is always defined and it is a permutation module that can be easily described in terms of the geometry.

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Finally, in Section 8, this theory is applied to the variety associated to the cohomology of a group to reprove a result of Nick Kuhn on the action of the outer automorphism group of a p-group G on the cohomology $H^*(G; \mathbb{F}_p)$.

2. General setup

Let $R = \bigoplus_{j=0}^{\infty} R_j$ be a commutative graded algebra over a field k. We suppose that R is a finitely generated k-algebra and that R_0 is finite-dimensional over k, so all the homogeneous components R_j are also finite-dimensional vector spaces over k. Let G be a finite group and let $M = \bigoplus_{i=N}^{\infty} M_i$ a finitely generated graded left RG-module, where the action of G preserves the grading and each M_i is a finite-dimensional k-vector space.

We recall some facts about the Hilbert series $H(M,t) = \sum_{i=N}^{\infty} \dim_k(M_i)\, t^i$ of M. The graded version of Noether normalization [Benson 1993, Theorem 2.2.7] guarantees the existence of homogeneous elements $d_1, d_2, \ldots, d_n$ of positive degrees in R that generate a polynomial subring $k[d_1, \ldots, d_n]$ of R and such that R is finitely generated as a $k[d_1, \ldots, d_n]$-module. We write $|d_i| := \deg d_i$ for the degree of d_i. The number n is equal to the Krull dimension of R. By the Hilbert–Serre Theorem [Benson 1993, 2.1.1] the Hilbert series $H(M,t)$ is of the form

$$H(M,t) = \frac{f(M,t)}{\prod_{i=1}^{n}(1 - t^{|d_i|})},$$

where $f(M,t)$ is a Laurent polynomial with integer coefficients. As in [Benson 1993, Section 2.4], for example, the rational number $\deg M$ is defined by the Laurent expansion of $H(M,t)$ about $t = 1$:

$$H(M,t) = \frac{\deg M}{(1-t)^n} + O\left(\frac{1}{(1-t)^{n-1}}\right). \tag{2-1}$$

Obviously the definition of the degree $\deg M$ ignores the action of G on M. In the next two sections, we shall define an equivariant analogue $\deg_G M$, which also incorporates the group action.

First, we define the degree of certain Laurent series. Let $p(t)$ be a Laurent series of the form

$$p(t) = \sum_{i=N}^{\infty} a_i t^i = \frac{g(t)}{\prod_{i=1}^{n}(1 - t^{|d_i|})},$$

where the a_i are rational numbers and $g(t)$ is a Laurent polynomial with rational coefficients. We define the rational number $\deg p(t)$ to be the coefficient of $\frac{1}{(1-t)^n}$ in the Laurent expansion of $p(t)$ about $t = 1$ and we call $\deg p(t)$ the *degree* of $p(t)$. If we want to emphasize the dependency on n, we write $\deg^n p(t)$ instead of $\deg p(t)$. In particular, we have $\deg H(M,t) = \deg M$ with $\deg M$ as in (2-1).

3. Equivariant degree over the Green ring

As usual, the Green ring $a(kG)$ is defined to be the ring with generators the isomorphism classes $|V|$ of kG-modules, and relations $|V| + |W| = |V \oplus W|$, $|V| \cdot |W| = |V \otimes_k W|$. We set $a(kG)_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} a(kG)$. The representation ring $\mathcal{R}(kG)$ is defined to be the quotient of $a(kG)$ by the ideal generated by the elements $|V_2| - |V_1| - |V_3|$, where $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of kG-modules. We set $\mathcal{R}(kG)_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{R}(kG)$.

We will consider two versions of the equivariant degree: one is an element of $a(kG)_{\mathbb{Q}}$, but is not always defined; the other is a weaker one, which is an element of $\mathcal{R}(kG)_{\mathbb{Q}}$, but it is always defined. The main tool used in the definition of the former is the following Weak Structure Theorem 3.1, so-called because it is a generalization of the Structure Theorem of [Symonds 2007].

Theorem 3.1. *For any finitely generated graded $k[d_1, \ldots, d_n]G$-module M,*

$$M \cong \bigoplus_{U \in \operatorname{Indecomp}(M)} \; \bigoplus_{I \subseteq \{1,\ldots,n\}} k[d_I] \otimes_k X_{U,I},$$

as a kG-module, where $X_{U,I}$ is a finite-dimensional graded kG-module that is a sum of U's (ignoring grading) and $k[d_I] = k[d_i \mid i \in I]$. The map from right to left is given by multiplication.

Proof. The only difference between this theorem and Proposition 4.4 of [Symonds 2007] is that there $\operatorname{Indecomp}(M)$ is supposed to be finite. But the same proof works, although it is better to keep the different indecomposables separate by using the double summation, as in the statement above, rather than combining them as $\bar{X}_I = \bigoplus_{U \in \operatorname{Indecomp}(M)} X_{U,I}$ as in [Symonds 2007]. □

Next we describe the definition of the degree with values in $a(kG)_{\mathbb{Q}}$. For each i, the kG-module M_i defines an element $|M_i|$ of $a(kG)$.

Definition 3.2. We call the Laurent series

$$[M] := \sum_{i=N}^{\infty} |M_i| \, t^i \tag{3-1}$$

with coefficients in $a(kG)$ the *equivariant Hilbert series of M with coefficients in the Green ring*.

Clearly, if $G = \{1\}$ is the trivial group, we can identify $|M_i|$ with the dimension of M_i as a k-vector space. So in this situation $[M]$ coincides with the usual Hilbert series of M. The equivariant Hilbert series has the following basic properties:

Lemma 3.3. *Suppose $M' = \bigoplus_{i=N'}^{\infty} M'_i$ is another finitely generated graded left RG-module, such that the action of G preserves the grading and every M'_i is a finite-dimensional k-vector space. Then*

$$[M \oplus M'] = [M] + [M'] \quad \textit{and} \quad [M \otimes_k M'] = [M] \cdot [M'].$$

Proof. Clear. □

Besides the Hilbert series $H(M,t)$, we can consider a Hilbert series that counts the multiplicity of some isomorphism class of indecomposable summands. Let $\mathrm{Indecomp}(M)$ be a set of representatives for the isomorphism classes of all indecomposable kG-modules which occur as a direct summand of some M_i and let $m_{U,i}$ be the multiplicity of $U \in \mathrm{Indecomp}(M)$ as a direct summand of M_i. We set $H_U(M,t) := \sum_{i=N}^{\infty} m_{U,i}\, t^i$. The Laurent series $H_U(M,t)$ can be written as a rational function too.

Proposition 3.4. *For each $U \in \mathrm{Indecomp}(M)$, the Laurent series $H_U(M,t)$ can be written as*

$$H_U(M,t) = \frac{f_U(M,t)}{\prod_{i=1}^{n}(1 - t^{|d_i|})},$$

where $f_U(M,t)$ is a Laurent polynomial in t with integer coefficients.

Proof. This is a consequence of the Weak Structure Theorem 3.1. □

Let F be an arbitrary finite subset of $\mathrm{Indecomp}(M)$. We consider the Laurent series with integer coefficients $q(t) := H(M,t) - \sum_{U \in F} \dim_k(U) H_U(M,t)$. By definition of the Hilbert series, all the coefficients of $q(t)$ are nonnegative integers, and $q(t)$ is of the form

$$q(t) = \frac{g(t)}{\prod_{i=1}^{n}(1 - t^{|d_i|})}$$

for some Laurent polynomial $g(t)$ with integer coefficients since something similar holds for $H(M,t)$ and $H_U(M,t)$ by Proposition 3.4. So we can take degrees and obtain

$$\deg M = \left(\sum_{U \in F} \dim_k(U) \deg H_U(M,t) \right) + \deg q(t). \tag{3-2}$$

It turns out that all the degrees occurring in (3-2) are nonnegative with bounded denominators by the following result.

Lemma 3.5. *Suppose that*

$$p(t) = \frac{h(t)}{\prod_{i=1}^{n}(1 - t^{|d_i|})} = \sum_{i=N}^{\infty} a_i\, t^i,$$

where $h(t)$ is a Laurent polynomial with rational coefficients and the a_i's are nonnegative integers. Then $\deg p(t) \geq 0$. If all the coefficients of $h(t)$ are integers then $\deg p(t)$ is of the form $\deg p(t) = d / \prod_{i=1}^{n} |d_i|$ for some nonnegative integer d.

Proof. We compute

$$\deg p(t) = \lim_{t\to 1}(1-t)^n p(t) = \lim_{t\to 1} \frac{h(t)}{\prod_{i=1}^n (1+t+\cdots+t^{|d_i|-1})} = \frac{h(1)}{\prod_{i=1}^n |d_i|}.$$

We still have to show that $\deg p(t) \geq 0$. Since multiplication with

$$\prod_{i=1}^n (1+t+\cdots+t^{|d_i|-1})$$

and a suitable power of t does not affect the sign of the degree or the sign of the a_i, we may assume that $p(t)$ is a Laurent *polynomial* in $1-t$ with rational coefficients, that is that

$$p(t) = \frac{b_{-n}}{(1-t)^n} + \frac{b_{1-n}}{(1-t)^{n-1}} + \cdots + b_{m-1}(1-t)^{m-1} + b_m(1-t)^m$$

for some rational numbers b_i and a nonnegative integer m. In particular, $b_{-n} = \deg p(t)$. Expanding the negative powers $(1-t)^{-j}$ as power series in t and comparing the coefficients of t^i we see that there exists a polynomial $r(i)$ in i of degree at most $n-2$ (or $r(i) = 0$ if $n = 1$) with coefficients depending on n and the b_j's such that $a_i = (1/(n-1)!)\, b_n\, i^{n-1} + r(i)$ for all large enough i. So the condition $a_i \geq 0$ implies that $\deg p(t) = b_{-n} \geq 0$. □

Corollary 3.6. *There are only finitely many* $U \in \operatorname{Indecomp}(M)$ *with*

$$\deg H_U(M,t) \neq 0$$

and we have

$$\sum_U \dim_k(U) \deg H_U(M,t) \leq \deg M,$$

where the sum means the sum over all $U \in \operatorname{Indecomp}(M)$ *with* $\deg H_U(M,t) \neq 0$.

Proof. This follows from (3-2) and Lemma 3.5. □

We can now define the equivariant degree with values in the Green ring.

Definition 3.7. We say that $\deg_{a(kG)} M$ *is defined* if

$$\sum_U \dim_k(U) \deg H_U(M,t) = \deg M.$$

In this case we call $\deg_{a(kG)} M := \sum_U \deg(H_U(M,t))\, |U| \in a(kG)_{\mathbb{Q}}$ the *equivariant degree of* M (*in the Green ring*). If we want to emphasize the dependency on n, we write $\deg^n_{a(kG)} M$ instead of $\deg_{a(kG)} M$.

The existence of the degree in the Green ring can be characterized as follows.

Lemma 3.8. *For* R, G, M *as above the following statements are equivalent.*

(1) $\deg_{a(kG)} M$ *is defined.*

(2) *There is a finite set F of indecomposable kG-modules such that*

$$\sum_{U \in F} \dim_k(U) \deg H_U(M,t) = \deg M.$$

(3) *There is a finite set F of indecomposable kG-modules such that*

$$\deg \sum_{U \notin F} \dim_k(U) H_U(M,t) = 0.$$

Here we have set $\sum_{U \notin F} \dim_k(U) H_U(M,t) := H(M,t) - \sum_{U \in F} \dim_k(U) H_U(M,t)$.

Proof. This is clear from the definition of $\deg_{a(kG)} M$. □

Certainly the equivariant degree $\deg_{a(kG)} M$ is defined if M has only finitely many isomorphism types of indecomposable summands. For example, this is the case if k is a finite field, M a polynomial ring in n variables over k, G a finite group acting on this polynomial ring by homogeneous linear substitutions and $R = M^G$ the ring of invariants [Karagueuzian and Symonds 2007, Theorem 17.1]. The following example shows that there are situations where $\deg_{a(kG)} M$ is not defined:

Example (see Example 4.4 in [Karagueuzian and Symonds 2004]). Let k be a field of two elements and $R = k[x,y]$ a polynomial ring in two variables over k. The Klein four group $G = \langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $M = k[x,y]\langle 1, z \rangle$ by $\alpha : z \mapsto z + x$ and $\beta : z \mapsto z + y$. We can regard M as a subset of $k[x,y,z]$ or as a free R-module of rank two.

If we attach a grading to R and the module M by assigning x, y and z grading 1, then M is the direct sum $M = \bigoplus_{i=0}^{\infty} M_i$. It is shown in [Karagueuzian and Symonds 2004] that $M_i \cong \Omega^i k$ as kG-modules, where $\Omega^i k$ is the i-th Heller translate of the trivial kG-module k. In particular, the M_i's are indecomposable and pairwise nonisomorphic.

We have $n = 2$, $\mathrm{Indecomp}(M) = \{\Omega^i k \mid i \in \mathbb{N}_0\}$ and $H_{\Omega^i k}(M,t) = t^i$. So we obtain $\deg H_{\Omega^i k}(M,t) = 0$ for all i. On the other hand we have

$$H(M,t) = \sum_{i=0}^{\infty} \dim_k(M_i)\, t^i = \sum_{i=0}^{\infty} (2i+1) t^i = \frac{2}{(1-t)^2} - \frac{1}{1-t},$$

and thus $\deg M = 2$. So $\deg_{a(kG)} M$ is not defined in this example.

4. Equivariant degree in the representation ring

One way to construct an equivariant degree which is defined for every module M (satisfying the assumptions in Section 2) is to work over the representation ring. In this section we will define the equivariant degree with values in the representation ring.

The first steps are very similar to those for the Green ring. Let R, G and M be as in Section 2. For each i, the kG-module M_i defines an element $|M_i|$ of $\mathcal{R}(kG)$.

Definition 4.1. We call the Laurent series $[M] := \sum_{i=N}^{\infty} |M_i|\, t^i$ with coefficients in $\mathcal{R}(kG)$ the *equivariant Hilbert series of M with coefficients in the representation ring*.

Clearly Lemma 3.3 carries over to equivariant Hilbert series with coefficients in the representation ring.

For each irreducible kG-module V, let $m_{V,i}$ be the multiplicity of V as a composition factor of M_i. We set $H_V(M,t) := \sum_{i=N}^{\infty} m_{V,i}\, t^i$. We choose a polynomial subring $k[d_1, \dots, d_n]$ of R as in Section 2. In fact the Laurent series $H_V(M,t)$ can be written as a rational function.

Lemma 4.2. *For each irreducible kG-module V, the Laurent series $H_V(M,t)$ can be written as*

$$H_V(M,t) = \frac{f_V(M,t)}{\prod_{i=1}^{n}(1-t^{|d_i|})},$$

where $f_V(M,t)$ is a Laurent polynomial in t with rational coefficients. If k is a splitting field for V, then all the coefficients of $f_V(M,t)$ are integers.

Proof. Let P_V be a projective cover of V. The graded $k[d_1, \dots, d_n]$-module

$$\mathrm{Hom}_{kG}(P_V, M)$$

is a direct summand of the graded $k[d_1, \dots, d_n]$-module $\mathrm{Hom}_{kG}(kG, M) \cong M$. This implies that $\mathrm{Hom}_{kG}(P_V, M)$ is finitely generated as a $k[d_1, \dots, d_n]$-module. Therefore, by the Hilbert–Serre Theorem [Benson 1993, 2.1.1], the Hilbert series $H(\mathrm{Hom}_{kG}(P_V, M), t)$ has the form

$$\frac{\tilde{f}_V(t)}{\prod_{i=1}^{n}(1-t^{|d_i|})}$$

for some Laurent polynomial $\tilde{f}_V(t)$ with integer coefficients. Since

$$\dim_k(\mathrm{Hom}_{kG}(P_V, M_i)) = \dim_k(\mathrm{End}_{kG}(V)) \cdot m_{V,i}$$

we get

$$\begin{aligned} H_V(M,t) &= \frac{1}{\dim_k(\mathrm{End}_{kG}(V))} H(\mathrm{Hom}_{kG}(P_V, M), t) \\ &= \frac{1}{\dim_k(\mathrm{End}_{kG}(V))} \cdot \frac{\tilde{f}_V(t)}{\prod_{i=1}^{n}(1-t^{|d_i|})}. \end{aligned} \tag{4-1}$$

If k is a splitting field for V then $\dim_k(\mathrm{End}_{kG}(V)) = 1$. □

Corollary 4.3. *The equivariant Hilbert series $[M]$ with coefficients in the representation ring is of the form*

$$[M] = \frac{[f](M,t)}{\prod_{i=1}^{n}(1-t^{|d_i|})},$$

where $[f](M,t)$ is a Laurent polynomial with coefficients in $\mathcal{R}(kG)_{\mathbb{Q}}$. If k is a splitting field for G then all the coefficients of $[f](M,t)$ are elements of $\mathcal{R}(kG)$.

Proof. This follows from Lemma 4.2. □

Now we can define the equivariant degree with values in the representation ring:

Definition 4.4. We call $\deg_{\mathcal{R}(kG)} M := \sum_V \deg(H_V(M,t))\,|V| \in \mathcal{R}(kG)_{\mathbb{Q}}$ the *equivariant degree of M (in the representation ring).* Here the sum varies over a set of representatives for the isomorphism classes of irreducible kG-modules. If we want to emphasize the dependency on n, we also write $\deg^n_{\mathcal{R}(kG)} M$ instead of $\deg_{\mathcal{R}(kG)} M$.

We use the same notation for the two degrees, specifying the ring in which the values lie explicitly when necessary. In any case the two versions are compatible in the following sense. Let $\pi : a(kG)_{\mathbb{Q}} \twoheadrightarrow \mathcal{R}(kG)_{\mathbb{Q}}$ denote the canonical map.

Proposition 4.5. *The map π takes the equivariant degree of M in the Green ring to the equivariant degree of M in the representation ring whenever the former is defined.*

Proof. Suppose that $\deg_{a(kG)} M$ is defined. For each $U \in \operatorname{Indecomp}(M)$ and each irreducible kG-module V, let $\mu_{U,V}$ be the multiplicity of V as a composition factor of U and choose a finite subset F of $\operatorname{Indecomp}(M)$ as in Lemma 3.8. We set

$$\sum_{U\notin F} \dim_k(U) H_U(M,t) := H(M,t) - \sum_{U\in F} \dim_k(U) H_U(M,t),$$
$$\sum_{U\notin F} \mu_{U,V} H_U(M,t) := H_V(M,t) - \sum_{U\in F} \mu_{U,V} H_U(M,t).$$

By Lemma 3.8 we get

$$\sum_V \Big(\dim_k(V) \deg \sum_{U\notin F} \mu_{U,V} H_U(M,t)\Big) = \deg \sum_{U\notin F} \dim_k(U) H_U(M,t) = 0, \quad (4\text{-}2)$$

where the first sum runs over a set of representatives for the isomorphism classes of the irreducible kG-modules. By Lemma 3.5 all degrees occurring in (4-2) are nonnegative. Hence $\deg\big(\sum_{U\notin F} \mu_{U,V} H_U(M,t)\big) = 0$ for all irreducible kG-modules V. The epimorphism π maps the equivariant degree

$$\deg_{a(kG)} M = \sum_{U\in F} \deg(H_U(M,t))|U| \in a(kG)_{\mathbb{Q}}$$

to

$$\begin{aligned}\sum_V \sum_{U\in F} \deg(H_U(M,t))\mu_{U,V}|V| &= \sum_V \sum_{U\in F} \deg(\mu_{U,V} H_U(M,t))\,|V| \\ &= \sum_V \Bigg(\sum_{U\in F} \deg(\mu_{U,V} H_U(M,t)) + \deg \sum_{U\notin F} \mu_{U,V} H_U(M,t)\Bigg)|V| \\ &= \sum_V \deg(H_V(M,t))\,|V|,\end{aligned}$$

which is, by definition, the equivariant degree of M over the representation ring. □

5. Basic properties of the equivariant degree

In this section we collect some of the basic properties of the equivariant degree. We always assume that R, G and M are as in Section 2 and that M' and M'' are finitely generated graded left RG-modules, where the action of G preserves the grading and every homogeneous component is finite dimensional as a k-vector space. We choose a polynomial subring $k[d_1, \dots, d_n]$ of R as in Section 2.

We begin with a trivial observation showing that the equivariant degree coincides with the usual degree if there is "no group action":

Lemma 5.1. *If $G = \{1\}$ is the trivial group then $\deg_{a(kG)} M$ is defined and*

$$\deg_{a(kG)} M = \deg(M)\,|k|$$

where k is the trivial kG-module. A similar statement holds for $\deg_{\mathcal{R}(kG)} M$.

Proof. This is clear from the definition of the degree. □

From now on G is again an arbitrary finite group. The next lemma holds both for the equivariant degree taking values in the Green ring as well as for the degree taking values in the representation ring.

Lemma 5.2. *If $\deg_{a(kG)} M$ is defined, then there is a positive integer c such that $c \cdot \deg_{a(kG)} M$ is a genuine module, that is, it is of the form $|V|$ for some kG-module V. A similar statement holds for $\deg_{\mathcal{R}(kG)} M$.*

Proof. By the definition of $\deg_{a(kG)} M$ and Lemma 3.5, we can take $c := \prod_{i=1}^n |d_i|$ for the degree with values in the Green ring. In the case of the representation ring, $c := \left(\prod_V \dim_k \mathrm{End}_{kG}(V)\right) \cdot \left(\prod_{i=1}^n |d_i|\right)$ does the job (where V runs through a set of representatives for the isomorphism classes of irreducible kG-modules). □

Lemma 5.3. *If $M' \hookrightarrow M \twoheadrightarrow M''$ is a short exact sequence of finitely generated graded RG-modules that is split over kG, then $\deg_{a(kG)} M$ is defined if and only if both $\deg_{a(kG)} M'$ and $\deg_{a(kG)} M''$ are defined. If this is the case then*

$$\deg_{a(kG)} M = \deg_{a(kG)} M' + \deg_{a(kG)} M''.$$

The same formula with $\deg_{a(kG)}$ replaced by $\deg_{\mathcal{R}(kG)}$ holds for any short exact sequence.

Proof. Let $U \in \operatorname{Indecomp}(M)$. Then $H_U(M,t) = H_U(M',t) + H_U(M'',t)$ because of the splitting. Thus

$$\sum_{U \in F} \dim_k(U) \deg H_U(M,t) = \sum_{U \in F} \dim_k(U) \deg H_U(M',t) + \sum_{U \in F} \dim_k(U) \deg H_U(M'',t) \quad (5\text{-}1)$$

with F as in Lemma 3.8. By additivity of the nonequivariant degree we have $\deg M = \deg M' + \deg M''$. Since all these degrees are nonnegative by Lemma 3.5 we get that $\deg_{a(kG)} M$ is defined if and only if $\deg_{a(kG)} M'$ and $\deg_{a(kG)} M''$ are defined. In this case we get

$$\begin{aligned} \deg_{a(kG)} M &= \sum_{U \in F} \deg(H_U(M,t))\,|U| \\ &= \sum_{U \in F} \deg(H_U(M',t))\,|U| + \sum_{U \in F} \deg(H_U(M'',t))\,|U| \qquad (5\text{-}2) \\ &= \deg_{a(kG)} M' + \deg_{a(kG)} M''. \end{aligned}$$

The statement about the degree over the representation ring follows from

$$H_V(M,t) = H_V(M',t) + H_V(M'',t)$$

for every irreducible kG-module V. □

For $W, W' \in \mathcal{R}(kG)_{\mathbb{Q}}$ we write $W \le W'$ if $W' - W$ is a linear combination of isomorphism classes of kG-modules with nonnegative rational coefficients. We write $W \ge W'$ if $W' \le W$.

Corollary 5.4. *For a finitely generated graded RG-module M, as at the beginning of this section, the following properties hold for the degree with values in the representation ring.*

(1) *If M' is a graded RG-submodule of M then $\deg_{\mathcal{R}(kG)} M' \le \deg_{\mathcal{R}(kG)} M$.*

(2) *If M' is a graded RG-epimorphic image of M then $\deg_{\mathcal{R}(kG)} M \ge \deg_{\mathcal{R}(kG)} M'$.*

Proof. This follows from Lemmas 5.2 and 5.3. □

For an integer d we write $M[d]$ for M with a degree shift of d, so that $M[d]_i = M_{i+d}$. For a positive integer q let $R^{[q]}$ be the graded k-algebra obtained from R by multiplying all degrees by q, that is, $(R^{[q]})_{iq} = R_i$ and $(R^{[q]})_i = 0$ for all i not divisible by q. Analogously, we can construct a graded $R^{[q]}G$-module $M^{[q]}$ with G-action from M by multiplying all degrees by q, that is, $(M^{[q]})_{iq} = M_i$ and $(M^{[q]})_i = 0$ for all i not divisible by q.

Lemma 5.5. *With the above notation, the equivariant degree has the following properties.*

(1) *If the Krull dimension of M is at most $n-1$ then $\deg_{a(kG)} M$ is defined and both $\deg_{a(kG)} M$ and $\deg_{\mathfrak{R}(kG)} M$ are equal to* 0.

(2) *$\deg_{a(kG)}(M[d])$ is defined if and only if $\deg_{a(kG)} M$ is defined. If this is the case then $\deg_{a(kG)}(M[d]) = \deg_{a(kG)} M$. We always have $\deg_{\mathfrak{R}(kG)}(M[d]) = \deg_{\mathfrak{R}(kG)} M$.*

(3) *$\deg_{a(kG)}(M^{[q]})$ is defined if and only if $\deg_{a(kG)} M$ is defined. If this is the case then $\deg_{a(kG)}(M^{[q]}) = q^{-n} \deg_{a(kG)} M$. We always have $\deg_{\mathfrak{R}(kG)}(M^{[q]}) = q^{-n} \deg_{\mathfrak{R}(kG)} M$.*

Proof. (1) follows from the corresponding property of the nonequivariant degree [Benson 1993, 2.4.1]. (2) and (3) are clear. □

Sometimes it is convenient to add an element z in degree 1 to R. Then $R[z] \otimes_k M$ is finitely generated over $R[z]$, which has dimension $n+1$.

Lemma 5.6. *The degree $\deg^{n+1}_{a(kG)}(R[z] \otimes_R M)$ is defined if and only if $\deg^{n}_{a(kG)} M$ is defined, and if this is the case then they are both equal. Equality always holds when $\deg_{a(kG)}$ is replaced by $\deg_{\mathfrak{R}(kG)}$.*

Proof. Clear. □

Sometimes it is convenient to change the field k.

Lemma 5.7. *Let ℓ be a field extension of k.*

(1) *If $\deg_{a(kG)} M$ is defined then so is $\deg_{a(\ell G)}(\ell \otimes_k M)$ and $\deg_{a(\ell G)}(\ell \otimes_k M) = \ell \otimes_k \deg_{a(kG)} M$.*

(2) *If ℓ/k is finite and L is a finitely generated graded $(\ell \otimes_k RG)$-module such that $\deg_{a(\ell G)} L$ is defined then $\deg_{a(kG)}(L \downarrow^{\ell}_{k}) = (\deg_{a(\ell G)} L) \downarrow^{\ell}_{k}$.*

(3) *If ℓ/k is finite and if $\deg_{a(\ell G)}(\ell \otimes_k M)$ is defined then so is $\deg_{a(kG)} M$ and we have $\deg_{a(kG)} M = |\ell : k|^{-1} (\deg_{a(\ell G)}(\ell \otimes_k M)) \downarrow^{\ell}_{k}$.*

Proof. Only (3) needs any comment. Since $(\ell \otimes_k M) \downarrow^{\ell}_{k} \cong M^{|\ell/k|}$, then by (2) we get $(\deg_{a(\ell G)}(\ell \otimes_k M)) \downarrow^{\ell}_{k} = \deg_{a(kG)}\big(M^{|\ell/k|}\big)$. But then $\deg_{a(kG)} M$ is defined and the formula holds, by 5.3. □

6. Further results

In this section $R = k[d_1, \ldots, d_n]$ is a graded polynomial ring with generators in positive degrees. Unless otherwise stated the degree will always take values in the Green ring.

We say that a map of R-modules *dominates* when the cokernel has dimension strictly less than n. This is not consistent with the customary use of *dominant* in algebraic geometry, but it is very convenient for us here.

Proposition 6.1. *The degree* $\deg_{a(kG)} M$ *of a finitely generated graded RG-module M is defined if and only if there is a finite-dimensional graded kG-submodule $X \subseteq M$ such that the multiplication map $R \otimes_k X \to M$ is injective and dominant and the image is a summand over kG. If this holds then* $\deg_{a(kG)} M = \deg R \cdot |X|$.

Proof. Suppose that such an X exists; then $\deg_{a(kG)}(M/(R \otimes_k X))$ is defined and equal to 0 by hypothesis (take $F = \varnothing$). We claim that $\deg_{a(kG)}(R \otimes_k X) = \deg R \cdot |X|$.

It is easy to see that $H(R \otimes X, t) = H(R, t)H(X, t)$, so

$$\deg_{a(kG)}(R \otimes_k X) = \lim_{t \to 1}\big((1-t)^n H(R,t)H(X,t)\big) = \deg R \cdot |X|.$$

By additivity (Lemma 5.3), $\deg_{a(kG)} M$ is defined and is equal to $\deg R \cdot |X|$. Conversely, suppose that $\deg_{a(kG)} M$ is defined using a finite set $F \subseteq \operatorname{Indecomp} M$. Then, using the notation of the Weak Structure Theorem 3.1, we must have

$$\deg\Big(\bigoplus_{U \notin F} \bigoplus_{I \subseteq \{1,\ldots,n\}} k[d_I] \otimes_k X_{U,I}\Big) = 0.$$

Thus we can take $X = \bigoplus_{U \in F} X_{U,\{1,\ldots,n\}}$. □

A lot of our work is made easier by the next easy, but surprising, result.

Proposition 6.2. *If M is a finitely generated graded RG-module and X is a finite dimensional graded kG-submodule such that the multiplication map $R \otimes_k X \to M$ is injective and dominates then the image is a summand over kG, so in particular* $\deg_{a(kG)} M$ *is defined and is equal to* $\deg R \cdot |X|$.

Proof. There is a homogeneous element $z \in R$ that annihilates the cokernel. Consider the composition of maps

$$R \otimes_k X \longrightarrow M \xrightarrow{z} zM \subseteq R \otimes_k X.$$

The image is $zR \otimes_k X$, and since zR is a k-summand of R it follows that the image is a kG-summand of $R \otimes_k X$. Thus the image of $R \otimes_k X$ in M is also a summand. □

Given a graded commutative ring S, let $Q(S)$ denote the graded ring of fractions, where we invert all the homogeneous elements. It is a $\mathbb{Z}$-graded ring and $Q(S)_0$ is a field. $Q(S) = Q(S)_0[z, z^{-1}]$, where z is an element of $Q(S)$ of least positive degree. $Q(S)$ is flat over S.

Notice that if M is a finitely generated graded RG-module then $Q(R)\otimes_R M$ is a finitely generated $Q(R)$-module and in each degree it is a finite-dimensional vector space over $Q(R)_0$. In addition, $Q(R) \otimes_R M = 0$ if and only if $\dim M < \dim R$.

Proposition 6.3. *Let M be a finitely generated graded RG-module. Then the degree $\deg_{a(kG)} M$ is defined if and only if there is a finite-dimensional graded kG-submodule $X \subseteq Q(R) \otimes_R M$ such that $Q(R) \otimes_R M = Q(R) \otimes_k X$. If this is the case then $\deg_{a(kG)} M = \deg R \cdot |X|$.*

Proof. If $\deg_{a(kG)} M$ is defined, we have a short exact sequence $R \otimes_k X \hookrightarrow M \twoheadrightarrow M/(R \otimes_k X)$ with $\dim(M/(R \otimes_k X)) < \dim R$, by Proposition 6.1. If we tensor this with $Q(R)$, we obtain $Q(R)\otimes_k X \hookrightarrow Q(R)\otimes_R M \twoheadrightarrow Q(R)\otimes_R (M/(R\otimes_k X))$. But the last term must be 0.

Conversely, suppose that we have an X satisfying the conditions of the statement of the proposition. Let $\{x_i\}$ be a k-basis for X and write $x_i = \sum_j (a_{i,j}/b_{i,j})m_j$, where $a_{i,j}, b_{i,j} \in R$ and $m_j \in M$, all homogeneous. Let $\bar{b}$ be the product of all the $b_{i,j}$. Then $\bar{b}X \subseteq M$, and we have a short exact sequence

$$R \otimes_k \bar{b}X \hookrightarrow M \twoheadrightarrow M/(R \otimes_k \bar{b}X).$$

But when we tensor with $Q(R)$ the first arrow becomes an isomorphism, so we must have $Q(R) \otimes_R (M/(R \otimes_k \bar{b}X)) = 0$ and thus $\dim(M/(R \otimes_k \bar{b}X)) < \dim R$, as required by 6.2. □

We now summarize the equivalent characterizations of the equivariant degree.

Theorem 6.4. *Let M be a finitely generated graded RG-module. The following conditions on M are equivalent.*

(1) *$\deg_{a(kG)} M$ is defined.*

(2) *There is a finite-dimensional graded kG-submodule $X \subseteq M$ such that the multiplication map $R \otimes_k X \to M$ dominates and is split injective over kG.*

(3) *There is a finite-dimensional graded kG-submodule $Y \subseteq M$ such that the multiplication map $R \otimes_k Y \to M$ dominates and is injective.*

(4) *There is a finite-dimensional graded kG-submodule $Z \subseteq Q(R) \otimes_R M$ such that $Q(R) \otimes_R M = Q(R) \otimes_k Z$.*

When these conditions hold we have $|X| = |Y| = |Z| = \dfrac{1}{\deg R} \deg_{a(kG)} M$.

Proof. Just combine 6.1, 6.2 and 6.3. □

Lemma 6.5. *Let R and R' be polynomial rings in n and n' variables respectively and let M and M' be finitely generated graded RG- and $R'G$-modules respectively. Let L be a finitely generated graded RH-module and let H be a subgroup of G. The degree commutes with the following operations* (*when the quantity on the right hand side is defined*):

(1) tensor product: $\deg^{n+n'}_{a(kG)}(M \otimes_k M') = \deg^n_{a(kG)}(M) \cdot \deg^{n'}_{a(kG)}(M')$.

(2) restriction: $\deg_{a(kH)}(M \downarrow^G_H) = (\deg_{a(kG)} M) \downarrow^G_H$.

(3) induction: $\deg_{a(kG)}(L \uparrow^G_H) = (\deg_{a(kH)} L) \uparrow^G_H$.

(4) fixed points: $\deg_{a(kG/H)} M^H = (\deg_{a(kG)} M)^H$ *if H is a normal subgroup of G.*

Proof. These all follow easily from property 6.4(3). □

In the remainder of this section we consider how Theorem 6.4 and Lemma 6.5 can be reformulated for the degree with values in the representation ring. Clearly if one of the conditions in Theorem 6.4 is satisfied then Proposition 4.5 implies that $|X| = |Y| = |Z| = (1/\deg R)\deg_{a(kG)} M$ also holds for the degree over the representation ring. The analogue of Lemma 6.5 is the following lemma.

Lemma 6.6. *With the same hypotheses as in the previous lemma*, *the degree with values in the representation ring commutes with the following operations*:

(1) tensor product: $\deg^{n+n'}_{a(kG)}(M \otimes_k M') = \deg^n_{a(kG)}(M) \cdot \deg^{n'}_{a(kG)}(M')$.

(2) restriction: $\deg_{a(kH)}(M \downarrow^G_H) = (\deg_{a(kG)} M) \downarrow^G_H$.

(3) induction: $\deg_{a(kG)}(L \uparrow^G_H) = (\deg_{a(kH)} L) \uparrow^G_H$.

Proof. This is straightforward and left to the reader. □

7. Rings

Throughout this section, S will be a graded ring in nonnegative degrees that is finitely generated over the field k and such that S_0 is finite-dimensional over k. We suppose that a finite group G acts on S by graded k-algebra automorphisms.

Geometrically, G acts as a group of automorphisms of the projective variety $V = \mathrm{Proj}(S)$, defined over k. Conversely, S could be the homogeneous coordinate ring of a variety over k on which G acts.

The invariant subring S^G is necessarily Noetherian and S is finitely generated over S^G [Benson 1993, 1.3.1]. By Noether normalization, we can find a graded polynomial subring $R \leq S^G$ such that S^G is finitely generated over R [Benson 1993, 2.2.7]. Thus S is finitely generated over R, and S and R have the same dimension. We need this ring R to exist in order for the preceding theory to apply, but it does not matter which ring R we choose.

Proposition 7.1. *If S is an integral domain and G acts faithfully, then* $\deg_{a(kG)} S$ *is defined and*

$$\deg_{a(kG)} S = \frac{\deg S}{|G|} \cdot kG$$

and the same equality holds with $\deg_{a(kG)} S$ *replaced by* $\deg_{\mathcal{R}(kG)} S$.

Proof. In [Symonds 2000], a graded submodule $F \leq S$ is produced such that $F \cong kG$ and such that the multiplication map $S^G \otimes_k F \hookrightarrow S$ dominates and is split over kG. It follows from the Additivity Lemma 5.3 that

$$\deg_{a(kG)} S = \deg_{a(kG)}(S^G \otimes F) = \deg S^G \cdot kG.$$

The proof for the degree with values in the representation ring is analogous.

There is an alternative proof that we sketch here. By Lemma 5.6, we may assume that R contains an element z of degree 1. But S is an integral domain, so it injects into $Q(S)$, thus G acts faithfully on $Q(S)$. Since $Q(S) = Q(S)_0[z, z^{-1}]$ and G acts trivially on z, G must act faithfully on $Q(S)_0$. By the Normal Basis Theorem there is a basis $\{x_g\}_{g\in G}$ for $Q(S)_0$ over $Q(S)_0^G$ that is freely permuted by G.

But $Q(S)_0$ is a finite-dimensional vector space over $Q(R)$; let $\{y_i\}$ be a basis. If we let X be the k-span of the set $\{y_i x_g\}$, then this is the module that we require. □

Let $\mathcal{P}_0$ denote the (finite) set of prime ideals in S of height 0.

Lemma 7.2. *The natural map $S \to \bigoplus_{\mathfrak{p}\in\mathcal{P}_0} S/\mathfrak{p}$ dominates and has* rad S *as kernel.*

Proof. The radical is equal to the intersection of all the prime ideals, which is equal to the intersection of the minimal ones.

We prove the claim of domination by labeling the distinct prime ideals of height 0 as $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ and showing by induction on r that the map $S \to \bigoplus_{i=1}^{r} S/\mathfrak{p}_i$ dominates.

This is clearly true when $r = 1$, and the induction step follows from considering the following diagram with exact rows and columns.

$$\begin{tikzcd}
S/\bigcap_{i=1}^{r+1}\mathfrak{p}_i \arrow[r] \arrow[d, equal] & S/\bigcap_{i=1}^{r}\mathfrak{p}_i \oplus S/\mathfrak{p}_{r+1} \arrow[r] \arrow[d] & S/\big(\bigcap_{i=1}^{r}\mathfrak{p}_i + \mathfrak{p}_{r+1}\big) \arrow[d] \\
S/\bigcap_{i=1}^{r+1}\mathfrak{p}_i \arrow[r] & \bigoplus_{i=1}^{r+1} S/\mathfrak{p}_i \arrow[r] \arrow[d] & X \arrow[d] \\
 & Y \arrow[r, equal] & Y
\end{tikzcd}$$

The induction hypothesis applied to the middle column shows that $\dim Y < \dim S$, and $\dim S/\big(\bigcap_{i=1}^{r}\mathfrak{p}_i + \mathfrak{p}_{r+1}\big) < \dim S$ by construction. Thus $\dim X < \dim S$ and the middle row yields the next stage in the induction. □

Given a prime $\mathfrak{p} < S$, let $G_\mathfrak{p}$ denote the stabilizer in G of $\mathfrak{p}$ and let $\bar{G}_\mathfrak{p}$ be the pointwise stabilizer of $S/\mathfrak{p}$. We can now state a decomposition theorem for the degree of S.

Theorem 7.3. *If S contains no nilpotent elements then $\deg_{a(kG)} S$ is defined and*

$$\deg_{a(kG)} S = \sum_{\substack{\mathfrak{p}\in\mathscr{P}_0/G \\ \dim S/\mathfrak{p}=\dim S}} \frac{\deg S/\mathfrak{p}}{|G_\mathfrak{p}/\bar{G}_\mathfrak{p}|} \cdot k[G/\bar{G}_\mathfrak{p}]$$

and the same equality holds with $\deg_{a(kG)} S$ replaced by $\deg_{\mathscr{R}(kG)} S$.

Proof. In view of Proposition 7.1, Lemma 7.2 and Theorem 6.4, all we need to do is to show that $\deg_{a(kG)}\big(\bigoplus_{\mathfrak{p}\in\mathscr{P}_0} S/\mathfrak{p}\big)$ is equal to the expression shown.

But

$$\bigoplus_{\mathfrak{p}\in\mathscr{P}_0} S/\mathfrak{p} \cong \bigoplus_{\mathfrak{p}\in\mathscr{P}_0/G} \bigoplus_{\mathfrak{q}\sim_G\mathfrak{p}} S/\mathfrak{q} \cong \bigoplus_{\mathfrak{p}\in\mathscr{P}_0/G} \operatorname{Ind}_{G_\mathfrak{p}}^G S/\mathfrak{p}.$$

So

$$\begin{aligned}
\deg_{a(kG)}\Big(\bigoplus_{\mathfrak{p}\in\mathscr{P}_0} S/\mathfrak{p}\Big) &\cong \bigoplus_{\mathfrak{p}\in\mathscr{P}_0/G} \deg_G \operatorname{Ind}_{G_\mathfrak{p}}^G S/\mathfrak{p} \\
&\cong \bigoplus_{\mathfrak{p}\in\mathscr{P}_0/G} \operatorname{Ind}_{G_\mathfrak{p}}^G \deg_{G_\mathfrak{p}} S/\mathfrak{p} && \text{by Lemma 6.5(3)} \\
&\cong \bigoplus_{\mathfrak{p}\in\mathscr{P}_0/G} \operatorname{Ind}_{G_\mathfrak{p}}^G \frac{\deg S/\mathfrak{p}}{|G_\mathfrak{p}/\bar{G}_\mathfrak{p}|} \cdot k[G_\mathfrak{p}/\bar{G}_\mathfrak{p}] && \text{by Proposition 7.1} \\
&\cong \bigoplus_{\mathfrak{p}\in\mathscr{P}_0/G} \frac{\deg S/\mathfrak{p}}{|G_\mathfrak{p}/\bar{G}_\mathfrak{p}|} \cdot k[G/\bar{G}_\mathfrak{p}].
\end{aligned}$$

We can omit from the sum the primes $\mathfrak{p}$ for which $\dim S/\mathfrak{p} \neq \dim S$, since for these $\deg S/\mathfrak{p} = 0$. □

Geometrically, the permutation modules that occur in the statement of the theorem correspond to the way that the group permutes the irreducible components of maximum dimension of the projective variety $\operatorname{Proj}(S)$.

Now suppose that the action of G on S can be written over a finite field $\mathbb{F}_q$. Recall from Lemma 5.5 that the operation of multiplying all degrees by q gives us a new ring $S^{[q]}$ with G-action and $\deg_{\mathscr{R}(kG)} S^{[q]} = q^{-n} \deg_{\mathscr{R}(kG)} S$ and $\deg_{a(kG)} S^{[q]} = q^{-n} \deg_{a(kG)} S$. Let $S^q < S$ denote the subring of q-th powers. There is a surjection $S^{[q]} \to S^q$ and this is an isomorphism if $\operatorname{rad} S = 0$.

Lemma 7.4. *We have $\deg_{\mathscr{R}(kG)} S^q \leq q^{-n} \deg_{\mathscr{R}(kG)} S$ and if S contains no nilpotents then $\deg_{a(kG)} S^q = q^{-n} \deg_{a(kG)} S$.*

Proof. This follows from the preceding remarks and the Additivity Lemma 5.3. □

8. Group cohomology

In this section we apply some of the theory that we have developed to a problem in group cohomology considered by Nick Kuhn [2008]. We fix a prime p and a finite group P (we do not yet require P to be a p-group). Then $G = \mathrm{Aut}(P)$ acts on the graded commutative ring $H^*(P) = H^*(P; \mathbb{F}_p)$.

By the Evens–Venkov theorem (see [Benson 1991, 3.10, 4.2], for example), $H^*(P)$ is Noetherian, hence so is $H^*(P)^G$, thus $H^*(P)$ is certainly finitely generated over some commutative polynomial ring R such that the action of G commutes with that of R; we can assume that $\dim R = \dim H^*(P)$.

Given a p-group P and a simple G-module V, Martino and Priddy [1992] asked whether the dimension of V as a composition factor of $H^*(P)$ is equal to $\dim H^*(P)$ (see also [Kuhn 2008]). It was already known from [Diethelm and Stammbach 1984; Harris and Kuhn 1988; Symonds 1999] that V does occur in $H^*(P)$.

Theorem 8.1 [Kuhn 2008]. *For p odd, the dimension of V as a composition factor of $H^*(P)$ is equal to the dimension of $H^*(P)$.*

The case of $p = 2$ is still undecided. Kuhn's methods used the nilpotent filtration of the category of unstable modules over the Steenrod algebra. We will show how this theorem can be proved using the equivariant degree. Clearly what we need to do is to show that V occurs as a composition factor of $\deg_{\mathcal{R}(\mathbb{F}_pG)} H^*(P)$.

For any finite elementary abelian p-group E, let $F^*(E) = H^*(E)/\operatorname{rad}$, which is just the symmetric algebra $\mathbb{F}_p[E] = S^*(E^*)$, where $E^* = \mathrm{Hom}(E, \mathbb{F}_p)$ is in degree 2 (or degree 1 if $p = 2$).

In general, let

$$F^*(P) = \varprojlim_{E \in \mathcal{A}_p(P)} F^*(E),$$

where $\mathcal{A}_p(P)$ denotes the category with objects the elementary abelian subgroups of P and morphisms the inclusions between them. G acts naturally on this.

Quillen [1971] (see also [Benson 1991, 5.6]) showed that the natural map induced by restrictions, $r : H^*(P) \to F^*(P)^P$ is a purely inseparable isogeny (or uniform F-isomorphism): that is that the kernel is nilpotent and there is an integer N such that $(F^*(P)^P)^{p^N} \subseteq \mathrm{Im}(r)$. From this he deduced that $\dim H^*(P)$ is equal to the p-rank of P, which we will denote by n.

Consider what this means for the degree with values in the representation ring. We have $\deg_{\mathcal{R}(\mathbb{F}_pG)} H^*(P) \geq \deg_{\mathcal{R}(\mathbb{F}_pG)} \mathrm{Im}(r) \geq \deg_{\mathcal{R}(\mathbb{F}_pG)}\big((F^*(P)^P)^{p^N}\big)$ using Lemma 5.3. By Lemma 7.4 we have

$$\deg_{a(\mathbb{F}_pG)}\big((F^*(P)^P)^{p^N}\big) = \frac{1}{p^{Nn}} \deg_{a(\mathbb{F}_pG)} F^*(P)^P,$$

since $F^*(P)$ contains no nilpotent elements.

Now we see that $\deg_{a(\mathbb{F}_pG)} F^*(P)^P = \big(\deg_{a(\mathbb{F}_pG)} F^*(P)\big)^P$ by Lemma 6.5(4).

We conclude that it is sufficient to show that $\deg_{a(\mathbb{F}_pG)} F^*(P)$ contains every simple G-module as a submodule. But the Decomposition Theorem 7.3 tells us that

$$\deg_{a(\mathbb{F}_pG)} F^*(P) = \sum_{\substack{E\in\mathcal{A}_p(P)/G \\ \operatorname{rank} E=n}} \frac{\deg S/\mathfrak{p}_E}{|N_G(E)/C_G(E)|} \cdot \mathbb{F}_p[G/C_G(E)],$$

where $\mathfrak{p}_E$ denotes the ideal corresponding to E. Since each E in the sum has maximal rank, $\deg(S/\mathfrak{p}_E) \neq 0$. Suppose that some $C_G(E)$ is a p-group. Then

$$\operatorname{Hom}_G(V, \mathbb{F}_p[G/C_G(E)]) \cong \operatorname{Hom}_{C_G(E)}(V, \mathbb{F}_p) \neq 0,$$

so V does occur in $\deg_{a(\mathbb{F}_pG)} F^*(P)$ and we are done. That this always happens when p is odd is the content of the next lemma, which appears as [Kuhn 2008, 2.3], although we first learnt it from Benson (private communication) in 1996. We include the proof for the convenience of the reader.

Lemma 8.2. *If p is odd and E is maximal then $C_G(E)$ is a p-group.*

Proof. Consider the composition of homomorphisms

$$C_G(E) \xrightarrow{\alpha} \operatorname{Aut}(C_P(E)) \xrightarrow{\beta} \operatorname{Aut}(E).$$

The composition is trivial, so it suffices to prove that the kernel of each map is a p-group. For β we use the result that if p is odd and Q is a p-group then the kernel of the map $\operatorname{Aut}(Q) \to \operatorname{Aut}(\Omega_1(Q))$ is a p-group [Gorenstein 1968, 5.3.10]. (This is the only place in this section where the argument requires p to be odd.)

For α we use Thompson's $A \times B$ Lemma [Gorenstein 1968, 5.3.4], which states that for any p-group P, if $A \times B \subseteq \operatorname{Aut}(P)$ with A a p'-group and B a p-group such that A acts trivially on $C_P(B)$, then $A = 1$. We apply this with A some p'-subgroup of $\operatorname{Ker}(\alpha)$ and B the image of E in G. □

9. Further results on the degree with values in the representation ring

We assume that k is a splitting field for the group G, but we do not need R to be polynomial.

Let V be a simple kG-module and let M be a finitely generated graded RG-module. Let M_V denote the part of M that is generated by submodules isomorphic to V.

Lemma 9.1. $\operatorname{Hom}_{kG}(V, M) \otimes_k V \cong M_V$ *by the map* $f \otimes v \mapsto f(v)$.

Proof. Since $\mathrm{Hom}_{kG}(V, M) \cong \mathrm{Hom}_{kG}(V, M_V)$ we may assume that $M = M_V$. But now the claimed isomorphism is additive in M_V, and M_V is just a direct sum of submodules isomorphic to V, so we are reduced to the case where $M_V = V$. But now it holds by the assumption that k is a splitting field, so $\mathrm{End}_{kG}(V) \cong k$ [Curtis and Reiner 1981, 7.14]. $\square$

The next result is an equivariant analogue of [Hartshorne 1977, I 7.4].

Proposition 9.2. *Let M be a finitely generated graded RG-module. Then M has a finite filtration $0 = M_0 \le M_1 \le \cdots \le M_m = M$ by graded RG-submodules such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i[\ell_i] \otimes_k V_i$, where $\mathfrak{p}_i$ is a homogeneous prime ideal of R and V_i is a simple kG-module.*

(1) *The minimal elements among the $\mathfrak{p}_i$ occurring are the minimal primes for M.*

(2) *For each minimal prime $\mathfrak{p}$ of M, let $k(\mathfrak{p})$ denote the quotient field of $R/\mathfrak{p}$. For each simple kG-module V, the number of times that $R/\mathfrak{p} \otimes_k V$ occurs as a composition factor of the filtration is equal to the number of times that the simple $R_\mathfrak{p}G$-module $k(\mathfrak{p}) \otimes_k V$ occurs as a composition factor of the localization $M_\mathfrak{p}$, hence is independent of the filtration.*

Proof. Let $\mathfrak{p}$ be an associated prime of $\mathrm{Hom}_{RG}(V, M)$, so it is the annihilator of some $\phi : V \to M$. Thus we have an injection of graded RG-modules $R/\mathfrak{p} \hookrightarrow \mathrm{Hom}_{RG}(V, M)$, $\quad r \mapsto r\phi$ and hence an injection $R/\mathfrak{p} \otimes_k V \hookrightarrow \mathrm{Hom}_{RG}(V, M) \otimes V$.

By Lemma 9.1 this leads to an injection $R/\mathfrak{p} \otimes_k V \hookrightarrow M$; denote its image by M_1.

Now repeat the process with M/M_1, and let M_2 be the inverse image in M of the resulting submodule. In this way we obtain an ascending sequence of graded RG-submodules of M, which must terminate since M is Noetherian.

Notice that this filtration can be refined to a nonequivariant one by filtering the V. Thus (1) follows from the nonequivariant case.

For (2), let $\mathfrak{q}$ be a minimal prime and consider what happens when we localize at $\mathfrak{q}$. If $\mathfrak{p}_i \ne \mathfrak{q}$ then $(R/\mathfrak{p}_i)_\mathfrak{q} = 0$, since $\mathfrak{q}$ is minimal in $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. If $\mathfrak{p}_i = \mathfrak{q}$ then $(R/\mathfrak{q})_\mathfrak{q} = k(\mathfrak{q})$ and $(R/\mathfrak{q} \otimes_k V)_\mathfrak{q} = k(\mathfrak{q}) \otimes_k V$. This is a simple $S_\mathfrak{q}G$-module since k is a splitting field. $\square$

Write $m(\mathfrak{p}, V, M)$ for the number of times that $R/\mathfrak{p} \otimes_k V$ occurs as a factor in a filtration of M of the type considered in the proposition above.

Corollary 9.3. $$\deg_{\mathfrak{R}(kG)} M = \sum_{\dim R/\mathfrak{p} = \dim M} m(\mathfrak{p}, V, M) \deg(R/\mathfrak{p}) \cdot |V|.$$

There are some straightforward reduction methods for calculating the degree with values in the representation ring.

Lemma 9.4. *Let $f \in R$ be homogeneous and let M be a finitely generated graded RG-module of dimension m. Suppose that the dimension of the kernel of the multiplication map $\phi_f : M \to M$, $m \mapsto fm$, has dimension at most $m-2$. Then $\deg^{m-1}_{\mathfrak{R}(kG)}(M/fM) = |f| \deg^m_{\mathfrak{R}(kG)} M$.*

Proof. There is a short exact sequence $\ker(\phi_f) \to M \xrightarrow{\phi_f[|f|]} fM\big[|f|\big]$, where $\big[|f|\big]$ denotes the degree shift needed to make all the maps degree preserving. Thus $\big[fM[|f|]\big] = [M] + O((1-t)^{-(m-2)})$ as a Laurent series in $1-t$ and so $[fM] = t^{|f|}[M] + O((1-t)^{-(m-2)})$.

There is also a short exact sequence $fM \to M \to M/fM$, so $[M/fM] = [M] - [fM]$.

Combining, we find that $[M/fM] = [M] - t^{|f|}[M] + O((1-t)^{-(m-2)})$. Thus

$$\begin{aligned}\deg^{m-1}_{\mathfrak{R}(kG)}[M/fM] &= \lim_{t\to 1}(1-t)^{m-1}\cdot(1-t^{|f|})[M]\\ &= \lim_{t\to 1}\frac{1-t^{|f|}}{1-t}\cdot(1-t)^m[M] = |f| \deg^m_{\mathfrak{R}(kG)} M. \qquad \square\end{aligned}$$

Our last result follows by repeated use of this lemma.

Proposition 9.5. *Let M be a finitely generated graded RG-module of dimension m and suppose that $f_1, \dots, f_r \in R$ is an M-regular sequence of homogeneous elements. Then*

$$\deg^m_{\mathfrak{R}(kG)} M = \prod |f_i| \cdot \deg^{m-r}_{\mathfrak{R}(kG)}\big(M/(f_1, \dots, f_r)M\big).$$

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himstedt@ma.tum.de *Technische Universität München, Zentrum Mathematik – M11, Boltzmannstr. 3, 85748 Garching, Germany*

psymonds@manchester.ac.uk *School of Mathematics, University of Manchester, Manchester M13 9PL, United Kingdom*