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A rooted-trees q-series lifting a one-parameter family of Lie idempotents

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We define and study a series indexed by rooted trees and with coefficients in $\mathbb{Q}(q)$. We show that it is related to a family of Lie idempotents. We prove that this series is a q-deformation of a more classical series and that some of its coefficients are Carlitz q-Bernoulli numbers.

1. Introduction

The aim of this article is to introduce and study a series Ω_q indexed by rooted trees, with coefficients that are rational functions of the indeterminate q.

The series Ω_q is in fact an element of the group G_{PL} of formal power series indexed by rooted trees, which is associated to the pre-Lie operad by a general functorial construction of a group from an operad. As there is an injective morphism of operads from the pre-Lie operad to the dendriform operad, there is an injection of groups from G_{PL} to the group $\mathsf{G}_{\mathrm{Dend}}$, which is a group of formal power series indexed by planar binary trees. This means that each series indexed by rooted trees can be mapped to a series indexed by planar binary trees, in a nontrivial way.

There is a conjectural description of the image of this injection of groups (see [Chapoton 2007, Corollary 5.4]). This can be stated roughly as the intersection in a bigger space (spanned by permutations) of the dendriform elements with the Lie elements. The inclusion of the image in the intersection is known, but the converse is not.

One starting point of this article was the existence of a one-parameter family of Lie idempotents belonging to the descent algebras of the symmetric groups [Duchamp et al. 1994; Krob et al. 1997]. As Lie idempotents, these are in particular Lie elements. As elements of the descent algebras, these are also dendriform elements. Therefore, according to the conjecture stated above, they should belong to the image of G_{PL} in $\mathsf{G}_{\mathrm{Dend}}$.

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Bypassing the conjecture, we prove this by exhibiting an element Ω_q of G_{PL} and then showing that its image is the expected sum of Lie idempotents.

We then obtain several results on Ω_q. First, we prove that the series Ω_q has only simple poles at nontrivial roots of unity and, in particular, can be evaluated at $q = 1$. Then we show that Ω_q is a q-deformation of a classical series Ω which is its value at $q = 1$. We also compute the value at $q = 0$ and the appropriate limit value when $q = \infty$.

We then consider the images of Ω_q in some other groups. There are two morphisms of groups from G_{PL} to usual groups of formal power series in one variable, both coming by functoriality from quotient operads of pre-Lie. Looking at linear trees only, that is, using the quotient map from pre-Lie to the associative operad, one gets a map from G_{PL} to the composition group of formal power series without constant term. The image of Ω_q is then a q-logarithm.

On the other hand, looking at corollas only, one gets a map from G_{PL} to the group of formal power series with constant term 1 for multiplication. The image of Ω_q is then the generating function of the q-Bernoulli numbers introduced by Carlitz. These numbers appear quite naturally here.

We recall in Appendix A the functorial definition of a group $\mathsf{G}_{\mathcal{P}}$ from an augmented operad $\mathcal{P}$. On this subject, the reader may also consult [Chapoton 2002a; 2007/08; van der Laan 2003; Chapoton and Livernet 2007].

In Appendix B, we give, for concreteness, the first few terms of the rooted-trees series that we consider.

Many useful computations and checks have been done using MuPAD.

2. General setting

We will work over the field $\mathbb{Q}$ of rational numbers and over the field $\mathbb{Q}(q)$ of fractions in the indeterminate q.

We have tried to avoid using operads as much as possible, but this language is necessary to define the ambient groups, and we will need it at some points in this article. The reader may consult [Loday 2001; Chapoton 2007/08] as references. The symbol $\circ_i$ will denote the (single) composition at position i in an operad and the symbols $\flat$ and $\natural$ will serve to note positions where composition is done.

Pre-Lie algebras. Recall (see for instance [Chapoton and Livernet 2001]) that a *pre-Lie algebra* is a vector space V endowed with a bilinear map $\curvearrowleft$ from $V \otimes V$ to V satisfying the axiom

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y). \tag{1}$$

This is sometimes called a *right pre-Lie algebra*.

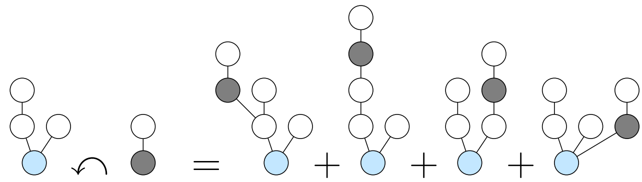


Figure 1. Example of operation in the free pre-Lie algebra PL.

The pre-Lie product $\curvearrowleft$ defines a Lie bracket on V as:

$$[x, y] = x \curvearrowleft y - y \curvearrowleft x. \tag{2}$$

One can easily check that the pre-Lie axiom (1) implies the Jacobi identity for the antisymmetric bracket [,]. The Lie algebra $(V, [\ ,\ ])$ will be called V_{Lie}.

The pre-Lie product $\curvearrowleft$ can also be considered as a right action $\curvearrowleft$ of the associated Lie algebra V_{Lie} on the vector space V. Indeed, one has

$$(x \curvearrowleft y) \curvearrowleft z - (x \curvearrowleft z) \curvearrowleft y = x \curvearrowleft [y, z]. \tag{3}$$

This should not be confused with the adjoint action of a Lie algebra on itself.

Free pre-Lie algebras and rooted trees. The free pre-Lie algebras have a simple description using rooted trees. Let us recall briefly this description and other properties. Details and proofs can be found in [Chapoton and Livernet 2001].

A *rooted tree* is a finite, connected and simply connected graph, together with a distinguished vertex called the *root*. We will picture rooted trees with their root at the bottom and orient (implicitly) the edges towards the root. There are two distinguished kinds of rooted trees: corollas, where every vertex other than the root is linked to the root by an edge; and linear trees, where at every vertex, there is at most one incoming edge. See Figure 4 on page 620 for examples. A *forest of rooted trees* is a finite graph whose connected components are rooted trees.

The free pre-Lie algebra $\mathsf{PL}(S)$ on a set S has a basis indexed by rooted trees decorated by S, that is, rooted trees together with a map from their set of vertices to S.

The pre-Lie product $T \curvearrowleft T'$ of a tree T' on another one T is given by the sum of all possible trees obtained from the disjoint union of T and T' by adding an edge from the root of T' to one of the vertices of T (the root of the resulting tree is the root of T). An example is depicted in Figure 1.

In particular, we will denote by PL the free pre-Lie algebra on one generator. This is the graded vector space $\mathsf{PL} = \oplus_{n \geq 1} \mathsf{PL}_n$ spanned by unlabeled rooted trees, where the degree of a tree T is the number $\#T$ of its vertices. The pre-Lie product obviously preserves this grading.

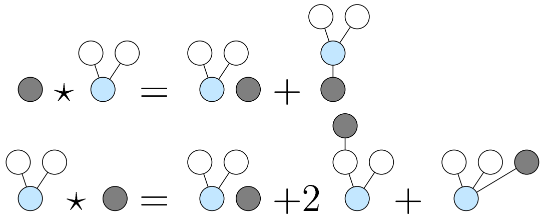


Figure 2. Products of trees in the universal enveloping algebra $U(\mathsf{PL})$.

Universal enveloping algebras of free pre-Lie algebras. The Lie algebras $\mathsf{PL}(S)_{\mathrm{Lie}}$ have the curious property that their universal enveloping algebras come naturally equipped with a basis, which depends on no choice and has nothing to do with a Poincaré–Birkhoff–Witt basis. Let us explain this in the case of $\mathsf{PL}_{\mathrm{Lie}}$.

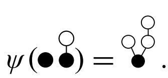
Let $U(\mathsf{PL})$ be the universal enveloping algebra of the Lie algebra $\mathsf{PL}_{\mathrm{Lie}}$. We will denote by $\star$ the associative product in $U(\mathsf{PL})$. We will freely identify right $\mathsf{PL}_{\mathrm{Lie}}$-modules with right $U(\mathsf{PL})$-modules. The crucial point is the following result [Chapoton and Livernet 2001, Theorem 3.3].

Theorem 2.1. *There exists a unique isomorphism ψ of graded right $\mathsf{PL}_{\mathrm{Lie}}$-modules between the free right $U(\mathsf{PL})$-module on one generator g of degree 1 and the $\mathsf{PL}_{\mathrm{Lie}}$-module $(\mathsf{PL}, \curvearrowleft)$ such that ψ maps the generator g to $\bullet$, the unique rooted tree with one vertex.*

This means that there is a commutative diagram as follows:

$$\begin{tikzcd} \mathbb{Q}g \otimes U(\mathsf{PL}) \otimes \mathsf{PL}_{\mathrm{Lie}} \arrow[r, "\psi\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\star"] & \mathsf{PL} \otimes \mathsf{PL}_{\mathrm{Lie}} \arrow[d, "\curvearrowleft"] \\ \mathbb{Q}g \otimes U(\mathsf{PL}) \arrow[r, "\psi"] & \mathsf{PL} \end{tikzcd} \tag{4}$$

As $\mathbb{Q}g$ has dimension 1, the map ψ can be considered as an isomorphism of vector spaces between $U(\mathsf{PL})$ and PL. One can therefore use ψ and the canonical basis of PL (indexed by rooted trees) to get a canonical basis of the enveloping algebra $U(\mathsf{PL})$. It is more convenient to index this basis by forests of rooted trees as follows. The inverse image $\psi^{-1}(T)$ in $U(\mathsf{PL})$ of a tree T in PL can be seen as an element of $U(\mathsf{PL})$. This element is defined to be the basis element corresponding to the forest F obtained from T by removal of its root. For example, one has



By using diagram (4) for the unit element 1 in $U(\mathsf{PL})$ (that is, the empty forest), one can show that the map from $\mathsf{PL}_{\mathrm{Lie}}$ to $U(\mathsf{PL})$ given by the universal property of the enveloping algebra corresponds to the inclusion of the set of rooted trees (as a

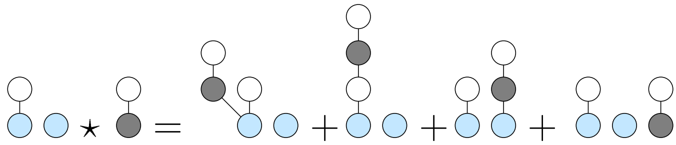


Figure 3. Product of a forest and a tree in $U(\mathsf{PL})$.

basis of PL) in the set of forests (as a basis of $U(\mathsf{PL})$). Indeed, for a rooted tree T, the top horizontal arrow maps $g \otimes 1 \otimes T$ to $\bullet \otimes T$. The right vertical arrow maps this to $\bullet \curvearrowleft T$. Then the inverse of ψ removes the root, which just gives T (seen as a forest with just one connected component). The image of the natural inclusion of $\mathsf{PL}_{\mathrm{Lie}}$ in $U(\mathsf{PL})$ is therefore the subspace spanned by rooted trees.

Note that one can use diagram (4) to compute the $\star$ product of a forest and a rooted tree in $U(\mathsf{PL})$, as a sum of forests (see Figure 3 for an example, compare with Figure 1). The usual spanning set of the universal enveloping algebra $U(\mathsf{PL})$ is the set of noncommutative monomials in rooted trees (for the $\star$ product). By induction on the length of the monomial, one can therefore map each such noncommutative monomial to a sum of forests. Examples are given in Figures 2 and 3.

In the basis of $U(\mathsf{PL})$ indexed by forests, there is a nice combinatorial description of the associative product $\star$. Let F and F' be forests in $U(\mathsf{PL})$. The product $F \star F'$ is the sum of all possible forests, obtained from the disjoint union of F and F' by the addition of some edges (possibly none), each of these new edges going from some root of F' to some vertex of F. Indeed, one can easily check that this operation is associative and coincide with $\star$ on rooted trees, hence the result.

There is a canonical projection π from $U(\mathsf{PL})$ to PL, defined using the canonical basis of $U(\mathsf{PL})$ by projection on the subspace spanned by rooted trees, annihilating the empty forest and all forests that are not trees.

Lemma 2.2. *Let F be a forest in $U(\mathsf{PL})$ and T be a rooted tree in PL. Then one has $\pi(F \star T) = \pi(F) \curvearrowleft T$.*

Proof. If F is not a tree, then each term of $F \star T$ is not a tree, therefore both sides vanish. If $F = \pi(F)$ is a tree, then $F \star T$ is the sum of $\pi(F) \curvearrowleft T$ with the disjoint union of F and T. Therefore $\pi(F \star T) = \pi(F) \curvearrowleft T$. □

The reader can check this statement on the examples of Figures 2 and 3.

Lemma 2.3. *For all $n \geq 1$, the maps $T \mapsto \bullet \curvearrowleft T$ and $T \mapsto T \curvearrowleft \bullet$ are injective from PL_n to PL_{n+1}.*

Proof. This is obvious for the first map, which is even an injection on the set of rooted trees. For the second map, this follows from the fact that enveloping algebras are domains, by restriction of the commutative diagram (4). □

Group associated to the pre-Lie operad. In the sequel, we will always work in the completed vector space $\widehat{\mathsf{PL}} = \prod_{n\geq 1} \mathsf{PL}_n$ and with its completed enveloping algebra $\widehat{U}(\mathsf{PL})$. All the results above are still true in this setting.

There is a group associated to each operad; see Appendix A and [Chapoton 2002a; 2007/08; van der Laan 2003; Chapoton and Livernet 2007]. We will need the group G_{PL} associated to the pre-Lie operad. Its elements are the elements of $\widehat{\mathsf{PL}}$ whose homogeneous component of degree 1 is $\bullet$. The product in G_{PL} is defined using the composition of the pre-Lie operad and $\bullet$ is the unit in G_{PL}. This group is contained in the bigger monoid $\widehat{\mathsf{PL}}$, on which it therefore acts on the right and on the left. The right action respects all the operations on $\widehat{\mathsf{PL}}$ induced by the product $\curvearrowleft$, including the product and the action of $\widehat{U}(\mathsf{PL})$.

Let us now introduce a special element of G_{PL} for later use. Let $\exp^* \in G_{\mathsf{PL}}$ be

$$\exp^* = \bullet \curvearrowleft ((\exp(\bullet) - 1)/\bullet), \tag{5}$$

where the rightmost factor is an element of $\widehat{U}(\mathsf{PL})$ defined using the formal power series $(\exp x - 1)/x$ and the $\star$ product.

The series $\exp^*$ is very classical, related to the flow of vector fields, and its coefficients are known as the Connes–Moscovici coefficients [Chapoton 2002a].

Consider the left action of $\exp^*$ on $\widehat{\mathsf{PL}}$. Let T be an element of $\widehat{\mathsf{PL}}$. Then $\exp^*(T)$ in $\widehat{\mathsf{PL}}$ is defined by

$$\exp^*(T) = \sum_{n\geq 1} \frac{1}{n!}((T \curvearrowleft T) \curvearrowleft \dots) \curvearrowleft T, \tag{6}$$

where there are n copies of T in the n-th term. As $\exp^*$ belongs to the group G_{PL}, the map $\exp^*$ defines a bijection from $\widehat{\mathsf{PL}}$ to itself.

Let us now relate the usual exponential map exp to the map $\exp^*$.

Let T be an element of $\widehat{\mathsf{PL}}$. Let $\exp T$ be the exponential of T in $\widehat{U}(\mathsf{PL})$ (which is defined by the usual series and using the $\star$ product). The map exp defines a bijection from $\widehat{\mathsf{PL}}$ to the set of group-like elements of $\widehat{U}(\mathsf{PL})$.

Therefore, the composite map $\exp^* \circ \exp^{-1}$ is a bijection from the set of group-like elements in $\widehat{U}(\mathsf{PL})$ to $\widehat{\mathsf{PL}}$. Let us show that this composite map is just a restriction of the canonical projection π.

Proposition 2.4. *Let T be an element of $\widehat{\mathsf{PL}}$. One has $\pi(\exp T) = \exp^*(T)$.*

Proof. Let F be in $\widehat{U}(\mathsf{PL})$. From Lemma 2.2 above, one knows that $\pi(F \star T)$ is exactly $\pi(F) \curvearrowleft T$. This implies that

$$\pi(T^{\star n}) = ((T \curvearrowleft T)\dots) \curvearrowleft T, \tag{7}$$

for all $n \geq 1$, hence the result. □

3. The classical case

Let us start by recalling the definition of a classical element Ω of $\widehat{\mathsf{PL}}$ with rational coefficients. It seems to have first appeared in [Agračev and Gamkrelidze 1980], was later considered under the name of $\log^*$ in [Chapoton 2002a] and has been since studied in [Murua 2006; Wright and Zhao 2003; Ebrahimi-Fard and Manchon 2009; Calaque et al. 2008].

Proposition 3.1. *There is a unique solution Ω in $\widehat{\mathsf{PL}}_{\mathbb{Q}}$ to the equation*

$$\bullet \curvearrowleft \left(\frac{\Omega}{\exp\Omega - 1}\right) = \Omega, \tag{8}$$

where $\Omega/(\exp\Omega - 1)$ is in the completed enveloping algebra $\widehat{U}(\mathsf{PL})$.

Proof. Let us write $\Omega = \sum_{n\geq 1} \Omega_n$ where each Ω_n is homogeneous of degree n.

Recall the Taylor expansion

$$\frac{x}{\exp x - 1} = \sum_{k\geq 0} \frac{B_k}{k!} x^k, \tag{9}$$

where the B_k are the Bernoulli numbers.

Then the homogeneous component of degree n of Equation (8) is

$$\Omega_n = \sum_{k\geq 0} \frac{B_k}{k!} \sum_{\substack{m_1\geq 1,\ldots,m_k\geq 1\\ m_1+\cdots+m_k=n-1}} ((\bullet \curvearrowleft \Omega_{m_k})\ldots) \curvearrowleft \Omega_{m_1}. \tag{10}$$

This gives a recursive definition of Ω_n, which implies the existence and uniqueness of Ω. □

Remark. One can use Equation (10) to compute Ω up to order n in a $O(n^3)$ number of pre-Lie operations.

As the element $\Omega/(\exp\Omega - 1)$ is invertible in the completed enveloping algebra, Equation (8) is also equivalent to the following equation:

$$\Omega \curvearrowleft \left(\frac{\exp\Omega - 1}{\Omega}\right) = \bullet. \tag{11}$$

One can interpret Equation (11) as follows.

Proposition 3.2. *The series Ω is the inverse of* $\exp^*$ *in the group G_{PL}.*

Proof. By right action by the inverse Ω^{-1} of Ω in G_{PL} on (11), one shows that Ω^{-1} satisfies the same Equation (5) as $\exp^*$. □

There is another equation for Ω.

Proposition 3.3. *The series* Ω *is the unique nonzero solution in* $\widehat{\mathsf{PL}}_{\mathbb{Q}}$ *to the equation*

$$\Omega \curvearrowleft (\exp \Omega - 1) = \bullet \curvearrowleft \Omega, \tag{12}$$

where $\exp \Omega - 1$ *is in the completed enveloping algebra* $\widehat{U}(\mathsf{PL})$.

Proof. First, by right action on (11) by Ω, one can see that the unique solution Ω of (8) is indeed a solution of (12).

Let us now prove uniqueness of a nonzero solution. Let Ω be any solution of (12). Let us write $\Omega = \sum_{n \geq 1} \Omega_n$ where each Ω_n is homogeneous of degree n.

Then the homogeneous component of degree n of Equation (12) is

$$\bullet \curvearrowleft \Omega_{n-1} = \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{m_1 \geq 1, \dots, m_k \geq 1, \ell \geq 1 \\ m_1 + \dots + m_k + \ell = n}} ((\Omega_\ell \curvearrowleft \Omega_{m_k}) \dots) \curvearrowleft \Omega_{m_1}. \tag{13}$$

If $n = 2$, this implies that Ω_1 is either 0 or $\bullet$.

Assume now that Ω is not zero. Let d be the degree of the first nonzero homogeneous component Ω_d of Ω. Assume that $d > 1$. Then Equation (13) in degree $d + 1$, together with Lemma 2.3, gives that $\Omega_d = 0$, a contradiction. Therefore necessarily, one has $d = 1$ and $\Omega_1 = \bullet$.

Let us look at the homogeneous component (13) in degree $n + 1 \geq 2$. The only terms involving Ω_n are $\bullet \curvearrowleft \Omega_n$ in the left-hand side and $\Omega_1 \curvearrowleft \Omega_n$, $\Omega_n \curvearrowleft \Omega_1$ in the right hand-side. As $\Omega_1 = \bullet$, two of them cancel out and one gets a recursive expression of $\Omega_n \curvearrowleft \bullet$ in terms of some Ω_j for $j < n$.

Using Lemma 2.3, this provides a recursive description of Ω (that may or may not possess a solution) and proves its uniqueness. □

The exponential of Ω has a simple shape.

Proposition 3.4. *In the enveloping algebra* $\widehat{U}(\mathsf{PL})$, *one has*

$$\exp \Omega = \sum_{n \geq 0} \frac{1}{n!} \bullet \ \bullet \dots \bullet, \tag{14}$$

where, in the n-th term, the forest has n nodes.

Proof. This is an equation for the exponential $\exp \Omega$ of the element Ω in the Lie algebra $\widehat{\mathsf{PL}}$. By Proposition 2.4, it is enough to prove that

$$\exp^*(\Omega) = \bullet, \tag{15}$$

because the image by π of the right side of (14) is $\bullet$.

But this amounts to saying that $\exp^*$ is the inverse of Ω in the group G_{PL}. This is none other than Proposition 3.2. □

It follows that

$$\Omega \curvearrowleft (\exp \Omega - 1) = \sum_{n\geq 2} \frac{1}{(n-1)!} \mathtt{Crl}_n^{\natural} \circ_{\natural} \Omega, \tag{16}$$

where $\mathtt{Crl}_n^{\natural}$ is the corolla with $n-1$ leaves and root labeled by $\natural$; see Figure 4.

Proposition 3.5. *The series Ω is the unique nonzero solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}}$ to the equation*

$$\sum_{n\geq 2} \frac{1}{(n-1)!} \mathtt{Crl}_n^{\natural} \circ_{\natural} \Omega = \bullet \curvearrowleft \Omega, \tag{17}$$

where $\sum_{n\geq 2} \frac{1}{(n-1)!} \mathtt{Crl}_n^{\natural}$ is in $\widehat{\mathsf{PL}}$.

Proof. This follows from (16) and Proposition 3.3. □

4. The quantum case

We will introduce now an element Ω_q in $\widehat{\mathsf{PL}}$ with coefficients in $\mathbb{Q}(q)$. We will show later that this is a q-deformation of Ω.

If $A = \sum_{n\geq 1} A_n$ is an element of $\widehat{\mathsf{PL}}$, let $A[q]$ be the q-shift of A defined by

$$A[q] = \sum_{n\geq 1} q^n A_n. \tag{18}$$

Proposition 4.1. *There exists a unique solution Ω_q in $\mathsf{PL}_{\mathbb{Q}(q)}$ to the equation*

$$\Omega_q[q] \curvearrowleft (\exp \Omega - 1) + \Omega_q[q] - \Omega_q = \bullet \curvearrowleft \Omega_q + (q-1)\bullet. \tag{19}$$

Moreover, the series Ω_q has coefficients in the ring of fractions with poles only at roots of unity.

Proof. Write $\Omega_q = \sum_{n\geq 1} \Omega_{q,n}$ where each $\Omega_{q,n}$ is homogeneous of degree n. The homogeneous component of degree 1 of (19) implies that $\Omega_{q,1} = \bullet$.

Then for $n \geq 2$, the homogeneous component of degree n of Equation (19) is

$$\begin{aligned}(q^n - 1)\Omega_{q,n} \\ = \bullet \curvearrowleft \Omega_{q,n-1} - \sum_{k\geq 1} \frac{1}{k!} \sum_{\substack{m_1\geq 1,\ldots,m_k\geq 1,\ell\geq 1 \\ m_1+\cdots+m_k+\ell=n}} q^{\ell}((\Omega_{q,\ell} \curvearrowleft \Omega_{m_k})\ldots) \curvearrowleft \Omega_{m_1}.\end{aligned} \tag{20}$$

This provides an explicit recursion for $\Omega_{q,n}$ in terms of $\Omega_{q,j}$ and Ω_j for $j < n$. This gives existence and uniqueness and also implies that Ω_q has coefficients with poles only at roots of unity. □

On can reformulate the equation for Ω_q.

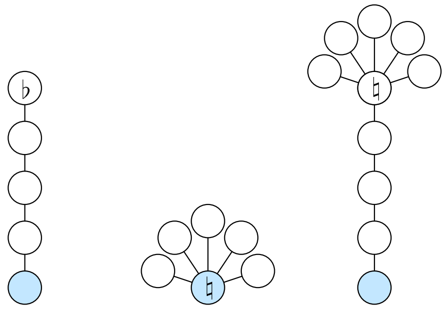


Figure 4. Rooted trees: $\mathtt{Lnr}_5^\flat$, $\mathtt{Crl}_6^\natural$ and $\mathtt{Frk}_{4,5}^\natural = \mathtt{Lnr}_5^\flat \circ_\flat \mathtt{Crl}_6^\natural$.

Proposition 4.2. *The series Ω_q is the unique solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ to the equation*

$$\sum_{n\geq 1} \frac{1}{(n-1)!}\mathtt{Crl}_n^\natural \circ_\natural \Omega_q[q] - \Omega_q = \bullet \curvearrowleft \Omega_q + (q-1)\bullet. \tag{21}$$

Proof. This follows from (16) and Proposition 4.1. □

Let $\mathtt{Frk}_{\ell,n}^\natural$ be the rooted tree with a linear trunk of ℓ vertices, a vertex $\natural$ on top of this trunk and a corolla with n leaves on top of the vertex $\natural$, see Figure 4. We will call this a *fork*. One has $\mathtt{Frk}_{\ell,n} = \mathtt{Lnr}_{\ell+1}^\flat \circ_\flat \mathtt{Crl}_{n+1}^\natural$.

Proposition 4.3. *The series Ω_q is the unique solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ to the equation*

$$\Omega_q = \sum_{\ell\geq 0}\sum_{n\geq 0} \frac{(-1)^\ell}{n!}\mathtt{Frk}_{\ell,n}^\natural \circ_\natural \Omega_q[q] + (1-q)\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}_\ell. \tag{22}$$

Proof. Let us compute the right-hand side of (22), using (21) for Ω_q, written as

$$\Omega_q + \bullet \curvearrowleft \Omega_q + (q-1)\bullet = \sum_{n\geq 1} \frac{1}{(n-1)!}\mathtt{Crl}_n^\natural \circ_\natural \Omega_q[q]. \tag{23}$$

One gets

$$\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}_\ell^\flat \circ_\flat (\Omega_q + \bullet \curvearrowleft \Omega_q + (q-1)\bullet) + (1-q)\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}_\ell. \tag{24}$$

As $\mathtt{Lnr}_\ell^\flat \circ_\flat \bullet = \mathtt{Lnr}_\ell$ and $\mathtt{Lnr}_\ell^\flat \circ_\flat (\bullet \curvearrowleft \Omega_q) = \mathtt{Lnr}_{\ell+1}^\flat \circ_\flat \Omega_q$, the two rightmost terms cancels, and the sum simplifies to

$$\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}_\ell^\flat \circ_\flat \Omega_q - \sum_{\ell\geq 2}(-1)^{\ell-1}\mathtt{Lnr}_\ell^\flat \circ_\flat \Omega_q, \tag{25}$$

which is just Ω_q. This proves that Ω_q does satisfy (22).

It is then easy to see that (22) has only one solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ by rewriting it as a recursion for the homogeneous components $\Omega_{q,n}$. □

5. Image in the free dendriform algebra

We describe in this section the image of Ω_q by the usual morphism from the free pre-Lie algebra to the free dendriform algebra. We show that this image is related to a family of Lie idempotents in the descent algebras of the symmetric groups. One deduces from that a nice explicit formula, that will be used later to get arithmetic information on Ω_q.

Dendriform algebra. Recall that a *dendriform algebra* [Loday 2001] is a vector space V endowed with two bilinear maps $\succ$ and $\prec$ from $V \otimes V$ to V satisfying the following axioms:

$$x \prec (y \prec z) + x \prec (y \succ z) = (x \prec y) \prec z, \tag{26}$$

$$x \succ (y \prec z) = (x \succ y) \prec z, \tag{27}$$

$$x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z. \tag{28}$$

Any dendriform algebra has the structure of a pre-Lie algebra given by

$$x \curvearrowleft y = y \succ x - x \prec y. \tag{29}$$

Any dendriform algebra has the structure of an associative algebra given by

$$x * y = x \succ y + x \prec y. \tag{30}$$

Remark. Equation (27) means that one can safely forget some parentheses. Equations (26) and (28) can be rewritten as

$$x \prec (y * z) = (x \prec y) \prec z, \tag{31}$$

$$x \succ (y \succ z) = (x * y) \succ z. \tag{32}$$

Let $\mathrm{Dend}(S)$ be the free dendriform algebra over a set S. This has an explicit basis indexed by *planar binary trees* with vertices decorated by S. For an example of a planar binary tree, see Figure 5. In particular, the free dendriform algebra on one generator, denoted by Dend, has a basis indexed by planar binary trees. This is a graded vector space, the degree $\#t$ of a planar binary tree t being the number of its inner vertices.

There is a unique morphism φ of pre-Lie algebras from PL to Dend that maps the rooted tree $\bullet$ to the planar binary tree $\curlyvee$. This extends uniquely to a continuous morphism φ from $\widehat{\mathsf{PL}}$ to the completion $\widehat{\mathrm{Dend}}$ of Dend.

Remark. With some care, one can add a unit 1 to the free dendriform algebra Dend. Then $1 * x = 1 \succ x = x = x \prec 1 = x * 1$, but one has to pay attention to never write either $1 \prec x$ or $x \succ 1$. We will use this convention in the sequel.

There are two kinds of special planar binary trees: the left combs and the right combs. They can be defined as follows. Let $L = \sum_{n\geq 1} L_n$ be the unique solution in $\widehat{\mathsf{Dend}}$ to the equation

$$L = \curlyvee + L \succ \curlyvee = (1+L) \succ \curlyvee, \tag{33}$$

and let $R = \sum_{n\geq 1} R_n$ be the unique solution in $\widehat{\mathsf{Dend}}$ to

$$R = \curlyvee + \curlyvee \prec R = \curlyvee \prec (1+R). \tag{34}$$

Then L_n is called the left comb with n vertices and R_n be the right comb with n vertices.

If $A = \sum_{n\geq 1} A_n$ is an element of $\widehat{\mathsf{PL}}$ or $\widehat{\mathsf{Dend}}$, the *suspension* of A is $\widetilde{A} = \sum_{n\geq 1}(-1)^{n-1}A_n$.

Proposition 5.1. *The inverse of* $1+R$ *with respect to the* $*$ *product is* $1-\widetilde{L}$.

Proof. One has $\widetilde{L} = \curlyvee - \widetilde{L} \succ \curlyvee$. We compute

$$(1-\widetilde{L}) * (1+R) = 1 + R - \widetilde{L} * (1+R). \tag{35}$$

By the definition of $*$ and the convention on the unit 1, this is

$$1 + R - \widetilde{L} \prec (1+R) - \widetilde{L} \succ R. \tag{36}$$

By (33), this becomes

$$1 + R - \curlyvee \prec (1+R) + \widetilde{L} \succ \curlyvee \prec (1+R) - \widetilde{L} \succ R. \tag{37}$$

The last two terms cancel by (34) and one gets

$$1 + R - \curlyvee \prec (1+R), \tag{38}$$

which is just 1, again by (34). □

Equation for the dendriform image of $\boldsymbol{\Omega_q}$. Define a series $E = \sum_{n\geq 1} nL_n$ in $\widehat{\mathsf{Dend}}$. One can easily show that

$$E = L + E \succ \curlyvee. \tag{39}$$

Lemma 5.2. *The series* $B^\flat = \varphi(\sum_{n\geq 1} \mathtt{Lnr}_n^\flat)$ *satisfies*

$$B^\flat = \curlyvee^\flat + B^\flat \succ \curlyvee - \curlyvee \prec B^\flat. \tag{40}$$

Proof. This comes from a similar equation in PL. Let $\mathtt{Lnr}^\flat = \sum_{n\geq 1} \mathtt{Lnr}_n^\flat$. Then

$$\mathtt{Lnr}^\flat = \bullet^\flat + \bullet \curvearrowleft \mathtt{Lnr}^\flat, \tag{41}$$

as one can easily check. □

These relations can be taken as definitions of the elements E and $B^\flat$ of $\widehat{\mathsf{Dend}}$. One can forget the marking $\flat$ in $B^\flat$ to define a series B.

Proposition 5.3. *The series $B = \varphi(\sum_{n\geq 1} \mathtt{Lnr}_n)$ satisfies*

$$E = (1+L) * B. \tag{42}$$

Proof. One has to show that $E = (1+L)*B$. It is enough to prove that $(1+L)*B$ does satisfy the defining relation (39) of E.

One computes, using (40) for B,

$$(1+L) * B = (1+L) * (\curlyvee + B \succ \curlyvee - \curlyvee \prec B). \tag{43}$$

Expanding the $*$ product, this is

$$(1+L) \succ \curlyvee + L \prec \curlyvee + (1+L) \succ (B \succ \curlyvee - \curlyvee \prec B) + L \prec (B \succ \curlyvee - \curlyvee \prec B). \tag{44}$$

Using (33) and the dendriform axioms, this becomes

$$L + L \prec \curlyvee + ((1+L)*B) \succ \curlyvee - (1+L) \succ \curlyvee \prec B + L \prec (B \succ \curlyvee - \curlyvee \prec B). \tag{45}$$

Using (33) again, one gets

$$L + ((1+L)*B) \succ \curlyvee + L \prec (\curlyvee - B + B \succ \curlyvee - \curlyvee \prec B). \tag{46}$$

This simplifies, by (40) for B, to

$$L + ((1+L)*B) \succ \curlyvee, \tag{47}$$

as expected. □

Lemma 5.4. *The image of $\sum_{n\geq 1} \frac{1}{(n-1)!} \mathtt{Crl}_n^{\natural}$ by φ is*

$$(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}). \tag{48}$$

Proof. This was proved in [Ronco 2000; 2001; Chapoton 2002b]. □

Proposition 5.5. *The image of $\sum_{\ell\geq 0}\sum_{n\geq 0} \frac{(-1)^{\ell}}{n!} \mathtt{Frk}_{\ell,n}^{\natural}$ by φ is*

$$(1+R) * \curlyvee^{\natural} * (1-\widetilde{L}), \tag{49}$$

where $\curlyvee^{\natural}$ is the planar binary tree $\curlyvee$ with vertex labeled by $\natural$.

Proof. Let $D = (1+R) * \curlyvee^{\natural} * (1-\widetilde{L})$. Let us first show that

$$D = (1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + \curlyvee \prec D - D \succ \curlyvee. \tag{50}$$

Expanding the $*$ product, one computes

$$D = ((1+R) * \curlyvee^{\natural}) \prec (1-\widetilde{L}) - ((1+R) * \curlyvee^{\natural}) \succ \widetilde{L}. \tag{51}$$

Then one gets, by expanding again,

$$(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + (R \prec \curlyvee^{\natural}) \prec (1-\widetilde{L}) - ((1+R) * \curlyvee^{\natural}) \succ \widetilde{L}. \tag{52}$$

Using the dendriform axioms, this is

$$(1+R) \succ \mathsf{Y}^{\natural} \prec (1-\widetilde{L}) + R \prec (\mathsf{Y}^{\natural} * (1-\widetilde{L})) - ((1+R) * \mathsf{Y}^{\natural}) \succ \widetilde{L}. \quad (53)$$

Then by (33) and (34), this can be rewritten

$$(1+R) \succ \mathsf{Y}^{\natural} \prec (1-\widetilde{L}) + (\mathsf{Y} \prec (1+R)) \prec (\mathsf{Y}^{\natural} * (1-\widetilde{L})) \\ - ((1+R) * \mathsf{Y}^{\natural}) \succ ((1-\widetilde{L}) \succ \mathsf{Y}). \quad (54)$$

One gets, using the dendriform axioms,

$$(1+R) \succ \mathsf{Y}^{\natural} \prec (1-\widetilde{L}) + \mathsf{Y} \prec ((1+R) * \mathsf{Y}^{\natural} * (1-\widetilde{L})) \\ - ((1+R) * \mathsf{Y}^{\natural} * (1-\widetilde{L})) \succ \mathsf{Y}. \quad (55)$$

This proves Equation (50) for D.

Let us show now that

$$D' = \left(\sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}^{\flat}_{\ell})\right) \circ_{\flat} \left(\sum_{n \geq 1} \frac{1}{(n-1)!} \varphi(\mathtt{Crl}^{\natural}_{n})\right)$$

does satisfy the same equation as D.

By Lemma 5.4, one has

$$D' = \widetilde{B}^{\flat} \circ \flat((1+R) \succ \mathsf{Y}^{\natural} \prec (1-\widetilde{L})). \quad (56)$$

By Lemma 5.2, one has $\widetilde{B}^{\flat} = \mathsf{Y}^{\flat} - \widetilde{B}^{\flat} \succ \mathsf{Y} + \mathsf{Y} \prec \widetilde{B}^{\flat}$, hence

$$D' = (1+R) \succ \mathsf{Y}^{\natural} \prec (1-\widetilde{L}) + \mathsf{Y} \prec D' - D' \succ \mathsf{Y}. \quad (57)$$

By uniqueness of the solution D of (50), one has $D = D'$, that is,

$$(1+R) * \mathsf{Y}^{\natural} * (1-\widetilde{L}) = \left(\sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}^{\flat}_{\ell})\right) \circ_{\flat} \left(\sum_{n \geq 1} \frac{1}{(n-1)!} \varphi(\mathtt{Crl}^{\natural}_{n})\right). \quad (58)$$

Therefore

$$(1+R) * \mathsf{Y}^{\natural} * (1-\widetilde{L}) = \varphi\left(\left(\sum_{\ell \geq 1} (-1)^{\ell-1} \mathtt{Lnr}^{\flat}_{\ell}\right) \circ_{\flat} \left(\sum_{n \geq 1} \frac{1}{(n-1)!} \mathtt{Crl}^{\natural}_{n}\right)\right), \quad (59)$$

which is exactly the expected image by φ of a sum over forks. □

One can now deduce a useful functional equation for the image of Ω_q by φ, using only the associative product $*$ of Dend.

Proposition 5.6. *The series $\varphi(\Omega_q)$ is the unique solution in $\widehat{\mathrm{Dend}}$ of*

$$\varphi(\Omega_q) = (1-\widetilde{L})^{-1} * \varphi(\Omega_q)[q] * (1-\widetilde{L}) + (1-q) \sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}_{\ell}). \quad (60)$$

Proof. Let us start from (22). By Proposition 5.5, we know the image by φ of the sum over forks. One gets

$$\varphi(\Omega_q) = ((1+R) * \curlyvee^{\natural} * (1-\widetilde{L})) \circ_\natural \varphi(\Omega_q)[q] + (1-q) \sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}_\ell). \quad (61)$$

Then one can use Proposition 5.1 to replace $1+R$ by the inverse of $1-\widetilde{L}$. □

Explicit formula. We will prove in this section that the image of Ω_q by φ coincides (in some sense) with a known family of Lie idempotents, and has an explicit description using q-binomial coefficients, descents and major indices of planar binary trees. To obtain this description, we use a result on noncommutative symmetric functions. We refer to [Gelfand et al. 1995; Duchamp et al. 1997; Krob et al. 1997] for background on this subject. We will use the notation of this last article.

The algebra **Sym** of noncommutative symmetric function is the free unital associative algebra on generators $S_1, S_2, \ldots$. It is a graded algebra (with S_i of degree i), with a basis $(S_I)_I$ indexed by compositions. There is another basis $(R_I)_I$ obtained from the basis $(S_I)_I$ by Möbius inversion on compositions ordered by refinement. By convention, S_0 is the unit of **Sym**.

As **Sym** is free, there is a unique morphism of associative algebras θ from **Sym** to Dend which maps S_i to the left comb L_i for each $i \geq 0$, with the convention that L_0 is the unit of Dend.

One can check that θ is the usual morphism from **Sym** to Dend, considered for instance in [Hivert et al. 2005, Section 4.8] and [Loday and Ronco 1998].

In **Sym**, there are elements Ψ_i for $i \geq 1$, uniquely defined by the conditions

$$n S_n = \sum_{i=0}^{n-1} S_i * \Psi_{n-i}, \quad (62)$$

for all $n \geq 1$.

Proposition 5.7. *The image of Ψ_i by θ is $\varphi(\mathtt{Lnr}_i)$.*

Proof. This is a corollary of Proposition 5.3. Indeed, one has

$$1 + L = \sum_{n \geq 0} \theta(S_n) \quad \text{and} \quad E = \sum_{n \geq 1} n\, \theta(S_n). \quad (63)$$

Therefore

$$B = \sum_{n \geq 1} \theta(\Psi_n).$$ □

The leaves of a planar binary tree with n vertices are labeled from 0 to n from left to right. The leaves with labels different from 0 and n are called *inner leaves*. A *descent* in a planar binary tree t is the label of an \-oriented inner leaf. The descent set $D(t)$ of t is the set of its descents.

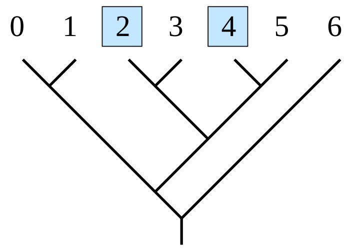


Figure 5. A planar binary tree t with descents at 2 and 4.

The number of descents of a planar binary tree t will be denoted $d(t)$. It satisfies $0 \leq d(t) \leq n-1$ for a tree t of degree n.

The *major index* $\mathrm{maj}(t)$ of t is the sum of its descents. For example, Figure 5 displays a planar binary tree with descent set $\{2,4\}$ and major index $2+4=6$.

Let us recall that the descent set $D(I)$ corresponding to a composition $I = (i_1, \dots, i_k)$ of n is the set $\{i_1, i_1+i_2, \dots, i_1+\cdots+i_k\}$.

Proposition 5.8. *The image by θ of R_I is the sum*

$$\sum_{\substack{\#t=n \\ D(t)=D(I)}} t \tag{64}$$

of all planar binary trees with n vertices and descent set $D(I)$.

Proof. This is a well-known property of the injection of **Sym** in Dend. □

In [Krob et al. 1997], elements $\Psi_n(A/(1-q))$, for $n \geq 1$, are defined by some "change of alphabet" applied to the elements Ψ_n. According to the proof of [Krob et al. 1997, Theorem 6.11], they are characterized by

$$\sum_{n\geq 1} \Psi_n\Big(\frac{A}{1-q}\Big) = \Big(\sum_{n\geq 0} S_n\Big)^{-1} \Big(\sum_{n\geq 1} q^n \Psi_n\Big(\frac{A}{1-q}\Big)\Big)\Big(\sum_{n\geq 0} S_n\Big) + \sum_{n\geq 1} \Psi_n. \tag{65}$$

There is a classical isomorphism of vector spaces α from **Sym** to the direct sum of all descent algebras of symmetric groups. By this morphism, each $\Psi_n(A/(1-q))$ is mapped, up to a multiplicative constant, to a Lie idempotent with coefficients in $\mathbb{Q}(q)$ in the descent algebra of the n-th symmetric group.

We can now state the precise relation between $\boldsymbol{\Omega}_q$ and these Lie idempotents, or rather with the elements $\Psi_n(A/(1-q))$.

Proposition 5.9. *The image of* $(1-q)\Psi(A/(1-q))$ *by θ is* $\varphi(\widetilde{\boldsymbol{\Omega}_q})$.

Proof. Indeed, by Proposition 5.6, one has

$$\sum_{n\geq 1} \varphi(\widetilde{\boldsymbol{\Omega}_q}) = (1+L)^{-1} * (\varphi(\widetilde{\boldsymbol{\Omega}_q})[q]) * (1+L) + (1-q) \sum_{n\geq 1} \varphi(\mathtt{Lnr}_n). \tag{66}$$

Then using Proposition 5.7 and (65), one gets that $\theta\big((1-q)\Psi(A/(1-q))\big)$ and $\varphi(\widetilde{\Omega_q})$ satisfy the same equation, hence they are equal. $\square$

Let $\Omega_{q,n}$ be the homogeneous component of degree n of Ω_q.

Proposition 5.10. *One has*

$$\varphi(\Omega_{q,n}) = \frac{(-1)^{n-1}}{[n]_q} \sum_{\#t=n} (-1)^{d(t)} \begin{bmatrix} n-1 \\ d(t) \end{bmatrix}_q^{-1} q^{\mathrm{maj}(t)-\binom{d(t)+1}{2}}\, t. \tag{67}$$

Proof. Theorem 6.11 of [Krob et al. 1997] says that the element $(1-q)\Psi_n\Big(\dfrac{A}{1-q}\Big)$ is

$$\frac{1}{[n]_q} \sum_{|I|=n} (-1)^{d(I)} \begin{bmatrix} n-1 \\ d(I) \end{bmatrix}_q^{-1} q^{\mathrm{maj}(I)-\binom{d(I)+1}{2}}\, R_I. \tag{68}$$

By Proposition 5.9, the image under θ of this formula is $(-1)^{n-1}\varphi(\Omega_{q,n})$. By Proposition 5.8, this becomes the expected formula. $\square$

6. Arithmetic properties

In this section, we obtain some properties of the denominators in Ω_q and consider what happens when q is specialized to $1, 0$ and ∞.

The case $q = 1$. First note that the morphism φ from $\widehat{\mathsf{PL}}$ to the completed free dendriform algebra $\widehat{\mathrm{Dend}}$ is defined over $\mathbb{Q}$ and injective. Hence one can deduce results on Ω_q from results on its image by φ.

Proposition 6.1. *The series Ω_q is regular at $q = 1$ and $\Omega_{q=1} = \Omega$.*

Proof. By Proposition 5.10, the image $\varphi(\Omega_q)$ is regular at $q = 1$, as q-binomial coefficients become usual binomial coefficients when $q = 1$. Therefore Ω_q itself is regular at $q = 1$.

At $q = 1$, (19) becomes (12). By uniqueness in Proposition 3.3, the value of Ω_q at $q = 1$ is Ω. $\square$

Remark. Knowing that $\Omega_{q=1} = \Omega$, one can use (20) to compute simultaneously Ω_q and Ω up to order n in a $O(n^3)$ number of pre-Lie operations.

There is a lot of cancellation in the coefficients of Ω_q, leading to a reduced complexity of the denominators. Note that the expected denominator of $\Omega_{q,n}$ (from recursion (20)) is the product $\prod_{d=2}^{n}(q^d - 1)$. Let Φ_d be the d-th cyclotomic polynomial.

Proposition 6.2. *The common denominator of the coefficients of the element $\Omega_{q,n}$ divides the product $\prod_{d=2}^{n} \Phi_d$.*

Proof. For the image of Ω_q by φ, this follows from 5.10 and a simple property of the q-binomial coefficients: their only roots are simple roots at roots of unity, see [Guo and Zeng 2006, Proposition 2.2]. This implies the same result for Ω_q. □

Remark. The true denominator of each individual coefficient in Ω_q is often smaller than the complete product $\prod_{d=2}^{n} \Phi_d$, see for instance the first few terms in Appendix B.

The case $q = 0$. Let us consider now what happens when $q = 0$. Then Ω_0 is well-defined, $\Omega_q[q]$ vanishes and (19) becomes

$$\Omega_0 = \bullet - \bullet \curvearrowleft \Omega_0. \tag{69}$$

It follows that Ω_0 is the alternating sum of linear trees.

The case $q = \infty$. Let us now consider what happens when $q = \infty$. Let $\omega_{q,T}$ be the coefficient of the rooted tree T in the expansion of Ω_q. We define the *valuation* at $q = \infty$ as the smallest exponent in the formal Laurent expansion in powers of q^{-1} of an element of $\mathbb{Q}(q)$.

Proposition 6.3. *The valuation of $\omega_{q,T}$ at $q = \infty$ is at least $\#T - 1$.*

Proof. This will follow from the recursion (20). This is true in degree $n = 1$. Let us assume that $n \geq 2$. Then the valuation of $\bullet \curvearrowleft \Omega_{q,n-1}$ is at least $n-2$ by induction and the valuation of each term of the rightmost sum in (20) is at least -1. Hence the valuation of $\Omega_{q,n}$ is at least $n - 1$. □

Hence there exists a limit Ω_∞ for $\Omega_q[q]/q$ when q goes to ∞ and the limit of Ω_q/q is zero.

Equation (19), divided by q, becomes, at $q = \infty$,

$$\Omega_\infty \curvearrowleft \exp \Omega = \bullet. \tag{70}$$

By right action by $\exp(-\Omega)$, this is equivalent to

$$\Omega_\infty = \bullet \curvearrowleft \exp(-\Omega). \tag{71}$$

The element $\exp(-\Omega)$ is the inverse of $\exp \Omega$ in $\widehat{U}(\mathsf{PL})$. This has been computed in [Chapoton and Livernet 2007, Section 6.4]. More precisely, the inverse of $\sum_{n\geq 1} \frac{1}{(n-1)!}\mathtt{Crl}_n$ in the group of characters of the Connes–Kreimer Hopf algebra was shown there to be

$$\sum_T \frac{(-1)^{\#T-1}}{\operatorname{aut}(T)} T, \tag{72}$$

where $\operatorname{aut}(T)$ is the cardinal of the automorphism group of the rooted tree T. But it is known [Chapoton and Livernet 2001] that this group of characters is isomorphic to the group of group-like elements in $\widehat{U}(\mathsf{PL})$. Going through the isomorphism, one gets the following result.

Proposition 6.4. *The series Ω_∞ is given by*

$$\Omega_\infty = \sum_T \frac{(-1)^{\#T-1}}{\mathrm{aut}(T)} T. \tag{73}$$

7. Morphisms and images

In this section, we consider several quotients of the free pre-Lie algebra PL and the images of Ω_q in some of these quotients. The first two quotients come from quotients operads of the pre-Lie operad.

As shown in [Chapoton and Livernet 2001], the pre-Lie operad can be described in terms of labeled rooted trees. We recall here briefly (see the article just cited for details) the definition of the composition of two labeled rooted trees T and S on the vertex sets I and J, respectively. Let $i \in I$; the composition of S at the vertex i of T is given by

$$T \circ_i S = \sum_f T \circ_i^f S, \tag{74}$$

where the sum runs over all maps f from the set of incoming edges of the vertex i of T to the set of vertices of S, and $T \circ_i^f S$ can be described as follows: replace the vertex i by the tree S, grafting back the subtrees of T previously attached to i, according to the map f.

Morphism to the free associative algebra.

Proposition 7.1. *The subspace of pre-Lie spanned by nonlinear labeled trees is an ideal. The quotient map κ coincides with the usual map from pre-Lie to the associative operad* As.

Proof. Using the description above of the composition map of the operad pre-Lie, it is clear that the composition of two labeled trees, at least one of which is nonlinear, is again nonlinear. The quotient operad, spanned by labeled linear trees, has dimension $n!$ in rank n. Its composition can be easily identified with the associative operad As. The quotient map is then checked on generators of pre-Lie to be the same as the usual map. □

Proposition 7.2. *The group G_{As} is isomorphic to the group of invertible formal power series in $x\mathbb{Q}[\![x]\!]$ for the composition product.*

Proof. It is more convenient here to work at the monoid level with $\widehat{\mathrm{As}}$ and $x\mathbb{Q}[\![x]\!]$. The vector space $\mathrm{As}(n)_{\mathfrak{S}_n}$ is one dimensional for all n, with a basis element θ_n. By left linearity of both monoids, it is sufficient to check the product rule for θ_m and $f = \sum_{n\geq 1} f_n\theta_n$. One finds that

$$\theta_m \times f = \sum_{n_1,\ldots,n_m\geq 1} f_{n_1} \ldots f_{n_m}\theta_{n_1+\cdots+n_m}, \tag{75}$$

which proves that the linear map defined by $x^n \mapsto \theta_n$ is an isomorphism between the monoids $\widehat{\mathsf{As}}$ and $x\mathbb{Q}[\![x]\!]$. The proposition follows by taking invertible elements.
□

The operad morphism κ induces a morphism of pre-Lie algebras from PL to the free (nonunital) associative algebra $\mathbb{Q}[x]_+$ on one generator x, sending $\bullet$ to x. This extends to a morphism from $\widehat{\mathsf{PL}}$ to the algebra $\mathbb{Q}[\![x]\!]_+$ of formal power series in x without constant term. This last morphism restricts to a group morphism from $\mathsf{G}_{\text{pre-Lie}}$ to G_{As}. We will denote by κ all these morphisms.

One can see that κ sends the linear trees $\mathtt{Lnr}_n = (\bullet \curvearrowleft \bullet)\dots \curvearrowleft \bullet$ to the monomials $(xx)\dots x = x^n$ and maps all other trees to 0.

From (8), one obtains that $\kappa(\Omega)$ satisfies

$$\exp(\kappa(\Omega)) - 1 = x. \tag{76}$$

Therefore $\kappa(\Omega)$ is the formal power series

$$\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n. \tag{77}$$

One then deduces from (19) that the image of Ω_q is the q-logarithm defined by

$$\log_q(x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{[n]_q} x^n, \tag{78}$$

which is the unique solution to the functional equation

$$x \log_q(qx) = x \log_q(x) + (q-1)\, x - \log_q(qx) + \log_q(x). \tag{79}$$

Morphism to corollas and point-wise multiplication. Let the *depth* of a rooted tree be the maximum number of vertices in a chain of adjacent vertices between the root and a leaf. Corollas are then the rooted trees of depth at most 2.

Proposition 7.3. *The subspace of pre-Lie spanned by labeled noncorollas is an ideal.*

Proof. Using the description above of the composition of pre-Lie, one shows that the depth of the composition of two labeled trees is greater or equal than the maximum of the depths of these labeled trees. Therefore, if one of the labeled trees has depth greater or equal to three, so does the composition. □

Let us denote by Mu the quotient operad and by λ the quotient map from pre-Lie to Mu. One can give a simple description of Mu. It has dimension n in rank n with basis given by the image of the labeled corollas with n nodes. Let us call μ_i^n the image of the corolla with n nodes and with root labeled by i for $i = 1, \dots, n$.

Then μ_1^1 is the unit of Mu and the composition is given by

$$\begin{cases} \mu_i^n \circ_i \mu_j^\ell = \mu_{i+j-1}^{n+\ell-1}, \\ \mu_i^n \circ_h \mu_j^\ell = 0 \qquad \text{for } h \neq i \text{ and } \ell \geq 2. \end{cases} \tag{80}$$

Let G_1 be the group of formal power series of the form $1+x\mathbb{Q}[[x]]$ for the pointwise multiplication product and G_2 be the multiplicative group $\mathbb{Q}^*$. There is an action of G_2 on G_1 by substitution: $\lambda \cdot f(x) = f(\lambda x)$.

From the description of Mu above, one deduces that

Proposition 7.4. *The group* G_{Mu} *is isomorphic to* $G_2 \ltimes G_1$.

Proof. The vector space $\mathrm{Mu}(n)_{\mathfrak{S}_n}$ is one-dimensional for all n, with basis given by the image of the corolla with n nodes. Let us denote this basis element by ν_{n-1}. Any element of G_{Mu} can be uniquely written as the product $\lambda\big(\sum_{m\geq 0} f_m \nu_m\big)$ of $\lambda \in \mathbb{Q}^*$ and $f = \sum_{m\geq 0} f_m \nu_m$ with $f_0 = 1$. Let us compute the product of $\lambda f = \lambda\big(\sum_{m\geq 0} f_m\nu_m\big)$ and $\theta g = \theta\big(\sum_{n\geq 0} g_n \nu_n\big)$ with the conventions $f_0 = 1$ and $g_0 = 1$. One finds that

$$\lambda f \times \theta g = \sum_{m\geq 0}\sum_{n\geq 0} \lambda f_m \theta^m (\theta g_n) \nu_{n+m} = \lambda\theta \sum_{m\geq 0}\sum_{n\geq 0} \theta^m f_m g_n \nu_{n+m}. \tag{81}$$

One defines a map from G_{Mu} to $G_2 \ltimes G_1$ by $\lambda\big(\sum_m f_m \nu_m\big) \mapsto (\lambda, f(x))$ with $f(x) = \sum_m f_m x^m$. The product in $G_2 \ltimes G_1$ is given by

$$(\lambda, f(x))(\theta, g(x)) = (\lambda\theta, f(\theta x) g(x)). \tag{82}$$

Hence the map is an isomorphism. $\square$

The quotient map λ induces a morphism of pre-Lie algebras from PL to the following pre-Lie algebra. Let us identify the image of the corolla $\mathtt{Crl}_{n+1}$ with n leaves to x^n for all $n \geq 0$. In particular, the tree $\bullet$ is mapped to 1. The underlying vector space is therefore identified with $\mathbb{Q}[x]$ and the quotient pre-Lie product is

$$x^p \curvearrowleft x^q = \begin{cases} x^{p+1} & \text{if } q = 0, \\ 0 & \text{else.} \end{cases} \tag{83}$$

One can show (using the description of the quotient $\curvearrowleft$ product given above) that the right action by $\lambda(\Omega)$ is given by the product by x. Therefore, the right action of the image of $\exp(\lambda(\Omega)) - 1$ is just given by the product by $\exp x - 1$. One sees as well that the right action by $\lambda(\Omega_q)$ is also given by the product by x.

Then one deduces from Proposition 3.3 that the image $\lambda(\Omega)$ is the generating function $x/(\exp x - 1)$ for the Bernoulli numbers.

Next, from (19), one gets that the image $F_q(x)$ of Ω_q satisfies the equation

$$(\exp x - 1)[q F_q(qx)] = x + q - 1 - q F_q(qx) + F_q(x). \tag{84}$$

This functional equation is known (see for instance [Satoh 1989]) to describe the generating function

$$F = \sum_{n \geq 0} \beta_n(q) \frac{x^n}{n!}, \tag{85}$$

where $\beta_q(n)$ are the q-Bernoulli numbers of [Carlitz 1948; 1954; 1958].

Therefore the coefficients of the corollas in Ω_q are the q-Bernoulli numbers of Carlitz. One may wonder whether it is possible to describe directly the coefficient of each rooted tree in Ω_q.

Morphism to a pre-Lie algebra of vector fields. There exists an interesting morphism from PL to a pre-Lie algebra of vector fields. We describe it here only as a side remark, as the image of Ω_q seems to have no special property.

Consider the vector space $V = \mathbb{Q}[x]_+$, endowed with the following pre-Lie product:

$$f \curvearrowleft g = xf\, \partial_x g. \tag{86}$$

Then there is a unique morphism from PL to V sending $\bullet$ to x.

This map has the following nice property: the coefficient of x^n in the image of a series A is the sum of the coefficients of the trees in the homogeneous component A_n of A. The proof is just a check that this sum-of-coefficients map defines a morphism of pre-Lie algebra from PL to V.

Appendix A. A group associated to an augmented operad

We briefly recall here the definition for each augmented operad of a group of formal power series. This can also be found in [Chapoton 2002a; 2007/08; van der Laan 2003; Chapoton and Livernet 2007]. We use in this section the definition of the notion operad by a multiple composition map γ, which is equivalent (using the unit) to the definition using the single compositions $\circ_i$ that we have used elsewhere in the article.

Let $\mathcal{P}$ be an operad in the category $\mathrm{Vect}_{\mathbb{Q}}$ of vector spaces over $\mathbb{Q}$ and assume that $\mathcal{P}(0) = \{0\}$ and that $\mathcal{P}(1) = \mathbb{Q}e$ where e is the unit of $\mathcal{P}$. Such an operad is called *augmented*.

Let $\mathsf{F}\mathcal{P} = \oplus_n \mathcal{P}(n)_{\mathfrak{S}_n}$ be the direct sum of the coinvariant spaces, which can be identified with the underlying vector space of the free $\mathcal{P}$-algebra on one generator, and $\widehat{\mathcal{P}} = \prod_n \mathcal{P}(n)_{\mathfrak{S}_n}$ be its completion.

Let $x = \sum_m x_m$, $y = \sum_n y_n$ be two elements of $\widehat{\mathcal{P}}$. Choose any representatives $\bar{x}_m$ of x_m (resp. $\bar{y}_n$ of y_n) in the operad $\mathcal{P}$. Then one can check that the following formula defines a product on $\widehat{\mathcal{P}}$:

$$x \times y = \sum_{m \geq 1} \sum_{n_1, \dots, n_m \geq 1} \langle \gamma\, (\bar{x}_m, \bar{y}_{n_1}, \dots, \bar{y}_{n_m}) \rangle, \tag{A.1}$$

where $\langle\ \rangle$ is the quotient map to the coinvariants and γ is the (multiple) composition map of the operad $\mathcal{P}$.

Proposition A.1. *The product $\times$ defines the structure of an associative monoid on the vector space $\widehat{\mathcal{P}}$. Furthermore, this product is $\mathbb{Q}$-linear on its left argument.*

Proof. We first prove associativity. On the one hand,

$$
\begin{aligned}
&(x \times y) \times z\\
&\quad = \sum_{m} \sum_{p_1,\dots,p_m} \langle \gamma\,(\overline{(x \times y)}_m, \bar{z}_{p_1}, \dots, \bar{z}_{p_m}) \rangle\\
&\quad = \sum_{m} \sum_{n_1,\dots,n_m} \sum_{p_1,\dots,p_{n_1+\dots+n_m}} \langle \gamma\,(\gamma\,(\bar{x}_m, \bar{y}_{n_1}, \dots, \bar{y}_{n_m}), \bar{z}_{p_1}, \dots, \bar{z}_{p_{n_1+\dots+n_m}}) \rangle. \qquad \text{(A.2)}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&x \times (y \times z)\\
&\quad = \sum_{m} \sum_{n_1,\dots,n_m} \big\langle \gamma\,(\bar{x}_m, \overline{(y \times z)}_{n_1}, \dots, \overline{(y \times z)}_{n_m}) \big\rangle\\
&\quad = \sum_{m} \sum_{n_1,\dots,n_m} \sum_{(q_{i,j})} \big\langle \gamma\,(\bar{x}_m, \gamma\,(\bar{y}_{n_1}, \bar{z}_{q_{1,1}}, \dots, \bar{z}_{q_{1,n_1}}), \dots, \gamma\,(\bar{y}_{n_m}, \bar{z}_{q_{m,1}}, \dots, \bar{z}_{q_{m,n_m}})) \big\rangle.
\end{aligned}
\qquad \text{(A.3)}
$$

Using then the "associativity" of the operad, one gets the associativity of $\times$. It is easy to check that the image of the unit e of the operad $\mathcal{P}$ is a two-sided unit for the $\times$ product. The left $\mathbb{Q}$-linearity is clear from the formula (A.1). □

Proposition A.2. *An element y of $\widehat{\mathcal{P}}$ is invertible for $\times$ if and only if the first component y_1 of y is nonzero.*

Proof. The direct implication is trivial. The reverse one is proved by a very standard recursive argument. □

Let us call $\mathsf{G}_{\mathcal{P}}$ the set of invertible elements of $\widehat{\mathcal{P}}$ for the $\times$ product.

Proposition A.3. G *is a functor from the category of augmented operads to the category of groups.*

Proof. The functoriality follows from inspection of the definitions of $\widehat{\mathcal{P}}$ and $\times$. □

In fact, one can see $\mathsf{G}_{\mathcal{P}}$ as the group of $\mathbb{Q}$-points of a proalgebraic group. The Lie algebra of this proalgebraic group is given by the usual linearization process on the tangent space $\widehat{\mathcal{P}}$, resulting in the formula

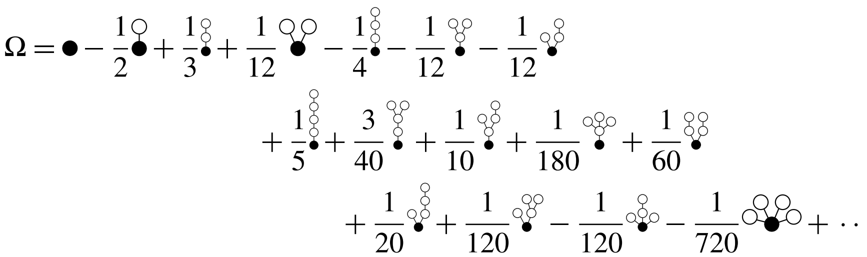
$$[x, y] = \sum_{m \geq 1} \sum_{n \geq 1} \langle \bar{x}_m \circ \bar{y}_n - \bar{y}_n \circ \bar{x}_m \rangle, \qquad \text{(A.4)}$$

where

$$\bar{x}_m \circ \bar{y}_n = \sum_{i=1}^{m} \gamma\,(\bar{x}_m, \underbrace{e, \dots, e}_{i-1 \text{ units}}, \bar{y}_n, e, \dots, e). \tag{A.5}$$

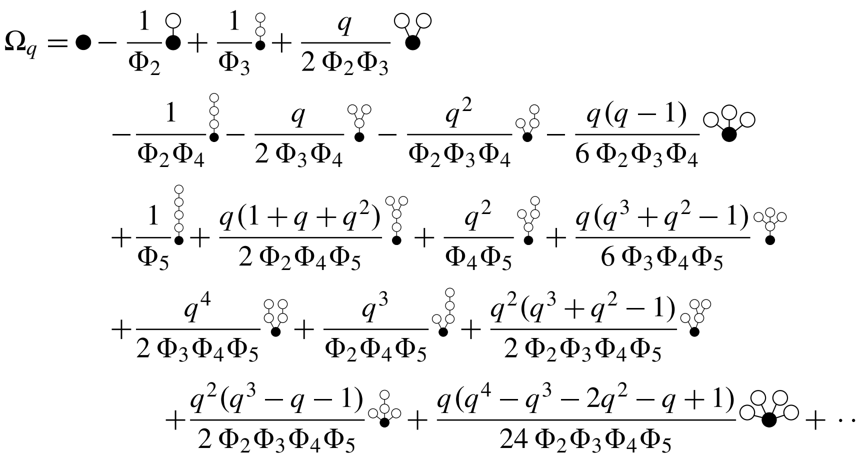
The graded Lie algebra structure on $\mathsf{F}\mathcal{P}$ defined by the same formulas has already appeared in the work of Kapranov and Manin on the category of right modules over an operad [Kapranov and Manin 2001, Theorem 1.7.3].

Appendix B. First terms of some expansions

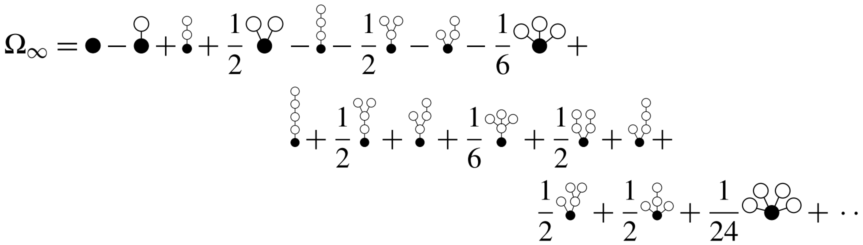


(B.1)

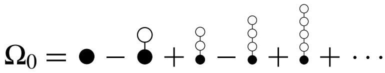
For $n \geq 1$, let Φ_n be the n-th cyclotomic polynomial.



(B.2)



(B.3)



(B.4)

References

[Agračev and Gamkrelidze 1980] A. A. Agračev and R. V. Gamkrelidze, "Chronological algebras and nonstationary vector fields", *Itogi Nauki Tekh. Ser. Probl. Geom.* **11** (1980), 135–176. In Russian; translated in *J. Sov. Math.* **17** (1981), 1650–1675. MR 82b:58075 Zbl 0473.58021

[Calaque et al. 2008] D. Calaque, K. Ebrahimi-Fard, and D. Manchon, "Two Hopf algebras of trees interacting", preprint, 2008. arXiv 0806.2238

[Carlitz 1948] L. Carlitz, "q-Bernoulli numbers and polynomials", *Duke Math. J.* **15** (1948), 987–1000. MR 10,283g Zbl 0032.00304

[Carlitz 1954] L. Carlitz, "q-Bernoulli and Eulerian numbers", *Trans. Amer. Math. Soc.* **76** (1954), 332–350. MR 15,686a Zbl 0058.01204

[Carlitz 1958] L. Carlitz, "Expansions of q-Bernoulli numbers", *Duke Math. J.* **25** (1958), 355–364. MR 20 #2480 Zbl 0102.03201

[Chapoton 2002a] F. Chapoton, "Rooted trees and an exponential-like series", preprint, 2002. arXiv 0209104

[Chapoton 2002b] F. Chapoton, "Un théorème de Cartier–Milnor–Moore–Quillen pour les bigèbres dendriformes et les algèbres braces", *J. Pure Appl. Algebra* **168**:1 (2002), 1–18. MR 2002k:18013 Zbl 0994.18006

[Chapoton 2007] F. Chapoton, "The anticyclic operad of moulds", *Int. Math. Res. Not.* **2007** (2007), Art. ID rnm078, 36. MR 2008k:18009

[Chapoton 2007/08] F. Chapoton, "Operads and algebraic combinatorics of trees", *Sém. Lothar. Combin.* **58** (2007/08), Art. B58c, 27. MR 2461995

[Chapoton and Livernet 2001] F. Chapoton and M. Livernet, "Pre-Lie algebras and the rooted trees operad", *Internat. Math. Res. Notices* **8** (2001), 395–408. MR 2002e:17003 Zbl 1053.17001

[Chapoton and Livernet 2007] F. Chapoton and M. Livernet, "Relating two Hopf algebras built from an operad", *Int. Math. Res. Not.* **2007** (2007), Art. ID rnm131, 27. MR 2377015

[Duchamp et al. 1994] G. Duchamp, D. Krob, B. Leclerc, and J.-Y. Thibon, "Déformations de projecteurs de Lie", *C. R. Acad. Sci. Paris Sér. I Math.* **319**:9 (1994), 909–914. MR 96a:05141 Zbl 0810.05067

[Duchamp et al. 1997] G. Duchamp, A. Klyachko, D. Krob, and J.-Y. Thibon, "Noncommutative symmetric functions, III: Deformations of Cauchy and convolution algebras", *Discrete Math. Theor. Comput. Sci.* **1**:1 (1997), 159–216. MR 99m:05161 Zbl 0930.05097

[Ebrahimi-Fard and Manchon 2009] K. Ebrahimi-Fard and D. Manchon, "A Magnus- and Fer-type formula in dendriform algebras", *Foundations Comp. Math.* **9**:3 (2009), 295–316. Zbl 05577220

[Gelfand et al. 1995] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon, "Noncommutative symmetric functions", *Adv. Math.* **112**:2 (1995), 218–348. MR 96e:05175 Zbl 0831.05063

[Guo and Zeng 2006] V. J. W. Guo and J. Zeng, "Some arithmetic properties of the q-Euler numbers and q-Salié numbers", *European J. Combin.* **27**:6 (2006), 884–895. MR 2007b:05023 Zbl 111.05009

[Hivert et al. 2005] F. Hivert, J.-C. Novelli, and J.-Y. Thibon, "The algebra of binary search trees", *Theoret. Comput. Sci.* **339**:1 (2005), 129–165. MR 2006a:05166 Zbl 1072.05052

[Kapranov and Manin 2001] M. Kapranov and Y. Manin, "Modules and Morita theorem for operads", *Amer. J. Math.* **123**:5 (2001), 811–838. MR 2002g:18013 Zbl 1001.18004

[Krob et al. 1997] D. Krob, B. Leclerc, and J.-Y. Thibon, "Noncommutative symmetric functions, II. Transformations of alphabets", *Internat. J. Algebra Comput.* **7**:2 (1997), 181–264. MR 98f:05148 Zbl 0907.05055

[van der Laan 2003] P. van der Laan, *Operads, Hopf algebras and coloured Koszul duality*, Ph.D. thesis, Utrecht University, 2003.

[Loday 2001] J.-L. Loday, "Dialgebras", pp. 7–66 in *Dialgebras and related operads*, Lecture Notes in Math. **1763**, Springer, Berlin, 2001. MR 2002i:17004 Zbl 0999.17002

[Loday and Ronco 1998] J.-L. Loday and M. O. Ronco, "Hopf algebra of the planar binary trees", *Adv. Math.* **139**:2 (1998), 293–309. MR 99m:16063 Zbl 0926.16032

[Murua 2006] A. Murua, "The Hopf algebra of rooted trees, free Lie algebras, and Lie series", *Found. Comput. Math.* **6**:4 (2006), 387–426. MR 2007h:17004 Zbl 1116.17004

[Ronco 2000] M. Ronco, "Primitive elements in a free dendriform algebra", pp. 245–263 in *New trends in Hopf algebra theory* (La Falda, 1999), edited by N. Andruskiewitsch et al., Contemp. Math. **267**, Amer. Math. Soc., Providence, RI, 2000. MR 2001k:16076 Zbl 0974.16035

[Ronco 2001] M. Ronco, "A Milnor–Moore theorem for dendriform Hopf algebras", *C. R. Acad. Sci. Paris Sér. I Math.* **332**:2 (2001), 109–114. MR 2001m:16065 Zbl 0978.16031

[Satoh 1989] J. Satoh, "q-analogue of Riemann's ζ-function and q-Euler numbers", *J. Number Theory* **31**:3 (1989), 346–362. MR 90d:11132 Zbl 0675.12010

[Wright and Zhao 2003] D. Wright and W. Zhao, "D-log and formal flow for analytic isomorphisms of n-space", *Trans. Amer. Math. Soc.* **355**:8 (2003), 3117–3141. MR 2004b:14103 Zbl 1030.58004

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