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Cox rings of degree one del Pezzo surfaces

Damiano Testa, Anthony Várilly-Alvarado and Mauricio Velasco

Let X be a del Pezzo surface of degree one over an algebraically closed field, and let $\text{Cox}(X)$ be its total coordinate ring. We prove the missing case of a conjecture of Batyrev and Popov, which states that $\text{Cox}(X)$ is a quadratic algebra. We use a complex of vector spaces whose homology determines part of the structure of the minimal free $\text{Pic}(X)$ -graded resolution of $\text{Cox}(X)$ over a polynomial ring. We show that sufficiently many Betti numbers of this minimal free resolution vanish to establish the conjecture.

1. Introduction

Let k be an algebraically closed field and let X be a smooth projective integral scheme over k . Assume that the Picard group $\text{Pic}(X)$ is freely generated by the classes of divisors D_0, D_1, \dots, D_r . The *total coordinate ring*, or *Cox ring* of X with respect to this basis is given by

$$\text{Cox}(X) := \bigoplus_{(m_0, \dots, m_r) \in \mathbb{Z}^{r+1}} H^0(X, \mathcal{O}_X(m_0 D_0 + \dots + m_r D_r)),$$

with multiplication induced by the multiplication of functions in $k(X)$. Different choices of bases yield (noncanonically) isomorphic Cox rings.

The first appearance of Cox rings was in the context of toric varieties in [Cox 1995]. In that paper Cox proves that if X is a toric variety then its total coordinate ring is a finitely generated multigraded polynomial ring, and that X can be recovered as a quotient of an open subset of $\text{Spec}(\text{Cox}(X))$ by the action of a torus.

Cox rings are finitely generated k -algebras in several other cases, including del Pezzo surfaces [Batyrev and Popov 2004], rational surfaces with big anticanonical divisor [Testa et al. 2009], blow-ups of \mathbb{P}^n at points lying on a rational normal curve [Castravet and Tevelev 2006] and wonderful varieties [Brion 2007]. All these varieties are examples of *Mori dream spaces* [Hu and Keel 2000], and for this

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class the Cox ring of X captures much of the birational geometry of the variety. For example, the effective and nef cones of X are finitely generated polyhedral cones and there are only finitely many varieties isomorphic to X in codimension one, satisfying certain mild restrictions. To add to the long list of consequences of the landmark paper [Birkar et al. 2008], we note that log Fano varieties are also Mori dream spaces.

Colliot-Thélène and Sansuc introduced universal torsors to aid the study of the Hasse principle and weak approximation on an algebraic variety X over a number field [Colliot-Thélène and Sansuc 1980; 1987]; see also [Colliot-Thélène et al. 1984]. If the Cox ring of X is finitely generated, then the universal torsor of X is an open subset of $\text{Spec}(\text{Cox}(X))$, an affine variety for which explicit presentations have been calculated in many cases [Hassett and Tschinkel 2004; Hassett 2004]. Starting with Salberger [1998], universal torsors have been successfully applied to the problem of counting points of bounded height on many classes of varieties. The expository article [Peyre 2004] has a very readable account of the ideas involved. The explicit descriptions of universal torsors via Cox rings have led to explicit examples of generalized del Pezzo surfaces that satisfy Manin's conjectures for points of bounded height [de la Bretèche and Browning 2007; de la Bretèche et al. 2007].

Batyrev and Popov [2004] systematically study the Cox rings of del Pezzo surfaces X/k . They show that $\text{Cox}(X)$ is a finitely generated k -algebra; moreover, if $\deg(X) \leq 6$ then $\text{Cox}(X)$ is generated by sections whose classes have anticanonical degree one [Batyrev and Popov 2004, Theorem 3.2]. Let \mathcal{G} denote a minimal set of homogeneous generators of $\text{Cox}(X)$, and denote by $k[\mathcal{G}]$ the polynomial ring whose variables are indexed by the elements of \mathcal{G} . As a result, $\text{Cox}(X)$ is a quotient of $k[\mathcal{G}]$:

$$\text{Cox}(X) \cong k[\mathcal{G}]/I_X.$$

Batyrev and Popov provided a conjectural description of the generators of I_X .

Conjecture 1.1 [Batyrev and Popov 2004]. Let X be a del Pezzo surface. The ideal I_X is generated by quadrics.

Quadratic relations have a clear geometric meaning: linear systems associated to degree two nef divisors on X have many more reducible elements than their dimension; thus there are linear dependence relations among degree two monomials in $k[\mathcal{G}]$. All quadratic relations arise in this way (see page 758).

Remark 1.2. Del Pezzo surfaces of degree at least six are toric varieties and hence their Cox rings are polynomial rings. The Cox ring of a del Pezzo surface of degree five is the homogeneous coordinate ring of the Grassmannian $Gr(2, 5)$ [Batyrev and Popov 2004].

Several partial results are known about this conjecture. Stillman, Testa, and Velasco established it for del Pezzo surfaces of degree four, and general del Pezzo surfaces of degree three [Stillman et al. 2007]. The conjecture was proved in [Laface and Velasco 2009] for surfaces of degree at least two. Recently, Serganova and Skorobogatov [2007; 2008] established [Conjecture 1.1](#) up to radical for surfaces of degree at least two, using representation theoretic methods, (see also [Derenthal 2007] for related work). In all cases, the conjecture for degree one surfaces eluded proof. The purpose of this paper is to fill in this gap for surfaces defined over a field of characteristic different from two, as well as for certain general surfaces in characteristic two that we call *sweeping* (see [Definition 4.3](#)).

Theorem 1.3. *Let X be a del Pezzo surface of degree one; if the characteristic of k is two, then assume that X is sweeping. The ideal I_X is generated by quadrics.*

Remark 1.4. Our method of proof is cohomological in nature and relies mainly on the Kawamata–Viehweg vanishing theorem. By [Terakawa 1998] and [Xie 2008], this theorem is independent of the characteristic of k for rational surfaces.

The argument we give works for del Pezzo surfaces of every degree. We include a short proof of the conjecture for del Pezzo surfaces of degree at least two in [Section 10](#). An alternative proof of [Conjecture 1.1](#) for general del Pezzo surfaces in characteristic zero, obtained independently by computational means, appears in a paper by Sturmfels and Xu [2008]. They apply the theory of sagbi bases to construct an initial toric ideal of I_X which is generated by quadrics. Geometrically this corresponds to a degeneration of the universal torsor on X to a suitable toric variety.

In order to prove [Theorem 1.3](#), we modify the approach taken in [Laface and Velasco 2009]. We use a complex of vector spaces whose homology determines part of the structure of the minimal free $\text{Pic}(X)$ -graded resolution of $\text{Cox}(X)$ over a polynomial ring. We show that sufficiently many Betti numbers of this minimal free resolution vanish to establish the theorem. We hope that similar techniques can be applied to obtain presentations of Cox rings of other classes of varieties, for example singular del Pezzo surfaces, blow-ups of \mathbb{P}^n at points lying on the rational normal curve of degree n and $\bar{M}_{0,n}$.

The paper is organized as follows. In [Section 2](#) we define del Pezzo surfaces, fix presentations for their Cox rings and establish notation for the rest of the paper. In [Section 3](#) we study the nef cone of del Pezzo surfaces and prove some geometric results. In [Section 4](#) we analyze low-degree linear systems on del Pezzo surfaces of degree at most two. Our proof of [Theorem 1.3](#) uses [Batyrev and Popov 2004, Proposition 3.4]. However, the proof given by those authors for their proposition has a gap: it applies only to general del Pezzo surfaces of degree one in characteristic not two [Popov 2004]. Thus we prove [Proposition 4.4](#) in order to establish

Theorem 1.3 for all del Pezzo surfaces of degree one in characteristic not two. In [Section 5](#) we review the approach of [[Laface and Velasco 2009](#)] to study the ideal of relations of finitely generated Cox rings, adapting it to the case of degree one del Pezzo surfaces. In [Section 6](#) we define the notions of *capturability* of a divisor and *stopping criterion*, and prove [Theorem 6.7](#), the main ingredients in our proof of the Batyrev–Popov conjecture. In [Section 7](#) we show the capturability of most divisors on del Pezzo surfaces of degree one. We then handle the remaining cases for del Pezzo surfaces of degree one ([Sections 8, 9](#)) and del Pezzo surfaces of higher degree ([Section 10](#)). In [Section 11](#) we finish the proof of the Batyrev–Popov conjecture and give the first multigraded Betti numbers of the Cox rings of del Pezzo surfaces.

2. Notation and background on del Pezzo surfaces

We briefly review some facts about del Pezzo surfaces and establish much of the paper’s notation along the way.

Definition 2.1. A *del Pezzo surface* X is a smooth, projective surface over k whose anticanonical divisor $-K_X$ is ample. The *degree* of X is the integer $(K_X)^2$.

Picard groups and Cox rings. A del Pezzo surface X not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to a blow-up of \mathbb{P}^2 centered at $r \leq 8$ points *in general position*: this means no three points on a line, no six on a conic and no eight on a singular cubic with a singularity at one of the points. Let L be the inverse image of a line in \mathbb{P}^2 and let E_1, \dots, E_r be the exceptional divisors corresponding to the blown-up points. Then (L, E_1, \dots, E_r) is a basis for $\text{Pic}(X)$, and

$$\text{Cox}(X) := \bigoplus_{(m_0, \dots, m_r) \in \mathbb{Z}^{r+1}} H^0(X, \mathcal{O}_X(m_0L + m_1E_1 + \dots + m_rE_r)).$$

If X has degree one, then with the above choice of basis, the classes in $\text{Pic}(X)$ of the 240 exceptional curves are given in [Table 1](#) [[Manin 1986](#)].

# of curves	Picard degree (up to a permutation of E_1, \dots, E_8)
8	E_1
28	$L - E_1 - E_2$
56	$2L - E_1 - E_2 - E_3 - E_4 - E_5$
56	$3L - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7$
56	$4L - 2E_1 - 2E_2 - 2E_3 - E_4 - E_5 - E_6 - E_7 - E_8$
28	$5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - E_7 - E_8$
8	$6L - 3E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7 - 2E_8$

Table 1. Exceptional curves on X .

For a del Pezzo surface X of any degree, we let \mathcal{C} denote the set of exceptional curves on X . If X has degree one, we let

$$\mathcal{G} := \mathcal{C} \cup \{K_1, K_2\},$$

where $K_1, K_2 \in |-K_X|$; if X has degree at least two, then we let

$$\mathcal{G} := \mathcal{C}.$$

In all cases, we write $k[\mathcal{G}]$ for the polynomial ring whose variables are indexed by the elements of \mathcal{G} . By [Batyrev and Popov 2004, Theorem 3.2], we know that $\text{Cox}(X)$ is a finitely generated k -algebra, generated by sections whose divisor classes have anticanonical degree one. For all $G \in \mathcal{G}$, we let $g \in H^0(X, \mathcal{O}_X(G))$ be a nonzero element. We typically use uppercase letters for divisors on X and we denote the generator of $k[\mathcal{G}]$ associated to an exceptional curve E by the corresponding lowercase letter e . If X has degree one, then we denote the generators of $k[\mathcal{G}]$ corresponding to K_1 and K_2 by k_1 and k_2 , respectively. Furthermore, given any exceptional curve $E \in \mathcal{C}$, we let E' denote the unique exceptional curve whose divisor class is $-2K_X - E$, and we denote accordingly the generator corresponding to E' by e' .

There is a surjective morphism

$$k[\mathcal{G}] \rightarrow \text{Cox}(X)$$

that maps g to g . We denote the kernel of this map by I_X .

For any divisor D on X , any integer n and any $\text{Pic}(X)$ -graded ideal $J \subset k[\mathcal{G}]$, denote by J_D the vector space of homogeneous elements of $\text{Pic}(X)$ -degree D , by J_n the vector space of the homogeneous elements of anticanonical degree n and call them, respectively, the degree D part of J and the degree n part of J .

Finally, let J_X be the ideal generated by $(I_X)_2$; since $J_X \subset I_X$, there is a surjection

$$(k[\mathcal{G}]/J_X)_D \rightarrow (k[\mathcal{G}]/I_X)_D.$$

This map plays a role in Section 8.

3. Remarks on the nef cone of del Pezzo surfaces

In this section we collect basic results on del Pezzo surfaces that we use in the paper. The following result is well known [Debarre 2001, page 148, 6.5].

Proposition 3.1. *Let X be a del Pezzo surface of degree $d \leq 7$. A divisor class $C \in \text{Pic}(X)$ is nef (respectively ample) if and only if $C \cdot E \geq 0$ (respectively $C \cdot E > 0$) for all (-1) -curves $E \subset X$.*

Definition 3.2. Let X be a del Pezzo surface. The *minimal ample divisor on X* is the ample divisor A_X defined in the following table:

X	A_X	$(K_X)^2$
\mathbb{P}^2	$-\frac{1}{3}K_X$	9
$\mathbb{P}^1 \times \mathbb{P}^1$	$-\frac{1}{2}K_X$	8
$Bl_p(\mathbb{P}^2)$	$2L - E$	8
$Bl_{p_1, \dots, p_r}(\mathbb{P}^2)$	$-K_X$	$9 - r$

We also extend the list to include $X = \mathbb{P}^1$ and we define the minimal ample divisor on \mathbb{P}^1 to be the class of a point. To simplify the notation, if $b: X \rightarrow Y$ is a morphism, then we may denote $b^*(A_Y)$ by A_Y . Minimal ample divisors allow us to give a geometric description of nef divisors.

Corollary 3.3. *Let X_r be a del Pezzo surface of degree $9 - r$. Let $D \in \text{Pic}(X_r)$ be a nef divisor. Then we can find nonnegative integers n_0, n_1, \dots, n_r and a sequence of morphisms $X_r \rightarrow X_{r-1} \rightarrow \dots \rightarrow X_1 \rightarrow X_0$ such that*

- each morphism is the contraction of a (-1) -curve, except for $X_1 \rightarrow X_0$, which is allowed to be a conic bundle; and
- D can be expressed as

$$D = n_r A_{X_r} + n_{r-1} A_{X_{r-1}} + \dots + n_0 A_{X_0}.$$

Proof. We proceed by induction on r , the cases $r \leq 1$ being immediate. Suppose that $r \geq 2$ and let $n := \min\{L \cdot D \mid L \subset X \text{ is a } (-1)\text{-curve}\}$. By assumption we have $n \geq 0$. Let $\bar{D} := D + nK_{X_r}$; note that \bar{D} is nef by Proposition 3.1. Choose a (-1) -curve $L_0 \subset X$ such that $\bar{D} \cdot L_0 = 0$. Thus \bar{D} is the pull-back of a nef divisor on the del Pezzo surface X_{r-1} obtained by contracting L_0 . The result follows by the inductive hypothesis. □

Observe that the integer n_r in the statement of the corollary is the nef threshold ([Reid 1997, page 126]) of the divisor D with respect to the minimal ample divisor of X_r . The minimal ample divisor is minimal in the sense that for every ample divisor A on X , the divisor $A - A_X$ is nef: this follows from Corollary 3.3.

By [Kollár 1996, Proposition III.3.4] and Corollary 3.3 we deduce that every nef divisor N on a del Pezzo surface is effective and that $|2N|$ is base-point free. Moreover, if N is a nef nonbig divisor, then N is a multiple of a conic bundle.

Let $X \rightarrow Y$ be a morphism with connected fibers and let A be the pull-back to X of the minimal ample divisor on Y . If $Y \simeq \mathbb{P}^1$, then we call A a *conic*. If $Y \simeq \mathbb{P}^2$, then we call A a *twisted cubic*, by analogy with the case of cubic surfaces. Finally, if Y is a del Pezzo surface of degree d , then we call A an *anticanonical divisor of degree d in X* .

We summarize and systematize the previous discussion in the following lemma.

Lemma 3.4. *Let X be a del Pezzo surface. The cone of nef divisors on X is generated by the following divisors:*

- (1) *the conics;*
- (2) *the twisted cubics;*
- (3) *the anticanonical divisors of degree $d \leq 3$ in X .*

In particular, if N is any nonzero nef divisor on X , then we may find $r := 8 - \deg(X)$ distinct exceptional curves E_1, E_2, \dots, E_r on X such that $N - E_1, \dots, N - E_r$ are either nef or the sum of a nef divisor and an exceptional curve, unless $\deg(X) = 1$ and $N = -K_X$.

Proof. The divisors in the list are clearly nef. Conversely, if D is any nef divisor, then either D is a multiple of a conic, and we are done, or a positive multiple of D induces a morphism with connected fibers $X \rightarrow Y$. It is clear that if A is the pull-back to X of the minimal ample divisor on Y , then $D - A$ is nef. By induction on $n := -K_X \cdot D$, we therefore reduce to showing that the anticanonical divisors of degree $d \geq 4$ in X are nonnegative linear combinations of the divisors listed. This is immediate.

For the second statement, note that it suffices to check it for the divisors in the list, and for $-2K_X$ if $\deg(X) = 1$, where the result is easy to verify. \square

The following lemma will be used in the proof of [Lemma 7.7](#).

Lemma 3.5. *Let X be a del Pezzo surface, let $b: X \rightarrow \mathbb{P}^1$ be a conic bundle with fiber class Q and let C be an exceptional curve such that $C \cdot Q = 2$. There are exactly five reducible fibers of b such that C intersects both components. In particular, if $\deg(X) = 1$, then there are two reducible fibers of b such that C is disjoint from one of the two components in each fiber.*

Remark 3.6. If $\deg(X) \geq 4$, then there are no exceptional curves C such that $C \cdot Q = 2$, and hence the lemma applies nontrivially only to the cases $\deg(X) \leq 3$.

Proof. Let S and T be the components of a reducible fiber of b . Since $C \cdot Q = 2$, there are only two possibilities for the intersection numbers $C \cdot S$ and $C \cdot T$: they are either both equal to one, or one of them is zero and the other is two. Suppose that there are k reducible fibers of b such that the curve E intersects both components. Thus contracting all the components in fibers of b disjoint from E and one component in each of the remaining reducible fibers, we obtain a relatively minimal ruled surface $b': X' \rightarrow \mathbb{P}^1$, together with a smooth rational curve C' in X' , the image of C , having anticanonical degree $k + 1$, square $k - 1$ and intersection number two with a fiber of b' . Since X' is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or $Bl_p(\mathbb{P}^2)$, a direct calculation shows that this is only possible if $X' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $k = 5$.

The last statement follows from the fact that any conic bundle structure on X contains exactly $8 - \deg(X)$ reducible fibers. \square

4. Low-degree linear systems

The lemmas in this section determine subsets of monomials that span $\text{Cox}(X)_D$ when $(\deg(X), D) \in \{(2, -K_X), (1, -2K_X)\}$.

Lemma 4.1. *Let X be a del Pezzo surface of degree two. The linear system $|-K_X|$ is spanned by any five of its reducible elements.*

Proof. The linear system $|-K_X|$ defines a separable morphism of degree two $\varphi: X \rightarrow \mathbb{P}^2$ such that every pair of exceptional curves E, E' on X with $E + E' = -K_X$ maps under φ to the same line in \mathbb{P}^2 . To prove the result it suffices to show that any five of these lines have no base-point. Suppose that $p \in \mathbb{P}^2$ is contained in $k_p \geq 4$ such lines; let $q \in X$ be a point in the inverse image of p . By construction, the point q is contained in k_p exceptional curves. Let E be an exceptional curve containing q and let E' be the exceptional curve such that $E + E' = -K_X$. The curve E' is disjoint from the exceptional curves F such that $F \cdot E = 1$, since $F \cdot E' = 0$. Thus contracting E' we obtain a del Pezzo surface Y of degree three with a point contained in at least $k_p - 1$ exceptional curves. The anticanonical linear system embeds Y as a smooth cubic surface in \mathbb{P}^3 , and exceptional curves through a point $y \in Y$ are lines contained in the tangent plane to Y at y . This implies that $k_p - 1 \leq 3$, and the result follows. □

To study the case of degree one del Pezzo surfaces we begin with a lemma.

Lemma 4.2. *Let $R \subset \mathbb{P}^3$ be a curve, let $r \in R$ be a smooth point and let $p \in \mathbb{P}^3$ be a closed point different from r . Let H be a plane through p intersecting R at r with order of contact $m \geq 2$. Let $\pi_R: R \rightarrow \mathbb{P}^2$ be the projection away from p , let $\bar{R} = \pi_R(R)$, and let $\gamma: R \dashrightarrow (\mathbb{P}^2)^\vee$ be the composition of π_R and the Gauss map of \bar{R} , i.e., the map sending a general point $q \in R$ to the tangent line to \bar{R} at $\pi_R(q)$. Then the map γ is defined at r and one of the following occurs:*

- (1) *the tangent line to R at r contains p , or*
- (2) *the length of the localization at r of the fiber of γ at $\gamma(r)$ equals $m - 1$ if $\text{char}(k) \nmid m$ and it equals m if $\text{char}(k) \mid m$.*

Proof. The rational map γ is defined at r since $r \in R$ is a smooth point and the range of γ is projective. Choose homogeneous coordinates X_0, X_1, X_2, X_3 on \mathbb{P}^3 so that $p = [1, 0, 0, 1]$ and $r = [0, 0, 0, 1]$. In these coordinates H is defined by the linear form $AX_1 + BX_2$, for some $A, B \in k$. In the affine coordinates $(x_0 = X_0/X_3, x_1 = X_1/X_3, x_2 = X_2/X_3)$ the curve R is defined near the origin r by a complete intersection (f_1, f_2) and thus its embedded tangent space is the kernel of the matrix $DF(r) = ((\partial f_i / \partial x_j)(r))$ with $i \in \{1, 2\}$ and $j \in \{0, 1, 2\}$. Therefore either the first column of $DF(r)$ vanishes—that is, $\partial f_1 / \partial x_0$ and $\partial f_2 / \partial x_0$ vanish at r —and the tangent line to R through r contains p , or some 2×2 minor of

$DF(r)$ containing the first column is nonzero, since otherwise all 2×2 minors of $DF(r)$ vanish, contradicting the assumption that R is nonsingular at r . In the latter case, the completion of the local ring of R at r is isomorphic to $k[[t]]$ via a parametrization in formal power series of the form $(x_0(t), x_1(t), t)$. Since H has order of contact m at r , the power series $Ax_1(t) + Bt$ vanishes to order m so $A \neq 0$ and we have

$$x_1(t) = -\frac{B}{A}t + c_m t^m + (\text{higher order terms})$$

with $c_m \neq 0$. The equation of the tangent line to \bar{R} at $\pi_R(x_0(t), x_1(t), t, 1)$ is

$$-(X_1 - x_1(t)) + (X_2 - t)x'_1(t) = 0,$$

so the morphism γ is given by

$$(x_0(t), x_1(t), t, 1) \mapsto [-1, x'_1(t), x_1(t) - tx'_1(t)].$$

The localization at r of the fiber of γ at $\gamma(r)$ is thus given by the ideal

$$I = (Ax'_1(t) + B, x_1(t) - tx'_1(t)).$$

Since we have

$$Ax'_1(t) + B = Ac_m m t^{m-1} + (\text{higher order terms}),$$

$$x_1(t) - tx'_1(t) = c_m(1 - m)t^m + (\text{higher order terms}),$$

it follows that either $\text{char}(k) \nmid m$ and $I = (t^{m-1})$, or $\text{char}(k) \mid m$ and $I = (t^m)$ as we wanted to show. □

Let X be a del Pezzo surface of degree one. The linear system $|-2K_X|$ is base-point free and the image of the associated morphism $\kappa: X \rightarrow |-2K_X|^* \simeq \mathbb{P}^3$ is a cone W over a smooth conic; the morphism κ is a separable double cover of W branched at the cone vertex $w \in W$ and along a divisor $R \subset W$. We call *tritangent planes of X* the elements of $|-2K_X|$ supported on exceptional curves, and think of them as planes in $|-2K_X|^* = \mathbb{P}^3$. There are 120 such planes and they do not contain the vertex w of the cone, since the planes containing w correspond to elements of $|-2K_X|$ supported on the sum of two effective anticanonical divisors (see [Kollár 1996, Section III.3] and [Manin 1986, Chapter IV] for details).

Definition 4.3. A del Pezzo surface X of degree one over a field k is *sweeping* if any 119 tritangent planes of X span $|-2K_X|$.

Our goal in the remainder of this section is to show that every del Pezzo surface of degree one in characteristic not two is sweeping; if the characteristic is two, then we only show that a general del Pezzo surface of degree one is sweeping. In the following sections we prove that if X is a sweeping del Pezzo surface of degree one, then [Conjecture 1.1](#) holds for X .

From here through page 740 we assume that k does not have characteristic 2. Then the curve R is smooth and it is the complete intersection of W with a cubic surface. Hence R is a canonical curve of genus four and degree six admitting a unique morphism of degree three to \mathbb{P}^1 (up to changes of coordinates on \mathbb{P}^1) obtained by projecting away from w . The tritangent planes to R are planes in \mathbb{P}^3 not containing w whose intersection with R is twice an effective divisor.

Proposition 4.4. *If X is a del Pezzo surface of degree one over a field k of characteristic different from two, then X is sweeping. More precisely, any 113 of the tritangent planes of X span $|-2K_X|$.*

Proof. Let $p \in \mathbb{P}^3$ be a closed point. We say that a tritangent plane H containing p is p -regular (resp. p -singular) if p does not belong to (resp. p belongs to) a tangent line to R at some point in $H \cap R$; denote by k_p the number of tritangent planes containing p and by k_p^r (resp. k_p^s) the number of p -regular (resp. p -singular) tritangent planes. Suppose that the point p belongs to R ; then there are no p -regular tritangent planes, since every tritangent plane containing p contains also the tangent line to R at p . Thus if q is any point on the tangent line to R at p , we have $k_p \leq k_q^s \leq k_q$, so it suffices to prove the proposition assuming that $p \in \mathbb{P}^3$ is a closed point not in R . Therefore let $p \in \mathbb{P}^3 \setminus R$ be a closed point; projecting away from p we obtain a morphism $\pi_R: R \rightarrow \mathbb{P}^2$; let $\bar{R} = \pi_R(R)$. Let $\bar{\gamma}: \bar{R} \dashrightarrow (\mathbb{P}^2)^\vee$ be the Gauss map, let $\gamma: R \rightarrow (\mathbb{P}^2)^\vee$ be the unique morphism that extends the composition $\bar{\gamma} \circ (\pi_R)$, and let $\check{R} = \gamma(R)$. The morphism γ factors as $R \xrightarrow{g} N \xrightarrow{\nu} \check{R}$, where $\nu: N \rightarrow \check{R}$ is the normalization of \check{R} . We summarize these definitions in a commutative diagram:

$$\begin{array}{ccc}
 \bar{R} & \xrightarrow{\bar{\gamma}} & \check{R} \\
 \pi_R \uparrow & \nearrow \gamma & \uparrow \nu \\
 R & \xrightarrow{g} & N
 \end{array} \tag{4-1}$$

The argument consists in identifying the contributions of the tritangent planes through p in terms of combinatorial invariants of Diagram 1. More precisely, the p -singular tritangent planes correspond (at most 7 : 1) to the points of \bar{R} where the morphism π_R is ramified; the p -regular tritangent planes correspond to points of \check{R} where (the separable part of) γ is ramified.

We now estimate the number k_p^s of p -singular tritangent planes.

Lemma 4.5. *As above, let R be the ramification divisor of κ .*

- (1) *If $\ell \subset \mathbb{P}^3$ is a tangent line to R , then there are at most 7 tritangent planes containing ℓ ;*
- (2) *Let $p \in \mathbb{P}^3$ be a closed point not in R . There are at most 42 p -singular tritangent planes.*

Proof of Lemma 4.5. (1) Let $r \in R$ be a point such that ℓ is tangent to R at r and let $\alpha: R \rightarrow \mathbb{P}^1$ be the morphism obtained by considering the pencil of planes in \mathbb{P}^3 containing ℓ . The morphism α has degree at most four since every plane containing ℓ is tangent to R at r and hence intersects R at r with multiplicity at least two. If α were not separable, then the characteristic of k would be three (recall that we assume $\text{char}(k) \neq 2$), α would be the Frobenius morphism, and the curve R would be rational, which is not the case. We deduce that α is separable; if α has degree three, then ℓ contains the vertex w of the cone W and in this case no tritangent plane contains ℓ . Moreover, α cannot have degree two since R is not hyperelliptic. Thus α is separable and we reduce to the case in which the degree of α is four. If H is a tritangent plane to R containing ℓ , then we have $R \cap H = 2((r) + (p) + (q))$, and hence the contribution of H to the ramification divisor of α is at least two. From the Hurwitz formula we deduce that the ramification divisor has degree 14 and r is contained in at most $14/2 = 7$ tritangent planes.

(2) First, we reduce to the case that the morphism π_R has degree at most two (and in particular it is separable). Indeed, the degree of π_R divides six, and since R is not contained in a plane, the image \bar{R} cannot be a line; therefore it suffices to analyze the cases in which the degree of π_R equals three. If π_R were not separable of degree three, then it would be purely inseparable and \bar{R} would have degree two but geometric genus four, which is not possible. If π_R is separable of degree three, then p is the vertex w of the cone W and hence $k_p^s = 0$, since there are no tritangent planes through w . This completes the reduction. Note that if the degree of π_R equals two, then the image of π_R is a smooth plane cubic, since R is not hyperelliptic.

Second, by [Hartshorne 1977, Proposition IV.3.4], the morphism π_R ramifies at every tangent line to R through p .

Finally, the image of a ramification point of π_R is either a singular point of \bar{R} or a ramification point of the morphism induced by π_R to the normalization of \bar{R} . Using the Hurwitz formula, the formula for the arithmetic genus of a plane curve, and the equality $\text{deg}(\bar{R}) \cdot \text{deg}(\pi_R) = 6$, we deduce that the number of points in \mathbb{P}^2 corresponding to ramification points of π_R is at most six. Thus there are at most six lines in \mathbb{P}^3 containing p such that every p -singular tritangent plane contains one of these lines. We conclude using Lemma 4.5(1). \square

By Lemma 4.5(2) we have $k_p^s \leq 42$. Thus to prove the proposition, it suffices to show that $k_p^r \leq 70$.

Define the torsion sheaf Δ on \check{R} by the exact sequence

$$0 \rightarrow \mathbb{C}_{\check{R}} \rightarrow v_* \mathbb{C}_N \rightarrow \Delta \rightarrow 0. \tag{4-2}$$

From the long exact cohomology sequence, we deduce that

$$h^1(N, \mathbb{C}_N) + h^0(\Delta) = h^1(\check{R}, \mathbb{C}_{\check{R}}). \tag{4-3}$$

Applying the Hurwitz formula to the composition of π_R with a projection of \bar{R} from a general point of \mathbb{P}^2 , we obtain $\text{deg}(\check{R}) \leq 18$, and hence $h^1(\check{R}, \mathbb{C}_{\check{R}}) \leq 136$.

For each tritangent plane H through p denote by $\ell_H \in \check{R} \subset (\mathbb{P}^2)^\vee$ the point corresponding to the image in \mathbb{P}^2 of H under the projection away from p . We want to estimate the contribution of the point $\ell_H \in \check{R}$ to $h^0(\Delta)$. Note that this contribution is at least $\text{length}(\mathbb{C}_{\nu^{-1}(\ell_H)}) - 1$, and that

$$\text{length}(\mathbb{C}_{\nu^{-1}(\ell_H)}) \cdot \text{deg}(g) = \text{length}(\mathbb{C}_{\gamma^{-1}(\ell_H)}). \tag{4-4}$$

Factor g as the composition of a power of the Frobenius morphism, followed by a separable morphism g_s . Let d_i be the inseparable degree of g and $d_s = \text{deg}(g_s)$ be the separable degree of g , let also \check{d} denote the degree of \check{R} ; recall that we have $d_i d_s \check{d} \leq 18$.

Suppose first that $d_i d_s = 1$. By Lemma 4.2 and (4-4) each p -regular tritangent plane contributes at least 2 to $h^0(\Delta)$; hence we have $h^0(\Delta) \geq 2k_p^r$, and we conclude that $k_p^r \leq 68$.

Suppose now that $d_i d_s > 1$ and hence $\check{d} \leq 9$. By the Hurwitz formula, the degree of the ramification divisor of the morphism g_s is at most $6 + 2 \text{deg}(g_s) \leq 6 + 36 = 42$. The curve \check{R} has degree at most nine, and hence it has at most 28 singular points.

Let $U \subset N \times \mathbb{P}^3$ be the universal family of planes through p tangent to R and let $Z := (U \cap N \times R)$. Thus Z defines a family of closed subschemes of $R \subset \mathbb{P}^3$ of dimension zero and degree six. The geometric generic fiber of $Z \rightarrow N$ therefore determines a partition of 6 that we call the generic splitting type. Alternatively, if H is a plane through p tangent to R , then $H \cap R$ determines an effective divisor of degree six on R ; if H is general with the required properties, then the multiplicities of the geometric points of $H \cap R$ are the generic splitting type.

By [Hefez and Kleiman 1985, Theorem 3.5] and [Kaji 1992, page 529], the generic splitting type of γ is of the form $(a, \dots, a, 1, \dots, 1)$, where either $a = 2$ and the morphism is separable or a is the inseparable degree of the rational map $\bar{R} \rightarrow \check{R}$. The possibilities are

$$(5, 1), (4, 1, 1), (3, 3), (3, 1, 1, 1), (2, 2, 2), (2, 2, 1, 1), (2, 1, 1, 1, 1).$$

If the generic splitting type is $(2, 2, 2)$, then the point p is the cone vertex w and in this case $k_p = 0$. The partition $(2, 1, 1, 1, 1)$ corresponds to the case $d_i d_s = 1$ and we already analyzed it. The partitions $(5, 1)$ and $(4, 1, 1)$ can also be excluded, since they imply that R is birational to a plane curve of degree at most $\lfloor 18/4 \rfloor = 4$ and hence cannot have arithmetic genus four.

Examining the remaining generic splitting types and using Lemma 4.2, we deduce that the p -regular tritangent planes correspond to ramification points of g_s . In particular, at most $42 + 28 = 70$ such points exist, and the proposition follows. \square

Remark 4.6. In characteristic two the curve R above has degree 3, contains the cone vertex w , and is not necessarily smooth [Cragnolini and Oliverio 2000]. While we are not able to prove that every del Pezzo surface of degree one in characteristic two is sweeping, we prove it for a general such surface. We write an explicit example of a del Pezzo surface S of degree two defined over $\overline{\mathbb{F}}_2$ and five tritangent planes to S no four of which share a common point. Thus S is sweeping; since the property of being sweeping is open, we conclude that a general del Pezzo surface of degree one in characteristic two is sweeping. We found this example using Magma [Bosma et al. 1997].

Let $\mathbb{P} := \mathbb{P}(1, 1, 2, 3)$ be the weighted projective space over $\overline{\mathbb{F}}_2$ with coordinates x, y, z, w and respective weights 1, 1, 2, 3. Let $S \subset \mathbb{P}$ be the surface defined by

$$S: \{w^2 + z^3 + wxz + wy^3 + x^6 = 0\}.$$

It is immediate to check that S is smooth. Let $t \in \overline{\mathbb{F}}_2$ be an element satisfying $t^2 + t + 1 = 0$; any four of the homogeneous forms

$$z + x^2, \quad z + y^2, \quad z + tx^2, \quad z + tx^2 + t^2xy + y^2, \quad z + tx^2 + xy + t^2y^2$$

are linearly independent. Moreover, they represent tritangent planes since we have

$$F(x, y, x^2, w) = w(w + x^3 + y^3),$$

$$F(x, y, y^2, w) = (w + (x^3 + txy^2 + x^2y + ty^3))(w + (x^3 + t^2xy^2 + x^2y + t^2y^3)),$$

$$F(x, y, tx^2, w) = w(w + tx^3 + y^3),$$

$$F(x, y, tx^2 + t^2xy + y^2, w) = (w + y(x^2 + xy + ty^2))(w + (tx^3 + t^2y^3 + tx^2y)),$$

$$F(x, y, tx^2 + xy + t^2y^2, w) = (w + ty(x^2 + txy + y^2))(w + t(x^3 + tx^2y + ty^3)),$$

and we deduce that S is sweeping.

The proof of Corollary 4.7 below given in [Batyrev and Popov 2004, Proposition 3.4] contains a gap pointed out in [Popov 2004]: the original argument implicitly assumes that the characteristic of the base field is not two, and reduces the proof to the fact that $H^0(X, \mathcal{O}_X(-2K_X))$ is spanned by the sections supported on exceptional curves. Proposition 4.4 fixes this gap.

Corollary 4.7. *Let X be a del Pezzo surface of degree one in characteristic not two. If $D \neq -K_X$ is an effective divisor, then $H^0(X, \mathcal{O}_X(D))$ is spanned by global sections supported on exceptional curves. \square*

Remark 4.8. A general del Pezzo surface of degree one in characteristic two is sweeping; therefore Corollary 4.7 also holds for such surfaces.

5. Betti numbers and the Batyrev–Popov conjecture

We review the approach of [Laface and Velasco 2009] to study the ideal of relations of Cox rings. As usual, let X be a del Pezzo surface, and let $R = k[\mathcal{G}]$. Throughout this section D denotes a divisor on X . Since R is positively graded (by anticanonical degree), every finitely generated $\text{Pic}(X)$ -graded R -module has a unique minimal $\text{Pic}(X)$ -graded free resolution. For the module $\text{Cox}(X) = R/I_X$ this resolution is of the form

$$\dots \rightarrow \bigoplus_{D \in \text{Pic}(X)} R(-D)^{b_{2,D}} \rightarrow \bigoplus_{D \in \text{Pic}(X)} R(-D)^{b_{1,D}} \rightarrow R \rightarrow 0,$$

where the rightmost nonzero map is given by a row matrix whose nonzero entries are a set of minimal generators of the ideal I_X . Since the differential of the resolution has degree 0, it follows that I_X has exactly $b_{1,D}(\text{Cox}(X))$ minimal generators of Picard degree D .

Let \mathbb{K} be the Koszul complex on \mathcal{G} . Consider the degree D part of the complex $\text{Cox}(X) \otimes_R \mathbb{K}$. Then

$$\begin{aligned} H_i((\text{Cox}(X) \otimes_R \mathbb{K})_D) &= (H_i(\text{Cox}(X) \otimes_R \mathbb{K}))_D \\ &= (\text{Tor}_i^R(\text{Cox}(X), k))_D = k^{b_{i,D}(\text{Cox}(X))}, \end{aligned}$$

where the equalities on the last line follow since $\text{Tor}_i^R(A, B)$ is symmetric in A and B and the Koszul complex is the minimal free resolution of k over R . Hence we have the equality

$$\dim_k(H_i((\text{Cox}(X) \otimes_R \mathbb{K})_D)) = b_{i,D}(\text{Cox}(X)).$$

Thus, Conjecture 1.1 is equivalent to the statement that $b_{1,D}(\text{Cox}(X)) = 0$ for all $D \in \text{Pic}(X)$ with $-K_X \cdot D \geq 3$. This is the form of the conjecture that we prove.

Let X be a del Pezzo surface of degree one. To compute the Betti numbers $b_{1,D}(\text{Cox}(X))$, denote by C_1, \dots, C_{240} the exceptional curves of X , and let

$$C_{240+i} := K_i$$

for $i \in \{1, 2\}$. With this notation, the part of the complex relevant to our task is

$$\bigoplus_{1 \leq i < j \leq 242} H^0(X, \mathcal{O}_X(D - C_i - C_j)) \xrightarrow{d_2} \bigoplus_{i=1}^{242} H^0(X, \mathcal{O}_X(D - C_i)) \xrightarrow{d_1} H^0(X, \mathcal{O}_X(D)),$$

where d_2 sends $\sigma_{ij} \in H^0(X, \mathcal{O}_X(D - C_i - C_j))$ to

$$(0, \dots, 0, \sigma_{ij}c_j, 0, \dots, 0, -\sigma_{ij}c_i, 0, \dots, 0)$$

and d_1 sends $\sigma_i \in H^0(X, \mathcal{O}_X(D - C_i))$ to $\sigma_i c_i$.

A *cycle* is an element of $\ker d_1$; a *boundary* is an element of $\operatorname{im} d_2$. The *support* of a cycle $\sigma = (\dots, \sigma_i, \dots)$ is

$$\|\sigma\| = \{C_i : \sigma_i \neq 0\}$$

and the size of the support is the cardinality of $\|\sigma\|$, denoted by $|\sigma|$.

Strategy. In order to show that $b_{1,D}(\operatorname{Cox}(X)) = 0$ whenever $-K_X \cdot D \geq 3$, we split the divisor classes D as follows:

- (1) D is ample and has anticanonical degree at least four (Section 7);
- (2) D is ample and has anticanonical degree three (Section 8);
- (3) D is not ample (Section 9).

The case when D is not ample follows by induction on the degree of the del Pezzo surface, and the ample divisors of anticanonical degree three are dealt with algebraically. In order to show that $b_{1,D}(\operatorname{Cox}(X)) = 0$ whenever D is ample and $-K_X \cdot D \geq 4$ we show that every cycle is a boundary by arguing as follows.

- We describe constructions through which a divisor may be removed from the support of a cycle using boundaries, at the cost of possibly introducing new divisors in the support of the cycle.
- We apply these constructions to all cycles in a systematic way to reduce their support to at most two elements.

We conclude using the following lemma.

Lemma 5.1. *Any cycle σ with $|\sigma| \leq 2$ is a boundary.*

Proof. We deduce from [Elizondo et al. 2004, Theorem 1.1] that the Cox rings of del Pezzo surfaces are unique factorization domains. The statement follows immediately. \square

6. Capturability and stopping criteria

We introduce the following terminology to aide us in our search of ways to reduce the support of a cycle. We keep the notation of the previous section.

Definition 6.1. A *capture move* for a divisor D is a pair (\mathcal{S}, C) , where $\mathcal{S} \subset \mathcal{G}$, $C \in \mathcal{G} \setminus \mathcal{S}$ and the map

$$\bigoplus_{S \in \mathcal{S}} H^0(X, \mathbb{C}_X(D - S - C)) \otimes H^0(X, \mathbb{C}_X(S)) \longrightarrow H^0(X, \mathbb{C}_X(D - C))$$

induced by tensor product of sections is surjective. We say that C is the *captured curve*, and that C is *capturable for D by \mathcal{S}* .

Let T be a set; for a direct sum $\bigoplus_{t \in T} V_t$ of vector spaces indexed by T and $T' \subset T$, we write $\sum_{t \in T'} a_t \hat{e}_t$ for the element $(b_t)_{t \in T}$ of $\bigoplus_{t \in T} V_t$ where $b_t = a_t$ if $t \in T'$ and $b_t = 0$ otherwise. As usual, if $S \in \mathcal{G}$, we denote by s the chosen global section of $\mathbb{O}_X(S)$. Let (\mathcal{S}, C) be a capture move for D , let $\sigma_C \in H^0(X, \mathbb{O}_X(D - C))$, and let $\sigma \in \bigoplus H^0(X, \mathbb{O}_X(D - C_i))$ be the element corresponding to σ_C . Then we have

$$\sigma_C = \sum_{S \in \mathcal{S}} p_S s, \quad p_S \in H^0(X, \mathbb{O}_X(D - S - C)),$$

and thus we obtain

$$\sigma = \sum_{S \in \mathcal{S}} p_S c \hat{e}_S + d_2 \left(\sum_{S \in \mathcal{S}} \varepsilon_{CS} p_S \hat{e}_{C+S} \right) \in \bigoplus_i H^0(X, \mathbb{O}_X(D - C_i)) \quad (6-1)$$

where $\varepsilon_{CS} \in \{\pm 1\}$.

Hence, if σ is a cycle and there is a capture move (\mathcal{S}, C) , then we can modify σ by a boundary so that $C \notin \|\sigma\|$. In this sense we have *captured* the curve C from the support of σ . Observe, however, that this modification adds a subset of \mathcal{S} to $\|\sigma\|$, so a priori the size of the support may not have decreased. We need to find and apply capture moves in an organized way to ensure that we are genuinely decreasing the size of the support of a cycle.

Finally note that if (\mathcal{S}, C) is a capture move and $\mathcal{S}' \supset \mathcal{S}$ with $C \notin \mathcal{S}'$, then (\mathcal{S}', C) is also a capture move. We frequently use this observation without explicitly mentioning it.

Lemma 6.2. *Let A, B and C be distinct exceptional curves on X , with A and B disjoint, and let D be any divisor. If $H^1(X, \mathbb{O}_X(D - A - B - C)) = 0$, then $(\{A, B\}, C)$ is a capture move for D . In particular, if $(-K_X + D - A - B - C)$ is nef, and if either it is big or it has anticanonical degree two, then $(\{A, B\}, C)$ is a capture move for D .*

Proof. Since A and B are disjoint, there is an exact sequence of sheaves

$$0 \rightarrow \mathbb{O}_X(-A - B) \rightarrow \mathbb{O}_X(-A) \oplus \mathbb{O}_X(-B) \rightarrow \mathcal{O}_X \rightarrow 0.$$

Tensoring with $\mathbb{O}_X(D - C)$ we obtain the short exact sequence

$$0 \rightarrow \mathbb{O}_X(D - A - B - C) \rightarrow \mathbb{O}_X(D - A - C) \oplus \mathbb{O}_X(D - B - C) \rightarrow \mathcal{O}_X(D - C) \rightarrow 0$$

and the desired surjectivity follows from the associated long exact sequence in cohomology and the assumption that

$$H^1(X, \mathbb{O}_X(D - A - B - C)) = 0.$$

The last statement follows from the Kawamata–Viehweg vanishing theorem when $-K_X + D - A - B - C$ is nef and big. Otherwise $-K_X + D - A - B - C = Q$ for

some conic Q and the statement follows from the exact sequence

$$0 \rightarrow \mathbb{O}_X(K_X) \rightarrow \mathbb{O}_X(K_X + Q) \rightarrow \mathbb{O}_Q(K_X + Q) \rightarrow 0,$$

by considering the associated long exact sequence in cohomology. □

Lemma 6.3. *If X is sweeping, then the pairs (\mathcal{C}, K_1) and (\mathcal{C}, K_2) are capture moves for all $D \neq -2K_X$.*

Proof. Since $D + K_X \neq -K_X$, it follows from [Corollary 4.7](#) and [Remark 4.8](#) that $H^0(X, \mathbb{O}_X(D + K_X))$ is spanned by global sections supported on exceptional curves. Thus the following map is surjective

$$\bigoplus_{S \in \mathcal{C}} H^0(X, \mathbb{O}_X(D - S + K_X)) \otimes H^0(X, \mathbb{O}_X(S)) \longrightarrow H^0(X, \mathbb{O}_X(D + K_X))$$

and (\mathcal{C}, K_1) and (\mathcal{C}, K_2) are capture moves for D . □

Definition 6.4. A *stopping criterion* for a divisor D is a set $\mathcal{S} \subset \mathcal{G}$ such that the following complex is exact:

$$\bigoplus_{\substack{C_i, C_j \in \mathcal{S} \\ i < j}} H^0(X, \mathbb{O}_X(D - C_i - C_j)) \xrightarrow{d_2} \bigoplus_{C_i \in \mathcal{S}} H^0(X, \mathbb{O}_X(D - C_i)) \xrightarrow{d_1} H^0(X, \mathbb{O}_X(D))$$

Remark 6.5. By [Lemma 5.1](#), a subset of \mathcal{G} of cardinality two is a stopping criterion for any divisor D .

The name stopping criterion is motivated by [Theorem 6.7](#): whenever we can capture all curves in a given degree D by curves contained in a stopping criterion, then there are no relations in degree D . In this case, we may *stop* looking for capture moves.

Definition 6.6. Let $D \in \text{Pic}(X)$ be a divisor class and let $\mathcal{M} := (E_1, \dots, E_n)$ be a sequence of elements of \mathcal{G} ; define $\mathcal{S}_i := \mathcal{G} \setminus \{E_1, E_2, \dots, E_i\}$ for all $i \in \{1, \dots, n\}$. We say that D is *capturable* (by \mathcal{M}) if

- (1) (\mathcal{S}_i, E_i) is a capture move for D for all $i \in \{1, \dots, n\}$,
- (2) \mathcal{S}_n is a stopping criterion for D .

Theorem 6.7. *Fix a divisor class $D \in \text{Pic}(X)$. If D is capturable then*

$$b_{1,D}(\text{Cox}(X)) = 0$$

and hence there are no minimal generators of the ideal I_X in degree D .

Proof. Let $\mathcal{M} := (E_1, \dots, E_n)$ be such that D is capturable by \mathcal{M} . Let $\mathcal{S}_0 = \mathcal{G}$ and, for $i \in \{1, \dots, n\}$ define $\mathcal{S}_i := \mathcal{G} \setminus \{E_1, E_2, \dots, E_i\}$. We want to show that every cycle σ is a boundary. Let $j(\sigma) := \max\{i \in \{0, \dots, n\} \mid \|\sigma\| \subset \mathcal{S}_i\}$. By definition if $\|\sigma\| \subset \mathcal{S}_n$, or equivalently if $j(\sigma) = n$, then σ is a boundary.

Suppose that $j = j(\sigma) < n$; since $(\mathcal{S}_{j+1}, E_{j+1})$ is a capture move for D , we deduce as in (6-1) that σ is homologous to a cycle τ such that $j(\tau) \geq j + 1$. Repeating this argument we conclude that σ is homologous to a cycle with support contained in \mathcal{S}_n and thus that σ is itself a boundary. \square

7. Ample divisors of anticanonical degree at least four

Let X be a del Pezzo surface of degree one. Assume throughout this section that if X is defined over a field of characteristic two, then X is sweeping. In this section we prove that there are no minimal generators of the ideal I_X in all sufficiently large ample degrees. More precisely, if D is an ample divisor such that $-K_X \cdot D \geq 4$ then we show that D is capturable.

The general strategy is the following. First we capture K_1 and K_2 using \mathcal{C} via Lemma 6.3. Next, assume C is an exceptional curve and we want to capture it using $\mathcal{S} \subset \mathcal{C}$. We prove that there exist disjoint exceptional curves S and T in \mathcal{S} such that $H^1(X, \mathcal{O}_X(K_X + L_{ST})) = 0$, where $L_{ST} := -K_X + D - C - S - T$. Then $(\{S, T\}, C)$ is a capture move for D by Lemma 6.2. Often we show that L_{ST} is nef and either big or of anticanonical degree two, omitting any reference to Lemma 6.2. In all cases we capture enough curves to conclude that D is a capturable divisor, using Lemma 5.1 as stopping criterion.

Lemma 7.1. *Let X be a del Pezzo surface of degree one and let $b: X \rightarrow Y$ be a birational morphism. Denote by K_Y the pull-back to X of a canonical divisor on Y and let N be a nef divisor on X . Assume that $\deg(Y) \geq 3$; then the divisor $D = -K_X - K_Y + N$ is capturable.*

Proof. Let \mathcal{S} be the set of exceptional curves contracted by b and note that \mathcal{S} consists of at least two disjoint exceptional curves and that $-K_Y - S - T$ is big and nef for all $S, T \in \mathcal{S}$, $S \neq T$. Let $C \in \mathcal{C} \setminus \mathcal{S}$ be any exceptional curve and let $C' := -2K_X - C$. First we capture the curves C such that $C' \in \mathcal{S}$ using \mathcal{S} : $L_{C'T} := -K_Y + N - T$ is big and nef for any choice of $T \in \mathcal{S} \setminus \{C'\}$.

Second we choose any two distinct $S, T \in \mathcal{S}$, we let $\mathcal{S}' := \{S, T\}$ and we capture all curves $C \in \mathcal{C} \setminus \mathcal{S}'$ such that $C' \notin \mathcal{S}'$: the divisor $L_{ST} = C' + (-K_Y - S - T) + N$ is big and nef provided $C' \cdot (-K_Y - S - T) > 0$; since the only curves orthogonal to $(-K_Y - S - T)$ are the curves in $\mathcal{S} \setminus \mathcal{S}'$, we conclude. \square

Lemma 7.2. *Let X be a del Pezzo surface of degree one and let $b: X \rightarrow Y$ be a birational morphism. Denote by A the pull-back to X of the minimal ample divisor on Y and let N be a nef divisor on X . Assume that $\deg(Y) \geq 8$; then the divisor $D = -K_X + A + N$ is capturable.*

Proof. Note that Y is isomorphic to \mathbb{P}^2 , $Bl_p(\mathbb{P}^2)$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

Case 1: $Y \not\cong \mathbb{P}^1 \times \mathbb{P}^1$. In this case $A = L + Q$, where L is a twisted cubic and Q is either zero or a conic. Thus it suffices to treat the case $D = -K_X + L + N$.

Let \mathcal{S} be the set of curves contracted by L ; thus \mathcal{S} consists of eight disjoint exceptional curves. Note that $\sum_{S \neq T \in \mathcal{S}} L - S - T = 7(-K_X + L)$ has intersection number at least 7 with every exceptional curve on X . Since the intersection number of two exceptional curves on X is at most 3, it follows that every exceptional curve on X intersects positively at least two of the curves $\{L - S - T \mid S \neq T \in \mathcal{S}\}$. For any $C \in \mathcal{C} \setminus \mathcal{S}$ let $S, T \in \mathcal{S}$ be such that $(-2K_X - C) \cdot (L - S - T) > 0$ and such that no (rational) multiple of N equals $(-2K_X - C) + (L - S - T)$. Note that the second condition ensures that N does not contract both $-2K_X - C$ and $(L - S - T)$ when $-2K_X - C + (L - S - T)$ is a conic. We have that $L_{ST} := -2K_X + L - C - S - T + N = (-2K_X - C) + (L - S - T) + N$ is nef and either it has anticanonical degree two or it is also big. Thus (\mathcal{S}, C) is a capture move for all $C \in \mathcal{C} \setminus \mathcal{S}$. Let $S, T \in \mathcal{S}$ be distinct elements and let $\mathcal{S}' := \{S, T\}$. For any $C \in \mathcal{S} \setminus \mathcal{S}'$ the divisor $L_{ST} := (-2K_X - C) + (L - S - T) + N$ is big and nef since $(-2K_X - C) \cdot (L - S - T) = 2$.

Case 2: $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let \mathcal{S} be the set of curves contracted by A ; thus \mathcal{S} consists of seven disjoint exceptional curves. Note that $\sum_{S \neq T \in \mathcal{S}} A - S - T = -6K_X + 9A$ has intersection number at least six with every exceptional curve on X and that the summands are conics. Since the intersection number of an exceptional curve and a conic on X is at most four, it follows that every exceptional curve on X intersects positively at least two of the conics $\{A - S - T \mid S \neq T \in \mathcal{S}\}$. For any $C \in \mathcal{C} \setminus \mathcal{S}$ let $S, T \in \mathcal{S}$ be such that $(-2K_X - C) \cdot (A - S - T) > 0$. We have that $L_{ST} := -2K_X + A - C - S - T + N = (-2K_X - C) + (A - S - T) + N$ is big and nef. Thus (\mathcal{S}, C) is a capture move for all $C \in \mathcal{C} \setminus \mathcal{S}$. Let $S, T \in \mathcal{S}$ be distinct elements and let $\mathcal{S}' := \{S, T\}$. For any $C \in \mathcal{S} \setminus \mathcal{S}'$ the divisor

$$L_{ST} := (-2K_X - C) + (A - S - T) + N$$

is big and nef since $(-2K_X - C) \cdot (A - S - T) = 4$. □

Lemma 7.3. *Let X be a del Pezzo surface of degree one, E an exceptional curve on X and $N \neq 0$ a nef divisor on X such that $N \cdot E = 0$. The divisor $-(n+2)K_X + E + N$ is capturable for $n \geq 0$.*

Proof. By Lemma 3.4, for $i \in \{1, 2\}$, there is an exceptional curve F_i on X such that $F_i \cdot E = 0$ and $N - F_i$ is either nef or the sum of a nef divisor and an exceptional curve, with $F_1 \neq F_2$. Let $\mathcal{S} := \{E, F_1, F_2\}$. For any $C \in \mathcal{C} \setminus \mathcal{S}$, let $F \in \{F_1, F_2\}$ be such that $(-2K_X - C)$ is not a fixed component of $(N - F)$; we have that $L_{EF} = -(n+3)K_X + E - C - E - F + N = -K_X + (-2K_X - C) + (N - F) - nK_X$ is big and nef and we conclude that we can capture all curves in $\mathcal{C} \setminus \mathcal{S}$, using only

the three curves in \mathcal{S} . Finally, by the same argument, we find $(\{E, F_1\}, F_2)$ is a capture move and we are done. \square

Lemma 7.4. *Let X be a del Pezzo surface of degree one, E an exceptional curve on X and N a nef divisor on X . The divisor $D := -(n+3)K_X + E + N$ is capturable for $n \geq 0$.*

Proof. Let E, E_2, \dots, E_8 be eight disjoint exceptional curves on X and let $\tilde{E}_i := -K_X + E - E_i$ for $2 \leq i \leq 8$. Let $\mathcal{S} := \{E, E_2, \dots, E_8, \tilde{E}_2, \dots, \tilde{E}_8\}$. It is clear from Table 1 that every exceptional curve $C \in \mathcal{C} \setminus \mathcal{S}$ intersects positively at least two of the curves in \mathcal{S} ; in particular for every $C \in \mathcal{C} \setminus \mathcal{S}$ there is $S \in \mathcal{S}$ such that $S \cdot E = 0$, $S \cdot C > 0$ and N is not a (rational) multiple of $-4K_X - (S + C)$. Thus $L_{ES} = -K_X + D - E - S - C = (-4K_X - C - S) + N - nK_X$ is nef and it either has anticanonical degree two or it is big. In either case $H^1(X, \mathbb{C}_X(K_X + L_{ES})) = 0$ and we may capture all the curves in $\mathcal{C} \setminus \mathcal{S}$ using the curves in \mathcal{S} .

Note that for all integers $i, j, k, l \in \{2, 3, \dots, 8\}$, $i \neq j, k \neq l$, we have

$$(-4K_X - E_i - \tilde{E}_j) \cdot (-4K_X - E_k - \tilde{E}_l) = 0$$

if and only if $i = k$ and $j = l$. In particular, if N is a multiple of a conic, it is proportional to at most one of the divisors $(-4K_X - E_i - \tilde{E}_j)$; relabeling the indices if necessary, we may assume that N is not proportional to $-4K_X - (E_i + \tilde{E}_j)$ for all $i, j \in \{2, 3, \dots, 8\}$ and $i \neq j, (i, j) \neq (7, 8)$. Thus for all $i \in \{3, 4, \dots, 8\}$ the divisor $L_{E\tilde{E}_2} = (-4K_X - E_i - \tilde{E}_2) + N - nK_X$ is either a conic or big and nef and we may capture E_3, E_4, \dots, E_8 using E, \tilde{E}_2 . Similarly we can capture $\tilde{E}_2, \tilde{E}_3, \dots, \tilde{E}_8$ using E, E_2 and we are done. \square

Lemma 7.5. *Let X be a del Pezzo surface of degree one. Let $D = -(n+4)K_X + N$ where $n \geq 0$ and N is a nef divisor. Let C, S and T be exceptional curves such that $S \cdot T = 0$ and that $C \cdot S, C \cdot T$ are at least 2. Then $(\{S, T\}, C)$ is a capture move for D if either*

- (1) $C \cdot (S + T) = 5$, or
- (2) $C \cdot (S + T) = 4$ and the divisor N is either 0 or not a multiple of the conic $T' + S' + C' + K_X$.

Proof. Consider $L_{ST} := -K_X + D - S - T - C = S' + T' + C' - (n-1)K_X + N$. Since the inequalities $C' \cdot S' \geq 2$ and $C' \cdot T' \geq 2$ hold, the curves S', C' and T' , taken together, are not all pullbacks of exceptional curves on a del Pezzo surface Y of degree at least two. It follows that every exceptional curve $V \notin \{S', T', C'\}$ intersects at least one curve in this set and thus L_{ST} is nef. If $C \cdot (S + T) = 5$ then either $C' + S' = -2K_X$ or $C' + T' = -2K_X$ and L_{ST} is big. If $N = 0$ then L_{ST} has anticanonical degree 2 and capturability follows from Lemma 6.2. Finally,

assume that $C \cdot S = C \cdot T = 2$. In this case $S' + T' + C' + K_X = Q$ is a conic and $L_{ST} = Q - nK_X + N$ is big unless $N = mQ$ for some $m \geq 1$. \square

Lemma 7.6. *Let X be a del Pezzo surface of degree one and let N be a nef divisor on X . The divisor $-(n + 4)K_X + N$ is capturable for $n \geq 0$.*

Proof. Let L be a twisted cubic and let E_1, \dots, E_8 be exceptional curves on X contracted by L . First we capture all curves $C \in \mathcal{C}$ such that $L \cdot C = 3$. From Table 1 it is clear that $C = -K_X + E_i - E_j$ for some $i \neq j$. Set

$$S = E_j, \quad T_1 = 2L - E_i - E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}, \quad T_2 = 2L - E_i - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_4},$$

with $\{i, j\} \cap \{i_1, \dots, i_4\} = \emptyset$, $\{i, j\} \cap \{j_1, \dots, j_4\} = \emptyset$, $\{i_1, \dots, i_4\} \neq \{j_1, \dots, j_4\}$. Note that $L \cdot S, L \cdot T_i \neq 3$, $C \cdot S = C \cdot T_i = 2$ and $S \cdot T_i = 0$ for $i = 1, 2$. Let T be an exceptional curve in $\{T_1, T_2\}$ such that $N \neq m(T' + S' + C' + K_X)$. By Lemma 7.5(2), $(\{S, T\}, C)$ is a capture move.

Next, we capture curves C with $L \cdot C \in \{4, 5\}$. It is clear from Table 1 that for any such curve there exists a pair of distinct exceptional curves $S, T \in \{E_1, \dots, E_8\}$ such that $(\{S, T\}, C)$ is a capture move via Lemma 7.5(2). Similarly we capture curves C with $L \cdot C \in \{1, 2\}$ using some pair $S, T \in \{E'_1, \dots, E'_8\}$. Next we capture E_3, \dots, E_8 using E'_1 and E'_2 via Lemma 7.5(1) and finally, we capture E'_1, \dots, E'_8 using E_1 and E_2 . This concludes the proof. \square

Lemma 7.7. *Let X be a del Pezzo surface of degree one and let Q be a conic on X . The divisor $D = -(n + 1)K_X + (m + 1)Q$ is capturable for $m, n \geq 0$ and $m + n \geq 1$.*

Proof. Suppose first that $n \geq 1$. Let \mathcal{S} be the set of curves contracted by Q and for any $S \in \mathcal{S}$, let \tilde{S} be the unique curve such that $S + \tilde{S} = Q$. If S, T, C are exceptional curves let $B_{ST} = -K_X + C' + Q - S - T$ and note that if B_{ST} is nef and big then so is $L_{ST} := -K_X + D - C - S - T = B_{ST} + mQ - (n - 1)K_X$ because $n \geq 1$. Moreover B_{ST} has anticanonical degree 2 and $h^2(X, \mathcal{O}_X(B_{ST})) = 0$ so by Riemann–Roch B_{ST} is nef if and only if $B_{ST}^2 = 2Q \cdot C' - 2C' \cdot S - 2C' \cdot T \geq 0$.

We first show that the curves C in $\mathcal{C} \setminus \mathcal{S}$ can be captured with the curves in \mathcal{S} . To do so, we split them into cases according to the value of $Q \cdot C'$.

If $Q \cdot C' = 0$ choose $T \in \mathcal{S}$ disjoint from C' . In this case $B_{C'T} = -K_X + \tilde{T}$ is nef and big, and $(\{C', T\}, C)$ is a capture move. If $Q \cdot C' = 1$ choose disjoint curves S and $T \in \mathcal{S}$ which are also disjoint from C' and note that B_{ST} has anticanonical degree 2 and square 2 so it is the pullback to X of the anticanonical divisor of a del Pezzo surface of degree 2 and thus it is nef and big; whence $(\{S, T\}, C)$ is a capture move. If $Q \cdot C' = 2$, by Lemma 3.5 there exist disjoint curves S, T in \mathcal{S} such that $S \cdot C' = T \cdot C' = 0$ so B_{ST} has square 4 and hence $B_{ST} = -2K$ is nef and big. Thus $(\{S, T\}, C)$ is a capture move in this case. Finally if $Q \cdot C' = 3$, then let S and T be disjoint curves in \mathcal{S} such that $S \cdot C' = T \cdot C' = 1$; then B_{ST} has square 2 and it is a nef and big divisor since it is the pullback to X of the anticanonical

divisor of a del Pezzo surface of degree 2. Thus $(\{S, T\}, C)$ is a capture move in this case.

Now choose S, T disjoint exceptional curves in \mathcal{S} . We show that all curves C in $\mathcal{S} \setminus \{S, T\}$ can be captured from $\{S, T\}$. In this case $Q \cdot C' = 4$ and $S \cdot C', T \cdot C' \leq 2$, so $B_{ST}^2 \geq 0$. If B_{ST} is big the statement follows since so is L_{ST} . If $B_{ST}^2 = 0$ there are two cases to consider, either $n = 1$ and $m = 0$ and the statement follows from [Lemma 6.2](#), or at least one of n, m is greater than one. In this case $L_{ST} = B_{ST} - (n - 1)K_X + mQ$ is big because $B_{ST} \cdot Q = 6 \neq 0$. It follows that we can capture every exceptional curve in $\mathcal{S} \setminus \{S, T\}$ from $\{S, T\}$.

Suppose now that $n = 0$ and $m \geq 1$. Note that if $Q \cdot C' \leq 1$ then

$$Q \cdot (D - C) = Q \cdot (K_X + C' + (m + 1)Q) = -2 + Q \cdot C' < 0,$$

so $D - C$ is not an effective divisor. In this case C is captured vacuously and thus we restrict our attention to curves C with $Q \cdot C' \geq 2$. Let \mathcal{S} be the set of exceptional curves contracted by Q and for any $S \in \mathcal{S}$ let \tilde{S} be the unique curve such that $S + \tilde{S} = Q$.

For any $C \in \mathcal{C} \setminus \mathcal{S}$ which satisfies $Q \cdot C' \geq 2$ there exist disjoint curves S, T in \mathcal{S} such that \tilde{S} and \tilde{T} intersect C' . By construction $\tilde{S} + \tilde{T} + C'$ is nef and big and so is $L_{ST} := -K_X + D - S - T - C = \tilde{S} + \tilde{T} + C' + (m - 1)Q$. It follows that we can capture $\mathcal{C} \setminus \mathcal{S}$ with \mathcal{S} .

Finally, fix two disjoint curves $S, T \in \mathcal{S}$ and let C be any curve in $\mathcal{S} \setminus \{S, T\}$. Since $Q \cdot C = 0$ we have $Q \cdot C' = 4$ so C' intersects both \tilde{S} and \tilde{T} . It follows that L_{ST} is nef and big and that \mathcal{S} can be captured from $\{S, T\}$. □

Theorem 7.8. *Let X be a sweeping del Pezzo surface of degree one and let D be an ample divisor. If $-K_X \cdot D \geq 4$, then D is capturable.*

Proof. Write $D = -nK_X + N$, where $n \geq 1$, the divisor N is nef and not ample and $n - K_X \cdot N \geq 4$. If $N = 0$, then $n \geq 4$ and we conclude using [Lemma 7.6](#). If $N \neq 0$, write $N = mA + N'$ where $X \rightarrow Y$ is a morphism with connected fibers, A is the pull-back to X of the minimal ample divisor on Y , N' is the pull-back of a nef divisor on Y and $m \geq 1$. If Y is a surface, then we conclude using [Lemmas 7.1–7.4](#). If $Y \simeq \mathbb{P}^1$, then we conclude using [Lemma 7.7](#). □

8. Ample divisors of anticanonical degree three

The only ample divisors D that seem to elude the strategy of [Section 7](#) are those of anticanonical degree three. These divisors are $-3K_X, -2K_X + E, -K_X + Q$, where E is any exceptional curve and Q is any conic. Recall that there is a surjection

$$(k[\mathcal{G}]/J_X)_D \rightarrow (k[\mathcal{G}]/I_X)_D;$$

see Section 2. We show the dimension of $(k[\mathcal{G}]/J_X)_D$ is at most $h^0(X, \mathcal{O}_X(D)) = \dim(k[\mathcal{G}]/I_X)$ and conclude that $(I_X)_D = (J_X)_D$.

Lemma 8.1. *Let X be a del Pezzo surface of degree one and let A, B, G be exceptional curves with $A + B = -K_X + G$. Then k_1g, k_2g and ab form a basis for $H^0(X, \mathcal{O}_X(-K_X + G))$. In particular, if C, D are exceptional curves with $C + D = -K_X + G$ then $cd = \alpha_1k_1g + \alpha_2k_2g + \alpha_3ab$ and the coefficient α_3 is nonzero.*

Proof. The morphism $X \rightarrow \mathbb{P}^2$ associated to $|-K_X + G|$ contracts G and is ramified along a smooth plane quartic R . The image of G is a point not lying on any bitangent to R since otherwise X would have had a (-2) -curve. The images of K_1 and K_2 in \mathbb{P}^2 are distinct lines through the image of G and therefore the three lines in \mathbb{P}^2 corresponding to $A + B, K_1 + G, K_2 + G$ have no common point and are independent. \square

Lemma 8.2. *Let X be a del Pezzo surface of degree one; then the ideal I_X has no minimal generators in degree $D = -3K_X$.*

Proof. The only ways of writing $-3K_X$ as a sum of three effective divisors are given in the following table.

Monomial	Description
$h_1h_2h_3$	$h_1, h_2, h_3 \in \{k_1, k_2\}$
haa'	$h \in \{k_1, k_2\}, A \in \mathcal{C}$
abc	$A, B, C \in \mathcal{C}$ $A \cdot B = A \cdot C = B \cdot C = 2$

(8-1)

Indeed, let $-3K_X = A + B + C$ be any expression of $-3K_X$ as a sum of three effective divisors. If one among A, B, C is in $|-K_X|$, then we are in one of the first two cases above. If $A, B, C \in \mathcal{C}$, then intersecting with A, B, C successively both sides of the equation $-3K_X = A + B + C$ we obtain the system

$$\begin{cases} A \cdot B + A \cdot C = 4, \\ A \cdot B + B \cdot C = 4, \\ A \cdot C + B \cdot C = 4, \end{cases}$$

whose only solution is $A \cdot B = A \cdot C = B \cdot C = 2$.

By definition $(I_X)_2 = (J_X)_2$; we deduce that the span of the monomials of the first two forms in (8-1) in $k[\mathcal{G}]/J_X$ has dimension at most six, being the image of $H^0(X, \mathcal{O}_X(-K_X)) \otimes H^0(X, \mathcal{O}_X(-2K_X))$ under the multiplication map (note that $\dim H^0(X, \mathcal{O}_X(-K_X)) = 2, \dim H^0(X, \mathcal{O}_X(-2K_X)) = 4$ and the linearly independent elements $k_1 \otimes k_2^2e - k_2 \otimes k_1k_2$ and $k_2 \otimes k_1^2e - k_1 \otimes k_1k_2$ are in the kernel).

Fix any monomial $p = abc$ of the third form in (8-1) and denote the span of p together with the monomials of the first two forms in (8-1) by V . Let $q = def$ be any monomial of the third form in (8-1); we prove that q can be written as the sum of an element of V and an element of J_X .

The result is clear if $c = f$, since in this case we can use the relation coming from degree $A + B$ involving the monomials k_1gc, k_2gc, abc and dec to conclude (see Lemma 8.1). By the same reasoning we reduce to the case in which $\{a, b, c\}$ and $\{d, e, f\}$ are disjoint.

Suppose that $A \cdot D = 2$; then the divisor $G := -3K_X - A - D$ satisfies $G^2 = K_X \cdot G = -1$, and hence G is an exceptional curve. Moreover from the fact that $D \cdot G = 2$, we deduce that G is an exceptional curve on the del Pezzo surface obtained by contracting D' and whose anticanonical divisor pulled-back to X is $E + F$; let $H := E + F - G$ and note that H is also an exceptional curve. In particular we can use a relation coming from degree $E + F$ involving the monomials dk_1d', dk_2d', dgh and def to reduce to the case $\{a, b, c\} \cap \{d, e, f\} \neq \emptyset$.

Finally, if $A \cdot D = A \cdot E = A \cdot F = 1$, then A is a conic in the del Pezzo surface obtained by contracting F' . From Table 1 it follows that there are exceptional curves intersecting any given conic twice; denote the strict transform in X of one such curve of X by H and the exceptional curve $E + F - H$ by J . Thus we can use a relation coming from degree $E + F$ involving the monomials dk_1d', dk_2d', dhj and def to reduce to the case in which $A \cdot D = 2$. □

Lemma 8.3. *Let E be an exceptional curve on X ; then the ideal I_X has no minimal generators in degree $D = -2K_X + E$.*

Proof. The monomials of degree D are of the following forms:

Monomial	Description
ks	$k \in \{k_1, k_2\}$, $s \in k[\mathcal{G}]_{-K_X+E}$
$aa'e$	$A \cdot A' = 3$
abc	$A \cdot E = 1$ $B \cdot E = C \cdot E = 0$ $A \cdot B = A \cdot C = 2$ $B \cdot C = 1$

Indeed let $m \in k[\mathcal{G}]_D$ be a monomial. If $k \in \{k_1, k_2\}$, or e divides m , then m is of one of the first two forms. Otherwise let $m = abc$, with $\{a, b, c\} \cap \{k_1, k_2, e\} = \emptyset$; if two of the curves A, B, C have intersection number three, then the remaining one is E , which we are excluding. If the intersection numbers among the curves A, B, C are all at most two, then the required conditions follow multiplying successively the equality $A + B + C = -2K_X + E$ by A, B, C .

We show that the monomials of degree D span a subspace of dimension at most six of $(k[\mathcal{G}]/J_X)_D$; since $\dim \text{Cox}(X)_D = 6$, the result follows.

First, the image of $H^0(X, \mathcal{O}_X(-K_X)) \otimes H^0(X, \mathcal{O}_X(-K_X + E))$ in $k[\mathcal{G}]_D$ has dimension at most five: $\dim H^0(X, \mathcal{O}_X(-K_X)) = 2$, $\dim H^0(X, \mathcal{O}_X(-K_X + E)) = 3$ and the element $k_1 \otimes k_2e - k_2 \otimes k_1e$ is in the kernel.

Second, the span of the monomials of the first two forms has dimension at most six. Let $aa'e, bb'e$ be any two monomials of the second form. There is a quadratic relation q involving $aa', bb', k_1^2, k_1k_2, k_2^2$ since these five vectors correspond to five elements of the four dimensional space $H^0(X, \mathcal{O}_X(-2K_X))$. Moreover aa' and bb' are independent from k_1^2, k_1k_2, k_2^2 , since the monomials in k_1, k_2 correspond to sections having a base-point, and neither of the remaining elements aa' and bb' vanishes at the base-point. Thus in the relation q the coefficients of aa' and bb' are both nonzero. We deduce that the span of the elements of the first two forms has dimension at most six in $(k[\mathcal{G}]/J_X)_D$.

Third, let abc be a monomial of the third form. The divisor $A + B$ contracts the unique exceptional curve $F := K_X + A + B$, and the divisor $G := A + B - E$ is an exceptional curve on the del Pezzo surface Y obtained from X by contracting F . Therefore there is a quadratic relation involving the monomials ab, ge, k_1f, k_2f and the monomials ab and ge are independent from k_1f, k_2f , by Lemma 8.1. Thus we may use this relation to write the image of abc in $k[\mathcal{G}]/J_X$ as a combination of divisors of the first two forms and the proof is complete. \square

Lemma 8.4. *Let Q be a conic on X ; then the ideal I_X has no minimal generators in degree $D = -K_X + Q$.*

Proof. The monomials of degree D are of the following forms:

Monomial	Description
ks	$k \in \{k_1, k_2\}$, $s \in k[\mathcal{G}]_Q$
efe_1	$E_1 \cdot Q = 0$ $E \cdot F = 2$ $E_1 \cdot (E + F) = 2$ $Q \cdot E = Q \cdot F = 1$
ee_1e_2	$E_1 \cdot Q = E_2 \cdot Q = E_1 \cdot E_2 = 0$ $Q \cdot E = E_1 \cdot E = E_2 \cdot E = 2$

Note that E_1, E_2 are disjoint components of reducible fibers of the conic bundle associated to Q . Indeed let

$$-K_X + Q \sim A + B + C. \tag{8-2}$$

Multiplying both sides of (8-2) by Q we find

$$2 = A \cdot Q + B \cdot Q + C \cdot Q;$$

thus at least one of the intersection products is zero. Suppose that $C \cdot Q = 0$, and let $\tilde{C} = Q - C$; then $A + B = -K_X + \tilde{C}$ is a reducible divisor in the linear system associated to the pull-back of the anticanonical divisor on the del Pezzo surface of degree two obtained from X by contracting \tilde{E} . The first case in (8-2) corresponds to the sections containing a divisor in the linear system $|-K_X|$; the second to one not containing it and containing two curves not in the linear system $|Q|$; the last one to one containing two elements of $|Q|$.

Since $(k[\mathcal{G}]/J_X)_2 = \text{Cox}(X)_2$, it follows that the dimension of the span of the monomials of the first form in (8-2) modulo $(J_X)_{-K_X+Q}$ is at most four. Let $p = ee_1e_2$ be a monomial in $k[\mathcal{G}]_{-K_X+Q}$ of the third form in (8-2) and let V denote the span of the monomials of the first form in (8-2) together with p in $(k[\mathcal{G}]/J_X)_{-K_X+Q}$. Since $\dim(\text{Cox}(X)_{-K_X+Q}) = 5$, the result follows if we show that

$$V = (k[\mathcal{G}]/J_X)_{-K_X+Q}.$$

Let $q = \bar{e}\bar{e}_1\bar{e}_2$ be a monomial in $k[\mathcal{G}]_{-K_X+Q}$ of the third form in (8-2).

If $\bar{e}_1 = e_1$, then let $E_3 = K_X + E + E_2 = K_X + \bar{E} + \bar{E}_2$ be the exceptional curve contracted by $E + E_2$. By Lemma 8.1, applied to the monomials $\bar{e}\bar{e}_2, ee_2, k_1e_3, k_2e_3$, we conclude that the image of q in $(k[\mathcal{G}]/J_X)_{-K_X+Q}$ belongs to V .

If $\{\bar{E}_1, \bar{E}_2\} \cap \{E_1, E_2\} = \emptyset$, then at least one curve in $\{\bar{E}_1, \bar{E}_2\}$ is disjoint from one curve in $\{E_1, E_2\}$ and relabeling the indices if necessary we may assume that $\bar{E}_1 \cdot E_1 = 0$. By the same reasoning above $q = \bar{e}\bar{e}_1\bar{e}_2$ is in the span of V and $\tilde{q} = \bar{e}\bar{e}_1e_1$ (note that $\tilde{E} := -K_X + Q - \bar{E}_1 - E_1$ is an exceptional curve, since it has anticanonical degree one and square negative one), and we conclude since \tilde{q} is in the span of V .

Finally, if $q = \bar{e}f\bar{e}_1$, then at least one curve in E_1, E_2 is disjoint from \bar{E}_1 and relabeling the indices if necessary we may assume that $\bar{E}_1 \cdot E_1 = 0$. Reasoning as above, q is in the span of V and $\tilde{e}e_1\bar{e}_1$, and we are done. □

Corollary 8.5. *Let X be a sweeping del Pezzo surface of degree one and let D be an ample divisor on X such that $-K_X \cdot D \geq 3$. Then the ideal I_X has no minimal generators in degree D .*

Proof. By Theorem 7.8, if $-K_X \cdot D \geq 4$ then D is capturable and the result follows from Theorem 6.7. Since the only ample divisors of anticanonical three are $-3K_X, -2K_X + E$ and $-K_X + Q$, where E is an exceptional curve and Q is a conic, the result is true if $-K_X \cdot D = 3$, by Lemmas 8.2–8.4. □

9. Nonample divisors

In this section we prove that the relations in degree D , where D is a nonample divisor on X of anticanonical degree at least three, coming from J_X are sufficient to show that the quotient $(k[\mathcal{G}]/J_X)_D$ is in fact spanned by monomials coming from a del Pezzo surface of smaller degree. This is the basis for the inductive procedure of [Section 11](#). Throughout this section we assume that del Pezzo surfaces X of degree one defined over fields of characteristic two are sweeping.

Lemma 9.1. *Let X be a del Pezzo surface of degree at most five and let D be a divisor on X . Suppose that E is an exceptional curve on X such that $D \cdot E = 0$. Then $(k[\mathcal{G}]/J_X)_D$ is spanned by products of variables corresponding to exceptional curves disjoint from E .*

Proof. Let $m \in k[\mathcal{G}]_D$ be a monomial and write $m = e^p \cdot s$, where $p \geq 0$ and s is a product of variables different from E . Note that if $p = 0$, then s is a product of variables corresponding to divisors disjoint from E , since by assumption $D \cdot E = 0$. We shall show that if $p \geq 1$, then using the quadratic relations we may decrease p ; the result then follows by induction on p .

Suppose that $p \geq 1$. Since $D \cdot E = 0$, there is a variable $c \in \mathcal{G}$ such that $E \cdot C > 0$ and c divides s . The monomial ce is a monomial of anticanonical degree two corresponding to a nef divisor. By definition of J_X , the vector space $(k[\mathcal{G}]/J_X)_{C+E}$ coincides with $\text{Cox}(X)_{C+E}$; thus it suffices to show that there is a basis of $\text{Cox}(X)_{C+E}$ consisting of sections vanishing along exceptional curves different from E . To conclude, we analyze all the possibilities for the divisor $C+E$.

Case 1. The divisor $C+E$ is a conic. The linear system $|C+E|$ contains $8 - \deg(X)$, i.e., at least three, distinct reducible elements and any two of these span it.

Case 2. The divisor $C+E$ is $-b^*K_Y$, where $b: X \rightarrow Y$ is a birational morphism, Y has degree two. The linear system $|C+E|$ contains 28 distinct reducible elements and any five of these span it ([Lemma 4.1](#)).

Case 3. The divisor $C+E$ is $-2K_X$ and the degree of X is one. The linear system $|-2K_X|$ contains 120 distinct reducible elements whose irreducible components are (-1) -curves and any 113 of these span it ([Proposition 4.4](#)).

In all these cases the sections supported on (-1) -curves distinct from E span $\text{Cox}(X)_{C+E}$, and the lemma follows. \square

Lemma 9.2. *Let X be a del Pezzo surface of degree at most five and let D be a divisor on X . Suppose that E is an exceptional curve on X such that $D \cdot E < 0$. Then $\dim(k[\mathcal{G}]/J_X)_D = \dim(k[\mathcal{G}]/J_X)_{D-E}$ and $\dim \text{Cox}(X)_D = \dim \text{Cox}(X)_{D-E}$.*

Proof. If D is not effective, then $D - E$ is not effective and $k[\mathcal{G}]_D = k[\mathcal{G}]_{D-E} = 0$. If D is effective, then $E \cdot D < 0$ implies that E is a component of D and thus every

monomial in $k[\mathcal{G}]_D$ is divisible by e ; the same argument also proves the statement about the Cox ring. \square

10. Del Pezzo surfaces of degree at least two

Lemma 10.1. *Let X be a del Pezzo surface of degree two. Any ample divisor is capturable except $-K_X$.*

Proof. Let $D \neq -K_X$ be an ample divisor and write $D = -nK_X + N$, with $n \geq 1$ and N nef and not ample; by assumption either $n \geq 2$ or $N \neq 0$.

Suppose first that $N \neq 0$; if N contracts more than one exceptional curve, then we let \mathcal{S} be the set of exceptional curves contracted by N . If N contracts exactly one exceptional curve E , then we let F be any exceptional curve disjoint from E and let $\mathcal{S} := \{E, F\}$.

Let $C \in \mathcal{C} \setminus \mathcal{S}$ and let $C' := -K_X - C$. If $C' \in \mathcal{S}$, then the divisor $L_{C'T} = N + (-K_X - T) - (n-1)K_X$ is big and nef for all $T \in \mathcal{S}$, $T \neq C'$ since

$$(-K_X - T) \cdot N = -K_X \cdot N > 0;$$

thus we can capture all such curves using \mathcal{S} . If $C' \notin \mathcal{S}$, then let $S, T \in \mathcal{S}$ be disjoint and note that $N - T$ is either nef or N is a multiple of a conic and it is the sum of a multiple of a conic and a single exceptional curve. Moreover

- $S' \cdot (N - T) = -K_X \cdot N - 1 > 0$;
- if N is a multiple of a conic, then S' also intersects the fixed component of $N - T$;
- $C' \cdot (N - T) = -K_X \cdot N - 1 - C \cdot T$ is zero if and only if N is a conic and $C \cdot T = 1$.

In this last case, note that $C \cdot N \neq 0$, and C intersects at least one component in every reducible fiber of the conic bundle determined by N ; hence we may choose T to be the other component if $C' \cdot (N - T) = 0$. In all cases we may choose $S, T \in \mathcal{S}$ disjoint such that L_{ST} is big and nef, and hence capture all $C \in \mathcal{C} \setminus \mathcal{S}$ using only curves in \mathcal{S} . In particular, we are done if N contracts at most two exceptional curves.

To conclude it suffices to treat the two cases $N = L + N'$ and $N = Q + N'$, where L is a twisted cubic, Q is a conic and N' is nef; note also that $S \cdot N' = 0$, for all $S \in \mathcal{S}$.

Choose any two disjoint $S, T \in \mathcal{S}$ and let $C \in \mathcal{S}$, $C \neq S, T$. If $N = L + N'$, we have that the divisor $L_{ST} = -K_X + (-K_X - C) + (L - S - T) + N'$ is big and nef since $(-K_X - C) \cdot (L - S - T) = 1$ and we conclude. If $N = Q + N'$, then the divisor $L_{ST} = (-K_X - S) + (-K_X - T) + (Q - C) + N'$ is big and nef since $(-K_X - S) \cdot (Q - C) = (-K_X - T) \cdot (Q - C) = 1$ and again we conclude.

Suppose now that $N = 0$ and hence $D = -nK_X$, where $n \geq 2$. Let \mathcal{S} be any set of seven disjoint exceptional curves. For all $C \in \mathcal{C} \setminus \mathcal{S}$, there are distinct $S, T \in \mathcal{S}$ such that $C \cdot S, C \cdot T > 0$: this is clear from the list of exceptional curves. Thus for such a choice of S, T , we have that the divisor $L_{ST} = -3K_X - C - S - T = C' + S' + T'$ is big and nef and we may capture all the curves in $\mathcal{C} \setminus \mathcal{S}$ using \mathcal{S} . Finally let $S, T \in \mathcal{S}$ be distinct elements and let $S' := -K_X - S, T' := -K_X - T$; for all $C \in \mathcal{S}$ the divisor $L_{S'T'} = -3K_X - C - S' - T' = C' + S + T$ is big and nef and we conclude. \square

Lemma 10.2. *Let X be a del Pezzo surface of degree three. Any ample divisor is capturable.*

Proof. Let D be an ample divisor and write $D = -K_X + N$, with N a nef divisor.

Let \mathcal{S} be a set of six disjoint exceptional curves. For all $C \in \mathcal{C} \setminus \mathcal{S}$ there are two distinct $S, T \in \mathcal{S}$ such that $C \cdot S = C \cdot T = 1$. With these choices of S, T we have that the divisor $L_{ST} = -2K_X + N - C - S - T = (-K_X - T) + (-K_X - C - S) + N$ is big and nef since $(-K_X - T) \cdot (-K_X - C - S) = 1$. Thus we may capture all the exceptional curves $C \in \mathcal{C} \setminus \mathcal{S}$ using the curves in \mathcal{S} . Let \mathcal{S}' be the set of six disjoint exceptional curves each intersecting all the curves in \mathcal{S} except for one (thus $\mathcal{S} \cup \mathcal{S}'$ is a *Schläfli double-six*). Choose $S, T \in \mathcal{S}$ and let $S', T' \in \mathcal{S}'$ be two corresponding curves, that is $S \cdot S' = T \cdot T' = 0$. Using the moves above we can capture $\{S', T'\}$ with \mathcal{S} and we can capture $\mathcal{S} \setminus \{S, T\}$ using $\mathcal{S}' \setminus \{S', T'\}$, and finally we capture $\mathcal{S}' \setminus \{S', T'\}$ using $\{S, T\}$. \square

Lemma 10.3. *Let X be a del Pezzo surface of degree four. Any ample divisor is capturable.*

Proof. Let D be an ample divisor and write $D = -K_X + N$, with N a nef divisor. The criterion to capture curves is that $(\{E, F\}, C)$ is a capture move if $C \notin \{E, F\}$, $E \cdot F = 0$ and $C \cdot E + C \cdot F > 0$: if $C \cdot E > 0$, then the divisor

$$L_{EF} = (-K_X - C - E) + (-K_X - F) + N$$

is the sum of a conic, a twisted cubic and a nef divisor and hence it is big and nef.

We refer to [Table 1](#). Use E_3, E_4 to capture $L - E_i - E_j$ for all $3 \leq i < j < 5$. Use $L - E_1 - E_3, L - E_1 - E_4, L - E_1 - E_5$ to capture E_3, E_4, E_5 and finally use E_1, E_2 to capture all the remaining curves. \square

Lemma 10.4. *Let X be a del Pezzo surface of degree five. Any ample divisor is capturable.*

Proof. Same criterion and strategy as in [Lemma 10.3](#), ignoring any reference to the index 5. \square

Lemma 10.5. *Let X be a del Pezzo surface of degree six. Any ample divisor is capturable.*

Proof. Same criterion as in [Lemma 10.3](#). Capture $L - E_1 - E_2$ using E_1, E_2 and then capture the remaining curves using $L - E_1 - E_3, L - E_2 - E_3$. \square

11. Quadratic generation

Main result. We collect all the information gathered in the previous sections to prove the main result of this paper: the ideal I_X is generated by its degree two part.

Proof of the Batyrev–Popov Conjecture. Let n be an integer; by induction on $r_X := 9 - \deg(X)$ and by induction on n we show that $(J_X)_n = (I_X)_n$. By definition, the statement is true if $n \leq 2$, for all del Pezzo surfaces X ; if $r_X \leq 4$, then the result is well known for all n .

Suppose that $r_X > 4, n \geq 3$, that for all del Pezzo surfaces Y such that $r_Y < r_X$ we have that $J_Y = I_Y$, and that $(J_X)_{n-1} = (I_X)_{n-1}$. Let D be a divisor on X of anticanonical degree n ; if there is an exceptional curve E such that $D \cdot E \leq 0$, then the result follows by [Lemma 9.2](#) or by [Lemma 9.1](#). Otherwise D is ample and the result follows from [Corollary 8.5](#) if $\deg(X) = 1$, or from the lemmas of [Section 10](#) if $\deg(X) \geq 2$. \square

Quadratic generators. Let X be a del Pezzo surface of degree d . We briefly explain how all generators of I_X arise. The nef divisors D with anticanonical degree two on X are:

- (1) the conics Q ,
- (2) $-K_X$ if $d = 2$,
- (3) $-K_X + E$ if $d = 1$, where E is an exceptional curve on X ,
- (4) $-2K_X$ if $d = 1$.

These divisors are precisely sums of pairs of intersecting exceptional curves on X .

We count the relations coming from conics as follows. Every conic Q has $8 - d$ reducible sections and $h^0(X, \mathcal{O}_X(Q)) = 2$; thus each conic gives rise to $6 - d$ quadratic relations. If there are B conics on X then there are $(6 - d)B$ generators of I_X induced by conic bundles. For example, when $d = 1$ we obtain $5 \times 2, 160 = 10,800$ relations.

When $d = 2$, we also have relations in degree $-K_X$. There are 28 monomials in $k[\mathcal{G}]_{-K_X}$, and $h^0(X, \mathcal{O}_X(-K_X)) = 3$, giving 25 linear dependence relations among these quadratic monomials. These relations yield the remaining generators of I_X .

When $d = 1$, there are 30 monomials in $k[\mathcal{G}]_{-K_X+E}$: k_1e, k_2e and the 28 monomials coming from the anticanonical divisor of the del Pezzo surface of degree two obtained by contracting E . This gives $(30 - 3) \times 240 = 6,480$ generators in I_X . Finally, we also have relations coming from $-2K_X$. There are 123 monomials in $k[\mathcal{G}]_{-2K_X}$: k_1^2, k_1k_2, k_2^2 and the 120 monomials of the form ee' , where $E, E' \in \mathcal{C}$ and

$\deg(X)$	D	Number of monomials in $k[\mathcal{G}]_D$	$h^0(X, \mathbb{O}_X(D))$	$b_{1,D}(\text{Cox}(X))$	Number of divisors of type D
1	Q	7	2	5	2160
	$-K_X + E$	$28 + 2$	3	27	240
	$-2K_X$	$120 + 3$	4	119	1
	(Total)	22443		17399	2401
2	Q	6	2	4	126
	$-K_X$	28	3	25	1
	(Total)	784		529	127
3	Q	5	2	3	27
	(Total)	135		81	27
4	Q	4	2	2	10
	(Total)	40		20	10
5	Q	3	2	1	5
	(Total)	15		5	5

Table 2. First Betti numbers $b_{1,D}(\text{Cox}(X))$ for del Pezzo surfaces.

$E + E' = -2K_X$. Since $h^0(X, \mathbb{O}_X(-2K_X)) = 4$, we obtain 119 linear dependence relations among these quadratic monomials. These relations yield the remaining generators of I_X . This information is summarized in [Table 2](#).

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Hilbert schemes of 8 points

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The Hilbert scheme H_n^d of n points in \mathbb{A}^d contains an irreducible component R_n^d which generically represents n distinct points in \mathbb{A}^d . We show that when n is at most 8, the Hilbert scheme H_n^d is reducible if and only if $n = 8$ and $d \geq 4$. In the simplest case of reducibility, the component $R_8^4 \subset H_8^4$ is defined by a single explicit equation, which serves as a criterion for deciding whether a given ideal is a limit of distinct points.

To understand the components of the Hilbert scheme, we study the closed subschemes of H_n^d which parametrize those ideals which are homogeneous and have a fixed Hilbert function. These subschemes are a special case of multi-graded Hilbert schemes, and we describe their components when the colength is at most 8. In particular, we show that the scheme corresponding to the Hilbert function $(1, 3, 2, 1)$ is the minimal reducible example.

1. Introduction

The Hilbert scheme H_n^d of n points in affine d -space parametrizes 0-dimensional, degree n subschemes of \mathbb{A}^d . Equivalently, the k -valued points of H_n^d parametrize ideals $I \subset S = k[x_1, \dots, x_d]$ such that S/I is an n -dimensional vector space over k . The smoothable component $R_n^d \subset H_n^d$ is the closure of the set of ideals of distinct points. The motivating problem of this paper is characterizing the ideals which lie in the smoothable component, that is, the 0-dimensional subschemes which are limits of distinct points. We determine the components of the schemes H_n^d for $n \leq 8$, and find explicit equations defining $R_8^4 \subset H_8^4$.

We assume that k is a field of characteristic not 2 or 3.

Theorem 1.1. *Suppose n is at most 8 and d is any positive integer. Then the Hilbert scheme H_n^d is reducible if and only if $n = 8$ and $d \geq 4$, in which case it consists of exactly two irreducible components: the smoothable component, of dimension $8d$, and a component denoted G_8^d , of dimension $8d - 7$, which consists of local algebras isomorphic to homogeneous algebras with Hilbert function $(1, 4, 3)$.*

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It is known that for d at least 3 and n sufficiently large the Hilbert scheme of points is always reducible [Iarrobino 1972]. The fact that the Hilbert scheme H_8^4 has at least two components appears in [Iarrobino and Emsalem 1978]. In contrast, for the Hilbert scheme of points in the plane ($d = 2$), the smoothable component is the only component [Fogarty 1968].

To show that the Hilbert scheme of n points is irreducible, it suffices to show that each isomorphism type of local algebras of rank at most n is smoothable, and for n at most 6 there are finitely many isomorphism types of local algebras. In contrast, there are infinitely many nonisomorphic local algebras of degree 7. Relying on a classification of the finitely many isomorphism types in degree 6, Mazzola [1980] proves the irreducibility of H_n^d for $n = 7$.

In our approach, a coarser geometric decomposition replaces most of the need for classification. We divide the local algebras in H_n^d into sets H_h^d by their Hilbert function h , and we determine which components of these sets are smoothable. The main advantage to this approach is that there are fewer Hilbert functions than isomorphism classes, and this enables us to extend the smoothability results of [Mazzola 1980] up to degree 8.

In order to determine the components of H_h^d , we first determine the components of the standard graded Hilbert scheme \mathcal{H}_h^d , which parametrizes homogeneous ideals with Hilbert function h . By considering the map $\pi_h : H_h^d \rightarrow \mathcal{H}_h^d$ which sends a local algebra to its associated graded ring, we relate the components of \mathcal{H}_h^d to those of H_h^d . The study of standard graded Hilbert schemes leads to the following analogue of Theorem 1.1:

Theorem 1.2. *Let \mathcal{H}_h^d be the standard graded Hilbert scheme for Hilbert function h , where $\sum h_i \leq 8$. Then \mathcal{H}_h^d is reducible if and only if*

$$h = (1, 3, 2, 1) \quad \text{or} \quad h = (1, 4, 2, 1),$$

in which case it has exactly two irreducible components. In particular, $\mathcal{H}_{(1,3,2,1)}^3$ is the minimal example of a reducible standard graded Hilbert scheme.

As in the ungraded case, all standard graded Hilbert schemes in the plane are smooth and irreducible [Evain 2004].

In the case when $d = 4$ and $n = 8$, we describe the intersection of the two components of H_8^4 explicitly. Let $S = k[x, y, z, w]$ and S_1 be the vector space of linear forms in S . Let S_2^* denote the space of symmetric bilinear forms on S_1 . Then, the component G_8^4 is isomorphic to $\mathbb{A}^4 \times \text{Gr}(3, S_2^*)$, where $\text{Gr}(3, S_2^*)$ denotes the Grassmannian of 3-dimensional subspaces of S_2^* .

Theorem 1.3. *The intersection $R_8^4 \cap G_8^4$ is a prime divisor on G_8^4 . We have the following equivalent descriptions of $R_8^4 \cap G_8^4 \subset G_8^4$:*

Set-theoretic: For a point $I \in G_8^4 \cong \mathbb{A}^4 \times \text{Gr}(3, S_2^*) \cong \mathbb{A}^4 \times \text{Gr}(7, S_2)$ let V be the corresponding 7-dimensional subspace of S_2 . Then $I \in G_8^4$ belongs to the intersection if and only if the skew-symmetric bilinear form

$$\langle \cdot, \cdot \rangle_I : (S_1 \otimes S_2/V)^{\otimes 2} \rightarrow \bigwedge^3 (S_2/V) \cong k$$

given by

$$\langle l_1 \otimes q_1, l_2 \otimes q_2 \rangle_I = (l_1 l_2) \wedge q_1 \wedge q_2$$

is degenerate.

Local equations: Around any $I \in G_8^4$, choose an open neighborhood $U_I \subset G_8^4$ such that the universal Grassmannian bundle over the U_I is generated by three sections. Since these sections are bilinear forms we may represent them as symmetric 4×4 matrices A_1, A_2 , and A_3 with entries in $\Gamma(U_I, \mathbb{O}_{G_8^4})$. The local equation for $R_8^4 \cap U_I$ is then the Pfaffian of the 12×12 matrix:

$$\begin{pmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{pmatrix}.$$

Note that specializing this equation to I gives the Pfaffian of $\langle \cdot, \cdot \rangle_I$.

The local equation from the previous theorem gives an effective criterion for deciding whether an algebra of colength 8 belongs to the smoothable component. Moreover, it can be lifted to equations which cut out $R_8^4 \subset H_8^4$. Recall that H_8^4 can be covered by open affines corresponding to monomial ideals in $k[x, y, z, w]$ of colength 8.

Theorem 1.4. On these monomial coordinate charts, $R_8^4 \subset H_8^4$ is cut out set-theoretically by

- (1) the zero ideal on charts corresponding to monomial ideals with Hilbert functions other than $(1, 4, 3)$, and
- (2) the pullback of the equations in [Theorem 1.3](#) along the projection to homogeneous ideals in charts corresponding to monomial ideals with Hilbert function $(1, 4, 3)$.

Remark 1.5. It is not known whether H_8^4 is reduced. If it is, then the equations in [Theorem 1.4](#) cut out the smoothable component scheme-theoretically.

Remark 1.6. Every point of the Hilbert scheme of n points in \mathbb{P}^d has an open neighborhood isomorphic to H_n^d . Therefore, analogues of [Theorems 1.1, 1.3, and 1.4](#) hold for the Hilbert scheme of n points in \mathbb{P}^d . However, the most natural setting for our methods is the affine case and the language of multigraded Hilbert schemes.

The material in this paper is organized as follows: [Section 2](#) contains background and definitions. [Section 3](#) describes the geometry of standard graded Hilbert schemes of degree at most 8. [Section 4](#) contains proofs of the smoothability of families of algebras and its main steps are collected in [Table 4.1](#). [Section 5](#) is devoted to the study of the components of H_8^4 and their intersection. [Section 6](#) ties together these results to give proofs of all theorems mentioned above. Finally, [Section 7](#) proposes some open questions.

2. Background

In this section, let k be a field and $S = k[x_1, \dots, x_d]$.

Multigraded Hilbert schemes. A grading of S by an abelian group A is a semi-group homomorphism $\deg : \mathbb{N}^d \rightarrow A$ which assigns to each monomial in S a degree in A . Let $h : A \rightarrow \mathbb{N}$ be an arbitrary function, which we will think of as a vector \mathbf{h} , with values h_a indexed by a in A . We say that a homogeneous ideal I in S has Hilbert function \mathbf{h} if S_a/I_a has k -dimension h_a for all $a \in A$. Multigraded Hilbert schemes, introduced in [[Haiman and Sturmfels 2004](#)], parametrize homogeneous ideals with a fixed Hilbert function. More precisely these are quasiprojective schemes over k which represent the following functors [[Haiman and Sturmfels 2004](#), Theorem 1.1]:

Definition 2.1. *For a fixed integer d , grading \deg , and Hilbert function \mathbf{h} , the multigraded Hilbert functor $\mathcal{H}_{\mathbf{h}} : k\text{-Alg} \rightarrow \underline{\text{Set}}$ assigns to each k -algebra T , the set of homogeneous ideals J in $S \otimes T$ such that the graded component $(S \otimes T/J)_a$ is a locally free T -module of rank h_a for all a in A . The multigraded Hilbert scheme is the scheme which represents the multigraded Hilbert functor.*

In particular, we will be interested in the following two special kinds of multigraded Hilbert scheme:

- *Let $\deg : \mathbb{N}^d \rightarrow 0$ be the constant function to the trivial group and define $h_0 = n$. In this case the multigraded Hilbert scheme is the Hilbert scheme of n points in \mathbb{A}^d and will be denoted H_n^d .*
- *Let $\deg : \mathbb{N}^d \rightarrow \mathbb{Z}$ be the summation function, which induces the standard grading $\deg(x_i) = 1$. We call the corresponding multigraded Hilbert scheme the standard graded Hilbert scheme for Hilbert function \mathbf{h} and denote it with $\mathcal{H}_{\mathbf{h}}^d$.*

If $n = \sum_{j \in \mathbb{N}} h_j$ there is a closed immersion $\mathcal{H}_{\mathbf{h}}^d \rightarrow H_n^d$ by [[Haiman and Sturmfels 2004](#), Proposition 1.5].

Coordinates for the Hilbert scheme of points. In this section we briefly discuss some coordinate systems on H_n^d . The reader should refer to [[Miller and Sturmfels](#)

2005, Chapter 18] for an extended treatment. For a monomial ideal M of colength n with standard monomials λ , let $U_\lambda \subset H_n^d$ be the set of ideals I such that the monomials in λ are a basis for S/I . Note that the U_λ form an open cover of H_n^d . An ideal $I \in U_\lambda$ has generators of the form $m - \sum_{m' \in \lambda} c_{m'}^m m'$. The $c_{m'}^m$ are local coordinates for U_λ which define a closed immersion into affine space.

Suppose $V(I)$ consists of n distinct points $q^{(1)}, \dots, q^{(n)}$ with coordinates $q_i^{(j)}$ for $1 \leq i \leq d$. Fix an order $\lambda = (m_1, \dots, m_n)$ on the set of monomials λ and define $\Delta_\lambda = \det([m_i(q^{(j)})]_{i,j})$. For example, if $\lambda = (1, x_1, \dots, x_1^{n-1})$, then Δ_λ is the determinant of the Vandermonde matrix on the $q_1^{(j)}$. If $I \in U_\lambda$, we can express the $c_{m'}^m$ in terms of the $q_i^{(j)}$, using Cramer's rule, as

$$c_{m'}^m = \frac{\Delta_{\lambda - m' + m}}{\Delta_\lambda},$$

where $\lambda - m' + m$ is the ordered set of monomials obtained from λ by replacing m' with m . Note that the right-hand side of this equality is only defined for ideals of distinct points. The quotient or product of two Δ_λ 's is S_n -invariant. Thus the formula does not depend on the order of λ . Gluing over the various U_λ , these quotients determine a birational map

$$(\mathbb{A}^d)^n // S_n \dashrightarrow R_n^d$$

that is regular when the points $q^{(j)}$ are all distinct. The rational functions $\Delta_{\eta_1} / \Delta_{\eta_2}$ are elements of the quotient field of either $(\mathbb{A}^d)^n // S_n$ or R_n^d . The expressions Δ_λ and their relationship to the local equations $c_{m'}^m$ were introduced in [Haiman 1998, Proposition 2.6].

Duality. First let us suppose that k has characteristic 0, and let S^* be the ring $k[y_1, \dots, y_d]$, with the structure of an S -module via formal partial differentiation $x_i \cdot f = \partial f / \partial y_i$. If we look at homogeneous polynomials of a fixed degree j in each of the two rings, we have a pairing $S_j \times S_j^* \rightarrow S_0^* = k$. Any vector subspace of S_j has an orthogonal subspace in S_j^* of complementary dimension. In particular, if I is a homogeneous ideal in S , we have subspaces $I_j^\perp \subset S_j^*$ and we set $I^\perp = \bigoplus I_j^\perp$, which is often called the Macaulay inverse system of I . The subspace I^\perp is closed under differentiation, that is, $\frac{\partial}{\partial y_i} I_j^\perp \subset I_{j-1}^\perp$ for all i and j . Conversely, any graded vector subspace $I^\perp \subset S^*$ which is closed under differentiation determines an orthogonal ideal $I \subset S$ with Hilbert function $h_j = \dim_k I_j^\perp$. Also, note that any linear change of variables in S induces a linear change of variables in S^* .

If k has positive characteristic p , then the same theory works for sufficiently small degree. Formal partial differentiation gives a perfect pairing $S_j \times S_j^* \rightarrow k$ if and only if j is less than p . Thus, we can associate orthogonal subspaces I_j^\perp to a homogeneous ideal I so long as I_p contains all of S_p . In this case, we define

$I_j^\perp = 0$ for all degrees j at least p , and $I = \bigoplus I_j$ as before. Conversely, for a graded vector subspace $I^\perp \subset S_*$ which is closed under differentiation and with $I_j^\perp = 0$ for j at least p , the orthogonal space is a homogeneous ideal $I \subset S$ with Hilbert function $h_j = \dim_k I_j^\perp$.

3. Components of the standard graded Hilbert schemes

In this section, k will denote an algebraically closed field of characteristic not 2 or 3.

We will study the components of the standard graded Hilbert schemes \mathcal{H}_h^d with Hilbert function h where $\sum h_i \leq 8$. These results will be important for the proofs of smoothability in the following section. From [Evain 2004, Theorem 1], we have that for $d = 2$, the standard graded Hilbert schemes are irreducible. Thus, we will only work with d at least 3. For the purposes of classifying irreducible components of \mathcal{H}_h^d , it is convenient to work with homogeneous ideals which contain no linear forms, and thus we assume that $h_1 = d$. The following lemma allows us to restrict our attention to this case:

Proposition 3.1. *The standard graded Hilbert scheme \mathcal{H}_h^d with $d \geq h_1$ is a $\mathcal{H}_h^{h_1}$ -bundle over $\text{Gr}(d - h_1, S_1)$. In particular, if $\mathcal{H}_h^{h_1}$ is irreducible of dimension D then \mathcal{H}_h^d is irreducible of dimension $D + (d - h_1)d$.*

Proof. The degree 1 summand of the universal ideal sheaf of $\mathbb{O}_{\mathcal{H}_h^d}[x_1, \dots, x_d]$ is locally free of rank $d - h_1$ and thus defines a morphism $\phi : \mathcal{H}_h^d \rightarrow \text{Gr}(d - h_1, S_1)$. Over an open affine $U \cong \mathbb{A}^{(d-h_1)h_1}$ in $\text{Gr}(d - h_1, S_1)$, we have an isomorphism $\phi^{-1}(U) \cong U \times H_h^{h_1}$ by taking a change of variables in $\mathbb{O}_U[x_1, \dots, x_d]$. \square

Lemma 3.2. *Let m be a positive integer such that $m!$ is not divisible by the characteristic of k . Let $f(y_1, \dots, y_d)$ be a homogeneous polynomial in S_m^* whose partial derivatives form an r -dimensional vector subspace of S_{m-1}^* . Then f can be written as a polynomial in terms of some r -dimensional subspace of S_1^* .*

Proof. There exists a linear map from $S_1 \rightarrow S_{m-1}^*$ which sends $x_i \mapsto \frac{\partial f}{\partial y_i}$. After a change of variables, we can assume that x_{r+1}, \dots, x_d annihilate f . Thus, any term of f contains only the variables y_1, \dots, y_r . \square

Throughout this section, N will denote $\dim_k S_2 = \binom{d+1}{2}$, the dimension of the vector space of quadrics.

Proposition 3.3. *Let $0 \leq e \leq N$. The standard graded Hilbert scheme for Hilbert function $(1, d, e)$ is isomorphic to the Grassmannian $\text{Gr}(N - e, S_2)$, and it is thus irreducible of dimension $(N - e)N$.*

Proof. We build the isomorphism via the functors of points of these schemes. For a k -algebra T let $\phi(T) : \mathcal{H}_h(T) \rightarrow \text{Gr}(N - e, S_2)(T)$ be the morphism of sets which

maps a homogeneous ideal $I \subset T \otimes_k S$ to I_2 . Let $\psi(T) : \text{Gr}(N - e, S_2)(T) \rightarrow \mathcal{H}_h(T)$ be the map which sends a k -submodule L of $T \otimes S_2$ to $L \oplus \bigoplus_{j \geq 3} (T \otimes_k S_j)$, which is an ideal of $T \otimes S$. The natural transformations ϕ and ψ are inverses of one another and the isomorphism follows from Yoneda's Lemma. \square

Proposition 3.4. *Let $\mathbf{h} = (1, d, 1, \dots, 1)$ and let $m \geq 3$ be the largest index such that h_m is nonzero. Then the standard graded Hilbert scheme for \mathbf{h} is irreducible of dimension $d - 1$.*

Proof. We claim that the scheme \mathcal{H}_h^d is parametrized by $\text{Gr}(1, S_1^*)$ by sending a vector space generated by $\ell \in S_1^*$ to the ideal generated by the quadrics orthogonal to ℓ^2 and all degree $m + 1$ polynomials. This ideal has the right Hilbert function and the parametrization is clearly surjective. \square

Theorem 3.5. *If d is at least 3, the standard graded Hilbert scheme for Hilbert function $(1, d, 2, 1)$ is reducible and consists of the following two components:*

- (1) *The homogeneous ideals orthogonal to ℓ^3, ℓ^2 , and q where ℓ is a linear form and q is a quadric linearly independent of ℓ^2 . We denote this component by \mathcal{Q}_d , and $\dim(\mathcal{Q}_d) = (d^2 + 3d - 6)/2$.*
- (2) *The closure of the homogeneous ideals orthogonal to a cubic c and its partial derivatives, where the degree 1 derivatives of c have rank 2. We denote this component by \mathcal{P}_d , and $\dim(\mathcal{P}_d) = 2d - 1$.*

Proof. We compute the dimension of the first component. It is parametrized by the 1-dimensional subspace of S_1^* generated by ℓ and a 2-dimensional subspace of S_2^* which contains ℓ^2 . These have dimensions $d - 1$ and $N - 1 - 1$ respectively, for a total of $(d^2 + 3d - 6)/2$.

Note also that an open subset of the second component, \mathcal{P}_d , is parametrized by a 2-dimensional subspace V of S_1^* and a cubic $c \in \text{Sym}_3(V)$ which is not a perfect cube. The parametrization is by taking the ideal whose components of degrees 3 and 2 are orthogonal to c and to its derivatives respectively. The space of derivatives is 2-dimensional by our construction of c . The dimension of \mathcal{P}_d is $3 + 2(d - 2) = 2d - 1$.

We claim that any homogeneous ideal with Hilbert function $(1, d, 2, 1)$ lies in one of these two components. Any such ideal is orthogonal to a cubic c , and the derivatives of c are at most 2-dimensional. If the derivatives are 1-dimensional, then c must be a perfect cube, so the ideal is in \mathcal{Q}_d . Otherwise, the ideal is in \mathcal{P}_d .

Finally, we will show that \mathcal{P}_d has a point that does not lie on \mathcal{Q}_d . Let I be the ideal orthogonal to $y_1 y_2^2$, and its partial derivatives, $2y_1 y_2, y_2^2$. Then I is generated by x_2^3 and all degree 2 monomials other than x_2^2 and $x_1 x_2$. We will study the homomorphisms $\phi : I \rightarrow S/I$ of degree 0, as these correspond to the tangent space of \mathcal{H}_h^d at I . For any quadric generator q , we can write $\phi(q) = a_q x_2^2 + b_q x_1 x_2$. Note

that $x_1\phi(q) = a_q x_1 x_2^2$ and $x_2\phi(q) = b_q x_1 x_2^2$. For any $i, j > 2$, ϕ must satisfy the conditions

$$\begin{aligned} x_1\phi(x_i x_j) &= x_j\phi(x_i x_1) = 0, & x_2\phi(x_i x_j) &= x_j\phi(x_i x_2) = 0, \\ x_1\phi(x_1 x_i) &= x_i\phi(x_1^2) = 0, & x_2\phi(x_1 x_i) &= x_1\phi(x_2 x_i). \end{aligned}$$

In matrix form, we see that ϕ must be in the form

$$x_2^2 \begin{pmatrix} x_1^2 & x_1 x_i & x_2 x_i & x_i x_j & x_2^3 \\ * & c_i & * & 0 & 0 \\ * & 0 & c_i & 0 & 0 \\ 0 & 0 & 0 & 0 & * \end{pmatrix},$$

where i and j range over all integers greater than 2. Thus there are at most $2(d - 2) + 3 = 2d - 1$ tangent directions, but since \mathcal{P}_d has dimension $2d - 1$, there are exactly $2d - 1$ tangent directions. On the other hand, \mathcal{Q}_d has dimension $(d^2 + 3d - 6)/2$, which is greater than $2d - 1$ for d at most 3, so I cannot belong to \mathcal{Q}_d and thus \mathcal{P}_d is a component. \square

Proposition 3.6. *The standard graded Hilbert scheme for Hilbert function $\mathbf{h} = (1, d, 2, 2)$ is irreducible of dimension $2d - 2$.*

Proof. The Hilbert scheme is parametrized by a 2-dimensional subspace L of S_1^* and a subspace of S_2 of dimension $N - 2$, and containing the $(N - 3)$ -dimensional subspace orthogonal to the square of L . The parametrization is by sending the subspace of S_2 to the ideal generated by that subspace, together with all degree 4 polynomials. The dimension of this parametrization is $2(d - 2) + 2 = 2d - 2$. \square

Proposition 3.7. *The standard graded Hilbert scheme for Hilbert function $\mathbf{h} = (1, 3, 3, 1)$ is irreducible of dimension 9.*

Proof. This Hilbert scheme embeds as a closed subscheme of the smooth 18-dimensional variety $\text{Gr}(3, S_2) \times \text{Gr}(9, S_3)$ by mapping an ideal to its degree 2 and 3 graded components. Furthermore, $\mathcal{H}_{\mathbf{h}}^3$ is defined by $9 = 3 \cdot 3$ equations, corresponding to the restrictions that the products of each of the 3 variables and each of the 3 quadrics in I_2 are in I_3 . In particular, the dimension of each irreducible component is at least 9.

Now we will look at the projection of $\mathcal{H}_{\mathbf{h}}^3$ onto the Grassmannian $\text{Gr}(9, S_3)$, which is isomorphic to $\text{Gr}(1, S_3^*)$. The orthogonal cubic in S_3^* can be classified according to the vector space dimension of its derivatives. For a generic cubic, its three derivatives will be linearly independent and therefore the cubic will completely determine the orthogonal space. Thus, the projection from $\mathcal{H}_{\mathbf{h}}^3$ is a bijection over this open set, so the preimage is 9-dimensional. In the case where the derivatives of the cubic are 2-dimensional, we have that, after a change of coordinates, the

cubic is written in terms of two variables. Thus, the parameter space of the cubic consists of a 2-dimensional choice of a subspace of S_1 and then a 3-dimensional choice of a cubic written in terms of this subspace. The fiber over any fixed cubic is isomorphic to the Grassmannian of 3-dimensional subspaces of the 4-dimensional subspace of S_2 orthogonal to the derivatives of the cubic. The dimension of the locus in \mathcal{H}_h^3 is therefore $2 + 3 + 3 = 8$. By a similar logic, the locus where the cubic has a 1-dimensional space of derivatives is $2 + 2 \cdot 3 = 8$. Therefore, \mathcal{H}_h^3 is the disjoint union of three irreducible sets, of dimensions 9, 8, and 8. We conclude that \mathcal{H}_h^3 is an irreducible complete intersection of dimension 9. \square

Proposition 3.8. *Let $1 \leq e \leq N$. The standard graded Hilbert scheme for Hilbert function $(1, d, e, 1, 1)$ is irreducible of dimension $d - 1 + (N - e)(e - 1)$.*

Proof. This Hilbert scheme is parametrized by a 1-dimensional subspace L of S_1^* , together with an e -dimensional subspace V of S_2^* which contains $\text{Sym}_2(L)$. The parametrization is by mapping (L, V) to the ideal whose summands of degrees 2, 3, and 4 are orthogonal to V , L^3 , and L^4 , respectively. Note that this has the desired dimension $(d - 1) + ((N - 1) - (e - 1))(e - 1)$. \square

Theorem 3.9. *With the exception of Hilbert functions $(1, 3, 2, 1)$ and $(1, 4, 2, 1)$, the standard graded Hilbert schemes with $\sum h_i \leq 8$ are irreducible.*

Proof. The cases when $d = 2$ follow from [Evain 2004, Theorem 1]. The cases when d is at least 3 are summarized in Table 4.1. \square

4. Smoothable 0-schemes of degree at most 8

In this section k will denote an algebraically closed field of characteristic not 2 or 3.

Recall that a point I in H_n^d is smoothable if I belongs to the smoothable component R_n^d . In this section, we first reduce the question of smoothability to ideals I in H_n^d where S/I is a local k -algebra and I has embedding dimension d . Then we define the schemes H_h^d which parametrize local algebras, and we use these to show that each 0-dimensional algebra of degree at most 8 is either smoothable or is isomorphic to a homogeneous local algebra with Hilbert function $(1, 4, 3)$.

We use two different methods to show that a subscheme H_h^d belongs to the smoothable component.

- (1) For each irreducible component of H_h^d , consider a generic ideal I from that component. Apply suitable isomorphisms to put I into a nice form. Then show I is smoothable. Since the set of ideals isomorphic to I are dense in the component and smoothable, the entire component of H_h^d containing I must belong to R_n^d .

- (2) Within each irreducible component of H_h^d , find an ideal I such that I is a smooth point in H_n^d and I belongs to R_n^d . Then the whole component of H_h^d containing I must belong to R_n^d .

In each method we need to show that a particular ideal I is smoothable. We do this by showing $I = \text{in } J$ with respect to some nonnegative weight vector for a smoothable ideal J . The corresponding Gröbner deformation induces a morphism $\mathbb{A}^1 \rightarrow R_n^d$ which maps 0 to I .

For $d = 2$, the Hilbert scheme H_n^2 is smooth and irreducible [Fogarty 1968, Theorem 2.4]. Thus, we will limit our analysis to algebras with embedding dimension at least 3.

For a finite rank k -algebra A_0 , we say that A_0 is smoothable if there exists a flat family $k[[t]] \rightarrow A$ such that the special fiber of A is isomorphic to A_0 and such that the generic fiber of A is smooth. This terminology is justified by the following result:

Lemma 4.1. *Let $I \subseteq S$ an ideal of colength n . Then I is smoothable if and only if S/I is smoothable (as a k -algebra).*

Proof. Let $I \subseteq S$ be a smoothable ideal, and let $U \subseteq R_n^d$ be the open set parametrizing smooth 0-schemes. Since $I \in R_n^d$, there exists a smooth curve C and a map $f : C \rightarrow R_n^d$ such that $f(P) = I$ for some point $P \in C$ and such that $f(C)$ meets U . By considering the completion of C at the point P , we obtain an induced map $\widehat{f} : \text{Spec}(k[[t]]) \rightarrow R_n^d$ which sends the closed point to I and the generic point into U . The flat family over $\text{Spec}(k[[t]])$ which corresponds to the map \widehat{f} induces an abstract smoothing of S/I .

Conversely, let's assume that $A_0 := S/I$ admits an abstract smoothing $k[[t]] \rightarrow A$. Let $\phi : k[[t]][x_1, \dots, x_d] \rightarrow A$ be defined by sending each x_i to any lift of x_i from $S/I = A_0$. Since A is a finitely generated $k[[t]]$ -module, the cokernel of ϕ is finitely generated, and so we can apply Nakayama's Lemma to show that ϕ is surjective. The map ϕ thus induces a morphism of schemes $f : \text{Spec}(k[[t]]) \rightarrow H_n^d$ which sends the closed point to I and the generic point into U . It follows that $I \in \overline{U} = R_n^d$. \square

If I is supported at multiple points, then S/I is the product of local Artin algebras and a smoothing of each factor over $k[[t]]$ yields a smoothing of S/I . Because of this observation and Lemma 4.1, we will now only consider ideals I in H_n^d which define local algebras with embedding dimension d .

The schemes H_h^d . If (A, \mathfrak{m}) is a local algebra, its Hilbert function is defined by $h_i = \dim_k \mathfrak{m}^i / \mathfrak{m}^{i+1}$, which is equivalently the Hilbert function of the associated graded ring of A . When A is both local and graded, the two notions of Hilbert function coincide. We now define the schemes H_h^d and explore their irreducible components for each Hilbert function h with $\sum h_i \leq 8$.

For each \mathbf{h} such that $\sum h_i = n$, the subscheme $H_{\mathbf{h}}^d \subset H_n^d$ consists set-theoretically of the ideals I defining a local algebra S/I with maximal ideal (x_1, \dots, x_d) whose Hilbert function equals \mathbf{h} . More precisely, let $\mathcal{A} = \mathbb{C}_{H_n^d}[x_1, \dots, x_d]/\mathcal{I}$ be the universal sheaf of algebras on H_n^d and let \mathcal{M} be the ideal $(x_1, \dots, x_d)\mathcal{A}$. The fiber at an ideal I of the quotient sheaf $\mathcal{A}/\mathcal{M}^i$ is isomorphic to $S/(I + (x_1, \dots, x_d)^i)$. For any fixed \mathbf{h} , there is a locally closed subset of H_n^d consisting of those points such that the fiber of $\mathcal{A}/\mathcal{M}^i$ has dimension $h_0 + \dots + h_{i-1}$ for all $i \geq 0$. Let $H_{\mathbf{h}}^d$ be the reduced subscheme on this subset, and then the restriction of each $\mathcal{A}/\mathcal{M}^i$ to $H_{\mathbf{h}}^d$ is locally free. Define \mathcal{B} to be the sheaf of graded algebras on $H_{\mathbf{h}}^d$ whose i -th component is

$$\ker((\mathcal{A}/\mathcal{M}^{i+1})|_{H_{\mathbf{h}}^d} \rightarrow (\mathcal{A}/\mathcal{M}^i)|_{H_{\mathbf{h}}^d}),$$

which is locally free of rank h_i because it is the kernel of a surjection of locally free sheaves. Note that the fiber of \mathcal{B} at I is the associated graded ring of S/I . There is a canonical surjection of graded algebras $\mathcal{O}_{H_n^d}[x_1, \dots, x_d] \rightarrow \mathcal{B}$ which defines a morphism $\pi_{\mathbf{h}} : H_{\mathbf{h}}^d \rightarrow \mathcal{H}_{\mathbf{h}}^d$ to the standard graded Hilbert scheme. The ideal I gets mapped to its initial ideal with respect to the weight vector $(-1, \dots, -1)$.

With the exception $\mathbf{h} = (1, 3, 2, 1, 1)$, we will show that the irreducible components of $H_{\mathbf{h}}^d$ and $\mathcal{H}_{\mathbf{h}}^d$ are in bijection via the map $\pi_{\mathbf{h}}$.

Proposition 4.2. *Each subscheme $H_{(1,d,e)}^d$ is irreducible.*

Proof. Since $H_{(1,d,e)}^d \cong \mathcal{H}_{(1,d,e)}^d$, this follows from Proposition 3.3. □

Proposition 4.3. *Fix $\mathbf{h} = (1, d, e, f)$. Let $m = (d + 1)d/2 - e = \dim_k S_2/I_2$. Then every fiber of $\pi_{\mathbf{h}}$ is irreducible of dimension mf . In particular, the irreducible components of $H_{\mathbf{h}}^d$ are exactly the preimages of the irreducible components of $\mathcal{H}_{\mathbf{h}}^d$.*

Proof. Fix a point in $\mathcal{H}_{\mathbf{h}}^d$, which corresponds to a homogeneous ideal I . Let q_1, \dots, q_m be quadratic generators of I , and let c_1, \dots, c_f be cubics which form a vector space basis for S_3/I_3 . Define a map $\phi : \mathbb{A}^{mf} \rightarrow H_n^d$ via the ideal

$$\left\langle q_i - \sum_{j=1}^f t_{ij}c_j \mid 1 \leq i \leq m \right\rangle + I_{\geq 3},$$

where the t_{ij} are the coordinate functions of \mathbb{A}^{mf} . Because a product of any variable x_{ℓ} with any of these generators is in I , this ideal has the right Hilbert function and maps to the fiber of $\pi_{\mathbf{h}}$ over I . Furthermore, ϕ is bijective on field-valued points, so the fiber is irreducible of dimension mf .

For the last statement, we have that for any irreducible component of $\mathcal{H}_{\mathbf{h}}^d$, the restriction of $\pi_{\mathbf{h}}$ has irreducible equidimensional fibers over an irreducible base, so the preimage is irreducible. These closed sets cover $H_{\mathbf{h}}^d$ and because each lies over a distinct component of $\mathcal{H}_{\mathbf{h}}^d$, they are distinct irreducible components. □

Combining [Theorem 3.5](#) with the above proposition, we see $H_{(1,d,2,1)}^d$ has exactly two components: $P_d := \pi^{-1}(\mathcal{P}_d)$ and $Q_d := \pi^{-1}(\mathcal{Q}_d)$. In addition, by [Propositions 3.6, 3.7, and 4.3](#), $H_{(1,d,2,2)}^d$ and $H_{(1,3,3,1)}^3$ are irreducible.

Proposition 4.4. *Let $\mathbf{h} = (1, d, 1, \dots, 1)$ and let $m \geq 3$ be the largest index such that h_m is nonzero. Then $H_{\mathbf{h}}^d$ is irreducible of dimension $(d + 2m - 2)(d - 1)/2$. At a generic point, after a change of coordinates, we can take the ideal to be*

$$\langle x_d^{m+1}, x_i^2 - x_d^m, x_j x_k \mid 1 \leq i < d, 1 \leq j < k \leq d \rangle.$$

Proof. Fix an ideal $I \in \mathcal{H}_{\mathbf{h}}^d$, and after a change of coordinates, we can assume

$$I = \langle x_d^{m+1}, x_i x_j \mid 1 \leq i \leq d - 1, 1 \leq j \leq d \rangle.$$

Let J be an ideal in the fiber above I . By assumption J contains an element of the form

$$x_i x_d - b_{i3} x_d^3 - \dots - b_{im} x_d^m,$$

for $1 \leq i < d$. Let J' be the image of J after the change of coordinates

$$x_i \mapsto x_i + b_{i3} x_d^2 + \dots + b_{im} x_d^{m-1}, \tag{4-1}$$

and note that J' contains $x_i x_d$ for $1 \leq i < d$ and also lies over I . Thus, for each $1 \leq i \leq j < d$, J' contains an element of the form $f = x_i x_j - a_{ij} x_d^k - \dots$ for some k . However, J' must also contain $x_j(x_i x_d) - x_d f = a_{ij} x_d^{k+1} + \dots$, so in order to have I as the initial ideal, k must equal m . Therefore, J' is of the form

$$J' = \langle x_d^{m+1}, x_i x_j - a_{ij} x_d^m, x_k x_d \mid 1 \leq i \leq j \leq d, 1 \leq k < d \rangle.$$

Conversely, for any choice of a_{ij} and b_{ij} , applying the change of variables in [\(4-1\)](#) to the ideal J' gives a unique ideal J with I as an initial ideal. Thus, the fiber is irreducible of dimension $(m - 2)(d - 1) + (d - 1)d/2 = (d - 1)(d + 2m - 4)/2$, which, together with [Proposition 3.4](#) proves the first statement.

For the second statement, note that the coefficients a_{ij} define a symmetric bilinear form. By taking the form to be generic and choosing a change of variables, we get the desired presentation of the quotient algebra. \square

The above propositions cover all Hilbert functions of length at most 8 except for $\mathbf{h} = (1, 3, 2, 1, 1)$. In this case the fibers of $\pi_{(1,3,2,1,1)}$ are not equidimensional. The dimension of the fiber depends on whether or not the homogeneous ideal requires a cubic generator.

Lemma 4.5. *No ideal in $\mathcal{H}_{(1,3,2,1,1)}^3$ requires a quartic generator.*

Proof. If I were such an ideal, then leaving out the quartic generator would yield an ideal with Hilbert function $(1, 3, 2, 1, 2)$. No such ideal exists, because no such monomial ideal exists. \square

Lemma 4.6. *There exists a 4-dimensional irreducible closed subset \mathcal{X} of $\mathcal{H} = \mathcal{H}_{(1,3,2,1,1)}^3$ where the corresponding homogeneous ideal requires a single cubic generator. On $\mathcal{U} = \mathcal{H} \setminus \mathcal{X}$, the ideal does not require any cubic generators.*

Proof. Let \mathcal{S}_j denote the j -th graded component of $\mathbb{O}_{\mathcal{H}}[x, y, z]$ and $\mathcal{I}_j \subset \mathcal{S}_j$ the j -th graded component of the universal family of ideals on \mathcal{H} . Consider the cokernel \mathcal{Q} of the multiplication map on the coherent sheaves $\mathcal{I}_2 \otimes_{\mathbb{O}_{\mathcal{H}}} \mathcal{S}_1 \rightarrow \mathcal{I}_3$ on \mathcal{H} . The dimension of \mathcal{Q} is upper semicontinuous. Furthermore, since it is not possible to have an algebra with Hilbert function $(1, 3, 2, 3)$, the dimension is at most 1. The set \mathcal{X} is exactly the support of \mathcal{Q} .

We claim that \mathcal{X} is parametrized by the data of a complete flag $V_1 \subset V_2 \subset S_1^*$, and a 2-dimensional subspace Q of V_2^2 which contains V_1^2 . The dimension of this parametrization is $2 + 1 + 1 = 4$. An ideal is formed by taking the ideal which is orthogonal to Q in degree 2 and to the powers V_1^3 and V_1^4 in degrees 3 and 4 respectively. After a change of variables, we can assume that the flag is orthogonal to $\langle x \rangle \subset \langle x, y \rangle \subset S_2$. Then the degree 2 generators of I are x^2, xy, xz and another quadric. It is easy to see that these only generate a codimension 2 subspace of S_3 . Conversely, for any ideal with this property, the orthogonal cubic has a 1-dimensional space of derivatives. Furthermore, there exists a homogeneous ideal with Hilbert function $(1, 3, 2, 2, 1)$ contained in the original ideal. The cubics orthogonal to these have a 2-dimensional space of derivatives, so we can write them in terms of a 2-dimensional space of the dual variables. These two vector spaces determine the flag, and the parametrization is bijective on closed points. In particular, \mathcal{X} is irreducible of dimension 4. □

Lemma 4.7. *The preimage $Z := \pi^{-1}(\mathcal{X})$ is irreducible of dimension 11.*

Proof. By Lemma 4.6, it suffices to prove that the fibers of π over \mathcal{X} are irreducible and 7-dimensional. Let I be a point in \mathcal{X} . As in the proof of Lemma 4.6, we can assume that the ideal corresponding to a point of \mathcal{X} is generated by x^2, xy, xz, q, c , where q and c are a homogeneous quadric and cubic respectively. A point J in the fiber must be generated by m^5 and

$$\begin{aligned} g_1 &:= x^2 + a_1z^3 + b_1z^4, & g_2 &:= xy + a_2z^3 + b_2z^4, \\ g_3 &:= xz + a_3z^3 + b_3z^4, & g_4 &:= q + a_4z^3 + b_4z^4, & g_5 &:= c + b_5z^4. \end{aligned}$$

The a_i, b_i are not necessarily free. We must impose additional conditions to ensure the initial ideal for the weight vector $(-1, -1, -1)$ is no larger than I . In particular, we must have

$$\begin{aligned} zg_1 - xg_3 &= a_1z^4 + b_1z^5 - a_3xz^3 - b_3xz^4 \in J, \\ zg_2 - yg_3 &= a_2z^4 + b_2z^5 - a_3yz^3 - b_3yz^4 \in J. \end{aligned}$$

This implies $a_1 = a_2 = 0$, because the final three terms of each expression are already in J . By Buchberger’s criterion, it is also sufficient for these conditions to be satisfied. Therefore the fibers are 7-dimensional. \square

Lemma 4.8. *The preimage $U := \pi^{-1}(\mathfrak{u})$ is irreducible of dimension 12.*

Proof. By Proposition 3.8, it suffices to show that the fibers of π over \mathfrak{u} are irreducible of dimension 6. Let I be an ideal in \mathfrak{u} . Let V be the 1-dimensional subspace of S_1^* such that I_3 is orthogonal to $\text{Sym}_3(V)$ and let q_1, \dots, q_4 be the degree 2 generators of I . Choose a basis x, y, z of S_1 such that x, y is a basis for V^\perp . Then any J in $\phi^{-1}(I)$ is of the form

$$\langle q_i + a_i z^3 + b_i z^4 \mid 1 \leq i \leq 4 \rangle + \mathfrak{m}^5.$$

We claim that forcing $\text{in}_{(-1,-1,-1)}(J) = I$ imposes two linear conditions on the a_i s. Using the table of isomorphism classes of $(1, 3, 2)$ algebras in [Poonen 2008], one can check that for any 4-dimensional subspace $\langle q_1, q_2, q_3, q_4 \rangle$ of $\text{Sym}_2(V)$, the intersection of $\langle zq_i \mid 1 \leq i \leq 4 \rangle_3$ and $\langle xq_j, yq_j \mid 1 \leq j \leq 4 \rangle_3$ is 2-dimensional. After choosing a different basis for Q , we may assume $zq_1, zq_2 \in \langle xq_j, yq_j \mid 1 \leq j \leq 4 \rangle$. By using a similar argument to the one in Lemma 4.7, we see $a_1 = a_2 = 0$. Since the only other linear syzygies among the q_i s have no z coefficients and xz^3, yz^3 and \mathfrak{m}^5 are in the ideal, these are the only conditions imposed. Therefore, the fiber is 6-dimensional. \square

Therefore, it suffices to show the following irreducible sets are contained in the smoothable component.

$$H^d_{(1,d,1,\dots,1)}, H^d_{(1,d,2)}, P_d, Q_d, H^d_{(1,d,2,2)}, H^3_{(1,3,4)}, H^3_{(1,3,3)}, H^3_{(1,3,3,1)}, U, Z.$$

Smoothable algebras. In this section we prove that the irreducible sets

$$H^d_{(1,d,1,\dots,1)}, H^d_{(1,d,2)}, P_d, Q_d, H^d_{(1,d,2,2)}, H^3_{(1,3,4)},$$

are in the smoothable component by showing that a generic algebra in each is smoothable. We then show that the remaining irreducible sets,

$$H^3_{(1,3,3)}, H^3_{(1,3,3,1)}, U, Z,$$

are in the smoothable component by finding a point in each which is smoothable and a smooth point on the Hilbert scheme.

Proposition 4.9. *All algebras in $H^d_{(1,d,1,\dots,1)}$ are smoothable.*

Proof. We prove this by induction on d . Note the $d = 1$ case is trivial. Let m be the greatest integer such that h_m is nonzero. Then, by Proposition 4.4 we can take a generic ideal to be

$$I = \langle x_1^2 - x_d^m, \dots, x_{d-1}^2 - x_d^m, x_d^{m+1} \rangle + \langle x_i x_j \mid i \neq j \rangle.$$

We define J to be

$$J = \langle x_1^2 + x_1 - x_d^m, \dots, x_{d-1}^2 - x_d^m, x_d^{m+1} \rangle + \langle x_i x_j \mid i \neq j \rangle.$$

Note that J admits a decomposition as $J = J_1 \cap J_2$ where $J_1 = \langle x_1 + 1, x_2, x_3, \dots, x_d \rangle$ and

$$J_2 = \langle x_1 - x_d^m, x_2^2 - x_d^m, \dots, x_{d-1}^2 - x_d^m, x_d^{m+1} \rangle + \langle x_i x_j \mid i \neq j \rangle.$$

As the Hilbert function of J_2 equals $(1, d - 1, 1, \dots, 1)$, the inductive hypothesis implies that J_2 is smoothable. Thus J itself is also smoothable. Next note that $I \subset \text{in}_{(m, \dots, m, 2)}(J)$. Since both I and J have the same colength, we obtain the equality $I = \text{in}_{(m, \dots, m, 2)}(J)$. The corresponding Gröbner degeneration induces a map $\mathbb{A}^1 \rightarrow R_n^d$ which sends 0 to I . Thus I is smoothable. \square

Proposition 4.10. *All algebras in $H_{(1,d,2)}^d$ are smoothable.*

Proof. The proof is by induction on d . The case $d = 2$ follows from Theorem 2.4 of [Fogarty 1968].

Assume d is at least 3. Note that I_2^\perp defines a pencils of quadrics in d -variables. It then follows from [Harris 1992, Lemma 22.42] that, up to isomorphism, a generic ideal in $H_{(1,d,2)}^d$ is of the form

$$I = \langle x_i x_j \mid i \neq j \rangle + \langle x_i^2 - a_i x_{d-1}^2 - b_i x_d^2 \mid 1 \leq i \leq d - 2 \rangle,$$

with a_i and b_i elements of k .

Define

$$J_1 = \langle x_i x_j \mid i \neq j \rangle + \langle x_1 + a_1 x_{d-1}^2 + b_1 x_d^2, x_i^2 - a_i x_{d-1}^2 - b_i x_d^2 \mid 2 \leq i \leq d - 2 \rangle,$$

$$J_2 = \langle x_1 - 1, x_2, \dots, x_d \rangle.$$

Since J_1 has Hilbert function $(1, d - 1, 2)$, it is thus smoothable by the induction hypothesis. One can check that $I = \text{in}_{(1, \dots, 1)}(J_1 \cap J_2)$. Therefore I is smoothable. \square

Proposition 4.11. *All algebras in P_d are smoothable.*

Proof. Let I be a generic ideal in P_d . After a change of variables we may assume

$$I = \langle x_i x_j, x_\ell^2 + x_1^3, x_1^3 - x_2^3 \mid 1 \leq i < j \leq d, 2 < \ell \leq d \rangle.$$

One can check

$$I = \text{in}_{(2,2,3, \dots, 3)}(\langle x_1 + 1, x_j \mid j > 1 \rangle \cap \langle x_i x_j, x_\ell^2 + x_1^2, x_1^2 - x_2^3 \mid 1 \leq i < j \leq d, 2 < \ell \leq d \rangle).$$

The second ideal in the intersection has Hilbert function $(1, d, 1, 1)$, hence is smoothable by Proposition 4.9. It follows that I is smoothable. \square

Proposition 4.12. *All algebras in \mathcal{Q}_d are smoothable.*

Proof. Let I be a generic ideal in \mathcal{Q}_d . After a change of variables, we may assume

$$I = \langle x_1 x_\ell, x_i x_j + b_{(i,j)} x_1^3, x_k^2 - x_{k+1}^2 + b_k x_1^3 \mid \ell \neq 1, 1 < i < j \leq d, 1 < k < d \rangle.$$

Define

$$J_1 = \langle x_1 x_\ell, x_i x_j + b_{(i,j)} x_1^2, x_k^2 - x_{k+1}^2 + b_k x_1^2 \mid \ell \neq 1, 1 < i < j \leq d, 1 < k < d \rangle,$$

$$J_2 = \langle x_1 + 1, x_2, \dots, x_d \rangle.$$

Then J_1 has Hilbert function $(1, d, 2)$ so is smoothable by [Proposition 4.10](#). One can check that $I = \text{in}_{(2,3,\dots,3)}(J_1 \cap J_2)$, and thus I is smoothable. \square

Proposition 4.13. *All algebras in $H_{(1,d,2,2)}^d$ are smoothable.*

Proof. Let I be a generic ideal with Hilbert function $(1, d, 2, 2)$. After a change of variable, we may assume $(\pi(I))_2^\perp = \langle y_1^2, y_2^2 \rangle$. Thus I must be of the form

$$\langle x_\ell^2 - a_\ell x_1^3 - b_\ell x_2^3, x_i x_j - a_{ij} x_1^3 - b_{ij} x_2^3 \mid i < j, 2 < \ell \rangle + \mathfrak{m}^4.$$

Note I determines a symmetric bilinear map

$$\begin{aligned} \phi : (\mathfrak{m} : \mathfrak{m}^3) / \mathfrak{m}^2 \times (\mathfrak{m} : \mathfrak{m}^3) / \mathfrak{m}^2 &\rightarrow \mathfrak{m}^3 \cong k^2, \\ (x_i, x_j) &\mapsto a_{ij} x_1^3 + b_{ij} x_2^3. \end{aligned}$$

By composing ϕ with projections onto the two coordinates, we get a pair of symmetric bilinear forms. For a generic ϕ , these are linearly independent and their span is invariant under a change of basis on \mathfrak{m}^3 . By [[Harris 1992](#), Lemma 22.42], there exists a basis for $(\mathfrak{m} : \mathfrak{m}^3) / \mathfrak{m}^2$ and \mathfrak{m}^3 such that these bilinear forms are represented by diagonal matrices. Thus I has the following form

$$\langle x_\ell^2 - a_\ell x_1^3 - b_\ell x_2^3, x_i x_j - a_{ij} x_1^3 - b_{ij} x_2^3 \mid i < j, 2 < \ell \rangle + \mathfrak{m}^4,$$

where $a_{ij} = b_{ij} = 0$ if i and j are both greater than 2 and a_ℓ, b_ℓ are nonzero for all $\ell > 2$. After suitable changes of variable, we may assume $b_{ij} = a_{ij} = 0$ for all i, j . This gives the ideal

$$I = \langle x_\ell^2 - a_\ell x_1^3 - b_\ell x_2^3, x_i x_j, x_1^4, x_2^4 \mid i < j, 2 < \ell \rangle.$$

Now consider the following ideals:

$$J_1 := \langle x_\ell^2 - a_\ell x_1^2 - b_\ell x_2^3, x_i x_j, x_1^3, x_2^4 \mid i < j, 2 < \ell \rangle,$$

$$J_2 := \langle x_1 + 1, x_2, \dots, x_d \rangle.$$

Note J_1 is a $(1, d, 2, 1)$ ideal and in fact lies in the component \mathcal{Q}_d , and therefore is smoothable by [Proposition 4.12](#). One can check that $I = \text{in}_{(2,2,3,\dots,3)}(J_1 \cap J_2)$, and therefore I is smoothable. \square

Proposition 4.14. *All algebras in $H^3_{(1,3,4)}$ are smoothable.*

Proof. Such algebras are given by a 2-dimensional subspace of the space of quadratic forms, with isomorphisms given by the action of GL_3 . Arguing as in Proposition 4.10, we conclude that, up to isomorphism, a generic 2-dimensional space of quadrics is spanned by $x^2 + z^2$ and $y^2 + z^2$. Adding the necessary cubic generators, we conclude that $I = \langle y^2 + z^2, x^2 + z^2, z^3, yz^2, xz^2, xyz \rangle$ is a generic point of $H^3_{(1,3,4)}$.

Consider

$$J = \langle y^2 + z^2, x + x^2 + z^2, z^3, yz^2, xz^2, xyz \rangle.$$

Note that J is the intersection of an ideal of colength 3 and an ideal of colength 5:

$$J = \langle x + 1, y^2, yz, z^2 \rangle \cap \langle x + z^2, y^2 + z^2, z^3, yz^2 \rangle.$$

Since both ideals in the above intersection are smoothable, J itself is smoothable. One can check that $I = \text{in}_{(1,1,1)}(J)$. Therefore I is smoothable. \square

To prove that the remaining families are smoothable, we show that they contain a smooth point which is also smoothable. For this, we use the following result, which is well known (see for example [Miller and Sturmfels 2005, Lemma 18.10] in characteristic 0), but we give the proof in arbitrary characteristic for the reader's convenience:

Proposition 4.15. *All monomial ideals are smoothable.*

Proof. Suppose we have a monomial ideal of colength n , written in multiindex notation $I = \langle \mathbf{x}^{\alpha^{(1)}}, \dots, \mathbf{x}^{\alpha^{(m)}} \rangle$. Since k is algebraically closed, we can pick an arbitrarily long sequence a_1, a_2, \dots consisting of distinct elements in k . Define

$$f_i = \prod_{j=1}^d ((x_j - a_1)(x_j - a_2) \cdots (x_j - a_{\alpha_j^{(i)}})).$$

Note that $\text{in}(f_i) = \mathbf{x}^{\alpha^{(i)}}$ with respect to any global term order. Let J be the ideal generated by the f_i for $1 \leq i \leq m$ and then $\text{in}(J) \supset I$ and so J has colength at most n . However, for any standard monomial \mathbf{x}^β in I , we have a distinct point $(a_{\beta_1}, \dots, a_{\beta_d})$ in \mathbb{A}^d , and each f_i vanishes at this point. Therefore, J must be the radical ideal vanishing at exactly these points and have initial ideal I . Thus, I is smoothable. \square

Furthermore, the tangent space of an ideal I in the Hilbert scheme is isomorphic to $\text{Hom}_S(I, S/I)$. We use this fact to compute the dimension of the tangent space of a point I .

Proposition 4.16. *All algebras in $H^3_{(1,3,3)}$ are smoothable.*

Proof. This irreducible set includes the smoothable monomial ideal I generated by x^2, y^2, z^2, xyz . A direct computation shows I has a 21-dimensional tangent space, so I is a smooth point in H_7^3 . Thus, any algebra in $H_{(1,3,3)}^3$ is smoothable. \square

Proposition 4.17. *All algebras in $H_{(1,3,3,1)}^3$ are smoothable.*

Proof. The ideal $I = \langle x^2, y^2, z^2 \rangle$ in this locus is smoothable by [Proposition 4.15](#), and one can check that the Hilbert scheme is smooth at this point as well. Therefore $H_{(1,3,3,1)}^3$ is contained in the smoothable component of the Hilbert scheme. \square

Proposition 4.18. *All algebras in Z are smoothable.*

Proof. Consider $I = \langle x^2, xy, xz, yz, z^3 - y^4 \rangle \in Z$ and note that

$$I = \text{in}_{(1,0,0)}(\langle x+1, y, z \rangle \cap \langle x, yz, z^3 - y^4 \rangle).$$

The second ideal is smoothable [[Fogarty 1968](#), Theorem 2.4], so I is smoothable. One can also check I is smooth in the Hilbert scheme by computing the dimension of $\text{Hom}_S(I, S/I)$. Therefore Z is contained in the smoothable component of the Hilbert scheme. \square

Proposition 4.19. *All algebras in $U \subset H_{(1,3,2,1,1)}^3$ are smoothable.*

Proof. Consider the ideal $I = \langle x^2, xy - z^4, y^2 - xz, yz \rangle \in U$. One can check that

$$I = \text{in}_{(7,5,3)}(\langle x, y, z-1 \rangle \cap \langle x^2, xy - z^3, y^2 - xz, yz \rangle).$$

The second ideal in the intersection is in Q_3 and therefore smoothable by [Proposition 4.12](#). Therefore I is smoothable by the same argument in the proof of [Proposition 4.14](#). One can also check I has a 24-dimensional tangent space in the Hilbert scheme and is thus smooth. Therefore U is contained in the smoothable component. \square

Theorem 4.20. *With the exception of local algebras with Hilbert function $(1, 4, 3)$, every algebra with $n \leq 8$ is smoothable.*

Proof. The possible Hilbert functions are exactly the Hilbert functions of monomial ideals, and for d at least 3, one can check that there are no possibilities other those listed in [Table 4.1](#). For d at most 2, smoothability follows from Theorem 2.4 of [[Fogarty 1968](#)]. \square

In particular, this implies that there are no components other than the ones listed in [Theorem 1.1](#).

degree	Hilbert function \mathbf{h}	$\mathcal{H}_{\mathbf{h}}^d$ c.d.	reference	$H_{\mathbf{h}}^d$ c.d.	smoothability reference
4	1, 3	0	Prop. 3.3	0	Prop. 4.15
5	1, 3, 1	5	Prop. 3.3	5	Prop. 4.9
	1, 4	0	Prop. 3.3	0	Prop. 4.15
6	1, 3, 1, 1	2	Prop. 3.4	7	Prop. 4.10
	1, 4, 1	9	Prop. 3.3	9	Prop. 4.9
	1, 5	0	Prop. 3.3	0	Prop. 4.15
7	1, 3, 1, 1, 1	2	Prop. 3.4	9	Prop. 4.9
	1, 3, 2, 1	5, 6	Thm. 3.5	9, 10	Prop. 4.12, 4.11
	1, 3, 3	9	Prop. 3.3	9	Prop. 4.16
	1, 4, 1, 1	3	Prop. 3.4	12	Prop. 4.9
	1, 4, 2	16	Prop. 3.3	16	Prop. 4.10
	1, 5, 1	14	Prop. 3.3	14	Prop. 4.9
	1, 6	0	Prop. 3.3	0	Prop. 4.15
	8	1, 3, 1, 1, 1, 1	2	Prop. 3.4	11
1, 3, 2, 1, 1	6	Prop. 3.8	11(?), 12	Prop. 4.19, 4.18	
1, 3, 2, 2	4	Prop. 3.6	12	Prop. 4.13	
1, 3, 3, 1	9	Prop. 3.7	12	Prop. 4.17	
1, 3, 4	8	Prop. 3.3	8	Prop. 4.14	
1, 4, 1, 1, 1	3	Prop. 3.4	15	Prop. 4.9	
1, 4, 2, 1	7, 11	Thm. 3.5	15, 19	Prop. 4.12, 4.11	
1, 4, 3	21	Prop. 3.3	21	*	
1, 5, 2	26	Prop. 3.3	26	Prop. 4.10	
1, 5, 1, 1	4	Prop. 3.4	18	Prop. 4.9	
1, 6, 1	20	Prop. 3.3	20	Prop. 4.9	
1, 7	0	Prop. 3.3	0	Prop. 4.15	

Table 4.1. Summary of the decomposition of Hilbert schemes by Hilbert function of the local algebra with $h_1 \geq 3$. Here c.d. stands for “component dimensions”. The component dimensions of $H_{\mathbf{h}}^d$ are computed using Propositions 4.3 and 4.4. In the case of $\mathbf{h} = (1, 3, 2, 1, 1)$, Lemmas 4.7 and 4.8 show that $H_{\mathbf{h}}^d$ is the union of two irreducible sets, but we don’t know whether the smaller set is contained in the closure of the larger one.

5. Characterization of smoothable points of H_8^4

In this section k will denote a field of characteristic not 2 or 3, except for Section 5D where $k = \mathbb{C}$.

We show that besides the smoothable component, the Hilbert scheme H_8^4 contains a second component parametrizing the local algebras with $\mathbf{h} = (1, 4, 3)$. We

prove that the intersection of the two components can be described as in [Theorem 1.3](#), and as a result we determine exactly which algebras with Hilbert function $(1, 4, 3)$ are smoothable. We will use G_0 to denote the standard graded Hilbert scheme $\mathcal{H}_{(1,4,3)}^4 \cong \text{Gr}(7, S_2)$, which we will think of as a closed subscheme of H_8^4 .

In [Section 5B](#) we introduce and investigate the Pfaffian which appears in [Theorem 1.3](#), and we prove the crucial fact that it is the unique GL_4 -invariant of minimal degree. In [Section 5C](#), we give a first approximation of the intersection locus. We then use the uniqueness results from [Section 5B](#) to prove [Theorem 1.3](#). We begin by proving reducibility,

Proposition 5.1. *For d at least 4, the Hilbert scheme H_8^d is reducible.*

Proof. It is sufficient to find a single ideal whose tangent space dimension is less than $8d = \dim R_8^d$. Consider the ideal

$$J = \langle x_1^2, x_1x_2, x_2^2, x_3^2, x_3x_4, x_4^2, x_1x_4 + x_2x_3 \rangle + \langle x_i \mid 4 < i \leq d \rangle.$$

The tangent space of J in H_8^d can be computed as $\dim_k \text{Hom}_S(J, S/J)$. A direct computation shows that an arbitrary element of $\text{Hom}_S(J, S/J)$ can be represented as a matrix

$$\begin{matrix} & x_1^2 & x_1x_2 & x_2^2 & x_3^2 & x_3x_4 & x_4^2 & x_1x_4+x_2x_3 & x_i \\ \begin{matrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_1x_3 \\ x_1x_4 \\ x_2x_4 \end{matrix} & \left(\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 2a_2 & a_1 & 0 & 0 & 0 & 0 & 0 & a_4 & * \\ 0 & a_2 & 2a_1 & 0 & 0 & 0 & 0 & a_3 & * \\ 0 & 0 & 0 & 2a_3 & a_4 & 0 & 0 & a_1 & * \\ 0 & 0 & 0 & 0 & a_3 & 2a_4 & 0 & a_2 & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{matrix} \right), \end{matrix}$$

where i again ranges over $4 < i \leq d$, the a_i are any elements in k , and each $*$ represents an independent choice of an element of k . Thus, $\dim_k \text{Hom}(J, S/J) = 4 + 21 + 8(d - 4) = 8d - 7$. The computation holds in all characteristics. Since $8d - 7 < 8d = \dim(R_8^d)$, we conclude that J is not smoothable and that H_8^d is reducible. \square

Remark 5.2. This proposition holds with the same proof even when $\text{char } k = 2, 3$.

5A. The irreducible components of H_8^4 . Consider H_8^4 with its universal ideal sheaf \mathcal{I} and let $\mathcal{A} = \mathbb{C}_{H_8^4}[x_1, \dots, x_4]/\mathcal{I}$. On each open affine $U = \text{Spec } B$ such that $\mathcal{A}|_U$ is free, define $f_i \in B$ to be $\frac{1}{8} \text{tr}(X_i)$ where X_i is the operator on the free B -module $\mathcal{A}(U)$ defined by multiplication by x_i . We think of the f_i as being the “center of mass” functions for the subscheme of \mathbb{A}_B^4 defined by $\mathcal{I}|_U$. Note that the

definitions of f_i commute with localization and thus they lift to define elements $f_i \in \Gamma(H_8^4, \mathcal{O}_{H_8^4})$, which determine a morphism $f : H_8^4 \rightarrow \mathbb{A}^4$.

Considered as an additive group, \mathbb{A}^4 acts on H_8^4 by translation. We define the “recentering” map r to be the composition

$$r : H_8^4 \xrightarrow{-f \times \text{id}} \mathbb{A}^4 \times H_8^4 \longrightarrow H_8^4.$$

By forgetting about the grading of ideals, we have a closed immersion ι of $G_0 \cong \mathcal{H}_{(1,4,3)}^4$ into H_8^4 [Haiman and Sturmfels 2004, Proposition 1.5]. We define G to be the preimage of this closed subscheme via the “recentering” map, that is, the fiber product

$$\begin{array}{ccc} G & \longrightarrow & H_8^4 \\ \downarrow & & \downarrow r \\ G_0 & \xrightarrow{\iota} & H_8^4 \end{array}$$

We define intersections $W := G \cap R_8^4$ and $W_0 := G_0 \cap R_8^4$. We will focus on W_0 , and the following lemma shows that this is sufficient for describing W .

Lemma 5.3. *We have isomorphisms $G \cong G_0 \times \mathbb{A}^4$ and $W \cong W_0 \times \mathbb{A}^4$.*

Proof. We have a map $G \rightarrow G_0$, and a map $f : H_8^4 \rightarrow \mathbb{A}^4$. We claim that the induced map $\phi : G \rightarrow \mathbb{A}^4 \times G_0$ is an isomorphism.

Define $\psi : \mathbb{A}^4 \times G_0 \rightarrow H_8^4$ to be the closed immersion ι followed by translation. We work with an open affine $U \cong \text{Spec } A \subset G_0$ such that the restriction $\iota|_U$ corresponds to an ideal $I \subset A[x_1, \dots, x_4]$ whose cokernel is a graded A -module with free components of ranks $(1, 4, 3)$. The map $\psi|_{\mathbb{A}^4 \times U}$ corresponds to the ideal $I' \subset A[t_1, \dots, t_4][x'_1, \dots, x'_4]$ where I' is the image of I under the homomorphism of A -algebras that sends x_i to $x'_i + t_i$. Then $A[t_1, \dots, t_4][x'_1, \dots, x'_4]/I' \cong (A[x_1, \dots, x_4]/I)[t_1, \dots, t_4]$ is a graded $A[t_1, \dots, t_4]$ -algebra with $x'_i - t_i = x_i$ homogeneous of degree 1. The key point is that as operators on a free $A[t_1, \dots, t_4]$ -module, the x_i have trace zero, so the trace of the x'_i is $8t_i$. Thus, $r \circ \psi : \mathbb{A}^4 \times G_0 \rightarrow H_8^4$ corresponds to an ideal $I'' \subset A[t_1, \dots, t_4][x''_1, \dots, x''_4]$ which is the image of I' under the homomorphism that takes x'_i to $x''_i - t_i$. This is of course the extension of $I \subset A[x_1, \dots, x_4]$ to $A[t_1, \dots, t_4][x''_1, \dots, x''_4]$ with $x''_i = x_i$, and so I'' has the required properties such that $r \circ \psi$ factors through the closed immersion ι . Thus, ψ maps to G . Furthermore, we see that $r \circ \psi$ is the projection onto the first coordinate of $\mathbb{A}^4 \times G_0$ and $f \circ \psi$ is projection onto the second coordinate. Thus, $\phi \circ \psi$ is the identity.

Second, we check that the composition $\psi \circ \phi$ is the identity on G . This is clear because $\psi \circ \phi$ amounts to translation by $-f$ followed by translation by f .

The isomorphism for W follows by the same argument. □

Lemma 5.4. *W and W_0 are prime divisors in G and G_0 respectively.*

Proof. The point $I = \langle x_1^2, x_1x_2, x_2^2, x_3^2, x_3x_4, x_4^2, x_1x_4 \rangle$ belongs to R_8^4 and to G and has a 33-dimensional tangent space in any characteristic. As a result, an open set U around I in the Hilbert scheme is a closed subscheme of a smooth 33-dimensional variety Y . By the subadditivity of codimension of intersections, as in [Harris 1992, Theorem 17.24], it follows that every component of W through I has codimension 1 in G .

To show integrality, fix a monomial ideal $M_\lambda \in G_0$, and let $U_\lambda \subset H_8^4$ be the corresponding open set, as in our discussion of coordinates for the Hilbert scheme of points (page 766). For any $I \in U_\lambda$ the initial ideal $\text{in}_{(1,1,1,1)}(I) \in G_0 \cap U_\lambda$ and is generated by the $(1, 1, 1, 1)$ -leading forms of the given generating set of I and all cubics. Thus we may define a projection morphism $\pi : U_\lambda \rightarrow G_0 \cap U_\lambda$ which corresponds to taking the $(1, 1, 1, 1)$ -initial ideal. Since R_8^4 is integral, so is the image $\pi(R_8^4 \cap U_\lambda) = W_0 \cap U_\lambda$. Thus W_0 and $W_0 \times \mathbb{A}^4 \cong W$ are integral. \square

5B. A GL_4 -invariant of a system of three quadrics. In this section we study the Pfaffian which appears in the statement of Theorem 1.3. Recall that any $I \in G_0$ defines a 3-dimensional subspace $I_2^\perp \subset S_2^*$.

Let Q_1, Q_2, Q_3 be a basis of quadrics for I_2^\perp , let A_1, A_2, A_3 be the symmetric 4×4 matrices which represent the Q_i via $\vec{y}^t A_i \vec{y} = Q_i$ where \vec{y} is the vector of formal variables (y_1, y_2, y_3, y_4) .

Definition 5.5. *The Salmon–Turnbull Pfaffian is the Pfaffian (that is, the square root of the determinant) of the skew-symmetric 12×12 matrix*

$$M_I = \begin{pmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{pmatrix}.$$

Lemma 5.6. *The Salmon–Turnbull Pfaffian of M_I coincides up to scaling with the Pfaffian of the skew-symmetric bilinear form $\langle \cdot, \cdot \rangle_I : (S_1 \otimes S_2/I_2)^{\otimes 2} \rightarrow \bigwedge^3 S_2/I_2 \cong k$ defined by*

$$\langle l_1 \otimes q_1, l_2 \otimes q_2 \rangle_I = (l_1 l_2) \wedge q_1 \wedge q_2.$$

In particular, the vanishing of the Salmon–Turnbull Pfaffian is independent of the choice of basis of I_2^\perp and invariant under the GL_4 action induced by linear change of coordinates on S .

Proof. Let m_1, m_2, m_3 be any basis of S_2/I_2 and let x_1, x_2, x_3, x_4 be a basis for S_1 . Let A_i be the matrix representation with respect to this basis of the symmetric bilinear form obtained by composing multiplication $S_1 \otimes S_1 \rightarrow S_2/I_2$ with projection onto m_i . Note that if m_1, m_2, m_3 form a basis dual to $\frac{1}{2}Q_1, \frac{1}{2}Q_2, \frac{1}{2}Q_3$ then this definition of A_i agrees with the definition of A_i above. Thus, $x_j x_{j'} =$

$\sum_i (A_i)_{jj'} m_i$ where $(A_i)_{jj'}$ is the (j, j') entry of A_i . Then we will use the basis $x_1 \otimes m_3, x_2 \otimes m_3, \dots, x_4 \otimes m_1$ for $S_1 \otimes S_2 / I_2$. We compute the matrix representation of $\langle \cdot, \cdot \rangle_I$ in this basis:

$$\begin{aligned} \langle x_j \otimes m_i, x_{j'} \otimes m_{i'} \rangle_I &= (x_j x_{j'}) \wedge m_i \wedge m_{i'} \\ &= \left(\sum_{1 \leq \ell \leq 3} (A_\ell)_{jj'} m_\ell \right) \wedge m_i \wedge m_{i'} \end{aligned}$$

If $i = i'$, this quantity will be zero. Otherwise, let i'' be the index which is not i or i' and then we get

$$= (A_{i''})_{jj'} m_{i''} \wedge m_i \wedge m_{i'} = \pm (A_{i''})_{jj'} m_1 \wedge m_2 \wedge m_3,$$

where \pm is the sign of the permutation which sends $1, 2, 3$ to i'', i, i' respectively. Thus, with $m_1 \wedge m_2 \wedge m_3$ as the basis for $\bigwedge^3 S_2 / I_2$, $\langle \cdot, \cdot \rangle_I$ is represented as

$$\begin{pmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{pmatrix}. \quad \square$$

Since the vanishing of the Salmon–Turnbull Pfaffian depends only on the vector subspace $I_2^\perp \subset S_2^*$, it defines a function P on G_0 which is homogeneous of degree 2 in the Plücker coordinates. We next show that the Salmon–Turnbull Pfaffian is irreducible and that, over the complex numbers, it is uniquely determined by its degree and GL_4 -invariance.

Lemma 5.7. *There are no polynomials of degree 1 in the Plücker coordinates of G_0 whose vanishing locus is invariant under the action of the algebraic group GL_4 . Therefore, the Salmon–Turnbull Pfaffian is irreducible.*

Proof. We may prove this lemma by passing to the algebraic closure, and we thus assume that k is algebraically closed. Let $W = \bigwedge^3 S_2^*$ and consider the Plücker embedding of $\text{Gr}(3, S_2^*)$ in $\mathbb{P}(W) = \text{Proj}(R)$ where R is the polynomial ring $k[p_{ij\ell}]$ where $\{i, j, \ell\}$ runs over all unordered triplets of monomials in S_2^* . The Plücker coordinate ring A is the quotient of R by a homogeneous ideal J . In each degree e , we obtain a split exact sequence of GL_4 -representations:

$$0 \rightarrow J_e \rightarrow \text{Sym}_e(W) \rightarrow A_e \rightarrow 0.$$

Since $J_1 = 0$ we have $\text{Sym}_1(W) = A_1$, and it suffices to show that this has no 1-dimensional subrepresentations. Given a monomial $i \in S_2^*$ let $\alpha_i \in \mathbb{N}^4$ be its multiindex. For $\theta = (\theta_1, \dots, \theta_4)$, let L be the diagonal matrix with $L_{mm} = \theta_m$. The action of L on the Plücker coordinate $p_{ij\ell}$ is to scale it by $\theta^{\alpha_i + \alpha_j + \alpha_\ell}$.

Suppose that there exists an invariant polynomial $F = \sum c_{ij\ell} p_{ij\ell}$ in $\text{Sym}_1(W)$. Then $L \cdot F = \lambda F$ for some $\lambda \in k^\times$. But since $L \cdot F = \sum c_{ij\ell} \theta^{\alpha_i + \alpha_j + \alpha_\ell} p_{ij\ell}$ it follows that whenever $c_{ij\ell}$ and $c_{i'j'\ell'}$ are both nonzero, then $\alpha_i + \alpha_j + \alpha_\ell = \alpha_{i'} + \alpha_{j'} + \alpha_{\ell'}$. However there are no multiindices of total degree 6 which are also symmetric in $\theta_1, \theta_2, \theta_3, \theta_4$. Thus each $c_{ij\ell} = 0$ and there are no nontrivial GL_4 -invariant polynomials of degree 1. In particular no product of linear polynomials is GL_4 -invariant, and thus the Salmon–Turnbull Pfaffian is irreducible. \square

Lemma 5.8. *If $k = \mathbb{C}$, then there is a polynomial of degree 2 in the Plücker coordinates, unique up to scaling, whose vanishing locus is GL_4 -invariant. This is the Salmon–Turnbull Pfaffian.*

Proof. We take the same notation as in the proof of Lemma 5.7 and recall that we have a split exact sequence of $\text{GL}_4(\mathbb{C})$ -representations:

$$0 \rightarrow J_2 \rightarrow \text{Sym}_2(W) \rightarrow A_2 \rightarrow 0.$$

We determine the irreducible subrepresentations of $\text{Sym}_2(W)$ by computing the following Schur function decomposition of its character χ :

$$\begin{aligned} \chi = & s_{(8,2,2)} + s_{(7,4,1)} + 2s_{(7,3,1,1)} + s_{(7,2,1,1)} + s_{(6,6)} + 3s_{(6,4,2)} + s_{(6,4,1,1)} \\ & + 2s_{(6,4,1,1)} + 2s_{(6,3,2,1)} + s_{(6,2,2,2)} + 2s_{(5,5,1,1)} + s_{(5,4,3)} + s_{(5,4,2,1)} + s_{(5,3,3,1)} \\ & + s_{(4,4,4)} + s_{(4,4,3,1)} + 2s_{(4,4,2,2)} + s_{(3,3,3,3)}. \end{aligned}$$

We conclude that $\text{Sym}_2(W)$ contains a unique 1-dimensional subrepresentation with character $s_{(3,3,3,3)}$. It follows from this and Lemma 5.6 that, over \mathbb{C} , the Salmon–Turnbull Pfaffian is the only GL_4 -invariant of degree 2 in the Plücker coordinates. \square

Remark 5.9. Salmon gives a geometric description of the Salmon–Turnbull Pfaffian [Salmon 1874, pp. 242–244], where he shows that the Pfaffian vanishes whenever there exists a cubic form C and three linear differential operators d_1, d_2, d_3 such that $d_i C = Q_i$. Turnbull also describes this invariant in his study of ternary quadrics [Turnbull 1922].

5C. A first approximation to the intersection locus. Any point I in W_0 is a singular point in the Hilbert scheme. In Lemma 5.11, we construct an equation that cuts out the singular locus over an open set of G_0 . The local equation defines a nonreduced divisor whose support contains W_0 . The following subsets of G_0 will be used in this section:

$$\begin{aligned} G'_0 &:= \{I \in G_0 \mid \text{the ideal } I \text{ is generated in degree 2}\}, \\ Z_1 &:= G_0 \setminus G'_0, \\ Z_2 &:= \{I \in G_0 \mid \text{Hom}(I, S/I)_{-2} \neq 0\}. \end{aligned}$$

Note that G'_0 is open in G_0 and that every ideal in G'_0 is generated by seven quadrics. The set Z_2 will be used in [Lemma 5.16](#). If I is any ideal in G_0 , the tangent space $\text{Hom}_S(I, S/I)$ is graded. The following lemma shows that if we want to determine whether I is a singular point in the Hilbert scheme, then it suffices to compute only the degree -1 component of the tangent space.

Lemma 5.10. *For any $I \in G'_0$ we have $\dim_k \text{Hom}_S(I, S/I)_{-1} \geq 4$, and I is singular in H^4_8 if and only if $\dim_k \text{Hom}_S(I, S/I)_{-1} \geq 5$.*

Proof. Since S/I is concentrated in degrees 0, 1 and 2, and $I \in G'_0$ is minimally generated only in degree 2, we have that $\text{Hom}_S(I, S/I)$ is concentrated in degrees 0, -1 , -2 . Furthermore, since $I \in G'_0$ we have that $\dim_k \text{Hom}_S(I, S/I)_0 = 21$, because any k -linear map $I_2 \rightarrow (S/I)_2$ will be S -linear. Next, note that the morphisms $t_i : I_2 \rightarrow (S/I)_1$ mapping q_j to the class of $\partial q_j / \partial x_i$ are S -linear morphisms, and thus we have $\text{Hom}_S(I, S/I)_{-1}$ is at least 4-dimensional.

Since the dimension of G'_0 is 25, I must be singular if $\dim_k \text{Hom}_S(I, S/I)_{-1} > 4$. Conversely, assume for contradiction that there exists an I such that I is singular and dimension of $\text{Hom}_S(I, S/I)_{-1}$ is exactly 4. Since I is singular, we have that $\text{Hom}_S(I, S/I)_{-2}$ is nontrivial. Let $\phi \in \text{Hom}_S(I, S/I)_{-2}$ be a nonzero map. By changing the generators of I we may assume that $\phi(q_i) = 0$ for $i = 1, \dots, 6$ and $\phi(q_7) = 1$. Now the vector space $\langle x_1\phi, x_2\phi, x_3\phi, x_4\phi \rangle$ is a 4-dimensional subspace of $\text{Hom}(I, S/I)_{-1}$. Since we have assumed that $\dim_k \text{Hom}(I, S/I)_{-1} = 4$ it must be the case that the space $\langle x_1\phi, x_2\phi, x_3\phi, x_4\phi \rangle$ equals the space $\langle t_1, \dots, t_4 \rangle$. However, this would imply that all partial derivatives of q_1 are zero, which is impossible. \square

Now we will investigate those ideals which have extra tangent vectors in degree -1 . If $\phi : I_2 \rightarrow (S/I)_1$ is a k -linear map then ϕ will be S -linear if and only if ϕ satisfies the syzygies of I modulo I . In other words, ϕ should belong to the kernel of

$$\begin{aligned} \text{Hom}_k(I_2, (S/I)_1) &\rightarrow \text{Hom}_k(\text{Syz}(I), (S/I)), \\ \phi &\mapsto (\sigma \mapsto \overline{\sigma(\phi)}). \end{aligned}$$

Since I contains m^3 and is generated by quadrics, it suffices to consider linear syzygies σ and we have an exact sequence

$$0 \rightarrow \text{Hom}_S(I, S/I)_{-1} \longrightarrow \text{Hom}_k(I_2, (S/I)_1) \xrightarrow{\psi} \text{Hom}_k(\text{Syz}(I)_1, (S/I)_2),$$

where $\text{Syz}(I)_1$ is the vector space of linear syzygies. We see that the t_i from the previous lemma span a 4-dimensional subspace T of the kernel of ψ . We obtain

$$\text{Hom}_k(I, S/I)_{-1}/T \xrightarrow{\overline{\psi}} \text{Hom}_k(\text{Syz}(I)_1, (S/I)_2),$$

and $I \in G'_0$ will be a singular point if and only if $\ker(\overline{\psi}) \neq 0$. Since I is generated by quadrics, it follows that $\text{Syz}(I)_1$ has dimension $4 \cdot 7 - 20 = 8$. Therefore $\overline{\psi}$ is a map between 24-dimensional spaces. Thus $\det(\overline{\psi})$ vanishes if and only if $I \in G'_0$ is a singular point in H^4 .

The global version of this determinant will give a divisor whose support contains W_0 . On G'_0 we have the $\mathbb{C}_{G'_0}$ -algebra $\mathcal{S} := \mathbb{C}_{G'_0}[x_1, x_2, x_3, x_4]$, which is graded in the standard way, $\mathcal{S} = \bigoplus_i \mathcal{S}_i$. We have a graded universal ideal sheaf $\mathcal{I} = \bigoplus \mathcal{I}_i$, and a universal sheaf of graded algebras $\mathcal{S}/\mathcal{I} = \bigoplus \mathcal{S}_i/\mathcal{I}_i$. For all i the sheaves \mathcal{S}_i , \mathcal{I}_i and $\mathcal{S}_i/\mathcal{I}_i$ are coherent locally free $\mathcal{O}_{G'_0}$ -modules.

Let $\mu : \mathcal{I}_2 \otimes \mathcal{I}_1 \rightarrow \mathcal{I}_3$ be the multiplication map. Surjectivity of this map follows from the definition of G'_0 . We define \mathcal{K}_1 to be the kernel of this map, so that we have the exact sequence

$$0 \rightarrow \mathcal{K}_1 \rightarrow \mathcal{I}_2 \otimes \mathcal{I}_1 \xrightarrow{\mu} \mathcal{I}_3 \rightarrow 0. \tag{5-1}$$

In other words, \mathcal{K}_1 is the sheaf of linear syzygies. Let U be an open subset of G'_0 such that $\mathcal{I}_2|_U$ is free. Denote generators of $\mathcal{I}_2(U)$ by q_1, \dots, q_7 and thus we have

$$\mathcal{K}_1(U) = \left\{ \left(\sum_{i=1}^7 q_i \otimes l_i \right) \mid l_i \in \mathcal{I}_1(U), \sum q_i l_i = 0 \in \mathcal{I}_3 \right\}.$$

To simplify notation in the following lemma we write $\mathcal{H}om$ to denote $\mathcal{H}om_{\mathbb{C}_{G'_0}}$.

Lemma 5.11. (1) *On G'_0 there is a morphism of locally free sheaves of ranks 28 and 24 respectively:*

$$h : \mathcal{H}om(\mathcal{I}_2, \mathcal{I}_1) \rightarrow \mathcal{H}om(\mathcal{K}_1, \mathcal{I}_2/\mathcal{I}_2),$$

such that for any $I \in G'_0$, we have $\ker(h \otimes k(I)) = \text{Hom}(I, S/I)_{-1}$.

(2) *There is a locally free subsheaf of rank four $\mathcal{T} \subset \ker(h)$ inducing a morphism:*

$$\overline{h} : \mathcal{H}om(\mathcal{I}_2, \mathcal{I}_1)/\mathcal{T} \rightarrow \mathcal{H}om(\mathcal{K}_1, \mathcal{I}_2/\mathcal{I}_2),$$

such that $\ker(\overline{h} \otimes k(I)) \neq 0$ if and only if $\dim_k \text{Hom}(I, S/I)_{-1} \geq 5$.

Proof. (1) We have a map of locally free $\mathbb{C}_{G'_0}$ -modules: $\mathcal{K}_1 \rightarrow \mathcal{I}_2 \otimes \mathcal{I}_1$. This induces the map $\mathcal{K}_1 \otimes \check{\mathcal{I}}_1 \rightarrow \mathcal{I}_2$. Applying $\mathcal{H}om(-, \mathcal{I}_1)$ to both sides we get

$$\mathcal{H}om(\mathcal{I}_2, \mathcal{I}_1) \rightarrow \mathcal{H}om(\mathcal{K}_1 \otimes \check{\mathcal{I}}_1, \mathcal{I}_1) \cong \mathcal{H}om(\mathcal{K}_1, \mathcal{I}_1 \otimes \mathcal{I}_1).$$

For the isomorphism above, we are using identities about $\mathcal{H}om$, tensor product of sheaves, and sheaf duality from [Hartshorne 1977, p. 123]. The sequence $\mathcal{I}_1 \otimes \mathcal{I}_1 \rightarrow \mathcal{I}_2 \rightarrow \mathcal{I}_2/\mathcal{I}_2$ gives a map from $\mathcal{H}om(\mathcal{K}_1, \mathcal{I}_1 \otimes \mathcal{I}_1) \rightarrow \mathcal{H}om(\mathcal{K}_1, \mathcal{I}_2/\mathcal{I}_2)$. By composition we obtain the desired map:

$$h : \mathcal{H}om(\mathcal{I}_2, \mathcal{I}_1) \rightarrow \mathcal{H}om(\mathcal{K}_1, \mathcal{I}_2/\mathcal{I}_2).$$

Let us take a moment and consider h in concrete terms, since this will be used for proving part (2) of the lemma. Let $U \subset G'_0$ be an open subset such that all relevant locally free sheaves are in fact free. Let q_1, \dots, q_7 be the generators of $\mathcal{F}_2(U)$ and let $\sigma_j := \sum_{i=1}^7 q_i \otimes l_{ij}$ for $1 \leq j \leq 8$ be the generators of $\mathcal{H}_1(U)$. Finally, let $\phi \in \text{Hom}(\mathcal{F}_2, \mathcal{S}_1)$ be a map ($q_i \mapsto m_i$). Then $h(\phi)$ is the map

$$\sigma_j \mapsto \sum \overline{m_i l_i},$$

where $\overline{m_i l_i}$ is the reduction of $m_i l_i$ modulo \mathcal{F}_2 .

(2) Over any U where \mathcal{F}_2 is free, let q_1, \dots, q_7 the global generators. Then we define $t_1 : q_i \mapsto \frac{\partial}{\partial x_1} q_i$, and we define t_2, t_3, t_4 similarly. This defines a locally free subsheaf $\mathcal{T}(U) := \langle t_1, \dots, t_4 \rangle \subset \mathcal{H}om(\mathcal{F}_2, \mathcal{S}_1)$ of rank 4. By the proof of [Lemma 5.10](#), the injection $\mathcal{T} \rightarrow \mathcal{H}om(\mathcal{F}_2, \mathcal{S}_1)$ remains exact under pullback to a point. It follows that the quotient $\mathcal{H}om(\mathcal{F}_2, \mathcal{S}_1)/\mathcal{T}$ is locally free of rank 24 [[Hartshorne 1977](#), Ex II.5.8].

It remains to show that $\mathcal{T} \subset \ker(h)$ and that $\ker(\overline{h} \otimes k(I))$ is nontrivial if and only if $\dim_k \text{Hom}(I, S/I)_1 \geq 5$. This is immediate from the discussion preceding this theorem. □

By the previous lemma, \overline{h} is a map between locally free sheaves of rank 24, and thus $\det(\overline{h})$ defines a divisor on G'_0 . To ensure that this is the restriction of a unique divisor on G_0 , we need to verify that Z_1 and Z_2 are not too large. For this, we construct the rational curve $\tau : \mathbb{P}^1 \rightarrow G_0$ defined for $t \neq \infty$ by

$$I_t = (x_1^2, x_2^2, x_3^2, x_4^2, x_1 x_2, x_2 x_3 + t x_3 x_4, x_1 x_4 + t x_3 x_4). \tag{5-2}$$

Lemma 5.12. $Z_1 \cup Z_2$ is a closed set of codimension at least 2 in G_0 .

Proof. Z_1 is closed because it is the support of the cokernel of the multiplication map $\mathcal{F}_2 \otimes \mathcal{S}_1 \rightarrow \mathcal{S}_3$. The intersection $Z_2 \cap G'_0$ is the degeneracy locus of

$$\mathcal{H}om(\mathcal{F}_2, \mathcal{S}_0) \rightarrow \mathcal{H}om(\mathcal{H}_1, \mathcal{S}_1) \oplus \mathcal{H}om(\mathcal{H}_2, \mathcal{S}_2/I_2),$$

which is the analogue of [Lemma 5.11](#) (1) for computing $\text{Hom}(I_2, S_0)$. Thus $Z_1 \cup Z_2$ is closed in G_0 .

Because $\text{Pic}(G_0) = \mathbb{Z}$ and G_0 is projective, checking that the 1-cycle τ does not intersect $Z_1 \cup Z_2$ will show that $Z_1 \cup Z_2$ has codimension at least 2. By passing to the algebraic closure, we can assume that k is algebraically closed. The group k^\times acts on \mathbb{A}^4 by $\alpha \cdot (x_1, x_2, x_3, x_4) = (x_1, x_2, \alpha x_3, \alpha x_4)$, and taking $\alpha = t$ maps I_1 to I_t , for any t other than 0 or ∞ . Thus, it suffices to check that the following three

ideals do not intersect $Z_1 \cup Z_2$:

$$\begin{aligned} I_0 &= (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3, x_1x_4), \\ I_1 &= (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3 + x_3x_4, x_1x_4 + x_3x_4), \\ I_\infty &= (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3 - x_1x_4, x_3x_4). \end{aligned}$$

It is obvious that these are generated in degree 2. A change of variables transforms I_∞ to the ideal J from [Proposition 5.1](#), which is smooth, so $\text{Hom}(I_t, S/I_t)_{-2} = 0$ for $t = 1, \infty$. One can also check that $\text{Hom}(I_0, S/I_0)_{-2} = 0$, which holds in all characteristics because I_0 is a monomial ideal. Therefore, $Z_1 \cup Z_2$ has codimension at least 2. □

Lemma 5.13. *Let D be the divisor on G_0 defined locally by $\det(\bar{h})$. Then W_0 belongs to the support of D .*

Proof. The Hilbert scheme is singular on W_0 , so $W_0 \cap G'_0 \subset V(\det(\bar{h}))$. Since W_0 is a divisor, [Lemma 5.12](#) tells us that W_0 intersects G'_0 , so the irreducibility of W_0 means that it is contained in D . □

5D. An equation for W_0 . In this section, except for the last paragraph, we restrict to the case $k = \mathbb{C}$ in order to use the representation theory of $\text{GL}_4(\mathbb{C})$.

We will use the result of [Lemma 5.13](#) to give an upper bound on the degree of W_0 in terms of Plücker coordinates. This leads to a proof of [Theorem 1.3](#) over \mathbb{C} . The restriction to \mathbb{C} will be removed in the next section.

Let H be an effective divisor which generates $\text{Pic}(G_0) = \mathbb{Z}$. First we compute the degree of D in Plücker coordinates, using the rational curve τ .

Lemma 5.14. *The curve τ has intersection multiplicities*

$$\tau \cdot H = 1 \quad \text{and} \quad \tau \cdot D = 16.$$

Proof. For the first statement, let p_1 and p_2 be the Plücker coordinates corresponding to the $(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3, x_3x_4)$ - and $(x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3, x_1x_4)$ -minors respectively. Then $L = V(p_1)$ does not meet I_t at infinity. For $t \neq \infty$, we see that $p_2(I_t) \neq 0$, so local equations for L valid at all common points of L and τ are given by $L = \frac{p_1}{p_2}$. Since this equation pulls back to t on $\mathbb{P}^1 - \infty$ the statement follows.

For the second statement, note from the proof of [Lemma 5.12](#) that I_∞ is a smooth point and τ does not intersect Z_1 or Z_2 . Therefore, it suffices to check the degree on the open affine defined by $t \neq \infty$.

For every $t \neq \infty$, I_t has the following 8 linear syzygies, where q_1, \dots, q_7 are the generators of I_t in the order in [Equation \(5-2\)](#).

$$\begin{array}{l|l}
 \sigma_1 = x_2q_1 - x_1q_5 & \sigma_2 = x_4q_1 - x_1q_7 + tx_3q_7 - t^2x_4q_3 \\
 \sigma_3 = x_1q_2 - x_2q_5 & \sigma_4 = x_3q_2 - x_2q_6 + tx_4q_6 - t^2x_3q_4 \\
 \sigma_5 = x_2q_3 - x_3q_6 + tx_4q_3 & \sigma_6 = x_1q_4 - x_4q_7 + tx_3q_4 \\
 \sigma_7 = x_3q_5 - x_1q_6 + tx_3q_7 - t^2x_4q_3 & \sigma_8 = x_4q_5 - x_2q_7 + tx_4q_6 - t^2x_3q_4
 \end{array}$$

The intersection number $\tau \cdot D$ equals the degree of $\tau^*(\det(\bar{h}))$, which we compute by writing out $\tau^*(h)$ as a matrix. Let $\phi \in \text{Hom}(I, S/I)_{-1}$ be written as $\phi(q_i) = c_{i,1}x_1 + c_{i,2}x_2 + c_{i,3}x_3 + c_{i,4}x_4$ and recall that, if $\sigma_j = \sum_i q_i \otimes l_{ij}$ then $h(\phi)(\sigma_j) = \sum \overline{\phi(q_i)l_{ij}}$ where the bar indicates that we are considering the image as an element of S_2/I_2 . The monomials x_1x_3 , x_2x_4 and x_1x_4 are a basis of S_2/I_2 for $t \neq \infty$, so we can explicitly express the $h(\phi)(\sigma_j)$ as follows:

$$\begin{array}{l|l}
 h(\phi)(\sigma_1) & -c_{5,3}x_1x_3 + c_{1,4}x_2x_4 + (-tc_{1,3} + tc_{5,4})x_3x_4 \\
 h(\phi)(\sigma_2) & (tc_{7,1} - c_{7,3})x_1x_3 + (c_{1,2} - t^2c_{3,2})x_2x_4 + \\
 & (-tc_{1,1} + c_{1,3} + t^3c_{3,1} - t^2c_{3,3} - t^2c_{7,2} + 2tc_{7,4})x_3x_4 \\
 h(\phi)(\sigma_3) & c_{2,3}x_1x_3 - c_{5,4}x_2x_4 + (-tc_{2,4} + tc_{5,3})x_3x_4 \\
 h(\phi)(\sigma_4) & (c_{2,1} - t^2c_{4,1})x_1x_3 + (tc_{6,2} - c_{6,4})x_2x_4 + \\
 & (-tc_{2,2} + c_{2,4} + t^3c_{4,2} - t^2c_{4,4} - t^2c_{6,1} + 2tc_{6,2})x_3x_4 \\
 h(\phi)(\sigma_5) & -c_{6,1}x_1x_3 + (tc_{3,2} + c_{3,4})x_2x_4 + (-t^2c_{3,1} + tc_{6,2} - c_{6,4})x_3x_4 \\
 h(\phi)(\sigma_6) & (tc_{4,1} + c_{4,3})x_1x_3 - c_{7,2}x_2x_4 + (-t^2c_{4,2} + tc_{7,1} - c_{7,3})x_3x_4 \\
 h(\phi)(\sigma_7) & (c_{5,1} - c_{6,3} + tc_{7,1})x_1x_3 - t^2c_{3,2}x_2x_4 + \\
 & (t^3c_{3,1} - t^2c_{3,3} - tc_{5,2} + c_{5,4} + tc_{6,4} - t^2c_{7,2} + tc_{7,4})x_3x_4 \\
 h(\phi)(\sigma_8) & -t^2c_{4,1}x_1x_3 + (c_{5,2} + tc_{6,2} - c_{7,4})x_2x_4 + \\
 & (t^3c_{4,2} - t^2c_{4,4} - tc_{5,1} + c_{5,3} - t^2c_{6,1} + tc_{6,3} + tc_{7,3})x_3x_4
 \end{array}$$

Each row of the above lines yields three linear equations so $\tau^*(h)$ is represented by a 24×28 matrix M as expected. Computation in Macaulay2 [Grayson and Stillman] shows that the ideal of 24×24 minors of M is (t^{16}) and the statement follows. □

Corollary 5.15. *The divisor D is linearly equivalent to $16H$.*

The following lemma allows us to determine the degree of W_0 .

Lemma 5.16. *The divisor D vanishes with multiplicity at least 8 on W_0 .*

Proof. By Lemma 5.13, we know that $|W_0| \subseteq |D|$. By Lemma 5.12, a general point of W_0 does not belong to $Z_1 \cup Z_2$. Let I be any such point. Since I is a singular point in R_8^4 , I has tangent space dimension at least $\dim(R_8^4) + 1 = 33$, and so the null space of $\bar{h} \otimes k(I)$ must have dimension at least 8.

Choose 8 vectors from the null space as basis vectors, and any other 16 to complete a basis of the source of $\bar{h} \otimes k(I)$. This basis in the quotient ring lifts to a basis in the local ring $\mathbb{C}_{G'_0, I}$. When we represent the localization of the map

$(\bar{h})_I$ as a matrix with respect to this basis we see that the first 8 columns belong to the maximal ideal \mathfrak{m}_I of $\mathbb{C}_{G'_0, I}$. Thus $\det(\bar{h})$ belongs to \mathfrak{m}_I^8 , and in turn D has multiplicity at least 8 at I . \square

Lemma 5.17. *The ideal sheaf of D is (P^8) , where P is the Salmon–Turnbull Pfaffian.*

Proof. Since D is a divisor on G_0 its defining ideal in the homogeneous coordinate ring of the Plücker embedding of G_0 is generated by a single element f of degree 16 in the Plücker coordinates. If g is the square-free part of f then Lemma 5.16 shows that g has degree at most 2. Since D is invariant under linear changes of variables, it follows from Lemmas 5.7 and 5.8 that $g = P$ and $f = P^8$. \square

By combining Lemmas 5.4, 5.13, and 5.17 we have now proven our descriptions of W_0 and W :

Theorem 5.18. *The subscheme W_0 is defined by P .*

For the rest of the section, we return to the case that k is a field, not necessarily algebraically closed, of characteristic not 2 or 3.

Recall that if M_λ is any monomial ideal in G_0 then there are local coordinates $c_{m'}^m$ on $U_\lambda \cap H_8^4$. Moreover there is a surjection $\pi : R_8^4 \cap U_\lambda \rightarrow W_0 \cap U_\lambda$, and there is a rational map $\phi : (\mathbb{A}^4)^8 // S_8 \dashrightarrow R_8^4 \cap U_\lambda$ given by $c_{m'}^m = \frac{\Delta_\lambda - m' + m}{\Delta_\lambda}$ whose image is dense in $R_8^4 \cap U_\lambda$.

Lemma 5.19. *With U_λ as above, the function $P \circ \pi$ vanishes identically on $R_8^4 \cap U_\lambda$ over an arbitrary field k .*

Proof. The composition $P \circ \pi \circ \phi$ is a rational function with integer coefficients. Theorem 5.18 proves that $P \circ \pi \circ \phi = 0$ in $\mathbb{C}[q_i^{(j)}][\Delta_\lambda^{-1}]$. Therefore, $P \circ \pi \circ \phi = 0$ in $\mathbb{Z}[q_i^{(j)}][\Delta_\lambda^{-1}]$. \square

Theorem 5.20. *The following irreducible subsets of G_0 coincide:*

- (1) W_0 ;
- (2) $V(P)$, the vanishing of the pullback to G of the Salmon–Turnbull Pfaffian;
- (3) the homogeneous ideals with Hilbert function $(1, 4, 3)$ which are flat limits of ideals of distinct points.

As a consequence, if we let $\pi|_G : G \rightarrow G_0$ be the restriction of the projection from Lemma 5.4 then $W = V(P \circ \pi|_G)$.

Proof. For other fields, note that for the ideal J of Proposition 5.1 with $d = 4$, we have that $P \circ \pi(J) = P(J) = 1$ and thus $P \circ \pi$ does not vanish uniformly on G in any characteristic. By the previous lemma, $P \circ \pi$ vanishes uniformly on R_8^4 for any k . Thus $W \subseteq V(P \circ \pi|_G)$. As both W and $V(P \circ \pi|_G)$ are integral closed subschemes of codimension 1 in G , they are equal. \square

6. Proofs of main results

In this section, k denotes a field of characteristic not 2 or 3. We have used our characteristic assumption in order to apply the theory of duality in Sections 3 and 4 and to define the trace map of Lemma 5.3.

Proof of Theorem 1.1. The irreducibility of H_n^d when d is at most 3 or n is at most 7 follows for an algebraically closed field from Theorem 4.20. For a nonalgebraically closed field, the Hilbert scheme is irreducible because it is irreducible after passing to the algebraic closure. Proposition 3.1 and the same argument as in Lemma 5.3 show that when d is at least 4, G_8^d is irreducible and $(8d - 7)$ -dimensional, and Proposition 5.1 shows that it is a separate component. Theorem 4.20 shows that there are no other components, again, by passing to the algebraic closure if necessary. \square

Proof of Theorem 1.2. This follows from Theorem 3.9. \square

Proof of Theorem 1.3. The statement that $R_8^4 \cap G_8^4$ is a prime divisor on G_8^4 is proved in Lemma 5.4. The equivalence of the set-theoretic description and the local equation description follows from Lemma 5.6. Theorem 5.20 proves that the Salmon–Turnbull Pfaffian is the correct local equation. \square

Proof of Theorem 1.4. Let M_λ be some monomial ideal and consider the monomial chart U_λ . If M_λ does not have Hilbert function $(1, 4, 3)$ then $U_\lambda \cap G_8^4 = \emptyset$ so that the zero ideals will cut out R_8^4 . If M_λ has Hilbert function $(1, 4, 3)$, then Lemma 5.19 and Theorem 5.20 show that the zero set of the pullback of the Pfaffian is precisely $R_8^4 \cap U_\lambda$. \square

7. Open questions

The motivating goal behind this work is understanding the smoothable component of the Hilbert scheme as explicitly as possible, and not just as the closure of a certain set. This can be phrased more abstractly by asking what functor the smoothable component represents, or, more concretely, by describing those algebras which occur in the smoothable component. In this paper we have done the latter for n at most 8. The following are natural further questions to ask:

- For d greater than 4, which algebras with Hilbert function $(1, d, 3)$ are smoothable? Generically, such algebras are not smoothable. Computer experiments lead us to conjecture that, for smoothable algebras, the analogue of the skew symmetric matrix in Theorem 1.3 has rank at most $2d + 2$. However, a dimension count shows that this rank condition alone is not sufficient for such an algebra to be smoothable. What are the other conditions?

- What is the smallest n such that H_n^3 is reducible? We have shown H_8^3 is irreducible and Iarrobino [1985, Example 3] has shown that H_{78}^3 is reducible.
- Is H_n^d ever nonreduced? What is the smallest example? Does it ever have generically nonreduced components?

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Discretely ordered groups

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We consider group orders and right-orders which are discrete, meaning there is a least element which is greater than the identity. We note that nonabelian free groups cannot be given discrete orders, although they do have right-orders which are discrete. More generally, we give necessary and sufficient conditions that a given orderable group can be endowed with a discrete order. In particular, every orderable group G embeds in a discretely orderable group. We also consider conditions on right-orderable groups to be discretely right-orderable. Finally, we discuss a number of illustrative examples involving discrete orderability, including the Artin braid groups and Bergman's nonlocally-indicable right orderable groups.

1. Introduction

Let G be a group and suppose $<$ is a strict total order relation on the set of its elements. Then $(G, <)$ is a right-ordered group if $f < g$ implies that $fh < gh$ for all $f, g, h \in G$. If in addition $f < g$ implies that $hf < hg$, then we say $(G, <)$ is an ordered group. If such an order exists for a given group G , we say that G is right-orderable or orderable, respectively. We call the order $<$ *discrete* if there is an element $a \in G$ such that $1 < a$, where 1 denotes the identity element of G , and there is no element of G strictly between these.

For a right-ordered group, the positive cone $P := \{g \in G \mid 1 < g\}$ satisfies

- (1) P is closed under multiplication and
- (2) for every $g \in G$, exactly one of $g = 1$, $g \in P$ or $g^{-1} \in P$ holds.

Conversely, if a group G has a subset P with properties (1) and (2), it is routine to verify that the order defined by $g < h$ if and only if $hg^{-1} \in P$ makes $(G, <)$ a right-ordered group. Similarly, a group G is orderable if and only if it admits a subset P satisfying (1), (2) and

- (3) $gPg^{-1} = P$ for all $g \in G$.

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A subset X of a (right-) ordered group G is *convex* if $x < y < z$ and $x, z \in X$ imply $y \in X$. We recall that the set of all convex subgroups of a (right-) ordered group is linearly ordered by inclusion. A convex *jump* $C \twoheadrightarrow D$ is a pair of distinct convex subgroups such that $C \subset D$ and there are no convex subgroups strictly between them. In particular, the convex jump determined by a nonidentity element g of G is defined by $C =$ the union of all convex subgroups not containing g and $D =$ the intersection of all convex subgroups which do contain g . If the group is orderable, then for any convex jump $C \twoheadrightarrow D$, C is normal in D and the quotient D/C embeds in \mathbb{R} , the additive reals, by an order-preserving isomorphism.

Lemma 1.1. *If $<$ is a discrete right-order on G and a is the least positive element under $<$, then $\langle a \rangle$ is convex. Moreover, for any $g \in G$, we have $a^{-1}g < g < ag$ and there is no element strictly between these elements of G . If the right-order $<$ is not discrete, then it is dense in the sense that for any $f, g \in G$ with $f < g$, there exists $h \in G$ with $f < h < g$.*

Proof. Since $a^{-1} < 1 < a$, we see that $a^{-1}g < g < ag$. If $g < x < ag$, then $1 < xg^{-1} < a$, a contradiction. Hence there is no element strictly between g and ag . Similarly there is no element between $a^{-1}g$ and g . In particular, for any integers $n < m$, $a^n < g < a^m$ implies $g \in \langle a \rangle$ and thus $\langle a \rangle$ is convex. If there exists $f < g$ with no element strictly between, a routine calculation shows gf^{-1} is the least positive element for the order. \square

Note that according to our definitions, the trivial group 1 has exactly one right order, and this order is dense but not discrete.

Situated strictly between the class of right-orderable groups and the class of orderable groups is the class of locally indicable groups. Recall that a group G is locally indicable if every finitely generated nontrivial subgroup of G has an infinite cyclic quotient. Such groups are right-orderable, as was shown by [Burns and Hale \[1972\]](#). On the other hand, a right-orderable group need not be locally indicable as was shown by [Bergman \[1991\]](#). However for a large class of groups the class of right-orderable groups coincides with the class of locally indicable groups. Further results on this topic are contained in [\[Linnell 2001\]](#), [\[Longobardi et al. 2000\]](#), and especially in [\[Morris 2006\]](#).

Our interest in considering locally indicable groups G is due to the fact that such groups have a series (defined below) with torsion-free abelian factors as shown in [\[Brodskii 1984\]](#). They possess a right-order in which the set of convex subgroups form a series with factors which are order isomorphic to subgroups of the additive group of reals; we shall refer to such orders as *lexicographic*. (Such orders are also called Conrad orders and are characterized by the condition that if $g > 1$ and $h > 1$, then there exists a positive integer n such that $(gh)^n > hg$; see [\[Botto Mura and Rhemtulla 1977, §7.4\]](#) for further details.) Note that for such an ordering, any

nontrivial element $g \in G$ is positive if and only if the cosets satisfy $Cg > C$ in the factor group D/C determined by g .

By a *series* for G we mean a set $\Sigma = \{H_\lambda \mid \lambda \in \Lambda\}$ of subgroups of G , where Λ is a totally ordered set of indices, satisfying:

- if $\lambda < \mu$ then $H_\lambda \subset H_\mu$,
- $\{1\}$ and G belong to Σ ,
- Σ is closed under arbitrary unions and intersections,
- if μ immediately follows λ in Λ , then H_λ is normal in H_μ and H_μ/H_λ is called a *factor* associated to the jump $H_\lambda \twoheadrightarrow H_\mu$.

In the next section we characterize groups that have discrete orders. We show that a group G has a discrete order if and only if it is an orderable group and its center $Z(G)$ contains an isolated infinite cyclic group. Recall that a subgroup H of a group G is said to be isolated if $g \in G$ and $g^n \in H$ for some $n > 0$ implies $g \in H$.

In [Section 3](#), we deal with groups possessing discrete lexicographic right-orders and discrete right-orders. It will follow, in particular, that any nontrivial finitely generated orderable group has discrete right-orders and if it has a central order (as is the case for free groups, pure braid groups and wreath products or free products of such groups), then it has discrete lexicographic right-orders. Recall that an order $<$ on G is called central if for every convex jump $C \twoheadrightarrow D$, we have $[D, G] \subseteq C$ where $[D, G]$ denotes the subgroup $\langle d^{-1}g^{-1}dg \mid d \in D, g \in G \rangle$.

The result is of course not true for orderable groups in general. The additive group of rational numbers has no discrete right-order.

The final section presents examples of discretely ordered groups which have nontrivial subgroups (for example, the commutator subgroup) upon which the restriction of the given order is dense. We also note that there exist finitely generated right-orderable groups, for example, the Artin braid groups B_n , $n \geq 5$, that are not locally indicable, yet have a discrete right-order.

2. Discrete orders

Theorem 2.1. *If $<$ is a discrete order on a group G , then there exists an element z in the center $Z(G)$ such that $\langle z \rangle$ is convex under $<$ and $1 \twoheadrightarrow \langle z \rangle$ is a jump. Conversely, if G is an orderable group and $Z(G)$ contains an isolated infinite cyclic group, then there is a discrete order on G .*

Proof. Let $<$ be a discrete order on G with $z > 1$ as the minimal positive element. Then $g^{-1}zg$ is positive for every $g \in G$. Moreover, $z < g^{-1}zg$ implies $1 < gzg^{-1} < z$, a contradiction. Thus $z \in Z(G)$. Also [Lemma 1.1](#) shows that $\langle z \rangle$ is convex under $<$ and $1 \twoheadrightarrow \langle z \rangle$ is a jump.

Conversely, let $\langle z \rangle$ be an isolated subgroup in the center $Z(G)$ of an orderable group G . Since G is orderable, so is $G/Z(G)$; see [Botto Mura and Rhemtulla 1977, Theorem 2.2.4]. Moreover, $Z(G)/\langle z \rangle$ is orderable since $\langle z \rangle$ is isolated in $Z(G)$. Order $\langle z \rangle$ (with z positive), $Z(G)/\langle z \rangle$ and $G/Z(G)$. Now order G as follows. If $1 \neq g \in G \setminus Z(G)$, then put g in the positive cone if $gZ(G)$ is positive; if $g \in Z(G) \setminus \langle z \rangle$, then put g in the positive cone if $g\langle z \rangle$ is positive; if $g = z^n$, then put g in the positive cone if $0 < n$. It is routine to verify that this gives a discrete order on G with z as the minimal positive element. \square

Corollary 2.2. *For any orderable group G , the group $\mathbb{Z} \times G$ has a discrete order. In particular, every orderable group embeds in a discretely orderable group, whose order extends the given order.*

3. Discrete right orders

We begin this section with the following result which is easy to prove. It is not required in the proofs of the other results.

Lemma 3.1. *If $(G, <)$ is a nontrivial right-ordered group such that the order $<$ is a well order on the set of positive elements of G , then G is infinite cyclic.*

Lemma 3.2. *If $<$ is a discrete right order on G and a the least positive element under $<$ then for any element $1 < g \in G$, we have $1 < aga^{-1}$ and $1 < a^{-1}ga$.*

Proof. Since $a \leq g$, $1 \leq ga^{-1}$. Thus aga^{-1} is a product of two positive elements and hence positive. By Lemma 1.1, there is no element of G strictly between $a^{-1}g$ and g . Since $1 < g$, $1 \leq a^{-1}g$ and so $a \leq a^{-1}ga$. \square

Definition 3.3. Let $<$ be a right order on a group G , C a subgroup of G and $a \in G$. We shall say that *conjugation by a preserves order on C* to mean that C is normalized by $\langle a \rangle$ and conjugation by a and by a^{-1} preserves the order on $(C, <)$.

Lemma 3.4. *Suppose $<$ is a right order on a group G , C is a subgroup of G , $1 \neq a \in G$ and $C \cap \langle a \rangle = 1$. If conjugation by a preserves order on C , then there is a discrete right order on the subgroup $\langle C, a \rangle$ with a as the minimal positive element. Moreover, this right order and the given right order agree on C . Finally if $aEa^{-1} = E$ for all convex subgroups E of C , then the convex subgroups of H under this new right order are $\{1\}$ and $\langle a, E \rangle$, where E is a convex subgroup of C .*

Proof. Set $H = \langle a, C \rangle$. An element $g \in H$ has a unique expression as $g = a^n c$ where $c \in C$ and $n \in \mathbb{Z}$. Define the set P as follows: $g \in P$ if $1 < c$ or $c = 1$ and $n > 0$. Note that $P \cup P^{-1} = H \setminus \{1\}$ and $P \cap P^{-1} = \emptyset$. Moreover if $g = a^n c$ and $h = a^m d$ are in P , then their product $gh = a^{n+m} (a^{-m} c a^m) d \in P$ as conjugation by a^m preserves order on C . Thus the order $<$ on H given by $g < h$ if and only if $hg^{-1} \in P$ is a right order on H . Furthermore if $g = a^n c$ and $h = a^m d$ are in H ,

then $g < h$ if and only if $c < d$ or $c = d$ and $n < m$. It is now clear that a is the least positive element under this order, and that $<$ and $<$ agree on C . Finally we verify that the convex subgroups are $\langle a, E \rangle$, where E is a convex subgroup of C .

Set $A = \langle a \rangle$, so $H = AC$. If K is a nontrivial convex subgroup of H , then $a \in K$ and $C \cap K$ is a convex subgroup of C , and we have $(K \cap C)A = K \cap CA = K$. Thus $K = \langle a, E \rangle$ where $E = C \cap K$. On the other hand if E is a convex subgroup of C , we claim that AE is a convex subgroup of H . Suppose $a^m b < a^n c < a^p d \in EA$, where $b, d \in E$, $c \in C$, and $m, n, p \in \mathbb{Z}$. Then $b \leq c \leq d$ and hence $c \in E$, and it follows that AE is a convex subgroup of $(H, <)$, as required. \square

Theorem 3.5. *Let $(G, <)$ be an ordered group, $1 \neq a \in G$ and $C \twoheadrightarrow D$ the convex jump determined by a (thus $a \in D \setminus C$ and D/C is torsion-free abelian). If $D/\langle a, C \rangle$ is torsion free, then there is a discrete right order on G with a as the minimal positive element. Moreover, if also $[a, F] \subseteq E$ for every jump $E \twoheadrightarrow F$, then there is a discrete lexicographic right order on G (with a as minimal positive element).*

Proof. Set $H = \langle a, C \rangle$. The hypothesis of [Lemma 3.4](#) applies and we right order H as described there. Next we order the factor group D/H . This is possible since D/H is torsion-free abelian. Define the set $Q \subset G$ as follows. If $g \in H$, then $g \in Q$ if g is positive in the order on H described above. If $g \in D \setminus H$, then put g in Q if gH is positive in the order on D/H given above. If $g \in G \setminus D$, then put g in Q if g is positive in $(G, <)$, the original order on G .

It is routine to verify that $Q \cup Q^{-1} = G \setminus \{1\}$, $Q \cap Q^{-1} = \emptyset$ and $QQ \subseteq Q$, thus giving a right order $<$ on G with $Q = \{g \in G \mid 1 < g\}$.

The same right order $<$ is lexicographic if $[a, F] \subseteq E$ for every jump $E \twoheadrightarrow F \subseteq C$. The convex subgroups are $\{1\}$, $\langle a \rangle$, and $\langle a, E \rangle$ for every subgroup E convex under the original order $<$. This follows from [Lemma 3.4](#): note that $E = \langle a, E \rangle$ if $E \geq D$ and $E \cap C$ is a convex subgroup of C . \square

Corollary 3.6. *Nontrivial free groups have discrete lexicographic right orders.*

Proof. This follows from the fact that the descending lower central series terminates in $\{1\}$ and the factors are free abelian groups. Thus any element may be made to be the least positive element so long as it is a primitive element of the factor group that is determined by the element. \square

[Corollary 3.6](#) can be generalized to free partially commutative groups. These are described in [[Duchamp and Krob 1992](#), §1.1], and the definition given there does not require these groups to be finitely generated. Free partially commutative groups are known under many other names, in particular they are also called right-angled Artin groups [[Charney 2007](#)], at least for finitely generated groups.

Corollary 3.7. *Nontrivial free partially commutative groups have discrete lexicographic right orders.*

Proof. Free partially commutative groups are residually nilpotent by [Duchamp and Krob 1992, Theorem 2.3]. Furthermore [Duchamp and Krob 1992, Theorems 1.1, 2.1] show that the quotients of the lower central series are free abelian groups. The result now follows from Theorem 3.5. \square

Note that a nonabelian free group does not have a discrete order. This follows from Theorem 2.1.

Finally, all surface groups (orientable or not) except the Klein bottle and projective plane are residually torsion-free nilpotent, by [Baumslag 1968, Theorem 1] (we would like to thank Warren Dicks for this reference). Thus these surface groups also have lexicographic discrete right orders. With the exception of the torus, these groups have trivial center and therefore do not enjoy discrete orders.

The pure braid groups P_n , like free groups and surface groups, are also residually torsion-free nilpotent. But, unlike those examples, the groups P_n do have discrete orders. The center $Z(P_n)$ is infinite cyclic, generated by $z =$ the full twist braid (often denoted Δ_n^2). Since $\langle z \rangle$ is trivially isolated in $Z(P_n)$, the second part of Theorem 2.1 provides a discrete order with Δ_n^2 as least positive element. In fact any discrete order of P_n must have Δ_n^2 (or its inverse) as least positive element.

4. Examples

A group may have a lexicographic right order and not have any discrete right order even when the factors formed by the convex jumps are all infinite cyclic. One example of this is the following.

Example 4.1. Let $G = \langle a_i \mid i \in \mathbb{Z} \rangle$ with defining relations $[a_i, a_j] = 1$ if $|i - j| > 1$ and $a_{i+1}a_i a_{i+1}^{-1} = a_i^{-1}$.

Every right order on G is lexicographic with the subgroups $\langle a_i \mid i < j \rangle$ forming the chain of convex subgroups, and every right order is determined by the a_i (that is, whether or not a_i is in the positive cone for each $i \in \mathbb{Z}$). This construction is just the expansion of the well known (Klein bottle) group $D = \langle a, b \rangle$ where $b^{-1}ab = a^{-1}$. There are exactly four right orders on D , every one discrete with a or a^{-1} as the minimal positive element.

We show next that any infinite cyclic extension of the group G of this example has a discrete right order if it is finitely generated. However we can have a metacyclic extension of G that is finitely generated and right orderable but without any discrete right order. These are given as Proposition 4.2 and Example 4.3. If x, t are elements of a group, then x^t will denote $t^{-1}xt$.

Proposition 4.2. Let $\Gamma = G\langle t \rangle$ be a finitely generated infinite cyclic extension of the group G given in Example 4.1. Then Γ has a discrete right order with t (or t^{-1}) as the minimal positive element.

Proof. Every nontrivial element $g \in G$ has unique expression of the form

$$g = a_{r_1}^{d_1} \dots a_{r_k}^{d_k},$$

where $r_1 < \dots < r_k$ and $d_i \neq 0$ for all i . Call $a_{r_k}^{d_k}$ the leading term of g and denote it by $\ell(g)$. Call r_k the leading suffix of g .

Note that $\ell(g^n) = (\ell(g))^n$ for all $n \in \mathbb{Z} \setminus \{0\}$. Moreover, if $\ell(g) = a_r^j$, $\ell(h) = a_s^k$ and $r < s$, then $\ell(gh) = \ell(hg) = \ell(h)$. Since $(a_{i+1}^t)^{-1}(a_i^t)(a_{i+1}^t) = (a_i^t)^{-1}$, we see that the leading suffix of $(a_{i+1}^t)^t$ is greater than that of $(a_i^t)^t$ by at least one. Thus also the leading suffix of $(a_{i+1}^t)^{t^{-1}}$ is greater than that of $(a_i^t)^{t^{-1}}$ by at least one, and we deduce that the leading suffix of $(a_{i+1}^t)^t$ is greater than that of $(a_i^t)^t$ by exactly one. It follows that $i > j$ implies that the leading suffix of $(a_i)^t$ is greater than that of $(a_j)^t$ by exactly $i - j$.

Since Γ is finitely generated, the leading suffix of a_i^t (or that of $a_i^{t^{-1}}$) is greater than i for at least one value of i — otherwise $\langle a_j \mid j \leq i \rangle$ is normal in Γ for every i , and hence Γ can not be finitely generated. Thus if the leading suffix of a_0^t is n , then the leading suffix of a_i^t is $i + n$ for every integer i , and we may assume that $n > 0$.

We now right order the group G by putting a_0, a_1, \dots, a_{n-1} in the positive cone P . Next, for all $n \leq r < 2n$, we put $a_r \in P$ if the exponent of $\ell(a_{r-n}^t)$ is positive and $a_r^{-1} \in P$ otherwise. Next put a_{r+n} or a_{r+n}^{-1} in P depending on whether the exponent of $\ell((\ell(a_{r-n}^t)^t)^t)$ is positive or negative. Continue this process. For every integer $i \geq 0$ we have determined whether a_i or its inverse is in P . Next, for $0 > r \geq -n$ put $a_r \in P$ if the exponent of $\ell(a_{r+n}^{t^{-1}})$ is positive. Put $a_r^{-1} \in P$ otherwise. Continue this process. This takes care of all a_i for $i \in \mathbb{Z}$. Note that the above order on G is $\langle t \rangle$ invariant. Hence by Lemma 3.4, Γ has a right discrete order with t as the minimal positive element. □

Example 4.3. Let G be the group in Example 4.1. Consider the map

$$\phi: \{a_i \mid i \in \mathbb{Z}\} \rightarrow \{a_i^{-1} \mid i \in \mathbb{Z}\}$$

given by $\phi(a_i) = a_i^{-1}$. Then ϕ extends uniquely to an automorphism of G that inverts every a_i . Let $\langle G, u \rangle$ be the infinite cyclic extension of G by $\langle u \rangle$ where $u^{-1}a_iu = a_i^{-1}$ for all $i \in \mathbb{Z}$. Note that $\langle G, u \rangle$ is right orderable because it is an infinite cyclic extension of the right orderable group G . However, it has no discrete right order. This can be seen as follows. Suppose $g \in G$ and $c := gu^j$ is a minimal positive element under some right order $<$ on $\langle G, u \rangle$. Since G has no discrete right order, $j \neq 0$ and it must be even otherwise $c^{-1}a_ic = a_i^{-1}$ for some i , contradicting Lemma 3.2.

Suppose $1 < u$. Then $a_i^r < u$ for every $i, r \in \mathbb{Z}$, for if $1 < u < a_i^r$, then $ua_i^{-r} < 1$. Hence $a_i^r = ua_i^{-r}u^{-1} < 1$, a contradiction. Thus $h < u$ for all $h \in G$. Since

$1 < c = gu^j = u^j g$, we see that j is positive, and then we have $1 < u^{j-1} u g$, which contradicts the hypothesis that c is the minimal positive element. The argument is similar if $u < 1$. We note in particular that if $1 < u$, then $h < u$ for every $h \in G$.

Next extend the group $\langle G, u \rangle$ by the infinite cyclic group $\langle v \rangle$ to get the group $J = \langle G, u, v \rangle$ where the action of v under conjugation is as follows: $v^{-1} a_i v = a_{i+1}$, the shift automorphism, and $v^{-1} u v = u^{-1}$. Note that $\langle G, v \rangle = \langle a_0, v \rangle$, $J = \langle a_0, u, v \rangle$ and J is right orderable.

We now show that J has no discrete right order. Suppose that $c := gu^j v^k$ is a minimal positive element under a right order $<$ on J . Then $k \neq 0$, since otherwise the restriction of the right order $<$ to $\langle G, u \rangle$ would be discrete with gu^j as the minimal positive element. We have seen that this is not possible. Next note that k must be even, otherwise assume without loss of generality that $u > 1$. Then $cuc^{-1} = gu^{-1}g^{-1} < 1$, which contradicts [Lemma 3.2](#). Since $vuv^{-1} = u^{-1}$, we see that $1 < v$ implies $x < v$ for every $x \in \langle G, u \rangle$, in particular $k > 0$ and $gu^j v^{k-1} > 1$. This contradicts the hypothesis that c is the minimal positive element. Similarly we cannot have $1 > v$, which finishes the verification that J has no discrete right order.

It is possible for a discretely (right-) ordered group to have a subgroup on which the same order is dense. Indeed, by [Corollary 2.2](#), any densely ordered group is a subgroup of a discretely ordered group, whose order extends the given order. Following is a natural example of this phenomenon for right-ordered groups.

Example 4.4. The Artin braid groups B_n have a discrete right-order, which becomes dense when restricted to the commutator subgroup. For each integer $n \geq 2$, B_n is the group generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$, subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1 \quad \text{and} \quad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1.$$

It was shown by Dehornoy (see [[Dehornoy 1994](#)] and [[Dehornoy et al. 2002](#)]) that each B_n is right-orderable (but not orderable, for $n > 2$). The positive cone consists of all elements expressible as a word in the σ_i such that the generator with the lowest subscript occurs with only positive exponents. This right-order is discrete, with smallest positive element σ_{n-1} . On the other hand, it is shown in [[Clay and Rolfsen 2007](#)] that the Dehornoy order, when restricted to the commutator subgroup $B'_n = [B_n, B_n]$, is a dense order for $n \geq 3$. For $n = 3$, B'_n is free (on two generators). For $n \geq 5$, B'_n is finitely-generated and perfect [[Gorin and Lin 1969](#)], so B_n is an example of a nonlocally indicable discretely right-orderable group for $n \geq 5$.

Now consider the braid group B_3 with its two generators σ_1 and σ_2 and let H be the subgroup generated by σ_1^2 and σ_2^2 . [Crisp and Paris \[2001\]](#) showed that H is a free group with free basis σ_1^2 and σ_2^2 . The Dehornoy order restricted to this

subgroup has the least positive element σ_2^2 . This gives an alternative construction of discrete right-orders on a free group.

[Bergman \[1991\]](#) published the first examples of groups which are right-orderable and not locally indicable; some of his examples are finitely generated and perfect. We shall argue that they can be given a discrete right-order.

If G is a group acting on a set and x_1, \dots, x_n are elements of the set, then $\text{Stab}_G(x_1, \dots, x_n)$ will denote the pointwise stabilizer of $\{x_1, \dots, x_n\}$ in G , namely $\{g \in G \mid gx_i = x_i \text{ for all } i\}$. Also I will denote the identity matrix of $\text{SL}_2(\mathbb{R})$. We have an action of $\text{SL}_2(\mathbb{R})$ on the one point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cong S^1$, the circle, given by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x) = \frac{ax + b}{cx + d}. \quad (4.5)$$

This induces a faithful action of $\text{PSL}_2(\mathbb{R})$ on $\overline{\mathbb{R}}$. Let H be a finitely generated subgroup of $\text{SL}_2(\mathbb{R})$ containing the center $\{\pm I\}$ of $\text{SL}_2(\mathbb{R})$, and let \bar{H} denote its image in $\text{PSL}_2(\mathbb{R})$. Since \mathbb{R} is the universal covering space of S^1 , we can lift the action of \bar{H} on S^1 to an action of a group G on \mathbb{R} by orientation preserving homeomorphisms. In this situation, G will have a central subgroup $Z \cong \mathbb{Z}$ such that $G/Z \cong \bar{H}$ and Z acts fixed point freely on \mathbb{R} . Also if $\pi : \mathbb{R} \rightarrow S^1$ is the associated covering map and $p \in \mathbb{R}$, then $\text{Stab}_{\bar{H}}(\pi p) = Z \text{Stab}_G(p)/Z \cong \text{Stab}_G(p)$.

Proposition 4.6. *Let H be a finitely generated subgroup of $\text{SL}_2(\mathbb{R})$ with $-I \in H$ and let G be its lift to orientation preserving homeomorphisms of \mathbb{R} (as described above). Suppose H contains a diagonal matrix other than $\pm I$. Then G has a discrete right order.*

To prove this, we need an auxiliary result:

Lemma 4.7 [[Linnell 2006](#), Lemma 2.2]. *Let G be a right ordered group, let H be a convex subgroup of G and let $<$ be any right order on H . Then there exists a right order on G whose restriction to H is $<$, and H is still a convex subgroup under this new right order.*

Proof of Proposition 4.6. Let us examine $\text{Stab}_H(0)$ and $\text{Stab}_H(0, \infty)$ with the action given by (4.5). The former is the lower triangular matrices, that is, the matrices above with $b = 0$; we shall denote by L those lower triangular matrices which lie in H . The latter is given by the diagonal matrices; we shall denote by D those diagonal matrices which lie in H .

Thus we have an action of H on S^1 and two points $p_1, p_2 \in S^1$ such that $\text{Stab}_H(p_1) = L$ and $\text{Stab}_H(p_1, p_2) = D$. Let $p_3 \in S^1$ be distinct from p_1, p_2 . Then $\text{Stab}_H(p_1, p_2, p_3) = \{\pm I\}$.

Now we can lift the action of H on S^1 to an action of G on \mathbb{R} by orientation preserving homeomorphisms; G will have a central subgroup $Z \cong \mathbb{Z}$ such that $G/Z \cong H/\{\pm I\}$. For $i = 1, 2, 3$, let $q_i \in \mathbb{R}$ be a lift of p_i . Then $Q := \text{Stab}_G(q_1, q_2) \cong D/\{\pm I\}$. We now define a right order on G in the usual way when we have a group acting on \mathbb{R} . The positive cone is the set of all $g \in G$ such that $g(q_i) > q_i$ for the smallest i such that $g(q_i) \neq q_i$. This right order will have the property that Q is a (smallest nontrivial) convex subgroup of G . If $Q \cong \mathbb{Z}$, then it would follow that the above defined right order will be discrete, but this is not true in general. However since H is finitely generated, $H \subseteq \text{SL}_2(R)$ for some finitely generated subring R of \mathbb{R} . By [Samuel 1966, Théorème 1], the group of units of a finitely generated integral domain is finitely generated, hence D is also finitely generated. We deduce that $D/\{\pm I\}$ is a finitely generated free abelian group. Thus Q is also a finitely generated free abelian group and hence has a discrete right order by Corollary 2.2. The result now follows from Lemma 4.7. \square

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The half-twist for $U_q(\mathfrak{g})$ representations

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We introduce the notion of a half-ribbon Hopf algebra, which is a Hopf algebra \mathcal{H} along with a distinguished element $t \in \mathcal{H}$ such that (\mathcal{H}, R, C) is a ribbon Hopf algebra, where $R = (t^{-1} \otimes t^{-1})\Delta(t)$ and $C = t^{-2}$. The element t is closely related to the topological “half-twist”, which twists a ribbon by 180 degrees. We construct a functor from a topological category of ribbons with half-twists to the category of representations of any half-ribbon Hopf algebra. We show that $U_q(\mathfrak{g})$ is a (topological) half-ribbon Hopf algebra, but that t^{-2} is not the standard ribbon element. For $U_q(\mathfrak{sl}_2)$, we show that there is no half-ribbon element t such that t^{-2} is the standard ribbon element. We then discuss how ribbon elements can be modified, and some consequences of these modifications.

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1. Introduction

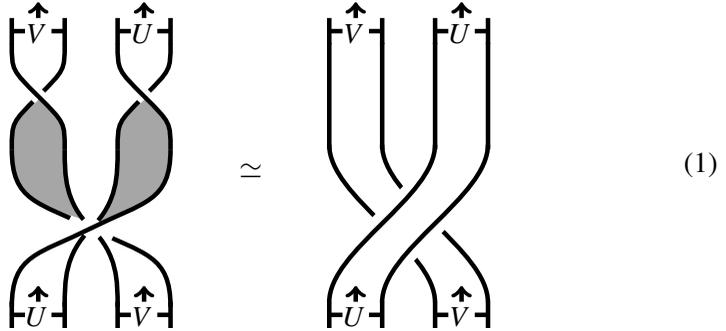
Let $\mathcal{RIBBON}(S)$ be the category whose morphisms consist of tangles of oriented directed ribbons up to isotopy, each labeled with an element of some set S . There is a notion of a ribbon Hopf algebra \mathcal{H} (see [Chari and Pressley 1994], for example), which is related to this topological category by the fact that there is a monoidal functor \mathcal{F}' from $\mathcal{RIBBON}(\mathcal{H}\text{-rep})$ to the category of representations $\mathcal{H}\text{-rep}$. This allows one to construct invariants of oriented framed links, and from there invariants of ordinary links.

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There is a morphism in $\mathcal{RIBBON}(S)$ which twists a ribbon by 360 degrees, but not one that twists a ribbon by 180 degrees (since negatively oriented objects are not allowed). We propose that one should consider a slightly larger category, denoted $\mathcal{HTRIB}(S)$, where this 180 degree twist, the “half-twist,” is allowed. The half-twist can be applied to several ribbons at once, and all morphisms in $\mathcal{HTRIB}(S)$ can be constructed out of half-twists, along with various caps and cups. The following isotopy shows how the crossing in $\mathcal{HTRIB}(S)$ is constructed out of the half-twist:



Recall that a ribbon Hopf algebra \mathcal{H} is a Hopf algebra along with two extra features:

- A universal R -matrix $R \in \mathcal{H} \otimes \mathcal{H}$. The functor \mathcal{F}' takes a simple crossing of ribbons labeled V and W to $\text{Flip} \circ R$ acting on $V \otimes W$.
- A central “ribbon” element $C \in \mathcal{H}$. The functor \mathcal{F}' takes a 360 degree twist of a ribbon labeled V to C acting on V .

The elements R and C must satisfy various compatibility conditions (see [Section 4A](#)).

In the present work we define a half-ribbon Hopf algebra to be a ribbon Hopf algebra along with a distinguished element t such that

- (i) $R = (t^{-1} \otimes t^{-1})\Delta(t)$,
- (ii) $C = t^{-2}$.

We show that, if \mathcal{H} is a half-ribbon Hopf algebra, then \mathcal{F}' can be extended to a functor \mathcal{F} from $\mathcal{HTRIB}(S)$ to \mathcal{H} -rep.

Our main interest is the case where \mathcal{H} is the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of a finite-dimensional complex simple Lie algebra \mathfrak{g} . In this case $U_q(\mathfrak{g})$ is actually a topological ribbon Hopf algebra, meaning that R and C only lie in some completion, not in $U_q(\mathfrak{g})$ itself. We define a topological half-ribbon Hopf algebra by allowing t to lie in a completion of \mathcal{H} , and show that $U_q(\mathfrak{g})$ has this structure. The main ingredient is a formula for the R matrix of $U_q(\mathfrak{g})$, due to Kirilov and Reshetikhin [1990] and Levendorskiĭ and Soibelman [1991], of the form

$$R = (X^{-1} \otimes X^{-1})\Delta(X). \tag{2}$$

The correspondence to condition (i) above should be clear, and we show that X is in fact a half-ribbon element for $U_q(\mathfrak{g})$. Interestingly, X^{-2} is not the standard ribbon element, and in fact we show that in certain cases it is not possible to find a half-ribbon element t for $U_q(\mathfrak{g})$ such that t^{-2} is the standard ribbon element.

In Section 5 we discuss the different ribbon elements for $U_q(\mathfrak{g})$, and in some small cases we describe exactly which ones arise from half-ribbon elements. We then discuss some consequences of using X^{-2} in place of the usual ribbon element. In particular, it simplifies the correspondence between certain skein theoretic constructions of link invariants and the quantum group constructions of the same objects, essentially by explaining some annoying negative signs that appear in, for example, [Ohtsuki 2002] or [Kuperberg 1996].

We feel it would be of considerable interest to study which ribbon Hopf algebras can be given the structure of half-ribbon Hopf algebras. It would also be nice to give straightforward conditions on an element t in a general Hopf algebra \mathcal{H} , such that \mathcal{H} along with t is a half-ribbon Hopf algebra (that is, \mathcal{H} is a ribbon Hopf algebra, with $R = (t^{-1} \otimes t^{-1})\Delta(t)$ and $C = t^{-2}$). At the end of the paper we discuss these and other possible future directions.

2. Conventions

We first fix some notation. For the most part we follow conventions from [Chari and Pressley 1994].

- \mathfrak{g} is a complex simple Lie algebra with Cartan algebra \mathfrak{h} , and $A = (a_{ij})_{i,j \in I}$ is its Cartan matrix.

- $\langle \cdot, \cdot \rangle$ denotes the pairing between \mathfrak{h} and \mathfrak{h}^* and (\cdot, \cdot) denotes the usual symmetric bilinear form on either \mathfrak{h} or \mathfrak{h}^* . Fix the usual bases α_i for \mathfrak{h}^* and H_i for \mathfrak{h} , and recall that $\langle H_i, \alpha_j \rangle = a_{ij}$.

- $d_i = (\alpha_i, \alpha_i)/2$, so that $(H_i, H_j) = d_j^{-1}a_{ij}$. Let B denote the matrix $(d_j^{-1}a_{ij})$.
- $q_i = q^{d_i}$.
- ρ is the element of \mathfrak{h}^* such that $(\alpha_i, \rho) = d_i$ for all i .
- ρ^\vee is the element of \mathfrak{h} such that $\langle \alpha_i, \rho^\vee \rangle = 1$ for all i .
- s_i is the element of the Weyl group which is defined by

$$s_i(\alpha_j) = \alpha_j - \langle \alpha_i, \alpha_j^\vee \rangle \alpha_i. \tag{3}$$

- θ is the diagram automorphism such that $w_0(\alpha_i) = -\alpha_{\theta(i)}$, where w_0 is the longest element in the Weyl group.

- $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra associated to \mathfrak{g} , generated over $\mathbb{C}(q)$ by E_i, F_i for all $i \in I$, and K_w for w in the coweight lattice of \mathfrak{g} . As usual, let $K_i = K_{H_i}$. We actually must adjoin a fixed k th root of q to the base field, for some k depending on \mathfrak{g} . This causes no real difficulty, and for the most part

we ignore it. For convenience, we recall the exact formula for the coproduct and antipode:

$$\begin{cases} \Delta E_i = E_i \otimes K_i + 1 \otimes E_i, \\ \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i, \\ \Delta K_i = K_i \otimes K_i, \end{cases} \quad \begin{cases} S(E_i) = -E_i K_i^{-1}, \\ S(F_i) = -K_i F_i, \\ S(K_i) = K_i^{-1}. \end{cases} \quad (4)$$

- $[n] = (q^n - q^{-n}) / (q - q^{-1})$, and $X^{(n)} = (X^n) / ([n][n-1] \cdots [2])$.
- V_λ is the irreducible representation of $U_q(\mathfrak{g})$ with highest weight λ , and v_λ is a highest weight vector.
- P is the weight lattice for \mathfrak{g} and Q is the root lattice.
- The standard ribbon element C for $U_q(\mathfrak{g})$ acts on V_λ as the constant

$$q^{-(\lambda, \lambda) - 2(\lambda, \rho)}.$$

- $U_q(\mathfrak{g})$ -rep is the category of finite-dimensional Type 1 representations.
- (\mathcal{H}, R) is a quasitriangular Hopf algebra over a field F , where
 - (i) $\mu : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ is multiplication;
 - (ii) $\iota : F \rightarrow \mathcal{H}$ is the unit;
 - (iii) $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is the comultiplication;
 - (iv) $\varepsilon : \mathcal{H} \rightarrow F$ is the counit;
 - (v) $S : \mathcal{H} \rightarrow \mathcal{H}$ is the antipode.
- Flip : $V \otimes W \rightarrow W \otimes V$ is defined by $\text{Flip}(v \otimes w) = w \otimes v$. For longer tensor products, define $\text{rev} : V_1 \otimes \cdots \otimes V_k \rightarrow V_k \otimes \cdots \otimes V_1$ by $v_1 \otimes \cdots \otimes v_k \mapsto v_k \otimes \cdots \otimes v_1$.
- If V is a representation of a Hopf algebra H , then we define the left dual V^* to be the dual vector space with the action $x \circ f(v) = f(S(x)v)$, and the right dual *V to be the dual vector space with the action $x \circ f(v) = f(S^{-1}(x)v)$. Note that $({}^*V)^* = V = {}^*(V^*)$.

3. Background

Much of the motivation for this paper comes from studying an expression for the R -matrix

$$R = (X^{-1} \otimes X^{-1}) \Delta(X), \quad (5)$$

where X is an element in some completion of $U_q(\mathfrak{g})$. This was first introduced by Kirilov and Reshetikhin [1990] and Levendorskiĭ and Soibelman [1991], and has recently proven useful in studying the relationship between the braiding and crystal bases [Kamnitzer and Tingley 2009]. In this section we review this formula, and also recall the definition and basic properties of a ribbon Hopf algebra.

3A. A completion of $U_q(\mathfrak{g})$. The element X from (5) is not actually in $U_q(\mathfrak{g})$, but only in a completion. In order to be precise, we briefly review the completion of $U_q(\mathfrak{g})$ that we use.

Definition 3.1. $\widetilde{U_q(\mathfrak{g})}$ is the completion of $U_q(\mathfrak{g})$ in the weak topology generated by all matrix elements of all (finite-dimensional Type 1) representations. Similarly, $\widetilde{U_q(\mathfrak{g})} \otimes \widetilde{U_q(\mathfrak{g})}$ is the completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ in the weak topology defined by all matrix elements of representations $V_\lambda \otimes V_\mu$, for all ordered pairs (λ, μ) .

Theorem 3.2 [Kamnitzer and Tingley 2009]. $\widetilde{U_q(\mathfrak{g})}$ is the direct product of the endomorphism rings of all irreducible representations of $U_q(\mathfrak{g})$. That is,

$$\widetilde{U_q(\mathfrak{g})} = \prod_{\lambda \in \Lambda_+} \text{End}(V_\lambda). \tag{6}$$

Comment 3.3. It is straightforward to see that $\widetilde{U_q(\mathfrak{g})}$ is a topological Hopf algebra.

By Theorem 3.2, specifying an element of $\widetilde{U_q(\mathfrak{g})}$ is exactly the same as specifying how it acts on each isomorphism class of irreducible representation. Similarly, if we want to specify an element of $\widetilde{U_q(\mathfrak{g})}^{\otimes 2}$ we just need to say how it acts on every tensor product of any two irreducible representations.

3B. A method of constructing commutativity constraints. The next result is a development of an idea introduced in [Henriques and Kamnitzer 2006] to study the crystal commutor; the proof is a straightforward exercise.

Proposition 3.4 [Kamnitzer and Tingley 2009, Proposition 3.11]. *Let Y be an invertible element in $\widetilde{U_q(\mathfrak{g})}$ such that the map $C_Y : X \rightarrow YXY^{-1}$ restricts to an algebra automorphism of $U_q(\mathfrak{g})$. Then C_Y is a coalgebra antiautomorphism if and only if, for every pair of representations V and W , the map*

$$\begin{aligned} \sigma_{V,W}^Y : V \otimes W &\rightarrow W \otimes V \\ v \otimes w &\rightarrow \text{Flip} \circ (Y^{-1} \otimes Y^{-1}) \Delta(Y)v \otimes w \end{aligned} \tag{7}$$

is an isomorphism.

Given an element Y satisfying the conditions of Proposition 3.4, we use the notation σ^Y to denote the system of isomorphisms $\{\sigma_{V,W}^Y\}$.

3C. The element X . We now explicitly describe the element X from (5). We will need a way to specify a lowest weight vector of V_λ , depending linearly on a chosen highest weight vector. We do this using the action of the braid group on V_λ . We very briefly review this theory, and refer the reader to, for example, [Chari and Pressley 1994, Chapter 8.1.2] or [Lusztig 1993] for more details.

Definition 3.5 [Lusztig 1993, 5.2.1]. T_i is the element of $\widetilde{U_q(\mathfrak{g})}$ that acts on a weight vector v by:

$$T_i(v) = \sum_{\substack{a,b,c \geq 0 \\ a-b+c = \langle \text{wt}(v), \alpha_i \rangle}} (-1)^b q_i^{ac-b} E_i^{(a)} F_i^{(b)} E_i^{(c)} v.$$

By [Lusztig 1993, Theorem 39.4.3], these T_i generate an action of the braid group on each V_λ , and thus a map from the braid group to $\widetilde{U_q(\mathfrak{g})}$. This realization of the braid group is often referred to as the quantum Weyl group. It is related to the classical Weyl group by the fact that, for any weight vector $v \in V$, $\text{wt}(T_i(v)) = s_i(\text{wt}(v))$.

Definition 3.6. Let $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ be a reduced expression for the longest word w_0 in the Weyl group. Then $T_{w_0} = T_{i_1} T_{i_2} \cdots T_{i_N}$.

It is clear that T_{w_0} is well defined because the elements T_i satisfy the braid relations. Note that T_{w_0} interchanges the highest and lowest weight spaces of V_λ .

Definition 3.7. v_λ^{low} is the element in the lowest weight space of V_λ defined by

$$T_{w_0}(v_\lambda^{\text{low}}) = v_\lambda,$$

where v_λ is the chosen highest weight vector.

Definition 3.8. J is the element of $\widetilde{U_q(\mathfrak{g})}$ defined by, for any weight vector $v \in V_\lambda$,

$$J(v) = q^{\langle \text{wt}(v), \text{wt}(v) \rangle / 2 + \langle \text{wt}(v), \rho \rangle} v.$$

Definition 3.9. $X = JT_{w_0}$.

Lemma 3.10 [Kamnitzer and Tingley 2009, Section 5.2]. *The element X has the following properties:*

- (i) $X(v_\lambda^{\text{low}}) = q^{(\lambda, \lambda) / 2 + (\lambda, \rho)} v_\lambda$.
- (ii) $X(v_\lambda) = (-1)^{\langle 2\lambda, \rho^\vee \rangle} q^{(\lambda, \lambda) / 2 + (\lambda, \rho)} v_\lambda^{\text{low}}$.
- (iii) X^2 is central, and acts on V_λ as multiplication by the scalar

$$(-1)^{\langle 2\lambda, \rho^\vee \rangle} q^{(\lambda, \lambda) + 2(\lambda, \rho)}.$$

(iv) C_X is given by

$$\begin{cases} C_X(E_i) = -F_{\theta(i)}, \\ C_X(F_i) = -E_{\theta(i)}, \\ C_X(K_i) = K_{\theta(i)}^{-1}. \end{cases} \tag{8}$$

We next present a key result:

Theorem 3.11 ([Kirillov and Reshetikhin 1990, Theorem 3; Levendorskiĭ and Soĭbel'man 1991, Theorem 1]; see [Kamnitzer and Tingley 2009] for this exact statement.). σ^X is the standard braiding. Equivalently, the standard R -matrix for $U_q(\mathfrak{g})$ can be realized as

$$R = (X^{-1} \otimes X^{-1})\Delta(X). \tag{9}$$

In fact, we will take (9) as the definition of the universal R -matrix.

3D. Ribbon Hopf algebras.

Definition 3.12 [Chari and Pressley 1994, Definition 4.2.8]. A ribbon Hopf algebra (\mathcal{H}, R, v) is a quasitriangular Hopf algebra (\mathcal{H}, R) equipped with an invertible central element v such that

- (i) $v^2 = uS(u)$, where $u = \mu(S \otimes id)R_{21}$,
- (ii) $S(v) = v$,
- (iii) $\varepsilon(v) = 1$,
- (iv) $\Delta(v) = (v \otimes v)(R_{21}R_{12})^{-1}$.

Proposition 3.13 [Chari and Pressley 1994]. The element $g := v^{-1}u$ is grouplike (where u is as in Definition 3.12(i)). □

The following four maps are crucial to studying ribbon Hopf algebras:

Definition 3.14 [Chari and Pressley 1994, page 163]. Let (\mathcal{H}, R, v) be a ribbon Hopf algebra, and (V, π) a representation of \mathcal{H} . Let $f \in V^*$ and $v \in V$. Let e_i and e^i be dual bases of V and V^* respectively. Then

- (i) $ev(f \otimes v) = f(v)$;
- (ii) $\widetilde{ev}(v \otimes f) = f(gv)$;
- (iii) $\text{coev}(1) = \sum e_i \otimes e^i$;
- (iv) $\widetilde{\text{coev}}(1) = \sum e^i \otimes g^{-1}e_i$.

Recall that $V \otimes V^*$ can be identified with $\text{End}(V)$. Under this identification \widetilde{ev} is the quantum trace of [Reshetikhin and Turaev 1991].

We will be working with $U_q(\mathfrak{g})$, which is not a ribbon Hopf algebra according to the above definition, since the elements v and R actually lie in the completions $\widetilde{U_q(\mathfrak{g})}$ and $U_q(\mathfrak{g}) \otimes \widetilde{U_q(\mathfrak{g})}$ discussed in Section 3A. This is known as a topological ribbon Hopf algebra. The theory goes through just as well in the topological case.

Definition 3.15. Let C be the element of $\widetilde{U_q(\mathfrak{g})}$ which acts on the representation V_λ as multiplication by $q^{-(\lambda, \lambda) - 2(\lambda, \rho)}$.

The following is well known (see [Chari and Pressley 1994, Corollary 8.3.16], for example).

Theorem 3.16. $(U_q(\mathfrak{g}), R, C)$ is a topological ribbon Hopf algebra. □

3E. The Temperley–Lieb category. The unoriented Temperley–Lieb category TL has as objects collections of points on the real line up to isotopy, and a morphism from A to B is a formal linear combination of “planar arc diagrams” each of which is a collection of nonintersecting segments in $\mathbb{R} \times [0, 1]$ whose boundary consists of $A \times 0$ and $B \times 1$ modulo isotopy in the plane and the following relation:

$$\bigcirc \longrightarrow -q - q^{-1}. \tag{10}$$

As discussed in, for example, [Goodman and Wenzl 2002], TL is a rigid monoidal category, where composition is given by vertical stacking, tensor product is given by disjoint union, and the dual is given by 180-degree rotation. Since the double dual acts trivially, this is a pivotal category with the trivial pivotal structure.

4. Half-ribbon Hopf algebras

This section contains the definition of a half-ribbon Hopf algebra. It also explains the relationship with the topological category $\mathcal{HRLB}(S)$.

4A. Definition and basic properties.

Definition 4.1. A half-ribbon Hopf algebra is a Hopf algebra, together with a distinguished element $t \in \mathcal{H}$, such that (\mathcal{H}, R, v) is a ribbon Hopf algebra, where $R = (t^{-1} \otimes t^{-1})\Delta(t)$ and $v = t^{-2}$.

In the case where t only exists in a completion of \mathcal{H} , we say \mathcal{H} is a topological half-ribbon Hopf algebra.

Proposition 4.2. *A half-ribbon element t in a half-ribbon Hopf algebra has the following properties:*

- (i) *The algebra automorphism $C_t : \mathcal{H} \rightarrow \mathcal{H}$ defined by $x \rightarrow txt^{-1}$ is also a coalgebra antiautomorphism.*
- (ii) *$\varepsilon(t) = 1$. Equivalently, t acts as the identity on the trivial representation.*
- (iii) *$S(t)^2 = t^2$.*

Proof. (i) follows from $R\Delta(x) = \Delta^{op}(x)R$.

(ii) follows from the fact that R acts as the identity on $\mathbf{1} \otimes V$, where $\mathbf{1}$ is the trivial representation.

(iii) follows because t^{-2} is a ribbon element so $S(t^{-2}) = t^{-2}$. □

Proposition 4.3. *The various important elements in a ribbon Hopf algebra can be written in terms of a half-ribbon elements t as follows:*

R	v	u	g	(11)
$(t^{-1} \otimes t^{-1})\Delta(t) = \Delta^{op}(t)(t^{-1} \otimes t^{-1})$	t^{-2}	$S(t^{-1})t^{-1}$	$S(t)t^{-1} = S(t^{-1})t$	

Proof. (R): The first formula is part of the definition of a half-ribbon element. The second formula follows by [Proposition 4.2\(i\)](#) since this implies that, for all $x \in \mathcal{H}$,

$$(t^{-1} \otimes t^{-1})\Delta(x)(t \otimes t) = \Delta^{op}(t^{-1}xt).$$

(v): This is part of the definition of a half-ribbon element.

(u): By definition, $u = \mu \circ (S \otimes 1)R_{21}$, where μ is multiplication. Thus:

$$u = \mu \circ (S \otimes 1)[(\Delta^{op}(t)(t^{-1} \otimes t^{-1})]_{21}) \tag{12}$$

$$= \mu[(S \otimes 1)(\Delta(t))(t^{-1} \otimes t^{-1})] \tag{13}$$

$$= \sum S(t^{-1})S(t_1)t_2t^{-1} \tag{14}$$

$$= S(t^{-1})[t \circ \varepsilon(t)]t^{-1} \tag{15}$$

$$= S(t^{-1})t^{-1}. \tag{16}$$

In (14) we have used Sweedler’s notation $\Delta(x) = \sum x_1 \otimes x_2$. The equality (15) follows from the antipode axiom of a Hopf algebra, and (16) follows by [Proposition 4.2\(ii\)](#).

(g): By definition, $g = v^{-1}u$. Recall that v is central. To get the second equation, apply [Proposition 4.2\(iii\)](#). □

Comment 4.4. For historical reasons the ribbon element v represents a negative twist. We use t for the positive half-twist, which explains the fact that $v = t^{-2}$.

4B. A topological category of ribbons with half-twists. There is a functor from a certain topological category to the category of representations of any ribbon Hopf algebra, which allows one to construct topological invariants. For a half-ribbon Hopf algebra, this functor can be extended to a larger topological category. In this section we define the two relevant categories, beginning with the large one.

Definition 4.5. The category $\mathcal{HRI\mathcal{B}}(S)$ of topological half-ribbons with labels in some set S is defined by:

- Objects in $\mathcal{HRI\mathcal{B}}(S)$ consist of a finite number of disjoint closed intervals on the real line, each labeled with an element of S , each with a choice of shading (shaded or unshaded), and each directed (up or down). These objects are considered up to isotopy of the real line. For example:

$$\begin{array}{ccccccc} \uparrow & \uparrow & \downarrow & \downarrow & \downarrow & \uparrow & \\ \boxed{A} & \boxed{B} & \boxed{B} & \boxed{C} & \boxed{A} & \boxed{D} & . \end{array} \tag{17}$$

- A morphism between two objects $A, B \in \mathcal{HRI\mathcal{B}}(S)$ consist of a “tangle of orientable, directed ribbons” in $\mathbb{R}^2 \times I$, whose loose ends are exactly

$$(A, 0, 0) \cup (B, 0, 1) \subset \mathbb{R} \times \mathbb{R} \times I,$$

along with a choice of direction and shading of each ribbon. These must be such that the direction (up or down) of each interval in $A \cup B$ agrees with the direction of the ribbon whose end lies at that interval, and the shading (light or dark) of each interval agrees with the shading on the visible side of the ribbon near that interval. These are considered up to isotopy.

- Composition of two morphisms is given by stacking them on top of each other, and then shrinking the vertical axis by a factor of two. Note that we read our diagrams bottom to top. For example:

Definition 4.6. $RIBBON(S)$ is the full subcategory of $\mathcal{H}RIB(S)$ consisting of those objects all of whose intervals are unshaded.

The definition of $\mathcal{H}RIB(S)$ can easily be made rigorous, in the same way as is done for the subcategory $RIBBON(S)$; see for example [Chari and Pressley 1994]. In fact, as with $RIBBON(S)$, $\mathcal{H}RIB(S)$ is a rigid monoidal category, where tensor products and duals are shown below:

$$\begin{aligned}
 (\uparrow A \downarrow \uparrow B \downarrow) \otimes (\uparrow A \downarrow \uparrow C \downarrow) &= \uparrow A \downarrow \uparrow B \downarrow \uparrow A \downarrow \uparrow C \downarrow, \\
 (\uparrow A \downarrow \uparrow B \downarrow \uparrow C \downarrow \uparrow B \downarrow)^* &= \uparrow B \downarrow \uparrow C \downarrow \uparrow B \downarrow \uparrow A \downarrow.
 \end{aligned}$$

4C. A functor from $\mathcal{H}RIB(S)$ to $\mathcal{H}\text{-rep}$. We will now show that the category of representations of a half-ribbon Hopf algebra admits a natural functor from $\mathcal{H}RIB(\mathcal{H}\text{-rep})$. This theorem is an extension of the corresponding result relating ribbon Hopf algebras and $RIBBON(\mathcal{H}\text{-rep})$, so we begin by stating this known result.

Theorem 4.7 (see [Reshetikhin 1989], [Shum 1994], or [Chari and Pressley 1994, Theorem 5.3.2]). *Let (\mathcal{H}, R, C) be a ribbon Hopf algebra. There is a unique monoidal functor \mathcal{F}' from $RIBBON(\mathcal{H}\text{-rep})$ to $\mathcal{H}\text{-rep}$ satisfying the following conditions:*

(i) $\mathcal{F}'(\uparrow V \downarrow) = V$ and $\mathcal{F}'(\uparrow V \downarrow) = V^*$,

$$(ii) \quad \begin{aligned} \mathcal{F}' \left(\begin{array}{c} \text{---} \uparrow \text{---} \\ \text{---} \downarrow \text{---} \end{array} \right) &= \text{ev}, & \mathcal{F}' \left(\begin{array}{c} \text{---} \downarrow \text{---} \\ \text{---} \uparrow \text{---} \end{array} \right) &= \text{coev}, \\ \mathcal{F}' \left(\begin{array}{c} \text{---} \downarrow \text{---} \\ \text{---} \uparrow \text{---} \end{array} \right) &= \tilde{\text{ev}}, & \mathcal{F}' \left(\begin{array}{c} \text{---} \uparrow \text{---} \\ \text{---} \downarrow \text{---} \end{array} \right) &= \widetilde{\text{coev}}. \end{aligned}$$

$$(iii) \quad \mathcal{F}' \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) = C,$$

thought of as a morphism from V to V or from V^* to V^* , depending on the orientation.

$$(iv) \quad \mathcal{F}' \left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right) = \text{Flip} \circ R$$

as a morphism from the tensor product of the bottom two objects to the tensor product of the top two objects, regardless of labeling and orientation.

Comment 4.8. Warning: this functor is not a strict rigid functor. Notice that $\mathcal{F}'(\text{---} \downarrow \text{---})^* = \mathcal{F}'(\text{---} \uparrow \text{---}) = V$, while $\mathcal{F}'(\text{---} \downarrow \text{---})^* = (V^*)^* = V^{**}$.

Our main result for this section is that \mathcal{F}' can be extended to a functor $\mathcal{F} : \mathcal{HRIB}(\mathcal{H}\text{-rep}) \rightarrow \mathcal{H}\text{-rep}$ as follows.

Definition 4.9. Let V be a representation of a half-ribbon Hopf algebra \mathcal{H} and for each $x \in \mathcal{H}$ let $\pi_V(x)$ be the element of $\text{End}(V)$ defined by x . Define a new representation V^{C_t} which is equal to V as a vector space, but with the action of $x \in \mathcal{H}$ defined by $\pi_{V^{C_t}}(x) = \pi_V(xt^{-1})$.

Definition 4.10. The *topological half-twist* is the morphism I_n in \mathcal{HRIB} which takes n ribbons of any shading and orientation, and twists them all together by 180 degrees:



Theorem 4.11. There is a unique monoidal functor \mathcal{F} from $\mathcal{HRIB}(\mathcal{H}\text{-rep})$ to $\mathcal{H}\text{-rep}$ satisfying:

- (i) $\mathcal{F}(\text{---} \uparrow \text{---}) = V$, $\mathcal{F}(\text{---} \downarrow \text{---}) = V^*$, $\mathcal{F}(\text{---} \uparrow \text{---}) = V^{C_t}$ and $\mathcal{F}(\text{---} \downarrow \text{---}) = (V^*)^{C_t}$.
- (ii) On unshaded caps and cups \mathcal{F} agrees with \mathcal{F}' .
- (iii) The topological half-twist on n ribbons is sent to $\text{rev} \circ \Delta^n(t)$ acting on \mathcal{F} of the bottom object, regardless of shading and orientation.

Furthermore, the restriction of \mathcal{F} to $\mathcal{RIBBON}(\mathcal{H}\text{-rep})$ agrees with \mathcal{F}' .

The proof of [Theorem 4.11](#) requires several lemmas, and will occupy the rest of this section.

Comment 4.12. Like \mathcal{F}' , \mathcal{F} is not a strict rigid functor. In particular, we have

$$\mathcal{F}(\text{HV}) = (V^*)^{C_t},$$

which is not equal to $(V^{C_t})^*$ (although these two representations are isomorphic). In fact, $(V^*)^{C_t}$ is equal to the right dual $*(V^{C_t})$. To see why, notice that, as vector spaces, both $(V^*)^{C_t}$ and $*(V^{C_t})$ are equal to $\text{Hom}_F(V, F)$. Each comes with a chosen action of \mathcal{H} , and, using the fact that C_t is a coalgebra antiautomorphism, one can show that these two actions are identical.

Comment 4.13. A good mnemonic for remembering what \mathcal{F} does to objects is to think of the ribbon as always being labeled with a representation on its light side, so that when you look at the dark side you see that label rotated 180 degrees around the y axis. For example, instead of $V_i^{C_t}$ you would see ${}_iV$. This works for remembering how duals interact with half twists since when you rotate V_i^* by 180 degrees about the y axis you see ${}_iV$.

Comment 4.14. There is more than one extension of \mathcal{F}' from $\mathcal{RIBBON}(\mathcal{H}\text{-rep})$ to the larger category $\mathcal{HTRIB}(\mathcal{H}\text{-rep})$, but only one satisfies [Theorem 4.11\(iii\)](#).

We first prove the following different characterization of \mathcal{F} . This is simpler to prove, and is also more general than [Theorem 4.11](#). However, this result is less satisfying in other ways. For instance, [Proposition 4.15](#) does not imply that all half twists are sent to t acting on \mathcal{F} of the bottom object.

Proposition 4.15. *Let (\mathcal{H}, R, v) be a ribbon Hopf algebra, and let $t \in \mathcal{H}$ be an invertible element. There is a unique monoidal functor \mathcal{F} from $\mathcal{HTRIB}(\mathcal{H}\text{-rep})$ to $\mathcal{H}\text{-rep}$ such that:*

- (i) \mathcal{F} agrees with \mathcal{F}' on the full subcategory $\mathcal{RIBBON}(\mathcal{H}\text{-rep})$.
- (ii) $\mathcal{F}(\text{HV}) = V^{C_t}$ and $\mathcal{F}(\text{VH}) = (V^*)^{C_t}$.
- (iii) $\mathcal{F}(\text{X}) = t$ as a morphism from V to V^{C_t} or from V^* to $(V^*)^{C_t}$ depending on orientation.
 $\mathcal{F}(\text{X}) = t^{-1}$ as a morphism from V^{C_t} to V or from $(V^*)^{C_t}$ to V^* depending on orientation (notice that this is a negative half-twist, hence the use of t^{-1}).

Proof. This is the simplest case of Reshetikhin and Turaev’s [1990] more general extension of \mathcal{F}' to ribbon graphs with coupons, where we think of the half-twists defined above as coupons on a single ribbon. So we just sketch the proof. Every morphism in $\mathcal{HTRIB}(\mathcal{H}\text{-rep})$ is isotopic to ribbon in the following standard form: every shaded object at the bottom has a positive half-twist next to it, then in the

middle of the diagram there's a ribbon tangle, then at the top of the diagram every shaded object has a negative half-twist next to it. Furthermore two such diagrams are isotopic if and only if the ribbon tangles in the middle of their standard diagrams are isotopic. Hence there is at most one possible functor \mathcal{F} . That \mathcal{F} is in fact a functor follows from the fact that t and t^{-1} are inverses of each other, so that composition of diagrams is just given by composing their middle parts. \square

The following lemma shows how \mathcal{F} acts on those elementary morphisms not already specified in the statement of [Proposition 4.15](#).

Lemma 4.16. *Let (\mathcal{H}, t) be a half-ribbon Hopf algebra, and define \mathcal{F} as in [Proposition 4.15](#).*

- (i) \mathcal{F} sends any positive half-twist to t acting on \mathcal{F} of the bottom object regardless of orientation or shading. \mathcal{F} sends any negative half-twist to t^{-1} regardless of orientation or shading.
- (ii) \mathcal{F} takes any simple crossing to $\text{Flip} \circ R$ acting on \mathcal{F} of the bottom object, regardless of orientations and shadings.
- (iii) \mathcal{F} sends shaded caps and cups to the following maps:

$$\mathcal{F} \left(\begin{array}{c} \text{shaded cap} \\ \uparrow \text{ on left, } \downarrow \text{ on right} \end{array} \right) = \text{ev}_{*(V^{C_t})} : v \otimes f \mapsto f(v), \tag{19}$$

$$\mathcal{F} \left(\begin{array}{c} \text{shaded cup} \\ \downarrow \text{ on left, } \uparrow \text{ on right} \end{array} \right) = \tilde{\text{ev}}_{*(V^{C_t})} : f \otimes v \mapsto f(\pi_V(g)v), \tag{20}$$

$$\mathcal{F} \left(\begin{array}{c} \text{unshaded cap} \\ \uparrow \text{ on left, } \downarrow \text{ on right} \end{array} \right) = \widetilde{\text{coev}}_{*(V^{C_t})} : 1 \mapsto \sum_i \pi_V(g)^{-1} v_i \otimes v^i, \tag{21}$$

$$\mathcal{F} \left(\begin{array}{c} \text{unshaded cup} \\ \downarrow \text{ on left, } \uparrow \text{ on right} \end{array} \right) = \text{coev}_{*(V^{C_t})} : 1 \mapsto \sum_i v^i \otimes v_i. \tag{22}$$

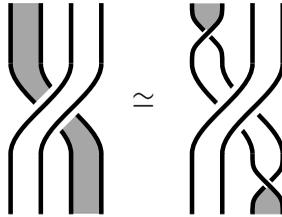
For the explicit formulas, we have used the fact that V^{C_t} and $(V^)^{C_t}$ are identical to V and V^* , respectively, as vector spaces, and chosen dual basis v_i and v^i for V and V^* .*

Comment 4.17. In order to apply the maps $\text{ev}_{*(V^{C_t})}$, etc. above, one must use the canonical isomorphism $(*(V^{C_t}))^* \cong V^{C_t}$, and the equality $(V^*)^{C_t} = *(V^{C_t})$ from [Comment 4.12](#). It is a straightforward exercise to show that these maps are given in coordinates by the above formulas.

Proof of Lemma 4.16. We prove each of these formulas by a direct calculation.

For the half-twists and ribbon full-twists we only use that $t^2 = v^{-1}$. For example, the positive half-twist going from unshaded to shaded is a composition of the negative half-twist with the full twist. So this result follows from $t^{-1}v^{-1} = t$.

The formulas for \mathcal{F} applied to a crossing with one or both of the ribbons shaded all follow from the naturality of the braiding. For example, to find \mathcal{F} of a crossing where one strand is shaded, consider the following isotopy:



Thus \mathcal{F} of the left side must be equal to $(t \otimes 1) \circ \text{Flip} \circ R \circ (1 \otimes t^{-1})$. By the naturality of the braiding,

$$(t \otimes 1) \circ \text{Flip} \circ R \circ (1 \otimes t^{-1}) = \text{Flip} \circ R \circ (1 \otimes t) \circ (1 \otimes t^{-1}) = \text{Flip} \circ R. \tag{23}$$

This holds independently of the orientations of the ribbons. A similar argument gives the same formula for crossings where the other ribbon is shaded, or where both are shaded.

It remains to compute the formula for shaded cups and caps. These equations use $g = S(t)t^{-1} = S(t^{-1})t$. We explicitly show two of the four cases. The other two are similar.

By the definition of \mathcal{F} , one has

$$\mathcal{F} \left(\begin{array}{c} \text{shaded cup} \\ \uparrow \quad \downarrow \\ \text{unshaded cup} \end{array} \right) (v \otimes f) = \mathcal{F} \left(\begin{array}{c} \text{unshaded cap} \\ \uparrow \quad \downarrow \\ \text{shaded cap} \end{array} \right) (v \otimes f) \tag{24}$$

$$= \tilde{e}v(tv \otimes t^{-1}f) \tag{25}$$

$$= t^{-1}f(gtv) \tag{26}$$

$$= f(S(t^{-1})gtv) \tag{27}$$

$$= f(v). \tag{28}$$

Here (24) is an isotopy, (25) follows from functoriality and our computation of \mathcal{F} on all half-twists, (26) is the definition of $\tilde{e}v$, (27) is the definition of the action on the dual space, and (28) uses the formula $g = S(t)t^{-1}$.

Now for a cup. Notice that, if $e^i \in V^*$ and $e_j \in V$ are dual bases and x is an invertible element in \mathcal{H} , then $v^i = xe^i$ and $v_i = S(x)^{-1}e_i$ are also dual bases. By the definition of \mathcal{F} , we have:

$$\mathcal{F} \left(\text{cup diagram with } v^i \text{ and } v_i \text{ labels} \right) (1) = \mathcal{F} \left(\text{cup diagram with } e^i \text{ and } e_j \text{ labels} \right) (1) \tag{29}$$

$$= \sum t e_i \otimes t^{-1} e^i \tag{30}$$

$$= \sum t S(t^{-1}) v_i \otimes v^i \tag{31}$$

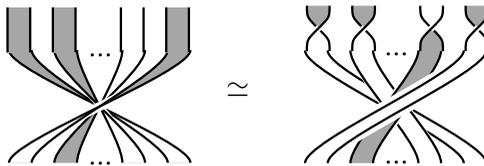
$$= \sum g^{-1} v_i \otimes v^i. \tag{32}$$

Here (29) is an isotopy, (30) follows from functoriality and our computation of \mathcal{F} on all half-twists, (31) is the change of dual bases described above with $x = t$, and (32) uses the formula $g = S(t)t^{-1}$. \square

Note that above we used the formulas relating t to g (which was derived using the formula relating R and t), but we did not use the formula for R directly. That formula is used in the following proposition which computes how \mathcal{F} acts on the half-twist applied to many strands.

Proposition 4.18. *Suppose that (\mathcal{H}, t) is a half-ribbon Hopf algebra, let \mathcal{F} be the unique functor guaranteed by Proposition 4.15. Then, $\mathcal{F}(I_n) = \text{rev} \circ \Delta^n(t)$.*

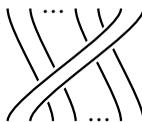
Proof. We proceed by induction on n , the case $n = 1$ being trivial. Consider the isotopy



Thus, by the definition of \mathcal{F} ,

$$\mathcal{F}(I_n) = (t \otimes t \otimes \cdots \otimes t) \mathcal{F}(T_{w_0}^{(n)}), \tag{33}$$

where $T_{w_0}^{(n)}$ is the braid group element corresponding to the longest element of S_n . Let $\sigma^{(n)}$ be the braid group element



Let $s^{(n)}$ be the image of $\sigma^{(n)}$ in S_n , and let $\text{rev}^{(n)}$ denote the longest element of S_n . Clearly $T_{w_0}^{(n)} = (T_{w_0}^{(n-1)} \otimes 1) \circ \sigma^{(n)}$. Hence

$$\mathcal{F}(I_n) = (t \otimes t \otimes \cdots \otimes t) \circ \mathcal{F}(T_{w_0}^{(n)}) \tag{34}$$

$$= (t \otimes t \otimes \cdots \otimes t) \circ (\mathcal{F}(T_{w_0}^{(n-1)}) \otimes 1) \circ \mathcal{F}(\sigma^{(n)}) \tag{35}$$

$$= (\mathcal{F}(I_{n-1}) \otimes t) \circ \mathcal{F}(\sigma^{(n)}) \tag{36}$$

$$= (\text{rev}^{(n-1)} \Delta^{n-1}(t) \otimes t) \circ s^{(n)} \circ (1 \otimes \Delta^{n-1})(R) \tag{37}$$

$$= (\text{rev}^{(n-1)} \Delta^{n-1}(t) \otimes t) \circ s^{(n)} \circ (1 \otimes \Delta^{n-1})((t^{-1} \otimes t^{-1})\Delta(t)) \tag{38}$$

$$= \text{rev}^{(n)} \Delta^n(t). \tag{39}$$

Here (36) follows from (33) in the case $n - 1$, (37) follows from the inductive assumption and the quasitriangularity of R , (38) holds because t is a half-ribbon element, and (39) is a straightforward calculation.

This proof works regardless of the orientations and shadings of the ribbons. \square

Proof of Theorem 4.11. Using the isotopy from (1), the crossing is a composition of the two strand topological half-twist and two copies of the inverse of the one strand topological half-twist. Thus the conditions in the statement uniquely determine \mathcal{F} on all the elementary morphisms listed in the condition of Proposition 4.15. So Proposition 4.15 shows that there is a unique candidate for \mathcal{F} . Proposition 4.18 shows that this \mathcal{F} satisfies the remaining condition of the theorem. \square

5. Ribbon and half-ribbon elements for $U_q(\mathfrak{g})$

We show that $U_q(\mathfrak{g})$ is always a topological half-ribbon Hopf algebra. That is, there exists an element X in a completion of $U_q(\mathfrak{g})$ such that $(X^{-1} \otimes X^{-1})\Delta(X)$ is the standard R -matrix, and X^{-2} is a ribbon element. Interestingly, X^{-2} is not the standard ribbon element C . We also classify the different ribbon elements for $U_q(\mathfrak{g})$, and discuss how one might decide which of these arise from half-ribbon elements. In particular, we show that the standard ribbon element for $U_q(\mathfrak{sl}_2)$ does not arise from a half-ribbon element. We then explain some consequences of varying the ribbon element. It turns out the ribbon element X^{-2} is in some ways particularly nice (see Lemma 5.7 and Section 5E).

5A. A half-ribbon Hopf algebra structure on $U_q(\mathfrak{g})$. We first need the following result relating different ribbon elements for the same quasitriangular Hopf algebra.

Lemma 5.1. *Let (\mathcal{H}, R, v) be a ribbon Hopf algebra, and $s \in \mathcal{H}$ be a central grouplike element that squares to 1. Then (\mathcal{H}, R, vs) is also a ribbon Hopf algebra. Furthermore all ribbon elements for (\mathcal{H}, R) are of the form vs for some such s .*

Proof. This is a straightforward application of the definition of a ribbon element [Barrett and Westbury 1999, Remark 3.4]. \square

Theorem 5.2. *Let X be the element from Theorem 3.11. Then $(U_q(\mathfrak{g}), X)$ is a topological half-ribbon Hopf algebra. Furthermore $(X^{-1} \otimes X^{-1})\Delta(X)$ is the standard R -matrix.*

Proof. By Theorem 3.11, $(U_q(\mathfrak{g}), R)$ is a quasitriangular Hopf algebra, where $R = (X^{-1} \otimes X^{-1})\Delta(X)$ is the standard R matrix. Thus it suffices to show that X^{-2} is a ribbon element for this quasitriangular Hopf algebra. We already know that $(U_q(\mathfrak{g}), R, C)$ is a ribbon Hopf algebra. Thus by Lemma 5.1 it suffices to show that (i) X^2C is central; (ii) X^2C is grouplike; and (iii) $(X^2C)^2 = 1$.

(i) By Lemma 3.10(iii) and Definition 3.15, X^2C acts on V_λ as multiplication by the scalar $(-1)^{\langle 2\lambda, \rho^\vee \rangle}$. Thus it is clearly central.

(ii) Any highest weight ν for $V_\lambda \otimes V_\mu$ has weight $\lambda + \mu - \gamma$ for some γ in the root lattice. Since $\langle 2\gamma, \rho^\vee \rangle$ is an even integer for any γ in the root lattice, it follows that X^2C is grouplike.

(iii) $(X^2C)^2$ acts on V_λ by $(-1)^{2 \cdot \langle 2\lambda, \rho^\vee \rangle} = 1$. □

5B. The Frobenius–Schur indicator. We must now discuss a tool for comparing ribbon elements. The following definition of the Frobenius–Schur indicator for a pivotal category was given by [Ng and Schauenburg 2007] (see also [Linchenko and Montgomery 2000] for the case of Hopf algebras and [Fuchs et al. 1999] for C^* sovereign categories).

Recall that a pivotal structure is a natural collection of maps $p_V : V \rightarrow V^{**}$ which defines a natural isomorphism $\text{Id} \rightarrow **$ of monoidal functors [Barrett and Westbury 1999]. The category of representations of a ribbon Hopf algebra is always pivotal with the pivotal structure being given by the grouplike element $g = v^{-1}u$ as follows: fix $v \in V$. Then $p_V(v)$ is the element of V^{**} defined by, for all $f \in V^*$, $p_V(v)(f) = f(gv)$ (see for example [Chari and Pressley 1994]).

Definition 5.3. Given an F -linear category with a chosen pivotal structure p , define the Frobenius–Schur indicator of an absolutely simple object V as follows: if $V \not\cong V^*$, then $FS_p(V) = 0$. Otherwise, choose an isomorphism $f : V \rightarrow V^*$. By Schur’s lemma there exists some constant, which we define to be $FS_p(V)$, such that $f = FS_p(V)f^* \circ p$.

Comment 5.4. As shown in [Ng and Schauenburg 2007], $FS_p(V)$ does not depend on the choice of f , and we have $FS_p(V) = \pm 1$ (or 0).

Comment 5.5. Notice that the Frobenius–Schur indicator depends on the pivotal structure, which in turn depends on the choice of ribbon element. As described in [Ng and Schauenburg 2007], we use $FS_v(V)$ to denote the Frobenius–Schur indicator for V , calculated with the ribbon element v .

Comment 5.6. The reason for the term Frobenius–Schur indicator is that in the case of groups (and indeed involutory Hopf algebras [Linchenko and Montgomery 2000]) with the trivial pivotal structure, it agrees with the usual Frobenius–Schur indicator, which is 1 when there is an invariant orthogonal form on $V \otimes V$, -1 when there is an invariant symplectic form on $V \otimes V$, and 0 otherwise.

Lemma 5.7. *Let V_λ be an irreducible representation of $U_q(\mathfrak{g})$ and assume V_λ is self-dual. Then $FS_{X^{-2}}(V_\lambda) = 1$.*

Proof. Let $f : V_\lambda \rightarrow V_\lambda^*$ be an isomorphism. It suffices to show that $f = f^* \circ p_{V_\lambda}$. Since both f and $f^* \circ p_{V_\lambda}$ are isomorphisms, it suffices to show that their actions on v_λ^{low} agree. But $f(v_\lambda^{\text{low}})$ is in the lowest weight space of V_λ^* , which is one-dimensional and pairs nondegenerately with v_λ . Thus, it in fact suffices to show that

$$f(v_\lambda^{\text{low}})(v_\lambda) = f^* \circ p_{V_\lambda}(v_\lambda^{\text{low}})(v_\lambda). \quad (40)$$

Using the formula for g given in Proposition 4.3,

$$S(X) = Xg^{-1} = gX. \quad (41)$$

Let $k = (-1)^{\langle 2\lambda, \rho^\vee \rangle} q^{(\lambda, \lambda)/2 + (\lambda, \rho)}$. By Lemma 3.10(ii), $X(v_\lambda) = k(v_\lambda^{\text{low}})$. Thus the left side of (40) can be simplified as follows

$$f(v_\lambda^{\text{low}})(v_\lambda) = f(k^{-1}X(v_\lambda))(v_\lambda) \quad (42)$$

$$= k^{-1}(Xf(v_\lambda))(v_\lambda) \quad (43)$$

$$= k^{-1}f(v_\lambda)(S(X)v_\lambda) \quad (44)$$

$$= k^{-1}f(v_\lambda)(Xg^{-1}v_\lambda) \quad (45)$$

$$= k^{-1}f(v_\lambda)(gXv_\lambda) \quad (46)$$

$$= k^{-1}f(v_\lambda)(kgv_\lambda^{\text{low}}) \quad (47)$$

$$= f(v_\lambda)(gv_\lambda^{\text{low}}). \quad (48)$$

Here (43) follows because f is an isomorphism, (45) and (46) follow by (41) and (47) by the above formula for the action of X on v_λ .

Now consider the right side of (40). By definition, $p_{V_\lambda}(v_\lambda^{\text{low}})$ is the element of V_λ^{**} which takes $\phi \in V_\lambda^*$ to $\phi(gv_\lambda^{\text{low}})$. Thus

$$f^* \circ p_{V_\lambda}(v_\lambda^{\text{low}})(v_\lambda) = f(v_\lambda)(gv_\lambda^{\text{low}}). \quad (49)$$

We have shown that the two sides of (40) agree, so the result follows. \square

Turaev [1994] calls the property that Lemma 5.7 holds unimodality and requires it when defining geometric 3j and 6j symbols.

Lemma 5.8. $FS_C(V)$ is the usual Frobenius–Schur indicator of the corresponding classical representation.

Proof. Since the Frobenius–Schur indicator is a discrete invariant, it doesn’t change under continuous deformation. \square

5C. Which ribbon elements for $U_q(\mathfrak{g})$ arise from half-ribbon elements? Here we classify ribbon elements for $U_q(\mathfrak{g})$. We then prove that the standard ribbon element for $U_q(\mathfrak{sl}_2)$ is not equal to t^{-2} for any half-ribbon element t .

Definition 5.9. Suppose that ϕ is a character of P/Q , where P is the weight lattice of \mathfrak{g} , and Q is the root lattice. Define $s(\phi) \in \widetilde{U_q(\mathfrak{g})}$ to act on V_λ as multiplication by the scalar $\phi(\lambda)$.

Theorem 5.10. The central grouplike elements of $\widetilde{U_q(\mathfrak{g})}$ are precisely the $s(\phi)$.

Proof. Suppose that s is a central grouplike element of $\widetilde{U_q(\mathfrak{g})}$. Since it is central, s acts by a scalar on any irreducible representation. Furthermore, since it is grouplike, s acts by the same scalar on any two irreducible representations that appear in the same arbitrary tensor product of irreducible representations. Thus such an s gives a function of P/\sim , where \sim is the equivalence relation on weights generated by $\lambda \sim \mu$ if V_λ and V_μ appear in the same tensor product of irreducible representations. It suffices to show that $\lambda \sim \mu$ if and only if $\lambda + Q = \mu + Q \in P/Q$. One can easily see that \sim respects the additive structure of the weight lattice, so it is enough to prove that V_λ appears in some tensor product also containing the trivial representation V_0 precisely when λ is in the root lattice.

Since the Littlewood–Richardson coefficients are the same in the quantum and classical cases, this reduces to the same question in the classical case. So let G be the simply connected Lie group attached to $U(\mathfrak{g})$, and let $Z(G)$ be the center of G . The central character map gives a pairing between $Z(G)$ and the weight lattice mod the root lattice. Since any tensor product of irreducible representations has a well-defined central character, we see that V_λ is equivalent to V_0 only if λ is in the root lattice.

To see the other direction, let V be a faithful representation of the compact adjoint form K of G .

Claim: For any λ in the root lattice, there exists N so that, for all $n > N$, V_λ occurs as a subrepresentation of $V^{\otimes n}$.

Proof of claim: since λ is in the root lattice, V_λ descends to a representation of K . On this compact form we can use character theory. Because K has no center, $|\chi_V(g)| < \dim V$ for any $g \neq 1$. Let

$$k_n = \left| \int_K \chi_V^n d\mu \right|, \tag{50}$$

where $d\mu$ is the normalized Haar measure. Since K is compact and $|\chi_V(g)| < \chi_V(1)$ for all $g \neq 1$, we see that

$$\lim_{n \rightarrow \infty} \frac{\langle \chi_{V^{\otimes n}}, \chi_{V_\lambda} \rangle}{k_n} = \lim_{n \rightarrow \infty} \frac{1}{k_n} \int_K \chi_V(g)^n \chi_{V_\lambda}(g^{-1}) d\mu = \chi_{V_\lambda}(1) = \dim V_\lambda. \quad (51)$$

In particular, for $n \geq N$ we see that $\langle \chi_{V^{\otimes n}}, \chi_{V_\lambda} \rangle$ is nonzero, so V_λ occurs in $V^{\otimes n}$.

Applying the above fact to 0 and an arbitrary root vector λ , we see that for a sufficiently high power $N' = \max(N_0, N_\lambda)$ the tensor power $V^{\otimes N'}$ contains both V_0 and V_λ , and hence $0 \sim \lambda$. □

Theorem 5.11. *Ribbon elements of $U_q(\widetilde{\mathfrak{g}})$ are exactly elements of the form*

$$s(\phi)X^{-2},$$

where ϕ is a character of the weight lattice mod the root lattice of order ≤ 2 .

Proof. This follows from [Lemma 5.1](#) and [Theorem 5.10](#). □

We have the following relationship between ribbon elements and the Frobenius–Schur indicator.

Proposition 5.12. *If V_λ is self dual, then $FS_{s(\phi)X^{-2}}(V_\lambda) = \phi(\lambda)$.*

Proof. The only part of the definition of the Frobenius–Schur indicator that changes when you change the ribbon element from X^{-2} to $s(\phi)X^{-2}$ is p_{V_λ} , and this is multiplied by $\phi(\lambda)$. Hence the proposition follows from [Lemma 5.7](#). □

We wish to understand which of these ribbon element extend to a half-ribbon elements on $U_q(\widetilde{\mathfrak{g}})$. As we shall see, the ratio of two half-ribbon elements is grouplike, so to classify all of them we need to understand the grouplike elements in $U_q(\widetilde{\mathfrak{g}})$. In general this seems to pose some technical challenges, but we can do the case of $U_q(\mathfrak{sl}_2)$.

Proposition 5.13. *Let (\mathcal{H}, R) be a quasitriangular Hopf algebra, and assume that t_1 and t_2 are two half-ribbon elements. Then $t_1 t_2^{-1}$ is grouplike. (Note that we do not assume here that $t_1^{-2} = t_2^{-2}$.)*

Proof. This follows from the equality $(t_1^{-1} \otimes t_1^{-1})\Delta(t_1) = (t_2^{-1} \otimes t_2^{-1})\Delta(t_2)$. □

Lemma 5.14. *Let V be the standard representation of $U_q(\mathfrak{sl}_2)$, with basis $\{v_+, v_-\}$ such that $E(v_-) = v_+$, $Kv_+ = qv_+$, and $Kv_- = q^{-1}v_-$. Then $V \otimes V$ contains a copy of the trivial representation V_0 . Furthermore, this is spanned by the element*

$$v_+ \otimes v_- - q^{-1}v_- \otimes v_+.$$

Proof. It suffices to check that $\Delta(E)(v_+ \otimes v_- - q^{-1}v_- \otimes v_+) = 0$. This follows from the definition of Δ . □

Definition 5.15. Let $a \in \mathbb{C}(q)$. Define K_a to be the element of $\widetilde{U_q(\mathfrak{sl}_2)}$ that acts on any weight vector $v \in V_\lambda$ by $K_a(v) = a^{\text{wt}(v)}$. Note that K_a is grouplike.

Lemma 5.16. All grouplike elements in $\widetilde{U_q(\mathfrak{sl}_2)}$ are of the form K_a , for $a \in \mathbb{C}(q)$.

Proof. A grouplike element is determined by its action on the fundamental representation V . Suppose that σ is grouplike. Define constants a, b, c and d by $\sigma v_+ = av_+ + bv_-$ and $\sigma v_- = cv_+ + dv_-$. Since σ is grouplike it must act trivially on the trivial subrepresentation of $V \otimes V$. Thus by Lemma 5.14,

$$v_+ \otimes v_- - q^{-1}v_- \otimes v_+ = \sigma \otimes \sigma(v_+ \otimes v_- - q^{-1}v_- \otimes v_+) \tag{52}$$

$$= ac(1 - q^{-1})v_+ \otimes v_+ + (ad - q^{-1}bc)v_+ \otimes v_- \tag{53}$$

$$- q^{-1}(ad - qbc)v_- \otimes v_+ + (1 - q^{-1})bdv_- \otimes v_-. \tag{54}$$

Comparing coefficients we see that $ac = bd = bc = 0$ and $ad = 1$. Hence σ acts on V in exactly the same way that K_a does. Since they are both grouplike and V is a tensor generator for the category of representations of $U_q(\mathfrak{sl}_2)$, we see that $\sigma = K_a$. □

Theorem 5.17. There is no topological half-ribbon element t for $U_q(\mathfrak{sl}_2)$ such that t^{-2} is the usual ribbon element C .

Proof. By Theorem 3.16, X is a half-ribbon element for $(U_q(\mathfrak{sl}_2), R)$. By Proposition 5.13, any other half-ribbon element is of the form $X\alpha$ for some grouplike element α . By Lemma 5.16, $\alpha = K_a$ for some $a \in \mathbb{C}(q)$. Since X sends the λ weight space to the $-\lambda$ weight, it follows that $K_a X K_a v = Xv$ for any weight space v . Therefore, we have that $(XK_a)^{-2} = X^{-2}$. Hence every half-ribbon element t for $U_q(\mathfrak{sl}_2)$ has $t^{-2} = X^{-2}$. But X^{-2} acts as $-C$ on the standard representation, so is not equal to C . □

5D. A summary of ribbon and half-ribbon elements for $U_q(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_3)$ and $U_q(\mathfrak{sl}_4)$. We now describe the possible ribbon elements of these three quantum groups, and describe which are the squares of half-ribbon elements. Notice that the three cases all behave differently.

- $U_q(\mathfrak{sl}_2)$. There are two ribbon elements C , and X^{-2} . Only X^{-2} can be realized using a half-ribbon element (see Theorem 5.17).
- $U_q(\mathfrak{sl}_3)$. In this case $P/Q \cong \mathbb{Z}/3$. Thus, there are no central grouplikes of order two, so there can be only one ribbon element. This unique ribbon element can be realized using the half-ribbon element X .
- $U_q(\mathfrak{sl}_4)$. In this case $P/Q \cong \mathbb{Z}/4$. This has two characters of order ≤ 2 , so there are two ribbon elements. In this case both come from half-ribbon elements. A straightforward calculation on the standard representation of $U_q(\mathfrak{sl}_4)$ shows that X^{-2} is not the standard ribbon element. To see the other half-ribbon element,

consider the central element $s \in \widetilde{U_q(\mathfrak{sl}_4)}$ which acts on V_λ as multiplication by $i^{4(\lambda, \lambda)}$, where i is the complex fourth root of unity. Recall that $(\lambda, \alpha) \in \mathbb{Z}$ for any root α , which implies that s is grouplike. Clearly we also have $s^4 = 1$, and from these two facts it follows that $X' := sX$ is a half-ribbon element for $(U_q(\mathfrak{sl}_4), R)$. One can check that s acts as multiplication by $-i$ on the standard representation, so $X'^{-2} = s^2 X^{-2} \neq X^{-2}$. Since X^{-2} is the nonstandard ribbon element, X'^{-2} must be the standard ribbon element.

5E. Varying the ribbon element and diagrammatic categories. Often you'll find a result in the literature saying that a certain diagram category "is the same as" as certain category coming from quantum groups (for example, [Ohtsuki 2002, Appendix H] and [Kuperberg 1996, page 11]). However, in the details of this claim there's an annoying sign difference between the diagram category and the quantum group category. The reason for this is that, although the two categories match up as braided tensor categories, they are different as pivotal categories. This can be fixed by changing the ribbon element on the quantum group side, thus changing the pivotal structure.

We illustrate this here by considering the case of $U_q(\mathfrak{sl}_2)$ -rep and the Temperley–Lieb category (see Section 3E). In the correspondence discussed in [Ohtsuki 2002, Appendix H], the standard representation of $U_q(\mathfrak{sl}_2)$ corresponds to the elementary object (i.e. a single \bullet) in the Temperley–Lieb category. We now use the Frobenius–Schur indicator to show that, for any such statement to hold on the level of pivotal categories, one must use the ribbon element X^{-2} .

Proposition 5.18. *Let \bullet denote a single point in Temperley–Lieb (with the trivial pivotal structure discussed in Section 3E). Then $FS(\bullet) = 1$.*

Proof. The single strand is an isomorphism between \bullet and $\bullet^* = \bullet$. The dual of the single strand (given by rotating 180 degrees about the z axis) is again the single strand. □

Proposition 5.19. *Let V be the standard representation of $U_q(\mathfrak{sl}_2)$. Then*

$$FS_C(V) = -1 \quad \text{and} \quad FS_{X^{-2}}(V) = 1.$$

Proof. This is a straightforward calculation using the definitions of C (Definition 3.15) and Lemma 3.10(iii). It also follows from the discussion in Section 5C. □

If \bullet in Temperley–Lieb is going to correspond to the standard representation, the Frobenius–Schur indicators should agree. Clearly this can only happen if we use the ribbon element X^{-2} .

Comment 5.20. A similar difficulty arises in relating the type C quantum group knot invariants with (a specialization of) the Kauffman polynomial. Once again, one can fix the problem by switching to the ribbon element X^{-2} .

Comment 5.21. This difficulty can also be addressed by changing the diagrammatic category. The Temperley–Lieb category has another pivotal structure p such that $p_\bullet = -\text{id}_\bullet$. With this pivotal structure $FS(\bullet) = -1$. This other pivotal structure is given diagrammatically by the disoriented Temperley–Lieb category of [Clark, Morrison and Walker 2009].

5F. The effect of varying the ribbon element on knot invariants. Suppose that (\mathcal{H}, R, v) is a ribbon Hopf algebra over a base field F . Fix a simple \mathcal{H} -module V . The functor from Theorem 4.7 sends a link with every component labeled with V to an element of F , denoted by $\mathcal{F}^V(L)$, which is an invariant of framed oriented links. Note that $\mathcal{F}^V(L)$ actually depends on the choice of ribbon element v , so when we need to be clear about which ribbon element we are using we will denote it by $\mathcal{F}_v^V(L)$. Since V is simple the ribbon element acts by a scalar $\theta_v(V)$ on V . It follows as in, for example, [Ohtsuki 2002, Section 3.3] that $\theta_v(V)^{w(L)}\mathcal{F}_v^V(L)$ is an invariant of oriented but unframed links (where $w(L)$ is the writhe of the link diagram, as described in, for example, [Ohtsuki 2002, page 11] and the current Wikipedia entry for “Writhe”).

Proposition 5.22. *Suppose that \mathcal{H} is a quasitriangular Hopf algebra with*

$$\text{End}(\mathbf{1}) = F,$$

and that V is a simple \mathcal{H} -module. Suppose that v_1 and v_2 are two different ribbon elements for \mathcal{H} . Let $\theta_{v_1}(V)$ and $\theta_{v_2}(V)$ be the scalars by which v_1 and v_2 act on V . Let $\mathcal{F}_{v_1}^V$ and $\mathcal{F}_{v_2}^V$ be the functors attached to these two ribbon Hopf algebras. Then for any link L ,

$$\theta_{v_1}^{w(L)}\mathcal{F}_{v_1}^V(L) = \left(\frac{FS_{v_1}(V)}{FS_{v_2}(V)}\right)^{\#L} \theta_{v_2}^{w(L)}\mathcal{F}_{v_2}^V(L),$$

where $\#L$ is the number of components of L .

Proof. By Theorem 5.11, $\alpha := v_1/v_2$ is central grouplike element of order 2. Let α_V be the scalar by which α acts on V . By Proposition 5.12,

$$\alpha_V = FS_{v_1}(V)/FS_{v_2}(V).$$

Let $g_i = v_i^{-1}u$ where u is the Drinfeld element. It follows immediately from definitions that $\widetilde{e}v_2 = \alpha_V^{-1}\widetilde{e}v_1$ and $\widetilde{\text{coe}}v_2 = \alpha_V^{-1}\widetilde{\text{coe}}v_1$.

The only elementary morphisms for which $\mathcal{F}_{v_1}^V$ and $\mathcal{F}_{v_2}^V$ disagree are left-going cups (corresponding to $\widetilde{\text{coe}}v$), left-going caps (corresponding to $\widetilde{e}v$), and full-twists. Thus,

$$\theta_{v_1}^{w(L)}\mathcal{F}_{v_1}^V(L) = (\alpha_V)^{-N_L}\theta_{v_2}^{w(L)}\mathcal{F}_{v_2}^V(L), \tag{55}$$

where

$$N_L := \#\{\text{left going caps and cups}\} + w(L). \tag{56}$$

It is easy to see that $N_L \pmod{2}$ is an invariant of oriented links (for example, check all the Turaev moves [Ohtsuki 2002, Section 3.2]). Furthermore, $N_L \pmod{2}$ doesn't change when you replace a positive crossing by a negative crossing. Since every link can be unknotted by replacing positive crossings with negative crossings, we see that $N_L \pmod{2}$ depends only on the number of components of L . By looking at the k -component unlink we see that $N_L \equiv \#L \pmod{2}$. \square

6. Questions

Question 1. Which ribbon Hopf algebras can be endowed with a half-ribbon element?

There are several aspects to this question. One could look for examples of half-ribbon Hopf algebras other than $U_q(\mathfrak{g})$. One could also try to find nonexamples, in the sense of finding ribbon Hopf algebras that do not contain the required element t . Such examples exist (at least for topological ribbon Hopf algebras). For instance we showed in Section 5C that $U_q(\mathfrak{sl}_2)$, with the standard ribbon element, cannot be made into a half-ribbon Hopf algebra. However, it can be modified by multiplying the ribbon element by a central grouplike element, and then it does have the required t . One could also ask if there are examples of ribbon Hopf algebras that cannot be made into half-ribbon Hopf algebras, even allowing this sort of modification. More ambitiously, one could look for a general method of determining when a ribbon Hopf algebra \mathcal{H} contains an element t such that (\mathcal{H}, t) is a half-twist Hopf algebra.

In the current work we have mainly considered $U_q(\mathfrak{g})$, which is infinite-dimensional, and only has a topological half-ribbon element in the sense that t only belongs to a completion of $U_q(\mathfrak{g})$. We feel it would be interesting to look at the case of finite-dimensional Hopf algebras as well.

Question 2. Fix a Hopf algebra \mathcal{H} . Is there a natural set of conditions one can impose on an element $t \in \mathcal{H}$ which guarantees that (\mathcal{H}, t) is a half-ribbon Hopf algebra?

We would like to be able to start with a Hopf algebra, which is not a priori quasitriangular, and endow it with a ribbon (and half-ribbon) structure by finding a certain $t \in \mathcal{H}$. One can of course write down the conditions t needs to satisfy by insisting that $(t^{-1} \otimes t^{-1})\Delta(t)$ is a quasitriangular structure, and t^{-2} is a ribbon element. However, these are very difficult to deal with, so the real question is to find nicer conditions on t . This would give a new method of constructing quasitriangular Hopf algebras.

Question 3. What happens if you weaken the conditions on t ?

Checking that $(t^{-1} \otimes t^{-1})\Delta t$ is a quasitriangular structure is difficult, but for some applications one can weaken this condition by insisting only that

(i) for any representation V and W of \mathcal{H} , the map

$$\text{Flip} \circ (t^{-1} \otimes t^{-1})\Delta(t) : V \otimes W \rightarrow W \otimes V \quad (57)$$

is an isomorphism; and

(ii) t^{-2} acts as a ribbon element, in the sense that

$$t^{-2} \circ \text{mult} \circ (S \otimes 1) \left((t^{-1} \otimes t^{-1})\Delta(t) \right) \quad (58)$$

is grouplike, and so can be used to develop a theory of quantum trace.

These conditions seem easier to check, and a Hopf algebra \mathcal{H} with such a t already has some nice structure. This could be used, for instance, in studying coboundary categories [Drinfel'd 1989].

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A 2-block splitting in alternating groups

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In 1956, Brauer showed that there is a partitioning of the p -regular conjugacy classes of a group according to the p -blocks of its irreducible characters with close connections to the block theoretical invariants. In a previous paper, the first explicit block splitting of regular classes for a family of groups was given for the 2-regular classes of the symmetric groups. Based on this work, the corresponding splitting problem is investigated here for the 2-regular classes of the alternating groups. As an application, an easy combinatorial formula for the elementary divisors of the Cartan matrix of the alternating groups at $p = 2$ is deduced.

1. Introduction

Richard Brauer [1956] introduced the idea of not only distributing characters into p -blocks but also of associating p -regular conjugacy classes to p -blocks. He showed that it is possible to distribute the p -regular classes into blocks in a way that fits with the blocks of irreducible Brauer characters (and suitable subsets of ordinary irreducible characters in the blocks); this is to say that the determinant of the corresponding block part of the Brauer character table (or a suitable part of the ordinary character table) is not congruent to 0 modulo \mathfrak{p} (a prime ideal over p). Given such a splitting of p -regular classes into blocks, Brauer showed that the elementary divisors of the Cartan matrix of a block are then exactly the p -parts in the orders of the centralizers of elements in the classes corresponding to the block. He also observed that in general there may be several such block splittings, and there did not seem to be any natural choice for a given finite group.

But while it is well known how to determine the p -blocks of irreducible characters, for the p -regular classes only the existence of such a block splitting is known by Brauer's work — concrete examples for providing such a distribution for families of groups were not known for a long time. Only recently, such an explicit block splitting in the sense of Brauer was exhibited for the conjugacy classes of odd order elements and the 2-blocks of the symmetric groups [Bessenrodt 2007];

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in fact, in this case the 2-block splitting of the 2-regular classes is unique. The proof exploited detailed information on the double covers of the symmetric groups, in particular results on the 2-powers in the spin character values of these groups [Bessenrodt and Olsson 2000] as well as on the 2-block distribution of the spin characters [Bessenrodt and Olsson 1997] turned out to be important ingredients.

Based on these results, the present paper investigates the corresponding problem of constructing a 2-block splitting of the 2-regular classes for the alternating groups. We provide a basic set of characters for the alternating groups, and find a natural choice for a block splitting of the classes. As an application, we deduce an easy combinatorial description of the invariants of the Cartan matrices for the 2-blocks of the alternating groups.

Here is a brief outline of the sections. In Section 2, we recall Brauer's results on block splittings for finite groups which motivated the present work. Then, in Section 3, some combinatorial notations needed in the representation theory of the symmetric groups is introduced, and we state some results from [Bessenrodt 2007] on the block splitting of 2-regular classes for the symmetric groups that are the basis for the new results on alternating groups. In particular, the class labels for the 2-block splitting of S_n are recalled. In Section 4, we first collect the necessary information on characters of the alternating groups, and prove some preliminary results towards the construction of a class splitting for the alternating groups. In the main Theorem 4.7 properties of the determinants of the corresponding block character tables are proved which imply that the construction gives indeed a block splitting of the classes. By Brauer's Theorem, our result then implies an easy combinatorial description of the Cartan invariants for the 2-blocks of the alternating groups (Corollary 4.9).

2. Brauer's block splitting

Let G be a finite group, p a prime, (\mathfrak{K}, R, F) a p -modular splitting system for G , and \mathfrak{p} a maximal ideal of R lying over p . Let $\ell(G)$ be the cardinality of the set $\text{Cl}_{p'}(G)$ of p -regular conjugacy classes in G . For each $K \in \text{Cl}_{p'}(G)$ we let x_K denote an element in K . A defect group of K is a Sylow p -subgroup of $C_G(x)$ for some $x \in K$; if this has order p^d , then d is called the p -defect of K . We let $\text{IBr}(G)$ denote the set of modular irreducible characters of G ; then

$$\Phi_G = (\varphi(x_K))_{\substack{\varphi \in \text{IBr}(G) \\ K \in \text{Cl}_{p'}(G)}}$$

is the Brauer character table of G . It is well known by Brauer's work that the Brauer character table is nonsingular modulo p , that is,

$$\det \Phi_G \not\equiv 0 \pmod{\mathfrak{p}}.$$

Furthermore, we let $D = (d_{\chi\varphi})_{\chi \in \text{Irr}(G), \varphi \in \text{IBr}(G)}$ denote the p -decomposition matrix for G , and we let $C = D^t D$ denote its Cartan matrix. Let $\text{Bl}_p(G)$ be the set of p -blocks of G . For $B \in \text{Bl}_p(G)$, $\text{Irr}(B)$ is the set of ordinary irreducible characters in B , $\text{IBr}(B)$ is the set of modular irreducible characters in B , $\ell(B) = |\text{IBr}(B)|$, $D(B) = (d_{\chi\varphi})_{\chi \in \text{Irr}(B), \varphi \in \text{IBr}(B)}$ denotes the p -decomposition matrix for B and $C(B)$ is the Cartan matrix for B .

Then C resp. D are the block direct sums of the matrices $C(B)$ resp. $D(B)$, $B \in \text{Bl}_p(G)$.

The following result was proved by Brauer.

Theorem 2.1. [Brauer 1956, Section 5] *There exists a disjoint decomposition of $\text{Cl}_{p'}(G)$ into blocks of p -regular conjugacy classes*

$$\text{Cl}_{p'}(G) = \bigcup_{B \in \text{Bl}_p(G)} \text{Cl}_{p'}(B)$$

and a selection of characters $\text{Irr}'(B) \subseteq \text{Irr}(B)$ for each p -block B of G such that the following conditions are fulfilled:

- (i) $|\text{Cl}_{p'}(B)| = |\text{Irr}'(B)| = \ell(B)$ for all $B \in \text{Bl}_p(G)$;
- (ii) For $X_B = (\chi(x_K))_{\substack{\chi \in \text{Irr}'(B) \\ K \in \text{Cl}_{p'}(B)}}$, we have $\det X_B \not\equiv 0 \pmod{p}$;
- (iii) For $\Phi_B = (\varphi(x_K))_{\substack{\varphi \in \text{IBr}(B) \\ K \in \text{Cl}_{p'}(B)}}$, we have $\det \Phi_B \not\equiv 0 \pmod{p}$;
- (iv) For $D_B = (d_{\chi\varphi})_{\substack{\chi \in \text{Irr}'(B) \\ \varphi \in \text{IBr}(B)}}$, we have $\det D_B \not\equiv 0 \pmod{p}$.

Furthermore, the elementary divisors of the Cartan matrix $C(B)$ are then exactly the orders of the p -defect groups of the conjugacy classes in $\text{Cl}_{p'}(B)$, for all B in $\text{Bl}_p(G)$.

The properties in (ii)–(iv) are not independent of each other, as $X_B = D_B \Phi_B$. In particular, if we have a suitable choice $\text{Irr}'(B)$ of characters that satisfies (iv), and a suitable choice of classes that satisfies (iii), then these together are a suitable choice for (ii). If we have a basic set of irreducible characters, that is, a subset $\text{Irr}'(G) \subseteq \text{Irr}(G)$ giving a \mathbb{Z} -basis for the character restrictions to the p -regular classes, then the p -block decomposition of this set will give a suitable choice of sets $\text{Irr}'(B)$ satisfying (iv).

3. The 2-block splitting for S_n

Let $n \in \mathbb{N}$. For the symmetric groups S_n , the corresponding combinatorial notions and their representation theory, we will follow mostly the usual notation in [James and Kerber 1981].

Let P be the set of partitions, $P(n)$ the partitions of n . For a partition λ of n , the number of its (nonzero) parts is called its length, and is denoted by $l(\lambda)$. The complex irreducible character of S_n corresponding to λ is denoted by $[\lambda]$. For any partition μ of n , we choose an element σ_μ in S_n of cycle type μ .

Let $\mu = (1^{m_1(\mu)}, 2^{m_2(\mu)}, \dots)$ be written in exponential notation; then we set

$$a_\mu = \prod_{i \geq 1} i^{m_i(\mu)}, \quad b_\mu = \prod_{i \geq 1} m_i(\mu)! .$$

Let $z_\mu = |C_{S_n}(\sigma_\mu)|$; then $z_\mu = a_\mu b_\mu$.

Let p be a prime; we will soon fix this to $p = 2$. A partition is called p -regular if no part is repeated p or more times, and a partition is called p -class regular if no part is divisible by p .

Let $\mathcal{D}(n)$ be the set of partitions of n into distinct parts; this is thus the set of 2-regular partitions of n . Let $\mathcal{O}(n)$ be the set of partitions of n into odd parts only; this is the set of 2-class regular partitions of n . Let $\mathcal{O} = \bigcup_{n \in \mathbb{N}} \mathcal{O}(n)$ and let $\mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}(n)$. It is well known that

$$\text{Irr}'(S_n) = \{[\lambda], \lambda \in \mathcal{D}(n)\}$$

forms a 2-basic set for S_n .

Then the 2-regular character table of the symmetric group S_n is defined to be

$$X_n = ([\lambda](\sigma_\alpha))_{\substack{\lambda \in \mathcal{D}(n) \\ \alpha \in \mathcal{O}(n)}}$$

where the partitions are ordered in a suitable way.

As a special case of a result by Olsson [2003], we know that $|\det(X_n)| = \prod_{\mu \in \mathcal{O}(n)} a_\mu$, and thus in particular,

$$2 \nmid \det(X_n) .$$

The main result in [Bessenrodt 2007] provides a block version of this property, by distributing not only the characters but also the 2-regular conjugacy classes into blocks in such a way that the corresponding block parts of the character table have odd determinants. This block distribution of conjugacy classes provided a block splitting in the sense of Brauer as described in the previous section.

We recall this 2-block splitting for the symmetric groups below. The reader is referred to [Bessenrodt 2007] for the full results; these involve more detailed information on spin characters which we omit here since it would require recalling a lot of notation on double cover groups and their characters.

For the combinatorics of the p -modular representation theory for S_n , and in particular the p -block distribution of its characters, we refer to [James and Kerber 1981].

Let B be a 2-block of S_n , with associated 2-core $\kappa(B)$; this is then a staircase partition $\rho_k = (k, k - 1, \dots, 2, 1)$, $k \in \mathbb{N}_0$. For any partition λ , we denote by $\lambda_{(2)}$ the 2-core of λ . Then we define

$$\mathfrak{D}_B = \{\lambda \in \mathfrak{D}(n) \mid \lambda_{(2)} = \kappa(B)\}.$$

This is the set of labels of irreducible characters in B in the basic set mentioned above, and we set

$$\text{Irr}'(B) = \{[\lambda] \mid \lambda \in \mathfrak{D}_B\}.$$

To define the splitting of the classes we need a few more definitions.

For a partition $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathfrak{D}(n)$ we set

$$\text{dbl}(\lambda) = \left(\left[\frac{\lambda_1 + 1}{2} \right], \left[\frac{\lambda_1}{2} \right], \left[\frac{\lambda_2 + 1}{2} \right], \left[\frac{\lambda_2}{2} \right], \dots, \left[\frac{\lambda_m + 1}{2} \right], \left[\frac{\lambda_m}{2} \right] \right),$$

the *doubling* of λ . For example, the staircase $\rho_k = (k, k - 1, \dots, 2, 1)$ is the doubling of the partition $\tau_k = (2k - 1, 2k - 5, \dots)$.

The most natural way of defining the blocks of classes is based on the Glaisher map which we consider next.

J. W. L. Glaisher [1883] defined a bijection between partitions with parts not divisible by a given number k on the one hand and partitions where no part is repeated k times on the other hand; in particular for $k = 2$ this gives a bijection between $\mathbb{O}(n)$ and $\mathfrak{D}(n)$. Here, Glaisher's map G is defined as follows. Let $\alpha = (1^{m_1}, 3^{m_3}, \dots) \in \mathbb{O}(n)$. Write each multiplicity m_i as a sum of distinct powers of 2, that is, in its 2-adic decomposition: $m_i = \sum_j 2^{a_{ij}}$. Then $G(\alpha) \in \mathfrak{D}(n)$ consists of the parts $(2^{a_{ij}} i)_{i,j}$, sorted in order to give a partition.

Let B be a 2-block of S_n , contained in a 2-block \tilde{B} of the double cover group \tilde{S}_n (see [Bessenrodt and Olsson 1997] for background and notation). Then we define the set

$$\mathbb{O}_B = \mathbb{O}_{\tilde{B}} = \{\alpha \in \mathbb{O}(n) \mid \text{dbl}(G(\alpha))_{(2)} = \kappa(B)\}.$$

Note that in [Bessenrodt 2007] we have used the language of $\bar{4}$ -combinatorics for the description of the 2-block distribution of spin characters, and the set \mathbb{O}_B is then the set of partitions of type \mathbb{O} such that the Glaisher image has as $\bar{4}$ -core the one associated to \tilde{B} .

The sets \mathbb{O}_B for the 2-blocks B of S_n give then a set partition $\mathbb{O}(n) = \bigcup_B \mathbb{O}_B$.

The set \mathbb{O}_B is the set of labels of the 2-regular classes we want to associate to B , that is, we set

$$\text{Cl}'(B) = \{\sigma_\alpha^{S_n} \mid \alpha \in \mathbb{O}_B\}.$$

Defining

$$\mathfrak{D}_{\tilde{B}} = \{\lambda \in \mathfrak{D}(n) \mid \text{dbl}(\lambda)_{(2)} = \kappa(B)\},$$

the Glaisher map then induces bijections $\mathbb{O}_B \rightarrow \mathfrak{D}_{\tilde{B}}$, for all 2-blocks B of S_n .

By [Bessenrodt and Olsson 1997], $|\mathcal{D}_B| = |\mathcal{D}_{\tilde{B}}| = p(w(B))$. Thus the following block parts of the character table are all square matrices:

$$X_B = ([\mu](\sigma_\alpha))_{\substack{\mu \in \mathcal{D}_B \\ \alpha \in \mathcal{O}_B}}.$$

Denoting the irreducible Brauer characters of S_n by φ^μ , $\mu \in \mathcal{D}(n)$, we also consider the corresponding block part of the Brauer character table:

$$\Phi_B = (\varphi^\mu(\sigma_\alpha))_{\substack{\mu \in \mathcal{D}_B \\ \alpha \in \mathcal{O}_B}}.$$

Theorem 3.1. [Bessenrodt 2007] *Let $\text{Irr}'(B)$ and $\text{Cl}'(B)$ for the 2-blocks B of S_n be defined as above. Then the determinants*

$$\det \Phi_B = \det X_B, \quad B \in \text{Bl}(S_n)$$

of the associated block parts of the character table and the Brauer character table are all odd.

Thus the sets $\text{Cl}'(B)$ define a 2-block splitting of the 2-regular classes for S_n .

Remarks 3.2. (i) More precisely, the determinant $\det X_B$ is (up to sign) the odd part of the determinant of the corresponding block part of the reduced spin character table for the 2-block \tilde{B} of the double cover group \tilde{S}_n containing B , that is,

$$Z_s(\tilde{B}) = ((\lambda)(\tilde{\sigma}_\alpha))_{\substack{\lambda \in \mathcal{D}_{\tilde{B}} \\ \alpha \in \mathcal{O}_B}}.$$

See [Bessenrodt 2007] for the notation used here and details on this result.

(ii) By the 2-block splitting for S_n given above and Brauer's Theorem, the elementary divisors of the Cartan matrix C_B of a 2-block B of S_n are exactly the 2-powers

$$2^{k_\alpha} = |C_{S_n}(\sigma_\alpha)|_2, \quad \alpha \in \mathcal{O}_B.$$

Here, the 2-defect of the class of type α in S_n may easily be computed as follows [Bessenrodt 2007]:

$$k_\alpha = l(\alpha) - l(G(\alpha)).$$

This is a restatement of a formula from [Uno and Yamada 2006] which is based on [Bessenrodt and Olsson 1997]; a corrected version of an earlier formula from [Olsson 1986] already appeared in [Bessenrodt and Olsson 1997]. One should note, though, that this formula was used in the confirmation of the block splitting for S_n in [Bessenrodt 2007], so this does not give an independent proof for the elementary divisors of the Cartan matrix.

4. A 2-block splitting for alternating groups

We also have to introduce some notation for the alternating group A_n .

We let $P^+(n) = \{\lambda \in P(n) \mid (-1)^{n-l(\lambda)} = 1\}$ denote the set of even partitions in $P(n)$; these are the cycle types of elements in A_n .

The conjugacy classes in A_n are then of two types. The classes labeled by partitions $\mu \in P^+(n) \setminus (\mathbb{O} \cap \mathbb{D})(n)$ are the nonsplit classes, that is, those conjugacy classes of S_n which are also A_n -classes; we note that the corresponding A_n -centralizer is then of order $z'_\mu = z_\mu/2$. For the partitions $\mu \in (\mathbb{O} \cap \mathbb{D})(n)$, the S_n -class of σ_μ splits into two conjugacy classes in A_n , for which we denote representatives by σ_μ^+ and σ_μ^- ; their centralizers are of order $z'_\mu = z_\mu$.

A set of representatives of the 2-regular classes of A_n is thus given by:

$$\mathcal{R}(n) = \{\sigma_\alpha \mid \alpha \in (\mathbb{O} \setminus \mathbb{O} \cap \mathbb{D})(n)\} \cup \{\sigma_\alpha^\pm \mid \alpha \in (\mathbb{O} \cap \mathbb{D})(n)\}.$$

Furthermore, we briefly have to recall some information on the irreducible A_n -characters [James and Kerber 1981, Section 2.5].

For a partition λ of n , let λ' denote the conjugate partition. Let

$$\mathcal{S}(n) = \{\lambda \in P(n) \mid \lambda = \lambda'\}$$

be the set of symmetric partitions of n .

If λ is nonsymmetric, then $[\lambda] \downarrow_{A_n} = [\lambda'] \downarrow_{A_n}$ is irreducible. Let $\{\lambda\} = \{\lambda'\}$ denote this irreducible character of A_n .

If $\lambda = \lambda'$, then $[\lambda] \downarrow_{A_n} = \{\lambda\}_+ + \{\lambda\}_-$ is a sum of two distinct irreducible A_n -characters (which are conjugate in S_n).

This gives all the irreducible complex characters of A_n , that is,

$$\text{Irr}(A_n) = \{\{\lambda\}_\pm \mid \lambda \in \mathcal{S}(n)\} \cup \{\{\lambda\} \mid \lambda \in (P \setminus \mathcal{S})(n)\}.$$

The characters $\{\lambda\}_\pm$, for $\lambda \in \mathcal{S}(n)$, are only distinguished by their values on the corresponding ‘‘critical’’ classes of cycle type $h(\lambda) = (h_1^\lambda, \dots, h_d^\lambda)$, where $h_1^\lambda, \dots, h_d^\lambda$ are the principal hook lengths in λ and $d = d(\lambda)$ is the diagonal length of λ . Note that $h(\lambda) \in (\mathbb{O} \cap \mathbb{D})(n)$, so the corresponding S_n -class splits.

Then we have $[\lambda](\sigma_{h(\lambda)}) = (-1)^{(n-d)/2} =: \varepsilon_\lambda$. We set $H_\lambda = \prod_{i=1}^d h_i^\lambda$. Then

$$\begin{aligned} \{\lambda\}_+(\sigma_{h(\lambda)}^\pm) &= \frac{1}{2}(\varepsilon_\lambda \pm \sqrt{\varepsilon_\lambda H_\lambda}), \\ \{\lambda\}_-(\sigma_{h(\lambda)}^\pm) &= \frac{1}{2}(\varepsilon_\lambda \mp \sqrt{\varepsilon_\lambda H_\lambda}). \end{aligned}$$

For any other irreducible A_n -character the values on these two classes coincide.

We have the following easy and well known property:

Lemma 4.1. *The map $h : \mathcal{S}(n) \rightarrow (\mathbb{O} \cap \mathbb{D})(n)$ with $h(\lambda) = (h_1^\lambda, \dots, h_{d(\lambda)}^\lambda)$, for $\lambda \in \mathcal{S}(n)$, is a bijection.*

Let $B \in \text{Bl}(S_n)$ with 2-core $\rho_k = (k, k - 1, \dots, 2, 1) = \text{dbl}(\tau_k)$, where, as before, $\tau_k = (2k - 1, 2k - 5, \dots)$; let $\mathcal{D}_{\tilde{B}}$ and $\mathcal{O}_B = \mathcal{O}_{\tilde{B}}$ as before in Section 3 and $(\mathcal{O} \cap \mathcal{D})_{\tilde{B}} = \mathcal{O}_{\tilde{B}} \cap \mathcal{D}_{\tilde{B}}$.

We set $\mathcal{S}_B = \{\lambda \in \mathcal{S}(n) \mid \lambda_{(2)} = \rho_k\}$.

In our context, we need the following refinement of Lemma 4.1:

Proposition 4.2. *The map h induces bijections $\mathcal{S}_B \rightarrow (\mathcal{O} \cap \mathcal{D})_{\tilde{B}}$.*

Proof. We have to show that for any $\lambda \in \mathcal{S}(n)$, we have $\lambda_{(2)} = \text{dbl}(h(\lambda))_{(2)}$. In the notation of $\bar{4}$ -combinatorics an easy reduction argument shows that $\lambda_{(2)} = \text{dbl}(h(\lambda))_{(\bar{4})}$; simultaneously removing 2-hooks from the diagram of λ that are symmetrically positioned in λ corresponds to removing 4-bars from $h(\lambda)$, namely, subtracting 4 from a part in $h(\lambda)$, and removing an inner 2×2 array corresponds to removing a pair 3, 1 (which is also a 4-bar). This ends at a staircase partition $\rho_k = \lambda_{(2)}$, and in parallel at the corresponding $\tau_k = h(\lambda)_{(\bar{4})}$.

By [Bessenrodt and Olsson 1997, Lemma 3.6], we obtain the equation on the 2-cores. □

Remark 4.3. It is not difficult to see [Olsson 1993, 12.5] that

$$|\mathcal{S}_B| = \begin{cases} 0 & \text{if } w(B) \text{ is odd,} \\ p(w/2) & \text{if } w(B) \text{ is even.} \end{cases}$$

For a character χ of G , let χ^o denote the restriction of χ to the 2-regular elements of G . The following useful proposition provides a good 2-basic set for the alternating groups. Note here that the set $(\mathcal{D} \cap \mathcal{S})(n)$ labeling the third subset of characters is nonempty only if n is a triangular number; in that case, if $n = \binom{k+1}{2}$, $(\mathcal{D} \cap \mathcal{S})(n) = \{\rho_k\}$, and thus both characters $\{\rho_k\}_{\pm}$ of defect 0 then belong to the basic set.

Proposition 4.4. *Set*

$$\mathcal{C}(n) = \{\{\lambda\} \mid \lambda \in (\mathcal{D} \setminus \mathcal{S})(n)\} \cup \{\{\lambda\}_+ \mid \lambda \in \mathcal{S}(n)\} \cup \{\{\lambda\}_- \mid \lambda \in (\mathcal{D} \cap \mathcal{S})(n)\}.$$

Then $\{\chi^o \mid \chi \in \mathcal{C}(n)\}$ is a basic set for A_n (at the prime $p = 2$).

Proof. If $\chi = \{\mu\} = [\mu] \downarrow_{A_n}$ for $\mu \in (P \setminus \mathcal{S})(n)$, then, since the 2-regular partitions label a basic set for S_n ,

$$\chi^o = [\mu]^o \downarrow_{A_n} = \sum_{\lambda \in \mathcal{D}(n)} c_{\mu \lambda} [\lambda]^o \downarrow_{A_n} = \sum_{\lambda \in (\mathcal{D} \setminus \mathcal{S})(n)} c_{\mu \lambda} \{\lambda\}^o + \sum_{\lambda \in (\mathcal{D} \cap \mathcal{S})(n)} c_{\mu \lambda} (\{\lambda\}_+^o + \{\lambda\}_-^o),$$

with integer coefficients $c_{\mu \lambda}$; as explained just before the proposition, the second sum above has at most one partition $\lambda \in (\mathcal{D} \cap \mathcal{S})(n)$ giving a contribution, namely when n is a triangular number.

For $\chi = \{\mu\}_+, \mu \in \mathcal{S}(n)$, or $\chi = \{\mu\}_-, \mu \in (\mathcal{D} \cap \mathcal{S})(n)$, there is nothing to prove. If $\chi = \{\mu\}_-, \mu \in \mathcal{S} \setminus (\mathcal{D} \cap \mathcal{S})(n)$, then, again since the 2-regular partitions label a basic set for S_n ,

$$\begin{aligned} \chi^o &= \{\mu\}_-^o = [\mu]^o \downarrow_{A_n} - \{\mu\}_+^o = \left(\sum_{\lambda \in \mathcal{D}(n)} c_{\mu\lambda} [\lambda]^o \downarrow_{A_n} \right) - \{\mu\}_+^o \\ &= \left(\sum_{\lambda \in (\mathcal{D} \setminus \mathcal{S})(n)} c_{\mu\lambda} \{\lambda\}^o + \sum_{\lambda \in (\mathcal{D} \cap \mathcal{S})(n)} c_{\mu\lambda} (\{\lambda\}_+^o + \{\lambda\}_-^o) \right) - \{\mu\}_+^o, \end{aligned}$$

an integral linear combination as desired.

Since $|\mathcal{C}(n)| = |\mathcal{D}(n)| + |\mathcal{S}(n)| = |\mathcal{O}(n)| + |(\mathcal{O} \cap \mathcal{D})(n)| = |\text{Cl}'(A_n)|$, the set $\{\chi^o \mid \chi \in \mathcal{C}(n)\}$ is a \mathbb{Z} -basis of $\langle \chi^o \mid \chi \in \text{Irr}(A_n) \rangle_{\mathbb{Z}}$, as claimed. \square

For any set $Q \subseteq P(n)$, we set

$$a_Q = \prod_{\lambda \in Q} a_\lambda.$$

Using the basic set above, we define the 2-regular character table for the alternating group A_n to be

$$X_n^A = (\chi(\sigma))_{\substack{\chi \in \mathcal{C}(n) \\ \sigma \in \mathcal{R}(n)}}.$$

Using the properties of the irreducible characters of A_n stated above (see also [Bessenrodt and Olsson 2004]), as well as the formula for the 2-regular character table for S_n from [Olsson 2003] we deduce:

Corollary 4.5. $|\det X_n^A| = |\det X_n| \cdot \sqrt{a_{(\mathcal{O} \cap \mathcal{D})(n)}} = a_{\mathcal{O}(n)} \cdot \sqrt{a_{(\mathcal{O} \cap \mathcal{D})(n)}}.$

In the main theorem stated below, we will give a block refinement of the first equation above.

Remark 4.6. The 2-blocks of S_n and A_n are closely related [Olsson 1993]. Let $B \in \text{Bl}(S_n)$. If $w(B) = 0$, then B covers two 2-blocks of A_n (of defect 0), say $B_\varepsilon^A, \varepsilon \in \{\pm\}$. This only occurs when n is a triangular number, say $n = \binom{k+1}{2}$, and $\kappa(B) = \rho_k = (k, k-1, \dots, 2, 1)$; then $\text{Irr}(B_\varepsilon^A) = \{\{\rho_k\}_\varepsilon\}, \varepsilon \in \{\pm\}$.

Note that there is then a suitable choice of signs $\bar{\varepsilon}$ for $\varepsilon \in \{\pm\}$ such that

$$\{\rho_k\}_\varepsilon(\sigma_{h(\rho_k), \bar{\varepsilon}}) \not\equiv 0 \pmod{\mathfrak{p}}.$$

If $w(B) > 0$, then B covers only one 2-block B^A of A_n , and this block B^A is only covered by B . We then have

$$\text{Irr}(B^A) = \{\{\lambda\}_{(\pm)} \mid \lambda \in P(n), \lambda_{(2)} = \kappa(B)\}.$$

Here, $\{\lambda\}_{(\pm)}$ means that we take the character $\{\lambda\}$ if λ is nonsymmetric, and both characters $\{\lambda\}_\pm$ if λ is symmetric.

Theorem 4.7. *Let $B \in \text{Bl}(S_n)$ with 2-core $\kappa(B) = \rho_k = \text{dbl}(\tau_k)$.*

If $w(B) = 0$, then $\mathbb{O}_B = \{\tau_k\}$ and $\mathcal{D}_B = \{\rho_k\}$, and we set

$$\text{Cl}'^A(B_\varepsilon^A) = \{\sigma_{\tau_k, \varepsilon}^{A_n}\}, \quad \text{Irr}'(B_\varepsilon^A) = \{\{\rho_k\}_\varepsilon\}, \quad \text{for } \varepsilon \in \{\pm\}.$$

If $w(B) > 0$, we set

$$\text{Cl}'^A(B^A) = \{\sigma_{\alpha(\pm)}^{A_n} \mid \alpha \in \mathbb{O}_B\}, \quad \text{Irr}'(B^A) = \{\{\lambda\} \mid \lambda \in \mathcal{D}_B\} \cup \{\{\lambda\}_+ \mid \lambda \in \mathcal{S}_B\}.$$

Let $X_{B^A} = (\chi(x_K))_{\chi \in \text{Irr}'(B^A), K \in \text{Cl}'^A(B^A)}$. Then

$$|\det X_{B^A}| = |\det X_B| \cdot \sqrt{a_{(\mathbb{O} \cap \mathcal{D})_{\tilde{B}}}}.$$

In particular, $|\det X_{B^A}| \not\equiv 0 \pmod{p}$. Hence, the sets above, taken for all $B \in \text{Bl}(S_n)$, define a 2-block splitting for A_n .

Proof. Using the notation above, let B^A be a 2-block of A_n . As seen above, the sets $\text{Cl}'^A(B^A)$ and $\text{Irr}'(B^A)$ are of the same cardinality. We have to show that all the block tables have a nonzero determinant modulo p .

For the case where $w(B) = 0$, we have already seen before that we can make a suitable choice (namely the one used in the statement of the Theorem) such that this holds for the two blocks of A_n covered by B .

Thus we may now assume that $w(B) > 0$; then we do not have an irreducible character labeled by a partition of type $\mathcal{D} \cap \mathcal{S}$ in B , that is, $\mathcal{D}_B \cap \mathcal{S}_B = \emptyset$. We consider the part of the character table of A_n corresponding to B^A , sorted such that among the character labels we first list the nonsymmetric ones and then the symmetric ones. The classes are ordered such that we first have the \mathbb{O}_B classes, and among these classes the $(\mathbb{O} \cap \mathcal{D})_{\tilde{B}}$ classes at the end, and here first the $(\mathbb{O} \cap \mathcal{D})_{\tilde{B}}^+$ classes (where the “+” indicates that we take the representatives σ_α^+), followed by the corresponding $(\mathbb{O} \cap \mathcal{D})_{\tilde{B}}^-$ classes. The classes of type $(\mathbb{O} \cap \mathcal{D})_{\tilde{B}}$ are taken in some ordering, and the \mathcal{S}_B characters are then taken in the corresponding order, that is, with the label $\mu \in \mathcal{S}_B$ corresponding to $h(\mu) \in (\mathbb{O} \cap \mathcal{D})_{\tilde{B}}$.

Recall that for any $\lambda \in (P \setminus \mathcal{S})(n)$ and $\alpha \in (\mathbb{O} \cap \mathcal{D})(n)$, we have $\{\lambda\}(\sigma_\alpha^+) = \{\lambda\}(\sigma_\alpha^-) = [\lambda](\sigma_\alpha)$. Now take $\mu \in \mathcal{S}(n)$; then for $\alpha = h(\mu)$ we have

$$\{\mu\}_+(\sigma_{h(\mu)}^\pm) = \frac{1}{2}(\varepsilon_\mu \pm \sqrt{\varepsilon_\mu H_\mu}) =: y_\mu^\pm,$$

while for $\beta \in (\mathbb{O} \cap \mathcal{D})(n)$, $\beta \neq h(\mu)$, we have $\{\mu\}_+(\sigma_\beta^+) = \{\mu\}_+(\sigma_\beta^-)$. Set $c_\mu = \sqrt{\varepsilon_\mu H_\mu} = y_\mu^+ - y_\mu^-$. Now for any $\alpha \in (\mathbb{O} \cap \mathcal{D})_{\tilde{B}}$, subtract the column of the block character table X_{B^A} to the class of σ_α^+ from the one to the class of σ_α^- . By the above, then the final columns to the $(\mathbb{O} \cap \mathcal{D})_{\tilde{B}}^-$ classes are transformed into an upper zero part, corresponding to the characters labeled by nonsymmetric partitions, and

below this a diagonal matrix with diagonal entries $-c_\mu$, $\mu \in \mathcal{S}_B$. The table

$$X_B^S = (\{\lambda\}(\sigma_\alpha^{(+)})_{\substack{\lambda \in \mathcal{D}_B \\ \alpha \in \mathcal{C}_B^{\bar{B}}}})$$

is the upper left hand block part of the table X_{B^A} . By the above and Section 3, this is exactly the block part of the 2-regular character table of the symmetric group corresponding to the block B of S_n , with the block splitting constructed for the symmetric groups, that is,

$$X_B^S = X_B = ([\lambda](\sigma_\alpha))_{\substack{\lambda \in \mathcal{D}_B \\ \alpha \in \mathcal{C}_B}}$$

Thus we have $\det X_B^S \not\equiv 0 \pmod{p}$. Hence

$$|\det X_{B^A}| = |\det X_B^S| \cdot \prod_{\mu \in \mathcal{S}_B} |c_\mu| = |\det X_B| \cdot \sqrt{a_{(\mathcal{C} \cap \mathcal{D})_{\bar{B}}}} \not\equiv 0 \pmod{p},$$

and we have proved that our construction provides a 2-block splitting for A_n . \square

Remark 4.8. In contrast to the case of symmetric groups, the block splitting of the 2-regular classes for the alternating groups as given above is not the only block splitting; already A_6 provides a counterexample. Indeed, instead of associating the classes to (1^6) , $(1^3, 3)$, (3^2) to the principal 2-block of A_6 , the choices (1^6) , $(1^3, 3)$, $(1, 5)_+$ or (1^6) , (3^2) , $(1, 5)_+$ are also possible.

Recall that for $\alpha \in \mathcal{C}(n) \setminus (\mathcal{C} \cap \mathcal{D})(n)$, the corresponding conjugacy class of σ_α is nonsplit in A_n , so we then have $|C_{A_n}(\sigma_\alpha)|_2 = 2^{k_\alpha - 1}$, with $k_\alpha = l(\alpha) - l(G(\alpha))$ as before.

By Brauer’s Theorem 2.1 we can now deduce from our 2-block splitting given in Theorem 4.7 the following result on the Cartan matrices of 2-blocks of alternating groups, providing a combinatorial formula for the elementary divisors which is easy to compute.

Corollary 4.9. *Let $B \in \text{Bl}(S_n)$ of weight $w(B) > 0$, covering the block $B^A \in \text{Bl}(A_n)$. Then the elementary divisors of the Cartan matrix C_{B^A} are*

$$|C_{A_n}(\sigma_\alpha)|_2 = 2^{k_\alpha - 1}, \quad \text{for } \alpha \in \mathcal{C}_B \setminus (\mathcal{C} \cap \mathcal{D})_{\bar{B}}; \quad 1^{2|(\mathcal{C} \cap \mathcal{D})_{\bar{B}}|}.$$

In particular,

$$\det C_B = 2^{2\ell(B) - \ell(B^A)} \det C_{B^A}.$$

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