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Jordan S. Ellenberg and Michael Larsen

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# Positive motivic measures are counting measures

Jordan S. Ellenberg and Michael Larsen

A *motivic measure* is a ring homomorphism from the Grothendieck group of a field $K$ (with multiplication coming from the fiber product over Spec $K$) to some field. We show that if a real-valued motivic measure $\mu$ satisfies $\mu([V]) \geq 0$ for all $K$-varieties $V$, then $\mu$ is a counting measure; that is, there exists a finite field $L$ containing $K$ such that $\mu([V]) = |V(L)|$ for all $K$-varieties $V$.

Let $K$ be a field. By a *$K$-variety*, we mean a geometrically reduced, separated scheme of finite type over $K$. Let $K_0(\mathrm{Var}_K)$ denote the Grothendieck group of $K$, that is, the free abelian group generated by isomorphism classes $[V]$ of $K$-varieties, with the scissors relations $[V] = [W] - [V \setminus W]$ whenever $W$ is a closed $K$-subvariety of $V$. There is a unique product on $K_0(\mathrm{Var}_K)$ characterized by the relation

$$[V] \cdot [W] = [V \times W],$$

where $\times$ denotes the fiber product over Spec $K$. This product gives $K_0(\mathrm{Var}_K)$ a commutative ring structure with identity [Spec $K$]. For every extension $L$ of $K$, extension of scalars gives a natural ring homomorphism $K_0(\mathrm{Var}_K) \to K_0(\mathrm{Var}_L)$. The map $K \mapsto K_0(\mathrm{Var}_K)$ can be regarded as a functor from fields to commutative rings. Throughout the paper, we follow the usual convention of writing $\mathbb{L}$ for $[\mathbb{A}^1_K]$.

Following the terminology of [Larsen and Lunts 2003], we call a ring homomorphism from $K_0(\mathrm{Var}_K)$ to a field $F$ a *motivic measure*. Note that the original meaning of this term [Hales 2005; Looijenga 2002] is different (though related). If $K$ is a finite field, the map $[V] \mapsto |V(K)|$ extends to a homomorphism $\mu_K : K_0(\mathrm{Var}_K) \to \mathbb{Z}$, and therefore to an $F$-valued measure for any field $F$. More generally, if $L$ is an extension of $K$ which is also a finite field, the composition of $\mu_L$ with the natural map $K_0(\mathrm{Var}_K) \to K_0(\mathrm{Var}_L)$ gives for each $F$ a motivic measure. We will call all such measures *counting measures*.

In this paper, we consider *positive* motivic measures, by which we mean $\mathbb{R}$-valued measures $\mu$ such that $\mu([V]) \geq 0$ for all $K$-varieties $V$. We now state our main result.

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**Theorem 1.** *Every positive motivic measure is a counting measure. In other words, if $K$ is any field and $\mu\colon K_0(\mathrm{Var}_K) \to \mathbb{R}$ is positive, there exists a finite field $L$ containing $K$ such that $\mu([V]) = |V(L)|$ for all $K$-varieties $V$.*

Of course, for other choices of $F$ there may still be motivic measures such that $\mu([V])$ lies in some interesting semiring of $F$ for all $K$-varieties $V$. For example, if $F$ is $\mathbb{C}(u, v)$ and $K = \mathbb{C}$, the measure sending $V$ to its Hodge–Deligne polynomial takes values in the semiring of polynomials in $u, v$ whose term of highest total degree is a positive multiple of a power of $uv$.

We begin with a direct proof of the following obvious corollary of Theorem 1.

**Proposition 2.** *If $K$ is infinite, there are no positive motivic measures on $K_0(\mathrm{Var}_K)$.*

*Proof.* Let $\mu$ be such a measure. For any finite subset $S$ of $K$, which we regard as a zero-dimensional subvariety of $\mathbb{A}^1$,

$$0 \le \mu(\mathbb{A}^1 \setminus S) = \mu(\mathbb{L}) - |S|.$$

Thus, $\mu(\mathbb{L}) \ge |S|$ for all subsets $S$ of $K$, which proves the proposition. □

For the remainder of the paper we may and do assume that $K$ is finite, of cardinality $q$. We write $\mathbb{F}_{q^n}$ for the degree $n$ extension of $K$.

**Proposition 3.** *Let $\Omega^n$ denote the variety obtained from $\mathbb{A}^n$ by removing all proper affine-linear subspaces defined over $\mathbb{F}_q$. Then*

$$[\Omega^n] = (\mathbb{L} - q)(\mathbb{L} - q^2)\cdots(\mathbb{L} - q^n).$$

*Proof.* For any $\mathbb{F}_q$-rational affine-linear subspace $A$ of $\mathbb{A}^n$, let $A^\circ$ denote the open subvariety of $A$ which is the complement of all proper $\mathbb{F}_q$-rational affine-linear subspaces of $A$. Then $[A^\circ] = [\Omega^{\dim A}]$, and one can write recursively

$$[\Omega^n] = \mathbb{L}^n - \sum_{i=1}^{n-1} a_{n,i}[\Omega^i],$$

where $a_{n,i}$ is the number of $\mathbb{F}_q$-rational $i$-dimensional affine linear subspaces of $\mathbb{A}^n$. Thus, $[\Omega^n]$ can be expressed as $P_n(\mathbb{L})$, where $P_n \in \mathbb{Z}[x]$ is monic and of degree $n$. It suffices to prove that $q^d$ is a root of $P_n(x)$ for all integers $d \in \{1, 2, \ldots, n\}$.

For any $d$ in this range $\Omega^n(\mathbb{F}_{q^d})$ is empty. Indeed, if $x \in \mathbb{A}^n(\mathbb{F}_{q^d})$, then the $n$ coordinates of $x$ together with 1 cannot be linearly independent over $\mathbb{F}_q$, which implies that $x$ lies in a proper $\mathbb{F}_q$-rational affine-linear subspace of $\mathbb{A}^n$. Thus,

$$0 = \mu_{\mathbb{F}_{q^d}}(\Omega^n) = P_n(q^d).$$ □

**Corollary 4.** *If $\mu$ is a positive measure on $K_0(\mathrm{Var}_{\mathbb{F}_q})$, there exists a positive integer $n$ such that $\mu(\mathbb{L}) = q^n$.*

*Proof.* If $q^{n-1} < \mu(\mathbb{L}) < q^n$ for some integer $n$, then $\mu(\Omega^n) < 0$, contrary to positivity. □

Our goal is then to prove that $\mu(\mathbb{L}) = q^n$ implies $\mu = \mu_{\mathbb{F}_{q^n}}$. We prove first that these measures coincide for varieties of the form $\operatorname{Spec}\mathbb{F}_{q^d}$ and deduce that they coincide for all affine varieties. As $K_0(\mathrm{Var}_{\mathbb{F}_q})$ is generated by the classes of affine varieties, this implies Theorem 1.

**Lemma 5.** *Let $\mu$ be a real-valued motivic measure of $K_0(\mathrm{Var}_{\mathbb{F}_q})$ and $m$ a positive integer. Then*

$$\mu(\operatorname{Spec}\mathbb{F}_{q^m}) \in \{0, m\}.$$

*If* $\operatorname{Spec}\mathbb{F}_{q^m}$ *has measure $m$, then* $\operatorname{Spec}\mathbb{F}_{q^d}$ *has measure $d$ whenever $d$ divides $m$.*

*Proof.* As

$$\mathbb{F}_{q^m} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m} = \mathbb{F}_{q^m}^m,$$

the class $x$ of $\operatorname{Spec}\mathbb{F}_{q^m}$ satisfies $x^2 = mx$. If $d$ divides $m$,

$$\mathbb{F}_{q^d} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m} = \mathbb{F}_{q^m}^d,$$

so $\mu(\operatorname{Spec}\mathbb{F}_{q^m}) = m$ implies $\mu(\operatorname{Spec}\mathbb{F}_{q^d}) = d$. □

Of course,

$$\mu_{\mathbb{F}_{q^n}}(\operatorname{Spec}\mathbb{F}_{q^m}) = \begin{cases} m & \text{if } m\,|\,n, \\ 0 & \text{otherwise.} \end{cases}$$

We will prove the same thing for the values of $\mu(\operatorname{Spec}\mathbb{F}_{q^m})$. We begin with:

**Proposition 6.** *If $\mu(\mathbb{L}) = q^n$ and $\mu(\operatorname{Spec}(\mathbb{F}_{q^k})) = k$ for some $k \geq n$, then*

$$\mu(\operatorname{Spec}\mathbb{F}_{q^m}) = \begin{cases} m & \textit{if } m\,|\,n, \\ 0 & \textit{otherwise.} \end{cases} \tag{1}$$

For any integer $k$, we denote by $X_k$ the complement in $\mathbb{A}^1$ of the set of all points with residue field contained in $\mathbb{F}_{q^k}$.

*Proof.* By Lemma 5, $\mu(\operatorname{Spec}\mathbb{F}_{q^d}) = d$ when $d$ divides $k$. Choose an $m$ not dividing $k$, and let $Y_{k,m}$ denote the complement in $X_k$ of the set of points with residue field $\mathbb{F}_{q^m}$. Then

$$\mu([Y_{k,m}]) = \mu(\mathbb{L}) - \sum_{d|k} c_d d - c_m \mu(\operatorname{Spec}\mathbb{F}_{q^m}),$$

where $c_i$ is the number of points in $\mathbb{A}^1$ with residue field $\mathbb{F}_{q^i}$. From the positivity of $\mu([Y_{k,m}])$ and the fact that

$$0 = \mu_{\mathbb{F}_{q^k}}([Y_{k,m}]) = q^k - \sum_{d|k} c_d d,$$

we see that $\mu(\mathbb{L}) - q^k = q^n - q^k$ must be nonnegative, which is to say $k = n$, and that $\mu(\operatorname{Spec}\mathbb{F}_{q^m}) = 0$. □

**Proposition 7.** *If* $\mu(\mathbb{L}) = q^n$, *then* $\mu(\operatorname{Spec} \mathbb{F}_{q^n}) = n$.

*Proof.* The assertion is clear for $n = 1$, so we assume $n > 1$. Let $c_i$ denote the number of points in $\mathbb{A}^1$ with residue field $\mathbb{F}_{q^i}$. Thus $ic_i \le q^i - 1$ for all $i > 1$. If $\mu(\operatorname{Spec} \mathbb{F}_{q^n}) = 0$, then $\mu(\operatorname{Spec}(\mathbb{F}_{q^i})) = 0$ for all $i \ge n$, so for all $k > 0$ we have

$$\mu([X_k]) \ge q^n - q - \sum_{i=2}^{n-1} (q^i - 1) \ge 2.$$

Now we consider all curves in $\mathbb{A}^2$ of the form $y = P(x)$ where $P(x) \in \mathbb{F}_q[x]$ has degree $\le 2n$. The total number of such curves is greater than $q^{2n}$, and for any intersection point $(\alpha, \beta)$ of any two distinct curves of this family, $\alpha$ satisfies a polynomial equation of degree $\le 2n$ over $\mathbb{F}_q$. Therefore, the open curves

$$C_P := \{(x, P(x)) \mid x \notin \mathbb{F}_{q^{(2n)!}}\},$$

indexed by polynomials $P$ of degree $\le 2n$, each isomorphic to $X_{(2n)!}$, are mutually disjoint. If $C$ denotes the closure of the union of the $C_P$ in $\mathbb{A}^2$, it follows that

$$\mu([C]) > q^{2n} \mu([X_{(2n)!}]) > q^{2n},$$

so $\mu([\mathbb{A}^2 \setminus C]) < 0$, which is absurd. □

Together, the two preceding propositions imply (1).

We can now prove Theorem 1. We assume $\mu(\mathbb{L}) = q^n$. It suffices to check that $\mu([V]) = |V(\mathbb{F}_{q^n})|$ for all affine $\mathbb{F}_q$-varieties $V$.

Each closed point of $V$ with residue field $\mathbb{F}_{q^d}$ corresponds to a $d$-element Galois orbit in $V(\mathbb{F}_{q^d})$. If $d$ divides $n$, it gives a $d$-element subset of $V(\mathbb{F}_{q^n})$ and the subsets arising from different closed points are mutually disjoint. Since $V(\mathbb{F}_{q^n})$ is the union of all these subsets, and $\mu(\operatorname{Spec} \mathbb{F}_{q^d}) = d$, we have

$$\mu([V]) \ge |V(\mathbb{F}_{q^n})| \tag{2}$$

for each $\mathbb{F}_q$-variety $V$. However, embedding $V$ as a closed subvariety of $\mathbb{A}^m$ for some $m$, the complement $W = \mathbb{A}^m \setminus V$ is again a variety, so

$$\mu([W]) \ge |W(\mathbb{F}_{q^n})|. \tag{3}$$

Since

$$\begin{aligned} q^{mn} = \mu([\mathbb{A}^m]) &= \mu([V]) + \mu([W]) \\ &\ge |V(\mathbb{F}_{q^n})| + |W(\mathbb{F}_{q^n})| \\ &= |\mathbb{A}^m(\mathbb{F}_{q^n})| = q^{mn}, \end{aligned}$$

we must have equality in (2) and (3).

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ellenber@math.wisc.edu *Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, United States*
http://math.wisc.edu/~ellenber

mjlarsen@indiana.edu *Department of Mathematics, Indiana University, Bloomington, IN 47405, United States*
http://mlarsen.math.indiana.edu/~larsen/