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of Jacobians**

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Let A be the Néron model of an abelian variety A_K over the fraction field K of a discrete valuation ring R . By work of Mazur and Messing, there is a functorial way to prolong the universal extension of A_K by a vector group to a smooth and separated group scheme over R , called the *canonical extension of A* . Here we study the canonical extension when $A_K = J_K$ is the Jacobian of a smooth, proper and geometrically connected curve X_K over K . Assuming that X_K admits a proper flat regular model X over R that has generically smooth closed fiber, our main result identifies the identity component of the canonical extension with a certain functor $\text{Pic}_{X/R}^{v,0}$ classifying line bundles on X that have partial degree zero on all components of geometric fibers and are equipped with a *regular connection*. This result is a natural extension of a theorem of Raynaud, which identifies the identity component of the Néron model J of J_K with the functor $\text{Pic}_{X/R}^0$. As an application of our result, we prove a comparison isomorphism between two canonical integral structures on the de Rham cohomology of X_K .

1. Introduction

Fix a discrete valuation ring R with field of fractions K and residue field k . Let A_K be an abelian variety over K and consider the universal extension $E(\widehat{A}_K)$ of the dual abelian variety \widehat{A}_K . This commutative algebraic K -group is an extension of \widehat{A}_K by the vector group of invariant differentials on A_K

$$0 \longrightarrow \omega_{A_K} \longrightarrow E(\widehat{A}_K) \longrightarrow \widehat{A}_K \longrightarrow 0 \quad (1-1)$$

and is universal among extensions of \widehat{A}_K by a vector group: for any vector group V over K , the natural homomorphism $\text{Hom}(\omega_{A_K}, V) \rightarrow \text{Ext}(\widehat{A}_K, V)$ arising by pushout from (1-1) is an isomorphism. The theory of the universal extension was

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initiated by Rosenlicht [1958], who defined the notion and showed its existence for abelian varieties, and was subsequently taken up in [Tate 1958; Murre 1962; Grothendieck 1974; Messing 1972; 1973; Mazur and Messing 1974]. It is central to the definition of the Mazur–Tate p -adic height pairing [Mazur and Tate 1983; Coleman 1991], to Deligne’s definition of the duality on the de Rham cohomology of A_K [Deligne 1974, Section 10.2.7.3] (see also [Coleman 1991; 1998]), and to certain proofs of the comparison isomorphism between the p -adic étale and de Rham cohomologies of A_K [Coleman 1984, Note added in proof; Wintenberger 1994].

As is well known, the Néron model \hat{A} of \hat{A}_K over R provides a functorial extension of \hat{A}_K to a smooth commutative group scheme over R , and it is natural to ask if (1-1) can be functorially extended to a short exact sequence of smooth commutative R -groups as well. Such an extension is provided by the “canonical extension” $\mathcal{E}(\hat{A})$ of \hat{A} , introduced by Mazur and Messing [1974, I, Section 5]. When \hat{A}_K has good reduction, $\mathcal{E}(\hat{A})$ coincides with the universal extension of (the abelian scheme) \hat{A} by a vector group, but in general, as an example of Breen and Raynaud shows (see Remarks 2.5), Néron models need not have universal extensions, and $\mathcal{E}(\hat{A})$ seems to be the best substitute in such cases. Although they seem to be of fundamental importance, canonical extensions of Néron models have been little studied, and as far as we know, do not appear anywhere in the literature beyond their introduction in [Mazur and Messing 1974] and [Gross 1990, Section 15].

In this paper, we study the canonical extension $\mathcal{E}(\hat{A})$ when $A_K = J_K$ is the Jacobian of a smooth proper and geometrically connected curve X_K over K . In this situation, a famous theorem of Raynaud [Bosch et al. 1990, Section 9.7, Theorem 1] relates the identity component \hat{J}^0 of \hat{J} to the relative Picard functor of any proper flat and normal model X of X_K that is “sufficiently nice”.

Theorem 1.1 (Raynaud). *Let $S = \text{Spec } R$ and fix a proper flat and normal model X of X_K over S . Denote by X_1, \dots, X_n the (reduced) irreducible components of the closed fiber X_k . Suppose that the greatest common divisor of the geometric multiplicities of the X_i in X_k is equal to 1, and assume either that k is perfect or that X admits an étale quasisection. Then $\text{Pic}_{X/S}^0$ is a smooth and separated S -group scheme and J_K admits a Néron model J of finite type. Moreover, the canonical morphism*

$$\text{Pic}_{X/S}^0 \longrightarrow \hat{J}^0 \tag{1-2}$$

*arising via the Néron mapping property from the canonical principal polarization of J_K is an isomorphism if and only if X has rational singularities.*¹

Our main result enhances Raynaud’s theorem by providing a similar description of the identity component $\mathcal{E}(\hat{J})^0$ of the canonical extension $\mathcal{E}(\hat{J})$ of \hat{J} .

¹ Recall that X is said to have *rational singularities* if it admits a resolution of singularities $\rho : X' \rightarrow X$ with $R^1\rho_*\mathcal{O}_{X'} = 0$. Trivially, any regular X has rational singularities.

Theorem 1.2. *Let X be a proper flat and normal model of X_K over $S = \text{Spec } R$. Suppose that the closed fiber of X is geometrically reduced and that either X is regular or that k is perfect. Then there is a canonical homomorphism of short exact sequences of smooth group schemes over S*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \omega_J & \longrightarrow & \mathcal{E}(\hat{J})^0 & \longrightarrow & \hat{J}^0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & f_*\omega_{X/S} & \longrightarrow & \text{Pic}_{X/S}^{\natural,0} & \longrightarrow & \text{Pic}_{X/S}^0 \longrightarrow 0
 \end{array} \tag{1-3}$$

which is an isomorphism of exact sequences if and only if X has rational singularities.

Here, $\omega_{X/S}$ is the relative dualizing sheaf of X over S ; it is a coherent sheaf of \mathbb{C}_X -modules that is flat over S and coincides with the sheaf of relative differentials over the smooth locus of f in X . We write $f_*\omega_{X/S}$ for the vector group attached to this locally free \mathbb{C}_S -module, and $\text{Pic}_{X/S}^{\natural,0}$ is the fppf sheaf associated to the functor on S -schemes that assigns to each S -scheme $\varphi : T \rightarrow S$ the set of isomorphism classes of pairs (\mathcal{L}, ∇) , where \mathcal{L} is a line bundle on X_T whose restriction to all components of each geometric fiber of X_T has degree zero and $\nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \varphi^*\omega_{X/S}$ is a *regular connection* on \mathcal{L} over T (Definition 3.5). We will show in Theorem 3.9 that under the hypotheses of Raynaud’s Theorem, $\text{Pic}_{X/S}^{\natural,0}$ is indeed a smooth and separated S -scheme, and that there is a short exact sequence of smooth groups over S as in the lower row of (1-3).

We note that when $f : X \rightarrow S$ is smooth, our notion of regular connection coincides with the familiar notion of connection, and we recover from Theorem 1.2 the “well known” description of the universal extension of a Jacobian of a smooth and proper curve as the representing object of the functor classifying degree zero line bundles on the curve that are equipped with connection.² Let us also point out that the hypotheses of Theorem 1.2 include not only all regular curves over K with semistable reduction but many regular curves which are quite far³ from having semistable reduction, such as the modular curves $X(N)$ and $X_1(N)$ over $K := \mathbb{Q}_p(\zeta_N)$ for arbitrary N (see [Katz and Mazur 1985, Theorems 13.7.6 and 13.11.4], which describe proper flat and regular models of $X(N)$ and $X_1(N)$, respectively, over $R = \mathbb{Z}_p[\zeta_N]$ that have geometrically reduced closed fibers).

It is well known that the exact sequence of Lie algebras arising from (1-1) is naturally isomorphic to the 3-term Hodge filtration exact sequence of the first de Rham

²Certainly this result appears in the literature — see for example [Coleman 1990, Section 2] — but we have been unable to find any proof of it. See, however [Mazur and Messing 1974, I, Section 4], which proves a result in a similar spirit.

³They achieve semistable reduction only after a large and wildly ramified extension of K .

cohomology of A_K ([Proposition 5.1](#)). Thus, the Lie algebra of the smooth R -group $\mathcal{E}(\widehat{A})$ provides a canonical R -lattice in the K -vector space $H_{\text{dR}}^1(A_K/K)$ which is functorial in K -morphisms of A_K (due to the Néron mapping property of A and the functorial dependence of $\mathcal{E}(\widehat{A})$ on A). When A is an abelian scheme and the maximal ideal of R has divided powers, Mazur and Messing proved [[1974](#), II, Section 15] Grothendieck’s conjecture [[1974](#), V Section 5] that this R -lattice is naturally isomorphic to the Dieudonné module of the associated Barsotti–Tate group $A_K[p^\infty]$. Thus, $\text{Lie}(\mathcal{E}(A))$ provides a natural generalization of the Dieudonné module when A is not an abelian scheme. In [[Cais 2009](#)], for a proper flat and normal R -curve X , we studied a canonical integral structure $H^1(X/R)$ on $H_{\text{dR}}^1(X_K/K)$ (that is, an R -lattice that is functorial in K -morphisms of X_K) defined in terms of relative dualizing sheaves. It is natural to ask how $H^1(X/R)$ compares with the lattice $\text{Lie}(\mathcal{E}(\widehat{J}))$ under the canonical identification $H_{\text{dR}}^1(X_K/K) \simeq H_{\text{dR}}^1(J_K/K)$. We will prove in [Corollary 5.6](#) that these two lattices coincide when X verifies the hypotheses of [Theorem 1.2](#).

We briefly explain the main ideas underlying the proof of [Theorem 1.2](#). Our first task is to reinterpret $\mathcal{E}(\widehat{J})^0$ as the representing object of the functor $\mathcal{E}xtrig_S(J, \mathbf{G}_m)$ on smooth S -schemes, à la Mazur and Messing [[1974](#)]. To do this, we must first show that the functor $\mathcal{E}xt_S(J, \mathbf{G}_m)$ is represented by \widehat{J}^0 on smooth S -schemes, and by [[Bosch 1997](#), Proposition 5.1] this holds if and only if Grothendieck’s pairing on component groups is perfect. It follows from results of Bosch and Lorenzini [[2002](#), Corollary 4.7] (see also [Proposition 2.8](#)) that the hypotheses of [Theorem 1.2](#) imply the perfectness of Grothendieck’s pairing. However, we note that Grothendieck’s pairing is *not* generally perfect (see [Remark 2.9](#)).

In [Section 3](#), we construct the exact sequence of smooth S -group schemes occurring in the bottom row of (1-3). This is accomplished by [Theorem 3.9](#), whose proof employs Čech-theoretical techniques to interpret the hypercohomology of the two-term complex $d \log : \mathbb{O}_X^\times \rightarrow \omega_{X/S}$ in terms of line bundles with regular connection, and makes essential use of the good cohomological properties of the relative dualizing sheaf and of Grothendieck duality. A key insight here is that the traditional notion of a connection on a line bundle on a scheme X over a base S is not well behaved when X is not S -smooth and must be suitably modified as in [Definition 3.5](#). With these preliminaries in place, we turn to the proof of [Theorem 1.2](#) in [Section 4](#). We must first construct a morphism of short exact sequences of smooth group schemes (1-3). Our strategy for doing this is as follows. Passing to an unramified extension of K if need be, we suppose that X_K has a rational point and use it to define an Albanese morphism $j_K : X_K \rightarrow J_K$. The Néron mapping property of J allows us to extend j_K to a morphism $j : X^{\text{sm}} \rightarrow J$ on the smooth locus of f in X . By (functorially) pulling back rigidified extensions of J by \mathbf{G}_m along j , we get line bundles on X^{sm} with connection. Via a careful analysis of the relative

dualizing sheaf, we show in [Lemma 4.1](#) that a line bundle with connection on X^{sm} is equivalent to a line bundle with regular connection on X ; this critically uses our hypothesis that the closed fiber of X is geometrically reduced (equivalently,⁴ that X^{sm} is fiber-wise dense in X). From this, we deduce the desired map (1-3). To complete the proof of [Theorem 1.2](#), we then “bootstrap” Raynaud’s [Theorem 1.1](#) using duality. Here, it is essential to know that the canonical evaluation duality between the Lie algebra of J and the sheaf of invariant differentials on J is compatible via j with the (Grothendieck) duality of $f_*\omega_{X/S}$ and $R^1f_*\mathbb{O}_X$. Such compatibility may be checked on generic fibers, where it is well known [[Coleman 1998](#), Theorem 5.1].

We remark that when k is perfect, both the short exact sequences of group schemes in the rows of (1-3) exist under the less restrictive hypotheses of [Theorem 1.1](#); this follows immediately from [Propositions 2.6](#) and [2.8](#) for the top row of (1-3), and from [Theorem 3.9](#) for the bottom row. It is natural to ask if [Theorem 1.2](#) holds in this generality as well. We do not know the answer to this question, as our construction of the map of short exact sequences of smooth groups in (1-3) seems to require the closed fiber of X to be generically smooth. Indeed, our construction of (1-3) relies on extending an Albanese morphism $X_K \rightarrow J_K$ to some open subscheme U of X with the property that line bundles with connection on U uniquely extend to line bundles with regular connection on X . On the one hand, this extension property seems to require U to be fiber-wise dense in X (see [Lemma 4.1](#) and [Remark 4.2](#)), while on the other hand one only expects to be able to extend the morphism $X_K \rightarrow J_K$ to $U = X^{\text{sm}}$. Thus, we are forced to require that $U = X^{\text{sm}}$ be fiber-wise dense in X , that is, that X_k be generically smooth (equivalently geometrically reduced). We note, however, that it is just our construction of the map (1-3) that requires X to have generically smooth closed fiber; the proof that this map is an isomorphism of exact sequences of group schemes relies only on the weaker hypotheses of Raynaud’s [Theorem 1.1](#).

Conventions and notation. Fix a base scheme S . If Y is any S -scheme and $S' \rightarrow S$ is any morphism, we will often write $Y_{S'} := Y \times_S S'$ for the base change. When $S' = \text{Spec}(F)$ is the spectrum of a field, we will sometimes abuse notation and write Y_F in place of $Y_{S'}$. We will work with the fppf topology on the categories of S -schemes and of smooth S -schemes (see [[SGA3-1 1970](#), Exposé IV, Section 6.3] or [[Bosch et al. 1990](#), Section 8.1]); if \mathcal{F} is any representable functor on one of these categories, we will also write \mathcal{F} for the representing object. By an *S -group scheme*

⁴ Indeed, if X_k is geometrically reduced, it is clearly generically smooth. Conversely, as X is normal by hypothesis, it must be S_2 by Serre’s criterion for normality, whence X_k is S_1 . Since X_k is also R_0 and “ $R_0 + S_1$ ” is equivalent to being reduced, we conclude that X_k is reduced and generically smooth, whence it must be geometrically reduced.

G we will always mean a finitely presented flat and separated commutative group scheme over S . As usual, we write \mathbf{G}_a and \mathbf{G}_m for the additive and multiplicative group schemes over S . A *vector group* on S is any S -group that is Zariski-locally isomorphic to a product of copies of \mathbf{G}_a . Associated to any quasicohherent \mathcal{O}_S -module \mathcal{M} is a sheaf for the fppf topology on S -schemes $\varphi : T \rightarrow S$ given by $\mathcal{M}(T) := \Gamma(T, \varphi^* \mathcal{M})$. When \mathcal{M} is locally free of finite rank, this fppf sheaf is represented by the vector group $\mathbf{Spec}(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{M}^*))$, where \mathcal{M}^* is the \mathcal{O}_S -linear dual of \mathcal{M} ; we will frequently abuse notation and write \mathcal{M} for both the locally free \mathcal{O}_S -module and the associated vector group on S . For any S -group G with identity section $e : S \rightarrow G$, we put $\omega_G := e^* \Omega_{G/S}^1$. As usual, for any S -scheme T we put $T[\epsilon] := T \times_{\mathbf{Z}} \mathrm{Spec}(\mathbf{Z}[\epsilon]/\epsilon^2)$, considered as a T -scheme via the first projection, and for any fppf sheaf G we write $\mathrm{Lie}(G)$ for the fppf sheaf of \mathcal{O}_S -modules defined (as in [Liu et al. 2004, Section 1]) by $\mathrm{Lie}(G)(T) := \ker(G(T[\epsilon]) \rightarrow G(T))$. When G is a smooth group, this agrees with the traditional notion of relative Lie algebra (as a sheaf of \mathcal{O}_S -modules). We set $\mathrm{Lie}(G) := \mathrm{Lie}(G)(S)$.

2. Canonical extensions of Néron models

In this section, following [Mazur and Messing 1974], we recall the construction and basic properties of the canonical extension of a Néron model, and we explain how to interpret its identity component via rigidified extensions.

Let S be any base scheme, and fix commutative S -group schemes F and G . A *rigidified extension* of F by G over S is a pair (E, σ) consisting of an extension E (of fppf sheaves of abelian groups over S) of F by G

$$0 \longrightarrow G \xrightarrow{i} E \longrightarrow F \longrightarrow 0 \tag{2-1}$$

and a section σ of S -pointed sheaves along the first infinitesimal neighborhood of the identity of F

$$\mathrm{Inf}_S^1(F) \xrightarrow{\sigma} E \tag{2-2}$$

that projects to the canonical closed immersion $\mathrm{Inf}_S^1(F) \rightarrow F$. Two rigidified extensions (E, σ) and (E', σ') of F by G are called *equivalent* if there is a homomorphism (necessarily an isomorphism) $\varphi : E \rightarrow E'$ that carries σ to σ' and makes the diagram (2-3) commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{i} & E & \longrightarrow & F & \longrightarrow & 0 \\ & & \parallel & & \downarrow \varphi & & \parallel & & \\ 0 & \longrightarrow & G & \xrightarrow{i'} & E' & \longrightarrow & F & \longrightarrow & 0 \end{array} \tag{2-3}$$

We denote by $\text{Extrig}_S(F, G)$ the set of equivalence classes of rigidified extensions of F by G over S . This set is equipped with a natural group structure via Baer sum of rigidified extensions [Mazur and Messing 1974, I, Section 2.1]) which makes the functor on S -schemes $T \rightsquigarrow \text{Extrig}_T(F_T, G_T)$ a group functor that is contravariant in the first variable via pullback (fibered product) and covariant in the second variable via pushout (fibered coproduct). We will write $\mathcal{E}\text{xtrig}_S(F, G)$ for the fppf sheaf of abelian groups associated to this functor.

We will exclusively be concerned with the special case that $G = \mathbf{G}_m$ is the multiplicative group over S . Note that (by fppf descent) any extension of F by \mathbf{G}_m is automatically representable as \mathbf{G}_m is affine (compare the proof of [Oort 1966, III, Proposition 17.4]). In this context, there is an alternate and more concrete functorial description of the group $\text{Extrig}_S(F, \mathbf{G}_m)$ that we will need for later use. Fix a choice of generator τ for the free rank-one \mathbf{Z} -module of invariant differentials $\omega_{\mathbf{G}_m}$ of \mathbf{G}_m over \mathbf{Z} . Note that τ is canonically determined up to multiplication by ± 1 . For any scheme S , we will denote the pullback of τ to a generator of $\omega_{\mathbf{G}_m}$ simply by τ . Write $\mathbf{E}_\tau(F)(S)$ for the set of equivalence classes of pairs (E, η) consisting of an extension E of F by \mathbf{G}_m over S and a global invariant differential $\eta \in \Gamma(S, \omega_E)$ which pulls back via the given morphism $\iota : \mathbf{G}_m \rightarrow E$ (realizing E as an extension of F by \mathbf{G}_m) to τ on \mathbf{G}_m . Two pairs (E, η) and (E', η') are declared to be equivalent if there is a morphism $\varphi : E \rightarrow E'$ inducing a diagram as in (2-3) and having the property that $\varphi^* \eta' = \eta$. We make $\mathbf{E}_\tau(F)(S)$ into an abelian group as follows. Let (E, η) and (E', η') be two pairs as above, and denote by E'' the Baer sum of E and E' . Writing pr, pr' for the projections from $E \times_F E'$ to E and E' , and denoting by $q : E \times_F E' \rightarrow E''$ the quotient map, we claim that there is a unique invariant differential η'' on E'' satisfying

$$q^* \eta'' = \text{pr}^* \eta + \text{pr}'^* \eta'.$$

Indeed, by definition, E'' is the cokernel of the skew-diagonal $(\iota, -\iota') : \mathbf{G}_m \rightarrow E \times_F E'$ under which $\text{pr}^* \eta + \text{pr}'^* \eta'$ pulls back to zero. Thus, via the short exact sequence

$$0 \longrightarrow \omega_{E''} \longrightarrow \omega_{E \times_F E'} \longrightarrow \omega_{\mathbf{G}_m} \longrightarrow 0$$

(which is left exact since $E \times_F E' \rightarrow E''$ is smooth due to [SGA3-1 1970, Exposé VI_B, Proposition 9.2 vii]), we obtain a unique invariant differential η'' on E'' as claimed. One easily checks that under the map $\mathbf{G}_m \rightarrow E''$ induced by either one of the inclusions $(\iota, 0), (0, \iota') : \mathbf{G}_m \rightrightarrows E \times_F E'$ (whose composites with q both coincide with the inclusion $\mathbf{G}_m \rightarrow E''$ realizing E'' as an extension of F by \mathbf{G}_m) the differential η'' pulls back to τ . We define the sum of the classes represented by (E, η) and (E', η') to be the class represented by (E'', η'') . It is straightforward to verify that this definition does not depend on the choice of representatives, and

makes $\mathbf{E}_\tau(F)(S)$ into an abelian group. This construction is obviously contravariantly functorial in S via pullback of extensions and of invariant differentials.

Lemma 2.1. *For any choice of basis τ of $\omega_{\mathbf{G}_m}$, there is a functorial isomorphism of abelian groups*

$$\mathrm{Extrig}_S(F, \mathbf{G}_m) \xrightarrow{\cong} \mathbf{E}_\tau(F)(S) .$$

Proof. Associated to the extension (2-1) with $G = \mathbf{G}_m$ is the short exact sequence of Lie algebras

$$0 \longrightarrow \mathcal{L}ie(\mathbf{G}_m) \longrightarrow \mathcal{L}ie(E) \longrightarrow \mathcal{L}ie(F) \longrightarrow 0 \quad (2-4)$$

(note that the map $\mathcal{L}ie(E) \rightarrow \mathcal{L}ie(F)$ is surjective by [Liu et al. 2004, Proposition 1.1 (c)], as $E \rightarrow F$ is smooth). We claim that the data of a rigidification on (2-1) is equivalent to a choice of a splitting of (2-4). Indeed, any map $\sigma : \mathrm{Inf}_S^1(F) \rightarrow E$ necessarily factors through $\mathrm{Inf}_S^1(E)$, so using the natural isomorphism $\mathrm{Inf}_S^1(H) \simeq \mathrm{Spec}(\mathbb{O}_S[\omega_H])$ for any smooth group scheme H over S we obtain a bijection between rigidifications of (2-1) and sections $\omega_E \rightarrow \omega_F$ to the pullback map $\omega_F \rightarrow \omega_E$. By the usual duality of the \mathbb{O}_S -modules $\mathcal{L}ie(H)$ and ω_H [SGA3-1 1970, Exposé 2, Section 4.11], this is equivalent to a section s as claimed.

Using τ to identify the free rank one \mathbb{O}_S -module $\mathcal{L}ie(\mathbf{G}_m)$ with \mathbb{O}_S and thinking of a splitting of (2-4) as a map $\mathcal{L}ie(E) \rightarrow \mathcal{L}ie(\mathbf{G}_m)$ restricting to the identity on $\mathcal{L}ie(\mathbf{G}_m)$, we see that any such splitting is by duality equivalent to a global section $\eta \in \Gamma(S, \omega_E)$ pulling back to τ in $\Gamma(S, \omega_{\mathbf{G}_m})$. One checks that the equivalence $(E, \sigma) \leftrightarrow (E, \eta)$ induces an isomorphism of abelian groups $\mathrm{Extrig}_S(F, \mathbf{G}_m) \rightarrow \mathcal{E}_\tau(F)(S)$ that is functorial in S , as claimed. \square

The following key result shows that the functor $\mathcal{E}xtrig$ allows one to realize the universal extension of an abelian scheme.

Proposition 2.2 (Mazur–Messing). *Let A be an abelian scheme over an arbitrary base scheme S and denote by \hat{A} the dual abelian scheme. Then the fppf sheaf $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ is a smooth and separated S -group scheme. It sits in a natural short exact sequence of smooth S -group schemes*

$$0 \longrightarrow \omega_A \longrightarrow \mathcal{E}xtrig_S(A, \mathbf{G}_m) \longrightarrow \hat{A} \longrightarrow 0 . \quad (2-5)$$

Moreover, (2-5) is the universal extension of \hat{A} by a vector group.

Proof. See [Mazur and Messing 1974], especially I, Section 2.6 and Proposition 2.6.7. \square

We now specialize to the case that $S = \mathrm{Spec} R$ is the spectrum of a discrete valuation ring R with field of fractions K . Fix an abelian variety A_K over K and denote by A the Néron model of A_K over S and by A^0 the relative identity

component of A . Let \widehat{A} be the Néron model of the dual abelian variety \widehat{A}_K . We have the following analogue of [Proposition 2.2](#):

Proposition 2.3 [[Mazur and Messing 1974](#), I, Corollary 5.2]. *As a functor on smooth S -schemes, the fppf abelian sheaf $\mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$ is represented by a smooth and separated S -group scheme. Moreover, there is a natural short exact sequence of smooth groups over S*

$$0 \longrightarrow \omega_A \longrightarrow \mathcal{E}xtrig_S(A^0, \mathbf{G}_m) \longrightarrow \widehat{A} \longrightarrow 0 . \tag{2-6}$$

Definition 2.4 (Mazur–Messing). The *canonical extension* of \widehat{A} is the smooth and separated S -group scheme

$$\mathcal{E}(\widehat{A}) := \mathcal{E}xtrig_S(A^0, \mathbf{G}_m).$$

Remarks 2.5. When A is an abelian scheme, the canonical extension $\mathcal{E}(\widehat{A})$ coincides with the universal extension of \widehat{A} by a vector group by [Proposition 2.2](#). When A is not an abelian scheme, an example of Breen and Raynaud [[Mazur and Messing 1974](#), I, 5.6] shows that A need not have a universal extension.

Note, however, that since the functor $\mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$ commutes with fppf base change, the smooth group scheme $\mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$ representing it on the category of smooth group schemes over S is of formation compatible with base change to a smooth S -scheme. In particular, the K -fiber of the canonical extension exact sequence (2-6) is the universal extension of \widehat{A}_K by a vector group, thanks to [Proposition 2.2](#).

In this paper, we work with $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ instead of $\mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$, as the former has better functorial properties due to the Néron mapping property of A (which is not enjoyed by A^0). Following the method of [[Mazur and Messing 1974](#), I, Corollary 5.2], we wish to show that $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ is representable, at least as a functor on smooth test objects. This is somewhat more subtle than the corresponding problem for $\mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$; in particular, denoting by Φ_A and $\Phi_{\widehat{A}}$ the component groups of A and \widehat{A} , we will need to know that Grothendieck’s pairing for A_K (see [[SGA7-1 1972](#), Exposés 7–9] or [[Bosch 1997](#), Section 4])

$$\Phi_A \times \Phi_{\widehat{A}} \longrightarrow \mathbf{Q}/\mathbf{Z} \tag{2-7}$$

is right nondegenerate.

Proposition 2.6. *Suppose that Grothendieck’s pairing on component groups is right nondegenerate. Then the fppf abelian sheaf $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ on the category of smooth S -schemes is represented by a smooth and separated S -group scheme.*

Moreover, there is a natural short exact sequence of smooth group schemes over S

$$0 \longrightarrow \omega_A \longrightarrow \mathcal{E}xtrig_S(A, \mathbf{G}_m) \longrightarrow \widehat{A}^0 \longrightarrow 0. \quad (2-8)$$

Proof. We follow the proof of [Mazur and Messing 1974, I, Corollary 5.2]. Let T be any smooth S -scheme and consider the natural map of abelian groups

$$\text{Extrig}_T(A, \mathbf{G}_m) \longrightarrow \text{Ext}_T(A, \mathbf{G}_m). \quad (2-9)$$

By Lemma 2.1, we see that when T is affine any extension E of A_T by \mathbf{G}_m admits a rigidification so (2-9) is surjective. By definition, the kernel of (2-9) consists of rigidifications on the trivial extension of A_T by \mathbf{G}_m , up to isomorphism. To give a rigidification $\text{Inf}_T^1(A_T) \rightarrow A_T \times_T \mathbf{G}_m$ of the trivial extension is obviously equivalent to giving a map of T -pointed T -schemes $\text{Inf}_T^1(A_T) \rightarrow \mathbf{G}_m$, which in turn is equivalent to giving a global section of ω_{A_T} (see [Mazur and Messing 1974, I, 1.2] or the proof of Lemma 2.1). If two sections η_1 and η_2 of ω_{A_T} give isomorphic rigidified extensions of the trivial extension, then there is an automorphism of the trivial extension, necessarily induced by a group map $\varphi : A_T \rightarrow \mathbf{G}_m$, with the property that η_1 and η_2 differ by

$$d\varphi \in \Gamma(T, \omega_{A_T}) \quad (\text{with } d : \text{Hom}_T(A_T, \mathbf{G}_m) \rightarrow \text{Hom}(\text{Inf}_T^1(A_T), \mathbf{G}_m),$$

the natural map induced by the canonical closed immersion $\text{Inf}_T^1(A_T) \rightarrow A_T$). Since A is flat with proper generic fiber and T is S -smooth, we have

$$\text{Hom}_T(A_T, \mathbf{G}_m) = 0,$$

so by passing to the associated fppf abelian sheaves, we thus obtain the short exact sequence of fppf sheaves

$$0 \longrightarrow \omega_A \longrightarrow \mathcal{E}xtrig_S(A, \mathbf{G}_m) \longrightarrow \mathcal{E}xt_S(A, \mathbf{G}_m) \longrightarrow 0.$$

By [Bosch 1997, Proposition 5.1] (or [Milne 1986a, III, Proposition C.14]), the canonical duality of abelian varieties extends to a natural map $\widehat{A}^0 \rightarrow \mathcal{E}xt_S(A, \mathbf{G}_m)$ which is an isomorphism of fppf abelian sheaves on the category of smooth S -schemes if and only if Grothendieck's pairing on component groups (2-7) is right nondegenerate. Thus, our hypotheses ensure that $\mathcal{E}xt_S(A, \mathbf{G}_m)$ is represented on the category of smooth S -schemes by the smooth and separated S -group scheme \widehat{A}^0 . Since ω_A is a vector group, it is clearly smooth and affine over S . Thus, the proof of [Oort 1966, III, Proposition 17.4], which is easily adapted from the situation considered there (fpqc topology on all S -schemes) to our situation (fppf topology on smooth S -schemes) since ω_A and \widehat{A}^0 are smooth, shows via fppf descent that $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ is represented (on smooth S -schemes) by a smooth and separated

S -group scheme, and that there is a short exact sequence of smooth S -schemes (2-8). \square

Remark 2.7. We note that Mazur and Messing [1974, I, Corollary 5.2] prove that the canonical map

$$\widehat{A} \longrightarrow \mathcal{E}xt_S(A^0, \mathbf{G}_m)$$

is an isomorphism of fppf abelian sheaves on smooth test objects for any Néron model A over any connected Dedekind scheme S by showing that $\mathcal{E}xt_S(A^0, \mathbf{G}_m)$ satisfies the Néron mapping property. In our situation, this method fails to generalize as \widehat{A}^0 does not satisfy any good mapping property on smooth S -schemes which do not have connected closed fiber.

In our applications, we will wish to apply Proposition 2.6 when A_K is the Jacobian of a smooth and proper curve over K . In this situation, it follows easily from the autoduality of J_K and the functoriality of the morphism $\widehat{J}^0 \rightarrow \mathcal{E}xt_S(J, \mathbf{G}_m)$ that Grothendieck's pairing is right nondegenerate if and only if it is left nondegenerate if and only if it is perfect. In order to apply Proposition 2.6, we will need the following criterion for the perfectness of Grothendieck's pairing.

Proposition 2.8. *Let X_K be a smooth and proper curve over K with Jacobian J_K over K . Fix a proper flat and normal model X of X_K over R , and denote by X_1, \dots, X_n the (reduced) irreducible components of the closed fiber X_k . Suppose that the greatest common divisor of the geometric multiplicities of the X_i in X_k is 1, and assume one of the following hypotheses holds:*

- (1) *The residue field k of R is perfect.*
- (2) *X is regular, each X_i is geometrically reduced and X admits an étale quasi-section.*

Then Grothendieck's pairing (2-7) for J_K is perfect.

Proof. As our hypotheses are preserved by and our conclusion may be checked after étale base change, we may replace R with a strict henselization of R and may thus assume that R is strictly henselian. In case (2), our hypotheses ensure that X_K has a K -rational point and admits a proper flat and regular model X over R all of whose (reduced) irreducible components are geometrically reduced. These are exactly the hypotheses of [Bosch and Lorenzini 2002, Corollary 4.7], which then ensures that Grothendieck's pairing for J_K is perfect.

In case (1), we first claim that our hypothesis on the gcd of the geometric multiplicities of the X_i in X_k imply the existence of a tamely ramified Galois extension K' of K (necessarily with trivial residue field extension) such that $X_{K'}$ has a K' -rational point. Indeed, by resolution of singularities for excellent surfaces [Deligne and Mumford 1969, Section 2; Lipman 1978] and descent arguments from the

completion of R (see [Conrad et al. 2003, Theorem 2.2.2]) there exists a proper birational morphism of proper and flat S -models $\tilde{X} \rightarrow X$ of X_K with \tilde{X} regular. Due to [Liu 2002, Corollary 9.2.30], we may assume that the closed fiber \tilde{X}_k is a normal crossings divisor on \tilde{X} . Observe that the proper and birational morphism $\tilde{X} \rightarrow X$ is an isomorphism over any point $\zeta \in X$ of codimension 1; this may be checked after the base change $\text{Spec}(\mathbb{O}_{X,\zeta}) \rightarrow X$, where it follows from the valuative criterion for properness applied to the discrete valuation ring $\mathbb{O}_{X,\zeta}$ (recall that X is normal). In particular, $\tilde{X} \rightarrow X$ is an isomorphism over the generic points of X_k and we deduce that our hypothesis on the gcd of the geometric multiplicities of the irreducible components of X_k is inherited by \tilde{X} . Thus, there exists an irreducible component Γ_0 of \tilde{X}_k whose multiplicity e in \tilde{X}_k is not divisible by $\text{char}(k)$. The proof of [Liu 2002, Theorem 10.4.6] (see also [Liu 2002, Corollary 10.4.7]) then shows that there is a Galois extension K' of K with ramification index e having the following property: letting R' denote the integral closure of R in K' , (which is again a discrete valuation ring, as R is henselian) and writing X' for the normalization of $\tilde{X} \times_S \text{Spec}(R')$, the closed fiber X'_k has an irreducible component Γ'_0 over Γ_0 whose geometric multiplicity in X'_k is 1; that is, Γ'_0 is generically smooth. As R' is strictly henselian, we conclude that there exists an R' -point of X' and hence a K' -point of $X'_{K'} = X_{K'}$, as claimed.

Now since k is perfect, $X_{K'}$ admits a proper flat and regular model over R' with the property that every (reduced) irreducible component of the closed fiber is geometrically reduced (any proper flat and regular model will do). We may therefore apply [Bosch and Lorenzini 2002, Corollary 4.7] to $X_{K'}$ to deduce that Grothendieck's pairing for $J_{K'}$ is perfect. As K'/K is tamely ramified, it now follows from [Bertapelle and Bosch 2000] that Grothendieck's pairing for J_K is perfect, as desired. \square

Remark 2.9. Assuming k to be perfect, it follows from work of Pépin [2008] (using the results of Bosch and Lorenzini [2002]) that Grothendieck's pairing for J_K is perfect whenever the index of X_K is not divisible by the characteristic of k .

Already in the case of Jacobians, Grothendieck's pairing may fail to be perfect. Indeed, working over R with imperfect residue fields, Bosch and Lorenzini give an explicit example of a Jacobian J_K for which Grothendieck's pairing is not perfect [Bosch and Lorenzini 2002, Example 6.2]. The first examples of abelian varieties for which Grothendieck's pairing is not perfect were given by [Bertapelle and Bosch 2000].

For an arbitrary abelian variety A_K over K , Grothendieck's pairing on component groups (2-7) is in addition known to be perfect under any of the following hypotheses:

- (1) R is of mixed characteristic $(0, p)$ and k is perfect.

- (2) k is finite.
- (3) k is perfect and A_K has potentially multiplicative reduction.
- (4) There exists a tamely ramified Galois extension K' of K having trivial residue field extension such that Grothendieck's pairing for the base change $A_{K'}$ is perfect.

For the proofs of these facts, see [Bégueri 1980] in the case of (1), [McCallum 1986] in case (2), [Bosch 1997] in case (3), and [Bertapelle and Bosch 2000] in the case of (4). See also [Milne 1986a, 3, Theorem 2.5] when R has mixed characteristic and finite residue field.

We end this section by relating the group $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ to the identity component of the canonical extension $\mathcal{E}(\hat{A}) := \mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$ of \hat{A} .

Lemma 2.10. *Let A_K be an abelian variety over K and A its Néron model over R . Suppose that Grothendieck's pairing (2-7) for A_K is right nondegenerate, so $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ is a smooth S -group. The canonical map of short exact sequences of S -groups*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \omega_A & \longrightarrow & \mathcal{E}xtrig_S(A, \mathbf{G}_m) & \longrightarrow & \hat{A}^0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \omega_A & \longrightarrow & \mathcal{E}xtrig_S(A^0, \mathbf{G}_m) & \longrightarrow & \hat{A} \longrightarrow 0
 \end{array} \tag{2-10}$$

furnished from the functoriality of $\mathcal{E}xtrig_S(\cdot, \mathbf{G}_m)$ by the inclusion $A^0 \hookrightarrow A$ identifies $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ with the identity component of $\mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$.

Proof. We first observe that $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ has connected fibers. More generally, we claim that any extension of (not necessarily commutative) finite type connected group schemes over a field must be connected. Indeed, suppose that

$$1 \longrightarrow G \longrightarrow E \longrightarrow F \longrightarrow 1$$

is such an extension. Since connectedness of any scheme with a rational point is preserved by ground field extension, the fibers of $E \rightarrow F$ are connected as they become isomorphic to G after passing to an extension field and G is connected. Thus, any separation $\{U, V\}$ of E is a union of fibers of $E \rightarrow F$. Since the quotient map $E \rightarrow F$ is faithfully flat and of finite type, it is open, so $\{U, V\}$ is the pullback of a separation of F ; by the connectedness of F we conclude that $\{U, V\}$ is trivial and E is connected.

To conclude, since $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$ has connected fibers it suffices to show that any homomorphism from a commutative S -group H with connected fibers to $\mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$ necessarily factors through $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$. By the functoriality of $\mathcal{E}xtrig_S(\cdot, \mathbf{G}_m)$, the top row of (2-10) is identified with the pullback of the bottom

row along the inclusion $\widehat{A}^0 \rightarrow \widehat{A}$; that is, we have a canonical isomorphism of smooth groups

$$\mathcal{E}xtrig_S(A, \mathbf{G}_m) = \mathcal{E}xtrig_S(A^0, \mathbf{G}_m) \times_{\widehat{A}} \widehat{A}^0.$$

Thus, since the composition of $H \rightarrow \mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$ with the projection to \widehat{A} necessarily factors through the inclusion of \widehat{A}^0 into \widehat{A} as H has connected fibers, we conclude that $H \rightarrow \mathcal{E}xtrig_S(A^0, \mathbf{G}_m)$ indeed factors through the fiber product $\mathcal{E}xtrig_S(A, \mathbf{G}_m)$, as desired. \square

3. An enhancement of the relative Picard functor

We continue to suppose that $S = \text{Spec } R$ with R a discrete valuation ring having field of fractions K . By a *relative curve* X over S we mean a flat finite type and separated S -scheme $f : X \rightarrow S$ of pure relative dimension 1 that is normal with smooth and geometrically connected generic fiber. In this section, we will introduce the functor $\text{Pic}_{X/S}^{\natural, 0}$ and prove that it is representable whenever $\text{Pic}_{X/S}^0$ is representable.

We begin by recalling some general facts about relative dualizing sheaves and Grothendieck duality that will be needed in what follows. Let X and Y be locally noetherian schemes and $f : X \rightarrow Y$ a Cohen–Macaulay morphism of pure relative dimension n . By [Conrad 2000, Theorem 3.5.1], the complex $f^! \mathbb{O}_Y$ has a unique nonzero cohomology sheaf, which is in degree $-n$, and the *relative dualizing sheaf of X over Y* is

$$\omega_{X/Y} := H^{-n}(f^! \mathbb{O}_Y).$$

It is flat over Y by [Conrad 2000, Theorem 3.5.1], and locally free if and only if the Cohen–Macaulay fibers of f are Gorenstein [Hartshorne 1966, V, Proposition 9.3, Theorem 9.1]. Furthermore, the formation of $\omega_{X/Y}$ is compatible with étale localization on X (see the discussion preceding [Conrad 2000, Corollary 4.4.5]) and with any base change $Y' \rightarrow Y$ where Y' is locally noetherian [Conrad 2000, Theorem 3.6.1]. When f is in addition proper, there is a natural \mathbb{O}_Y -linear trace map

$$\gamma_f : R^n f_* \omega_{X/Y} \rightarrow \mathbb{O}_Y, \tag{3-1}$$

which is compatible with any base change $Y' \rightarrow Y$ with Y' locally noetherian [Conrad 2000, Corollary 3.6.6]. By Grothendieck–Serre duality [Conrad 2000, Theorem 4.3.1] the canonical map

$$\mathbf{R}f_* \mathbf{R}\mathcal{H}om_X^\bullet(\mathcal{F}^\bullet, \omega_{X/Y}[n]) \longrightarrow \mathbf{R}\mathcal{H}om_Y^\bullet(\mathbf{R}f_* \mathcal{F}^\bullet, \mathbb{O}_Y), \tag{3-2}$$

induced by (3-1) is a quasi-isomorphism for any complex \mathcal{F}^\bullet in the derived category of sheaves of \mathbb{O}_X -modules whose cohomology is coherent and vanishes in sufficiently negative and positive degrees.

For arbitrary base schemes Y and Cohen–Macaulay morphisms $f : X \rightarrow Y$ of pure relative dimension, one defines $\omega_{X/Y}$ (and γ_f when f is proper) via direct limits and base change from the locally noetherian case (see [Conrad 2000, page 174]); this makes sense due to the aforementioned base-change compatibility in the locally noetherian context and yields a coherent sheaf of \mathbb{O}_X -modules $\omega_{X/Y}$ and a trace map γ_f when f is proper that are compatible with arbitrary base change on Y .

Let us apply these considerations to the case of a relative curve $f : X \rightarrow S$. Since X is normal and of pure relative dimension one, it is Cohen–Macaulay by Serre’s criterion for normality. Theorem 23.3 in [Matsumura 1989]. Thus, the complex $f^! \mathbb{O}_S$ is a coherent sheaf $\omega_{X/S}$ concentrated in degree -1 . By our discussion, $\omega_{X/S}$ is S -flat, and of formation compatible with étale localization on X and arbitrary base change on S . When f is S -smooth, the theory of the dualizing sheaf provides a canonical identification of the relative dualizing sheaf with the sheaf of relative differential 1-forms on X over S . It is natural to ask how these two sheaves are related in general.

Proposition 3.1. *There is a canonical \mathbb{O}_X -linear morphism*

$$c_{X/S} : \Omega_{X/S}^1 \longrightarrow \omega_{X/S}, \tag{3-3}$$

whose restriction to any S -smooth open subset of X is the canonical isomorphism.

Proof. See [Cais 2009, Proposition 5.1]. □

In fact, we can realize $\omega_{X/S}$ as a subsheaf of differentials on X which are regular on the generic fiber. Precisely, if $i : U \hookrightarrow X$ is any open subscheme of X containing the generic fiber then the canonical map $\omega_{X/S} \rightarrow i_* i^* \omega_{X/S}$ is injective as it is an isomorphism over X_K and $\omega_{X/S}$ is S -flat. Since the formation of $\omega_{X/S}$ is compatible with étale localization on X , we thus obtain a natural injective map

$$\omega_{X/S} \hookrightarrow i_* \omega_{U/S}. \tag{3-4}$$

In particular, taking $U = X_K$ we have $\omega_{U/S} \simeq \Omega_{X_K/K}^1$ by the general theory of the dualizing sheaf (or by Proposition 3.1), so $\omega_{X/S}$ is a subsheaf of $i_* \Omega_{X_K/K}^1$. When U is large enough, the map (3-4) is also surjective.

Lemma 3.2. *Suppose that the complement of U in X consists of finitely many closed points of codimension 2 (necessarily in the closed fiber). Then the canonical injective map (3-4) is an isomorphism.*

Proof. We follow the proof given right after (5.2.7) in [Conrad 2000]. By standard arguments, it suffices to show that the local cohomology groups $H_x^1(X, \omega_{X/S})$

vanish for all $x \in X - U$. Due to [SGA2 1968, Exposé III, Example 3.4], such vanishing is equivalent to

$$\text{depth}_{\mathbb{O}_{X,x}}(\omega_{X/S,x}) \geq 2.$$

If $x \in X - U$ is a regular point, this inequality is trivially verified, since $\omega_{X,x}$ is a free $\mathbb{O}_{X,x}$ -module of rank 1 for such x (regular local rings are Gorenstein) and $\mathbb{O}_{X,x}$ is two-dimensional and normal (whence it has depth 2 by Serre’s criterion for normality).

In general, by [SGA2 1968, Exposé III, Corollary 2.5] it is enough to show that for each nonregular point x of the closed fiber X_k of X we have

$$\text{depth}_{\mathbb{O}_{X_k,x}}(\omega_{X_k/k,x}) \geq 1. \tag{3-5}$$

If this is not the case, then the maximal ideal \mathfrak{m}_x of $\mathbb{O}_{X_k,x}$ consists entirely of zero-divisors for the finite $\mathbb{O}_{X_k,x}$ -module $\omega_{X_k/k,x}$, so it must be an associated prime of $\omega_{X_k/k,x}$. We would then have $\mathfrak{m}_x = \text{Ann}(s)$ for some nonzero $s \in \omega_{X_k/k,x}$ whence $\text{Hom}_{X_k}(k(x), \omega_{X_k/k}) \neq 0$. However,

$$\text{Hom}_{X_k}(k(x), \omega_{X_k/k}) = H^1(X_k, k(x))^\vee \tag{3-6}$$

by Grothendieck duality for the k -scheme X_k (see Corollary 5.1.3 and the bottom half of page 224 in [Conrad 2000]), and we know that the right side of (3-6) is zero (since $k(x)$ is a skyscraper sheaf supported at the point x), which is a contradiction. Thus, \mathfrak{m}_x contains an $\omega_{X_k/k,x}$ -regular element, so (3-5) holds, as desired. \square

When $f : X \rightarrow S$ is in addition proper, so we have a trace map (3-1), we may apply the machinery of Grothendieck duality. For our purposes, we need only the following.

Proposition 3.3. *If $f : X \rightarrow S$ is a proper relative curve then the canonical map of flat \mathbb{O}_S -modules*

$$f_*\omega_{X/S} \longrightarrow (R^1f_*\mathbb{O}_X)^\vee \tag{3-7}$$

induced by cup product and the trace map (3-1) is an isomorphism. Furthermore, there is a natural short exact sequence of \mathbb{O}_S -modules

$$0 \longrightarrow \mathcal{E}xt_S^1(R^1f_*\mathbb{O}_X, \mathbb{O}_S) \longrightarrow R^1f_*\omega_{X/S} \longrightarrow (f_*\mathbb{O}_X)^\vee \longrightarrow 0. \tag{3-8}$$

In particular, if f is cohomologically flat (in dimension 0) then $R^1f_\omega_{X/S}$ is a locally free \mathbb{O}_S -module.*

Proof. Since $\mathcal{H}om_X(\mathbb{O}_X, \cdot)$ is naturally isomorphic to the identity functor, (3-2) with $\mathcal{F}^\bullet = \mathbb{O}_X$ (thought of as a complex in degree zero) yields a quasi-isomorphism

$$Rf_*\omega_{X/S}[1] \simeq R\mathcal{H}om_S^\bullet(Rf_*\mathbb{O}_X, \mathbb{O}_S). \tag{3-9}$$

Applying H^{-1} to (3-9) and using the spectral sequence

$$E_2^{m,n} = \mathcal{E}xt_S^m(H^{-n}(\mathbf{R}f_*\mathbb{O}_X), \mathbb{O}_S) \implies H^{m+n}(\mathbf{R}\mathcal{H}om_S^\bullet(\mathbf{R}f_*\mathbb{O}_X, \mathbb{O}_S)) \quad (3-10)$$

(whose only nonzero terms occur when $m = 0, 1$ and $n = 0, -1$) to calculate the right side, we obtain a natural isomorphism $f_*\omega_{X/S} \simeq (R^1f_*\mathbb{O}_X)^\vee$. To know that this map coincides with the map (3-7) induced by cup product and the trace map γ_f , one proceeds as in the proof of [Conrad 2000, Theorem 5.1.2]. Similarly, applying H^0 to (3-9) and using (3-10), we arrive at the short exact sequence (3-8). For the final statement of the proposition, recall that by definition f is cohomologically flat in dimension 0 if $f_*\mathbb{O}_X$ commutes with arbitrary base change, which holds if and only if $R^1f_*\mathbb{O}_X$ is locally free. Thus, when f is cohomologically flat, the sheaf $\mathcal{E}xt_S^1(R^1f_*\mathbb{O}_X, \mathbb{O}_S)$ vanishes and it follows easily from (3-8) that $R^1f_*\omega_{X/S}$ is locally free over S . \square

We record here a corollary showing that the relative dualizing sheaf is in general much better behaved than the sheaf of relative differential 1-forms:

Corollary 3.4. *Let $f : X \rightarrow S$ be a proper relative curve, and assume that f is cohomologically flat in dimension 0. Then for all $i \geq 0$, the \mathbb{O}_S -module $R^if_*\omega_{X/S}$ is locally free and commutes with arbitrary base change on S .*

Proof. By standard arguments on base change, it is enough to show that $R^if_*\omega_{X/S}$ is locally free for $i \geq 0$. This holds for $i \geq 2$ by the theorem on formal functions (as then $R^if_*\omega_{X/S} = 0$), and for $i = 0$ since $\omega_{X/S}$ is S -flat. For $i = 1$, it follows immediately from Proposition 3.3. \square

For a relative curve $f : X \rightarrow S$, we now wish to apply the preceding considerations to define a natural enhancement $\text{Pic}_{X/S}^1$ of the relative Picard functor classifying invertible sheaves with the additional data of a “regular connection”.

Let T be any S -scheme. Since both the sheaf of relative differential 1-forms and the relative dualizing sheaf are compatible with base change, via pullback along $T \rightarrow S$ we obtain from (3-3) a natural morphism $\Omega_{X_T/T}^1 \rightarrow \omega_{X_T/T}$, and hence an \mathbb{O}_T -linear derivation

$$d_T : \mathbb{O}_T \longrightarrow \omega_{X_T/T} .$$

Fix a line bundle \mathcal{L} on X_T . Recall that a *connection* on \mathcal{L} over T is an \mathbb{O}_T -linear homomorphism $\nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes_{\mathbb{O}_T} \Omega_{X_T/T}^1$ satisfying the usual Leibnitz rule. When X is not S -smooth, this notion is not generally well behaved, and it is often desirable to allow connections to have certain types of poles along the singularities of X . For our purposes, the right notion of a connection is:

Definition 3.5. *A regular connection on \mathcal{L} over T is an \mathbb{O}_T -linear homomorphism*

$$\nabla : \mathcal{L} \longrightarrow \mathcal{L} \otimes_{\mathbb{O}_{X_T}} \omega_{X_T/T}$$

satisfying the Leibnitz rule: $\nabla(h\eta) = \eta \otimes d_T(h) + h\nabla\eta$ for any sections h of \mathbb{C}_{X_T} and η of \mathcal{L} . A *morphism* of line bundles with regular connection over T is an \mathbb{C}_{X_T} -linear morphism of the underlying line bundles that is compatible with the given connections.

Remark 3.6. Observe that any connection $\nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega^1_{X_T/T}$ on \mathcal{L} over T gives rise to a regular connection on \mathcal{L} over T via composition with the map induced by $\Omega^1_{X_T/T} \rightarrow \omega_{X_T/T}$.

If \mathcal{L} and \mathcal{L}' are two line bundles on X_T equipped with regular connections ∇ and ∇' over T , the tensor product $\mathcal{L} \otimes_{\mathbb{C}_{X_T}} \mathcal{L}'$ is naturally equipped with the tensor product regular connection $\nabla \otimes \nabla'$ induced by decreeing

$$(\nabla \otimes \nabla')(\eta \otimes \eta') := \eta \otimes \nabla'(\eta') + \eta' \otimes \nabla(\eta),$$

for any sections η of \mathcal{L} and η' of \mathcal{L}' . Observe that with respect to this operation, the pair (\mathbb{C}_{X_T}, d_T) serves as an identity element. Thus, the set of isomorphism classes of line bundles on X_T with a regular connection over T has a natural abelian group structure which is obviously compatible with our definition of a morphism of line bundles with connection. Furthermore, if $T' \rightarrow T$ is any morphism of S -schemes, then since the formation of $\omega_{X/S}$ is compatible with base change, any line bundle on X_T with regular connection over T pulls back to a line bundle on $X_{T'}$ with regular connection over T' .

Definition 3.7. Denote by $P^{\natural}_{X/S}$ the contravariant functor from the category of S -schemes to the category of abelian groups given on an S -scheme T by

$$P^{\natural}_{X/S}(T) := \left\{ \begin{array}{l} \text{Isomorphism classes of pairs } (\mathcal{L}, \nabla) \text{ consisting of a line bundle} \\ \mathcal{L} \text{ on } X_T \text{ equipped with a regular connection } \nabla \text{ over } T \end{array} \right\},$$

and write $\text{Pic}^{\natural}_{X/S}$ for the fppf sheaf associated to $P^{\natural}_{X/S}$.

As is customary, we will denote by $P_{X/S}$ the contravariant functor on the category of S -schemes which associates to an S -scheme T the set of isomorphism classes of line bundles on X_T , and by $\text{Pic}_{X/S}$ the fppf sheaf on the category of S -schemes associated to $P_{X/S}$. For any S -scheme T , there is an obvious homomorphism of abelian groups $P^{\natural}_{X/S}(T) \rightarrow P_{X/S}(T)$ given by “forgetting the connection”, and hence a map of fppf abelian sheaves

$$\text{Pic}^{\natural}_{X/S} \longrightarrow \text{Pic}_{X/S} . \tag{3-11}$$

We wish to define a certain subfunctor of $\text{Pic}^{\natural}_{X/S}$ which will play the role of “identity component” and which will enjoy good representability properties. We adopt the following definition:

Definition 3.8. Let $\text{Pic}_{X/S}^{\natural,0}$ be the fppf abelian sheaf on the category of S -schemes given by

$$\text{Pic}_{X/S}^{\natural,0} := \text{Pic}_{X/S}^{\natural} \times_{\text{Pic}_{X/S}} \text{Pic}_{X/S}^0.$$

Here, $\text{Pic}_{X/S}^0$ denotes the identity component of the group functor $\text{Pic}_{X/S}$ (whose fibers are representable; see [Liu et al. 2004, page 459] and compare [Bosch et al. 1990, page 233]). Alternately, $\text{Pic}_{X/S}^0$ the open subfunctor of $\text{Pic}_{X/S}$ classifying line bundles of partial degree zero on each irreducible component of every geometric fiber [Bosch et al. 1990, Section 9.2, Corollary 13].

Theorem 3.9. *Let $f : X \rightarrow S$ be a proper relative curve and suppose that the greatest common divisor of the geometric multiplicities of the irreducible components of the closed fiber X_k of X is 1. Then $\text{Pic}_{X/S}^{\natural,0}$ is a smooth S -scheme and there is a short exact sequence of smooth group schemes over S*

$$0 \longrightarrow f_*\omega_{X/S} \longrightarrow \text{Pic}_{X/S}^{\natural,0} \longrightarrow \text{Pic}_{X/S}^0 \longrightarrow 0. \quad (3-12)$$

To prove [Theorem 3.9](#), we will first construct (3-12) as an exact sequence of fppf abelian sheaves. By work of Raynaud [1970, Theorem 8.2.1] (or [Bosch et al. 1990, Section 9.4, Theorem 2]), the hypotheses on X imply that $\text{Pic}_{X/S}^0$ is a separated S -group scheme which is smooth by [Bosch et al. 1990, Section 8.4, Proposition 2]. On the other hand, our hypotheses ensure that X is cohomologically flat in dimension zero, whence $f_*\omega_{X/S}$ is a vector group (in particular, it is smooth and separated) by [Corollary 3.4](#). A straightforward descent argument will complete the proof.

We will begin by constructing the exact sequence (3-12). Fix an S -scheme T and consider the natural map (3-11). The kernel of this map consists of all isomorphism classes represented by pairs of the form $(\mathbb{O}_{X_T}, \nabla)$, where ∇ is a regular connection on \mathbb{O}_{X_T} over T . By the Leibnitz rule, ∇ is determined up to isomorphism by the value $\nabla(1) \in \Gamma(X_T, \omega_{X_T/T})$. Since two pairs $(\mathbb{O}_{X_T}, \nabla)$ and $(\mathbb{O}_{X_T}, \nabla')$ are isomorphic precisely when there is a unit $u \in \Gamma(X_T, \mathbb{O}_{X_T}^\times)$ satisfying

$$\nabla(1) = \nabla'(1) + u^{-1} \cdot d_T u$$

we see that the kernel of (3-11) is naturally identified with $H^0(X_T, \omega_{X_T/T})$ modulo the image of the map

$$d_T \log : H^0(X_T, \mathbb{O}_{X_T}^\times) \longrightarrow H^0(X_T, \omega_{X_T/T}) \quad (3-13)$$

that sends a global section u of \mathbb{O}_{X_T} to $u^{-1} \cdot d_T u$. Since pushforward by the base change f_{T*} of f is left exact, we know that $f_{T*}\mathbb{O}_{X_T}^\times$ is a subsheaf of $f_{T*}\mathbb{O}_{X_T}$. By

[Raynaud 1970, Théorème 7.2.1], the hypotheses on X ensure that f is cohomologically flat, so $f_{T*}\mathbb{O}_{X_T} \simeq \mathbb{O}_T$. Since d_T annihilates $\Gamma(T, \mathbb{O}_T)$, we conclude that the map (3-13) is zero.

We thus arrive at a short exact sequence of abelian groups

$$0 \longrightarrow H^0(X_T, \omega_{X_T/T}) \longrightarrow P_{X/S}^\natural(T) \longrightarrow P_{X/S}(T) \quad (3-14)$$

that is easily seen to be functorial in T . In order to construct the exact sequence of fppf abelian sheaves (3-12), we need to extend (3-14). To do this, we use Čech theory to interpret (3-14) as part of a long exact sequence of cohomology groups.

Consider the two-term complex (in degrees 0 and 1) $d_T \log : \mathbb{O}_{X_T}^\times \rightarrow \omega_{X_T/T}$ given by sending a section u of $\mathbb{O}_{X_T}^\times$ to $u^{-1} \cdot d_T u$; we will denote this complex by $\omega_{X_T/T}^{\times, \bullet}$. The evident short exact sequence of complexes

$$0 \longrightarrow \omega_{X_T/T}[-1] \longrightarrow \omega_{X_T/T}^{\times, \bullet} \longrightarrow \mathbb{O}_{X_T}^\times \longrightarrow 0$$

yields (since $d_T \log : H^0(X_T, \mathbb{O}_{X_T}^\times) \rightarrow H^0(X_T, \omega_{X_T/T})$ is the zero map) a long exact sequence in hypercohomology that is clearly functorial in T :

$$\begin{aligned} 0 \longrightarrow H^0(X_T, \omega_{X_T/T}) &\longrightarrow \mathbf{H}^1(X_T, \omega_{X_T/T}^{\times, \bullet}) \\ &\longrightarrow H^1(X_T, \mathbb{O}_{X_T}^\times) \xrightarrow{d_T \log} H^1(X_T, \omega_{X_T/T}). \end{aligned} \quad (3-15)$$

Lemma 3.10. *For affine T , the exact sequence (3-14) is identified with the first three terms of (3-15) in a manner that is functorial in T .*

Proof. By [EGA III 1961, Section 12.4, pp. 406–407], there is a natural identification of derived-functor (hyper)cohomology with Čech (hyper)cohomology which is δ -functorial in degrees 0 and 1. We thus have a natural identification of (3-15) with the corresponding exact sequence of Čech (hyper)cohomology groups, so it suffices to interpret (3-14) Čech-theoretically in a manner that is natural in T .

For (\mathcal{L}, ∇) representing a class in $P_{X/S}^\natural(T)$, let $\{U_i\}$ be a Zariski open cover of X_T that trivializes \mathcal{L} , and denote by $f_{ij} \in \Gamma(U_i \cap U_j, \mathbb{O}_{X_T}^\times)$ the transition functions. Because of the Leibnitz rule, $\nabla|_{U_i}$ is determined by a unique “connection form” $\omega_i \in \Gamma(U_i, \omega_{X_T/T})$, and the relation

$$\omega_i - \omega_j = f_{ij}^{-1} \cdot d_T f_{ij}$$

holds on $U_i \cap U_j$. We thus obtain a Čech 1-hypercocycle for the complex $\omega_{X_T/T}^{\times, \bullet}$:

$$(\{f_{ij}\}, \{\omega_i\}) \in C^1(\{U_i\}, \omega_{X_T/T}^{\times, \bullet}) := C^1(\{U_i \cap U_j\}, \mathbb{O}_{X_T}^\times) \oplus C^0(\{U_i\}, \omega_{X_T/T}).$$

It is straightforward to check that any two such trivializations over open covers $\{U_i\}$ and $\{V_j\}$ yield hyper 1-cocycles which differ by a hyper coboundary when viewed as hyper 1-cocycles for the common refining open cover $\{U_i \cap V_j\}$, and likewise that two different representatives of the same isomorphism class in $P_{X/S}^{\natural}(T)$ yield hyper 1-cocycles that differ by a hyper-coboundary (after passing to a common refining cover of the associated cocycles). We therefore obtain a well defined Čech hypercohomology class. This procedure is easily reversed, and so we have a bijection

$$P_{X/S}^{\natural}(T) \simeq \check{H}^1(X_T, \omega_{X_T/T}^{\times, \bullet}).$$

To check that this is in fact a homomorphism of abelian groups that is functorial in T is straightforward (albeit tedious).

We identify $P_{X/S}(T)$ with $\check{H}^1(X_T, \mathbb{O}_{X_T}^{\times})$ in the usual way, by sending a class represented by \mathcal{L} to the 1-cocycle $\{f_{ij}\}$ given by the transition functions associated to a trivializing open cover $\{U_i\}$ and choice of trivializations of $\mathcal{L}|_{U_i}$. Similarly, we use the natural isomorphism of $H^0(X_T, \omega_{X_T/T})$ with $\check{H}^0(X_T, \omega_{X_T/T})$, and we thus obtain a functorial diagram of homomorphisms of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X_T, \omega_{X_T/T}) & \longrightarrow & P_{X/S}^{\natural}(T) & \longrightarrow & P_{X/S}(T) \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & \check{H}^0(X_T, \omega_{X_T/T}) & \longrightarrow & \check{H}^1(X_T, \omega_{X_T/T}^{\times, \bullet}) & \longrightarrow & \check{H}^1(X_T, \mathbb{O}_{X_T}^{\times}) \end{array}$$

That this diagram commutes is easily verified by appealing to the explicit descriptions of the maps involved. □

By Raynaud’s *critère de platitude cohomologique* [Raynaud 1970, Théorème 7.2.1], our hypotheses X ensure that f is cohomologically flat in dimension zero. Thus, due to Corollary 3.4 and the fact that the formation of $\omega_{X/S}$ commutes with any base change on S , for each $i \geq 0$ the fppf sheaf associated to functor on S -schemes

$$T \rightsquigarrow H^i(X_T, \omega_{X_T/T})$$

is represented by the vector group $R^i f_* \omega_{X/S}$. By Lemma 3.10, we therefore have an exact sequence of fppf sheaves of abelian groups on the category of S -schemes whose first and last (nonzero) terms are smooth affine S -groups:

$$0 \longrightarrow f_* \omega_{X/S} \longrightarrow \text{Pic}_{X/S}^{\natural} \longrightarrow \text{Pic}_{X/S} \longrightarrow R^1 f_* \omega_{X/S}. \tag{3-16}$$

With (3-16) at hand, we can now prove Theorem 3.9.

Proof of Theorem 3.9. Consider the identity component $\text{Pic}_{X/S}^0$ and the composition of its inclusion into $\text{Pic}_{X/S}$ with the map of fppf sheaves $\text{Pic}_{X/S} \rightarrow R^1 f_* \omega_{X/S}$. We claim that this composite is the zero map. Indeed, by [Raynaud 1970, Exemples

6.1.6 and Théorème 8.2.1] (or [Bosch et al. 1990, Section 9.4, Theorem 2]) and [Bosch et al. 1990, Section 8.4, Proposition 2], our hypotheses on X ensure that $\text{Pic}_{X/S}^0$ is a smooth and separated S -scheme, so the composite map

$$\text{Pic}_{X/S}^0 \longrightarrow R^1 f_* \omega_{X/S} \tag{3-17}$$

is a map of S -group schemes. Since the generic fiber of $\text{Pic}_{X/S}^0$ is an abelian variety and $R^1 f_* \omega_{X/S}$ is affine over S , the closed kernel of (3-17) contains the generic fiber, and hence (3-17) is the zero map. Thus, the inclusion $\text{Pic}_{X/S}^0 \rightarrow \text{Pic}_{X/S}$ factors through the image of (3-11). By pullback, we obtain a short exact sequence (3-12) of fppf abelian sheaves on the category of S -schemes. As we have observed, the leftmost term in (3-12) is a vector group (in particular it is a smooth and affine S -group), and the rightmost term is a smooth and separated S -group scheme. It follows from this by fppf descent, as in the proof of Proposition 2.6, that $\text{Pic}_{X/S}^{h,0}$ is a smooth and separated S -group scheme, and that we have a short exact sequence (3-12) of smooth and separated group schemes over S . \square

4. Proof of the main theorem

In this section, we prove Theorem 1.2, following the outline sketched in the introduction (in particular, we will keep our notation from that section). Throughout this section, we fix a proper relative curve $f : X \rightarrow S$ over $S = \text{Spec } R$ which we suppose satisfies the hypotheses of Theorem 1.2. Note that these hypotheses ensure that Grothendieck’s pairing on component groups for J_K is perfect, by Proposition 2.8. In particular, there is a natural short exact sequence of smooth S -groups:

$$0 \longrightarrow \omega_J \longrightarrow \mathcal{E}xtrig_S(J, \mathbf{G}_m) \longrightarrow \widehat{J}^0 \longrightarrow 0 .$$

We begin our proof of Theorem 1.2 by constructing a canonical map of short exact sequences of smooth S -group schemes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega_J & \longrightarrow & \mathcal{E}xtrig_S(J, \mathbf{G}_m) & \longrightarrow & \widehat{J}^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & f_* \omega_{X/S} & \longrightarrow & \text{Pic}_{X/S}^{h,0} & \longrightarrow & \text{Pic}_{X/S}^0 \longrightarrow 0 \end{array} \tag{4-1}$$

which we do in three steps.

Step 1. We initially suppose there exists a rational point $x \in X_K(K)$ and will later explain how to reduce the general case to this one. Associated to x is the usual Albanese mapping $j_{x,K} : X_K \rightarrow J_K$ given by the functorial recipe

$$y \mapsto \mathbb{O}(y) \otimes \mathbb{O}(x)^{-1} .$$

Letting $i : X^{\text{sm}} \hookrightarrow X$ denote the S -smooth locus of $f : X \rightarrow S$ in X , we denote by $j_x : X^{\text{sm}} \rightarrow J$ the morphism obtained from $j_{x,K}$ by the Néron mapping property of J . By abuse of notation, we will also write j_x for any base change of j_x . For each smooth and affine S -scheme T , we will show that “pullback along j_x ” yields a commutative diagram of exact sequences of abelian groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma(T, \omega_{J_T}) & \longrightarrow & \text{Extrig}_T(J_T, \mathbf{G}_m) & \longrightarrow & \text{Ext}_T(J_T, \mathbf{G}_m) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Gamma(X_T, \omega_{X_T/T}) & \longrightarrow & P_{X/S}^{\natural}(T) & \longrightarrow & P_{X/S}(T)
 \end{array} \tag{4-2}$$

that is functorial in T . To do this, we will need to apply the following lemma with $U = X^{\text{sm}}$; that this choice of U satisfies the hypotheses of the lemma crucially uses our hypothesis that the closed fiber of X is generically smooth.

Lemma 4.1. *Let U be any open subscheme of X whose complement in X consists of points of codimension at least 2. For each smooth S -scheme T , pushforward along $i_T : U_T \rightarrow X_T$ yields a natural isomorphism of short exact sequences of abelian groups*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma(U_T, \Omega_{U_T/T}^1) & \longrightarrow & P_{U/S}^{\natural}(T) & \longrightarrow & P_{U/S}(T) . \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 0 & \longrightarrow & \Gamma(X_T, \omega_{X_T/T}) & \longrightarrow & P_{X/S}^{\natural}(T) & \longrightarrow & P_{X/S}(T)
 \end{array}$$

Proof. To minimize notation, we will simply write i for i_T . Since the dualizing sheaf is compatible with étale localization, it suffices to show that for any pair (\mathcal{L}, ∇) consisting of a line bundle \mathcal{L} on X_T with regular connection ∇ over T , the canonical commutative diagram

$$\begin{array}{ccc}
 \mathcal{L} & \longrightarrow & i_* i^* \mathcal{L} \\
 \nabla \downarrow & & \downarrow i_* i^*(\nabla) \\
 \mathcal{L} \otimes \omega_{X_T/T} & \longrightarrow & i_* i^* \mathcal{L} \otimes i_* i^* \omega_{X_T/T}
 \end{array} \tag{4-3}$$

has horizontal arrows that are isomorphisms. By hypothesis, X is normal and the complement of U in X consists of points of codimension at least two. Since $T \rightarrow S$ is smooth, the base change X_T is also normal and the complement of U_T in X_T has codimension at least 2 (see part (ii) of the corollary to Theorem 23.9 and Theorem 15.1 in [Matsumura 1989]). As \mathcal{L} is locally free, it follows that the top horizontal map of (4-3) is an isomorphism. By Lemma 3.2 the canonical map $\omega_{X/S} \rightarrow i_* i^* \omega_{X/S}$ is an isomorphism; since this map and the sheaves in question

are compatible with base change, we conclude that the bottom horizontal arrow in (4-3) is also an isomorphism. \square

Remark 4.2. Note that Lemma 4.1 is generally false if the complement of U in X has codimension strictly less than 2.

We deduce from Lemma 4.1 applied to $U := X^{\text{sm}}$ that it suffices to construct (4-2) with X replaced by U in the bottom row. Note that since U_T is T -smooth, the notions of regular connection and connection coincide (see Proposition 3.1). Thus, we wish to associate to any element of $\text{Extrig}(J_T, \mathbf{G}_m)$ an invertible sheaf on U_T with connection over T in a manner that is Zariski-local on (and functorial in) T , and so globalizes from the case of affine T . To do this, we proceed as follows.

Fix a choice τ of generator for $\omega_{\mathbf{G}_m}$ and (functorially) identify $\text{Extrig}_T(J_T, \mathbf{G}_m)$ with $\mathbf{E}_\tau(J_T)(T)$ via Lemma 2.1. Let (E, η) be a representative of a class in $\mathbf{E}_\tau(J_T)(T)$. Viewing E as a \mathbf{G}_m -torsor over J_T , we choose a Zariski open cover $\{V_i\}$ of J_T and local sections $s_i : V_i \rightarrow E$ to the projection $E \rightarrow J_T$ that trivialize E over V_i . Set $\omega_i := s_i^* \eta \in \Gamma(V_i, \Omega_{J_T/T}^1)$ and let \mathcal{L} be the invertible sheaf on J_T corresponding to the \mathbf{G}_m -torsor E . There are two canonical ways to associate transition functions to \mathcal{L} and the sections s_i depending on whether we consider the section $s_i - s_j : V_i \cap V_j \rightarrow \mathbf{G}_m$ or its inverse $s_j - s_i$. However, since any two choices of τ differ by multiplication by ± 1 , there is a *unique* choice $f_{ij} : V_i \cap V_j \rightarrow \mathbf{G}_m$ with the property that $f_{ij}^* \tau = f_{ij}^{-1} df_{ij}$ (interpreting f_{ij} as a section of \mathbf{G}_m over $V_i \cap V_j$), and we consistently make this choice of transition function.

Define

$$\nabla_i : \mathcal{L}|_{V_i} \longrightarrow \mathcal{L}|_{V_i} \otimes_{\mathbb{C}_{V_i}} \Omega_{V_i/T}^1$$

by $\nabla_i(ts_i) := ts_i \otimes \omega_i + s_i \otimes dt$ for any section t of \mathbb{C}_{V_i} . Using the definition of ω_i and the fact that η pulls back to τ on \mathbf{G}_m , it is straightforward to check that

$$\omega_i - \omega_j = f_{ij}^* \tau = f_{ij}^{-1} df_{ij}$$

(by our choice of f_{ij}) in $\Gamma(V_i \cap V_j, \Omega_{J_T/T}^1)$ and hence that the ∇_i uniquely glue to give a connection ∇ on \mathcal{L} over T . By passing to a common refining open cover, one checks that any other choice of trivialization (V'_i, s'_i) yields the same connection on \mathcal{L} , so the pair (\mathcal{L}, ∇) is independent of our choices of cover $\{V_i\}$ and sections $\{s_i\}$.

By pullback along $j_x : U_T \rightarrow J_T$, we thus obtain a line bundle on U_T with a connection. If (E', η') is another choice of representative for the same class in $\mathbf{E}_\tau(J_T)(T)$ then by definition there is an isomorphism of extensions $\varphi : E \rightarrow E'$ with the property that $\varphi^* \eta' = \eta$. One easily checks that φ induces an isomorphism of the invertible sheaves on J_T with connection corresponding to (E, η) and to (E', η') , and hence that we have a well defined map of sets $\mathbf{E}_\tau(J_T)(T) \rightarrow P_{U/S}^\natural(T)$ which is readily seen to be functorial in T . That this map is in fact a homomorphism of abelian groups follows easily from the definition using the description of the group

law on $\mathbf{E}_\tau(J_T)(T)$ as in Section 2. By Lemma 2.1, we thus obtain a homomorphism of abelian groups

$$\text{Extrig}_T(J_T, \mathbf{G}_m) \longrightarrow P_{U/S}^\natural(T) \tag{4-4}$$

that is functorial in T . It is straightforward to check that this map is moreover independent of our initial choice of τ (but may *a priori* depend on our choice of rational point x) and so provides the desired functorial map.

We similarly define $\text{Ext}_T(J_T, \mathbf{G}_m) \rightarrow P_{U/S}(T)$ by associating to an extension E of J_T by \mathbf{G}_m the pull back along $j_x : U_T \rightarrow J_T$ of the line bundle \mathcal{L} on J_T obtained by viewing E as a \mathbf{G}_m -torsor over J_T . This is readily seen to be a homomorphism of abelian groups (using Baer sum on $\text{Ext}_T(J_T, \mathbf{G}_m)$) and is obviously functorial in T .

Finally, we define $\Gamma(T, \omega_{J_T}) \rightarrow \Gamma(U_T, \Omega_{U_T/T}^1)$ as follows. By [Bosch et al. 1990, Section 4.1, Proposition 1], any global section ω_0 of $\omega_{J_T} = e_T^* \Omega_{J_T/T}^1$ can be uniquely propagated to an invariant differential form ω on J_T over T satisfying $e_T^* \omega = \omega_0$. Pulling ω back along $j_x : U_T \rightarrow J_T$, we obtain a section of $\Omega_{U_T/T}^1$ over U_T . This association is clearly a homomorphism and functorial in T .

We thus obtain (via Lemma 4.1) a diagram (4-2) with all maps homomorphisms of abelian groups, functorially in smooth affine S -schemes T . That this diagram commutes follows immediately from the explicit definition of all the maps involved (morally, each vertical map is simply “pullback by j_x ”).

Step 2. Passing from (4-2) to the corresponding diagram of associated fppf sheaves and recalling the construction of the exact sequence of fppf sheaves (3-16) in Section 3, we obtain a commutative diagram of fppf sheaves of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega_J & \longrightarrow & \mathcal{E}\text{xtrig}_S(J, \mathbf{G}_m) & \longrightarrow & \mathcal{E}\text{x}t_S(J, \mathbf{G}_m) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & f_*\omega_{X/S} & \longrightarrow & \text{Pic}_{X/S}^\natural & \longrightarrow & \text{Pic}_{X/S} \end{array}$$

From Proposition 2.8, we thus deduce the following commutative diagram of fppf abelian sheaves on smooth S -schemes with each term in the top row a smooth S -group:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega_J & \longrightarrow & \mathcal{E}\text{xtrig}_S(J, \mathbf{G}_m) & \longrightarrow & \widehat{J}^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & f_*\omega_{X/S} & \longrightarrow & \text{Pic}_{X/S}^\natural & \longrightarrow & \text{Pic}_{X/S} \end{array}$$

Since the map $\widehat{J}^0 \rightarrow \text{Pic}_{X/S}$ is homomorphism of group functors (on smooth S -schemes) and \widehat{J}^0 has connected fibers, for topological reasons this map necessarily

factors through the open subfunctor $\text{Pic}_{X/S}^0$ (thinking of $\text{Pic}_{X/S}^0$ as the union of all identity components of the fibers of $\text{Pic}_{X/S}$ [EGA IV₃ 1966, Corollaire 15.6.5, p. 238] and arguing fiber-by-fiber). By the definition of $\text{Pic}_{X/S}^{\text{h},0}$ (Definition 3.8) as a fiber product, we thus have a commutative diagram of fppf abelian sheaves on smooth S -schemes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \omega_J & \longrightarrow & \mathcal{E}xtrig_S(J, \mathbf{G}_m) & \longrightarrow & \widehat{J}^0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & f_*\omega_{X/S} & \longrightarrow & \text{Pic}_{X/S}^{\text{h},0} & \longrightarrow & \text{Pic}_{X/S}^0 \longrightarrow 0
 \end{array} \tag{4-5}$$

Step 3. By Proposition 2.6 and Theorem 3.9, both rows of (4-5) are short exact sequences of smooth S -group schemes, and we claim that the commutative diagram (4-5) of fppf sheaves on smooth S -schemes can be enhanced to a corresponding commutative diagram of maps between smooth group schemes over S . Indeed, this follows from Yoneda’s lemma, which ensures that the natural “restriction to the smooth site” map

$$\text{Hom}_S(\mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}_{S_{\text{sm}}}(\mathcal{F}, \mathcal{G})$$

is bijective for any fppf abelian sheaves \mathcal{F}, \mathcal{G} on S -schemes with \mathcal{F} represented by a smooth S -group scheme.

We have therefore constructed (4-1) using our initial choice of rational point $x \in X_K(K)$. If x' is any other choice of rational point, we claim that the resulting maps (4-1) obtained from x and x' coincide. Since $j_{x,K}, j_{x',K} : X_K \rightarrow J_K$ differ by a translation on J_K , it is enough to show that for any translation $\tau : J_K \rightarrow J_K$, the induced map

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \omega_J & \longrightarrow & \mathcal{E}xtrig_S(J, \mathbf{G}_m) & \longrightarrow & \widehat{J}^0 \longrightarrow 0 \\
 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 \\
 0 & \longrightarrow & \omega_J & \longrightarrow & \mathcal{E}xtrig_S(J, \mathbf{G}_m) & \longrightarrow & \widehat{J}^0 \longrightarrow 0
 \end{array} \tag{4-6}$$

(using the Néron mapping property) is the identity map. Since each term in the bottom row is separated and each term in the top row is flat, whether or not (4-6) coincides with the identity map may be checked on generic fibers. Now $\tau^* : \omega_{J_K} \rightarrow \omega_{J_K}$ is the identity map as ω_{J_K} is identified with the sheaf of (translation) invariant differentials [Bosch et al. 1990, Section 4.2 Proposition 1]. That $\tau^* : \widehat{J}_K \rightarrow \widehat{J}_K$ is the identity is well known, and follows from the fact that the line-bundles classified by $\widehat{J}_K := \text{Pic}_{J_K/K}^0$ are translation invariant (or equivalently

that the classes in $\mathcal{E}xt_K(J_K, \mathbf{G}_m)$ are primitive).⁵ Thus $\varphi_1 = \varphi_3 = \text{id}$ so on K -fibers, $\varphi_3 - \text{id}$ uniquely factors through a map $\widehat{J}_K \rightarrow \omega_{J_K}$ which takes the identity to the identity. As any map from an abelian variety to a vector group is constant, we conclude that $\varphi_3 - \text{id}$ is identically zero on K -fibers, and hence that $\varphi_3 = \text{id}$ as well. Thus, the map (4-1) which we have constructed is independent of the choice of rational point $x \in X_K(K)$.

In the general case when $X_K(K)$ may be empty, we proceed as follows. Denote by Y any one of the three schemes occurring in the top row of (4-1) and by Z the corresponding scheme in the bottom row. We first claim that we have a canonical map $Y_K \rightarrow Z_K$. Indeed, X_K has a K' -rational point for some finite Galois extension K' of K , and we may use this point to define a K' -map $Y_{K'} \rightarrow Z_{K'}$ as we have explained. Since this map is independent of the choice of K' -rational point by what we have said above, via Galois descent we have a canonical K -map $\varphi_K : Y_K \rightarrow Z_K$ as claimed.

We now appeal to the following general lemma.

Lemma 4.3. *Fix an integral scheme T with generic point η and let $Y \rightarrow T$ and $Z \rightarrow T$ be any flat T -schemes, with Z separated over T . Suppose given a map $\varphi_\eta : Y_\eta \rightarrow Z_\eta$. Then there is at most one extension of φ_η to a T -map $\varphi : Y \rightarrow Z$, and φ exists if and only if the schematic closure in $Y \times_T Z$ of the graph of φ_η maps isomorphically onto Y by the first projection. In particular, φ exists if and only if there is an fpqc morphism $T' \rightarrow T$ and a map $\varphi' : Y_{T'} \rightarrow Z_{T'}$ with $\varphi'_{\eta'} = \varphi_\eta$ where $\eta' = T' \times_T \eta$.*

Proof. The proof of Lemma 4.3 proceeds via standard arguments with schematic closures of graphs; due to lack of a reference, we sketch how this goes. The uniqueness of an extension is clear, as T is integral, Z is separated over T , and Y is T -flat. For existence, we proceed as follows. Let $\Gamma \subseteq Y \times_T Z$ be the schematic closure in $Y \times_T Z$ of the graph $\Gamma_{\varphi_\eta} \subseteq Y_\eta \times_\eta Z_\eta$ of φ_η , and note that $\Gamma_\eta = \Gamma_{\varphi_\eta}$ as Z is T -separated. Now if the first projection $\Gamma \rightarrow Y$ is an isomorphism then it is clear that φ_η extends to a T -morphism. Conversely, given $\varphi : Y \rightarrow Z$ extending φ_η and denoting by Γ_φ the graph of φ , we claim that necessarily $\Gamma = \Gamma_\varphi$. Indeed,

⁵More precisely, for any abelian variety A over K we have a homomorphism of group functors

$$\phi : \text{Pic}_{A/K} \rightarrow \mathbf{Hom}(A, \text{Pic}_{A/K}^0)$$

given functorially on K -schemes T by sending a line bundle \mathcal{L} on A_T to the map $x \mapsto \tau_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$ with τ_x translation by a T -point x of A_T . Since A and $\text{Pic}_{A/K}^0$ are projective, Grothendieck's theory of Hom-schemes ensures that $\mathbf{Hom}(A, \text{Pic}_{A/K}^0)$ is a finite-type K -scheme which we claim is étale. Working over \overline{K} , our claim follows from the fact that there are no nonzero liftings to $K[\epsilon]$ of the zero map $A \rightarrow \text{Pic}_{A/K}^0$ (due to [Mumford et al. 1994, Theorem 6.1]), so the tangent space of $\mathbf{Hom}(A, \text{Pic}_{A/K}^0)$ at the origin is zero. Again passing to \overline{K} , we conclude that the group map ϕ maps connected components of $\text{Pic}_{A/K}$ to individual points, so in particular restricts to the zero map on the connected component of the identity $\text{Pic}_{X/K}^0$.

the canonical closed immersion $\Gamma \rightarrow Y \times_T Z$ factors through a closed immersion $\Gamma \rightarrow \Gamma_\varphi$ as Γ_φ is closed in $Y \times_T Z$ (due to T -separatedness of Z) and contains Γ_{φ_η} . Since the closed immersion $\Gamma \rightarrow \Gamma_\varphi$ is an isomorphism over η (using that $\Gamma_\eta \simeq \Gamma_{\varphi_\eta}$) it must be an isomorphism, since Γ_{φ_η} is dense in Γ_φ as Γ_φ is T -flat and T is integral. We conclude that $\Gamma = \Gamma_\varphi$ maps isomorphically onto Y via the first projection. Finally, whether or not the first projection $\Gamma \rightarrow Y$ is an isomorphism is insensitive to fpqc base change; since the formation of Γ commutes with such base change (as $\eta \rightarrow T$ is quasicompact and separated), we deduce the last statement of the lemma. \square

Applying the lemma with $T = S = \text{Spec}(R)$, Y, Z as above, and $T' = \text{Spec}(R^{\text{sh}})$ for R^{sh} a strict henselization of R , we see that it remains to construct a T' -morphism $Y_{T'} \rightarrow Z_{T'}$ recovering the base change of φ_K to $K^{\text{sh}} := \text{Frac}(R^{\text{sh}})$ on generic fibers. Now X has generically smooth closed fiber, so X_K has a K^{sh} -point. As our hypotheses on X are unaltered by base change along local-étale extensions of discrete valuation rings (such as $R \rightarrow R^{\text{sh}}$) and the formation of the top and bottom rows of (4-1) commute with such base change we may use this K^{sh} -point as in the construction of (4-1) to define the desired T' -map $Y_{T'} \rightarrow Z_{T'}$. We conclude that φ_K uniquely extends to an S -map, and thus we obtain (4-1) over S , as desired.

Now that we have constructed the canonical map of short exact sequences of smooth S -groups (4-1), we can show that it is an isomorphism. We reiterate here that only the *construction* of this map uses the hypothesis that the closed fiber of X is generically smooth; as we will see below, the proof that (4-1) is an isomorphism requires only the weaker hypotheses of Raynaud’s Theorem 1.1.

Proof of Theorem 1.2. By passing to a finite étale extension if necessary, we may assume that $X_K(K)$ is nonempty, and we select $x \in X_K(K)$ and use it to define (4-1). Note that since X is normal with generically smooth closed fiber, X satisfies the hypotheses of Theorem 1.1.

Consider the composite mapping

$$\text{Pic}_{X/S}^0 \longrightarrow J^0 \xrightarrow{\simeq} \widehat{J}^0, \tag{4-7}$$

where the first map is deduced via the Néron mapping property from the canonical identification $J_K = \text{Pic}_{X_K/K}^0$ and the second map is similarly obtained from the canonical principal polarization $J_K \rightarrow \widehat{J}_K$ induced by the Θ -divisor [Milne 1986b, Section 6]. We claim that the composite $\widehat{J}^0 \rightarrow \widehat{J}^0$ of (4-7) with the right vertical map of (4-1) coincides with negation on \widehat{J}^0 . Since \widehat{J}^0 is flat and separated, it suffices to check this claim on generic fibers, so we wish to show that the map $\text{Pic}^0(j_{x,K}) : \widehat{J}_K \rightarrow J_K$ is the negative of the inverse of the canonical principal polarization $J_K \rightarrow \widehat{J}_K$. This is the content of [Milne 1986b, Lemma 6.9]. It follows from Theorem 1.1 that the right vertical map of (4-1) is an isomorphism

if and only if X has rational singularities; in particular, this settles the “only if” direction of [Theorem 1.2](#).

We henceforth suppose that X has rational singularities and we wish to show that (4-1) is an isomorphism of exact sequences of smooth group schemes over S . By [Theorem 1.1](#) and our discussion, we know that the right vertical map of (4-1) is an isomorphism, and we will “bootstrap” Raynaud’s theorem using duality; more precisely, we will show that the left vertical map in (4-1) is dual to the map on Lie algebras obtained from (1-2) and must therefore be an isomorphism as well. Indeed, consider the dual of the map on Lie algebras obtained from (1-2):

$$\mathcal{L}ie(J^0)^\vee \xrightarrow{\cong} \mathcal{L}ie(\text{Pic}_{X/S}^0)^\vee .$$

For any commutative group functor G over S with representable fibers, the canonical inclusion $G^0 \hookrightarrow G$ induces an isomorphism on Lie algebras [[Liu et al. 2004](#), Proposition 1.1(d)], so we obtain a natural isomorphism of \mathbb{O}_S -modules $\mathcal{L}ie(J)^\vee \simeq \mathcal{L}ie(\text{Pic}_{X/S})^\vee$. The canonical identification $R^1f_*\mathbb{O}_X \simeq \mathcal{L}ie(\text{Pic}_{X/S})$ ([[Bosch et al. 1990](#), Section 8.4, Theorem 1] or [[Liu et al. 2004](#), Proposition 1.3(b)]) then gives a natural isomorphism

$$\mathcal{L}ie(J)^\vee \xrightarrow{\cong} (R^1f_*\mathbb{O}_X)^\vee . \tag{4-8}$$

Using the canonical duality $\omega_J \simeq \mathcal{L}ie(J)^\vee$ (see [[SGA3-1 1970](#), II, Section 4.11] or [[Liu et al. 2004](#), Proposition 1.1(b)]) and Grothendieck duality ([Proposition 3.3](#)) yields a natural isomorphism of \mathbb{O}_S -modules

$$\omega_J \xrightarrow{\cong} \mathcal{L}ie(J)^\vee \xrightarrow[(4-8)]{\cong} (R^1f_*\mathbb{O}_X)^\vee \xleftarrow[(3-7)]{\cong} f_*\omega_{X/S} \tag{4-9}$$

and hence an isomorphism of the corresponding vector groups over S . We claim that the left vertical map in (4-1) coincides with the negative of (4-9). Since the source of both maps is flat and the target is separated over S , it suffices to check such agreement on generic fibers.

To do this, we consider the following diagram, in which we simply write j for $j_{x,K}$ and $\varphi : J_K \rightarrow \widehat{J}_K$ for the canonical principal polarization:

$$\begin{array}{ccccccc}
 \Gamma(\text{Spec } K, \omega_{J_K}) & \xrightarrow{\text{Pic}^0(j)^*} & \Gamma(\text{Spec } K, \omega_{\widehat{J}_K}) & \xrightarrow[\text{ev}]{\cong} & \text{Lie}(\widehat{J}_K)^\vee & \xrightarrow[\text{Lie}(-\varphi)^\vee]{\cong} & \text{Lie}(J_K)^\vee \\
 \text{can} \downarrow \cong & & \text{can} \downarrow \cong & & \theta_{J_K}^\vee \downarrow \cong & & \theta_{X_K}^\vee \downarrow \cong \\
 H^0(J_K, \Omega_{J_K/K}^1) & \xleftarrow[-\varphi^*]{\cong} & H^0(\widehat{J}_K, \Omega_{\widehat{J}_K/K}^1) & \xrightarrow[\psi_{J_K}^0]{\cong} & H^1(J_K, \mathbb{O}_{J_K}) & \xleftarrow[H^1(j)^\vee]{\cong} & H^1(X_K, \mathbb{O}_{X_K})^\vee \\
 & \searrow^{j^*} & & & & \nearrow_{(3-7)} & \\
 & & H^0(X_K, \Omega_{X_K}^1) & & & &
 \end{array} \tag{4-10}$$

Here, $\psi_{J_K}^0$ is the usual duality (defined using the Künneth formula and the first Chern class of the Poincaré bundle [Berthelot et al. 1982, 5.1.3 and Lemme 5.1.4]; the map ev is the canonical evaluation pairing, and can is the canonical map obtained by extending sections of ω_{J_K} to invariant differential forms on J_K [Bosch et al. 1990, Section 4.2, Propositions 1 and 2]. We claim that each of the three small squares in (4-10) commute, and that the bottom “sector” anticommutes. For the first square, this follows from the fact that the composite $\text{Pic}^0(j) \circ (-\varphi) : J \rightarrow J$ is the identity map [Milne 1986b, Lemma 6.9], together with the fact that the canonical map can is functorial. The same reasoning shows that the third square commutes, using the functoriality of the identification θ_K [Liu et al. 2004, Proposition 1.3(c)]. The commutativity of the middle square follows immediately from 5.1.1 and the proof of Théorème 5.1.6 in [Berthelot et al. 1982]. That the bottom sector region anticommutes is the content of [Coleman 1998, Theorem 5.1]. Note, as a particular consequence of these commutativity statements, that every map occurring in (4-10) is an isomorphism.

Using the functoriality in J_K of the canonical duality $\omega_{J_K} \simeq \text{Lie}(J_K)$ [Liu et al. 2004, Proposition 1.1(b)] and the agreement of $-\varphi^{-1}$ with $\text{Pic}^0(j) : \widehat{J}_K \rightarrow J$, as above, we conclude that the composite isomorphism $\Gamma(\text{Spec } K, \omega_{J_K}) \rightarrow \text{Lie}(J_K)^\vee$ along the top row of (4-10) is the canonical evaluation duality for J_K . Thus, on generic fibers, the map (4-9) is none other than the map induced by the top, right, and bottom-right edges in (4-10). But by definition, the left vertical map of (4-1) coincides with the composite of the left and bottom-left edges of (4-10) on generic fibers, and is thus the negative of (4-9), as claimed.

Now that we know that the left and right vertical maps in (4-1) are isomorphisms when X has rational singularities, it follows that the same is true of all three vertical maps, as desired. \square

Remark 4.4. That (4-9) coincides with the left vertical map in (4-1) over generic fibers is essentially Theorem B.4.1 of [Conrad 2000]. We have chosen here to present a different proof because [Conrad 2000, Theorem B.4.1] rests upon knowing a priori that the natural pullback map $\Omega_{J_K/K}^1 \rightarrow j_* \Omega_{\widehat{X}_K/K}^1$ is an isomorphism, while we prefer to deduce this fact as a corollary of our main result.

5. Comparison of integral structures

In this section, we use Theorem 1.2 to prove a comparison result for integral structures in de Rham cohomology. As usual, we fix a discrete valuation ring R with field of fractions K .

Let A_K be an abelian variety over K . It is well known that the Lie algebra of the universal extension $E(\widehat{A}_K)$ of the dual abelian variety \widehat{A}_K is naturally isomorphic to the first de Rham cohomology of A_K over K , compatibly with Hodge filtrations.

Proposition 5.1. *There is a canonical isomorphism of short exact sequences of finite dimensional K -vector spaces*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Lie}(\omega_{A_K}) & \longrightarrow & \text{Lie}(E(\widehat{A}_K)) & \longrightarrow & \text{Lie}(\widehat{A}_K) \longrightarrow 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 0 & \longrightarrow & H^0(A_K, \Omega_{A_K/K}^1) & \longrightarrow & H_{\text{dR}}^1(A_K/K) & \longrightarrow & H^1(A, \mathbb{O}_A) \longrightarrow 0
 \end{array}$$

Proof. See [Mazur and Messing 1974, I, Section 4]. □

Note that since ω_{A_K} is a vector group, we have a canonical identification of $\text{Lie}(\omega_{A_K})$ with the global sections of ω_{A_K} . We deduce from Proposition 5.1 and Proposition 2.3 the following corollary, which equips the de Rham cohomology of A_K with a canonical integral structure.

Corollary 5.2. *Let A and \widehat{A} be the Néron models over R of A_K and \widehat{A}_K , respectively, and let $\mathcal{E}(\widehat{A})$ be the canonical extension of \widehat{A} (Definition 2.4). The sequence of Lie algebras*

$$0 \longrightarrow \text{Lie}(\omega_A) \longrightarrow \text{Lie}(\mathcal{E}(\widehat{A})) \longrightarrow \text{Lie}(\widehat{A}) \longrightarrow 0 \tag{5-1}$$

associated to the canonical extension (2-6) of \widehat{A} over R is a short exact sequence of finite free R -modules that is contravariantly functorial in K -morphisms of abelian varieties $A_K \rightarrow B_K$ over K and recovers the 3-term Hodge filtration of $H_{\text{dR}}^1(A_K/K)$ after extending scalars to K . That is, (5-1) provides a canonical integral structure on the 3-term Hodge filtration of $H_{\text{dR}}^1(A_K/K)$.

Proof. Each term in (2-6) is a smooth S -scheme; in particular the map

$$\mathcal{E}\text{trig}_S(A^0, \mathbf{G}_m) \rightarrow \widehat{A}$$

is smooth [SGA3-1 1970, Exposé VI_B, Proposition 9.2 vii]. Thus, by [Liu et al. 2004, Proposition 1.1(c)], applying the left exact functor Lie to (2-6) yields a short exact sequence of finite R -modules which are free by smoothness. Since any homomorphism of Néron models $A \rightarrow B$ induces a map on identity components $A^0 \rightarrow B^0$, it follows from the Néron mapping property and the functoriality of the canonical extension (2-6) that (5-1) is contravariantly functorial in K -morphisms of abelian varieties $A_K \rightarrow B_K$ over K . Since the formation of Lie algebras commutes with the scalar extension $R \rightarrow K$, we deduce from Proposition 5.1 and the fact that K -fiber of (2-6) is the universal extension of \widehat{A}_K by a vector group (see Remarks 2.5) that (5-1) recovers the Hodge filtration of $H_{\text{dR}}^1(A_K/K)$ after extending scalars to K . □

Remark 5.3. Supposing that Grothendieck’s pairing (2-7) is right nondegenerate, so $\mathcal{E}\text{trig}_S(A, \mathbf{G}_m)$ is a smooth and separated S -scheme by Proposition 2.6, the

natural map of short exact sequences (2-10) induces an isomorphism on associated exact sequences of Lie algebras by Lemma 2.10 and [Liu et al. 2004, Proposition 1.1(d)].

For the remainder of this section, we suppose that $A_K = J_K$ is the Jacobian of a smooth proper and geometrically connected curve X_K over K . Recall that the 3-term Hodge filtration

$$0 \longrightarrow H^0(X_K, \Omega_{X_K/K}^1) \longrightarrow H_{\text{dR}}^1(X_K/K) \longrightarrow H^1(X_K, \mathbb{O}_{X_K}) \longrightarrow 0 \quad (5-2)$$

is autodual with respect to the cup product pairing on $H_{\text{dR}}^1(X_K/K)$ and is contravariantly and covariantly functorial in finite morphisms of smooth and proper curves $g : X_K \rightarrow X'_K$ via pullback g^* and pushforward g_* of differentials. Attached to any such morphism, we have associated homomorphisms of abelian varieties

$$\text{Pic}^0(g) : J'_K \rightarrow J_K \quad \text{and} \quad \text{Alb}(g) : J_K \rightarrow J'_K$$

by Picard and Albanese functoriality (where J'_K is the Jacobian of X'_K). The following proposition is well known, but we have been unable to find a proof in the literature so we include one here for the convenience of the reader.

Proposition 5.4. *There is a canonical isomorphism of short exact sequences of K -vector spaces*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(J_K, \Omega_{J_K/K}^1) & \longrightarrow & H_{\text{dR}}^1(J_K/K) & \longrightarrow & H^1(J_K, \mathbb{O}_{J_K}) \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & H^0(X_K, \Omega_{X_K/K}^1) & \longrightarrow & H_{\text{dR}}^1(X_K/K) & \longrightarrow & H^1(X_K, \mathbb{O}_{X_K}) \longrightarrow 0 \end{array} \quad (5-3)$$

This isomorphism respects the autodualities of the top and bottom rows. Furthermore, for any finite morphism $g : X_K \rightarrow X'_K$, the map (5-3) intertwines $\text{Alb}(g)^$ with g^* and $\text{Pic}^0(g)^*$ with g_* .*

Proof. We first suppose that $X(K)$ is nonempty and select $x_0 \in X(K)$. Let $j : X_K \rightarrow J_K$ be the associated Albanese morphism. By pullback along j , we obtain a morphism on de Rham cohomology that yields a commutative diagram (5-3). Clearly this map commutes with extension of K (using the same x_0) and we claim that it is independent of our choice x_0 . Each term in the Hodge filtration of $H_{\text{dR}}^1(J_K/K)$ is clearly (the global sections of) a vector group over K ; denoting any one of them by V it suffices to show that the natural map $J_K \rightarrow \mathbf{Aut}_K(V)$ given by translations is the zero map. Since the target is affine of finite type over K and the source is an abelian variety, this map factors through a section of the target and must therefore be identically zero, as claimed. It follows from Galois descent that we have a canonical map (5-3) even when $X(K)$ is empty.

Let us denote by $H(J_K)$ (respectively $H(X_K)$) the 3-term exact sequence of K -vector spaces given by the top (respectively bottom) row of (5-3). By passing to an extension of K if need be, we may suppose that $X_K(K)$ is nonempty and that (5-3) is given by pullback along an Albanese morphism $j : X_K \rightarrow J_K$ associated to some $x_0 \in X_K(K)$. To show that (5-3) is an isomorphism, we will exploit the natural autodualities on $H(J_K)$ and $H(X_K)$. For this to be successful, it is essential to know that these dualities are compatible.

Lemma 5.5. *The canonical autodualities of the short exact sequences $H(J_K)$ and $H(X_K)$ are compatible via j^* . That is, the following diagram commutes:*

$$\begin{array}{ccc} H(J_K) & \xleftarrow{\cong} & H(J_K)^\vee \\ j^* \downarrow & & \uparrow (j^*)^\vee \\ H(X_K) & \xrightarrow{\cong} & H(X_K)^\vee \end{array}$$

Proof. This is Theorem 5.1 of [Coleman 1998]. □

Continuing with the proof of Proposition 5.4, observe that the functoriality of the canonical identification $H^1(X_K, \mathcal{O}_{X_K}) \simeq \text{Lie}(\text{Pic}_{X_K/K}^0)$ yields a commutative diagram

$$\begin{array}{ccc} H^1(J_K, \mathcal{O}_{J_K}) & \xrightarrow{\cong} & \text{Lie}(\text{Pic}_{J_K/K}^0) \\ j^* \downarrow & & \downarrow \text{Lie}(\text{Pic}^0(j)) \\ H^1(X_K, \mathcal{O}_{X_K}) & \xrightarrow{\cong} & \text{Lie}(\text{Pic}_{X_K/K}^0) \end{array}$$

(see [Liu et al. 2004, Proposition 1.3(c)]). Due to [Milne 1986b, Lemma 6.9], the map $\text{Pic}^0(j) : \widehat{J}_K \rightarrow J_K$ is the negative of the inverse of the canonical principal polarization $J_K \rightarrow \widehat{J}_K$, so in particular it is an isomorphism. Thus, the map $j^* : H^1(J_K, \mathcal{O}_{J_K}) \rightarrow H^1(X_K, \mathcal{O}_{X_K})$ is an isomorphism. Taking K -linear duals and using the autoduality of $H(J_K)$ and $H(X_K)$, it follows from Lemma 5.5 that the map $j^* : H^0(J_K, \omega_{J_K}) \rightarrow H^0(X_K, \Omega_{X_K/K}^1)$ is also an isomorphism. We conclude that all three vertical maps of (5-3) are isomorphisms, as desired.

It remains to check our claims concerning the functoriality of (5-3) in finite morphisms of smooth proper and geometrically connected curves $g : X_K \rightarrow X'_K$. Denote by J'_K the Jacobian of X'_K and by $j' : X'_K \rightarrow J'_K$ the Albanese map attached to $g(x_0)$. Albanese functoriality gives a commutative diagram

$$\begin{array}{ccc} X_K & \xrightarrow{g} & X'_K \\ j \downarrow & & \downarrow j' \\ J_K & \xrightarrow{\text{Alb}(g)} & J'_K \end{array} \tag{5-4}$$

from which we easily obtain the commutative diagram of short exact sequences

$$\begin{array}{ccc}
 H(J'_K) & \xrightarrow{\text{Alb}(g)^*} & H(J_K) \\
 j'^* \downarrow & & \downarrow j^* \\
 H(X'_K) & \xrightarrow{g^*} & H(X_K)
 \end{array} \tag{5-5}$$

which shows that (5-3) intertwines the maps g^* and $\text{Alb}(g)^*$. Dualizing (5-5) and using Lemma 5.5 gives a commutative diagram

$$\begin{array}{ccccccc}
 H(J_K) & \xleftarrow{\cong} & H(J_K)^\vee & \xrightarrow{(\text{Alb}(g)^*)^\vee} & H(J'_K)^\vee & \xrightarrow{\cong} & H(J'_K) \\
 j^* \downarrow & & \uparrow (j^*)^\vee & & \uparrow (j'^*)^\vee & & \downarrow j'^* \\
 H(X_K) & \xrightarrow{\cong} & H(X_K)^\vee & \xrightarrow{(g^*)^\vee} & H(X'_K)^\vee & \xleftarrow{\cong} & H(X'_K)
 \end{array} \tag{5-6}$$

By [Cais 2009, Theorem 5.11(3)], the maps g^* and g_* are adjoint with respect to the cup-product pairing on $H_{\text{dR}}^1(X_K/K)$, so the composite map on the bottom row of (5-6) coincides with g_* . We claim that $\text{Alb}(g)^*$ and $\text{Pic}^0(g)^*$ are adjoint with respect to the pairing on $H_{\text{dR}}^1(J_K/K)$, so the top row of (5-6) coincides with $\text{Pic}^0(g)^*$. By definition, the duality pairing on $H_{\text{dR}}^1(J_K/K)$ is deduced from the natural perfect pairing

$$H_{\text{dR}}^1(J_K/K) \times H_{\text{dR}}^1(\widehat{J}_K/K) \longrightarrow K \tag{5-7}$$

(defined as in [Berthelot et al. 1982, Section 5]) by identifying the de Rham cohomology of \widehat{J}_K with that of J_K via the principal polarization $J_K \simeq \widehat{J}_K$. Now if $u : J'_K \rightarrow J_K$ is any morphism with dual $\widehat{u} : \widehat{J}_K \rightarrow \widehat{J}'_K$, then the induced maps u^* and \widehat{u}^* on de Rham cohomology are adjoint with respect to (5-7) by [Berthelot et al. 1982, 5.1.3.3]. Applying this to $u = \text{Pic}^0(g)$, our claim that $\text{Alb}(g)^*$ and $\text{Pic}^0(g)^*$ are adjoint then follows from the assertion that the composite map

$$J_K \xrightarrow[\varphi]{\cong} \widehat{J}_K \xrightarrow{\widehat{\text{Pic}^0(g)}} \widehat{J}'_K \xleftarrow[\varphi']{\cong} J'_K$$

coincides with $\text{Alb}(g)$, where φ and φ' are the canonical principal polarizations. But this follows by applying the functor Pic^0 to the diagram (5-4) and using the fact that $\text{Pic}^0(j)$ and $\text{Pic}^0(j')$ coincide with $-\varphi^{-1}$ and $-\varphi'^{-1}$, respectively, thanks to [Milne 1986b, Lemma 6.9]. \square

Fix a proper flat and normal model $f : X \rightarrow S$ of X_K over $S = \text{Spec } R$, and denote by $\omega_{X/S}^\bullet$ the two-term \mathbb{O}_S -linear complex of \mathbb{O}_X -modules $d : \mathbb{O}_X \rightarrow \omega_{X/S}$

furnished by [Proposition 3.1](#). We will say that X is an *admissible model* of X_K if X has rational singularities and f is cohomologically flat in dimension zero.

Define $H^1(X/R) := \mathbf{H}^1(X, \omega_{X/S}^\bullet)$. When X is cohomologically flat, there is a natural short exact sequence of finite free R -modules

$$0 \longrightarrow H^0(X, \omega_{X/S}) \longrightarrow H^1(X/R) \longrightarrow H^1(X, \mathbb{C}_X) \longrightarrow 0 \quad (5-8)$$

whose scalar extension to K is identified with the 3-term Hodge filtration exact sequence $H(X_K)$. Moreover, (5-8) is self-dual with respect to the usual cup-product autoduality of $H(X_K)$; see Proposition 5.8 of [[Cais 2009](#)]. By Theorem 5.11 of the same paper, when X is admissible, the integral structure provided by (5-8) is canonical: this short exact sequence is independent of the choice of admissible model X of X_K and is both contravariantly and covariantly functorial via pullback and trace in finite K -morphisms $X_K \rightarrow X'_K$ of curves having admissible models over R .

Via [Corollary 5.2](#) and the identification of Hodge filtrations (5-3), when X is admissible we thus have two canonical integral structures on the de Rham cohomology of X_K , and it is natural to ask how these R -lattices compare.

Corollary 5.6. *With the notation and hypotheses of [Theorem 1.2](#), when X has rational singularities there is a canonical isomorphism of short exact sequences of finite free R -modules*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Lie}(\omega_J) & \longrightarrow & \text{Lie}(\mathcal{E}(\hat{J})) & \longrightarrow & \text{Lie}(\hat{J}) \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & H^0(X, \omega_{X/S}) & \longrightarrow & H^1(X/R) & \longrightarrow & H^1(X, \mathbb{C}_X) \longrightarrow 0 \end{array} \quad (5-9)$$

that recovers the identification (5-3) after extending scalars to K .

Remark 5.7. Let $g : X_K \rightarrow X'_K$ be any finite map of smooth and geometrically connected curves over K and suppose that X_K and X'_K admit proper flat and normal models over R which have rational singularities and generically smooth closed fibers. (Such models are automatically admissible due to Raynaud’s *critère de platitude cohomologique* [[Raynaud 1970](#), Théorème 7.2.1]). By our discussion above, g induces maps g^* and g_* on the canonical integral structure (5-8) via pullback and trace, and induces maps $\text{Alb}(g)^*$ and $\text{Pic}^0(g)^*$ on the canonical integral structure (5-1) by Albanese and Picard functoriality via the Néron mapping property. The R -isomorphism (5-9) necessarily intertwines $\text{Alb}(g)^*$ with g^* and $\text{Pic}^0(g)^*$ with g_* as such agreement of maps between free R -modules may be checked after the flat scalar extension $R \rightarrow K$, where it follows from [Proposition 5.4](#).

Question 5.8. As an interesting consequence of [Corollary 5.6](#), the duality statement of [Proposition 5.4](#), and the fact that the integral structure (5-8) is autodual with respect to cup-product pairing, we deduce that the autoduality of the Hodge

filtration of $H_{\text{dR}}^1(J_K/K)$ preserves the integral structure (5-1). It seems natural to ask if this is true in greater generality, that is, if for any abelian variety A_K over K , the natural duality isomorphism

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^1(\widehat{A}_K, \mathbb{C}_{\widehat{A}_K})^\vee & \longrightarrow & H_{\text{dR}}^1(\widehat{A}_K/K)^\vee & \longrightarrow & H^0(\widehat{A}_K, \Omega_{\widehat{A}_K}^1) \longrightarrow 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 0 & \longrightarrow & H^0(A_K, \Omega_{A_K/K}^1) & \longrightarrow & H_{\text{dR}}^1(A_K/K) & \longrightarrow & H^1(A, \mathbb{C}_A) \longrightarrow 0
 \end{array}$$

(see [Berthelot et al. 1982, Lemme 5.1.4 and Théorème 5.1.6]) identifies the corresponding canonical integral structures provided by (5-1). It is also natural to wonder how such an identification might come about at the level of canonical extensions and Néron models or more precisely if the definition of the duality for the de Rham cohomology of an abelian scheme (or more generally a 1-motive) in terms of its universal extension (see [Coleman 1991, page 636] for the case of abelian schemes, [Bertapelle 2008] for 1-motives and [Deligne 1974, 10.2.7.2] for abelian varieties over \mathbb{C}) can be extended to the case of Néron models and their canonical extensions.

Proof of Corollary 5.6. By Theorem 1.2, we have an isomorphism of short exact sequences of smooth groups as in (1-3). Applying the functor Lie and using the fact that for any group functor G over S with representable fibers, the inclusion $G^0 \hookrightarrow G$ of the identity component induces an isomorphism on Lie algebras [Liu et al. 2004, Proposition 1.1 (d)], we deduce a canonical isomorphism of finite free R -modules

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Lie}(\omega_J) & \longrightarrow & \text{Lie}(\mathcal{E}(\widehat{J})) & \longrightarrow & \text{Lie}(\widehat{J}) \longrightarrow 0 . \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 0 & \longrightarrow & \text{Lie}(\omega_{X/S}) & \longrightarrow & \text{Lie}(\text{Pic}_{X/S}^{\natural,0}) & \longrightarrow & \text{Lie}(\text{Pic}_{X/S}) \longrightarrow 0
 \end{array}$$

By Definition 3.8 we have the canonical identifications

$$\begin{aligned}
 \text{Lie}(\text{Pic}_{X/S}^{\natural,0}) &= \text{Lie}(\text{Pic}_{X/S}^{\natural} \times_{\text{Pic}_{X/S}} \text{Pic}_{X/S}^0) \\
 &= \text{Lie}(\text{Pic}_{X/S}^{\natural}) \times_{\text{Lie}(\text{Pic}_{X/S})} \text{Lie}(\text{Pic}_{X/S}^0) = \text{Lie}(\text{Pic}_{X/S}^{\natural}),
 \end{aligned}$$

so it suffices to identify the left exact sequence of Lie algebras attached to the exact sequence of fppf abelian sheaves

$$0 \longrightarrow f_*\omega_{X/S} \longrightarrow \text{Pic}_{X/S}^{\natural} \longrightarrow \text{Pic}_{X/S}, \tag{5-10}$$

of (3-16) with the integral structure on $H(X_K)$ provided by (5-8). As in Section 3, let $\omega_{X_T/T}^{\times,\bullet}$ be the two-term complex $d_T \log : \mathbb{C}_{X_T}^{\times} \rightarrow \omega_{X_T/T}$ defined by $d_T \log(u) =$

$u^{-1} \cdot d_T(u)$, and write $\mathbf{R}^1 f_* \omega_{X/S}^\times$ and $R^1 f_* \mathbb{O}_X^\times$, respectively, for the fppf sheaves associated to the group functors on S -schemes

$$T \rightsquigarrow \mathbf{H}^1(X_T, \omega_{X_T/T}^{\times, \bullet}) \quad \text{and} \quad T \rightsquigarrow H^1(X_T, \mathbb{O}_{X_T}^\times).$$

By [Lemma 3.10](#), the exact sequence (5-10) is naturally isomorphic to the exact sequence of fppf abelian sheaves

$$0 \longrightarrow f_* \omega_{X/S} \longrightarrow \mathbf{R}^1 f_* \omega_{X/S}^{\times, \bullet} \longrightarrow R^1 f_* \mathbb{O}_X^\times$$

obtained by sheafifying (3-15). Thus, the proof of [Corollary 5.6](#) is completed by:

Lemma 5.9. *There is a natural isomorphism of exact sequences of free R -modules*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X, \omega_{X/S}) & \longrightarrow & H^1(X) & \longrightarrow & H^1(X, \mathbb{O}_X) \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & \text{Lie}(f_* \omega_{X/S}) & \longrightarrow & \text{Lie}(\mathbf{R}^1 f_* \omega_{X/S}^{\times, \bullet}) & \longrightarrow & \text{Lie}(R^1 f_* \mathbb{O}_X^\times) \end{array} \quad (5-11)$$

Proof. By construction, the exact sequence (5-8) results from the Hodge to de Rham spectral sequence attached to the evident filtration of $\omega_{X/S}^\bullet$. Now the canonical section $\mathbf{Z} \rightarrow \mathbf{Z}[\epsilon]/(\epsilon^2)$ to the quotient map $\epsilon \mapsto 0$ induces a canonically split exact sequence of filtered two-term (vertical) complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{O}_X & \xrightarrow{h \mapsto 1 + \epsilon h} & \mathbb{O}_{X_{S[\epsilon]}}^\times & \xrightleftharpoons{\epsilon \mapsto 0} & \mathbb{O}_X^\times \longrightarrow 1 \\ & & \downarrow d & & \downarrow d \log & & \downarrow d \log \\ 0 & \longrightarrow & \omega_{X/S} & \xrightarrow{\cdot \epsilon} & \omega_{X_{S[\epsilon]}/S[\epsilon]} & \xrightleftharpoons{\epsilon \mapsto 0} & \omega_{X/S} \longrightarrow 0 \end{array}$$

so passing to cohomology yields the commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(X, \omega_{X/S}) & \longrightarrow & H^0(X_{S[\epsilon]}, \omega_{X_{S[\epsilon]}/S[\epsilon]}) & \longrightarrow & H^0(X, \omega_{X/S}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{H}^1(X, \omega_{X/S}^\bullet) & \longrightarrow & \mathbf{H}^1(X_{S[\epsilon]}, \omega_{X_{S[\epsilon]}/S[\epsilon]}^{\times, \bullet}) & \longrightarrow & \mathbf{H}^1(X, \omega_{X/S}^\bullet) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(X, \mathbb{O}_X) & \longrightarrow & H^1(X_{S[\epsilon]}, \mathbb{O}_{X_{S[\epsilon]}}^\times) & \longrightarrow & H^1(X, \mathbb{O}_X^\times) \end{array} \quad (5-12)$$

with exact rows and columns, where the zeroes in the left column result from the splitting (that is, \mathbf{H}^1 is left exact on *split* short exact sequences). We conclude

that we have an isomorphism of exact sequences of abelian groups as in (5-11). It remains to show that this is in fact an R -linear isomorphism. Recall that for any group functor G on S -schemes, the multiplication on $\mathrm{Lie}(G)$ by $\mathbb{O}_S(S)$ is induced by the functoriality of G from the map $\mathbb{O}_S(S) \rightarrow \mathrm{End}_S(S[\epsilon])$ sending $s \in \mathbb{O}_S(S)$ to the self-map u_s of $S[\epsilon]$ that is induced by $\epsilon \mapsto s \cdot \epsilon$. Thus, the fact that the map (5-11) defined by (5-12) is a map of R -modules amounts to the assertion that for any $s \in \mathbb{O}_S(S)$ the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \omega_{X/S}^\bullet & \longrightarrow & \omega_{X_{S[\epsilon]}/S[\epsilon]}^{\times, \bullet} & \longrightarrow & \omega_{X/S}^{\times, \bullet} \longrightarrow 0 \\
 & & \downarrow \cdot s & & \downarrow u_s^* & & \downarrow \mathrm{id} \\
 0 & \longrightarrow & \omega_{X/S}^\bullet & \longrightarrow & \omega_{X_{S[\epsilon]}/S[\epsilon]}^{\times, \bullet} & \longrightarrow & \omega_{X/S}^{\times, \bullet} \longrightarrow 0
 \end{array} \tag{5-13}$$

of filtered complexes with exact rows commutes. This is easily checked. \square

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