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We study the topology of the prime spectrum of an algebra supporting a rational torus action. More precisely, we study inclusions between prime ideals that are torus-invariant using the H -stratification theory of Goodearl and Letzter on the one hand, and the theory of deleting derivations of Cauchon on the other. We also give a formula for the dimensions of the H -strata described by Goodearl and Letzter. We apply the results obtained to the algebra of $m \times n$ generic quantum matrices to show that the dimensions of the H -strata are bounded above by the minimum of m and n , and that all values between 0 and this bound are achieved.

1. Introduction

We denote by $R = \mathbb{C}_q(M_{m,n})$ the standard quantization of the ring of regular functions on $m \times n$ matrices with entries in a field \mathbb{K} ; it is the \mathbb{K} -algebra generated by the $m \times n$ indeterminates $Y_{i,\alpha}$, $1 \leq i \leq m$ and $1 \leq \alpha \leq n$, subject to the relations

$$\begin{aligned} Y_{i,\beta}Y_{i,\alpha} &= q^{-1}Y_{i,\alpha}Y_{i,\beta}, & \alpha < \beta, \\ Y_{j,\alpha}Y_{i,\alpha} &= q^{-1}Y_{i,\alpha}Y_{j,\alpha}, & i < j, \\ Y_{j,\beta}Y_{i,\alpha} &= Y_{i,\alpha}Y_{j,\beta}, & i < j, \alpha > \beta, \\ Y_{j,\beta}Y_{i,\alpha} &= Y_{i,\alpha}Y_{j,\beta} - (q - q^{-1})Y_{i,\beta}Y_{j,\alpha}, & i < j, \alpha < \beta, \end{aligned}$$

where $q \in \mathbb{K}^*$ is not a root of unity. We note that the torus $(\mathbb{K}^*)^{m+n}$ acts on R by \mathbb{K} -algebra automorphisms via the action

$$(a_1, \dots, a_m, b_1, \dots, b_n) \cdot Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad \text{for } 1 \leq i \leq m, 1 \leq \alpha \leq n.$$

Understanding this torus action has been responsible for most of the important advances that have been made in the study of quantum matrices. The most important object of study is the prime spectrum of R . In analogy with algebraic

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geometry, where great understanding of commutative rings comes from the study of their prime spectra, one seeks to understand the prime spectrum of R and its topology. The noncommutativity introduced by the parameter q in quantum matrices makes the prime spectrum of R harder to understand than the prime spectrum of the coordinate ring of the variety of $m \times n$ matrices and much work has been done in understanding the structure of this topological object. The most important of these advances is the stratification theory of Goodearl and Letzter [2000].

To describe the work of those authors, we give a few basic definitions. Let A be a \mathbb{K} -algebra with a group H acting on it by \mathbb{K} -algebra automorphisms. A two-sided ideal I of A is said to be H -invariant if $h \cdot I = I$ for all $h \in H$. An H -prime ideal of A is a proper H -invariant ideal J of A such that whenever J contains the product of two H -invariant ideals of A , J contains at least one of them. We denote by $H\text{-Spec}(A)$ the set of all H -prime ideals of A . Observe that if P is a prime ideal of A then

$$(P : H) := \bigcap_{h \in H} h \cdot P \quad (1.1)$$

is an H -prime ideal of A . This observation allowed Goodearl and Letzter [2000] (see also [Brown and Goodearl 2002]) to construct a stratification of the prime spectrum of A that is indexed by the H -spectrum. Indeed, let J be an H -prime ideal of A . We denote by $\text{Spec}_J(A)$ the H -stratum associated to J ; that is,

$$\text{Spec}_J(A) = \{P \in \text{Spec}(A) \mid (P : H) = J\}. \quad (1.2)$$

Then the H -strata of $\text{Spec}(A)$ form a partition of $\text{Spec}(A)$ [Brown and Goodearl 2002, Chapter II.2]; that is,

$$\text{Spec}(A) = \bigsqcup_{J \in H\text{-Spec}(A)} \text{Spec}_J(A). \quad (1.3)$$

This partition is the so-called H -stratification of $\text{Spec}(A)$.

When the H -spectrum of A is finite this partition is a powerful tool in the study of the prime spectrum of A .

As we work in the generic case where q is not a root of unity, the ring R of $m \times n$ quantum matrices has a finite $\mathcal{H} = (\mathbb{K}^*)^{m+n}$ -spectrum. Remarkably, for each \mathcal{H} -prime J , the space $\text{Spec}_J(R)$ is homeomorphic to $\text{Spec}(\mathbb{K}[z_1^{\pm 1}, \dots, z_d^{\pm 1}])$ for some d which depends on J . This d is simply the (Krull) dimension of the \mathcal{H} -stratum $\text{Spec}_J(R)$.

The work of Goodearl and Letzter spurred much research into the structure of $\text{Spec}(R)$ in terms of the \mathcal{H} -spectrum. Some of the main themes in the study of the \mathcal{H} -spectrum have been computing its size, computing the structure of the poset of \mathcal{H} -primes under inclusion, and computing the dimensions of the \mathcal{H} -strata and how they are distributed.

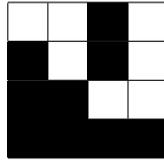


Figure 1. An example of a 4×4 Cauchon diagram.

The question of the size of the \mathcal{H} -spectrum of R was answered in [Cauchon 2003b]. For many years the finiteness of the \mathcal{H} -spectrum of R was known, but no formula for its size was known — except for small values of m and n — due to the complicated nature of the relations in R . Cauchon used his theory of deleting derivations to compute the size of the \mathcal{H} -spectrum of R . In particular, the set of \mathcal{H} -primes is in one-to-one correspondence with a set of combinatorial objects called Cauchon diagrams. In fact, Cauchon’s method applies to a much broader class of algebras, the so-called CGL (Cauchon–Goodearl–Letzter) extensions, and the term “Cauchon diagram” has now acquired a more general meaning than the one we now describe for quantum matrices.

Definition. An $m \times n$ Cauchon diagram C is simply an $m \times n$ grid consisting of mn boxes in which certain boxes are coloured black. We require that if a box is black, then either every box strictly to its left is black or every box strictly above it is black. We let $\mathcal{C}_{m,n}$ denote the collection of $m \times n$ Cauchon diagrams.

Cauchon showed that the \mathcal{H} -primes of the ring R of $m \times n$ quantum matrices are parameterized by the collection of $m \times n$ Cauchon diagrams, and he also gave a closed formula for the size of this set. Moreover it is known that the poset $\mathcal{H}\text{-Spec}(R)$ (under inclusion) is isomorphic to a subposet of the symmetric group S_{m+n} endowed with the Bruhat order [Launois 2007].

Some of the major questions that remain are to determine the possible dimensions of \mathcal{H} -strata that can occur in R and to give a formula for the dimension of a stratum in terms of the associated Cauchon diagram. We answer these questions. In particular, we prove the following result.

Theorem 1.1. *Let m and n be natural numbers. Then the dimensions of \mathcal{H} -strata in $\mathbb{C}_q(M_{m,n})$ are all at most $\min(m, n)$; moreover, for each $d \in \{0, 1, \dots, \min(m, n)\}$ there exists a d -dimensional \mathcal{H} -stratum.*

Previously, the best known bound for the dimensions of \mathcal{H} -strata in $\mathbb{C}_q(M_{m,n})$ was $m + n - 1$, so this result represents a significant improvement.

Regarding the dimension of the stratum associated with a given Cauchon diagram, we give a formula which only relies on the Cauchon diagram; see Proposition 4.2. In fact, we are able to give a formula for a much broader class of algebras,

called *uniparameter CGL extensions*. This class of algebras includes in particular the so-called quantum Schubert cells $U_q[w]$ defined by De Concini, Kac and Procesi (see Section 3C). The algebra $U_q[w]$ supports a rational torus action and the theory of Cauchon and Goodearl–Letzter can be applied to this algebra. The torus-invariant primes of this algebra have been studied recently and independently in [Cauchon and Mériaux 2009] and [Yakimov 2009]. As a consequence of our formula, we are able to give a formula for the dimension of the (0)-stratum of $U_q[w]$ which only depends on the Weyl group element w .

Regarding the \mathcal{H} -strata in $\mathbb{O}_q(M_{m,n})$, we show even more. We say that an $m \times n$ Cauchon diagram C contains another $m \times n$ Cauchon diagram C' if whenever a square is coloured black in C' , the corresponding square is also coloured black in C . After additional investigations we are able to prove the following result.

Theorem 1.2. *Let P be an \mathcal{H} -prime of $\mathbb{O}_q(M_{m,n})$ whose associated \mathcal{H} -stratum is d -dimensional. Then there exists a chain*

$$P = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d$$

of \mathcal{H} -primes such that the dimension of the \mathcal{H} -stratum associated to P_i is $d - i$ and such that

$$C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_d,$$

where C_i is the Cauchon diagram associated to P_i .

To prove this chain result, we need to understand the relation between inclusion of Cauchon diagrams and inclusion of the corresponding H -primes. One might naively expect these two posets to be isomorphic, but this is not the case. For instance, consider the algebra of 2×2 quantum matrices $\mathbb{O}_q(M_2)$ generated by four indeterminates $Y_{1,1}, Y_{1,2}, Y_{2,1}, Y_{2,2}$, subject to the relations given in the beginning of this section. It is well known that the ideal $(Y_{1,1}Y_{2,2} - qY_{1,2}Y_{2,1})$ generated by the quantum determinant and $(Y_{2,1}, Y_{2,2})$ are \mathcal{H} -invariant prime ideals in $\mathbb{O}_q(M_2)$. Clearly, $(Y_{1,1}Y_{2,2} - qY_{1,2}Y_{2,1}) \subsetneq (Y_{2,1}, Y_{2,2})$, but the corresponding Cauchon diagrams, which can be represented by the pictures in Figure 2, are not comparable.

Thus two \mathcal{H} -primes can be comparable (for the inclusion) and yet their corresponding Cauchon diagrams may fail to be comparable. Interestingly, if we consider things from the other direction, we see there is a definite relation between these two posets.



Figure 2. Cauchon diagrams representing the ideal generated by the quantum determinant and by $Y_{2,1}, Y_{2,2}$ respectively in $\mathbb{O}_q(M_2)$.

Theorem 1.3. *If C and C' are two $m \times n$ Cauchon diagrams with $C \subsetneq C'$, then $J_C \subsetneq J_{C'}$, where J_C and $J_{C'}$ denote respectively the \mathcal{H} -primes of $\mathbb{O}_q(M_{m,n})$ associated to C and C' .*

Again, we are able to prove Theorem 1.3 for a much broader class of algebras, called CGL extensions. This class of algebras contains quantum affine spaces, the algebra of $m \times n$ quantum matrices, positive parts of quantized enveloping algebras, and many other interesting families of algebras. These algebras support a rational action by a torus H and have the property that they have only finitely many H -strata in the Goodearl–Letzter stratification.

The deleting derivations theory of Cauchon applies to CGL extensions and this gives new insights into the H -stratification in these cases. The H -primes are, just as in the case of quantum matrices, in one-to-one correspondence with combinatorial objects called Cauchon diagrams. These diagrams depend on the algebra, and a CGL extension other than quantum matrices has a different collection of Cauchon diagrams than the ones described earlier for $m \times n$ quantum matrices. Cauchon's original description [2003b] was just for quantum matrices, but other authors have since applied his deleting derivations theory to other classes of algebras. In [Launois et al. 2008] we gave a description of Cauchon diagrams for the quantum Grassmannian. Mériaux [2010] gave a description of these diagrams for the positive part of the quantized enveloping algebra of a simple Lie algebra, while Cauchon and Mériaux [2009] have recently described Cauchon diagrams in quantum Schubert cells. The set of Cauchon diagrams has a natural poset structure under inclusion; likewise the set of H -primes can be viewed as a poset under inclusion.

In this broader context, we are able to show the following result, of which Theorem 1.3 is a special case:

Theorem 1.4. *If w and w' are two Cauchon diagrams of a CGL extension R with $w \subsetneq w'$, then $J_w \subsetneq J_{w'}$, where J_w and $J_{w'}$ denote respectively the (unique) H -primes associated to w and w' .*

The outline of this paper is as follows. In Section 2 we give the necessary background on CGL extensions and we prove Theorem 1.4. In Section 3, we give a formula for the dimension of a stratum in a uniparameter CGL extension. Then we use this formula to compute the dimension of the (0)-stratum in a quantum Schubert cell. In Section 4 we describe the results obtained in the previous sections in the particular case of quantum matrices. Then we use these results in order to prove Theorems 1.1 and 1.2. In Section 5, after having seen the possible values that can occur as the dimension of an \mathcal{H} -stratum in quantum matrices, we give a conjecture about the number of d -dimensional \mathcal{H} -strata in $\mathbb{O}_q(M_{m,n})$.

Throughout this paper, we use the following conventions.

- (i) If I is a finite set, $|I|$ denotes its cardinality.
- (ii) $\llbracket a, b \rrbracket := \{i \in \mathbb{N} \mid a \leq i \leq b\}$.
- (iii) \mathbb{K} denotes a field and we set $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.
- (iv) If A is a \mathbb{K} -algebra, then $\text{Spec}(A)$ and $\text{Prim}(A)$ denote respectively its prime and primitive spectra.

2. H -primes in CGL extensions

In this section, we recall the notion of CGL extensions that was introduced in [Launois et al. 2006]. Examples include various quantum algebras in the generic case such as quantum affine spaces, quantum matrices, the positive part of quantized enveloping algebras of semisimple complex Lie algebras, etc. As we will see, the advantage of this class of algebras is that one can use both the stratification theory of Goodearl and Letzter and the theory of deleting derivations of Cauchon in order to study their prime and primitive spectra. This will allow us to investigate the topology of the H -spectrum of such algebras and prove Theorem 1.4.

2A. CGL extensions and the H -stratification theory of Goodearl and Letzter. In this subsection, N denotes a positive integer and R is an iterated Ore extension; that is,

$$R = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_N; \sigma_N, \delta_N], \quad (2.4)$$

where σ_j is an automorphism of the \mathbb{K} -algebra

$$R_{j-1} := \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}]$$

and δ_j is a \mathbb{K} -linear σ_j -derivation of R_{j-1} for all $j \in \llbracket 2, N \rrbracket$. In other words, R is a skew polynomial ring whose multiplication is defined by

$$X_j a = \sigma_j(a) X_j + \delta_j(a)$$

for all $j \in \llbracket 2, N \rrbracket$ and $a \in R_{j-1}$. Thus R is a noetherian domain. Henceforth, we assume that R is a CGL extension:

Definition [Launois et al. 2006]. An iterated Ore extension R is said to be a CGL extension if

- (1) for all $j \in \llbracket 2, N \rrbracket$, δ_j is locally nilpotent;
- (2) for all $j \in \llbracket 2, N \rrbracket$, there exists $q_j \in \mathbb{K}^*$ such that $\sigma_j \circ \delta_j = q_j \delta_j \circ \sigma_j$ and, for all $i \in \llbracket 1, j-1 \rrbracket$, there exists $\lambda_{j,i} \in \mathbb{K}^*$ such that $\sigma_j(X_i) = \lambda_{j,i} X_i$;
- (3) none of the q_j ($2 \leq j \leq N$) is a root of unity; and
- (4) there exists a torus $H = (\mathbb{K}^*)^d$ that acts rationally by \mathbb{K} -automorphisms on R such that

- X_1, \dots, X_N are H -eigenvectors,
- the set $\{\lambda \in \mathbb{K}^* \mid (\exists h \in H)(h \cdot X_1 = \lambda X_1)\}$ is infinite, and
- for all $j \in \llbracket 2, N \rrbracket$, there exists $h_j \in H$ such that $h_j \cdot X_i = \lambda_{j,i} X_i$ if $1 \leq i < j$ and $h_j \cdot X_j = q_j X_j$.

It follows from [Goodearl and Letzter 2000] that every H -prime ideal of R is completely prime, so $H\text{-Spec}(R)$ coincides with the set of H -invariant completely prime ideals of R . Moreover there are at most 2^N H -prime ideals in R . As a corollary, the H -stratification (1.3) breaks down the prime spectrum of R into a finite number of parts, the H -strata. The geometric nature of the H -strata is well known: each H -stratum is homeomorphic to the scheme of irreducible subvarieties of a \mathbb{K} -torus [Brown and Goodearl 2002, Theorems II.2.13 and II.6.4]. However, the dimensions of these schemes are unknown in general.

2B. Quantum affine spaces. Let N be a positive integer and let $\Lambda = (\Lambda_{i,j}) \in M_N(\mathbb{K}^*)$ be a multiplicatively antisymmetric matrix; that is, $\Lambda_{i,j}\Lambda_{j,i} = \Lambda_{i,i} = 1$ for all $i, j \in \llbracket 1, N \rrbracket$. The quantum affine space associated to Λ is denoted by $\mathbb{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}_\Lambda[T_1, \dots, T_N]$; this is the \mathbb{K} -algebra generated by N indeterminates T_1, \dots, T_N subject to the relations $T_j T_i = \Lambda_{j,i} T_i T_j$ for all $i, j \in \llbracket 1, N \rrbracket$. It is well known that $\mathbb{O}_\Lambda(\mathbb{K}^N)$ is an iterated Ore extension:

$$\mathbb{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}[T_1][T_2; \sigma_2] \cdots [T_N; \sigma_N],$$

where σ_j is the automorphism defined by $\sigma_j(T_i) = \Lambda_{j,i} T_i$ for all $1 \leq i < j \leq N$. Observe that the torus $H = (\mathbb{K}^*)^N$ acts by automorphisms on $\mathbb{O}_\Lambda(\mathbb{K}^N)$ via:

$$(a_1, \dots, a_N) \cdot T_i = a_i T_i \text{ for all } i \in \llbracket 1, N \rrbracket \text{ and } (a_1, \dots, a_N) \in H.$$

Moreover, it is well known (see for instance [Launois et al. 2006, Corollary 3.8]) that $\mathbb{O}_\Lambda(\mathbb{K}^N)$ is a CGL extension with this action of H . Hence $\mathbb{O}_\Lambda(\mathbb{K}^N)$ has at most 2^N H -prime ideals and they are all completely prime.

The H -stratification of $\text{Spec}(\mathbb{O}_\Lambda(\mathbb{K}^N))$ was entirely described in [Brown and Goodearl 1996] when the group $\langle \Lambda_{i,j} \rangle$ is torsion-free and in [Goodearl and Letzter 1998] in the general case. We now recall these results.

Let W denote the set of subsets of $\llbracket 1, N \rrbracket$. If $w \in W$, then we denote by K_w the (two-sided) ideal of $\mathbb{O}_\Lambda(\mathbb{K}^N)$ generated by the indeterminates T_i with $i \in w$. It is easy to check that K_w is an H -invariant completely prime ideal of $\mathbb{O}_\Lambda(\mathbb{K}^N)$.

Proposition 2.1 [Goodearl and Letzter 1998, Proposition 2.11].

(1) *The ideals K_w with $w \in W$ are exactly the H -prime ideals of $\mathbb{O}_\Lambda(\mathbb{K}^N)$. Hence there are exactly 2^N H -prime ideals in this case.*

(2) *For all $w \in W$, the H -stratum associated to K_w is given by*

$$\text{Spec}_{K_w}(\mathbb{O}_\Lambda(\mathbb{K}^N)) = \{P \in \text{Spec}(\mathbb{O}_\Lambda(\mathbb{K}^N)) \mid P \cap \{T_i \mid i \in \llbracket 1, N \rrbracket\} = \{T_i \mid i \in w\}\}.$$

2C. The canonical partition of $\text{Spec}(R)$. Let R denote a CGL extension, as in Section 2A. Following [Cauchon 2003a], we construct the canonical partition of $\text{Spec}(R)$, which gives new insight into the H -stratification of $\text{Spec}(R)$. *All references in this subsection, through page 184, are to the article just mentioned.*

To describe the prime spectrum of R , Cauchon introduced the *deleting derivations algorithm* (Section 3.2). This algorithm constructs, for each $j \in \llbracket N + 1, 2 \rrbracket$, a family $\{X_1^{(j)}, \dots, X_N^{(j)}\}$ of elements of the division ring of fractions $\text{Fract}(R)$ of R defined as follows:

- (1) When $j = N + 1$, we set $(X_1^{(N+1)}, \dots, X_N^{(N+1)}) = (X_1, \dots, X_N)$.
- (2) Assume that $j < N + 1$ and that the $X_i^{(j+1)}$ ($i \in \llbracket 1, N \rrbracket$) are already constructed. Then it follows from Cauchon's Théorème 3.2.1 that $X_j^{(j+1)} \neq 0$ and that, for each $i \in \llbracket 1, N \rrbracket$, we have

$$X_i^{(j)} = \begin{cases} X_i^{(j+1)} & \text{if } i \geq j, \\ \sum_{k=0}^{+\infty} \frac{(1 - q_j)^{-k}}{[k]!_{q_j}} \delta_j^k \circ \sigma_j^{-k} (X_i^{(j+1)}) (X_j^{(j+1)})^{-k} & \text{if } i < j, \end{cases}$$

where $[k]!_{q_j} = [0]_{q_j} \times \dots \times [k]_{q_j}$ with $[0]_{q_j} = 1$ and $[i]_{q_j} = 1 + q_j + \dots + q_j^{i-1}$ when $i \geq 1$.

Fix $j \in \llbracket 2, N + 1 \rrbracket$. We denote by $R^{(j)}$ the subalgebra of $\text{Fract}(R)$ generated by the $X_i^{(j)}$:

$$R^{(j)} := \mathbb{K}\langle X_1^{(j)}, \dots, X_N^{(j)} \rangle.$$

The following results are Cauchon's Théorème 3.2.1 and Lemme 4.2.1.

- (1) $R^{(j)}$ is isomorphic to an iterated Ore extension of the form

$$\mathbb{K}[Y_1] \dots [Y_{j-1}; \sigma_{j-1}, \delta_{j-1}] [Y_j; \tau_j] \dots [Y_N; \tau_N]$$

by an isomorphism that sends $X_i^{(j)}$ to Y_i ($1 \leq i \leq N$), where τ_j, \dots, τ_N denote the \mathbb{K} -linear automorphisms such that $\tau_l(Y_i) = \lambda_{l,i} Y_i$ ($1 \leq i \leq l$).

- (2) Assume that $j \neq N + 1$ and set $S_j := \{(X_j^{(j+1)})^n : n \in \mathbb{N}\} = \{(X_j^{(j)})^n : n \in \mathbb{N}\}$. This is a multiplicative system of regular elements of $R^{(j)}$ and $R^{(j+1)}$, that satisfies the Ore condition in $R^{(j)}$ and $R^{(j+1)}$. Moreover

$$R^{(j)} S_j^{-1} = R^{(j+1)} S_j^{-1}.$$

It follows from these results that $R^{(j)}$ is a noetherian domain.

Notation. We set $\bar{R} := R^{(2)}$ and $T_i := X_i^{(2)}$ for all $i \in \llbracket 1, N \rrbracket$.

It follows from Cauchon's Proposition 3.2.1 that \bar{R} is a quantum affine space in the indeterminates T_1, \dots, T_N ; hence his expression "effacement des dérivations".

More precisely, let $\Lambda = (\mu_{i,j}) \in M_N(\mathbb{K}^*)$ be the multiplicatively antisymmetric matrix whose entries are defined by

$$\mu_{j,i} = \begin{cases} \lambda_{j,i} & \text{if } i < j, \\ 1 & \text{if } i = j, \\ \lambda_{i,j}^{-1} & \text{if } i > j, \end{cases}$$

where the $\lambda_{j,i}$ with $i < j$ come from the CGL extension structure of R (see page 180 for the definition). Then

$$\bar{R} = \mathbb{K}_\Lambda[T_1, \dots, T_N] = \mathbb{O}_\Lambda(\mathbb{K}^N). \tag{2.5}$$

The deleting derivations algorithm is used to relate the prime spectrum of a CGL extension R to the prime spectrum of the associated quantum affine space \bar{R} . More precisely, Cauchon used this algorithm to construct embeddings

$$\varphi_j : \text{Spec}(R^{(j+1)}) \longrightarrow \text{Spec}(R^{(j)}) \quad \text{for } j \in \llbracket 2, N \rrbracket, \tag{2.6}$$

as follows (Section 4.3; see especially his Lemme 4.3.2).

Let $P \in \text{Spec}(R^{(j+1)})$. Then

$$\varphi_j(P) = \begin{cases} PS_j^{-1} \cap R^{(j)} & \text{if } X_j^{(j+1)} \notin P, \\ g_j^{-1}(P/(X_j^{(j+1)})) & \text{if } X_j^{(j+1)} \in P, \end{cases}$$

where g_j denotes the surjective homomorphism

$$g_j : R^{(j)} \rightarrow R^{(j+1)}/(X_j^{(j+1)})$$

defined by

$$g_j(X_i^{(j)}) := X_i^{(j+1)} + (X_j^{(j+1)}).$$

Cauchon's Proposition 4.3.1 states that φ_j induces an increasing homeomorphism from the topological space

$$\{P \in \text{Spec}(R^{(j+1)}) \mid X_j^{(j+1)} \notin P\} \quad \text{onto} \quad \{Q \in \text{Spec}(R^{(j)}) \mid X_j^{(j)} \notin Q\},$$

whose inverse is also an increasing homeomorphism; also, φ_j induces an increasing homeomorphism from

$$\{P \in \text{Spec}(R^{(j+1)}) \mid X_j^{(j+1)} \in P\}$$

onto its image by φ_j , whose inverse similarly is an increasing homeomorphism. Note however that, in general, φ_j is not an homeomorphism from $\text{Spec}(R^{(j+1)})$ onto its image.

Composing these embeddings, we get an embedding

$$\varphi := \varphi_2 \circ \dots \circ \varphi_N : \text{Spec}(R) \longrightarrow \text{Spec}(\bar{R}), \tag{2.7}$$

which is called the *canonical embedding* from $\text{Spec}(R)$ into $\text{Spec}(\bar{R})$. This canonical embedding allows the construction of a partition of $\text{Spec}(R)$ as follows.

We keep the notation of the previous sections. In particular, W still denotes the set of all subsets of $\llbracket 1, N \rrbracket$. If $w \in W$, then we set

$$\text{Spec}_w(R) := \varphi^{-1}(\text{Spec}_{K_w}(\bar{R})).$$

Moreover, we denote by W' the set of those $w \in W$ such that $\text{Spec}_w(R) \neq \emptyset$. The elements of W' are called the *Cauchon diagrams* of the CGL extension R . It follows from Cauchon's Proposition 4.4.1 that

$$\text{Spec}(R) = \bigsqcup_{w \in W'} \text{Spec}_w(R) \text{ and } |W'| \leq |W| = 2^N.$$

This partition is called the *canonical partition* of $\text{Spec}(R)$, and it gives another way to understand the H -stratification, since, by Cauchon's Théorème 5.5.2, these two partitions coincide. As a consequence, we obtain another description of the H -prime ideals of R :

Proposition 2.2 [Cauchon 2003a, Lemme 5.5.8 and Théorème 5.5.2].

- (1) Let $w \in W'$. There exists a (unique) H -invariant (completely) prime ideal J_w of R such that $\varphi(J_w) = K_w$, where K_w denotes the ideal of \bar{R} generated by the T_i with $i \in w$.
- (2) $H\text{-Spec}(R) = \{J_w \mid w \in W'\}$.
- (3) $\text{Spec}_{J_w}(R) = \text{Spec}_w(R)$ for all $w \in W'$.

2D. The map $w \mapsto J_w$ is increasing. In this section, we prove Theorem 1.4.

Theorem 2.3. Let R be a CGL extension and let $w, w' \in W'$ be two Cauchon diagrams of R . If $w \subseteq w'$ then $J_w \subseteq J_{w'}$.

Proof. For $j \in \llbracket 2, N + 1 \rrbracket$ and $P \in \text{Spec}(R)$, we set $P^{(j)} := \varphi_j \circ \dots \circ \varphi_N(P)$. We prove by induction on j that

$$J_w^{(j)} \subseteq J_{w'}^{(j)} \quad \text{for } j \in \llbracket 2, N + 1 \rrbracket.$$

When $j = 2$, we have $J_w^{(2)} = K_w$ and $J_{w'}^{(2)} = K_{w'}$. As $w \subseteq w'$, we have $K_w \subseteq K_{w'}$, so $J_w^{(2)} \subseteq J_{w'}^{(2)}$, as desired.

We assume $j \leq N$ and $J_w^{(j)} \subseteq J_{w'}^{(j)}$. We need to prove that $J_w^{(j+1)} \subseteq J_{w'}^{(j+1)}$. Observe first that if $X_j^{(j)} \notin J_{w'}^{(j)}$, then

$$J_w^{(j)}, J_{w'}^{(j)} \in \{Q \in \text{Spec}(R^{(j)}) \mid X_j^{(j)} \notin Q\}.$$

As φ_j induces an increasing homeomorphism, still denoted φ_j , from

$$\{P \in \text{Spec}(R^{(j+1)}) \mid X_j^{(j+1)} \notin P\}$$

onto

$$\{Q \in \text{Spec}(R^{(j)}) \mid X_j^{(j)} \notin Q\},$$

whose inverse is also an increasing homeomorphism, we obtain

$$J_w^{(j+1)} = \varphi_j^{-1}(J_w^{(j)}) \subseteq \varphi_j^{-1}(J_{w'}^{(j)}) = J_{w'}^{(j+1)},$$

as desired.

Similarly, if $X_j^{(j)} \in J_w^{(j)}$, then both $J_w^{(j)}$ and $J_{w'}^{(j)}$ belong to

$$\varphi_j(\{P \in \text{Spec}(R^{(j+1)}) \mid X_j^{(j+1)} \in P\}).$$

As φ_j induces an increasing homeomorphism, still denoted φ_j , from

$$\{P \in \text{Spec}(R^{(j+1)}) \mid X_j^{(j+1)} \in P\}$$

onto its image whose inverse is also an increasing homeomorphism, we get

$$J_w^{(j+1)} = \varphi_j^{-1}(J_w^{(j)}) \subseteq \varphi_j^{-1}(J_{w'}^{(j)}) = J_{w'}^{(j+1)},$$

as desired.

It only remains to deal with the case where $X_j^{(j)} \in J_{w'}^{(j)}$ and $X_j^{(j)} \notin J_w^{(j)}$, so

$$J_w^{(j)} = J_w^{(j+1)}S_j^{-1} \cap R^{(j)}, \quad J_w^{(j+1)} = J_w^{(j)}S_j^{-1} \cap R^{(j+1)}, \quad J_{w'}^{(j)} = g_j^{-1}(J_{w'}^{(j+1)}/(X_j^{(j+1)})).$$

In order to simplify the notation we set

$$Y_i := X_i^{(j)} \text{ and } Z_i := X_i^{(j+1)}. \tag{2.8}$$

Also we let A denote the subalgebra of $R^{(j+1)}$ generated by the Z_i with $i \neq j$:

$$A := \mathbb{K}\langle Z_1, \dots, Z_{j-1}, Z_{j+1}, \dots, Z_N \rangle. \tag{2.9}$$

Observe that $R^{(j+1)} = A[Z_j; \sigma, \delta]$, where σ denotes the automorphism of A defined by $\sigma(Z_i) = \lambda_{j,i}Z_i = \sigma_j(Z_i)$ if $i < j$ and $\sigma(Z_i) = \lambda_{i,j}^{-1}Z_i$ otherwise; and where δ denotes the σ -derivation of A defined by $\delta(Z_i) = \delta_j(Z_i)$ if $i < j$ and $\delta(Z_i) = 0$ otherwise. One can easily check that this Ore extension satisfies the conditions of [Cauchon 2003a, Section 2], so the map $\theta : A \rightarrow R^{(j+1)}S_j^{-1} = R^{(j)}S_j^{-1}$ defined by

$$\theta(a) = \sum_{k=0}^{+\infty} \frac{(1-q_j)^{-k}}{[k]!_{q_j}} \delta^k \circ \sigma^{-k}(a) Z_j^{-k} \text{ for } a \in A$$

is an homomorphism. Observe that by the definition of the deleting derivations algorithm we have $\theta(Z_i) = Y_i$ for all $i \neq j$. Hence $\theta(A) \subset R^{(j)}$ and $R^{(j)}$ is the subalgebra of $R^{(j+1)}S_j^{-1}$ generated by $\theta(A)$ and $Z_j = Y_j$.

Again to simplify the notation, we set

$$P := J_w^{(j+1)}, \quad P' := J_w^{(j)}, \quad Q = J_{w'}^{(j+1)}, \quad Q' := J_{w'}^{(j)}. \tag{2.10}$$

Let $z \in P$ with $z \neq 0$. We need to prove that $z \in Q$. First we can write

$$z = \sum_{t=0}^d a_t Z_j^t,$$

where $a_t \in A$ and $a_d \neq 0$; moreover, this expression for z in this form is unique. If $a_0 = 0$, then $z \in (Z_j) \subseteq Q$. So we assume that $a_0 \neq 0$.

For every t , there exists k_t minimal such that $\delta^{k_t+1}(a_t) = 0$ (recall that δ is locally nilpotent), so

$$\theta(a_t) = \sum_{k=0}^{k_t} \frac{(1 - q_j)^{-k}}{[k]!_{q_j}} \delta^k \circ \sigma^{-k}(a_t) Z_j^{-k}.$$

By induction on the degree of local nilpotency we get:

$$a_t = \theta(a_t) + \sum_{k=1}^{k_t} \mu_{k,t} \theta(\delta^k \circ \sigma^{-k}(a_t)) Z_j^{-k} = \sum_{k=0}^{k_t} \mu_{k,t} \theta(\delta^k \circ \sigma^{-k}(a_t)) Z_j^{-k},$$

where $\mu_{0,t} = 1$ and $\mu_{k,t} \in \mathbb{K}$. Let m be the maximum of the $k_t - t$. Then

$$z Z_j^m = \sum_{t=0}^d a_t Z_j^{t+m} = \sum_{t=0}^d \sum_{k=0}^{k_t} \mu_{k,t} \theta(\delta^k \circ \sigma^{-k}(a_t)) Z_j^{t+m-k} \in R^{(j)}.$$

Thus $z Z_j^m \in R^{(j)} \cap P S_j^{-1} = P'$. Hence

$$z Z_j^m = \sum_{t=0}^d \sum_{k=0}^{k_t} \mu_{k,t} \theta(\delta^k \circ \sigma^{-k}(a_t)) Z_j^{t+m-k} \in P'. \tag{2.11}$$

We let A' denote the subalgebra of $R^{(j)}$ generated by Y_i with $i \neq j$, or equivalently, the image of θ . As P' is an H -prime ideal of

$$R^{(j)} = A'[Z_j; \sigma],$$

it follows from [Launois et al. 2006, Corollary 2.4] that the coefficient of Z_j^l in the previous sum belongs to P' for every nonnegative integer l . In particular, the coefficient of degree m is in P' . Hence, by setting $k = t$ in (2.11), we obtain

$$\sum_{t=0}^d \mu_{t,t} \theta(\delta^t \circ \sigma^{-t}(a_t)) \in P' \subseteq Q'.$$

As $Q' = g_j^{-1}(Q/(Z_j))$ and $(Z_j) \subseteq Q$, we get that

$$\sum_{t=0}^d \mu_{t,t} \delta^t \circ \sigma^{-t}(a_t) \in Q. \tag{2.12}$$

Since $Z_j \in Q$, we see that $\delta(a) = Z_j a - \sigma(a)Z_j \in Q$ for every $a \in A$. Hence we deduce from (2.12) that $a_0 = \mu_{0,0}a_0 \in Q$. Since $(Z_j) \subseteq Q$, we obtain, as desired,

$$z = a_0 + \left(\sum_{t=1}^d a_t Z_j^{t-1} \right) Z_j \in Q. \quad \square$$

3. Dimension of H -strata of uniparameter CGL extensions

In this section, we obtain a formula for the dimension of a stratum of a uniparameter CGL extension, and apply it to compute the dimension of the (0)-stratum of a quantum Schubert cell.

3A. Uniparameter CGL extensions. In this section, we assume that R is a *uniparameter CGL extension*, that is, R is a CGL extension such that there exist an antisymmetric matrix $(a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$ and $q \in \mathbb{K}^*$ not a root of unity such that $\lambda_{j,i} = q^{a_{j,i}}$ for all $1 \leq i < j \leq N$.

3B. Dimension of H -strata of uniparameter CGL extensions. The aim of this section is to give a formula for the dimension of the H -stratum in R associated to a Cauchon diagram $w \in W'$. We need to introduce the following definition.

Definition. Let $w \in W'$ be a Cauchon diagram of R . Let $\{l_1 < \dots < l_d\} := \llbracket 1, N \rrbracket \setminus w$ be the complement of w . We define the *skew-adjacency matrix*, $M_R(w)$, of w to be the $d \times d$ matrix whose (i, j) entry is $a_{l_i l_j}$.

Theorem 3.1. *Let $w \in W'$. Then the H -stratum associated to J_w is homeomorphic to the prime spectrum of a commutative Laurent polynomial ring over \mathbb{K} in $\dim_{\mathbb{Q}}(\ker(M_R(w)))$ indeterminates.*

Proof. Let $w \in W'$ be a Cauchon diagram of R . Let $\{l_1 < \dots < l_d\} := \llbracket 1, N \rrbracket \setminus w$ be the complement of w . From [Cauchon 2003a, Théorèmes 5.1.1 and 5.5.1] we know that the canonical embedding induces an inclusion-preserving homeomorphism from the H -stratum $\text{Spec}_{J_w}(R)$ of R associated to J_w onto the H -stratum $\text{Spec}_{K_w}(\bar{R})$ of \bar{R} associated to K_w . Hence we deduce from Proposition 2.1 that

$$\begin{aligned} \text{Spec}_{J_w}(R) &\simeq \text{Spec}_{K_w}(\bar{R}) \\ &\simeq \{P \in \text{Spec}(\bar{R}) \mid P \cap \{T_1, \dots, T_N\} = \{T_i \mid i \in w\}\}. \end{aligned} \quad (3.13)$$

Recall that $\bar{R} = \mathbb{K}_{\Lambda}[T_1, T_2, \dots, T_N]$, where Λ denotes the $N \times N$ matrix whose entries are defined by $\Lambda_{k,l} = q^{a_{k,l}}$ for all $k, l \in \llbracket 1, N \rrbracket$. Let Λ_w denote the multiplicatively antisymmetric $d \times d$ matrix whose entries are defined by $(\Lambda_w)_{i,j} = q^{M_R(w)_{i,j}} = q^{a_{l_i l_j}}$.

As K_w is the prime ideal generated by the indeterminates T_i such that $i \in w$, the algebra \bar{R}/K_w is isomorphic to the quantum affine space $\mathbb{K}_{\Lambda_w}[t_1, \dots, t_d]$ by an isomorphism that sends $T_i + K_w$ to t_i and T_k to 0 if $k \neq i$.

To finish the proof, we use the same idea as in [Launois and Lenagan 2007, Corollary 1.3].

We denote by $P(\Lambda_w)$ the quantum torus associated to $\mathbb{K}_{\Lambda_w}[t_1, \dots, t_d]$:

$$P(\Lambda_w) := \mathbb{K}_{\Lambda_w}[t_1, \dots, t_d]\Sigma^{-1},$$

where Σ denotes the multiplicative system of $\mathbb{K}_{\Lambda_w}[t_1, \dots, t_d]$ generated by the normal elements t_1, \dots, t_d .

It follows from (3.13) that

$$\text{Spec}_{K_w}(\bar{R}) \simeq \text{Spec}_{(0)}(\mathbb{K}_{\Lambda_w}[t_1, \dots, t_d]) \simeq \text{Spec}(P(\Lambda_w)).$$

Next, $\text{Spec}(P(\Lambda_w))$ is Zariski-homeomorphic via extension and contraction to the prime spectrum of the centre $Z(P(\Lambda_w))$ of $P(\Lambda_w)$, by [Goodearl and Letzter 1998, Corollary 1.5]. Further, as we shall see, $Z(P(\Lambda_w))$ is a Laurent polynomial ring. To make this result precise, we need to introduce the following notation.

If $\underline{s} = (s_1, \dots, s_d) \in \mathbb{Z}^d$, then we set $t^{\underline{s}} := t_1^{s_1} \dots t_d^{s_d} \in P(\Lambda_w)$. As in [Goodearl and Letzter 1998], we denote by $\sigma : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{K}^*$ the antisymmetric bicharacter defined by

$$\sigma(\underline{s}, \underline{t}) := \prod_{i,j=1}^d (\Lambda_w)_{i,j}^{s_i t_j} = q^{\sum_{i,j=1}^d a_{i,j} s_i t_j} \quad \text{for all } \underline{s}, \underline{t} \in \mathbb{Z}^d.$$

Then it follows from [Goodearl and Letzter 1998, 1.3] that the centre $Z(P(\Lambda_w))$ of $P(\Lambda_w)$ is a Laurent polynomial ring over \mathbb{K} in the variables $(t^{\underline{b}_1})^{\pm 1}, \dots, (t^{\underline{b}_r})^{\pm 1}$, where $(\underline{b}_1, \dots, \underline{b}_r)$ is any basis of

$$V := \{\underline{s} \in \mathbb{Z}^d \mid \sigma(\underline{s}, -) \equiv 1\}.$$

Since q is not a root of unity, easy computations show that $\underline{s} \in V$ if and only if $M_R(w)^t \underline{s}^t = 0$. Hence the centre $Z(P(\Lambda_w))$ of $P(\Lambda_w)$ is a Laurent polynomial ring in $\dim_{\mathbb{Q}}(\ker(M_R(w)^t)) = \dim_{\mathbb{Q}}(\ker(M_R(w)))$ indeterminates.

To summarize, we have

$$\text{Spec}_{J_w}(R) \simeq \text{Spec}_{K_w}(\bar{R}) \simeq \text{Spec}(P(\Lambda_w)) \simeq \text{Spec}(Z(P(\Lambda_w))),$$

and $Z(P(\Lambda_w))$ is a Laurent polynomial ring in $\dim_{\mathbb{Q}}(\ker(M_R(w)))$ indeterminates, as desired. □

3C. Application to quantum Schubert cells. We now use Theorem 3.1 to compute the dimension of the (0)-stratum of quantum Schubert cells. We first recall the definition of quantum Schubert cells.

Let \mathfrak{g} be a simple Lie \mathbb{C} -algebra of rank n . We denote by $\pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that π is a basis of a Euclidean vector space E over \mathbb{R} , whose inner product is denoted by (\cdot, \cdot) (E is usually denoted by $\mathfrak{h}_{\mathbb{R}}^*$ in Bourbaki.) We denote by W the Weyl group of \mathfrak{g} , that is, the subgroup of the orthogonal group of E generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1, \dots, n\}$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$, $i \in \{1, \dots, n\}$. The length of $w \in W$ is denoted by $l(w)$. Further, we denote by w_0 the longest element of W . Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data. As \mathfrak{g} is simple, $a_{ij} \in \{0, -1, -2, -3\}$ for all $i \neq j$.

Recall that the scalar product of two roots (α, β) is always an integer. We assume that the short roots have length $\sqrt{2}$.

For all $i \in \{1, \dots, n\}$, set $q_i := q^{(a_i, a_i)/2}$ and

$$\begin{bmatrix} m \\ k \end{bmatrix}_i := \frac{(q_i - q_i^{-1}) \dots (q_i^{m-1} - q_i^{1-m})(q_i^m - q_i^{-m})}{(q_i - q_i^{-1}) \dots (q_i^k - q_i^{-k})(q_i - q_i^{-1}) \dots (q_i^{m-k} - q_i^{k-m})}$$

for all integers $0 \leq k \leq m$. By convention,

$$\begin{bmatrix} m \\ 0 \end{bmatrix}_i := 1.$$

The quantised enveloping algebra $U_q(\mathfrak{g})$ of \mathfrak{g} over \mathbb{C} associated to the previous data is the \mathbb{k} -algebra generated by the indeterminates $E_1, \dots, E_n, F_1, \dots, F_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$ subject to the following relations:

$$K_i K_j = K_j K_i,$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j \text{ and } K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i E_i^{1-a_{ij}-k} E_j E_i^k = 0 \text{ (} i \neq j \text{)}, \tag{3.14}$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i F_i^{1-a_{ij}-k} F_j F_i^k = 0 \text{ (} i \neq j \text{)}.$$

We refer the reader to [Brown and Goodearl 2002; Jantzen 1996; Joseph 1995] for more details on this (Hopf) algebra. Further, as usual, we denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by E_1, \dots, E_n (resp. F_1, \dots, F_n) and by U^0 the subalgebra of $U_q(\mathfrak{g})$ generated by $K_1^{\pm 1}, \dots, K_n^{\pm 1}$.

To each reduced decomposition of the longest element w_0 of the Weyl group W of \mathfrak{g} , Lusztig has associated a PBW basis of $U_q^+(\mathfrak{g})$; see, for instance, [Lusztig 1993, Chapter 37; Jantzen 1996, Chapter 8; Brown and Goodearl 2002, I.6.7]. The construction relates to a braid group action by automorphisms on $U_q^+(\mathfrak{g})$. We use the convention of [Jantzen 1996, Chapter 8]. In particular, for any $\alpha \in \pi$, we define the braid automorphism T_α of the algebra $U_q(\mathfrak{g})$ as in [Jantzen 1996, p. 153]. We set $T_i := T_{\alpha_i}$. It was proved by Lusztig that the automorphisms T_i satisfy the braid relations, that is, if $s_i s_j$ has order m in W , then

$$T_i T_j T_i \dots = T_j T_i T_j \dots,$$

where there are exactly m factors on each side of this equality.

Consider any $w \in W$, and set $t := l(w)$. Let $w = s_{i_1} \circ \dots \circ s_{i_t}$ ($i_j \in \{1, \dots, n\}$) be a reduced decomposition of w . It is well known that $\beta_1 = \alpha_{i_1}$, $\beta_2 = s_{i_1}(\alpha_{i_2})$, \dots , $\beta_t = s_{i_1} \circ \dots \circ s_{i_{t-1}}(\alpha_{i_t})$ are distinct positive roots and that the set $\{\beta_1, \dots, \beta_t\}$ does not depend on the chosen reduced expression of w . Similarly, we define elements E_{β_k} of $U_q(\mathfrak{g})$ by

$$E_{\beta_k} := T_{i_1} \dots T_{i_{k-1}}(E_{i_k}).$$

Note that the elements E_{β_k} depend on the reduced decomposition of w . The following well-known results were proved by Lusztig and Levendorskiĭ–Soibelman.

Theorem 3.2 [Levendorskiĭ and Soibelman 1991].

- (1) For all $k \in \{1, \dots, t\}$, the element E_{β_k} belongs to $U_q^+(\mathfrak{g})$.
- (2) If $\beta_k = \alpha_i$, then $E_{\beta_k} = E_i$.
- (3) For all $1 \leq i < j \leq t$, we have

$$E_{\beta_j} E_{\beta_i} - q^{-(\beta_i, \beta_j)} E_{\beta_i} E_{\beta_j} = \sum a_{k_{i+1}, \dots, k_{j-1}} E_{\beta_{i+1}}^{k_{i+1}} \dots E_{\beta_{j-1}}^{k_{j-1}},$$

where each $a_{k_{i+1}, \dots, k_{j-1}}$ belongs to \mathbb{K} .

We denote by $U_q[w]$ the subalgebra of $U_q^+(\mathfrak{g})$ generated by $E_{\beta_1}, \dots, E_{\beta_t}$. It is well known that $U_q[w]$ does not depend on the reduced decomposition of w . Moreover, the monomials $E_{\beta_1}^{k_1} \dots E_{\beta_t}^{k_t}$, with $k_1, \dots, k_t \in \mathbb{N}$, form a linear basis of $U_q[w]$. As a consequence of this result, $U_q[w]$ can be presented as a skew-polynomial algebra:

$$U_q[w] = \mathbb{K}[E_{\beta_1}][E_{\beta_2}; \sigma_2, \delta_2] \dots [E_{\beta_t}; \sigma_t, \delta_t],$$

where each σ_i is a linear automorphism and each δ_i is a σ_i -derivation of the appropriate subalgebra. In particular, $U_q[w]$ is a noetherian domain and its group of invertible elements is reduced to nonzero elements of the base-field.

The torus $\mathcal{H} := (\mathbb{K}^*)^n$ acts rationally by automorphisms on $U_q^+(\mathfrak{g})$ via

$$(h_1, \dots, h_n) \cdot E_i = h_i E_i \quad \text{for all } i \in \{1, \dots, n\}.$$

(It is easy to check that the quantum Serre relations are preserved by the group \mathcal{H} .) It is also well known that this action of \mathcal{H} on $U_q^+(\mathfrak{g})$ restricts to a rational action of \mathcal{H} on $U_q[w]$. Observe that (0) is an \mathcal{H} -prime in $U_q[w]$ as this algebra is a domain.

It was proved by Cauchon [2003a, Proposition 6.1.2 and Lemme 6.2.1] that $U_q[w]$ is a uniparameter CGL extension with the following associated antisymmetric matrix:

$$\begin{pmatrix} 0 & (\beta_1, \beta_2) & \cdots & \cdots & (\beta_1, \beta_t) \\ -(\beta_1, \beta_2) & 0 & (\beta_2, \beta_3) & & (\beta_2, \beta_t) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & (\beta_{t-1}, \beta_t) \\ -(\beta_1, \beta_t) & \cdots & \cdots & -(\beta_{t-1}, \beta_t) & 0 \end{pmatrix}.$$

The kernel of this matrix has been described by De Concini and Procesi [1993, Lemma 10.4 and 10.6] who proved that the kernel of this matrix identifies with $\ker(\text{id}_E + w)$. So we deduce from Theorem 3.1 the following result.

Proposition 3.3. *The dimension of the stratum associated to (0) in $U_q[w]$ is*

$$\dim \ker(\text{id}_E + w).$$

It follows from [Cauchon and Mériaux 2009, Proposition 2.2.1] that the algebra $\mathbb{C}_q(M_{m,n})$ of quantum matrices can be presented as a quantum Schubert cell. So one can use the previous proposition in order to retrieve the dimension of the (0) -stratum of $\mathbb{C}_q(M_{m,n})$ that was first obtained in [Launois and Lenagan 2007].

4. Quantum matrices

In this section we investigate in more detail the dimensions of the strata occurring in the Goodearl–Letzter stratification of the ring $R = \mathbb{C}_q(M_{m,n})$ of $m \times n$ quantum matrices, and we prove Theorems 1.1 and 1.2. Throughout this section, q denotes a nonzero element of \mathbb{K} that is not a root of unity. The ring R is known to be a quantum Schubert cell by [Cauchon and Mériaux 2009, Proposition 2.2.1], so we know (see the previous section) that $\mathbb{C}_q(M_{m,n})$ is a uniparameter CGL extension. Nevertheless we start by explicitly describing the results of the previous sections in this situation.

4A. Quantum matrices as a CGL extension. This section serves to show that quantum matrix rings give examples of CGL extensions and that we can therefore draw upon the background given in Section 2.

It is well known that $R = \mathbb{C}_q(M_{m,n})$ can be presented as an iterated Ore extension over \mathbb{K} with the generators $Y_{i,\alpha}$ given in the beginning of Section 1 adjoined in

lexicographic order. Thus the ring R is a noetherian domain; we denote by F its skew-field of fractions. Since q is not a root of unity, it follows from [Goodearl and Letzter 1994, Theorem 3.2] that all prime ideals of R are completely prime.

The algebras $\mathbb{O}_q(M_{m,n})$ and $\mathbb{O}_q(M_{n,m})$ are isomorphic. Because of this, we can and will assume that $n \leq m$.

$\mathcal{H} := (\mathbb{K}^*)^{m+n}$ acts on R by \mathbb{K} -algebra automorphisms via

$$(a_1, \dots, a_m, b_1, \dots, b_n) \cdot Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad \text{for all } (i, \alpha) \in \llbracket 1, m \rrbracket \times \llbracket 1, n \rrbracket.$$

Moreover, since q is not a root of unity, R endowed with this action of \mathcal{H} is a uniparameter CGL extension (see for instance [Launois et al. 2006]). Before going any further let us describe the antisymmetric matrix associated to the uniparameter CGL extension $\mathbb{O}_q(M_{m,n})$; it is given by

$$B = (b_{k,l}) := \begin{pmatrix} A & I_m & I_m & \dots & I_m \\ -I_m & A & I_m & \dots & I_m \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -I_m & \dots & -I_m & A & I_m \\ -I_m & \dots & \dots & -I_m & A \end{pmatrix} \in M_{mn}(\mathbb{Q}),$$

where I_m denotes the identity matrix of $M_m(\mathbb{Q})$ and

$$A := \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & \dots & -1 & 0 & 1 \\ -1 & \dots & \dots & -1 & 0 \end{pmatrix} \in M_m(\mathbb{Z}) \subseteq M_m(\mathbb{Q}).$$

That $R = \mathbb{O}_q(M_{m,n})$ is a (uniparameter) CGL extension implies in particular that $\mathcal{H}\text{-Spec}(R)$ is finite and that every \mathcal{H} -prime is completely prime. Also, as $R = \mathbb{O}_q(M_{m,n})$ is a CGL extension, one can apply the results of Section 2 to this algebra. In particular, using the theory of deleting derivations, Cauchon has given a combinatorial description of $\mathcal{H}\text{-Spec}(R)$. More precisely, in the case of the algebra $R = \mathbb{O}_q(M_{m,n})$, he has described the set W' that appeared in Section 2C as follows.

First, it follows from [Cauchon 2003b, Section 2.2] that the quantum affine space \bar{R} that appears in Section 2C is in this case $\bar{R} = \mathbb{K}_\Lambda[T_{1,1}, T_{1,2}, \dots, T_{m,n}]$, where Λ denotes the $mn \times mn$ matrix whose entries are defined by $\Lambda_{k,l} = q^{b_{k,l}}$ for all $k, l \in \llbracket 1, mn \rrbracket$. Using the canonical embedding (see Section 2C), Cauchon [2003b] produced a bijection between $\mathcal{H}\text{-Spec}(\mathbb{O}_q(M_{m,n}))$ and the collection $\mathcal{C}_{m,n}$ of $m \times n$ Cauchon diagrams as defined on page 177. Roughly speaking, the set W' defined on page 184 coincides with $\mathcal{C}_{m,n}$.

Let us make this precise. If C is an $m \times n$ Cauchon diagram in the sense of the definition on page 177, we denote by K_C the (completely) prime ideal of \bar{R} generated by the indeterminates $T_{i,\alpha}$ such that the box in position (i, α) is a black box of C . Then, with $\varphi : \text{Spec}(R) \rightarrow \text{Spec}(\bar{R})$ denoting the canonical embedding, it follows from [Cauchon 2003b, Corollaire 3.2.1] that there exists a unique \mathcal{H} -invariant (completely) prime ideal J_C of R such that $\varphi(J_C) = K_C$; moreover there are no other \mathcal{H} -primes in $\mathbb{O}_q(M_{m,n})$; that is,

$$\mathcal{H}\text{-Spec}(\mathbb{O}_q(M_{m,n})) = \{J_C \mid C \in \mathcal{C}_{m,n}\}.$$

This last equality justifies the terminology “ $m \times n$ Cauchon diagrams” for the combinatorial objects defined on page 177.

In light of this, the containment rule for $m \times n$ Cauchon diagrams given in the Introduction coincides exactly with set-theoretic containment for the more general description of Cauchon diagrams in terms of sets. So, in the case of quantum matrices, Theorem 2.3 can be rephrased as follows.

Theorem 4.1. *If C and C' are two $m \times n$ Cauchon diagrams with $C \subsetneq C'$, then $J_C \subsetneq J_{C'}$, where J_C and $J_{C'}$ denote respectively the \mathcal{H} -primes of $\mathbb{O}_q(M_{m,n})$ associated to C and C' .*

4B. Dimension of \mathcal{H} -strata. We now give some results about the dimension of the \mathcal{H} -stratum of an \mathcal{H} -prime of $\mathbb{O}_q(M_{m,n})$ corresponding to an $m \times n$ Cauchon diagram C . We give two related definitions.

A Cauchon diagram C is *labeled* if each white box in C is labeled with a positive integer such that: the labels are strictly increasing from left to right along rows, and if $i < j$ then the label of each white box in row i is strictly less than the label of each white box in row j .

Let C be an $m \times n$ labeled Cauchon diagram with d white boxes and labels $l_1 < \dots < l_d$. We define the *skew-adjacency matrix*, $M(C)$, of C to be the $d \times d$ matrix whose (i, j) entry is:

- (1) 1 if the box labeled l_i is strictly to the left and in the same row as the box labeled l_j or is strictly above and in the same column as the box labeled l_j ;
- (2) -1 if the box labeled l_i is strictly to the right and in the same row as the box labeled l_j or is strictly below and in the same column as the box labeled l_j ;
- (3) 0 otherwise.

Observe that $M(C)$ is independent of the set of labels which appear in C .

See, for example, Figure 3.

As a particular case of Theorem 3.1, we get the following result for the uniparameter CGL extension $\mathbb{O}_q(M_{m,n})$.

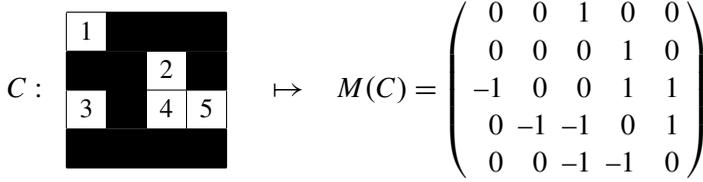


Figure 3. A labeled Cauchon diagram C and its corresponding skew-adjacency matrix $M(C)$.

Proposition 4.2. *Let C be an $m \times n$ Cauchon diagram. The \mathcal{H} -stratum associated to J_C is homeomorphic to the prime spectrum of a commutative Laurent polynomial ring over \mathbb{K} in $\dim_{\mathbb{Q}}(\ker(M(C)))$ indeterminates.*

We now use this result in order to prove the first part of Theorem 1.1. In order to achieve this aim, we need the following lemma.

Lemma 4.3. *Let C be an $m \times n$ labeled Cauchon diagram with $n \leq m$ and with d white boxes with labels $\llbracket 1, d \rrbracket$. Assume that C has no all-black columns. For $1 \leq j \leq n$, let a_j denote the smallest label which appears in column j of C . Then there is a $d \times d$ lower triangular matrix S such that the matrix obtained by deleting columns a_1, \dots, a_n and rows a_1, \dots, a_n from $S \cdot M(C)$ is invertible.*

Proof. We let d_i denote the number of white boxes in the i -th row of C . Then

$$d = d_1 + \dots + d_m$$

and we can write $M(C)$ in block form as

$$M(C) = \begin{pmatrix} A_1 & J_{1,2} & \dots & J_{1,m} \\ -J_{1,2}^T & A_2 & \dots & J_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ -J_{1,m}^T & -J_{2,m}^T & \dots & A_m \end{pmatrix}, \tag{4.15}$$

where A_i is the $d_i \times d_i$ matrix whose diagonal entries are zero, whose entries above the diagonal are 1, and whose entries below the diagonal are -1 ; and $J_{i,j}$ is the $d_i \times d_j$ 0, 1-matrix whose (k, l) entry is 1 if the k -th white element in row i of C (looking from left to right) is directly above the l -th white element in row j of C (again, looking left to right) and is 0 otherwise.

We now define a $d \times d$ lower-triangular matrix S , whose diagonal entries are all 1 and for $i > j$, whose (i, j) entry is -1 if the white box labeled i is in the k -th column of C and $j = a_k$; and whose (i, j) entry is 0 otherwise. We now consider the product $S \cdot M(C)$.

Claim. *If $i < j$ and the boxes labeled i and j are not in the same row of C and $i \notin \{a_1, \dots, a_n\}$ then the (i, j) entry of $S \cdot M(C)$ is zero.*

Proof of the claim. Suppose $i < j$ and $i \notin \{a_1, \dots, a_n\}$. Let l be the column in which the box labeled i in C sits. Then

$$\begin{aligned} (S \cdot M(C))_{i,j} &= \sum_{k=1}^d S_{i,k} M(C)_{k,j} \\ &= S_{i,i} M(C)_{i,j} + S_{i,a_l} M(C)_{i,j} = M(C)_{i,j} - M(C)_{a_l,j}. \end{aligned}$$

By assumption, the boxes labeled i and j are not in the same row of C and hence if $M(C)_{i,j} = 1$, then j must also be in the l -th column of C . But then $M(C)_{a_l,j} = 1$, and so $(S \cdot M(C))_{i,j} = 0$. Similarly, if $M(C)_{i,j} = 0$, then j must be in a different column than i and so $M(C)_{a_l,j} = 0$ as well. The claim follows. \square

Let D denote the matrix obtained from $S \cdot M(C)$ by deleting the rows indexed by a_1, \dots, a_n and the columns indexed by a_1, \dots, a_n . Let e_i denote the number of labels $\{a_1, \dots, a_n\}$ which appear in the i -th row of C . By the claim above, D is a block lower-triangular matrix; that is,

$$D = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ * & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & D_m \end{pmatrix},$$

where D_i is a $(d_i - e_i) \times (d_i - e_i)$ matrix.

Claim. For $1 \leq i \leq m$, we have $D_i = -I_{d_i - e_i} + D'_i$ for some skew-symmetric matrix D'_i .

Proof of the claim. We first consider the matrix $S \cdot M(C)$. Let $\{a + 1, a + 2, \dots, a + d_k\}$ be the set of labels in the k -th row of $M(C)$. (Here $a = d_1 + d_2 + \dots + d_{k-1}$.) Consider $(S \cdot M(C))_{a+i, a+j}$. Since D is obtained by deleting the rows and columns of $S \cdot M(C)$ indexed by a_1, \dots, a_n , we may assume that $a + i \notin \{a_1, \dots, a_n\}$. In this case, the box labeled $a + i$ appears in the l -th column of C for some l . Then $(S \cdot M(C))_{a+i, a+j} = M(C)_{a+i, a+j} - M(C)_{a_l, a+j}$. Observe that $M(C)_{a_l, a+j}$ is nonzero if and only if $a + j$ is also in the l -th column of C ; but by assumption, $a + i$ is in the l -th column and the boxes labeled $a + i$ and $a + j$ are in the same row, so this is impossible unless $i = j$. Hence

$$(S \cdot M(C))_{a+i, a+j} = M(C)_{a+i, a+j} - \delta_{i,j},$$

if $a + i \notin \{a_1, \dots, a_n\}$. To obtain D , we simply delete the rows and columns indexed by $\{a_1, \dots, a_n\}$. In particular, if we let A_k denote the $d_k \times d_k$ submatrix of A whose rows and columns are indexed by $a + 1, \dots, a + d_k$ (see (4.15)), then D_k is obtained by deleting the e_i rows and columns indexed by the labels $\{a_1, \dots, a_n\}$

which occur in the i -th row of C and then subtracting the identity. Since each A_k is skew-symmetric, the claim follows. \square

We have seen that the matrix D is indeed a block lower-triangular matrix; thus to finish the proof, it is enough to show that each D_i is invertible. To see this, note that it is of the form $-I + D'_i$ for some real skew-symmetric matrix D'_i . Since the nonzero eigenvalues of a real skew-symmetric matrix are purely imaginary, we see that D_i cannot have any eigenvalue equal to zero and hence is invertible. It follows that the matrix D is invertible. \square

We are now in position to prove the first part of Theorem 1.1.

Theorem 4.4. *Let C be an $m \times n$ Cauchon diagram with $n \leq m$. Then*

$$\dim(\ker(M(C))) \leq n.$$

As a consequence the dimension of the \mathcal{H} -stratum associated to J_C is at most n .

Proof. Assume first that C has no all-black columns. By Lemma 4.3, there exists an invertible matrix S such that deleting n rows and n columns from $S \cdot M(C)$ gives an invertible matrix. This means that $\dim(\ker(S \cdot M(C))) \leq n$. Moreover $\dim(\ker(M(C))) = \dim(\ker(S \cdot M(C)))$, since S is invertible. The result follows in this case.

Assume now that C has at least one all-black column. Let \widehat{C} be the Cauchon diagram obtained from C by removing every all-black column of C . Then $M(C) = M(\widehat{C})$, and so the result follows from the previous case. \square

We can now prove Theorem 1.1.

Theorem 4.5. *Let m and n be natural numbers. Then the dimensions of \mathcal{H} -strata in $\mathbb{C}_q(M_{m,n})$ are all at most $\min(m, n)$; moreover, for each $d \in \{0, 1, \dots, \min(m, n)\}$ there exists a d -dimensional \mathcal{H} -stratum.*

Proof. We assume that $n \leq m$. By Theorem 4.4, the dimensions of the \mathcal{H} -strata are all at most n and so it is sufficient to show that each of these values can occur. Let $d \leq n$. We take P to be the \mathcal{H} -prime corresponding to the $m \times n$ Cauchon diagram whose (i, j) square is white if and only if i and j are both at most d . Then $\mathbb{C}_q(M_{m,n})/P$ is isomorphic to the ring of $d \times d$ quantum matrices. It follows from [Launois and Lenagan 2007, Theorem 2.5] that the dimension of the stratum associated to P is exactly d . This completes the proof. \square

4C. Proof of Theorem 1.2. In this section we use Theorem 4.1 along with Theorem 4.5 to prove Theorem 1.2.

We first make a remark that will be useful in proving the next proposition.

Remark. Let A be an $n \times n$ real skew-symmetric matrix. Then the dimension of the kernel of A has the same parity as n .

1	3		4
	8		10
		15	16

Figure 4. An example of a 4×4 labeled Cauchon diagram.

Proposition 4.6. *Let C be an $m \times n$ Cauchon diagram with $n \leq m$. Suppose that the kernel of $M(C)$ has dimension $e \geq 1$. Then there is a Cauchon diagram $C' \supseteq C$ obtained by adding exactly one black box to C such that $M(C')$ has an $(e - 1)$ -dimensional kernel.*

Proof. Clearly we can assume that C has no all-black columns. Let d be the number of white boxes in C . We make C into an $m \times n$ labeled Cauchon diagram with labels $\llbracket 1, d \rrbracket$.

Let $T = \{i_1, \dots, i_k\}$ denote the set of all labels of white boxes of C with the property that if one of these labels is coloured black and the remaining boxes of C are left unchanged then the result is still a Cauchon diagram.

For example, in Figure 4 the labels of the white boxes which can be coloured black to obtain a Cauchon diagram are 1, 3, 4, 8 and 15.

Given $i \in T$, we let C_i denote the Cauchon diagram obtained by colouring the white box with label i black.

If $\dim(\ker(M(C_i))) \geq e$ for every $i \in T$, then by parity considerations (see remark immediately above) we must have $\dim(\ker(M(C_i))) \geq e + 1$, for $i \in T$. Let

$$\{v_1^{(i)}, \dots, v_{e+1}^{(i)}\}$$

be a linearly independent set of vectors in the kernel of $M(C_i)$. We construct $e + 1$ vectors as follows. For $1 \leq j \leq e + 1$, let $w_j^{(i)}$ be the $d \times 1$ column vector whose l -th coordinate is the l -th coordinate of $v_j^{(i)}$ if $l < i$, is 0 if $l = i$, and is the $(l - 1)$ -th coordinate of $v_j^{(i)}$ if $l > i$. By construction, every row of $M(C)$ is orthogonal to the linearly independent set $\{w_1^{(i)}, \dots, w_{e+1}^{(i)}\}$ except for possibly the i -th row. Let r denote the i -th row of $M(C)$. Then r is an $1 \times d$ row vector. We then have a map

$$\text{Span}(\{w_1^{(i)}, \dots, w_{e+1}^{(i)}\}) \rightarrow \mathbb{Q}$$

in which a vector w in the span is sent to $r \cdot w \in \mathbb{Q}$. This map is surjective since otherwise the dimension of the kernel of $M(C)$ would be at least $(e + 1)$ -dimensional. Thus the kernel of this map is an e -dimensional subspace of

$$\text{Span}(\{w_1^{(i)}, \dots, w_{e+1}^{(i)}\})$$

which lies in the kernel of $M(C)$. Since the kernel of $M(C)$ is exactly e -dimensional, and every vector in $\text{Span}(\{w_1^{(i)}, \dots, w_{e+1}^{(i)}\})$ has a zero in the i -th coordinate,

we see that every vector in $\ker(M(C))$ has a zero in the i -th coordinate; moreover, this is the case for every $i \in T$. For $1 \leq j \leq n$, recall that a_j denotes the smallest label which appears in column j of C . Then $\{a_1, \dots, a_n\} \subseteq T$, as they are the labels of the uppermost white boxes in each column. By Lemma 4.3 there is a lower-triangular matrix S such that $S \cdot M(C)$ has the property that if columns a_1, \dots, a_n and rows a_1, \dots, a_n are deleted then the resulting matrix is invertible. Let $\{v_1, \dots, v_e\}$ be a basis for $\ker(M(C))$. For each $i \leq e$, let u_i denote the $(d - n) \times 1$ column vector obtained by taking the $d \times 1$ vector v_i and simply removing coordinates a_1 through a_n . Since the a_j -th coordinate of v_i is 0 for each i and j , we see that u_1, \dots, u_e are linearly independent and are in the kernel of the matrix obtained by removing columns a_1, \dots, a_n from $S \cdot M(C)$. Thus these vectors are in the kernel of the matrix obtained by deleting columns a_1, \dots, a_n and rows a_1, \dots, a_n of $S \cdot M(C)$. But this contradicts the fact that the matrix obtained by deleting columns a_1, \dots, a_n and rows a_1, \dots, a_n of $S \cdot M(C)$ is invertible. It follows that there is some $i \in T$ such that the Cauchon diagram C_i has $\dim(\ker(M(C_i))) \leq e - 1$. We claim that

$$\dim(\ker(M(C_i))) = e - 1.$$

To see this, observe that it is no loss of generality to assume that in the basis $\{v_1, \dots, v_e\}$ of $\ker(M(C))$ the vectors v_1, \dots, v_{e-1} have a zero i -th coordinate. Then the vectors v'_1, \dots, v'_{e-1} obtained by deleting the i -th coordinate from v_1, \dots, v_{e-1} are in the kernel of $M(C_i)$ and are linearly independent. The result follows. □

Recall that an \mathcal{H} -prime ideal in $R = \mathbb{O}_q(M_{m,n})$ is primitive if and only if its associated \mathcal{H} -stratum is 0-dimensional [Brown and Goodearl 2002, Theorem II.8.4].

Theorem 4.7. *Let J_C be the \mathcal{H} -prime of $R = \mathbb{O}_q(M_{m,n})$ associated to the Cauchon diagram C . Suppose that the dimension of the \mathcal{H} -stratum associated to J_C is equal to e . Then there is a chain $C = C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_e$ of $m \times n$ Cauchon diagrams such that $J_C = J_{C_0} \subsetneq J_{C_1} \subsetneq \dots \subsetneq J_{C_e}$ and, for all i , the dimension of the \mathcal{H} -stratum associated to J_{C_i} is $e - i$. In particular, J_{C_e} is a primitive \mathcal{H} -prime ideal in $\mathbb{O}_q(M_{m,n})$.*

Proof. We prove this by induction on e . If $e = 0$, then J_C is primitive and there is nothing to prove.

Suppose now that $e > 0$. By Proposition 4.6, there exists a Cauchon diagram C_1 obtained by turning a single white box of C black such that $\ker(M(C_1))$ has dimension $e - 1$. As $C = C_0 \subsetneq C_1$, it follows from Theorem 4.1 that $J_C = J_{C_0} \subsetneq J_{C_1}$. Moreover it follows from Proposition 4.2 that the dimension of the \mathcal{H} -stratum associated to J_{C_1} is equal to $\dim(\ker(M(C_1))) = e - 1$. The result is obtained by applying the induction hypothesis to J_{C_1} . □

5. A conjecture

Our results show that the possible dimensions of the \mathcal{H} -strata of $\mathbb{O}_q(M_{m,n})$ are $\{0, 1, 2, \dots, \min(m, n)\}$; moreover, every one of these values occurs as the dimension of some \mathcal{H} -stratum. What is still unresolved, however, is how exactly the dimensions of \mathcal{H} -strata of $\mathbb{O}_q(M_{m,n})$ are distributed. Earlier, we investigated the enumeration of 0-dimensional \mathcal{H} -strata in $\mathbb{O}_q(M_{m,n})$ [Bell et al. 2009; Bell et al. 2010]; this is the same as enumerating the primitive \mathcal{H} -primes of $\mathbb{O}_q(M_{m,n})$. We make a general conjecture about the proportion of i -dimensional \mathcal{H} -strata in $\mathbb{O}_q(M_{m,n})$ with $i \leq m$ and $n \geq m$.

Conjecture 5.1. Let m be a natural number. Then for $0 \leq i \leq m$ we have

$$\lim_{n \rightarrow \infty} \frac{\#i\text{-dimensional } \mathcal{H}\text{-strata in } \mathbb{O}_q(M_{m,n})}{\#\mathcal{H}\text{-strata in } \mathbb{O}_q(M_{m,n})} = 2^{1-\delta_{i,0}} \binom{2m}{m+i} 4^{-m}.$$

We have shown this for $(i, m) \in \{(0, 1), (1, 1), (0, 2), (1, 2), (2, 2), (0, 3), (2, 3)\}$; moreover, extensive computer computations suggest this is true in general.

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