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Sherman, Zelevinsky and Cerulli constructed canonically positive bases in cluster algebras associated to affine quivers having at most three vertices. Their constructions involve cluster monomials and normalized Chebyshev polynomials of the first kind evaluated at a certain "imaginary" element in the cluster algebra. Using this combinatorial description, it is possible to define for any affine quiver Q a set $\mathcal{B}(Q)$, which is conjectured to be the canonically positive basis of the acyclic cluster algebra $\mathcal{A}(Q)$.

In this article, we provide a geometric realization of the elements in $\mathcal{B}(Q)$ in terms of the representation theory of Q. This is done by introducing an analogue of the Caldero–Chapoton cluster character, where the usual quiver Grassmannian is replaced by a constructible subset called the *transverse quiver Grassmannian*.

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [2002; 2003; 2007; Berenstein et al. 2005] in order to define a combinatorial framework for studying positivity in algebraic groups and canonical bases in quantum groups. Since then, cluster algebras have found applications in various areas of mathematics, such as Lie theory, combinatorics, Teichmüller theory, Poisson geometry and quiver representations.

A (coefficient-free) cluster algebra $\mathcal{A}$ is a commutative $\mathbb{Z}$-algebra equipped with a distinguished set of generators, called *cluster variables*, gathered into possibly overlapping sets of fixed cardinality, called *clusters*. Monomials in variables belonging all to the same cluster are called *cluster monomials*. According to the *Laurent phenomenon* [Fomin and Zelevinsky 2002], it is known that $\mathcal{A}$ is a subalgebra of $\mathbb{Z}[\boldsymbol{c}^{\pm 1}]$ for any cluster $\boldsymbol{c}$ in $\mathcal{A}$. A nonzero element $y \in \mathcal{A}$ is called *positive* if y belongs to $\mathbb{Z}_{\geq 0}[\boldsymbol{c}^{\pm 1}]$ for any cluster $\boldsymbol{c}$ in $\mathcal{A}$. Following [Cerulli 2009], a $\mathbb{Z}$-basis $\mathcal{B} \subset \mathcal{A}$ is called *canonically positive* if the semiring of positive elements in $\mathcal{A}$ coincides with the set of $\mathbb{Z}_{\geq 0}$-linear combinations of elements of $\mathcal{B}$. Note that if such a basis exists, it is unique.

The problems of existence and description of a canonically positive basis in an arbitrary cluster algebra are still wide open. Both problems were first solved in the particular case of cluster algebras of finite type $\mathbb{A}_2$ and affine type $\tilde{\mathbb{A}}_{1,1}$ by Sherman and Zelevinsky [2004]. It was later extended by Cerulli [2009] for cluster algebras of affine type $\tilde{\mathbb{A}}_{2,1}$. To the best of the author's knowledge, these are the only known constructions of canonically positive bases in cluster algebras.

Using categorifications of acyclic cluster algebras with cluster categories and cluster characters, it is possible to rephrase Sherman–Zelevinsky and Cerulli constructions in order to place them in the more general context of acyclic cluster algebras associated to arbitrary affine quivers.

If Q is an acyclic quiver and $\boldsymbol{u}$ is a Q_0-tuple of indeterminates over $\mathbb{Z}$, we denote by $\mathcal{A}(Q)$ the acyclic cluster algebra with initial seed $(Q, \boldsymbol{u})$. We denote by $\mathscr{C}_Q$ the associated *cluster category* (over the field $\boldsymbol{k}$ of complex numbers) and by $X_? : \mathrm{Ob}(\mathscr{C}_Q) \to \mathbb{Z}[\boldsymbol{u}^{\pm 1}]$ the *Caldero–Chapoton map* on $\mathscr{C}_Q$, also called the (canonical) *cluster character* (see Section 2 for details). When Q is an affine quiver with positive minimal imaginary root δ, we set

$$\mathscr{B}(Q) = \mathcal{M}(Q) \sqcup \{F_n(X_\delta)X_R \mid n \geq 1 \text{ and } R \text{ is a regular rigid } \boldsymbol{k}Q\text{-module}\},$$

where $\mathcal{M}(Q)$ denotes the set of cluster monomials in $\mathcal{A}(Q)$, F_n denotes the n-th normalized Chebyshev polynomial of the first kind and X_δ is the evaluation of $X_?$ at any quasisimple module in a homogeneous tube of the Auslander–Reiten quiver $\Gamma(\boldsymbol{k}Q\text{-mod})$ of $\boldsymbol{k}Q$-mod.

If Q is of type $\tilde{\mathbb{A}}_{1,1}$ or $\tilde{\mathbb{A}}_{2,1}$, the set $\mathscr{B}(Q)$ coincides with the canonically positive basis constructed in [Sherman and Zelevinsky 2004] and [Cerulli 2009], respectively. It was conjectured in [Dupont 2010, Conjecture 7.10] that, for any affine quiver Q, the set $\mathscr{B}(Q)$ is the canonically positive basis of $\mathcal{A}(Q)$. Using the *generic basis*, it is possible to prove that, for any affine quiver Q, the set $\mathscr{B}(Q)$ is a $\mathbb{Z}$-basis in $\mathcal{A}(Q)$ [Dupont 2008; Ding et al. 2009]. Nevertheless, it is not known if this basis is the canonically positive basis in general.

An essential problem in investigating this question is that the elements of the form $F_n(X_\delta)X_R$ are defined combinatorially, and as yet have no representation-theoretic or geometric interpretation. The aim of this article is to provide such an interpretation.

Extending the idea of Caldero and Chapoton [2006], for any integrable bundle $\mathscr{F}$ on $\mathrm{rep}_k(Q)$ (see Section 4 for definitions), we define a map $\theta_{\mathscr{F}}$, called the *character associated to $\mathscr{F}$*, from the set of objects in $\mathscr{C}_Q$ to the ring $\mathbb{Z}[\boldsymbol{u}^{\pm 1}]$. With this termi-nology, the Caldero–Chapoton map $X_?$ is the character θ_{Gr}, where $\mathrm{Gr}: M \mapsto \mathrm{Gr}(M)$ denotes the integrable bundle of quiver Grassmannians.

For any indecomposable kQ-module M, we introduce a constructible subset $\mathrm{Tr}(M) \subset \mathrm{Gr}(M)$, called the *transverse quiver Grassmannian*. We prove that the bundle $\mathrm{Tr}: M \mapsto \mathrm{Tr}(M)$ is integrable on $\mathrm{rep}_k(Q)$ and that the elements in $\mathscr{B}(Q)$ can be described using the associated character θ_{Tr}. More precisely, we prove that for any $l \geq 1$,

$$F_l(X_\delta) = \theta_{\mathrm{Tr}}(M),$$

where M is any indecomposable kQ-module with dimension vector $l\delta$. It turns out that θ_{Tr}, unlike θ_{Gr}, is independent of the tube containing M. In particular, it takes the same values if M belongs to a homogeneous or to an exceptional tube. This is surprising since the usual quiver Grassmannians of two indecomposable modules of dimension $l\delta$ belonging to tubes of different ranks are in general completely different.

Moreover, if R is an indecomposable regular rigid kQ-module, then

$$F_l(X_\delta)X_R = \theta_{\mathrm{Tr}}(M),$$

where M is the unique indecomposable kQ-module of dimension $l\delta + \mathbf{dim}\, R$.

As a consequence, we obtain the following description of the set $\mathscr{B}(Q)$:

$$\mathscr{B}(Q) = \left\{ \theta_{\mathrm{Tr}}(M \oplus R) \,\middle|\, \begin{array}{l} M \text{ is an indecomposable (or zero) regular } kQ\text{-module,} \\ R \text{ is any rigid object in } \mathscr{C}_Q \text{ such that } \mathrm{Ext}^1_{\mathscr{C}_Q}(M, R) = 0 \end{array} \right\}.$$

This paper is organized as follows. In Section 2, we start by recalling several results concerning Chebyshev and generalized Chebyshev polynomials. Then we recall necessary background on cluster categories and cluster characters associ-ated to acyclic and especially affine quivers. Finally, we recall the known results concerning constructions of bases in affine cluster algebras.

In Section 3, we use the combinatorics of generalized Chebyshev polynomials to prove relations for cluster characters associated to regular kQ-modules when Q is an affine quiver with minimal imaginary root δ. These relations are generalizations of the *difference property* [Dupont 2008], used to compute the difference between cluster characters evaluated at indecomposable modules of dimension vector δ in different tubes.

Section 4 introduces the notions of integrable bundles on $\mathrm{rep}_{\boldsymbol{k}}(Q)$ and associated characters for any acyclic quiver. With this terminology, the Caldero–Chapoton map is the character associated to the quiver Grassmannian bundle. For affine quivers, we introduce the integrable bundle Tr of the Grassmannian of transverse submodules and see that it coincides with the Caldero–Chapoton map on rigid objects in the cluster category.

In Section 5, we prove that the elements in $\mathcal{B}(Q)$ can be expressed as values of the character θ_{Tr} associated to the integrable bundle Tr of $\mathrm{rep}_{\boldsymbol{k}}(Q)$. This provides a geometrization of the set $\mathcal{B}(Q)$.

In Section 6, we illustrate some of our results for quivers of affine types $\tilde{\mathbb{A}}_{1,1}$ and $\tilde{\mathbb{A}}_{2,1}$, putting [Sherman and Zelevinsky 2004; Cerulli 2009] into context.

2. Background, notation and terminology

Given a quiver Q, we denote by Q_0 its set of arrows and by Q_1 its set of vertices. We always assume that Q_0, Q_1 are finite sets and that the underlying unoriented graph of Q is connected. A quiver is called *acyclic* if it does not contain any oriented cycles.

We now fix an acyclic quiver Q and a Q_0-tuple $\boldsymbol{u} = (u_i \mid i \in Q_0)$ of indeterminates over $\mathbb{Z}$. We denote by $\mathcal{A}(Q)$ the coefficient-free cluster algebra with initial seed $(Q, \boldsymbol{u})$.

Chebyshev polynomials and their generalizations. Chebyshev (respectively generalized Chebyshev) polynomials are orthogonal polynomials in one variable (respectively several variables) playing an important role in the context of cluster algebras associated to representation-infinite quivers [Sherman and Zelevinsky 2004; Caldero and Zelevinsky 2006] (respectively [Dupont 2009; 2010]). We recall some basic results concerning these polynomials.

For any $l \geq 0$, the *l-th* (*normalized*) *Chebyshev polynomial of the first kind* is the polynomial F_l in $\mathbb{Z}[x]$ defined inductively by

$$F_0(x) = 2, \quad F_1(x) = x, \quad \text{and} \quad F_l(x) = x\,F_{l-1}(x) - F_{l-2}(x) \quad \text{for any } l \geq 2.$$

F_l is characterized by the following identity in $\mathbb{Z}[t, t^{-1}]$:

$$F_l(t + t^{-1}) = t^l + t^{-l}.$$

These polynomials first appeared in the context of cluster algebras in [Sherman and Zelevinsky 2004].

For any $l \geq 0$, the *l-th* (*normalized*) *Chebyshev polynomial of the second kind* is the polynomial S_l in $\mathbb{Z}[x]$ defined inductively by

$$S_0(x) = 1, \quad S_1(x) = x, \quad \text{and} \quad S_l(x) = x\,S_{l-1}(x) - S_{l-2}(x) \quad \text{for any } l \geq 2.$$

S_l is characterized by the following identity in $\mathbb{Z}[t, t^{-1}]$:

$$S_l(t + t^{-1}) = \sum_{k=0}^{n} t^{n-2k}.$$

Chebyshev polynomials of the Second kind first appeared in the context of cluster algebras in [Caldero and Zelevinsky 2006]. For any $l \geq 1$, $S_l(x)$ is the polynomial given by

$$S_l(x) = \det \begin{bmatrix} x & 1 & & & (0) \\ 1 & x & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ (0) & & & 1 & x \end{bmatrix},$$

where the matrix is tridiagonal in $M_l(\mathbb{Z}[x])$. The two kinds of Chebyshev polynomials are related by

$$F_l(x) = S_l(x) - S_{l-2}(x)$$

for any $l \geq 1$, with the convention that $S_{-1}(x) = 0$.

Fix a family $\{x_i \mid i \geq 1\}$ of indeterminates over $\mathbb{Z}$. For any $l \geq 0$, the *l-th generalized Chebyshev polynomial* is the polynomial in $\mathbb{Z}[x_1, \ldots, x_l]$ defined inductively by $P_0 = 1$, $P_1(x_1) = x_1$, and

$$P_l(x_1, \ldots, x_l) = x_l P_{l-1}(x_1, \ldots, x_{l-1}) - P_{l-2}(x_1, \ldots, x_{l-2}) \quad \text{for any } l \geq 2.$$

Equivalently,

$$P_l(x_1, \ldots, x_l) = \det \begin{bmatrix} x_l & 1 & & & (0) \\ 1 & x_{l-1} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ (0) & & & 1 & x_1 \end{bmatrix},$$

where the matrix is tridiagonal in $M_l(\mathbb{Z}[x_1, \ldots, x_l])$. These polynomials first appeared in the context of cluster algebras in [Dupont 2009] under the name of *generalized Chebyshev polynomials of infinite rank*, and similar polynomials also arose in the context of cluster algebras in [Yang and Zelevinsky 2008; Dupont 2010].

Cluster categories and cluster characters. Let $k Q$-mod be the category of finitely generated left-modules over the path algebra $k Q$ of Q. As usual, this category will be identified with the category $\text{rep}_k(Q)$ of finite-dimensional representations of Q over k.

For any vertex $i \in Q_0$, we denote by S_i the simple module associated to i, by P_i its projective cover, and by I_i its injective hull. We denote by $\langle -, - \rangle$ the

homological Euler form defined on kQ-mod by

$$\langle M, N \rangle = \dim \operatorname{Hom}_{kQ}(M, N) - \dim \operatorname{Ext}^1_{kQ}(M, N)$$

for any two kQ-modules M, N. Since Q is acyclic, kQ is a finite-dimensional hereditary algebra, and therefore $\langle -, - \rangle$ is well-defined on the Grothendieck group $K_0(kQ\text{-mod})$.

For any kQ-module M, the dimension vector of M is

$$\mathbf{dim}\, M = (\dim \operatorname{Hom}_{kQ}(P_i, M))_{i \in Q_0} \in \mathbb{N}^{Q_0}.$$

Viewed as a representation of Q, $\mathbf{dim}\, M = (\dim M(i))_{i \in Q_0}$ where $M(i)$ is the k-vector space at vertex i in the representation M of Q. The dimension vector map $\mathbf{dim}$ induces an isomorphism of abelian groups

$$\mathbf{dim} : K_0(kQ\text{-mod}) \xrightarrow{\sim} \mathbb{Z}^{Q_0}$$

sending the class of the simple S_i to the i-th vector of the canonical basis of $\mathbb{Z}^{Q_0}$.

The *cluster category* was introduced in [Buan et al. 2006] (see also [Caldero et al. 2006] for Dynkin type $\mathbb{A}$) in order to define a categorical framework for studying the cluster algebra $\mathcal{A}(Q)$. Let $D^b(kQ\text{-mod})$ be the bounded derived category of kQ-mod with shift functor [1] and Auslander–Reiten translation τ. The *cluster category* is the orbit category $\mathcal{C}_Q$ of the auto-functor $\tau^{-1}[1]$ in $D^b(kQ\text{-mod})$. It is a 2-Calabi–Yau triangulated category. The set of isoclasses of indecomposable objects in $\mathcal{C}_Q$ can be identified with the union of the set of isoclasses of indecomposable kQ-modules and the set of isoclasses of shifts of indecomposable projective kQ-modules [Keller 2005; Buan et al. 2006]. In particular, every object M in $\mathcal{C}_Q$ can be uniquely (up to isomorphism) decomposed into

$$M = M_0 \oplus P_M[1],$$

where M_0 is a kQ-module and P_M is a projective kQ-module.

Given a representation M of Q, the *quiver Grassmannian of M* is the set $\operatorname{Gr}(M)$ of all subrepresentations of M. For any element $\boldsymbol{e} \in \mathbb{Z}^{Q_0}$, the set

$$\operatorname{Gr}_{\boldsymbol{e}}(M) = \{N \text{ submodule of } M \mid \mathbf{dim}\, N = \boldsymbol{e}\}$$

is a projective variety. We denote by $\chi(\operatorname{Gr}_{\boldsymbol{e}}(M))$ its Euler characteristic with respect to the singular cohomology with rational coefficients.

Definition 2.1 [Caldero and Chapoton 2006]. The *Caldero–Chapoton map* is the map

$$X_? : \operatorname{Ob}(\mathcal{C}_Q) \to \mathbb{Z}[\boldsymbol{u}^{\pm 1}]$$

defined by:

- For any $i \in Q_0$,
$$X_{P_i[1]} = u_i.$$

- If M is an indecomposable kQ-module, then
$$X_M = \sum_{e \in \mathbb{N}^{Q_0}} \chi(\mathrm{Gr}_e(M)) \prod_{i \in Q_0} u_i^{-\langle e, S_i \rangle - \langle S_i, \dim M - e \rangle}. \tag{2-1}$$

- For any two objects M, N in $\mathscr{C}_Q$,
$$X_{M \oplus N} = X_M X_N.$$

Note that (2-1) also holds for decomposable modules.

Caldero and Keller [2006, Theorem 4] proved that $X_?$ induces a 1-1 correspondence between the set of isoclasses of indecomposable rigid objects (that is, without self-extensions) in $\mathscr{C}_Q$ and the set of cluster variables in $\mathscr{A}(Q)$. Moreover, $X_?$ induces a 1-1 correspondence between the set of isoclasses of cluster-tilting objects in $\mathscr{C}_Q$ and the set of clusters in $\mathscr{A}(Q)$. In particular, we have the following description of cluster monomials in $\mathscr{A}(Q)$:

$$\mathcal{M}(Q) = \{X_M \mid M \text{ is rigid in } \mathscr{C}_Q\}.$$

For any $\boldsymbol{d} = (d_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$, we set $\boldsymbol{u}^{\boldsymbol{d}} = \prod_{i \in Q_0} u_i^{d_i}$. For any Laurent polynomial $L \in \mathbb{Z}[\boldsymbol{u}^{\pm 1}]$, the *denominator vector* of L is the Q_0-tuple $\mathrm{den}(L) \in \mathbb{Z}^{Q_0}$ such that there exists a polynomial $P(u_i \mid i \in Q_0)$ not divisible by any u_i such that

$$L = \frac{P(u_i \mid i \in Q_0)}{\boldsymbol{u}^{\mathrm{den}(L)}}.$$

We define the dimension vector map $\dim_{\mathscr{C}_Q}$ on $\mathscr{C}_Q$ by setting $\dim_{\mathscr{C}_Q} M = \dim M$ if M is a kQ-module and $\dim_{\mathscr{C}_Q} P_i[1] = -\dim S_i$ and extending by additivity. Note that, for any kQ-module M, we have $\dim M = \dim_{\mathscr{C}_Q}(M)$; we will abuse notation and write $\dim M$ for any object in $\mathscr{C}_Q$. Caldero and Keller's denominator theorem [2006, Theorem 3] relates the denominator vector of the character with the dimension vector of the corresponding object in the cluster category:

$$\mathrm{den}(X_M) = \dim M,$$

for any object M in $\mathscr{C}_Q$.

Representation theory of affine quivers. We shall briefly recall some well-known facts concerning the representation theory of affine quivers. We refer the reader to [Simson and Skowroński 2007; Ringel 1984] for details.

We now fix an affine quiver Q, that is, an acyclic quiver of type $\tilde{\mathbb{A}}_n$, $(n \geq 1)$, $\tilde{\mathbb{D}}_n$, $(n \geq 4)$, $\tilde{\mathbb{E}}_n$, $(n = 6, 7, 8)$. We will say that a quiver is of affine type $\tilde{\mathbb{A}}_{r,s}$ if it is an orientation of an affine diagram of affine type $\tilde{\mathbb{A}}_{r+s-1}$, with r arrows going

clockwise and s arrows going counterclockwise. Let $\mathfrak{g}_Q$ denote the Kac–Moody algebra associated to Q.

We denote by $\Phi_{>0}$ the set of *positive roots* of $\mathfrak{g}_Q$, by $\Phi^{\mathrm{re}}_{>0}$ the set of positive *real roots* and by $\Phi^{\mathrm{im}}_{>0}$ the set of positive *imaginary roots*. Since Q is affine, there exists a unique $\delta \in \Phi_{>0}$ such that $\Phi^{\mathrm{im}}_{>0} = \mathbb{Z}_{>0}\delta$. We always identify the root lattice of $\mathfrak{g}_Q$ with $\mathbb{Z}^{Q_0}$ by sending the i-th simple root of $\mathfrak{g}_Q$ to the i-th vector of the canonical basis of $\mathbb{Z}^{Q_0}$.

According to Kac's theorem, for any $\boldsymbol{d} \in \mathbb{N}^{Q_0}$, there exists an indecomposable representation M such that $\mathbf{dim}\, M = \boldsymbol{d}$ if and only if $\boldsymbol{d} \in \Phi_{>0}$. Moreover, this representation is unique up to isomorphism if and only if $\boldsymbol{d} \in \Phi^{\mathrm{re}}_{>0}$. A positive root $\boldsymbol{d}$ is called a *Schur root* if there exists a (necessarily indecomposable) representation M of Q such that $\mathbf{dim}\, M = \boldsymbol{d}$ and $\mathrm{End}_{\boldsymbol{k}Q}(M) \simeq \boldsymbol{k}$.

We define a partial order $\leq$ on the root lattice by setting

$$\boldsymbol{e} \leq \boldsymbol{f} \quad \Longleftrightarrow \quad e_i \leq d_i \text{ for any } i \in Q_0,$$

and we set

$$\boldsymbol{e} \lneq \boldsymbol{f} \quad \text{if } \boldsymbol{e} \leq \boldsymbol{f} \text{ and } \boldsymbol{e} \neq \boldsymbol{f}.$$

The Auslander–Reiten quiver $\Gamma(\boldsymbol{k}Q\text{-mod})$ of $\boldsymbol{k}Q$-mod contains infinitely many connected components. There exists a connected component containing all the projective (resp. injective) modules, called the *preprojective* (resp. *preinjective*) component of $\Gamma(\boldsymbol{k}Q\text{-mod})$ and denoted by $\mathscr{P}$ (resp. $\mathscr{I}$). The other components are called *regular*. A $\boldsymbol{k}Q$-module M is called *preprojective*, *preinjective*, or *regular* if each indecomposable direct summand of M belongs to a component with the same property.

It is convenient to introduce the so-called *defect form* on $\mathbb{Z}^{Q_0}$. It is given by

$$\partial_? : \mathbb{Z}^{Q_0} \to \mathbb{Z}, \quad \boldsymbol{e} \mapsto \partial_{\boldsymbol{e}} = \langle \delta, \boldsymbol{e} \rangle.$$

By definition, the defect ∂_M of a $\boldsymbol{k}Q$-module M is the defect $\partial_{\mathbf{dim}\, M}$ of its dimension vector. It is well-known that an indecomposable $\boldsymbol{k}Q$-module M is preprojective, preinjective, or regular depending on whether ∂_M is negative, positive or zero.

The regular components in $\Gamma(\boldsymbol{k}Q\text{-mod})$ form a $\mathbb{P}^1(\boldsymbol{k})$-family of tubes. Thus, for every tube $\mathscr{T}$, there exists an integer $p \geq 1$, called the *rank* of $\mathscr{T}$, such that $\mathscr{T} \simeq \mathbb{Z}\mathbb{A}_\infty/(\tau^p)$. The tubes of rank 1 are called *homogeneous*, while those of rank $p > 1$ are called *exceptional*. At most three tubes are exceptional in $\Gamma(\boldsymbol{k}Q\text{-mod})$. It is well-known that the full subcategory of $\boldsymbol{k}Q$-mod formed by the objects in any tube $\mathscr{T}$ is standard, that is, isomorphic to the mesh category of $\mathscr{T}$. It is also known that there are neither morphisms nor extensions between pairwise distinct tubes.

An indecomposable regular $\boldsymbol{k}Q$-module M is called *quasisimple* if it is at the mouth of the tube, or equivalently, if it does not contain any proper regular submodule. For any quasisimple module R in a tube $\mathscr{T}$ and any integer $l \geq 1$, we denote by

$R^{(l)}$ the unique indecomposable kQ-module with quasisocle R and quasilength l. For any indecomposable regular kQ-module $R^{(l)}$, we denote by

$$\text{qsoc } R^{(l)} = R$$

the *quasisocle* of M and by

$$\text{qrad } R^{(l)} = R^{(l-1)}$$

the quasiradical of M with the convention that $R^{(0)} = 0$.

For any indecomposable regular kQ-module M, we have

$$M \text{ is rigid} \iff \mathbf{dim}\, M \lneqq \delta; \quad \text{End}_{kQ}(M) \simeq k \iff \mathbf{dim}\, M \leq \delta.$$

Cluster characters associated to modules in tubes are known to be governed by the combinatorics of generalized Chebyshev polynomials. More precisely, it is proved in [Dupont 2009, Theorem 5.1] that for any quasisimple module M in a tube $\mathcal{T}$, we have

$$X_{M^{(l)}} = P_l(X_M, X_{\tau^{-1}M}, \ldots, X_{\tau^{-l+1}M}).$$

In particular, if $\mathcal{T}$ is homogeneous, we get $X_{M^{(l)}} = S_l(X_M)$, recovering a result of [Caldero and Zelevinsky 2006].

In [Dupont 2009, Theorem 7.2], generalized Chebyshev polynomials provide multiplication formulas for cluster characters associated to indecomposable regular kQ-modules. The following theorem will be essential in the proofs.

Theorem 2.2 [Dupont 2009]. *Let Q be an affine quiver and $\mathcal{T}$ be a tube of rank p in $\Gamma(kQ\text{-mod})$. Let $R_i, i \in \mathbb{Z}$ denote the quasisimple modules in $\mathcal{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ and $R_{i+p} \simeq R_i$ for any $i \in \mathbb{Z}$. Let $m, n > 0$ be integers and $j \in [0, p-1]$. Then, for every $k \in \mathbb{Z}$ such that $0 < j + kp \leq n$ and $m \geq n - j - kp$, we have the identity*

$$X_{R_j^{(m)}} X_{R_0^{(n)}} = X_{R_0^{(m+j+kp)}} X_{R_j^{(n-j-kp)}} + X_{R_0^{(j+kp-1)}} X_{R_{n+1}^{(m+j+kp-n-1)}}.$$

Bases in affine cluster algebras. We shall now review some results concerning the construction of $\mathbb{Z}$-bases in cluster algebras associated to affine quivers. In this section, Q still denotes an affine quiver with positive minimal imaginary root δ.

If M, N are quasisimple modules in distinct homogeneous tubes, then $X_M = X_N$; see [Dupont 2008], for example. We denote this common value by X_δ, and call it the *generic variable of dimension δ*, as in that earlier paper.

Theorem 2.3 [Dupont 2008; Ding et al. 2009]. *Let Q be an affine quiver. Then*

$$\mathcal{G}(Q) = \mathcal{M}(Q) \sqcup \{X_\delta^l X_R \mid l \geq 1, R \text{ is a regular rigid } kQ\text{-module}\}$$

is a $\mathbb{Z}$-basis of $\mathcal{A}(Q)$.

Moreover, den *induces a 1-1 correspondence from $\mathcal{G}(Q)$ to $\mathbb{Z}^{Q_0}$.*

The set $\mathscr{G}(Q)$ is called the *generic basis* of $\mathscr{A}(Q)$.

Since F_l and S_l are monic polynomials of degree l, it follows that, for any affine quiver Q, the sets

$$\mathscr{B}(Q) = \mathcal{M}(Q) \sqcup \{F_l(X_\delta)X_R \mid l \geq 1, \ R \text{ is a regular rigid } \boldsymbol{k}Q\text{-module}\}$$

and

$$\mathscr{C}(Q) = \mathcal{M}(Q) \sqcup \{S_l(X_\delta)X_R \mid l \geq 1, \ R \text{ is a regular rigid } \boldsymbol{k}Q\text{-module}\}$$

are $\mathbb{Z}$-bases of the cluster algebra $\mathscr{A}(Q)$.

When Q is the Kronecker quiver, $\mathscr{B}(Q)$ coincides with the canonically positive basis constructed in [Sherman and Zelevinsky 2004] and $\mathscr{C}(Q)$ coincides with the basis constructed in [Caldero and Zelevinsky 2006]. When Q is a quiver of affine type $\tilde{\mathbb{A}}_{2,1}$, the basis $\mathscr{B}(Q)$ is the canonically positive basis of $\mathscr{A}(Q)$ constructed in [Cerulli 2009].

Since $X_\delta = X_M$ for any quasisimple module M in a homogeneous tube, it follows that $S_l(X_\delta) = X_{M^{(l)}}$ for any $l \geq 0$, so the set $\mathscr{C}(Q)$ has an interpretation in terms of the cluster character $X_?$. No such interpretation had been known for the set $\mathscr{B}(Q)$; this paper provides one.

The map $\phi : \mathscr{G}(Q) \to \mathscr{B}(Q)$ preserving cluster monomials and sending $X_\delta^l X_R$ to $F_l(X_\delta)X_R$ for any $l \geq 1$ and any rigid regular module R is a 1-1 correspondence. We denote by

$$\mathfrak{b}_? : \mathbb{Z}^{Q_0} \xrightarrow{1:1} \mathscr{B}(Q), \quad \boldsymbol{d} \mapsto \mathfrak{b}_{\boldsymbol{d}}$$

the 1-1 correspondence obtained by composing the bijection above with the one provided in Theorem 2.3.

3. Difference properties of higher orders

Here Q still denotes an affine quiver with positive minimal imaginary root δ.

The difference property. In [Dupont 2008] we introduced the *difference property*, which relates the possibly different values of cluster characters evaluated at different indecomposable representations of dimension δ. The difference property was crucial in that article. It is also an essential ingredient in this one, since transverse Grassmannians will arise precisely from difference properties of higher orders.

This difference property was established in [Dupont 2008] for affine type $\tilde{\mathbb{A}}$ and in [Ding et al. 2009] in general. It can be expressed as follows:

Theorem 3.1 [Dupont 2008; Ding et al. 2009]. *If Q is an affine quiver and M is any indecomposable module of dimension δ, then*

$$\mathfrak{b}_\delta = X_\delta = X_M - X_{\mathrm{qrad}\,M/\,\mathrm{qsoc}\,M},$$

with the convention that $X_{\mathrm{qrad}\,M/\,\mathrm{qsoc}\,M} = 0$ if M is quasisimple.

Higher-difference properties. The aim of this section is to provide an analogue of Theorem 3.1 for $\mathfrak{b}_d$ when d is any positive root with zero defect. We will first consider the imaginary roots and then the real roots of defect zero.

We fix a tube $\mathcal{T}$ in $\Gamma(kQ\text{-mod})$ of rank $p \geq 1$. The quasisimples of $\mathcal{T}$ are denoted by R_i, with $i \in \mathbb{Z}/p\mathbb{Z}$, ordered so that $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. Note that for any $l \geq 1$ and $0 \leq k \leq p-1$ and any $i \in \mathbb{Z}/p\mathbb{Z}$, we have $\mathbf{dim}\, R_i^{(lp)} = l\delta \in \Phi_{>0}^{\mathrm{im}}$ and $\mathbf{dim}\, R_i^{(lp+k)} \in \Phi_{>0}^{\mathrm{re}}$ if $k \neq 0$.

The following technical lemma will be used in the proof of Proposition 3.3:

Lemma 3.2. *With the notation above, for any $l \geq 1$,*

$$X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} = X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}}.$$

Proof. We first notice that generalized Chebyshev polynomials are symmetric in the sense that for every $i \in \mathbb{Z}$ and $n \geq 1$,

$$P_n(x_i, \ldots, x_{i+n-1}) = P_n(x_{i+n-1}, \ldots, x_i).$$

If $l = 1$, the result is obvious. We thus fix some $l \geq 2$. For technical convenience, we denote by R_i, $i \in \mathbb{Z}$ the quasisimple modules in $\mathcal{T}$ and we assume that $R_i \simeq R_{i+p}$ for every $i \in \mathbb{Z}$. Consider the morphism of $\mathbb{Z}$-algebras

$$\phi : \mathbb{Z}[X_{R_0}, \ldots, X_{R_{lp-1}}] \to \mathbb{Z}[X_{R_0}, \ldots, X_{R_{lp-1}}]$$
$$X_{R_i} \mapsto X_{R_{lp-1-i}} \qquad\qquad \text{for all } i = 0, \ldots, lp-1.$$

It is well-defined since $X_{R_0}, \ldots, X_{R_{p-1}}$ are known to be algebraically independent over $\mathbb{Z}$ (see for example [Dupont 2009]).

According to Theorem 2.2, we have

$$X_{R_1^{(p-1)}} X_{R_0^{(lp-1)}} = X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_{p+1}^{((l-1)p-2)}}.$$

According to [Dupont 2009, Theorem 5.1], each of the $X_{R_j}^{(k)}$ appearing above lies in $\mathbb{Z}[X_{R_0}, \ldots, X_{R_{lp-1}}]$. We can thus apply ϕ and we get

$$\phi(X_{R_1^{(p-1)}})\phi(X_{R_0^{(lp-1)}}) = \phi(X_{R_0^{(p)}})\phi(X_{R_1^{(lp-2)}}) - \phi(X_{R_{p+1}^{((l-1)p-2)}}). \qquad (3\text{-}1)$$

We now compute these images under ϕ.

$$\begin{aligned}
\phi(X_{R_1^{(p-1)}}) &= \phi(P_{p-1}(X_{R_1}, \ldots, X_{R_{p-1}})) \\
&= P_{p-1}(\phi(X_{R_1}), \ldots, \phi(X_{R_{p-1}})) = P_{p-1}(X_{R_{lp-2}}, \ldots, X_{R_{(l-1)p}}) \\
&= P_{p-1}(X_{R_{(l-1)p}}, \ldots, X_{R_{lp-2}}) = X_{R_{(l-1)p}^{(p-1)}} = X_{R_0^{(p-1)}};
\end{aligned}$$

$$\phi(X_{R_0^{(lp-1)}}) = \phi(P_{lp-1}(X_{R_0}, \ldots, X_{R_{lp-2}}))$$

$$= P_{lp-1}(\phi(X_{R_0}), \ldots, \phi(X_{R_{lp-2}})) = P_{lp-1}(X_{R_{lp-1}}, \ldots, X_{R_1})$$

$$= P_{lp-1}(X_{R_1}, \ldots, X_{R_{lp-1}}) = X_{R_1^{(lp-1)}};$$

$$\phi(X_{R_0^{(p)}}) = \phi(P_p(X_{R_0}, \ldots, X_{R_{p-1}})) = P_p(\phi(X_{R_0}), \ldots, \phi(X_{R_{p-1}}))$$

$$= P_p(X_{R_{lp-1}}, \ldots, X_{R_{(l-1)p}}) = P_p(X_{R_{(l-1)p}}, \ldots, X_{R_{lp-1}})$$

$$= X_{R_{(l-1)p}^{(p)}} = X_{R_0^{(p)}};$$

$$\phi(X_{R_1^{(lp-2)}}) = \phi(P_{lp-2}(X_{R_1}, \ldots, X_{R_{lp-2}}))$$

$$= P_{lp-2}(\phi(X_{R_1}), \ldots, \phi(X_{R_{lp-2}})) = P_{lp-2}(X_{R_{lp-2}}, \ldots, X_{R_1})$$

$$= P_{lp-2}(X_{R_1}, \ldots, X_{R_{lp-2}}) = X_{R_1^{(lp-2)}};$$

$$\phi(X_{R_{p+1}^{((l-1)p-2)}}) = \phi(P_{(l-1)p-2}(X_{R_{p+1}}, \ldots, X_{R_{lp-2}}))$$

$$= P_{(l-1)p-2}(\phi(X_{R_{p+1}}), \ldots, \phi(X_{R_{lp-2}}))$$

$$= P_{(l-1)p-2}(X_{R_{(l-1)p-2}}, \ldots, X_{R_1})$$

$$= P_{(l-1)p-2}(X_{R_1}, \ldots, X_{R_{(l-1)p-2}}) = X_{R_1^{((l-1)p-2)}}.$$

Substituting in (3-1), we get

$$X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} = X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_1^{((l-1)p-2)}}$$

$$= X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_{p+1}^{((l-1)p-2)}} = X_{R_1^{(p-1)}} X_{R_0^{(lp-1)}}. \qquad \square$$

We can now prove some higher-difference properties for imaginary roots.

Proposition 3.3. *Fix $l \geq 1$. Then for any indecomposable representation M in* $\mathrm{rep}_k(Q, l\delta)$, *we have*

$$\mathfrak{b}_{l\delta} = F_l(X_\delta) = X_M - X_{\mathrm{qrad}\,M/\,\mathrm{qsoc}\,M},$$

with the convention that $X_{\mathrm{qrad}\,M/\,\mathrm{qsoc}\,M} = 0$ if M is quasisimple.

Proof. We first treat the case where M is an indecomposable representation of dimension $l\delta$ in a homogeneous tube. It is not necessary to prove it separately but in this particular case, the proof is straightforward. We write $M = R^{(l)}$ for some quasisimple module R in a homogeneous tube. If $l = 1$, the proposition follows from Theorem 3.1. If $l \geq 2$, $\mathrm{qrad}\,M/\,\mathrm{qsoc}\,M \simeq R^{(l-2)}$, so that

$$X_M - X_{\mathrm{qrad}\,M/\,\mathrm{qsoc}\,M} = X_{R^{(l)}} - X_{R^{(l-2)}}$$

$$= S_l(X_R) - S_{l-2}(X_R) = F_l(X_R) = F_l(X_\delta).$$

We now assume that M is an indecomposable representation of dimension $l\delta$ in an exceptional tube $\mathcal{T}$ of rank $p \geq 2$. We denote by $R_0, \dots, R_{p-1}$ the quasisimples in $\mathcal{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. We can thus write $M \simeq R_i^{(lp)}$ for some $i \in \mathbb{Z}/p\mathbb{Z}$. Without loss of generality, we assume that $i = 0$. In order to simplify the notation, for any $l \geq 1$, we write

$$\Delta_l = X_{R_0^{(lp)}} - X_{R_1^{(lp-2)}}.$$

We thus have to prove that for any $l \geq 1$,

$$\Delta_l = F_l(X_\delta).$$

The central tool in this proof is Theorem 2.2. According to Theorem 3.1, we have

$$X_\delta = X_{R_0^{(p)}} - X_{R_1^{(p-2)}},$$

so the proposition holds for $l = 1$.

We now prove it for $l = 2$. We have

$$\begin{aligned}
F_2(X_\delta) = X_\delta^2 - 2 &= (X_{R_0^{(p)}} - X_{R_1^{(p-2)}})^2 - 2 \\
&= X_{R_0^{(p)}}^2 + X_{R_1^{(p-2)}}^2 - 2X_{R_0^{(p)}}X_{R_1^{(p-2)}} - 2,
\end{aligned}$$

but according to the almost split multiplication formula [Caldero and Chapoton 2006, Proposition 3.10], we have

$$X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} = X_{R_0^{(p)}}X_{R_1^{(p-2)}},$$

so

$$F_2(X_\delta) = X_{R_0^{(p)}}^2 - 2X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} + X_{R_1^{(p-2)}}^2.$$

But, according to Theorem 2.2, we have

$$X_{R_0^{(p)}}^2 = X_{R_0^{(p)}}X_{R_0^{(p)}} = X_{R_0^{(2p)}} + X_{R_0^{(p-1)}}X_{R_1^{(p-1)}},$$

so that finally

$$F_2(X_\delta) = X_{R_0^{(2p)}} - X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} + X_{R_1^{(p-2)}}^2.$$

Thus,

$$\begin{aligned}
F_2(X_\delta) = \Delta_2 &\iff -X_{R_1^{(2p-2)}} = X_{R_1^{(p-2)}}^2 - X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} \\
&\iff X_{R_1^{(2p-2)}} + X_{R_1^{(p-2)}}^2 - X_{R_0^{(p-1)}}X_{R_1^{(p-1)}} = 0.
\end{aligned}$$

But

$$X_{R_1^{(2p-2)}} = -X_{R_1^{(p-3)}}X_{R_0^{(p-1)}} + X_{R_1^{(p-2)}}X_{R_{p-1}^{(p)}},$$

so

$$F_2(X_\delta) = \Delta_2 \iff X^2_{R_1^{(p-2)}} + X_{R_1^{(p-2)}} X_{R_{p-1}^{(p)}} - X_{R_0^{(p-1)}} X_{R_1^{(p-1)}} - X_{R_0^{(p-1)}} X_{R_1^{(p-3)}} = 0$$

$$\iff X_{R_1^{(p-2)}} \left[X_{R_1^{(p-2)}} + X_{R_{p-1}^{(p)}} \right] - X_{R_0^{(p-1)}} \left[X_{R_1^{(p-3)}} + X_{R_1^{(p-1)}} \right] = 0.$$

Theorem 2.2 gives

$$X_{R_{p-1}} X_{R_0^{(p-1)}} = X_{R_1^{(p-2)}} + X_{R_{p-1}^{(p)}} \quad \text{and} \quad X_{R_1^{(p-2)}} X_{R_{p-1}} = X_{R_1^{(p-3)}} + X_{R_1^{(p-1)}},$$

so

$$F_2(X_\delta) = \Delta_2$$

and the proposition is proved for $l = 2$.

For $l > 2$, we will use the three-term relations for first kind Chebyshev polynomials

$$F_l(x) = x F_{l-1}(x) - F_{l-2}(x).$$

Thus, it is enough to prove that for any $l \geq 2$,

$$\Delta_{l+1} = \Delta_1 \Delta_l - \Delta_{l-1}.$$

In order to simplify our notation, we denote by LHS the left side of the equality above and by RHS the right side. We thus have

$$\text{RHS} = (X_{R_0^{(p)}} - X_{R_1^{(p-2)}})(X_{R_0^{(lp)}} - X_{R_1^{(lp-2)}}) - (X_{R_0^{((l-1)p)}} - X_{R_1^{((l-1)p-2)}})$$

$$= X_{R_0^{(p)}} X_{R_0^{(lp)}} - X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_1^{(p-2)}} X_{R_0^{(lp)}}$$

$$+ X_{R_1^{(p-2)}} X_{R_1^{(lp-2)}} - X_{R_0^{((l-1)p)}} + X_{R_1^{((l-1)p-2)}}.$$

But, according to the multiplication theorem, we get

$$X_{R_0^{(p)}} X_{R_0^{(lp)}} = X_{R_0^{((l+1)p)}} + X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}},$$

so

$$\text{LHS} = \text{RHS} \iff X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_1^{(p-2)}} X_{R_0^{(lp)}}$$

$$+ X_{R_1^{(p-2)}} X_{R_1^{(lp-2)}} - X_{R_0^{((l-1)p)}} + X_{R_1^{((l-1)p-2)}} + X_{R_1^{((l+1)p-2)}} = 0.$$

Applying the multiplication theorem, we get

$$X_{R_0^{(lp-2)}} X_{R_{lp-2}^{(p)}} = X_{R_0^{((l+1)p-2)}} + X_{R_0^{(lp-3)}} X_{R_{lp-1}^{(p-1)}},$$

so

$$X_{R_1^{((l+1)p-2)}} = X_{R_1^{(lp-2)}} X_{R_{p-1}^{(p)}} - X_{R_1^{(lp-3)}} X_{R_0^{(p-1)}},$$

and thus

$$\text{LHS} = \text{RHS} \iff X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(p)}} X_{R_1^{(lp-2)}} - X_{R_1^{(p-2)}} X_{R_0^{(lp)}} + X_{R_1^{(p-2)}} X_{R_1^{(lp-2)}}$$

$$- X_{R_0^{((l-1)p)}} + X_{R_1^{((l-1)p-2)}} + X_{R_1^{(lp-2)}} X_{R_{p-1}^{(p)}} - X_{R_1^{(lp-3)}} X_{R_0^{(p-1)}} = 0.$$

But

$$X_{R_0^{(p)}} X_{R_1^{(lp-2)}} = X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} + X_{R_1^{((l-1)p-2)}};$$

hence,

$$\text{LHS} = \text{RHS} \iff X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}}$$
$$- X_{R_1^{(p-2)}} X_{R_0^{(lp)}} + X_{R_1^{(p-2)}} X_{R_1^{(lp-2)}}$$
$$- X_{R_0^{((l-1)p)}} + X_{R_1^{(lp-2)}} X_{R_{p-1}^{(p)}} - X_{R_1^{(lp-3)}} X_{R_0^{(p-1)}} = 0$$
$$\iff X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}}$$
$$- X_{R_1^{(p-2)}} X_{R_0^{(lp)}} - X_{R_0^{((l-1)p)}} - X_{R_1^{(lp-3)}} X_{R_0^{(p-1)}}$$
$$+ X_{R_1^{(lp-2)}} \left(X_{R_1^{(p-2)}} + X_{R_{p-1}^{(p)}} \right) = 0.$$

Theorem 2.2 gives

$$X_{R_{p-1}} X_{R_0^{(p-1)}} = X_{R_1^{(p-2)}} + X_{R_{p-1}^{(p)}},$$

so

$$\text{LHS} = \text{RHS} \iff X_{R_0^{(p-1)}} \left(X_{R_1^{(lp-1)}} - X_{R_1^{(lp-3)}} + X_{R_1^{(lp-2)}} X_{R_{p-1}} \right)$$
$$- X_{R_0^{((l-1)p)}} - X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} - X_{R_1^{(p-2)}} X_{R_0^{(lp)}} = 0.$$

The three-term relation for generalized Chebyshev polynomials gives

$$X_{R_1^{(lp-1)}} = X_{R_{p-1}} X_{R_1^{(lp-2)}} - X_{R_1^{(lp-3)}}.$$

Thus

$$\text{LHS} = \text{RHS}$$
$$\iff 2X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - \left(X_{R_0^{((l-1)p)}} + X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} + X_{R_0^{(lp)}} X_{R_1^{(p-2)}} \right) = 0.$$

Theorem 2.2 gives

$$X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} = X_{R_0^{(lp)}} X_{R_1^{(p-2)}} + X_{R_0^{((l-1)p)}},$$

so we finally get

$$\text{LHS} = \text{RHS} \iff X_{R_0^{(p-1)}} X_{R_1^{(lp-1)}} - X_{R_0^{(lp-1)}} X_{R_1^{(p-1)}} = 0.$$

The second equality holds by Lemma 3.2, so we have proved that for any $l \geq 2$,

$$\Delta_{l+1} = \Delta_1 \Delta_l - \Delta_{l-1}.$$

Since we know that $\Delta_1 = X_\delta$ and $\Delta_2 = F_2(X_\delta)$, it follows that $\Delta_l = F_l(X_\delta)$ for any $l \geq 1$. □

We are now able to prove the general difference property:

Theorem 3.4. *Let Q be an affine quiver, $\mathcal{T}$ be a tube of rank $p \geq 1$ in $\Gamma(kQ\text{-mod})$. For any $l \geq 1$ and any $0 \leq k \leq p-1$, we have*

$$\mathfrak{b}_{l\delta + \mathbf{dim}\, R_0^{(k)}} = X_{R_0^{(k)}} F_l(X_\delta) = X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}},$$

with the convention that $X_{R_0^{(-1)}} = 0$.

Proof. The first equality follows from

$$\mathrm{den}(X_{R_0^{(k)}} X_\delta^l) = \mathbf{dim}\, R_0^{(k)} + l\delta,$$

so

$$\mathfrak{b}_{l\delta + \mathbf{dim}\, R_0^{(k)}} = X_{R_0^{(k)}} F_l(X_\delta).$$

We now prove that

$$X_{R_0^{(k)}} F_l(X_\delta) = X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}},$$

with the convention that $X_{R_0^{(-1)}} = 0$.

We denote by LHS the left side and by RHS the right side of this equation.

$$\text{LHS} = X_{R_0^{(k)}}\left(X_{R_k^{(lp)}} - X_{R_{k+1}^{(lp-2)}}\right) = X_{R_0^{(k)}} X_{R_k^{(lp)}} - X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}}$$

$$= X_{R_0^{(lp+k)}} + X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-1)}} - X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}}.$$

If $l = 1$ and $k = p-1$, we get

$$\text{LHS} = X_{R_0^{(lp+k)}} + X_{R_0^{(p-2)}} X_{R_0^{(p-1)}} - X_{R_0^{(p-1)}} X_{R_0^{(p-2)}} = X_{R_0^{(lp+k)}} = \text{RHS}.$$

Otherwise, LHS $=$ RHS if and only if

$$X_{R_{k+1}^{(lp-k-2)}} = X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}} - X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-1)}}. \tag{3-2}$$

Using the three-term recurrence relations for generalized Chebyshev polynomials, we have

$$X_{R_0^{(k)}} = X_{R_{k-1}} X_{R_0^{(k-1)}} - X_{R_0^{(k-2)}} \quad \text{and} \quad X_{R_{k+1}^{(lp-1)}} = X_{R_{lp+k-1}} X_{R_{k+1}^{(lp-2)}} - X_{R_{k+1}^{(lp-3)}},$$

so that, replacing in the right side of (3-2), we get

$$X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}} - X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-1)}} = X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-3)}} - X_{R_0^{(k-2)}}.$$

Thus, by induction, we get

$$X_{R_0^{(k)}} X_{R_{k+1}^{(lp-2)}} - X_{R_0^{(k-1)}} X_{R_{k+1}^{(lp-1)}} = X_{R_0} X_{R_{k+1}^{(lp-k-1)}} - X_{R_{k+1}^{(lp-k)}}.$$

Now, the three-term recurrence relation gives

$$X_{R_{k+1}^{(lp-k)}} = X_{R_{k+1+lp-k-1}} X_{R_{k+1}^{(lp-k-1)}} - X_{R_{k+1}^{(lp-k-2)}} = X_{R_0} X_{R_{k+1}^{(lp-k-1)}} - X_{R_{k+1}^{(lp-k-2)}},$$

and thus

$$X_{R_0} X_{R_{k+1}^{(lp-k-1)}} - X_{R_{k+1}^{(lp-k)}} = X_{R_{k+1}^{(lp-k-2)}},$$

proving that (3-2) holds.   □

As a corollary, for any positive root d with defect zero, we obtain a description of $\mathfrak{b}_d$ as a certain difference of cluster characters:

Corollary 3.5. *Let Q be an affine quiver and d be a positive root with defect zero. Let M be any indecomposable representation of dimension d. Then there exists a quasisimple module R_0 in a tube of rank $p \geq 1$, an integer $0 \leq k \leq p-1$, and an integer $l \geq 0$ such that $d = l\delta + \mathbf{dim}\, R_0^{(k)}$. Moreover, for any such R_0, k, l, we have*

$$\mathfrak{b}_d = X_{R_0^{(k)}} F_l(X_\delta) = X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}},$$

where R_i, with $i \in \mathbb{Z}/p\mathbb{Z}$, are the quasisimple modules in $\mathcal{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ for every $i \in \mathbb{Z}/p\mathbb{Z}$.

4. Integrable bundles on $\mathrm{rep}_k(Q)$ and their characters

In the previous section, we obtained a realization of the elements $\mathfrak{b}_d$ associated to defect zero roots as differences of cluster characters. The aim of this section is to introduce a new map θ_{Tr} such that these elements correspond precisely to values of θ_{Tr}.

Unless otherwise specified, Q denotes an arbitrary acyclic quiver in this section.

Integrable bundles. For any $d \in \mathbb{N}^{Q_0}$, the representation variety $\mathrm{rep}_k(Q, d)$ of dimension d is the set of all representations M of Q with dimension vector d. Note that

$$\mathrm{rep}_k(Q, d) \simeq \prod_{i \to j \in Q_1} \mathrm{Hom}_k(k^{d_i}, k^{d_j}),$$

so that $\mathrm{rep}_k(Q, d)$ is an affine irreducible variety.

Definition 4.1. Let Q be any acyclic quiver. An *integrable bundle* on $\mathrm{rep}_k(Q)$ is a map

$$\mathscr{F} : M \mapsto \mathscr{F}(M) \subset \mathrm{Gr}(M)$$

defined on the set of indecomposable objects in $\mathrm{rep}_k(Q)$ such that for any $M \in \mathrm{rep}_k(Q)$,

- for any $e \in \mathbb{N}^{Q_0}$, $\mathscr{F}_e(M) = \mathscr{F}(M) \cap \mathrm{Gr}_e(M)$ is constructible, and
- if $M \simeq N$ in $\mathrm{rep}_k(Q)$, then $\chi(\mathscr{F}_e(M)) \simeq \chi(\mathscr{F}_e(N))$ for any $e \in \mathbb{N}^{Q_0}$.

Remark 4.2. Note that, if $\mathscr{F}$ is an integrable bundle on $\mathrm{rep}_{\boldsymbol{k}}(Q)$, then the family $(\chi(\mathscr{F}_{\boldsymbol{e}}(M)))_{\boldsymbol{e} \in \mathbb{N}^{Q_0}}$ has finite support.

Example 4.3. The map $M \mapsto \mathrm{Gr}(M)$ is an integrable bundle called the *quiver Grassmannian bundle*.

For any $\boldsymbol{k}Q$-module M and any submodule $U \subset M$, we set

$$\mathrm{Gr}^U(M) = \{N \in \mathrm{Gr}(M) \mid U \text{ is a submodule of } N\}.$$

This is a constructible subset in the quiver Grassmannian $\mathrm{Gr}(M)$.

If Q is an affine quiver, we define another integrable bundle Tr as follows. Let M be an indecomposable $\boldsymbol{k}Q$-module. If M is rigid, we set $\mathrm{Tr}(M) = \mathrm{Gr}(M)$. If M is not rigid, it is regular and we can thus write $M = R_0^{(lp+k)}$ for some quasisimple module R_0 in a tube of rank $p \geq 1, l \geq 1$ and $0 \leq k \leq p-1$. There exists a nonzero monomorphism $\imath : R_0^{(lp-1)} \to R_0^{(lp)}$ such that

$$\mathrm{Hom}_{\boldsymbol{k}Q}(R_0^{(lp-1)}, R_0^{(lp)}) \simeq \boldsymbol{k}\imath.$$

The set $\imath(\mathrm{Gr}^{R_0^{(k+1)}}(R_0^{(lp-1)}))$ is a constructible subset of $\mathrm{Gr}(R_0^{(lp)})$, and since

$$\mathrm{Hom}_{\boldsymbol{k}Q}(R_0^{(lp-1)}, R_0^{(lp)}) \simeq \boldsymbol{k}\imath,$$

it does not depend on the choice of $\imath$. We can thus identify $\mathrm{Gr}^{R_0^{(k+1)}}(R_0^{(lp-1)})$ with a constructible subset of $\mathrm{Gr}(R_0^{(lp)})$. With these notations and identifications, we set

$$\mathrm{Tr}(M) = \mathrm{Gr}(M) \setminus \mathrm{Gr}^{R_0^{(k+1)}}(R_0^{(lp-1)}).$$

Note that if $l = 0$, M is rigid and we recover the equality $\mathrm{Tr}(M) = \mathrm{Gr}(M)$.

For every dimension vector $\boldsymbol{e} \in \mathbb{N}^{Q_0}$ and any indecomposable $\boldsymbol{k}Q$-module M, the *transverse quiver Grassmannian of M* (of dimension $\boldsymbol{e}$) is the constructible subset of $\mathrm{Gr}_{\boldsymbol{e}}(M)$

$$\mathrm{Tr}_{\boldsymbol{e}}(M) = \{N \in \mathrm{Tr}(M) \mid \mathbf{dim}\, N = \boldsymbol{e}\}.$$

The map

$$\mathrm{Tr} : M \mapsto \mathrm{Tr}(M) \subset \mathrm{Gr}(M)$$

is an integrable bundle on $\mathrm{rep}_{\boldsymbol{k}}(Q)$.

Character associated to an integrable bundle. Extending an idea of Caldero and Chapoton, we associate to any integrable bundle on $\mathrm{rep}_{\boldsymbol{k}}(Q)$ a map from the set of objects in $\mathscr{C}_Q$ to the ring $\mathbb{Z}[\boldsymbol{u}^{\pm 1}]$ of Laurent polynomials in the initial cluster of $\mathscr{A}(Q)$.

Definition 4.4. Let $\mathscr{F}$ be an integrable bundle on $\mathrm{rep}_{\boldsymbol{k}}(Q)$. The *character associated to $\mathscr{F}$* is the map

$$\theta_{\mathscr{F}}(?) : \mathrm{Ob}(\mathscr{C}_Q) \to \mathbb{Z}[\boldsymbol{u}^{\pm 1}]$$

given by:

- If $M \simeq P_i[1]$ for some $i \in Q_0$, then $\theta_{\mathscr{F}}(P_i[1]) = u_i$.

- If M is an indecomposable kQ-module, then

$$\theta_{\mathscr{F}}(M) = \sum_{e \in \mathbb{N}^{Q_0}} \chi(\mathscr{F}_e(M)) \prod_{i \in Q_0} u_i^{-\langle e, S_i \rangle - \langle S_i, \mathbf{dim}\, M - e \rangle}.$$

- $\theta_{\mathscr{F}}(M \oplus N) = \theta_{\mathscr{F}}(M)\theta_{\mathscr{F}}(N)$ for any two objects M, N in $\mathscr{C}_Q$.

We now prove that θ_{Tr} coincides with $X_?$ on the set of rigid objects in $\mathscr{C}_Q$. In particular, this will allow us to realize cluster monomials in terms of θ_{Tr}.

Lemma 4.5. *Let Q be an affine quiver. Then, for any rigid object M in $\mathscr{C}_Q$, we have $\theta_{\mathrm{Tr}}(M) = X_M$. In particular,*

$$\mathscr{M}(Q) = \{\theta_{\mathrm{Tr}}(M) \mid M \text{ is rigid in } \mathscr{C}_Q\}.$$

Proof. Let M be a rigid object in $\mathscr{C}_Q$. We write

$$M = P_{i_1}[1] \oplus \cdots \oplus P_{i_r}[1] \oplus M_1 \oplus \cdots \oplus M_s,$$

where each P_{i_j} is an indecomposable projective kQ-module and each M_i is an indecomposable module. Also, since M is rigid, each M_i is a rigid kQ-module and thus $\mathrm{Tr}(M_i) = \mathrm{Gr}(M_i)$ for any $i \in \{1, \ldots, s\}$. In particular, it follows that $\theta_{\mathrm{Tr}}(M_i) = X_{M_i}$ for any $i \in \{1, \ldots, s\}$. Then

$$\begin{aligned}
\theta_{\mathrm{Tr}}(M) &= \theta_{\mathrm{Tr}}(P_{i_1}[1] \oplus \cdots \oplus P_{i_r}[1] \oplus M_1 \oplus \cdots \oplus M_s) \\
&= \theta_{\mathrm{Tr}}(P_{i_1}[1]) \cdots \theta_{\mathrm{Tr}}(P_{i_r}[1])\theta_{\mathrm{Tr}}(M_1) \cdots \theta_{\mathrm{Tr}}(M_s) \\
&= u_{i_1} \cdots u_{i_r} X_{M_1} \cdots X_{M_s} \\
&= X_{P_{i_1}[1] \oplus \cdots \oplus P_{i_r}[1] \oplus M_1 \oplus \cdots \oplus M_s} = X_M.
\end{aligned}$$

The second assertion follows directly from Caldero and Keller's realization of cluster monomials:

$$\mathscr{M}(Q) = \{X_M \mid M \text{ is rigid in } \mathscr{C}_Q\} = \{\theta_{\mathrm{Tr}}(M) \mid M \text{ is rigid in } \mathscr{C}_Q\}. \qquad \square$$

5. A geometrization of $\mathscr{B}(Q)$

We now relate the character θ_{Tr} with the difference properties obtained in Section 3. This will provide a realization of the elements in $\mathscr{B}(Q)$ in terms of θ_{Tr}.

From difference properties to θ_{Tr}. Using Theorem 3.4, we first deduce a realization in terms of θ_{Tr} of the elements in $\mathscr{B}(Q)$ corresponding to positive roots.

Theorem 5.1. *Let d be any positive root. Then $\mathfrak{b}_d = \theta_{\mathrm{Tr}}(M)$, where M is any indecomposable representation of dimension d.*

Proof. If $\boldsymbol{d}$ is a positive root with nonzero defect, then $\boldsymbol{d}$ is real and there exists a unique indecomposable representation M in $\mathrm{rep}_{\boldsymbol{k}}(Q, \boldsymbol{d})$. Moreover, this representation has to be preprojective or preinjective. In both cases, it is rigid and thus $\mathfrak{b}_{\boldsymbol{d}} = X_M = \theta_{\mathrm{Tr}}(M)$. We can thus assume that $\boldsymbol{d} \in \mathbb{N}^{Q_0}$ is a root with defect zero.

Let M be an indecomposable representation in $\mathrm{rep}_{\boldsymbol{k}}(Q, \boldsymbol{d})$. It is necessarily contained in a tube $\mathscr{T}$ of rank $p \geq 1$. We denote by R_i, with $i \in \mathbb{Z}/p\mathbb{Z}$, the quasisimple modules in $\mathscr{T}$ ordered such that $\tau R_i \simeq R_{i-1}$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. We can write $\boldsymbol{d} = l\delta + \boldsymbol{n}$, where $\boldsymbol{n}$ is either a real Schur root or zero. If $\boldsymbol{n} \neq 0$, there exists a unique indecomposable representation N in $\mathrm{rep}_{\boldsymbol{k}}(Q, \boldsymbol{n})$. In any case, if $M \simeq R_0^{(lp+k)}$ with $l \geq 0$ and $0 \leq k \leq p-1$, N is the rigid representation $R_0^{(k)}$ (still with the convention that $R_0^{(0)} = 0$) and

$$\mathfrak{b}_{\boldsymbol{d}} = X_{R_0^{(k)}} F_l(X_\delta).$$

Now, according to Theorem 3.4, we have

$$X_{R_0^{(k)}} F_l(X_\delta) = X_{R_0^{(lp+k)}} - X_{R_{k+1}^{(lp-k-2)}}.$$

For any $\boldsymbol{e} \in \mathbb{N}^{Q_0}$, the map $\mathrm{Gr}_{\boldsymbol{e}}^{R_0^{(k+1)}}(R_0^{(lp-1)}) \to \mathrm{Gr}_{\boldsymbol{e}-\mathbf{dim}\, R_0^{(k+1)}}(R_{k+1}^{(lp-k-2)})$ given by

$$U \mapsto U/R_0^{(k+1)}$$

is an algebraic isomorphism, and we denote by $c_{\boldsymbol{e}} \in \mathbb{Z}$ the common value of the Euler characteristics of these constructible sets. Fix now some $\boldsymbol{e} \in \mathbb{N}^{Q_0}$; the monomial corresponding to $\boldsymbol{e}$ in $X_{R_0^{(lp+k)}}$ is

$$c_{\boldsymbol{e}} \prod_i u_i^{-\langle \boldsymbol{e}, S_i \rangle - \left\langle S_i, \mathbf{dim}\, R_0^{(lp+k)} - \boldsymbol{e} \right\rangle},$$

and the monomial corresponding to $\boldsymbol{e} - \mathbf{dim}\, R_0^{(k+1)}$ in $X_{R_{k+1}^{(lp-k-2)}}$ is

$$c_{\boldsymbol{e}} \prod_i u_i^{-\left\langle \boldsymbol{e}-\mathbf{dim}\, R_0^{(k+1)}, S_i \right\rangle - \left\langle S_i, \mathbf{dim}\, R_{k+1}^{(lp-k-2)} + \mathbf{dim}\, R_0^{(k+1)} - \boldsymbol{e} \right\rangle}.$$

We now prove that these monomials are the same. For any $i = 0, \ldots, p-1$, we set $r_i = \mathbf{dim}\, R_i$ and we denote by c the Coxeter transformation on $\mathbb{Z}^{Q_0}$ induced by the Auslander–Reiten translation. We recall that for any $\beta, \gamma \in \mathbb{Z}^{Q_0}$, we have $\langle \gamma, c(\beta) \rangle = -\langle \beta, \gamma \rangle$. With this notation, we have

$$\mathbf{dim}\, R_0^{(k+1)} = r_0 + \cdots + r_k \quad \text{and} \quad \mathbf{dim}\, R_{k+1}^{(lp-k-2)} = (l-1)\delta + r_{k+1} + \cdots r_{p-2},$$

so

$$\mathbf{dim}\, R_0^{(k+1)} + \mathbf{dim}\, R_{k+1}^{(lp-k-2)} = l\delta - r_{p-1}.$$

We now compute the exponents:

$$
\begin{aligned}
-\langle e, S_i\rangle - \left\langle S_i, \mathbf{dim}\, R_0^{(lp+k)} - e\right\rangle &= -\langle e, S_i\rangle - \langle S_i, l\delta + r_0 + \cdots + r_{k-1} - e\rangle \\
&= -\langle e, S_i\rangle - \langle S_i, l\delta - e\rangle - \langle S_i, r_0 + \cdots + r_{k-1}\rangle \\
&= -\langle e, S_i\rangle - \langle S_i, l\delta - e\rangle - \langle r_1 + \cdots + r_k, S_i\rangle\,;
\end{aligned}
$$

$$
\begin{aligned}
-\left\langle e - \mathbf{dim}\, R_0^{(k+1)}, S_i\right\rangle - \left\langle S_i, \mathbf{dim}\, R_{k+1}^{(lp-k-2)} + \mathbf{dim}\, R_0^{(k+1)} - e\right\rangle & \\
&\hspace{-9cm} = -\langle e, S_i\rangle + \langle r_0 + \cdots + r_k, S_i\rangle + \langle S_i, r_{p-1}\rangle - \langle S_i, l\delta - e\rangle \\
&\hspace{-9cm} = -\langle e, S_i\rangle - \langle S_i, l\delta - e\rangle + \langle r_1 + \cdots + r_k, S_i\rangle\,.
\end{aligned}
$$

Thus the two monomials are the same. It follows that

$$
\begin{aligned}
X_{R_0^{(lp+k)}} &- X_{R_{k+1}^{(lp-k-2)}} \\
&= \sum_e \chi(\mathrm{Gr}_e(R_0^{(lp+k)})) \prod_i u_i^{-\langle e, S_i\rangle - \langle S_i, l\delta - e\rangle - \langle r_1 + \cdots + r_k, S_i\rangle} \\
&\quad - \sum_e \chi\left(\mathrm{Gr}_{e - \mathbf{dim}\, R_0^{(k+1)}}(R_{k+1}^{(lp-k-2)})\right) \prod_i u_i^{-\langle e, S_i\rangle - \langle S_i, l\delta - e\rangle - \langle r_1 + \cdots + r_k, S_i\rangle} \\
&= \sum_e \chi\left(\mathrm{Gr}_e(R_0^{(lp+k)}) \setminus \mathrm{Gr}_e^{R_0^{(k+1)}}(R_0^{(lp-1)})\right) \prod_i u_i^{-\langle e, S_i\rangle - \langle S_i, l\delta - e\rangle - \langle r_1 + \cdots + r_k, S_i\rangle} \\
&= \sum_e \chi(\mathrm{Tr}_e(R_0^{(lp+k)})) \prod_i u_i^{-\langle e, S_i\rangle - \left\langle S_i, \mathbf{dim}\, R_0^{(lp+k)} - e\right\rangle}. \qquad \square
\end{aligned}
$$

Realization of $\mathcal{B}(Q)$ in terms of θ_{Tr}. Summing up the previous results, we deduce the following geometric description of $\mathcal{B}(Q)$:

Theorem 5.2. *Let Q be an affine quiver. Then*

$$
\mathcal{B}(Q) = \left\{ \theta_{\mathrm{Tr}}(M \oplus R) \,\middle|\, \begin{array}{l} M \text{ is an indecomposable (or zero) regular } \mathbf{k}Q\text{-module,} \\ R \text{ is any rigid object in } \mathscr{C}_Q \text{ such that } \mathrm{Ext}^1_{\mathscr{C}_Q}(M, R) = 0 \end{array} \right\}.
$$

Proof. We denote by $\mathscr{S}$ the right side of the claimed equality. By definition,

$$
\mathcal{B}(Q) = \mathcal{M}(Q) \sqcup \{F_l(X_\delta) X_R \mid l \geq 1, R \text{ is a regular rigid } \mathbf{k}Q\text{-module}\}.
$$

We first prove that $\mathscr{S} \subset \mathcal{B}(Q)$. Let R be a rigid object in $\mathscr{C}_Q$; then $\theta_{\mathrm{Tr}}(R)$ is a cluster monomial by Lemma 4.5. Fix now M to be an indecomposable regular $\mathbf{k}Q$-module in a tube $\mathscr{T}$ such that $\mathrm{Ext}^1_{\mathscr{C}_Q}(M, R) = 0$. If M is rigid, then $M \oplus R$ is rigid in $\mathscr{C}_Q$, and $\theta_{\mathrm{Tr}}(M)\theta_{\mathrm{Tr}}(R) = \theta_{\mathrm{Tr}}(M \oplus R)$ is a cluster monomial by Lemma 4.5. Now, if M is nonrigid, then $d = \mathbf{dim}\, M$ is a positive root of defect zero, and thus $\theta_{\mathrm{Tr}}(M) = \mathfrak{b}_d$ by Theorem 5.1. Thus, there exists $l \geq 1$ and N an indecomposable

rigid (or zero) module in $\mathcal{T}$ such that $\mathbf{d} = l\delta + \mathbf{dim}\, N$. According to Theorem 5.1,

$$\theta_{\mathrm{Tr}}(M \oplus R) = \theta_{\mathrm{Tr}}(M)\theta_{\mathrm{Tr}}(R) = \mathfrak{b}_{l\delta + \mathbf{dim}\, N}\theta_{\mathrm{Tr}}(R) = F_l(X_\delta)X_N\theta_{\mathrm{Tr}}(R)$$

$$= F_l(X_\delta)\theta_{\mathrm{Tr}}(N)\theta_{\mathrm{Tr}}(R) = F_l(X_\delta)\theta_{\mathrm{Tr}}(N \oplus R).$$

Since $\mathrm{Ext}^1_{\mathscr{C}_Q}(M, R) = 0$, we have $\mathrm{Ext}^1_{kQ}(M, R) = 0$ and $\mathrm{Ext}^1_{kQ}(R, M) = 0$. Thus, it follows easily that $\mathrm{Ext}^1_{kQ}(N, R) = 0$ and $\mathrm{Ext}^1_{kQ}(R, N) = 0$, so $N \oplus R$ is a rigid regular kQ-module. In particular, $\theta_{\mathrm{Tr}}(N \oplus R) = X_{N \oplus R}$ and thus

$$\theta_{\mathrm{Tr}}(M \oplus R) = F_l(X_\delta)X_{N \oplus R} \in \mathscr{B}(Q).$$

Conversely, fix an element in $\mathscr{B}(Q)$. If x is a cluster monomial, then according to Lemma 4.5, there exists some rigid object M in $\mathscr{C}_Q$ such that $x = \theta_{\mathrm{Tr}}(M)$. Thus, $x \in \mathcal{S}$. Fix now some regular rigid kQ-module R and some integer $l \geq 1$. Then the direct summands of R belong to exceptional tubes. We fix an indecomposable kQ-module M of dimension vector $l\delta$ in a homogeneous tube. Then $\mathrm{Ext}^1_{\mathscr{C}_Q}(M, R) = 0$. According to Theorem 5.1, we have $F_l(X_\delta)X_R = \theta_{\mathrm{Tr}}(M)X_R$, but R is rigid, so $X_R = \theta_{\mathrm{Tr}}(R)$. Thus,

$$F_l(X_\delta)X_R = \theta_{\mathrm{Tr}}(M)\theta_{\mathrm{Tr}}(R) = \theta_{\mathrm{Tr}}(M \oplus R) \in \mathcal{S}. \qquad \square$$

6. Examples

We shall now study two examples corresponding to cases where it is known that $\mathscr{B}(Q)$ is the canonically positive basis in $\mathscr{A}(Q)$.

The $\tilde{\mathbb{A}}_{1,1}$ case. Let Q be the Kronecker quiver, that is, the affine quiver of type $\tilde{\mathbb{A}}_{1,1}$ with the orientation

$$Q: \qquad 1 \Longrightarrow 2$$

and minimal imaginary root $\delta = (11)$.

For any $\lambda \in k$, we set

$$M_\lambda: \qquad k \overset{1}{\underset{\lambda}{\Longrightarrow}} k$$

and

$$M_\infty: \qquad k \overset{0}{\underset{1}{\Longrightarrow}} k.$$

It is well-known that every tube in $\Gamma(kQ\text{-mod})$ is homogeneous and that the family $\{M_\lambda \mid \lambda \in k \sqcup \{\infty\}\}$ is a complete set of representatives of pairwise nonisomorphic quasisimple kQ-modules.

For any $n \geq 1$, the indecomposable representations of quasilength n are given by

$$M_\lambda^{(n)}: \qquad k^n \overset{1}{\underset{J_n(\lambda)}{\Longrightarrow}} k^n$$

for any $\lambda \in \boldsymbol{k}$ and

$$M_{\infty}^{(n)}: \qquad \boldsymbol{k}^n \underset{1}{\overset{J_n(0)}{\rightrightarrows}} \boldsymbol{k}^n \ ,$$

where $J_n(\lambda) \in M_n(\boldsymbol{k})$ denotes the Jordan block of size n associated to the eigenvalue λ. Quiver Grassmannians and transverse quiver Grassmannians of indecomposable representations with quasilength 2 are described in Table 1.

Note that $\boldsymbol{k}Q$-mod contains no regular rigid modules. It follows that in this case

$$\mathcal{B}(Q) = \mathcal{M}(Q) \sqcup \{\theta_{\mathrm{Tr}}(M) \mid M \text{ is an indecomposable regular } \boldsymbol{k}Q\text{-module}\}.$$

According to [Sherman and Zelevinsky 2004], this set is the canonically positive basis of $\mathcal{A}(Q)$.

From Table 1, we see that for any $\lambda \in \boldsymbol{k} \sqcup \{\infty\}$,

$$\theta_{\mathrm{Tr}}(M_{\lambda}^{(2)}) = \theta_{\mathrm{Gr}}(M_{\lambda}^{(2)}) - 1 = X_{M_{\lambda}^{(2)}} - 1 = S_2(X_{M_{\lambda}}) - 1 = F_2(X_{M_{\lambda}}) = \mathfrak{b}_{2\delta}.$$

This illustrates Theorem 5.1.

The $\tilde{\mathbb{A}}_{2,1}$ case. We now consider the quiver Q of affine type $\tilde{\mathbb{A}}_{2,1}$ equipped with the orientation
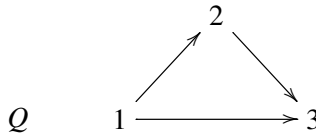
$\boldsymbol{e}$	$\mathrm{Gr}_{\boldsymbol{e}}(M_{\lambda}^{(2)})$	$\mathrm{Tr}_{\boldsymbol{e}}(M_{\lambda}^{(2)})$	$\boldsymbol{u}^{\langle -\boldsymbol{e}, S_i \rangle - \langle S_i, 2\delta - \boldsymbol{e} \rangle}$
(00)	$\{0\}$	$\{0\}$	u_1^2/u_2^2
(01)	$\mathbb{P}^1 \times \{S_2\}$	$\mathbb{P}^1 \times \{S_2\}$	$1/u_2^2$
(02)	$\{S_2 \oplus S_2\}$	$\{S_2 \oplus S_2\}$	$1/(u_1^2 u_2^2)$
(11)	$\{M_{\lambda}\}$	$\varnothing$	1
(12)	see caption		$1/u_1^2$
(22)	$\{M_{\lambda}^{(2)}\}$	$\{M_{\lambda}^{(2)}\}$	u_2^2/u_1^2

Table 1. Grassmannians and transverse Grassmannians of indecomposable modules of quasilength 2 in type $\tilde{\mathbb{A}}_{1,1}$. Here λ takes values in $\boldsymbol{k} \cup \{0, \infty\}$. For $\boldsymbol{e} = (12)$, the values of both $\mathrm{Gr}_{(12)}(M_{\lambda}^{(2)})$ and $\mathrm{Tr}_{(12)}(M_{\lambda}^{(2)})$ are $\{P_1, M_{\lambda} \oplus S_2\}$ if $\lambda \in \{0, \infty\}$, and $\mathbb{P}^1 \times \{M_{\lambda} \oplus S_2\}$ otherwise.

The minimal imaginary root of Q is $\delta = (111)$. For any $\lambda \in \boldsymbol{k}$, we set

$$M_\lambda \qquad
\begin{array}{c}
\boldsymbol{k} \\
{}^1\nearrow \qquad \searrow^\lambda \\
\boldsymbol{k} \xrightarrow{\ 1\ } \boldsymbol{k}
\end{array}$$

and

$$M_\infty \qquad
\begin{array}{c}
\boldsymbol{k} \\
{}^1\nearrow \qquad \searrow^1 \\
\boldsymbol{k} \xrightarrow{\ 0\ } \boldsymbol{k}
\end{array}$$

Further, $\Gamma(\boldsymbol{k}Q\text{-mod})$ contains exactly one exceptional tube $\mathscr{T}$ of rank 2, whose quasisimples are

$$R_0 \qquad
\begin{array}{c}
0 \\
{}^0\nearrow \qquad \searrow \\
\boldsymbol{k} \xrightarrow{\ 1\ } \boldsymbol{k}
\end{array}$$

and

$$R_1 \simeq S_2 \qquad
\begin{array}{c}
\boldsymbol{k} \\
\nearrow \qquad \searrow^0 \\
0 \xrightarrow{\quad} 0
\end{array}$$

The set $\{M_\lambda \mid \lambda \in \boldsymbol{k} \sqcup \{\infty\}\} \sqcup \{R_0^{(2)}\}$ is a complete set of representatives of pairwise nonisomorphic indecomposable representations in $\mathrm{rep}_{\boldsymbol{k}}(Q, \delta)$. For any $\lambda \neq 0, \infty$, M_λ is a quasisimple $\boldsymbol{k}Q$-module in a homogeneous tube. Moreover, $M_0 = R_1^{(2)}$ and M_∞ is quasisimple in a homogeneous tube.

Quiver Grassmannians and transverse quiver Grassmannians of indecomposable representations of dimension δ are described in Table 2. For simplicity, we only listed the dimension vectors that lead to nonempty quiver Grassmannians.

In Table 2, we observe that $X_{M_\lambda} = X_{M_0} - 1 = X_{M_\infty} - 1$, illustrating Theorem 3.1. Also, we see that $\theta_{\mathrm{Tr}}(M_\lambda) = \theta_{\mathrm{Tr}}(M_0) = \theta_{\mathrm{Tr}}(M_\infty)$ for any $\lambda \in \boldsymbol{k} \setminus \{0\}$ so that the transverse character does not depend on the chosen tube. Moreover,

$$\theta_{\mathrm{Tr}}(M_\lambda) = X_{M_\lambda} = F_1(X_\delta),$$

illustrating Theorem 5.1.

Remark 6.1. Table 2 justifies the terminology *transverse submodule*. Indeed, we see that, given two indecomposable regular modules M and N having the same dimension vectors, the submodules U in $\mathrm{Tr}(M)$ are those having a corresponding submodule in $\mathrm{Gr}(N)$. In some sense, we can see U as a submodule "common" to M and N. This is why we call it *transverse*.

e	$\mathrm{Gr}_e(M_\lambda)$	$\mathrm{Tr}_e(M_\lambda)$	$\mathrm{Gr}_e(R_0^{(2)})$	$\mathrm{Tr}_e(R_0^{(2)})$	$u^{\langle -e, S_i\rangle - \langle S_i, \delta - e\rangle}$
(000)	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	u_1/u_3
(001)	$\{S_3\}$	$\{S_3\}$	$\{S_3\}$	$\{S_3\}$	$1/(u_2 u_3)$
(010)	see caption		$\varnothing$	$\varnothing$	1
(011)	see caption		$\{P_2\}$	$\{P_2\}$	$1/(u_1 u_2)$
(101)	$\varnothing$	$\varnothing$	$\{R_0\}$	$\varnothing$	1
(111)	$\{M_\lambda\}$	$\{M_\lambda\}$	$\{R_0^{(2)}\}$	$\{R_0^{(2)}\}$	u_3/u_1

Table 2. Grassmannians and transverse Grassmannians for quasilength 2 in type $\tilde{\mathbb{A}}_{2,1}$. Here λ takes values in $\boldsymbol{k} \cup \{0, \infty\}$. For $e = (010)$, all entries are $\varnothing$ except that $\mathrm{Gr}_{010}(M_0) = \{S_2\}$. For $e = (011)$, all entries equal $\{P_2\}$, apart from that $\mathrm{Gr}_{(011)}(M_0)$ and $\mathrm{Tr}_{(011)}(M_0)$, which equal $\{S_2 \oplus S_3\}$.

As suggested by Bernhard Keller, this notion of transversality should have a more precise meaning in the context of deformation theory. Some connections are known at this time, and this will be discussed in a forthcoming article.

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