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Parabolic induction and Hecke modules in characteristic p for p-adic GL_n

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We classify the simple supersingular modules for the pro-p-Iwahori Hecke algebra $\mathscr{H}$ of p-adic GL_n by proving a conjecture by Vignéras about a *mod p numerical Langlands correspondence* on the side of the Hecke modules. We define a process of induction for $\mathscr{H}$-modules in characteristic p that reflects the parabolic induction for representations of the p-adic general linear group and explore the semisimplification of the standard nonsupersingular $\mathscr{H}$-modules in light of this process.

1. Introduction

Let F be a p-adic field and let $n \geq 1$ be an integer. When exploring the category of smooth mod p representations of $\mathrm{GL}_n(F)$, it is natural to consider the functor that associates to such a representation its subspace of invariant vectors under the action of the pro-p-Iwahori subgroup of $\mathrm{GL}_n(F)$. It has values in the category of right modules in characteristic p over the pro-p Hecke algebra $\mathscr{H}$. The structure of this Hecke algebra has been studied by Vignéras [2005], and the classification of the simple modules in the case $n = 3$ is given in [Ollivier 2006b]. Three families of $\mathscr{H}$-modules appear, namely, the *regular, singular*, and *supersingular* ones. This

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definition resonates with the idea that, just as the regular modules should be related to the principal series and the supersingular modules to the supersingular representations, likewise the singular modules should be related to the hybrid case where one induces a supersingular representation from a strict Levi subgroup of $\mathrm{GL}_n(F)$. The first link has been explored and proves fruitful [Ollivier 2006a; 2006c; Grosse-Klönne 2009; Vignéras 2008]. Except for the isolated case of $\mathrm{GL}_2(\mathbb{Q}_p)$, the link between supersingular modules and representations does not seem tight enough to give substantial information about the supersingular representations [Breuil and Paskunas 2007]. However, a striking numerical coincidence occurs: in this article (Section 7), we prove Conjecture 1 of [Vignéras 2005], which says that any nonzero simple supersingular module contains a character for the affine Hecke subalgebra of $\mathscr{H}$. It implies the following result, which can be seen as a numerical Langlands correspondence on the side of the Hecke modules.

Theorem 1.1. *The number of n-dimensional simple supersingular modules* (*with fixed action of the uniformizer*) *over the pro-p-Hecke algebra of* $\mathrm{GL}_n(F)$ *is equal to the number of smooth irreducible n-dimensional mod p representations of the absolute Galois group of F* (*with fixed determinant of a Frobenius*).

The aim of Sections 5 and 6 is to investigate the nonsupersingular Hecke modules. We define a process of *induction* for Hecke modules in characteristic p and relate it to the parabolic induction on the side of the representations of $\mathrm{GL}_n(F)$. In characteristic zero, one of the ingredients for the construction of types by covers consists in embedding a Hecke algebra relative to a Levi subgroup into a Hecke algebra relative to $\mathrm{GL}_n(F)$ using Iwahori decomposition and the notion of *positive subalgebra*. This allows a reading of the parabolic induction of representations in terms of induction on the side of the Hecke modules [Bushnell and Kutzko 1998, §6]. Some of these results can be adapted to the case of mod ℓ representations when $\ell \neq p$ [Vignéras 1998; Dat 1999]. In characteristic p, one cannot expect an injection of the pro-p Hecke algebra $\mathscr{H}(L)$ relative to a strict standard Levi subgroup L into the pro-p Hecke algebra of $\mathrm{GL}_n(F)$. Nevertheless, it is still true for the *positive* part $\mathscr{H}(L^+)$ of $\mathscr{H}(L)$. We now provide a summary of the results proved in this article, keeping in mind that all the modules have mod p coefficients.

Let $\mathfrak{M}$ be a right $\mathscr{H}(L)$-module with scalar action of the uniformizers. The $\mathscr{H}$-module *induced* from $\mathfrak{M}$ is defined in Section 5A by the tensor product over $\mathscr{H}(L^+)$ of $\mathfrak{M}$ by $\mathscr{H}$. This process of induction defines an exact functor from the category of $\mathscr{H}(L)$-modules with scalar action of the uniformizers into the category of right $\mathscr{H}$-modules.

In Section 5B, we recall the definition of a standard $\mathscr{H}$-module: a regular, singular or supersingular character (with values in a field with characteristic p) of the commutative part $\mathscr{A}$ of $\mathscr{H}$ gives rise to a standard module. This standard module and any of its quotients are then called regular, singular or supersingular respectively. Any simple $\mathscr{H}$-module is a quotient of a standard module. We show in Section 5C that the standard modules relative to *L-adapted* characters of $\mathscr{A}$ are *induced* from $\mathscr{H}(L)$-modules in the sense defined above. These are a special case of nonsupersingular standard modules. Owing to intertwining operators defined in Section 5D, any nonsupersingular standard module can be related to a standard module of this kind. We then give sufficient conditions for these operators to be isomorphisms, from which we deduce:

• Assuming that Conjecture 5.20 is true, we bolster the definition of *nonsupersingular modules* with the proof that any simple nonsupersingular $\mathscr{H}$-module appears in the semisimplification of a standard module that is *induced* from a $\mathscr{H}(L)$-module, where L is a strict Levi subgroup of $\mathrm{GL}_n(F)$. We prove the conjecture and its consequence for the simple modules that are actually modules over the Iwahori–Hecke algebra. The key to this proof is a theorem by Rogawski [1985] which relies on the Kazhdan–Lusztig polynomials for the Iwahori–Hecke algebra in characteristic zero (Section 5E).

• We show that if an irreducible $\mathscr{H}(L)$-module $\mathfrak{M}$ satisfies Hypothesis $(\star)$, it gives rise by induction to an irreducible $\mathscr{H}$-module (Section 5F).

• In Section 6B, we consider the compact induction $\mathscr{U}$ (resp. $\mathscr{U}_L$) of the trivial character of the pro-p-Iwahori subgroup of $\mathrm{GL}_n(F)$ (resp. L), and relate the representation $\mathfrak{M}\otimes_{\mathscr{H}(L^+)}\mathscr{U}$ to the one which is parabolically induced from $\mathfrak{M}\otimes_{\mathscr{H}(L)}\mathscr{U}_L$. Denote the latter representation of $\mathrm{GL}_n(F)$ by $\rho_{\mathfrak{M}}$.

We compare the $\mathscr{H}$-module induced from $\mathfrak{M}$ with the pro-p-invariant subspace of $\rho_{\mathfrak{M}}$. So far we have made no specific hypothesis about the p-adic field F, the Levi subgroup L, or the $\mathscr{H}(L)$-module $\mathfrak{M}$ with scalar action of the uniformizers.

In Section 6D we give some examples in the case where $F = \mathbb{Q}_p$ and the standard Levi subgroup L is isomorphic to a product of $\mathrm{GL}_1(\mathbb{Q}_p)$'s and $\mathrm{GL}_2(\mathbb{Q}_p)$'s. In these cases, the irreducible representations of L and the corresponding Hecke modules are thoroughly understood. Our process of induction describes explicitly the pro-p-invariant subspace of $\rho_{\mathfrak{M}}$, which is irreducible as a Hecke module in the chosen examples. After the first version of this article was written, however, Herzig announced that he could prove that these representations $\rho_{\mathfrak{M}}$ are actually irreducible.

While this article does not draw on Herzig's work [2010, Theorem 8.1], it is noticeable that Hypothesis $(\star)$ reflects parallel conditions. Our approach, which focuses on the Hecke modules, does not require any further hypotheses on F and L. A barrier to further investigation of the pro-p-invariant subspace of the irreducible induced representations classified in [Herzig 2010] is the lack of knowledge of the (pro-p-invariants of) supersingular representations of L, for general L and F.

In Section 8, we work with the Iwahori–Hecke algebra. Using [Schneider and Teitelbaum 2006], which deals with p-adic Hecke algebras, we make an integral Satake transform for the generic Iwahori–Hecke algebra of $\mathrm{GL}_n(F)$ explicit. By analyzing the map (8-7), Barthel and Livné's method for producing unramified representations [1995] can then be related to the construction of representations arising from the natural left adjoint of the functor of the Iwahori-invariants.

2. Affine root system and Weyl groups

2A. We consider an affine root datum $(\Lambda, \check{\Lambda}, \Phi, \check{\Phi}, \Pi, \check{\Pi})$; for this notion and the facts in the subsequent review, see [Lusztig 1989, 1]. An element of the free abelian group Λ is called a *weight*. We will denote by $\langle\, .\, , .\, \rangle$ the perfect pairing on $\Lambda \times \check{\Lambda}$. The elements of $\check{\Lambda}$ are the *coweights*. The elements in $\Phi \subset \Lambda$ are the *coroots*, while those in $\check{\Phi} \subset \check{\Lambda}$ are the *roots*. There is a correspondence $\alpha \leftrightarrow \check{\alpha}$ between roots and coroots satisfying $\langle \alpha, \check{\alpha}\rangle = 2$. The set Π of simple coroots is a basis for Φ, and the corresponding set $\check{\Pi}$ of simple roots is a basis for $\check{\Phi}$. Let $\check{\Phi}^+$ and $\check{\Phi}^-$ denote, respectively the set of roots which are positive and negative with respect to $\check{\Pi}$. There is a partial order on $\check{\Phi}$ given by $\check{\alpha} \leq \check{\beta}$ if and only if $\check{\beta} - \check{\alpha}$ is a linear combination with (integral) nonnegative coefficients of elements in $\check{\Pi}$. Denote by Π_m the set of coroots such that the associated root is a minimal element in $\check{\Phi}$ for $\leq$.

To the (simple) root $\check{\alpha}$ corresponds the (simple) reflection $s_\alpha : \lambda \mapsto \lambda - \langle \lambda, \check{\alpha}\rangle\, \alpha$, which leaves Φ stable. Reciprocally, we will denote by $\check{\alpha}_s$ the simple root associated to the simple reflection s. The finite Weyl group W_0 is the subgroup of $\mathrm{GL}(\Lambda)$ generated by the simple reflections s_α for $\alpha \in \Pi$. It is a Coxeter system with generating set $S_0 = \{s_\alpha,\ \alpha \in \Pi\}$. We will denote by $(w_0, \lambda) \mapsto {}^{w_0}\lambda$ the natural action of W_0 on the set of weights and by $W_0(\lambda)$ the stabilizer of a weight λ under the action of W_0. This action induces a natural action of W_0 on the coweights which stabilizes the set of roots. The set Λ acts on itself by translations: for any weight λ, we denote by e^λ the associated translation. The Weyl group W is the semidirect product of W_0 and Λ. For $w_0 \in W_0$ and $\lambda \in \Lambda$, observe that $w_0 e^\lambda = e^{{}^{w_0}\lambda} w_0$. The affine Weyl group W_{aff} is the semidirect product of W_0 and Φ.

The Weyl group acts on $\check{\Phi} \times \mathbb{Z}$ by

$$w_0 e^\lambda : (\check{\alpha}, k) \mapsto (w_0\check{\alpha},\ k - \langle \lambda, \check{\alpha}\rangle),$$

where we denote by $(w_0, \check{\alpha}) \mapsto w_0\check{\alpha}$ the natural action of W_0 on the roots. Define the set of affine roots by $\check{\boldsymbol{\Phi}} = \check{\boldsymbol{\Phi}}^+ \cup \check{\boldsymbol{\Phi}}^- \subset \check{\Phi} \times \mathbb{Z}$, where

$$\check{\boldsymbol{\Phi}}^+ := \{(\check{\alpha}, k),\ \check{\alpha} \in \Phi,\ k > 0\} \cup \{(\check{\alpha}, 0),\ \check{\alpha} \in \Phi^+\},$$
$$\check{\boldsymbol{\Phi}}^- := \{(\check{\alpha}, k),\ \check{\alpha} \in \Phi,\ k < 0\} \cup \{(\check{\alpha}, 0),\ \check{\alpha} \in \Phi^-\},$$

and the set of simple affine roots by $\check{\mathbf{\Pi}} := \{(\check{\alpha}, 0),\ \alpha \in \Pi\} \cup \{(\check{\alpha}, 1),\ \check{\alpha} \in \Pi_m\}$. Identifying $\check{\alpha}$ with $(\check{\alpha}, 0)$, we will often consider Π a subset of $\check{\mathbf{\Pi}}$.

For $A \in \check{\mathbf{\Pi}}$, denote by s_A the associated reflection $s_A = s_\alpha$ if $A = (\check{\alpha}, 0)$ and $s_A = s_\alpha e^\alpha$ if $A = (\check{\alpha}, 1)$. The affine Weyl group is a Coxeter system with generating set

$$S_{\mathrm{aff}} = \{s_A,\ A \in \check{\mathbf{\Pi}}\}.$$

The length on the Coxeter group W_{aff} extends to W in such a way that, for any $w \in W$,

$$\ell(w) := \#\{A \in \check{\mathbf{\Phi}}^+,\ w(A) \in \check{\mathbf{\Phi}}^-\}.$$

The Weyl group is the semidirect product of W_{aff} by the subgroup Ω of the elements with length zero. The Bruhat order $\leq$ inflates from W_{aff} to W [Vignéras 2005, Proposition 1].

2B. The length on W has the following properties [Lusztig 1989; Vignéras 2006, appendice]. Let $\lambda, \lambda' \in \Lambda$, $w_0, w_0' \in W_0$, $w \in W$, $A \in \check{\mathbf{\Phi}}$.

2B1. $\ell(ws_A) = \begin{cases} \ell(w)+1 & \text{if } wA \in \check{\mathbf{\Phi}}^+, \\ \ell(w)-1 & \text{if } wA \in \check{\mathbf{\Phi}}^-. \end{cases}$

2B2. The quantity $\ell(w_0) + \ell(w_0' e^\lambda) - \ell(w_0 w_0' e^\lambda)$ is twice the number of positive roots $\check{\alpha} \in \check{\Phi}^+$ satisfying

$$w_0'\check{\alpha} \in \check{\Phi}^-, \quad w_0 w_0'\check{\alpha} \in \check{\Phi}^+, \quad \langle \lambda, \check{\alpha}\rangle \geq 0 \quad \text{or}$$
$$w_0'\check{\alpha} \in \check{\Phi}^+, \quad w_0 w_0'\check{\alpha} \in \check{\Phi}^-, \quad \langle \lambda, \check{\alpha}\rangle < 0.$$

2B3. Set $n(\check{\alpha}, w_0 e^\lambda) = \langle \lambda, \check{\alpha}\rangle$ if $w_0\check{\alpha} \in \check{\Phi}^+$ and $n(\check{\alpha}, w_0 e^\lambda) = 1 + \langle \lambda, \check{\alpha}\rangle$ otherwise. If the integers $n(\check{\alpha}, w_0 e^\lambda)$ and $n(\check{\alpha}, e^{\lambda'})$ have the same sign (or one of them vanishes) for all $\check{\alpha} \in \check{\Phi}^+$, then

$$\ell(w_0 e^{\lambda+\lambda'}) = \ell(w_0 e^\lambda) + \ell(e^{\lambda'}).$$

2C. ***The root datum associated to p-adic*** $\mathbf{GL}_n$**.**

2C1. We denote by F a nonarchimedean locally compact field with ring of integers $\mathcal{O}$, maximal ideal $\mathcal{P}$ and residue field $\mathbb{F}_q$, where q is a power of p. We choose a uniformizer π and fix the valuation (denoted by val) normalized by $\mathrm{val}(\pi) = 1$ and the corresponding absolute value $|\,.\,|$ such that $|\pi| = q^{-1}$.

Let $n \in \mathbb{N}$, $n \geq 2$. Denote by G the group of F-valued points of the general linear group GL_n, by K_0 the maximal compact $\mathrm{GL}_n(\mathcal{O})$, by I the standard upper Iwahori subgroup of K_0 and by $I(1)$ its unique pro-p-Sylow. It contains the first congruent subgroup K_1 of the matrices in K_0 which are congruent to the identity modulo π. The element

$$\varpi = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & 0 & 1 \\ \pi & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

normalizes the Iwahori subgroup and $\varpi^n = \pi.\mathrm{Id}$ is central in G. Let B denote the upper triangular Borel subgroup of G with Levi decomposition $B = UT$ and modulus character $\delta : B \to \mathbb{Z}[q^{\pm 1}]$.

Consider the affine root datum associated to (G, B, T). The set of cocharacters of T identifies with $\Lambda \simeq T/(T \cap K_0) \simeq \mathbb{Z}^n$. We will also consider it a multiplicative subgroup of G by lifting $T/(T \cap K_0)$ to the subgroup of diagonal matrices with coefficients in $\pi^{\mathbb{Z}}$. The simple positive roots are

$$\check{\alpha}_i : \mathrm{diag}(\pi^{x_1}, \pi^{x_2} \ldots, \pi^{x_n}) \mapsto x_{i+1} - x_i, \quad \text{for} \quad i = 1, \ldots, n-1.$$

Identifying the reflection s_i associated to $\check{\alpha}_i$ with the transposition $(i, i+1)$ gives an isomorphism between the finite Weyl group W_0 and the symmetric group $\mathfrak{S}_n$. We see $W = W_0 \Lambda$ as a subgroup of G. It is a system of representatives of the double cosets $I \backslash G / I$.

There is a unique coroot in Π_m and the associated root is $-\check{\alpha}_0$, where $\check{\alpha}_0$ denotes the positive root

$$\check{\alpha}_0 = \check{\alpha}_1 + \cdots + \check{\alpha}_{n-1}.$$

The reflection associated to $(-\check{\alpha}_0, 1)$ is $s_0 = \varpi s_1 \varpi^{-1}$. A generating set for the affine Weyl group is $S_{\text{aff}} = \{s_0, s_1, \ldots, s_{n-1}\}$. The subgroup Ω of W of the elements with length zero is generated by ϖ.

For $s \in S_{\text{aff}}$, denote by $\Phi_s : \mathrm{GL}_2(F) \to G$ the associated morphism [Iwahori and Matsumoto 1965]. Recall that the cocharacter associated to s is the map $F^* \to T$, $x \mapsto \Phi_s\left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$. Denote by $\mathbb{T}_s$ the image of $\mathbb{F}_q^*$ by this cocharacter and set $\phi_s = \Phi_s\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Define the *dominant* and *antidominant* weights respectively by

$$\Lambda_{\text{dom}} = \{\lambda \in \Lambda,\ \langle \lambda, \check{\alpha} \rangle \geq 0 \text{ for any } \check{\alpha} \in \check{\Phi}^+\},$$

$$\Lambda_{\text{anti}} = \{\lambda \in \Lambda,\ \langle \lambda, \check{\alpha} \rangle \leq 0 \text{ for any } \check{\alpha} \in \check{\Phi}^+\}.$$

A weight $\mu \in \Lambda$ is said to be *minuscule* if $\langle \mu, \check{\alpha} \rangle \in \{0, \pm 1\}$ for any positive root $\check{\alpha} \in \check{\Phi}^+$. To any subset $J \subset \{1, \ldots, n\}$ corresponds a minuscule weight μ_J defined by $(\mu_J)_i = \pi$ if $i \in J$, $(\mu_J)_i = 1$ otherwise. The semigroup Λ_{anti} of the antidominant weights is generated by the minuscule antidominant weights

$$\{\mu_1, \ldots, \mu_{n-1}, \mu_n^{\pm 1}\},$$

where, for $i \in \{1, \ldots, n\}$, we denote by μ_i the minuscule weight associated to $\{1, \ldots, i\}$. Set $\mu_0 := \mu_\varnothing$.

2C2. The Weyl group W of G identifies with the quotient of the normalizer $N_G(T)$ of T in G by $T \cap K_0$. The extended Weyl group $W^{(1)}$ of G is defined to be the quotient $N_G(T)/(T \cap K_1)$. We have an exact canonically split sequence

$$0 \to \mathbb{T} \to W^{(1)} \to W \to 0,$$

where $\mathbb{T}$ denotes the finite diagonal torus of the Chevalley group $\mathrm{GL}_n(\mathbb{F}_q)$. For any subset X of W we will denote by $X^{(1)}$ its inverse image in $W^{(1)}$. In particular, the set of extended weights $\Lambda^{(1)}$, which identifies with the direct product of Λ by $\mathbb{T}$, is seen as the set of translations on itself. Again, for any extended weight λ, we denote by e^λ the associated translation. An extended weight is said to be dominant, antidominant, or minuscule if its component in Λ is so. The action of the extended Weyl group on $\Lambda^{(1)}$ and on $\check{\Phi} \times \mathbb{Z}$ is the one inflated from the action of W. By Teichmüller lifting, we identify $\Lambda^{(1)}$ and $W^{(1)} = W_0 \Lambda^{(1)}$ with subgroups of G. The extended affine Weyl group $W_{\mathrm{aff}}^{(1)}$ is generated by $S_{\mathrm{aff}}^{(1)}$. The length function on W extends to $W^{(1)}$ in such a way that the elements of $\mathbb{T}$ have length zero.

The extended Weyl group $W^{(1)}$ is a system of representatives of the double cosets $I(1)\backslash G/I(1)$.

2C3. Throughout, we fix a standard Levi subgroup $L = L_1 \times \cdots \times L_m$ in G, where $L_j \simeq \mathrm{GL}_{n_j}(F)$ for $j \in \{1, \ldots, m\}$ with $n_1 + \cdots + n_m = n$. Set $\Delta := \{1, \ldots, n-1\}$ and define its subset Δ_L to be the set of i such that $s_i \in L$. Denote by $W_{0,L}$ the finite Weyl group of L. It is a Coxeter group generated by $\{s_i, i \in \Delta_L\}$. Denote by $\check{\Phi}_L \subset \check{\Phi}$ the set of associated roots, and by $\check{\Phi}_L^+ = \check{\Phi}_L \cap \check{\Phi}^+$ the set of positive ones. The Weyl group W_L of L is the semidirect product of $W_{0,L}$ by Λ. The extended Weyl group $W_L^{(1)}$ of L is the semidirect product of $W_{0,L}$ by $\Lambda^{(1)}$.

Proposition 2.1. *There exists a system D_L of representatives of the right cosets $W_{0,L}\backslash W_0$ such that*

$$\ell(w_0 d) = \ell(w_0) + \ell(d) \quad \text{for all } w_0 \in W_{0,L},\ d \in D_L. \tag{2-1}$$

Any $d \in D_L$ is the unique element with minimal length in $W_{0,L}d$.

Proof. The proposition is proved in [Carter 1985, 2.3.3], where D_L is explicitly given by

$$D_L := \{d \in W_0,\ d^{-1}\check{\Phi}_L^+ \subset \check{\Phi}^+\}. \tag{2-2}$$

This concludes the proof. □

Proposition 2.2. *Let $d \in D_L$ and $s \in S_0$.*

(1) *If $\ell(ds) = \ell(d) - 1$ then $ds \in D_L$.*

(2) *If $\ell(ds) = \ell(d) + 1$ then either $ds \in D_L$ or $W_{0,L}ds = W_{0,L}d$.*

Proof. Suppose $ds \notin D_L$. Let $i \in \{1, \dots, n-1\}$ be such that $s = s_i$. Since $d \in D_L$ and $ds_i \notin D_L$, there is an element $\check{\beta} \in \check{\Phi}_L^+$ such that $d^{-1}\check{\beta} \in \check{\Phi}^+$ and $s_i d^{-1}\check{\beta} \notin \check{\Phi}^+$. But $\check{\alpha}_i$ is the only positive root made negative by s_i [Carter 1985, Proposition 2.2.6], so $d^{-1}\check{\beta} = \check{\alpha}_i$. This implies in particular that $d\check{\alpha}_i \in \check{\mathbf{\Phi}}^+$, and so $\ell(ds_i) = \ell(d) + 1$ by 2B1. The fact that $d\check{\alpha}_i$ belongs to $\check{\Phi}_L$ ensures that $ds_i d^{-1} \in W_{0,L}$. $\square$

2C4. We denote the upper standard parabolic subgroup associated to L by P. It has Levi decomposition $P = LN$, and $\overline{N}$ will denote the opposite unipotent subgroup. The Iwahori subgroup decomposes into $I = I^+ I_L I^-$, where

$$I^+ = I \cap N, \quad I_L = I \cap L, \quad I^- = I \cap \overline{N}.$$

We also set $I_L(1) := I(1) \cap L$. As in [Vignéras 1998, II.4] and [Bushnell and Kutzko 1998, 6], we consider the semigroup L^+ of *L-positive elements*: an element $w \in L$ is called L-positive if it contracts I^+ and dilates I^-, that is,

$$wI^+w^{-1} \subset I^+ \quad \text{and} \quad w^{-1}I^-w \subset I^-.$$

The elements w in $W_L^{(1)}$ which are L-positive are the ones satisfying

$$w(\check{\Phi}^+ - \check{\Phi}_L^+) \subset \check{\mathbf{\Phi}}^+. \tag{2-3}$$

A weight $\lambda \in \Lambda^{(1)}$ is said to be L-positive if the associated translation in $W^{(1)}$ is L-positive. It means that $\langle \lambda, \check{\alpha} \rangle \le 0$ for any $\check{\alpha} \in \check{\Phi}^+ - \check{\Phi}_L^+$. For example, if L is the diagonal torus, a weight λ is T-positive if and only if it is antidominant.

The set D_L is also a system of representatives of the right cosets $W_L \backslash W$, and we have a weak analog of (2-1):

Lemma 2.3. *For any $w \in W_L^{(1)}$ which is L-positive and any $d \in D_L$,*

$$\ell(wd) = \ell(w) + \ell(d). \tag{2-4}$$

Proof. Let $d \in D_L$, and let $w \in W_L^{(1)}$ be a L-positive element. Write $w = e^{\lambda} w_0$. Equality (2-4) is equivalent to $\ell(d^{-1}) + \ell(w_0^{-1}e^{-\lambda}) - \ell(d^{-1}w_0^{-1}e^{-\lambda}) = 0$.

Let $\check{\alpha} \in \check{\Phi}^+$ be a positive root. Suppose $w_0^{-1}\check{\alpha} \in \check{\Phi}^+$ and $d^{-1}w_0^{-1}\check{\alpha} \in \check{\Phi}^-$. Then by (2-2) and (2-3), one has $w_0^{-1}\check{\alpha} \in \check{\Phi}^+ - \check{\Phi}_L^+$ and $w(w_0^{-1}\check{\alpha}, 0) = (\check{\alpha}, -\langle \lambda, \check{\alpha} \rangle) \in \check{\mathbf{\Phi}}^+$, so $\langle -\lambda, \check{\alpha} \rangle \ge 0$. In the same way, one gets $\langle -\lambda, \check{\alpha} \rangle < 0$ if $w_0^{-1}\check{\alpha} \in \check{\Phi}^-$ and $d^{-1}w_0^{-1}\check{\alpha} \in \check{\Phi}^+$. Applying the length property 2B2 then gives the required equality. $\square$

Lemma 2.4. *The set $I(1)L^+K_0$ is the disjoint union of the sets $I(1)L^+dI(1)$ where d runs over D_L.*

Proof. Lemma 2.3 implies that $I(1)w^+dI(1) = I(1)w^+I(1)dI(1)$ for any $d \in D_L$ and any L-positive $w^+ \in W_L^{(1)}$. So the set $I(1)L^+I(1)dI(1)$ is the disjoint union of the sets $I(1)w^+dI(1)$, where w^+ runs over the L-positive elements of $W_L^{(1)}$. It is equal to $I(1)L^+dI(1)$. In particular, the sets $I(1)L^+dI(1)$ are pairwise disjoint for $d \in D_L$.

The set $I(1)L^+K_0$ is the union of the sets $I(1)w^+I(1)w_0dI(1)$, where d runs over D_L, w_0 over $W_{0,L}$ and w^+ over the L-positive elements in $W_L^{(1)}$. By Proposition 2.1, we have $I(1)w_0dI(1) = I(1)w_0I(1)dI(1)$, so $I(1)w^+I(1)w_0dI(1) = I(1)w^+I(1)w_0I(1)dI(1)$, and, since w^+ and w_0 are L-positive,

$$I(1)w^+I(1)w_0dI(1) = I(1)w^+I_L(1)w_0I(1)dI(1) \subset I(1)L^+I(1)dI(1). \qquad \square$$

Proposition 2.5. *There is a system $\mathfrak{D}$ of representatives of the right cosets $W_0\backslash W$ such that*

$$\ell(w_0d) = \ell(w_0) + \ell(d), \quad \text{for all } w_0 \in W_0,\ d \in \mathfrak{D}. \tag{2-5}$$

Any $d \in \mathfrak{D}$ is the unique element with minimal length in W_0d.

Proof. Set

$$\mathfrak{D} := \{d \in W,\ d^{-1}\check{\Phi}^+ \subset \check{\mathbf{\Phi}}^+\}.$$

First check that the cosets W_0d are pairwise disjoint for $d \in \mathfrak{D}$. Let $d, d' \in \mathfrak{D}$, $w_0, w_0' \in W_0$ be such that $w_0d = w_0'd'$. If $d \neq d'$, then $w_0 \neq w_0'$ and there exists a simple root $\check{\beta} \in \check{\Pi}$ such that $\ell(s_\beta w_0^{-1}w_0') = \ell(w_0^{-1}w_0') - 1$, that is, $(w_0'^{-1}w_0)\check{\beta} = (d'd^{-1})\check{\beta} \in \check{\Phi}^-$. But $d' \in \mathfrak{D}$, and hence $d^{-1}\check{\beta} \in d'^{-1}(\check{\Phi}^-) \subset \check{\mathbf{\Phi}}^-$, which contradicts the fact that $d \in \mathfrak{D}$.

For $w \in W$, we prove by induction on the length of w that there exists an (obviously unique) $(w_0, d) \in W_0 \times \mathfrak{D}$ such that $w = w_0d$ and $\ell(w_0d) = \ell(w_0) + \ell(d)$.

By 2B1, saying that w does not belong to $\mathfrak{D}$ means that there exists a simple root $\check{\alpha} \in \check{\Pi}$ such that $\ell(s_\alpha w) = \ell(w) - 1$. In particular, if w has length 0, it belongs to $\mathfrak{D}$. Suppose now that $\ell(w) > 0$ and that it does not belong to $\mathfrak{D}$. Then, by induction, there exists $(w_0, d) \in W_0 \times \mathfrak{D}$ with $s_\alpha w = w_0d$ and $\ell(s_\alpha w) = \ell(w_0) + \ell(d)$, where α is chosen as before. So $w = s_\alpha w_0 d$ and

$$\ell(w) = \ell(s_\alpha w) + 1 = \ell(w_0) + \ell(d) + 1.$$

Verifying that $\ell(w) = \ell(s_\alpha w_0) + \ell(d)$ is just verifying that $\ell(s_\alpha w_0) = \ell(w_0) + 1$, which is true, since otherwise $\ell(s_\alpha w_0) < \ell(w_0)$ and $\ell(w) \leq \ell(s_\alpha w_0) + \ell(d) < \ell(w_0) + \ell(d) = \ell(w) - 1$.

We have proved that $\mathfrak{D}$ is a system of representatives of the right cosets $W_0\backslash W$ and that it satisfies (2-5). In particular, any $d \in \mathfrak{D}$ is the unique element with minimal length in W_0d, since $w_0 \in W_0$ has length zero if and only if $w_0 = 1$. $\square$

Lemma 2.6. *Any $d \in \mathfrak{D}$ can be written $d = e^\lambda w_0 \in W$, with $w_0 \in W_0$ and $\lambda \in \Lambda$ a dominant weight such that*

$$\ell(e^\lambda w_0) + \ell(w_0^{-1}) = \ell(e^\lambda).$$

Proof. By definition of the set $\mathfrak{D}$, we have $(w_0^{-1}\check{\alpha}, \langle\lambda, \check{\alpha}\rangle) \in \check{\mathbf{\Phi}}^+$ for every $\check{\alpha} \in \check{\Phi}^+$. Then λ is dominant and $w_0^{-1}\check{\alpha} \in \check{\Phi}^+$ if $\check{\alpha} \in \check{\Phi}^+$ satisfies $\langle\lambda, \check{\alpha}\rangle = 0$. Applying the length property 2B2, one gets the required equality. □

Proposition 2.7. *Let $d \in \mathfrak{D}$ and $s \in S_{\mathrm{aff}}$.*

(1) *If $\ell(ds) = \ell(d) - 1$ then $ds \in \mathfrak{D}$.*

(2) *If $\ell(ds) = \ell(d) + 1$ then either $ds \in \mathfrak{D}$ or $W_0 ds = W_0 d$.*

Proof. Write $d = e^\lambda w_0 \in W$.

(A) We first prove the proposition for s in the finite Weyl group; write $s = s_i$ with $1 \le i \le n-1$. Saying that $ds_i \notin \mathfrak{D}$ means that there exists $\check{\beta} \in \check{\Phi}^+$ such that $d^{-1}\check{\beta} = (\check{\alpha}_i, 0)$, since $(\check{\alpha}_i, 0)$ is the only positive affine root made negative by s_i. This implies in particular that $d\check{\alpha}_i \in \check{\mathbf{\Phi}}^+$, so $\ell(ds_i) = \ell(d) + 1$. We have

$$\check{\beta} = w_0\check{\alpha}_i, \quad \langle\lambda, w_0\check{\alpha}_i\rangle = 0.$$

The latter equality means that $w_0 s_i w_0^{-1}$ fixes λ, so

$$ds_i = e^\lambda w_0 s_i = w_0 s_i w_0^{-1} e^\lambda w_0 \in W_0 d.$$

(B) Now suppose $s = s_0$. Recall that the associated affine simple root is $(-\check{\alpha}_0, 1)$. The coroot α_0 can be seen as the diagonal matrix $(\pi^{-1}, 1, \dots, 1, \pi)$. Write $s_0 = \rho e^{-\alpha_0}$, where ρ denotes the reflection sending α_0 to its opposite. Saying that $\ell(ds_0) = \ell(d) + 1$ means that $d(-\check{\alpha}_0, 1) \in \check{\mathbf{\Phi}}^+$, that is, we are either in case (a) or in case (b):

(a) $\langle\lambda, w_0\check{\alpha}_0\rangle \ge 0$,

(b) $w_0\check{\alpha}_0 \in \check{\Phi}^-$ and $\langle\lambda, w_0\check{\alpha}_0\rangle = -1$.

Saying that $\ell(ds_0) = \ell(d) - 1$ means that $d(-\check{\alpha}_0, 1) \in \check{\mathbf{\Phi}}^-$, so we are in case (c) (note that since λ is dominant, it is impossible to simultaneously have the conditions $w_0\check{\alpha}_0 \in \Phi^+$ and $\langle\lambda, w_0\check{\alpha}_0\rangle = -1$):

(c) $\langle\lambda, w_0\check{\alpha}_0\rangle < -1$.

By definition of the reflection ρ, hypothesis (b) says that ${}^{w_0\rho w_0^{-1}}\lambda = \lambda + {}^{w_0}\alpha_0$, so that we have $ds_0 = e^\lambda w_0 \rho e^{-\alpha_0} = w_0 \rho w_0^{-1} e^\lambda w_0 \in W_0 d$.

Suppose that we are under hypothesis (a) or (c), that is, $\langle\lambda, w_0\check{\alpha}_0\rangle \neq -1$. Take $\check{\beta} \in \check{\Pi}$. Under the action of $s_0 d^{-1}$, it becomes the affine root

$$s_0 d^{-1}\check{\beta} = (\rho w_0^{-1}\check{\beta}, \langle\lambda, \check{\beta}\rangle + \langle\alpha_0, w_0^{-1}\check{\beta}\rangle).$$

Let us check that it belongs to $\check{\mathbf{\Phi}}^+$, which will prove that $ds_0 \in \mathfrak{D}$. Recall that $d \in \mathfrak{D}$, so

$$d^{-1}\check{\beta} = (w_0^{-1}\check{\beta}, \langle\lambda, \check{\beta}\rangle) \in \check{\mathbf{\Phi}}^+.$$

First we verify that $\langle \alpha_0, w_0^{-1}\check{\beta}\rangle + \langle \lambda, \check{\beta}\rangle \geq 0$. Since $\langle \alpha_0, w_0^{-1}\check{\beta}\rangle \in \{0, \pm 1, \pm 2\}$, the required inequality is true if $\langle \lambda, \check{\beta}\rangle \geq 2$. If $\langle \lambda, \check{\beta}\rangle = 0$, then $w_0^{-1}\check{\beta} \in \Phi^+$ and $\langle \alpha_0, w_0^{-1}\check{\beta}\rangle \geq 0$. If $\langle \lambda, \check{\beta}\rangle = 1$ then, by the chosen hypotheses, $w_0^{-1}\check{\beta} \neq -\check{\alpha}_0$, so $\langle \alpha_0, w_0^{-1}\check{\beta}\rangle \neq -2$.

Finally, we have to show that $\langle \alpha_0, w_0^{-1}\check{\beta}\rangle + \langle \lambda, \check{\beta}\rangle = 0$ implies $\rho w_0^{-1}\check{\beta} \in \check{\Phi}^+$. A positive root $\check{\gamma}$ becomes a positive root under the action of ρ if and only if it is fixed by the action of ρ, or in other words, if $\langle \alpha_0, \check{\gamma}\rangle = 0$. Suppose that

$$\langle \lambda, \check{\beta}\rangle = \langle \alpha_0, w_0^{-1}\check{\beta}\rangle = 0;$$

then $w_0^{-1}\check{\beta} \in \check{\Phi}^+$, and so, by the preceding remark, $\rho w_0^{-1}\check{\beta} \in \check{\Phi}^+$. Suppose that

$$\langle \lambda, \check{\beta}\rangle = -\langle \alpha_0, w_0^{-1}\check{\beta}\rangle > 0;$$

then $w_0^{-1}\check{\beta} \in \check{\Phi}^-$, and by the preceding remark, $\rho w_0 \check{\beta} \in \check{\Phi}^+$. □

3. Hecke algebras and universal modules

3A. Consider the Chevalley group $\mathbb{G} = \mathrm{GL}_n(\mathbb{F}_q)$ and its standard upper Borel subgroup $\mathbb{B}$ with Levi decomposition $\mathbb{B} = \mathbb{T}\mathbb{U}$. We denote by $\overline{\mathbb{U}}$ the opposite unipotent subgroup. The double cosets $\mathbb{U}\backslash\mathbb{G}/\mathbb{U}$ are represented by the extended Weyl group of $\mathbb{G}$, which is isomorphic to the extended finite Weyl group $W_0^{(1)}$ of G. The *finite universal module* $\mathbb{Z}[\mathbb{U}\backslash\mathbb{G}]$ of $\mathbb{Z}$-valued functions with support on the right cosets $\mathbb{U}\backslash\mathbb{G}$ is endowed with a natural action of $\mathbb{G}$. The ring $\mathscr{H}(\mathbb{G}, \mathbb{U})$ of its $\mathbb{Z}[\mathbb{G}]$-endomorphisms will be called the *finite Hecke ring*. By Frobenius reciprocity, a $\mathbb{Z}$-basis of the latter identifies with the characteristic functions of the double cosets $\mathbb{U}\backslash\mathbb{G}/\mathbb{U}$.

We call the space $\mathbb{Z}[I(1)\backslash G]$ of $\mathbb{Z}$-valued functions with finite support on the right cosets $I(1)\backslash G$ the *pro-p-universal module*. It is endowed with an action of G. The subspace of the functions that are actually left invariant under the Iwahori subgroup constitute a G-subspace that is isomorphic to the universal module $\mathbb{Z}[I\backslash G]$.

The $\mathbb{Z}$-ring of the $\mathbb{Z}[G]$-endomorphisms of $\mathbb{Z}[I(1)\backslash G]$ will be called the *pro-p-Hecke ring* and denoted by $\mathscr{H}(G, I(1))$. By Frobenius reciprocity, $\mathscr{H}(G, I(1))$ is seen as the convolution ring of the functions with finite support on the double cosets of G modulo $I(1)$. Among these functions, the ones that are actually biinvariant under the Iwahori subgroup constitute a ring that is isomorphic to the Iwahori–Hecke ring $\mathscr{H}(G, I)$ of the $\mathbb{Z}[G]$-endomorphisms of $\mathbb{Z}[I\backslash G]$.

A $\mathbb{Z}$-basis for $\mathscr{H}(G, I(1))$ (resp. $\mathscr{H}(G, I)$) is given by the characteristic functions of the double cosets $I(1)\backslash G/I(1)$ (resp. $I\backslash G/I$).

For $w \in W^{(1)}$, we denote by τ_w the element of $\mathscr{H}(G, I(1))$ corresponding to the associated double coset. The subalgebra generated by the elements τ_w for $w \in W_{\mathrm{aff}}^{(1)}$ is called the affine Hecke algebra.

The subspace of $\mathbb{Z}[I(1)\backslash G]$ of the functions with support in K_0 identifies with the finite universal module. Among the $\mathbb{Z}[G]$-endomorphisms of the pro-p-universal module, those stabilizing this subspace form a subring that identifies with the finite Hecke algebra. It is the subring generated by the elements τ_w for $w \in W_0^{(1)}$.

Fix k an algebraic closure of $\mathbb{F}_q$. The space $\mathbb{Z}[I(1)\backslash G] \otimes_{\mathbb{Z}} k$ is endowed with a smooth action of G and is isomorphic to the compact induction $\mathrm{ind}_{I(1)}^{G} \mathbf{1}_k$ of the trivial character with values in k of the pro-p-Iwahori subgroup. We will denote by $\mathscr{U}$ this representation of G.

3B. The pro-p-Hecke ring is the ring with $\mathbb{Z}$-basis $(\tau_w)_{w\in W^{(1)}}$ satisfying the *braid* and *quadratic* relations, namely

- $\tau_w \tau_{w'} = \tau_{ww'}$ for any $w,\ w' \in W^{(1)}$ such that $\ell(ww') = \ell(w) + \ell(w')$, and
- $\tau_s^2 = q + \left(\sum_{t\in\mathbb{T}_s} \tau_{\phi_s}\tau_t\right)\tau_s$ for $s \in S_{\mathrm{aff}}$,

in the notation of 2C1. From now on, we consider q an indeterminate and work with the $\mathbb{Z}[q]$-algebra $\mathscr{H}$ with generators $(\tau_w)_{w\in W^{(1)}}$ satisfying the relations above. It will be called the *generic pro-p-Hecke algebra*.

For $w \in W^{(1)}$, set
$$\tau_w^* := q^{\ell(w)}\tau_w^{-1}. \tag{3-1}$$
The map $\mu : \tau_w \mapsto (-1)^{\ell(w)}\tau^*_{w^{-1}}$ defines an involutive algebra endomorphism of $\mathscr{H}$ [Vignéras 2005, Corollary 2].

Remark 3.1. For $s \in S_{\mathrm{aff}}$, one checks that the following equalities hold in $\mathscr{H}\otimes_{\mathbb{Z}[q]}k$:
$$(\tau_s^*)^2 = (\tau_s + \nu_s)^2 = \tau_s^*\nu_s = \nu_s\tau_s^*,$$
where $\nu_s := -\sum_{t\in\mathbb{T}_s} \tau_{\phi_s}\tau_t$.

4. Pro-p-Iwahori Hecke algebra relative to a Levi subgroup of G

The generic pro-p-Hecke algebra $\mathscr{H}(L)$ of the Levi subgroup L is the tensor product of the generic pro-p-Hecke algebras of the L_j's, for $j \in \{1, \dots, m\}$. For any element $w = (w_1, \dots, w_m)$ in the extended Weyl group $W_L^{(1)}$ of L, we will denote by
$$\tau_w^{\otimes} := \bigotimes_{j=1}^{m} \tau_{w_j}$$
the corresponding element of $\mathscr{H}(L)$. Denote by $\mathscr{H}(L^+)$ the subspace of $\mathscr{H}(L)$ generated over $\mathbb{Z}[q]$ by the elements $\tau_w^{\otimes}$ corresponding to L-positive elements w in $W_L^{(1)}$. From [Bushnell and Kutzko 1998, 6.12] and [Vignéras 1998, II], we know that $\mathscr{H}(L^+)$ is a $\mathbb{Z}[q]$-algebra and the following holds.

Proposition 4.1. *The natural injective map* θ_L^+

$$\mathcal{H}(L^+) \to \mathcal{H}, \quad \tau_w^{\otimes} \mapsto \tau_w, \tag{4-1}$$

where $w \in W_L^{(1)}$ *is* L*-positive, respects the product. It extends uniquely into an injective morphism* θ_L *of* $\mathbb{Z}[q^{\pm 1}]$*-algebras*

$$\theta_L : \mathcal{H}(L) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1}] \to \mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1}].$$

The proof of the second assertion [Bushnell and Kutzko 1998; Vignéras 1998] makes use of the following (strongly) L-positive central element in L:

$$a_L = e^{\lambda_L}, \quad \text{where} \quad \lambda_L = \sum_{j \in \Delta - \Delta_L} \mu_j, \tag{4-2}$$

and the fact that for any $w \in W_L^{(1)}$ there exists $r \in \mathbb{N}$ such that $a_L^r w$ is L-positive. Then $\theta_L(\tau_w^{\otimes})$ is given by $\tau_{a_L}^{-r} \tau_{a_L^r w}$, which is well-defined in $\mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1}]$ (and does not depend on the choice of r).

We will call $\mathcal{H}(L^+)$ the *positive subalgebra* of $\mathcal{H}(L)$. We will sometimes identify it with its image in $\mathcal{H}$ without further notice.

4A. ***Classical Bernstein presentation.*** In the case where the Levi subgroup L is the diagonal torus T, the map θ_T is simply denoted by θ and called the Bernstein embedding. It is more traditional to consider its renormalization

$$\begin{aligned} \tilde{\theta} : \mathbb{Z}[q^{\pm 1/2}][\Lambda^{(1)}] &\to \mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1/2}], \\ \lambda &\mapsto \delta^{1/2}(\lambda)\theta(\lambda), \end{aligned} \tag{4-3}$$

whose image is denoted by $\mathcal{A}[q^{\pm 1/2}]$, where δ is the modulus character of the Borel subgroup defined in 2C1. The following well-known properties of this commutative subalgebra are proved in, for example, [Lusztig 1989, 3] (and [Vignéras 2005, 1.4] for the extension to the pro-p case). The center of $\mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1/2}]$ is the image under $\tilde{\theta}$ of the subspace $\mathbb{Z}[q^{\pm 1/2}][\Lambda^{(1)}]^{W_0}$ of the invariants in $\mathbb{Z}[q^{\pm 1/2}][\Lambda^{(1)}]$ under the natural action of W_0. The Hecke algebra $\mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1/2}]$ is a free right module over $\mathcal{A}[q^{\pm 1/2}]$ with basis $\{\tau_{w_0},\ w_0 \in W_0\}$.

4B. ***Integral Bernstein presentation.*** In this section, we recall the results obtained by Vignéras [2005] concerning an integral version of the previous Bernstein presentation. We present them in the light of [Schneider and Teitelbaum 2006].

4B1. Following [Schneider and Teitelbaum 2006, p. 10 and Example 2], we consider the action of W_0 on $\mathbb{Z}[q^{\pm 1/2}][\Lambda^{(1)}]$ twisted by the map

$$\gamma : W_0 \times \Lambda^{(1)} \to \mathbb{Z}[q^{\pm 1/2}], \quad (w_0, \lambda) \mapsto \frac{\delta^{1/2}({}^{w_0}\lambda)}{\delta^{1/2}(\lambda)}. \tag{4-4}$$

This map is a cocycle in the sense that it satisfies

(a) $\gamma(v_0 w_0, \lambda) = \gamma(v_0, {}^{w_0}\lambda)\gamma(w_0, \lambda)$, for $v_0, w_0 \in W_0$ and $\lambda \in \Lambda^{(1)}$,

so we have a well-defined action of W_0 on $\Lambda^{(1)}$ denoted by $(w_0, \lambda) \mapsto w_0 \centerdot \lambda$ and given by

$$w_0 \centerdot \lambda = \gamma(w_0, \lambda)^{w_0}\lambda. \tag{4-5}$$

The map γ also satisfies the following conditions:

(b) $\gamma(w_0, \lambda\mu) = \gamma(w_0, \lambda)\gamma(w_0, \mu)$, for $w_0 \in W_0$ and $\lambda, \mu \in \Lambda^{(1)}$,

(c) $\gamma(w_0, \lambda) = 1$ for $w_0 \in W_0$, $\lambda \in \Lambda^{(1)}$ such that ${}^{w_0}\lambda = \lambda$,

so the twisted action (4-5) extends into an action on $\mathbb{Z}[q^{\pm 1/2}][\Lambda^{(1)}]$, which is compatible with the structure of $\mathbb{Z}[q^{\pm 1/2}]$-algebra.

Lemma 4.2 [Schneider and Teitelbaum 2006, Example 2 and Lemma 4.2]. (1) *For $w_0 \in W_0$, $\lambda \in \Lambda^{(1)}$, one has*

$$\gamma(w_0, \lambda) = \prod_{\check{\alpha} \in \check{\Phi}^+ \cap w_0^{-1}(\check{\Phi}^-)} |\check{\alpha}(\lambda)|,$$

so γ actually takes values in $\mathbb{Z}[q^{\pm 1}]$.

(2) *Any $\lambda \in \Lambda^{(1)}$ can be written $\lambda_1 - \lambda_2$ with λ_1, λ_2 antidominant weights. Let $w_0 \in W_0$ such that ${}^{w_0}\lambda$ is antidominant. Then*

$$\gamma(w_0, \lambda) = q^{-(\ell(\lambda) - \ell(\lambda_1) + \ell(\lambda_2))/2}$$

and it does not depend on the choice of $w_0, \lambda_1, \lambda_2$.

4B2. Let $\lambda \in \Lambda^{(1)}$ and $w_0 \in W_0$ such that ${}^{w_0}\lambda$ is antidominant. Define the element $E(\lambda)$ in $\mathscr{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1}]$ by

$$E(\lambda) := \gamma(w_0, \lambda)^{-1}\theta(\lambda) = \gamma(w_0^{-1}, {}^{w_0}\lambda)\theta(\lambda). \tag{4-6}$$

It is proved in [Vignéras 2005] that $E(\lambda)$ actually lies in $\mathscr{H}$ (see Theorem 4.5 below for the precise statement). Hence, we have an injective $\mathbb{Z}[q]$-equivariant map

$$E : \mathbb{Z}[q][\Lambda^{(1)}] \to \mathscr{H}, \tag{4-7}$$

but it does not respect the product. The natural action of W_0 on $\Lambda^{(1)}$ induces an action of W_0 on the image $\mathscr{A}$ of E.

Proposition 4.3 (integral Bernstein relations). *Let $\lambda \in \Lambda^{(1)}$ be a weight, $\check{\alpha} \in \check{\Pi}$ a simple root and s the associated reflection. The following holds in $\mathscr{H}$:*

(1) *If $\langle \lambda, \check{\alpha} \rangle = 0$, then $E(\lambda)$ and τ_s commute.*

(2) *If $\langle \lambda, \check{\alpha} \rangle = 1$, then $\tau_s E(\lambda) = E({}^s\lambda)\tau_s^*$ and $E(\lambda)\tau_s = \tau_s^* E({}^s\lambda)$.*

Proof. This is a direct corollary of the classical Bernstein relations proved in [Lusztig 1989, Proposition 3.6] and [Vignéras 2005, Proposition 5]. An integral version of these is proved in [Ollivier 2006a, 4.4.1] (use the involution μ defined in 3B to pass from the definition of the Bernstein map in the latter to the present situation). □

Lemma 4.4. *Let $\lambda \in \Lambda$ be antidominant. Then $E(\lambda) = \tau_{e^\lambda}$ and $E(\lambda^{-1}) = \tau^*_{e^\lambda}$. Suppose also that λ is minuscule. Let $d \in W_0$ with minimal length in $W_0(\lambda)d$. Then*

$$E({}^{d^{-1}}\lambda) = \tau_{d^{-1}e^\lambda}\tau^*_{d^{-1}} \quad \textit{and} \quad \tau_d E({}^{d^{-1}}\lambda) = E(\lambda)\tau^*_{d^{-1}}.$$

Proof. First recall that an element $\lambda \in \Lambda$ is T-positive if and only if it is antidominant. So $\theta(\lambda) = \theta^+_T(\lambda) = \tau_{e^\lambda}$. Then, by Lemma 4.2(2) and since θ respects the product, one has $E(\lambda^{-1}) = q^{\ell(\lambda)}\tau^{-1}_{e^\lambda} = \tau^*_{e^\lambda}$. We have proved the first statement, which gives the second one for the case $d = 1$. Suppose λ is minuscule and show the second one by induction on $\ell(d)$. Let $d \in W_0$ with minimal length in $W_0(\lambda)d$ and $\ell(d) > 0$. Let $s \in S_0$ such that $\ell(ds) = \ell(d) - 1$. Then $\tau^*_{sd^{-1}}\tau^*_s = \tau^*_{d^{-1}}$ and $d\check{\alpha}_s \in \check{\Phi}^-$. The stabilizer $W_0(\lambda)$ is a Coxeter subgroup of W_0, so Proposition 2.2 applies: ds has minimal length in $W_0(\lambda)ds$. In particular, this implies that dsd^{-1} does not stabilize λ, so $\langle \lambda, d\check{\alpha}_s\rangle > 0$. The length property 2B2 then gives $\ell(s) + \ell(d^{-1}e^\lambda) = \ell(sd^{-1}e^\lambda)$. By induction, $E({}^{sd^{-1}}\lambda) = \tau_{sd^{-1}e^\lambda}\tau^*_{sd^{-1}} = \tau_s\tau_{d^{-1}e^\lambda}\tau^*_{sd^{-1}}$. Now work in $\mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1}]$ and apply the Bernstein relations (2) to ${}^{d^{-1}}\lambda$:

$$E({}^{d^{-1}}\lambda) = \tau_s^{-1}E({}^{sd^{-1}}\lambda)\tau^*_s = \tau_{d^{-1}e^\lambda}\tau^*_{d^{-1}}.$$

The last equality of the lemma easily follows using 2B2 and the fact that $\langle\lambda, \check{\alpha}\rangle = 0$ implies $d^{-1}\check{\alpha} \in \check{\Phi}^+$ for any $\check{\alpha} \in \check{\Phi}^+$. □

Theorem 4.5 [Vignéras 2005, Theorems 2, 3, and 4]. *The image $\mathcal{A}$ of E is a $\mathbb{Z}[q]$-algebra. It coincides with the intersection $\mathcal{A}[q^{\pm 1/2}] \cap \mathcal{H}$. The action of W_0 on $\mathcal{A}$ is compatible with the structure of $\mathbb{Z}[q]$-algebra.*

A $\mathbb{Z}[q]$-basis for $\mathcal{A}$ is given by $(E(\lambda))_{\lambda\in\Lambda^{(1)}}$.

As a $\mathbb{Z}[q]$-algebra, $\mathcal{A}$ is generated by elements corresponding to minuscule weights, that is, by the elements τ_t for $t \in \mathbb{T}$ and

$$(E(\mu_I))_{I\subsetneq\{1,\dots,n\}},\ E(\mu_{\{1,\dots,n\}})^{\pm 1}$$

with the relations

$$E(\mu_I)E(\mu_J) = q^{bc}E(\mu_{I\cup J})E(\mu_{I\cap J}) \tag{4-8}$$

for any $I, J \subset \{1, \dots, n\}$ with $|I \cap J| = a$, $|I| = a + b$, $|J| = a + c$.

The center of $\mathcal{H}$ is the space of W_0-invariants in $\mathcal{A}$.

As an $\mathcal{A}$-module, $\mathcal{H}$ is finitely generated; as a module over the center, $\mathcal{A}$ is finitely generated.

The proof of the theorem relies on the more general definition of an element $E(w) \in \mathcal{H}$ associated to any $w = e^{\lambda} w_0 \in W^{(1)}$:

$$E(w) := q^{(\ell(w)-\ell(w_0)-\ell(e^{\lambda}))/2} E(\lambda)\tau_{w_0} \tag{4-9}$$

in $\mathcal{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1/2}]$, and the fact that the elements $(E(w))_{w\in W^{(1)}}$ constitute a $\mathbb{Z}[q]$-basis for $\mathcal{H}$ called the *integral Bernstein basis*.

Remark 4.6. Note that (4-8) implies that in $\mathcal{H} \otimes_{\mathbb{Z}[q]} k$, the product $E(\mu_I)E(\mu_J)$ is zero unless either $I \subset J$ or $J \subset I$.

4C. For $w = (w_1, \dots, w_m) \in W_L^{(1)}$, we denote by $E^{\otimes}(w) \in \mathcal{H}(L)$ the tensor product of the Bernstein elements corresponding to the elements w_j in the generic pro-p-Hecke algebras of the L_is. The Hecke algebra $\mathcal{H}(L)$ contains the commutative subring $\mathcal{A}_L$ with $\mathbb{Z}[q]$-basis $(E^{\otimes}(\lambda))_{\lambda\in\Lambda^{(1)}}$.

Proposition 4.7. *A $\mathbb{Z}[q]$-basis for the positive subalgebra $\mathcal{H}(L^+)$ is given by*

$$(E^{\otimes}(w))_w,$$

where w runs over the L-positive elements in $W_L^{(1)}$. For any such w, one has

$$\theta_L^+(E^{\otimes}(w)) = E(w). \tag{4-10}$$

Proof. **(A)** We first check that $E^{\otimes}(\lambda)$ lies in the positive subalgebra $\mathcal{H}(L^+)$ for any L-positive weight $\lambda \in \Lambda^{(1)}$. It is enough to show the property for λ minuscule. In this case, using Lemma 4.4, one easily computes $E^{\otimes}(\lambda)$ and checks that the elements of the Iwahori–Matsumoto basis appearing in its decomposition correspond to L-positive elements in $W_L^{(1)}$.

Now consider $w = (w_1, \dots, w_m) \in W_L^{(1)}$. Write $w = e^{\lambda}v$ with $\lambda \in \Lambda^{(1)}$, $v \in W_{0,L}$. Since $W_{0,L}$ normalizes I^- and I^+, the element w is L-positive if and only if λ is an L-positive weight. Decompose $\lambda = (\lambda_1, \dots, \lambda_m)$ and $v = (v_1, \dots, v_m)$ in the Levi L and recall that, after extending the scalars to $\mathbb{Z}[q^{\pm 1/2}]$,

$$E^{\otimes}(w) = \prod_{j=1}^{m} q^{(\ell(w_j)-\ell(v_j)-\ell(e^{\lambda_j}))/2} E^{\otimes}(\lambda)\tau_v^{\otimes}. \tag{4-11}$$

The element $\tau_v^{\otimes}$ lies in the positive subalgebra, and $E^{\otimes}(\lambda)$ does too if w is L-positive, so the property also holds for $E^{\otimes}(w)$.

Once we know that $E^{\otimes}(w)$ lies in the positive subalgebra $\mathcal{H}(L^+)$ for any L-positive element $w \in W_L^{(1)}$, it is clear that these elements constitute a $\mathbb{Z}[q]$-basis of $\mathcal{H}(L^+)$ by using [Vignéras 2006, 1.5].

(B) Let us show Equality (4-10) for L-positive elements of the form e^{λ} with $\lambda \in \Lambda^{(1)}$. The weight λ can be written $\lambda = \mu - \nu$, where $\mu, \nu \in \Lambda^{(1)}$ are antidominant weights which decompose into $\mu = (\mu_1, \dots, \mu_m)$, $\nu = (\nu_1, \dots, \nu_m)$, so

$\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i = \mu_i - \nu_i$ for $i = 1, \dots, m$. By definition,

$$E^{\otimes}(\lambda) = \prod_{i=1}^{m} q^{(\ell(e^{\lambda_i})+\ell(e^{\nu_i})-\ell(e^{\mu_i}))/2}\theta^{\otimes}(\lambda)$$

and

$$E(\lambda) = q^{(\ell(e^{\lambda})+\ell(e^{\nu})-\ell(e^{\mu}))/2}\theta(\lambda).$$

Note that uniqueness in Proposition 4.1 gives $\theta_L \circ \theta^{\otimes} = \theta$, where $\theta^{\otimes}$ denotes the tensor product of the Bernstein maps, so the required equality will be proved once we have checked that

$$\ell(e^{\lambda}) + \ell(e^{\nu}) - \ell(e^{\mu}) = \sum_{i=1}^{m} (\ell(e^{\lambda_i}) + \ell(e^{\nu_i}) - \ell(e^{\mu_i})). \tag{4-12}$$

By the definition of the length on $\Lambda^{(1)}$,

$$\begin{aligned}\ell(e^{\lambda}) + \ell(e^{\nu}) - \ell(e^{\mu}) &= \sum_{\check\alpha \in \check\Phi^+} |\langle \mu - \nu, \check\alpha\rangle| + |\langle \nu, \check\alpha\rangle| - |\langle \mu, \check\alpha\rangle| \\ &= \sum_{\check\alpha \in \check\Phi^+} |\langle \mu - \nu, \check\alpha\rangle| - \langle \nu, \check\alpha\rangle + \langle \mu, \check\alpha\rangle.\end{aligned}$$

A positive root $\check\alpha$ will give a zero contribution to this sum if and only if $\langle \nu, \check\alpha\rangle \geq \langle \mu, \check\alpha\rangle$. According to (2-3), the fact that λ is L-positive ensures that it is the case for every $\check\alpha \in \check\Phi^+ - \check\Phi^+_L$. Hence the sum can be restricted to the roots $\check\alpha \in \check\Phi^+_L$, which proves that (4-12) holds.

We return to the general case of an L-positive element of the form $w = e^{\lambda}v$. By the previous case, applying θ_L to (4-11) gives

$$\theta_L(E^{\otimes}(w)) = \prod_{j=1}^{k} q^{(\ell(w_j)-\ell(v_j)-\ell(e^{\lambda_j}))/2} E(\lambda)\tau_v.$$

Since $E(w) = q^{(\ell(w)-\ell(v)-\ell(e^{\lambda}))/2} E(\lambda)\tau_v$, it remains to check that

$$\sum_{j=1}^{k} (\ell(e^{\lambda_j}) + \ell(v_j) - \ell(e^{\lambda_j}v_j)) = \ell(e^{\lambda}) + \ell(v) - \ell(e^{\lambda}v).$$

By 2B2, the right side of this equality is twice the number of roots $\check\alpha \in \check\Phi^+$ such that $v\check\alpha \in \check\Phi^-$ and $\langle \lambda, v\check\alpha\rangle < 0$. But $v \in W_{0,L}$, so any $\check\alpha \in \check\Phi^+$ satisfying $v\check\alpha \in \check\Phi^-$ belongs to $\check\Phi^+_L$. Now applying 2B2 to each summand of the left hand side, this remark ensures that the equality holds. □

Proposition 4.7 says in particular that the $\mathbb{Z}[q]$-algebra

$$\mathcal{A}_{L^+} := \mathcal{A}_L \cap \mathcal{H}(L^+) \tag{4-13}$$

has $\mathbb{Z}[q]$-basis $E^{\otimes}(\lambda)$, where λ runs over the L-positive weights λ in $\Lambda^{(1)}$.

Proposition 4.8. *For any $h \in \mathcal{H}$, there is $r \in \mathbb{N}$ such that*

$$\tau_{a_L}^r h \in \sum_{d \in D_L} \mathcal{H}(L^+)\tau_d.$$

Proof. Let $w \in W^{(1)}$. Write $w = e^\lambda w_0 d$ with $w_0 \in W_{0,L}$, $d \in D_L$ and $\lambda \in \Lambda^{(1)}$ a weight that decomposes into $\lambda = \mu - \nu$ where μ and ν are antidominant. There is $r \in \mathbb{N}$ such that $a_L^r e^\lambda w_0$ is a L-positive element and $\ell(a_L^r w) = \ell(a_L^r e^\lambda w_0) + \ell(d)$ by Property (2-4). Note that $a_L^r e^\lambda = e^{r\lambda_L + \mu - \nu}$ and that $r\lambda_L + \mu$ is antidominant. The elements $E(w)$ and $E(a_L^r e^\lambda w_0)$ of the integral Bernstein basis of $\mathcal{H}$ can be written respectively

$$E(w) = q^{(\ell(w) - \ell(w_0) - \ell(d) + \ell(e^\nu) - \ell(e^\mu))/2} \tau_\mu \tau_\nu^{-1} \tau_{w_0} \tau_d$$

and

$$E(a_L^r e^\lambda w_0) = q^{(\ell(a_L^r e^\lambda w_0) - \ell(w_0) + \ell(e^\nu) - \ell(a_L^r) - \ell(e^\mu))/2} \tau_{a_L}^r \tau_\mu \tau_\nu^{-1} \tau_{w_0},$$

so the element

$$\tau_{a_L}^r E(w) = q^{(\ell(w) + \ell(a_L^r) - \ell(a_L^r w))/2} E(a_L^r e^\lambda w_0) \tau_d$$

belongs to $\mathcal{H}(L^+)\tau_d$. □

5. Inducing Hecke modules

5A. We consider the category $\mathscr{C}_L$ of the k-vector spaces $\mathfrak{M}$ endowed with a structure of right $\mathcal{H}(L)$-module such that the central invertible elements $\tau_{\mu_j}^\otimes$, $j \in \Delta - \Delta_L$ act by multiplication by nonzero scalars. This category is closed relative to subquotients.

Proposition 5.1. *Let $\mathfrak{M}$ be a k-vector space endowed with a right action of the positive algebra $\mathcal{H}(L^+)$. Suppose that the central invertible elements $\tau_{\mu_j}^\otimes$, $j \in \Delta - \Delta_L$ act by multiplication by nonzero scalars. Then there is a unique structure of right module over $\mathcal{H}(L)$ on $\mathfrak{M}$ extending the action of $\mathcal{H}(L^+)$.*

Proof. The element $\tau_{a_L}^\otimes$ defined by (4-2) is the product of the $\tau_{\mu_j}^\otimes$, $j \in \Delta - \Delta_L$. Denote by ζ the scalar action of $\tau_{a_L}^\otimes$ on $\mathfrak{M}$. The Hecke algebra $\mathcal{H}(L)$ is generated by $\mathcal{H}(L^+)$ and by the central elements $(\tau_{a_L}^\otimes)^{\pm 1}$. So, if $\mathfrak{M}$ is endowed with an action of $\mathcal{H}(L)$, it is unique and the natural map $\mathfrak{M} \to \mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H}(L)$ is surjective. Define the map $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H}(L) \to \mathfrak{M}$, $v \otimes \tau_h^\otimes \mapsto \zeta^{-r}\, v\, \tau_{a_L^r h}^\otimes$, where $h \in W_L^{(1)}$ and $r \in \mathbb{N}$ is chosen so that $a_L^r h$ is L-positive. One checks that this map is well-defined and factors into an inverse for the previous one. □

Proposition 5.2. *Let $\mathfrak{M}$ in $\mathscr{C}_L$. As a vector space, $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$ decomposes into the direct sums*

$$\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H} = \bigoplus_{d \in D_L} \mathfrak{M} \otimes \tau_d \tag{5-1}$$

and

$$\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H} = \bigoplus_{d \in D_L} \mathfrak{M} \otimes \tau^*_{d^{-1}}. \tag{5-2}$$

Each subspace in these decompositions is isomorphic to $\mathfrak{M}$ via the natural maps $\mathfrak{M} \to \mathfrak{M} \otimes \tau_d$ and $\mathfrak{M} \to \mathfrak{M} \otimes \tau^*_{d^{-1}}$.

The decomposition (5-2) is a decomposition into eigenspaces for the action of τ_{a_L}: it acts by zero on each $\mathfrak{M} \otimes \tau^*_{d^{-1}}$ with $d \neq 1$ and by ζ on $\mathfrak{M} \otimes \tau_1$.

Corollary 5.3. *Let $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ in $\mathscr{C}_L$ be such that there is an exact sequence of right $\mathcal{H}(L)$-modules $0 \to \mathfrak{L} \to \mathfrak{M} \to \mathfrak{N} \to 0$. Then one has an exact sequence of $\mathcal{H}$-modules*

$$0 \to \mathfrak{L} \otimes_{\mathcal{H}(L^+)} \mathcal{H} \to \mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H} \to \mathfrak{N} \otimes_{\mathcal{H}(L^+)} \mathcal{H} \to 0.$$

Corollary 5.4. *Suppose that $\mathfrak{N}$ and $\mathfrak{L}$ in $\mathscr{C}_L$ are finite-dimensional over k and that they have the same semisimplification as $\mathcal{H}(L)$-modules. Then any irreducible quotient of the $\mathcal{H}$-module $\mathfrak{N} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$ is also an irreducible subquotient of $\mathfrak{L} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$.*

Corollary 5.5. *Let $\mathfrak{M}$ in $\mathscr{C}_L$ be such that $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$ is an irreducible $\mathcal{H}$-module. Then $\mathfrak{M}$ is an irreducible $\mathcal{H}(L)$-module.*

Corollaries 5.3 and 5.5 easily follow from Proposition 5.2.

Proof of Corollary 5.4. Let N be an irreducible quotient of $\mathfrak{N} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$. Let $\mathfrak{N}_0$ be a subquotient of the $\mathcal{H}(L)$-module $\mathfrak{N}$ with minimal dimension over k such that N is a quotient of $\mathfrak{N}_0 \otimes_{\mathcal{H}(L^+)} \mathcal{H}$. Using Corollary 5.3 and the irreducibility of N, one sees that $\mathfrak{N}_0$ is irreducible as an $\mathcal{H}(L)$-module. Hence $\mathfrak{N}_0$ is an irreducible subquotient of $\mathfrak{L}$, so that N appears in the semisimplification of $\mathfrak{L} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$. □

Proof of Proposition 5.2.

(A) Proposition 4.8 ensures that

$$\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H} = \sum_{d \in D_L} \mathfrak{M} \otimes \tau_d. \tag{5-3}$$

Since $\tau^*_{d^{-1}}$ decomposes with respect to the Iwahori–Matsumoto basis into the sum of τ_d and of other terms corresponding to elements with strictly smaller length [Vignéras 2005, Lemma 13], we also have

$$\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H} = \sum_{d \in D_L} \mathfrak{M} \otimes \tau^*_{d^{-1}}. \tag{5-4}$$

(B) Let $\mu \in \Lambda^{(1)}$ be a minuscule weight and $m \in \mathfrak{M}$. If μ is not L-positive, then $E(\mu)$ acts by zero on $m \otimes 1$ (Because of relations (4-8), there is $j \in \Delta - \Delta_L$ such that $E(\mu)E(\mu_j)=0$; since $E(\mu_j)=\tau_{e^{\mu_j}}$ acts by a nonzero scalar on $m\otimes 1$, the element $E(\mu)$ acts by zero.) We show by induction on $\ell(d)$ that, for any $d\in D_L$,

$$m\otimes \tau^*_{d^{-1}}E(\mu)=m\otimes E({}^d\mu)\tau^*_{d^{-1}}. \tag{5-5}$$

Let $d\in D_L$ and let $s\in S_0$ be such that $ds\in D_L$ and $\ell(ds)=\ell(d)+1$. These hypotheses imply $d\check{\alpha}_s\in\check{\Phi}^+-\check{\Phi}^+_L$. Suppose that (5-5) holds. We have to show that

$$m\otimes \tau^*_{d^{-1}}\tau^*_s E(\mu)=m\otimes E({}^{ds}\mu)\tau^*_{d^{-1}}\tau^*_s. \tag{5-6}$$

If $\langle\mu,\check{\alpha}_s\rangle=0$, then $\mu={}^s\mu$ and $E(\mu)$ and τ^*_s commute by Proposition 4.3(1), so we have the required equality.

If $\langle\mu,\check{\alpha}_s\rangle>0$, then

$$\begin{aligned} m\otimes\tau^*_{d^{-1}}\tau^*_s E(\mu) &= m\otimes\tau^*_{d^{-1}}(\tau_s+\nu_s)E(\mu)\\ &= m\otimes\tau^*_{d^{-1}}E({}^s\mu)\tau^*_s+m\otimes\tau^*_{d^{-1}}E(\mu)\nu_s \quad \text{by the Bernstein relations}\\ &= m\otimes E({}^{ds}\mu)\tau^*_{d^{-1}}\tau^*_s+m\otimes E({}^{d}\mu)\tau^*_{d^{-1}}\nu_s \quad \text{by induction.}\end{aligned}$$

The hypothesis on μ implies that $\langle{}^d\mu,d\check{\alpha}_s\rangle>0$, so ${}^d\mu$ is not L-positive. Hence the second part of the preceding sum is zero, which gives the required equality.

If $\langle\mu,\check{\alpha}_s\rangle<0$, then

$$\begin{aligned} m\otimes\tau^*_{d^{-1}}\tau^*_s E(\mu) &= m\otimes\tau^*_{d^{-1}}E({}^s\mu)\tau_s \quad \text{by the Bernstein relations}\\ &= m\otimes E({}^{ds}\mu)\tau^*_{d^{-1}}\tau_s \quad \text{by induction.}\end{aligned}$$

But $\langle{}^{ds}\mu,d\check{\alpha}_s\rangle>0$, so ${}^{ds}\mu$ is not L-positive. Hence we have proved that both sides of (5-6) are zero.

By Proposition 2.2, we have proved (5-6) by induction.

(C) Result (B) shows that the right action of $E({}^{d^{-1}}\lambda_L)$ on $\mathfrak{M}\otimes\tau^*_{d'^{-1}}$ is zero for any $d'\in D_L$ $d'\neq d$ and that it is a multiplication by ζ on $\mathfrak{M}\otimes\tau^*_{d^{-1}}$. Hence, the decomposition (5-4) is a direct sum.

(D) Let us prove that

$$\mathfrak{M}\to\mathfrak{M}\otimes\tau^*_{d_0^{-1}}$$

is injective for any $d_0\in D_L$. Let $m\in\mathfrak{M}$ such that

$$m\otimes\tau^*_{d_0^{-1}}=0. \tag{5-7}$$

Let $(m_\nu)_{\nu\in\mathcal{N}}$ be a family of generators of the $\mathcal{H}(L^+)$-module $\mathfrak{M}$ that contains m, say $m_{\nu_0}=m$. By [Bourbaki 1961, Chapitre 1, §2, n° 11], (5-7) implies that there is a finite family $(k_\iota)_{\iota\in\mathcal{I}}$ of elements in $\mathcal{H}$ and a finitely supported family $(b_{\iota,\nu})_{\iota\in\mathcal{I},\nu\in\mathcal{N}}$ of elements in $\mathcal{H}(L^+)$ such that

- $\sum_{\nu\in\mathcal{N}}m_\nu b_{\iota,\nu}=0$ for any $\iota\in\mathcal{I}$,

- $\sum_{\iota\in\mathcal{I}} b_{\iota,\nu_0}k_\iota = \tau^*_{d_0^{-1}}$,
- $\sum_{\iota\in\mathcal{I}} b_{\iota,\nu}k_\iota = 0$ for any $\nu \neq \nu_0$.

By Proposition 4.8, there exists $r \in \mathbb{N}$ such that $\tau^r_{a_L}k_\iota = \sum_{d\in\mathcal{D}_L} c_{\iota,d}\tau_d$ with $c_{\iota,d} \in \mathcal{H}(L^+)$ for any $\iota \in \mathcal{I}$. The component of

$$\tau^r_{a_L}\tau^*_{d_0^{-1}} = \sum_{d\in D_L}\sum_{\iota} b_{\iota,\nu_0}c_{\iota,d}\tau_d$$

with support in $I(1)L^+d_0I(1)$ is equal to $\tau^r_{a_L}\tau_{d_0}$ on one hand, and to $\sum_\iota b_{\iota,\nu_0}c_{\iota,d_0}\tau_{d_0}$ on the other hand. So, by Lemma 2.4, we get $\tau^r_{a_L}\tau_{d_0} = \sum_\iota b_{\iota,\nu_0}c_{\iota,d_0}\tau_{d_0}$ and then $\tau^r_{a_L} = \sum_\iota b_{\iota,\nu_0}c_{\iota,d_0}$.

The same argument applied to $0 = \sum_{\iota\in\mathcal{I}} b_{\iota,\nu}k_\iota$ shows that $0 = \sum_\iota b_{\iota,\nu}c_{\iota,d_0}$ for $\nu \neq \nu_0$.

Multiplying $0 = \sum_{\nu\in\mathcal{N}} m_\nu b_{\iota,\nu}$ by c_{ι,d_0} for any $\iota \in \mathcal{I}$, and then summing over ι, gives $0 = m_{\nu_0}\tau^r_{a_L}$, and hence $m = 0$.

This proves the remaining assertions of Proposition 5.2, also using again the argument of [Vignéras 2005, Lemma 13] to deduce the direct sum (5-1) from the direct sum (5-2). □

5B. ***Standard modules.*** The field k is naturally a $\mathbb{Z}[q]$-module via the specialization $q \mapsto 0$. A k-character of $\mathcal{A}$ is a morphism of unitary rings $\chi : \mathcal{A} \to k$ which is compatible with the structures of $\mathbb{Z}[q]$-modules. The set of k-characters of $\mathcal{A}$ inherits a natural action of W_0 given by $(w_0, \chi) \mapsto {}^{w_0}\chi$.

Because of (4-8), one has $E(\mu_J)E(\mu_K) = 0$ for any $J, K \subset \{1, \dots, n\}$, unless either $J \subset K$ or $K \subset J$. So, a k-character χ of $\mathcal{A}$ is completely determined by its values on $\{\tau_t,\ t \in \mathbb{T}\}$, the flag

$$J_0 = \varnothing \subsetneq J_1 \subsetneq \cdots \subsetneq J_r = \{1, \dots, n\}$$

of the subsets $J_i \subset \{1, \dots, n\}$ such that $\chi(E(\mu_{J_i}))$ is nonzero, and these nonzero values. The standard module induced by χ is the right $\mathcal{H}$-module

$$\chi \otimes_{\mathcal{A}} \mathcal{H}.$$

The set of minuscule weights $(\mu_{J_i})_{i\in\{1,\dots,r\}}$ we call the *support* of χ. We say that χ has dominant or antidominant support if every weight in the support is so.

Recall that any k-vector space which is a simple $\mathcal{H}$-module is a quotient of a standard module [Vignéras 2005, 1.4].

Definition 5.6. The character χ, the associated standard module, and any quotient of the latter are said to be *regular* if the flag is maximal, that is, $r = n$; *supersingular* if the flag is minimal, that is, $r = 1$; and *singular* otherwise.

If $n = 1$, we make the convention that any character of $\mathcal{A}$ is supersingular.

5C. *Inducing standard modules.*

5C1. A k-character $\chi : \mathcal{A} \to k$ is called *adapted* to L (or L-*adapted*) if $\chi(E(\lambda_L))$ is nonzero, where λ_L is defined by (4-2). This implies that χ has L-positive support, that is, any weight in its support is L-positive. A k-character $\chi_L : \mathcal{A}_L \to k$ of the integral Bernstein subalgebra of $\mathcal{H}(L)$ is the tensor product of k-characters of the integral Bernstein algebras corresponding to the L_j's, $j \in \{1, \dots, m\}$. The value of χ_L on the invertible element $\tau_{a_L}^{\otimes}$ being nonzero, χ_L is completely determined by its restriction to $\mathcal{A}_{L^+}$ and we have an isomorphism of $\mathcal{H}(L)$-modules:

$$\chi_L \otimes_{\mathcal{A}_L} \mathcal{H}(L) \simeq \chi_L \otimes_{\mathcal{A}_{L^+}} \mathcal{H}(L).$$

There is a one-to-one correspondence between the k-characters χ_L of $\mathcal{A}_L$ and the k-characters of $\mathcal{A}$ adapted to L: it associates the character $\chi : \mathcal{A} \to k$ adapted to L with the character χ_L given on $\mathcal{A}_{L^+}$ by

$$\chi_L(E^{\otimes}(\lambda)) := \chi \circ \theta_L(E^{\otimes}(\lambda)) = \chi(E(\lambda))$$

for any L-positive weight $\lambda \in \Lambda^{(1)}$.

The algebra $\mathcal{A}_L$ is endowed not only with an action of the finite Weyl group $W_{0,L}$, but also of the normalizer of $W_{0,L}$ in W_0. Nevertheless, the previous correspondence is only compatible with the action of $W_{0,L}$ which preserves the set of L-positive weights in $\Lambda^{(1)}$.

5C2. With Proposition 5.1, the previous paragraph gives the following result.

Proposition 5.7. *Given $\chi_L : \mathcal{A}_L \to k$, let $\chi : \mathcal{A} \to k$ be the associated L-adapted character of $\mathcal{A}$. The standard module relative to χ is induced by the standard module relative to χ_L in the sense that the following isomorphisms of $\mathcal{H}$-modules hold*:

$$\chi \otimes_{\mathcal{A}} \mathcal{H} \simeq \chi_L \otimes_{\mathcal{A}_L} \mathcal{H}(L) \otimes_{\mathcal{H}(L^+)} \mathcal{H} \simeq \chi_L \otimes_{\mathcal{A}_{L^+}} \mathcal{H}.$$

5D. *Intertwining operators between standard modules.*

Let $\chi : \mathcal{A} \to k$ be a character. We assume that L is a strict Levi subgroup of G and that χ is adapted to L. Then its support contains at least $\{\mu_j,\ j \in \Delta - \Delta_L\}$.

Let $d \in W_0$ and $s \in S_0$ be a simple reflection such that $d, ds \in D_L$ and $\ell(ds) = \ell(d) + 1$. Let ξ be the k-character $\xi = {}^{d^{-1}}\chi$. Denote respectively by φ and φ_s the canonical generators of the standard modules induced by ξ and ${}^{s}\xi$.

5D1. *Definition of the intertwiners.*

Remark 5.8. The fact that $\ell(ds) = \ell(d) + 1$ implies that $d\check{\alpha}_s$ is a positive root.

That both ds and d belong to D_L implies that $ds \notin W_{0,L}d$, so there exists j in $\Delta - \Delta_L$ such that ${}^{dsd^{-1}}\mu_j \neq \mu_j$: the weight ${}^{d^{-1}}\mu_j$ lies in the support of ξ and satisfies $\langle {}^{d^{-1}}\mu_j, \check{\alpha}_s \rangle = \langle \mu_j, d\check{\alpha}_s \rangle < 0$. Because of relations (4-8), any other minuscule weight μ in the support of ξ will then satisfy $\langle \mu, \check{\alpha}_s \rangle \leq 0$.

Lemma 5.9. *The vector $\varphi\tau_s^*$ is an eigenvector for the character ${}^s\xi$ of $\mathscr{A}$.*

Proof. It is easy to check that $\varphi\tau_s^*\tau_t = {}^s\xi(\tau_t)\varphi\tau_s^*$ for any $t \in \mathbb{T}$ (or see [Ollivier 2006a, 4.4.2]). We have yet to show that

$$\varphi\tau_s^* E(\mu_J) = {}^s\xi(E(\mu_J))\,\varphi\tau_s^* \tag{5-8}$$

for any minuscule weight μ_J associated to $J \subset \{1, \dots, n-1\}$.

If μ_J is fixed by s, the Bernstein relations ensure that τ_s^* and $E(\mu_J)$ commute and (5-8) holds.

If $\langle \mu_J, \check{\alpha}_s\rangle > 0$, the Bernstein relations give

$$\varphi\tau_s^* E(\mu_J) = \varphi\tau_s E(\mu_J) + \varphi E(\mu_J)\nu_s = \varphi E({}^s\mu_J)\tau_s^* = {}^s\xi(E(\mu_J))\,\varphi\tau_s^*,$$

because μ_J is not in the support of ξ by Remark 5.8 .

If $\langle \mu_J, \check{\alpha}_s\rangle < 0$, the Bernstein relations give $\varphi\tau_s^* E(\mu_J) = \varphi E({}^s\mu_J)\tau_s = 0$, because ${}^s\mu_J$ is not in the support of ξ, and (5-8) holds. ☐

We choose a weight ${}^{d^{-1}}\mu_j$ as in Remark 5.8. It is a minuscule weight in the support of ξ. It can be denoted by μ_K for some $K \subset \{1, \dots, n\}$. Recall that $\langle \mu_K, \check{\alpha}_s\rangle < 0$. Set

$$\beta := \xi(E(\mu_{K\cup {}^sK}))\xi(E(\mu_{K\cap {}^sK}))\xi(E(\mu_K))^{-1},$$

where ${}^sK$ denotes the image of K under the natural action of s.

Remark 5.10. Because of the relations (4-8), this scalar β is zero as soon as there exists a minuscule weight μ_J different from μ_K in the support of ξ such that $\langle \mu_J, \check{\alpha}_s\rangle < 0$.

Lemma 5.11. *The vector $\varphi_s(E(se^{\mu_K}) - \beta\nu_s)$ is an eigenvector for the character ξ of $\mathscr{A}$.*

Proof. Note that ν_s lies in $\mathscr{A}$ and commutes with τ_s. See [Ollivier 2006a, 4.4.2] to check that $\varphi_s(E(se^{\mu_K}) - \beta\nu_s)\tau_t = \xi(\tau_t)\varphi_s(E(se^{\mu_K}) - \beta\nu_s)$ for any $t \in \mathbb{T}$. We have yet to prove that

$$\varphi_s(E(se^{\mu_K}) - \beta\nu_s)E(\mu_J) = \xi(E(\mu_J))\,\varphi_s(E(se^{\mu_K}) - \beta\nu_s) \tag{5-9}$$

for any minuscule weight μ_J associated to $J \subset \{1, \dots, n-1\}$.

We use the fact that after extending the scalars to $\mathbb{Z}[q^{\pm 1/2}]$, we have

$$E(se^{\mu_K}) = q^{-1}E({}^s\mu_K)\tau_s = q^{-1}\tau_s^* E(\mu_K). \tag{5-10}$$

If μ_J is fixed by s, then (5-9) holds.

If $\langle \mu_J, \check{\alpha}_s\rangle < 0$, then μ_J is not in the support of ${}^s\xi$ by Remark 5.8, and the left side of (5-9) is $\varphi_s E(se^{\mu_K})E(\mu_J)$. The Bernstein relations and (5-10) give

$$
\begin{aligned}
&E(se^{\mu_K})E(\mu_J)\\
&\quad = E({}^s\mu_J)E(se^{\mu_K}) - \nu_s q^{(|K|-|{}^sK\cap J|)(|J|-|{}^sK\cap J|)-1} E(\mu_{{}^sK\cup J})E(\mu_{{}^sK\cap J}). \quad (5\text{-}11)
\end{aligned}
$$

- If $J \neq K$, the power of q in the preceding equality is at least 1, so
$$\varphi_s E(se^{\mu_K})E(\mu_J) = \xi(E(\mu_J))\,\varphi_s E(se^{\mu_K}).$$
If J is in the support of ξ, then $\beta = 0$ by Remark 5.10, and (5-9) holds. If J is not in the support of ξ, we have proved that both sides of (5-9) are zero.
- If $J = K$, then (5-11) gives equality (5-9).

If $\langle \mu_J, \check{\alpha}_s\rangle > 0$, then μ_J is not in the support of ξ and the right side of (5-9) is zero. The Bernstein relations give

$$
\begin{aligned}
&E(se^{\mu_K})E(\mu_J)\\
&\quad = E({}^s\mu_J)E(se^{\mu_K}) + \nu_s q^{(|K|-|K\cap J|)(|J|-|K\cap J|)-1} E({}^s\mu_{K\cup J})E({}^s\mu_{K\cap J}), \quad (5\text{-}12)
\end{aligned}
$$

so $\varphi_s E(se^{\mu_K})E(\mu_J) = \varphi_s \nu_s q^{(|K|-|K\cap J|)(|J|-|K\cap J|)-1} E({}^s\mu_{K\cup J})E({}^s\mu_{K\cap J})$.

- If $J \neq {}^sK$, the latter power of q is at least 1, so the only remaining term in the left side of (5-9) is equal to $-{}^s\xi(E(\mu_J))\beta\,\varphi_s\nu_s$: if μ_J is in the support of ${}^s\xi$, then $\beta = 0$ by Remark 5.10; if μ_J is not in the support of ${}^s\xi$, then ${}^s\xi(E(\mu_J)) = 0$.
- If $J = {}^sK$, then $\varphi_s E(se^{\mu_K})E(\mu_J) = \xi(E(\mu_K))\beta\varphi_s\nu_s$, so the left side of (5-9) is zero. $\square$

The preceding lemmas allow us to define an $\mathcal{H}$-equivariant morphism Φ from the standard module induced by ξ into the one induced by ${}^s\xi$, and another, Ψ, going the other way around. They are fully determined by $\Phi(\varphi) = \varphi_s(E(se^{\mu_K}) - \beta\nu_s)$ and $\Psi(\varphi_s) = \varphi\tau_s^*$.

Lemma 5.12. *The composition of Φ and Ψ is the homothety with ratio*

$$\xi(E(\mu_K) - \beta\nu_s^2).$$

Proof. Any $d_0 \in W_0$ such that ${}^s\mu_K = {}^{d_0^{-1}}\mu_j$ satisfies $\langle \mu_j, d_0\check{\alpha}_s\rangle = -\langle \mu_K, \check{\alpha}_s\rangle > 0$, so $d_0\check{\alpha}_s \in \check{\Phi}^-$ and $\ell(d_0 s) = \ell(d_0) - 1$. Hence

$$\tau^*_{d_0^{-1}}\tau_s = 0$$

in $\mathcal{H}\otimes_{\mathbb{Z}[q]}k$, and Lemma 4.4 ensures that $E({}^s\mu_K)\tau_s = 0$ in $\mathcal{H}\otimes_{\mathbb{Z}[q]}k$. Thus $\varphi_s\tau_s = 0$ and $\varphi_s(E(se^{\mu_K}) - \beta\nu_s)\tau_s^* = \xi(E(\mu_K) - \beta\nu_s^2)\varphi_s$, and $\Phi\circ\Psi$ is a homothety with ratio $\xi(E(\mu_K) - \beta\nu_s^2)$. Using the equalities $E(\mu_K)E(se^{\mu_K}) = \tau_s E(\mu_{K\cap{}^sK})E(\mu_{K\cup{}^sK})$ and $\tau_s E(se^{\mu_K}) = E(\mu_K)$, one checks that $\Psi\circ\Phi$ is a homothety with the same ratio. $\square$

5D2. *Conditions of isomorphism.*

5D2.1. Suppose that χ_L is a tensor product of supersingular characters. Then the support of χ is exactly $\{\mu_j,\ j \in \Delta - \Delta_L\}$.

Recall that the standard Levi subgroup L decomposes into $L = L_1 \times \cdots \times L_m$, where L_i is isomorphic to $\mathrm{GL}_{n_i}(F)$ for $i \in \{1, \ldots, m\}$. There exists a simple reflection not belonging to $W_{0,L}$ but normalizing $W_{0,L}$ if and only if one can find two consecutive L_i and L_{i+1} with $i \in \{1, \ldots, m-1\}$ such that $n_i = n_{i+1} = 1$.

We will say that χ_L *satisfies Hypothesis* $(\star)$ if for any simple reflection s_j not belonging to $W_{0,L}$ but normalizing $W_{0,L}$, the characters ${}^{s_j}\chi_L$ and χ_L differ.

Lemma 5.13. *Let $j \in \Delta$ and suppose that the simple reflection s_j does not belong to $W_{0,L}$ but normalizes $W_{0,L}$. The k-character χ_L and its conjugate by s_j coincide if and only if two conditions are satisfied*:

- $\chi(\nu_{s_j}^2) \neq 0$, *that is*, $\chi(\nu_{s_j}^2) = 1$,
- $\chi(E(\mu_j))^2 = \chi(E(\mu_{j-1}))\chi(E(\mu_{j+1}))$.

Proof. First note that $\nu_{s_j}^2 = \sum_{t \in \mathbb{T}_{s_j}} \tau_t$. One then easily checks that $\chi(\nu_{s_j}^2) = 1$ if the characters χ_L and its conjugate by s_j coincide on the space generated by $\{\tau_t^{\otimes}, t \in \mathbb{T}\}$, and that $\chi(\nu_{s_j}^2) = 0$ otherwise (see also [Ollivier 2006a, Remarque 7]).

Saying that s_j does not belong to $W_{0,L}$ means that $e^{\mu_{j-1}}$, e^{μ_j}, $e^{\mu_{j+1}}$ are central elements in L, so $\chi(E(\mu_{j-1}))$, $\chi(E(\mu_j))$, $\chi(E(\mu_{j+1}))$ are nonzero elements in k. The characters χ_L and its conjugate by s_j coincide if and only if they coincide on the space generated by $\{\tau_t^{\otimes}, t \in \mathbb{T}\}$, and

$$\frac{\chi(E(\mu_j))}{\chi(E(\mu_{j-1}))} = \frac{\chi(E(\mu_{j+1}))}{\chi(E(\mu_j))}. \qquad \square$$

By Lemma 5.12, it is clear that if $\beta = 0$, then $I(\xi)$ and $I({}^s\xi)$ are isomorphic. Saying that β is nonzero means that $\mu_{K \cup {}^sK}$ and $\mu_{K \cap {}^sK}$ both belong to the support of ξ. Because of the hypothesis on the support of χ, this implies that $\mu_{j+1} = {}^d\mu_{K \cup {}^sK}$, $\mu_j = {}^d\mu_K$, $\mu_{j-1} = {}^d\mu_{K \cap {}^sK}$ belong to the support of χ and that $dsd^{-1} = s_j$ is a simple reflection not belonging to $W_{0,L}$ and normalizing $W_{0,L}$. By Hypothesis $(\star)$, Lemma 5.13 then proves that $\xi(E(\mu_K) - \beta\nu_s^2)$ is nonzero, so $I(\xi)$ and $I({}^s\xi)$ are isomorphic.

By induction and using Proposition 2.2, we get the following result.

Proposition 5.14. *Let $\chi : \mathcal{A} \to k$ be an L-adapted character.*

Suppose that the associated $\chi_L : \mathcal{A}_L \to k$ is a tensor product of supersingular characters and that it satisfies Hypothesis $(\star)$. Then the standard module induced by χ is isomorphic to the standard module induced by any conjugate ${}^{d^{-1}}\chi$ of χ under the action of the inverse of an element $d \in \mathcal{D}_L$.

5D2.2. Let $\chi_0 : \mathscr{A} \to k$ be a character with antidominant support, and L be the maximal Levi subgroup such that the associated character $\chi_{0,L} : \mathscr{A} \to k$ is a tensor product of supersingular or regular characters. This Levi subgroup can be described in the following way: any $j \in \Delta$ lies in $\Delta - \Delta_L$ if and only if μ_j lies in the support of χ_0 and at least one of μ_{j+1} or μ_{j-1} does not lie in the support of χ_0.

We suppose now that $\chi = {}^{w_0}\chi_0$, where $w_0 \in W_{0,L}$. It is adapted to L and we can apply the results of Section 5D1.

Consider as before the weight μ_K in the support of ξ and the element $j \in \Delta - \Delta_L$ such that $\mu_K = {}^{d^{-1}}\mu_j$. Then $\mu_{K \cup {}^sK}$ and $\mu_{K \cap {}^sK}$ cannot be simultaneously in the support of ξ; otherwise μ_{j-1}, μ_j, μ_{j+1} would be in the support of χ_0, which contradicts the definition of $\Delta - \Delta_L$. Hence $\beta = 0$ and $I(\xi)$ and $I({}^s\xi)$ are isomorphic. By induction (using Proposition 2.2), the following proposition is proved.

Proposition 5.15. *Let $\chi_0 : \mathscr{A} \to k$ be a character with antidominant support and L the maximal Levi subgroup such that the associated character $\chi_{0,L} : \mathscr{A}_L \to k$ is a tensor product of supersingular or regular characters. Let $w_0 \in W_{0,L}$. The standard module induced by $\chi := {}^{w_0}\chi_0$ is isomorphic to the standard module induced by any conjugate ${}^{d^{-1}}\chi$ of χ under the action of the inverse of an element $d \in \mathscr{D}_L$.*

5E. *Nonsupersingular Hecke modules.*

5E1. *Regular standard modules.*

Proposition 5.16. *The standard module induced by a character $\chi : \mathscr{A} \to k$ with regular support is a k-vector space with dimension $n!$.*

5E1.1. Our proof relies on further ingredients relative to root data and Coxeter systems. Let $R \subset \check{\Pi}$ be a set of simple roots and denote by $\langle R \rangle$ the subset of Φ^+ generated by R. Define $W_0(R)$ to be the subset of W_0 whose elements w satisfy $w(R) \subset \check{\Phi}^-$ and $w(\check{\Pi} - R) \subset \check{\Phi}^+$.

Lemma 5.17. *In $W_0(R)$ there is a unique element w_R with minimal length. It is an involution and its length is equal to the cardinality of $\langle R \rangle$.*

Proof. The length of an element w in W_0 being the number of positive roots $\alpha \in \Phi^+$ such that $w\alpha \in \Phi^-$ (Section 2A), any element in $W_0(R)$ has length larger than the cardinality of $\langle R \rangle$. The subgroup of W_0 generated by the simple reflections corresponding to the simple roots in R has a unique maximal length element w_R, with length the cardinality of $\langle R \rangle$. It is an involution satisfying $w_R(R) = -R$ and $w_R(\Phi^+ - \langle R \rangle) \subset \Phi^+$ [Bourbaki 1968, Chapitre VI, §1, n° 1.6, corollaire 3]. This element belongs to $W_0(R)$.

Let $w \in W_0(R)$. Suppose that $\ell(w) = \ell(w_R)$. Then the roots in $\langle R \rangle$ are the only positive ones made negative by w. Applying the length property 2B2 and the definition of $W_0(R)$, we then see that $\ell(w) = \ell(ww_R) + \ell(w_R)$, so $ww_R = 1$ and $w = w_R$. □

Lemma 5.18. *Let $w \in W_0(R)$. Suppose that $w \neq w_R$ and consider $s_j \in S_0$ such that $\ell(s_j w w_R) = \ell(w w_R) - 1$. Then $\ell(s_j w) = \ell(w) - 1$, the element $s_j w$ lies in $W_0(R)$ and the positive root $-w^{-1}\check{\alpha}_j$ is not a simple root.*

Proof. The hypothesis on the length ensures that $w_R w^{-1}\check{\alpha}_j \in \check{\Phi}^-$. Because of the properties of w and w_R, it implies $w^{-1}\check{\alpha}_j \in \check{\Phi}^-$ and $\ell(s_j w) = \ell(w) - 1$. More precisely, one checks that the only possibility is $-w^{-1}\check{\alpha}_j \in \check{\Phi}^+ - \langle R\rangle$. So, if $-w^{-1}\check{\alpha}_j$ were a simple root, it would be an element in $\check{\Pi} - R$, which would contradict $w(\check{\Pi} - R) \subset \check{\Phi}^+$. It remains to check that $s_j w$ lies in $W_0(R)$. Let $\check{\alpha} \in R$. Since $w\check{\alpha} \in \check{\Phi}^- - \{-\check{\alpha}_j\}$, we have $s_j w\check{\alpha} \in \check{\Phi}^-$. Let $\check{\alpha} \in \check{\Pi} - R$. Since $w\check{\alpha} \in \check{\Phi}^+ - \{\check{\alpha}_j\}$, we have $s_j w\check{\alpha} \in \check{\Phi}^+$. □

Lemma 5.19. *Denote by $\sigma \in W_0$ the cycle $(n, n-1, \dots, 1)$. Let $\check{\alpha} \in \check{\Pi} - R$. There exists $j \in \{1, \dots, n-1\}$ such that $\sigma^j w_R \in W(R \cup \{\check{\alpha}\})$.*

Proof. We first make some remarks.

(1) Let $\check{\beta} \in \check{\Pi} - R$ be a simple root. Then $w_R\check{\beta}$ is a positive root. Also, s_β appears in any reduced decomposition of the transposition $w_R s_\beta w_R$ according to the set S_0. From this, one easily deduces that $w_R\check{\beta} \geq \check{\beta}$, where $\geq$ denotes the partial order on $\check{\Phi}$ described in 2A. Conversely, let $\check{\alpha} \in \check{\Pi} - R$. If $w_R\check{\beta} \geq \check{\alpha}$, this means that s_α appears in any reduced decomposition of $w_R s_\beta w_R$, so $\check{\beta} = \check{\alpha}$.

(2) Let $j \in \{1, \dots, n-1\}$ and $\check{\beta} \in \check{\Phi}^+$. Then $\sigma^j\check{\beta} \in \check{\Phi}^-$ if and only if $\beta \geq \check{\alpha}_j$.

Let $\check{\alpha} \in \check{\Pi} - R$ as in the lemma and $j \in \{1, \dots, n-1\}$ such that $\check{\alpha} = \check{\alpha}_j$. We check that $\sigma^j w_R \in W_0(R \cup \{\check{\alpha}_j\})$. Any $\check{\beta} \in R$ is sent by w_R to an element in $-R$, which in turn is sent by σ^j to an element in $\check{\Phi}^-$ by (2). Let $\check{\beta} \in \check{\Pi} - R$. Then $w_R\check{\beta} \in \Phi^+$ and using (2), $\sigma^j w_R\check{\beta} \in \check{\Phi}^-$ if and only if $w_R\check{\beta} \geq \check{\alpha}_j$, which by (1) is equivalent to $\check{\beta} = \check{\alpha}_j$. □

Proof of Proposition 5.16. Let $\chi : \mathscr{A} \to k$ be a character with regular antidominant support.

(A) Let $R \subset \check{\Pi}$ be as in 5E1.1. We prove by induction on the length of $w \in W_0(R)$ that the standard modules induced by ${}^{w}\chi$ and ${}^{w_R}\chi$ are isomorphic as $\mathscr{H}$-modules.

Let $w \in W_0(R)$. Suppose $w \neq w_R$; then there is $s_j \in S_0$ such that $\ell(s_j w w_R) = \ell(w w_R) - 1$. By Lemma 5.18, this implies $\ell(s_j w) = \ell(w) - 1$ and the element $s_j w$ also lies in $W_0(R)$. Set $\xi = {}^{s_j w}\chi$. We prove that ξ and ${}^{s_j}\xi$ induce isomorphic standard modules. We are in the situation of Section 5D; the Levi subgroup here is simply the diagonal torus. So we have two well-defined intertwining operators between the standard modules in question. By Remark 5.10, there is an easy sufficient condition for these operators to be isomorphisms: it suffices to check that there is more than one minuscule weight μ in the support of ξ satisfying $\langle \mu, \check{\alpha}_j\rangle < 0$; that is, that there is more than one antidominant minuscule weight λ

such that $\langle \lambda, w^{-1}\check{\alpha}_j \rangle > 0$. This is true, because $w^{-1}\check{\alpha}_j \in \Phi^-$ and $-w^{-1}\check{\alpha}_j$ is not a simple root, by Lemma 5.18.

(B) For $w \in W_0$, the standard modules induced by ${}^{w}\chi$ and ${}^{\sigma w}\chi$ have the same dimension, as proved in [Ollivier 2006a, Proposition 2].

(C) Let $R \subset \check{\Pi}$ be a set of simple roots. We prove by induction on the cardinality of R that the standard module induced by ${}^{w}\chi$ is $n!$-dimensional for any $w \in W_0(R)$. If $R = \varnothing$, then $W_0(R) = \{1\}$, and the result is given by Propositions 5.2 and 5.7. Suppose that the property holds for some set of simple roots $R \subsetneq \check{\Pi}$. Let $\check{\alpha} \in \check{\Pi} - R$ and $w \in W_0(R \cup \{\check{\alpha}\})$. By Lemma 5.19, there is a power σ^j of the cycle σ such that $\sigma^j w_R \in W_0(R \cup \{\check{\alpha}\})$. We conclude using (A) and (B). $\square$

5E1.2. The motivation for Proposition 5.16 is this:

Conjecture 5.20. *Let $\chi : \mathcal{A} \to k$ be a character with regular support and $w_0 \in W_0$. The standard modules induced by ${}^{w_0}\chi$ and χ have the same semisimplification as modules over $\mathcal{H}$.*

We can prove the conjecture if we consider characters of $\mathcal{A}$ which are totally degenerate on the finite torus, that is, for $t \in \mathbb{T}$, the value $\chi(\tau_t)$ only depends on the orbit of t under the action of W_0. By twisting, we can consider that χ is trivial on the finite torus. Then the standard module induced by χ can be seen as a module over the Iwahori–Hecke algebra (see for example Section 8). One can then apply the arguments listed in [Ollivier 2006b, 2.4] (for the case of GL_3) to show that χ and its conjugates induce standard modules which have the same semisimplification. The first argument comes from [Vignéras 2006, théorème 6]: the character χ can be lifted to a character χ_0 with values in $\overline{\mathbb{Z}}_p$, and we see the latter as a character with values in $\overline{\mathbb{Q}}_p$. Since the standard module induced by χ is $n!$-dimensional over $\overline{\mathbb{F}}_p$, [Vignéras 2006, théorème 5] says that it is isomorphic to the reduction of the canonical integral structure of the $\mathcal{H} \otimes_{\mathbb{Z}[q]} \overline{\mathbb{Q}}_p$ standard module induced by χ_0. To conclude, we recall Proposition 2.3 of [Rogawski 1985]: two standard modules for the Iwahori–Hecke algebra in characteristic zero have the same semisimplification if they are induced by conjugate characters. The proof is based on the description of an explicit basis for the standard modules owing to the Kazhdan–Lusztig polynomials for the Iwahori–Hecke algebra.

Proposition 5.21. *Conjecture 5.20 is true for the standard modules over the Iwahori–Hecke algebra, that is, for characters χ that are trivial on the finite torus.*

5E2. *Nonsupersingular simple modules and induction.* Recall that a nonsupersingular character $\chi : \mathcal{A} \to k$ with antidominant support is adapted to some strict Levi subgroup L of G. So the associated standard module is *induced* from a $\mathcal{H}(L)$-module by Proposition 5.7. In the light of this, the following proposition bolsters the definition of a *nonsupersingular* module.

Proposition 5.22. *Assume that Conjecture 5.20 is true. Any simple nonsupersingular $\mathcal{H}$-module appears in the semisimplification of a standard module for $\mathcal{H}$ relative to a nonsupersingular character with antidominant support.*

Proof. Let M be a simple nonsupersingular module: it is a quotient of a standard module induced by some nonsupersingular character $\xi : \mathcal{A} \to k$. Let $w \in W_0$ with minimal length such that $\chi := {}^{w}\xi$ has antidominant support. We want to prove that M appears in the semisimplification of the standard module induced by χ. Let L be the standard Levi subgroup associated to χ as in Proposition 5.15. Let $(w_0, d) \in W_{0,L} \times D_L$ be such that $w = w_0 d$. Recall that $\ell(w) = \ell(w_0) + \ell(d)$. By Proposition 5.15, the standard modules induced by ${}^{w_0^{-1}}\chi$ and ξ are isomorphic. So M is an irreducible quotient of the standard module induced by ${}^{w_0^{-1}}\chi$. We have yet to check that it is a subquotient of the standard module induced by χ.

- If $L = G$, then χ is a regular character and the claim comes from Conjecture 5.20.
- Suppose $L \neq G$. Decompose $L \simeq L_1 \times \cdots \times L_m$ and $w_0^{-1} = (w_1, \dots, w_m) \in L_1 \times \cdots \times L_m$. Both χ and ${}^{w_0^{-1}}\chi$ are L-adapted: denote by $\chi_L = \chi_{L_1} \otimes \cdots \otimes \chi_{L_m}$ the character of $\mathcal{A}_L$ corresponding to χ. Then ${}^{w_0^{-1}}\chi_L = {}^{w_1}\chi_{L_1} \otimes \cdots \otimes {}^{w_m}\chi_{L_m}$ corresponds to ${}^{w_0^{-1}}\chi$. If χ_{L_i} is a supersingular character for an $i \in \{1, \dots, m\}$, then ${}^{w_i}\chi$ and χ have the same support, so by minimality of the length of w, we must have $w_i = 1$. In other words, if $w_i \neq 1$, then χ_{L_i} is a regular character of $\mathcal{A}_{L_i}$. So Conjecture 5.20 says that the standard modules for $\mathcal{H}(L)$ induced by χ_L and ${}^{w_0^{-1}}\chi_L$ have the same semisimplification. Then applying Proposition 5.7 and Corollary 5.4, one gets that M is an irreducible subquotient of the standard module induced by χ. $\square$

Proposition 5.23. *The statement of Proposition 5.22 holds without further hypothesis for modules over the Iwahori–Hecke algebra.*

5F. *Irreducible induced modules.* Let $\mathfrak{M}$ be a k-vector space endowed with a structure of right $\mathcal{H}(L)$-module. Let $\mathfrak{M}$ be irreducible as an $\mathcal{H}(L)$-module. Then it is finite-dimensional and has a central character [Vignéras 2007, 5.3], so $\mathfrak{M}$ is a quotient of some standard module for $\mathcal{H}(L)$ induced by a character $\chi_L : \mathcal{A}_L \to k$. In particular, $\mathfrak{M}$ belongs to the category $\mathcal{C}_L$ defined in 5A. Suppose that χ_L is the tensor product of supersingular characters and consider as before its associated L-adapted character $\chi : \mathcal{A} \to k$.

Proposition 5.24. *Let χ' be a k-character for $\mathcal{A}$ contained in $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$. There is $d \in D_L$ such that ${}^{d}\chi'$ is the L-adapted character associated to some $W_{0,L}$-conjugate of χ_L.*

Proof. First note, using the braid relations in $\mathcal{H}(L)$ and the fact that χ_L is a product of supersingular characters, that any k-character for $\mathcal{A}_L$ contained in $\mathfrak{M}$ is a $W_{0,L}$-conjugate of χ_L. Then, using Proposition 5.7, note that $\mathfrak{M}\otimes_{\mathcal{H}(L^+)}\mathcal{H}$ is a quotient of the standard module for $\mathcal{H}$ induced by χ. So it has a central character given by the restriction of χ to the center of $\mathcal{H}$. Any k-character χ' for $\mathcal{A}$ contained in $\mathfrak{M}\otimes_{\mathcal{H}(L^+)}\mathcal{H}$ has the same restriction to the center, which ensures that the supports of χ' and χ are conjugate, and more precisely, that there is an element $d\in D_L$ such that χ and ${}^d\chi'$ coincide on $(E(\lambda))_{\lambda\in\Lambda}$. In particular, $\chi'(E({}^{d^{-1}}\lambda_L))\neq 0$, so the character χ' is supported by an element in $\mathfrak{M}\otimes\tau^*_{d^{-1}}$ by Proposition 5.2 and its proof. With the braid relations in $\mathcal{H}$, our first remark then shows that ${}^d\chi'$ is the L-adapted character associated to some $W_{0,L}$-conjugate of χ_L. □

Corollary 5.25. *Suppose that χ_L satisfies Hypothesis $(\star)$. Then $\mathfrak{M}\otimes_{\mathcal{H}(L^+)}\mathcal{H}$ is an irreducible $\mathcal{H}$-module.*

Proof. A nontrivial irreducible submodule of $\mathfrak{M}\otimes_{\mathcal{H}(L^+)}\mathcal{H}$ is a quotient of a standard module for $\mathcal{H}$. By Proposition 5.24, the latter is induced by a k-character χ' such that ${}^d\chi'$ is the L-adapted character associated to ${}^{w_0}\chi_L$ for some $d\in D_L$ and $w_0\in W_{0,L}$. It is clear that ${}^{w_0}\chi_L$ satisfies Hypothesis $(\star)$ since χ_L does, so Proposition 5.14 ensures that the standard module induced by χ' is isomorphic to the one induced by ${}^d\chi'$. In particular, any nonzero submodule of $\mathfrak{M}\otimes_{\mathcal{H}(L^+)}\mathcal{H}$ contains an L-adapted character, and hence a nonzero eigenvector for τ_{a_L} and the value ζ. By Proposition 5.2 and by the irreducibility of $\mathfrak{M}$, any nonzero submodule contains $\mathfrak{M}\otimes\tau_1$, and hence it is the whole $\mathfrak{M}\otimes_{\mathcal{H}(L^+)}\mathcal{H}$. □

6. Parabolic induction and compact induction

Recall that the universal module $\mathcal{U}$ is the compact induction to G of the trivial character of $I(1)$ with values in k. We will denote by $\mathcal{U}_L$ the compact induction to L of the trivial character of $I_L(1)$ with values in k. These representations of G and L are respectively generated by the characteristic functions of the pro-p-Iwahori subgroups $I(1)$ and $I_L(1)$. We will denote both of these by $\mathbf{1}$ when there is no possible ambiguity.

We consider a module $\mathfrak{M}$ in the category $\mathcal{C}_L$ defined in 5A. Let $(\pi(\mathfrak{M}), V)$ be the representation of G on $\mathfrak{M}\otimes_{\mathcal{H}(L^+)}\mathcal{U}$ and $(\pi_L(\mathfrak{M}), V_L)$ the representation of L on $\mathfrak{M}\otimes_{\mathcal{H}(L)}\mathcal{U}_L$.

6A. The parabolic induction $\mathrm{Ind}_P^G\,\pi_L(\mathfrak{M})$ is the smooth part of the space of functions $f: G\to V_L$ satisfying $f(lng)=l.f(g)$ for $g\in G$, $(l,n)\in L\times N$, endowed with the action of G by right translation.

6A1. The set D_L is a system of representatives of the double cosets $\mathbb{P}\backslash\mathbb{G}/\mathbb{U}$ in the Chevalley group. For $d\in D_L$, set $\mathbb{U}_d=\mathbb{U}\cap d^{-1}\mathbb{U}d$ and $\overline{\mathbb{U}}_d=\mathbb{U}\cap d^{-1}\overline{\mathbb{U}}d$. Any element in $\mathbb{U}$ can be written as a product of an element of $\mathbb{U}_d$ and of an element of $\overline{\mathbb{U}}_d$ and this decomposition is unique. From this, one deduces that any element in $\mathbb{P}d\mathbb{U}$ decomposes uniquely in $\mathbb{P}d\overline{\mathbb{U}}_d$ [Carter 1985, 2.5.12].

The set D_L is also a system of representatives of the double cosets $P\backslash G/I(1)$. For any $d \in D_L$, one has

$$PdI(1) = \coprod_y PI(1)dy, \tag{6-1}$$

where dy runs over a system of representatives of $I(1)\backslash I(1)dI(1)$.

For any $d \in D_L$ and any $I_L(1)$-invariant element v in $\pi_L(\mathfrak{M})$, the $I(1)$-invariant function $f_{PdI(1),v}$ with support $PdI(1)$ and value v at d is a well-defined element of $\mathrm{Ind}_P^G \pi_L(\mathfrak{M})$. Any $I(1)$-invariant function in the latter representation is a linear combination of such functions.

6A2. The right action of τ_d maps $f_{PI(1),v}$ to an $I(1)$-invariant element with support $PdI(1)$, which is completely determined by its value at d. Using (6-1), one easily checks that this value is v, so

$$(f_{PI(1),v})\tau_d = f_{PdI(1),v} \tag{6-2}$$

6A3. Let $w \in W_L^{(1)}$. Suppose it is a L-positive element.

According to [Vignéras 1998, II.4], there is a system of representatives of the right cosets $I(1)\backslash I(1)wI(1)$ respecting the decomposition of $I_L(1)wI_L(1)$ into right cosets mod $I_L(1)$. Explicitly, from the decomposition

$$I_L(1)wI_L(1) = \coprod_x I_L(1)wx$$

one gets

$$I(1)wI(1) = \coprod_x I(1)wxI(1)^-$$

and a decomposition $I(1)wI(1) = \coprod_{x,u_x} I(1)wxu_x$, where u_x belong to $I(1)^-$.

From arguments analogous to [Schneider and Stuhler 1991, Proposition 7], one shows that $PI(1)wx \cap PI(1)wxu_x \neq \varnothing$ implies $I(1)wxu_x = I(1)wx$: the hypothesis can be written $Pw^{-1}I(1)^-wx \cap Pw^{-1}I(1)^-wxu_x \neq \varnothing$, and we recall that $I(1)^-$ is normalized by $x \in I_L(1)$. So there exists an element $\kappa_1 xu_x x^{-1}\kappa_2$ in P with $\kappa_1, \kappa_2 \in w^{-1}I(1)^-w \subset I(1)^-$. Since $P \cap I(1)^- = \{1\}$, one deduces that $xu_x x^{-1} \in w^{-1}I(1)w$ and $I(1)wxu_x = I(1)wx$.

The right action of $\tau_w \in \mathscr{H}$ on $f_{PI(1),v}$ gives the $I(1)$-invariant function with support $PI(1)$ and value at 1_G given by $\sum_{x,u_x} f_{PI(1),v}((wxu_x)^{-1})$. But $(wxu_x)^{-1} \in PI(1)$ implies $1 \in PI(1)wx \cap PI(1)wxu_x$; therefore this value is $\sum_x (wx)^{-1}v = v\tau_w^{\otimes}$, and

$$(f_{PI(1),v})\tau_w = f_{PI(1),v\tau_w^{\otimes}}. \tag{6-3}$$

6B. For any $m \in \mathfrak{M}$, there is a well-defined G-equivariant map

$$\mathcal{F}_m : \mathcal{U} \to \mathrm{Ind}_P^G \, \pi_L(\mathfrak{M})$$

sending the characteristic function of $I(1)$ on $f_{PI(1), m\otimes \mathbf{1}}$. The computation of 6A3 shows that we then have a G-equivariant morphism

$$\mathcal{F} : \pi(\mathfrak{M}) \to \mathrm{Ind}_P^G \, \pi_L(\mathfrak{M}), \quad m \otimes u \mapsto \mathcal{F}_m(u). \tag{6-4}$$

Remark 6.1. In the case where L is the diagonal torus T and $\mathfrak{M}$ is a character of $\mathcal{A}_T$, the map $\mathcal{F}$ is an isomorphism [Schneider and Stuhler 1991; Vignéras 2004].

6C. In the tensor product $\mathfrak{M} \otimes_{\mathcal{H}(L)} \mathcal{U}_L$, the group L only acts on $\mathcal{U}_L$, so there is a natural morphism of $\mathcal{H}(L)$-modules

$$\mathfrak{M} \to (\mathfrak{M} \otimes_{\mathcal{H}(L)} \mathcal{U}_L)^{I_L(1)}, \tag{6-5}$$

and a natural morphism of $\mathcal{H}$-modules

$$\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H} \to (\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{U})^{I(1)}, \tag{6-6}$$

which composes with $\mathcal{F}$ to give the morphism of $\mathcal{H}$-modules

$$\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H} \to (\mathrm{Ind}_P^G \pi_L(\mathfrak{M}))^{I(1)}. \tag{6-7}$$

6C1. If (6-5) is not trivial, then (6-7) is not trivial and neither is (6-6). By adjunction, if there exists a representation (π_L, V_L) of L and a nonzero $\mathcal{H}(L)$-equivariant map $\mathfrak{M} \to V_L^{I_L(1)}$, then (6-5) is not trivial.

6C2. Suppose (6-5) is surjective. Then (6-7) is surjective.

6C3. Using Proposition 5.2, one sees that (6-7) is injective if (6-5) is injective. In this case, (6-6) is also injective.

In 5F, we gave sufficient conditions for certain irreducible $\mathcal{H}(L)$-modules $\mathfrak{M}$ to induce irreducible $\mathcal{H}$-modules. Under these conditions, and if (6-5) is nonzero, then (6-7) allows us to describe an irreducible subspace $\mathfrak{M} \otimes_{\mathcal{H}(L)} \mathcal{H}$ of the pro-p-invariants of $\mathrm{Ind}_P^G \pi_L(\mathfrak{M})$.

If $\mathcal{H}(L)$ is a direct factor of $\mathcal{U}_L$ as a left $\mathcal{H}(L)$-module, then (6-5) is injective for any $\mathfrak{M}$ in $\mathcal{C}_L$. This is the case if F has residue field $\mathbb{F}_p$ and L is isomorphic to a product of $\mathrm{GL}_1(F)$'s and $\mathrm{GL}_2(F)$'s [Ollivier 2007, 2.1.3].

6D. ***Examples.***

6D1. If L is the diagonal torus T, then $\mathfrak{M}$ identifies with a character $\chi_T : \mathcal{A}_T \to k$. By Remark 6.1 and previous results, the representation $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{U}$ is isomorphic to the principal series induced by the character $T \to k^*$, $t \mapsto \chi_T(t^{-1})$. The semisimplification of this representation and of its space of pro-p-invariants is well-understood [Grosse-Klönne 2009; Ollivier 2006a; Ollivier 2006c; Vignéras 2008].

6D2. We consider the case where $F = \mathbb{Q}_p$. Suppose that L is isomorphic to a product of $\mathrm{GL}_1(\mathbb{Q}_p)$'s and $\mathrm{GL}_2(\mathbb{Q}_p)$'s. There is an equivalence of categories between the right $\mathcal{H}(L)$-modules (with scalar action of the uniformizers) and the representations of L generated by their $I_L(1)$-invariants (with scalar action of the uniformizers). In particular, (6-5) is an isomorphism for any $\mathfrak{M}$. If L is the diagonal torus, it is clear. Otherwise, the result is given by [Ollivier 2009]. So, for any $\mathfrak{M}$ in $\mathscr{C}_L$, the map (6-7) is an isomorphism.

6D2.1. Suppose that $G = \mathrm{GL}_3(\mathbb{Q}_p)$ and L is isomorphic to $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$.

Let $\chi_L : \mathcal{A}_L \to k$ be the tensor product of two supersingular characters. It satisfies Hypothesis $(\star)$. Denote by $\mathfrak{M}$ the standard module for $\mathcal{H}(L)$ induced by χ_L. It is irreducible and 2-dimensional. Because of the above-mentioned equivalence of categories, the representation $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{U}_L$ is the tensor product of a supersingular representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ by a character of $\mathrm{GL}_1(\mathbb{Q}_p)$.

By Corollary 5.25, the $\mathcal{H}$-module $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$ is irreducible. By the remarks of 6C, it is isomorphic to the subspace of $I(1)$-invariants of the representation which is parabolically induced from $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{U}_L$. Hence, this subspace generates an irreducible subrepresentation for $\mathrm{GL}_3(\mathbb{Q}_p)$. By the results of Herzig, this subrepresentation is actually the whole $\mathrm{Ind}_P^G \pi_L(\mathfrak{M})$.

6D2.2. Suppose that G is $\mathrm{GL}_4(\mathbb{Q}_p)$ and L is isomorphic to $\mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{Q}_p)$. Let $\chi_L : \mathcal{A}_L \to k$ be the tensor product of two supersingular characters. It satisfies Hypothesis $(\star)$. Denote by $\mathfrak{M}$ the standard module for $\mathcal{H}(L)$ induced by χ_L. It is irreducible and 4-dimensional. The same arguments as before ensure that $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{U}_L$ is the tensor product of two supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, and that the $\mathcal{H}$-module $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{H}$ is irreducible and isomorphic to the space of $I(1)$-invariants of the representation which is parabolically induced from $\mathfrak{M} \otimes_{\mathcal{H}(L^+)} \mathcal{U}_L$. The latter is an irreducible representation by the results of Herzig.

7. Supersingular modules

Fix a supersingular character $\chi : \mathcal{A} \to k$. It is defined by its restriction to $\{\tau_t,\ t \in \mathbb{T}\}$, its value $\zeta \in k^*$ on $E(\mu_{\{1,\dots,n\}})$ and by the fact that for any $\lambda \in \Lambda^{(1)}$ such that $\ell(e^{\lambda}) > 0$, the scalar $\chi(E(\lambda))$ is zero.

Let M be a nonzero quotient of the standard module for $\mathcal{H}$ induced by χ. Denote by M_χ the sum of the equivariant subspaces in M for $\mathcal{A}$ and the W_0-conjugates of χ (it is nonzero).

Proposition 7.1. *M_χ is stable under the action of the finite Hecke algebra.*

Proof. This is a direct consequence of the integral Bernstein relations. $\square$

Recall that the simple modules for the finite Hecke algebra are the characters [Cabanes and Enguehard 2004, Theorem 6.12]. So the proposition says in particular that M_χ contains a character for the finite Hecke algebra. Denote by $m \in M_\chi$ its support. The set $\mathfrak{D}$ was introduced by Proposition 2.5 and one has the following result.

Proposition 7.2. *The set of the lengths $\ell(d)$, where d runs over the elements of $\mathfrak{D}$ such that $mE(d) \neq 0$, is bounded.*

Proof. Let $d \in \mathfrak{D}$. Write $d = e^\lambda w_0 \in W$. According to Lemma 2.6, the weight λ is dominant, so (after a suitable twist of d by a power of the central element ϖ^n) it decomposes into a linear combination

$$\lambda = \sum_{1 \le i \le n-1} -n_i \mu_i$$

with nonnegative integral coefficients. Suppose that one of the coefficients, say n_j, is at least 2. Then $\lambda + \mu_j$ is still dominant and we show that

(1) $d' := e^{\lambda+\mu_j} w_0 \in \mathfrak{D}$,

(2) $\ell(d) = \ell(e^{-\mu_j}) + \ell(d')$, which easily implies that $E(d) = E(-\mu_j)E(d')$ and $mE(d) = 0$.

Since $\lambda+\mu_j$ is dominant, the only thing one has to check to make sure that $d' \in \mathfrak{D}$ is the following: for any $\check{\alpha} \in \check{\Phi}^+$, if $\langle \lambda+\mu_j, \check{\alpha}\rangle = 0$ then $w_0^{-1}\check{\alpha} \in \check{\Phi}^+$. Since $d = e^\lambda w_0$ is already in $\mathfrak{D}$, the only tricky case is $\langle \lambda, \check{\alpha}\rangle = -\langle \mu_j, \check{\alpha}\rangle > 0$. By definition of the weight μ_j, this condition implies that $\check{\alpha} \ge \check{\alpha}_j$ and $2 \le n_j = \langle \lambda, \check{\alpha}_j\rangle \le \langle \lambda, \check{\alpha}\rangle$, which contradicts the fact $\langle \lambda, \check{\alpha}\rangle = -\langle \mu_j, \check{\alpha}\rangle = 1$, since μ_j is minuscule.

Now for the second assertion, recall from 2B3 that this equality holds if and only if, for any $\check{\alpha} \in \check{\Phi}^+$,

$$\langle \mu_j, \check{\alpha}\rangle\, n(\check{\alpha}, w_0^{-1} e^{-\lambda-\mu_j}) \ge 0, \tag{7-1}$$

where the integer $n(\check{\alpha}, w_0^{-1} e^{-\lambda-\mu_j})$ is $\langle -\lambda - \mu_j, \check{\alpha}\rangle$ in the case $w_0^{-1}\check{\alpha} \in \check{\Phi}^+$ and $1 - \langle \lambda + \mu_j, \check{\alpha}\rangle$ if $w_0^{-1}\check{\alpha} \in \check{\Phi}^-$. In the case $w_0^{-1}\check{\alpha} \in \check{\Phi}^+$, inequality (7-1) obviously holds. Suppose now that $w_0^{-1}\check{\alpha} \in \check{\Phi}^-$ and that $\langle \mu_j, \check{\alpha}\rangle = -1$. Then $\check{\alpha} \ge \check{\alpha}_j$, so again, $2 \le \langle \lambda, \check{\alpha}\rangle$ and $n(\check{\alpha}, w_0^{-1} e^{-\lambda-\mu_j}) \le 0$. □

Choose $d \in \mathfrak{D}$ an element with maximal length such that $mE(d) \neq 0$.

Theorem 7.3. *The element $mE(d)$ is an eigenvector for the action of the affine Hecke algebra.*

Proof. With Lemmas 2.6 and 4.4 we compute $E(d) = \tau^*_{d^{-1}}$ for any $d \in \mathfrak{D}$. First note that the braid relations in $\mathcal{H}$ ensure that $mE(d)$ is an eigenvector for the elements of the form τ_t with $t \in \mathbb{T}$. Let $s \in S_{\text{aff}}$. We have to show that $mE(d)\tau^*_s$ is proportional to $mE(d)$.

- If $\ell(ds) = \ell(d) - 1$, then $\tau^*_{d^{-1}} = \tau^*_{(sd)^{-1}}\tau^*_s$. In $\mathcal{H} \otimes_{\mathbb{Z}} k$, where $(\tau^*_s)^2 = \tau^*_s \nu_s$ (Remark 3.1), we have $\tau^*_{d^{-1}}\tau^*_s = \tau^*_{d^{-1}}\nu_s$, so $mE(d)\tau^*_s = mE(d)\nu_s$, which is proportional to $mE(d)$ by our first remark.
- If $\ell(ds) = \ell(d) + 1$, then $\tau^*_{(ds)^{-1}} = \tau^*_{d^{-1}}\tau^*_s$. If $ds \in \mathfrak{D}$, then $0 = mE(ds) = m\tau^*_{(ds)^{-1}} = mE(d)\tau^*_s$ by the maximal property of $\ell(d)$. If $ds \notin \mathfrak{D}$, then Proposition 2.7 says that there exists $w_0 \in W_0$ such that $ds = w_0 d$ with $\ell(w_0) + \ell(d) = \ell(ds)$. So
$$E(d)\tau^*_s = \tau^*_{w_0^{-1}} E(d).$$
Since m is a character for the finite Hecke algebra, $m\tau^*_{w_0^{-1}}$ is proportional to m, so $mE(d)\tau^*_s$ is proportional to $mE(d)$. □

The statement of the theorem is exactly the claim of [Vignéras 2005, Conjecture 1], where it is proven that it implies the numerical correspondence described by Theorem 1.1 in our introduction.

8. Generic spherical Hecke algebra and Iwahori–Hecke algebra

8A. Denote by $*$ the convolution operator in the generic pro-p-Hecke algebra $\mathcal{H}$ and by $e_I \in \mathcal{H}$ the characteristic function of the Iwahori subgroup. The generic Iwahori–Hecke algebra $\mathbf{H}$ coincides with the algebra $e_I * \mathcal{H} * e_I$ with unit e_I, so all the results of Sections 3 and 4 have (well-known) analogs in the Iwahori case. The generic Iwahori–Hecke algebra $\mathbf{H}$ has $\mathbb{Z}[q]$-basis $(\mathbf{T}_w)_{w\in W}$, where $\mathbf{T}_w = e_I * \tau_w * e_I$ corresponds to the double coset IwI, satisfying the following *braid* and *quadratic* relations.

- $\mathbf{T}_w\mathbf{T}_{w'} = \mathbf{T}_{ww'}$ for any $w,\ w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$,
- $\mathbf{T}_s^2 = q + (q-1)\mathbf{T}_s$ for $s \in S_{\mathrm{aff}}$.

Denote by Θ the classic Bernstein embedding
$$\Theta : \mathbb{Z}[q^{\pm 1/2}][\Lambda] \to \mathbf{H} \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q^{\pm 1/2}]$$
naturally arising from the Bernstein map θ of Section 4A and satisfying $\Theta(\lambda) = \mathbf{T}_{e^\lambda}$ for any antidominant weight $\lambda \in \Lambda$. For $w \in W$, define $\mathbf{E}(w) := e_I * E(w) * e_I$. It is explicitly given by the formula
$$\mathbf{E}(w) = q^{(\ell(w)-\ell(w_0)-\ell(e^{\lambda_1})+\ell(e^{\lambda_2}))/2}\Theta(\lambda)\mathbf{T}_{w_0}$$
for $\lambda \in \Lambda$ and $w_0 \in W_0$ such that $w = e^\lambda w_0$ and $\lambda_1, \lambda_2 \in \Lambda$ are antidominant weights satisfying $\lambda = \lambda_1 - \lambda_2$. Theorem 4.5, translated to the Iwahori case, gives the following results (see also [Vignéras 2006, Chapitre 3]). The image $\mathbf{A}$ of $\mathbf{E} : \mathbb{Z}[q][\Lambda] \to \mathbf{H}$ coincides with the intersection of $\mathbf{H}$ with the image of Θ. It has

$\mathbb{Z}[q]$-basis $(\mathbf{E}(\lambda))_{\lambda\in\Lambda}$. As a $\mathbb{Z}[q]$-algebra, it is generated by the elements

$$(\mathbf{E}(\lambda_I))_{I\subsetneq\{1,\dots,n\}},\ \mathbf{E}(\lambda_{\{1,\dots,n\}})^{\pm 1}$$

with the relations

$$\mathbf{E}(\lambda_I)\mathbf{E}(\lambda_J)=q^{bc}\mathbf{E}(\lambda_{I\cup J})\mathbf{E}(\lambda_{I\cap J}) \tag{8-1}$$

for any $I, J\subset\{1,\dots,n\}$ with $|I\cap J|=a$, $|I|=a+b$, $|J|=a+c$. The center of $\mathbf{H}$ is the space of W_0-invariants in $\mathbf{A}$. It is equal to the $\mathbb{Z}[q]$-algebra of polynomials in the variables

$$Z_1,\dots,Z_{n-1},Z_n^{\pm 1},$$

where, for $i\in\{1,\dots,n\}$, we denote by Z_i the central element

$$Z_i=\sum_{w_0\in W_0/W_0(\mu_i)}\mathbf{E}({}^{w_0}\mu_i).$$

8B. ***Integral Satake isomorphism.*** We closely follow the work of Schneider and Teitelbaum [2006], who introduce a renormalized version of the classic Satake map in order to get a p-adic Satake isomorphism, and check that their description provides us in addition with an *integral Satake isomorphism*.

8B1. In Section 4B, we defined a twisted action of W_0 on the weights. Denote by $\mathbb{Z}[q^{\pm 1/2}][\Lambda]^{W_0,\gamma}$ the space of invariants of $\mathbb{Z}[q^{\pm 1/2}][\Lambda]$ under this action. It has $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{\sigma_\lambda\}_\lambda$ with

$$\sigma_\lambda=\sum_{w_0\in W_0/W_0(\lambda)}w_0\centerdot\lambda=\sum_{w_0\in W_0/W_0(\lambda)}\gamma(w_0,\lambda)^{w_0}\lambda,$$

where λ runs over the set Λ_{anti} of antidominant weights. Note that σ_λ is well-defined for any weight λ thanks to property (c) (of Section 4B) of the cocycle γ.

We call *the generic spherical Hecke algebra* and denote by $\mathscr{H}_{\mathbb{Z}[q]}(G,K_0)$ the $\mathbb{Z}[q]$-algebra $\mathbb{Z}[q][K_0\backslash G/K_0]$ of the functions with finite support on the double cosets of G modulo K_0, with the usual convolution product. The $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathscr{H}_{\mathbb{Z}[q]}(G,K_0)\otimes_{\mathbb{Z}[q]}\mathbb{Z}[q^{\pm 1/2}]$ will be denoted by $\mathscr{H}_{\mathbb{Z}[q^{\pm 1/2}]}(G,K_0)$.

A system of representatives for the double cosets $K_0\backslash G/K_0$ is given by the set Λ_{anti} of antidominant weights. For $\lambda\in\Lambda$, denote by ψ_λ the characteristic function of $K_0e^\lambda K_0$. The results of [Schneider and Teitelbaum 2006, p. 23] with $\xi=1$ give the next theorem, the proof of which involves the subsequent lemma.

Theorem 8.1. *There is an injective morphism of $\mathbb{Z}[q^{\pm 1/2}]$-algebras*

$$\begin{aligned}\mathrm{S}:\mathscr{H}_{\mathbb{Z}[q^{\pm 1/2}]}(G,K_0)&\to\mathbb{Z}[q^{\pm 1/2}][\Lambda],\\ \psi_\lambda,\ \lambda\in\Lambda_{\text{anti}}&\mapsto\sum_{\eta\in\Lambda_{\text{anti}}}c(\eta,\lambda)\sigma_\eta,\end{aligned}\tag{8-2}$$

where $c(\eta,\lambda)=[(Ue^\eta K_0\cap K_0e^\lambda K_0)/K_0]$. Its image is equal to $\mathbb{Z}[q^{\pm 1/2}][\Lambda]^{W_0,\gamma}$.

Lemma 8.2. *If $\eta, \lambda \in \Lambda$ are antidominant weights, then*

(1) $c(\lambda, \lambda) = 1$, *and*

(2) $c(\eta, \lambda) = 0$ *unless $\lambda - \eta$ is an antidominant weight.*

Note that, the coefficient $c(\eta, \lambda)$ being integral, the image of $\mathcal{H}_{\mathbb{Z}[q]}(G, K_0)$ by the map S lies in $\mathbb{Z}[q][\sigma_\lambda, \lambda \in \Lambda_{\mathrm{anti}}]$. From this lemma, one also deduces the following result.

Lemma 8.3. *The image of $\mathcal{H}_{\mathbb{Z}[q]}(G, K_0)$ by the map* S *is* $\mathbb{Z}[q][\sigma_\lambda, \lambda \in \Lambda_{\mathrm{anti}}]$.

Proof. One has to check that any σ_λ with $\lambda \in \Lambda_{\mathrm{anti}}$ lies in the image of $\mathcal{H}_{\mathbb{Z}[q]}(G, K_0)$ by the map S. Recall that the element $\varpi^n = e^{\mu_n}$ is central in G, so if the weight λ has the form $k\mu_n$ with $k \in \mathbb{Z}$, then σ_λ is the image by S of $\psi_{k\mu_n}$, which is invertible in $\mathcal{H}_{\mathbb{Z}[q]}(G, K_0)$. So it remains to prove the property for nontrivial weights λ that can be written $\lambda = \sum_{i=1}^{n-1} k_i \mu_i$, with $k_i \in \mathbb{N}$, and we do it by induction on $\sum_{i=1}^{n-1} k_i$. The only antidominant weights η such that $\lambda - \eta$ is antidominant are the $\sum_{i=1}^{n-1} m_i \mu_i$ with $0 \le m_i \le k_i$. By induction, if such an η satisfies $\eta \ne \lambda$, then σ_η is in the image of $\mathcal{H}_{\mathbb{Z}[q]}(G, K_0)$ by S. Lemma 8.2(1) then ensures that it is also true for σ_λ. □

We have checked that the map in Theorem 8.1 actually defines an integral version of a Satake isomorphism: the restriction of S to the generic spherical algebra $\mathcal{H}_{\mathbb{Z}[q]}(G, K_0)$ defines an isomorphism

$$\mathrm{S} : \mathcal{H}_{\mathbb{Z}[q]}(G, K_0) \xrightarrow{\sim} \mathbb{Z}[q][\sigma_\lambda, \lambda \in \Lambda_{\mathrm{anti}}]. \tag{8-3}$$

An important consequence of Lemma 4.2 and property (a) of the cocycle γ is the fact that for any $w_0 \in W_0$, the coefficient $\gamma(w_0, \lambda)$ belongs to $\mathbb{Z}[q]$ if λ is antidominant. So σ_λ actually lies in $\mathbb{Z}[q][\Lambda]$. The supports of the elements σ_λ being disjoint for $\lambda \in \Lambda_{\mathrm{anti}}$ and each coefficient $\gamma(1, \lambda)$ being 1, one obtains

$$\mathbb{Z}[q^{\pm 1/2}][\Lambda]^{W_0, \gamma} \cap \mathbb{Z}[q][\Lambda] = \mathbb{Z}[q][\sigma_\lambda, \lambda \in \Lambda_{\mathrm{anti}}]. \tag{8-4}$$

8C. ***Compatibility of Bernstein and Satake transforms.*** Note that for any antidominant weight λ, the element

$$\Theta(\sigma_\lambda) = \sum_{w_0 \in W_0 / W_0(\lambda)} \mathbf{E}({}^{w_0}\lambda) \tag{8-5}$$

belongs to the center of $\mathbf{H}$. The description of the center of $\mathbf{H}$ in Section 8A implies the following.

Proposition 8.4. *Composing Θ with the isomorphism (8-3) gives an isomorphism between $\mathcal{H}_{\mathbb{Z}[q]}(G, K_0)$ and the center of* $\mathbf{H}$.

For $1 \leq i \leq n$, denote by T_i the element ψ_{μ_i}. The generic spherical algebra $\mathcal{H}_{\mathbb{Z}[q]}(G, K_0)$ is an algebra of polynomials in the variables $T_1, \dots, T_{n-1}, T_n^{\pm 1}$. Consider the G-equivariant map

$$\begin{aligned} \mathbb{Z}[q][I\backslash G] &\to \mathbb{Z}[q][K_0\backslash G], \\ f &\mapsto e_{K_0} * f, \end{aligned} \tag{8-6}$$

where e_{K_0} denotes the characteristic function of K_0 and the convolution product is given by

$$e_{K_0} * f(x) = \sum_{t \in G/I} e_{K_0}(t)\, f(t^{-1}x) = \sum_{t \in I\backslash K_0} f(tx) \quad \text{for} \quad x \in G.$$

Proposition 8.5. *Composing the maps*

$$\mathcal{H}_{\mathbb{Z}[q]}(G, K_0) \xrightarrow{\Theta\circ S} \mathbf{A} \xrightarrow{e_{K_0} * \cdot} \mathbb{Z}[q][K_0\backslash G]$$

gives the identity on $\mathcal{H}_{\mathbb{Z}[q]}(\mathrm{G}, K_0)$.

Proof. See [Schneider and Stuhler 1991, p. 32]. □

Note that the compatibility refers to the *classic* Bernstein map and the *integral* Satake transform.

8D. Denote by $\mathbf{R}$ the mod p reduction of the map (8-6), that is, the G-equivariant map

$$\begin{aligned} \mathrm{ind}_I^G \mathbf{1}_k &\to \mathrm{ind}_{K_0}^G \mathbf{1}_k, \\ f &\mapsto e_{K_0} * f, \end{aligned} \tag{8-7}$$

where $\mathrm{ind}_I^G \mathbf{1}_k$ and $\mathrm{ind}_{K_0}^G \mathbf{1}_k$ denote respectively the compact induction of the trivial character with values in k of the Iwahori subgroup I and of the maximal compact subgroup K_0.

Proposition 8.6. *Let $\mu \in \Lambda$ be a minuscule weight. The image by $\mathbf{R}$ of $\mathbf{E}(\mu) \in \mathbf{A} \otimes_{\mathbb{Z}[q]} k$ is equal to $\psi_\mu \in \mathcal{H}_{\mathbb{Z}[q]}(G, K_0) \otimes_{\mathbb{Z}[q]} k$ if μ is a dominant weight*, *and to zero otherwise.*

The proof will be a consequence of the following lemmas.

Lemma 8.7. *For $\mu \in \Lambda$ dominant and minuscule*,

$$K_0 e^\mu K_0 = \coprod_{d \in \mathcal{D},\, d \preceq e^\mu} K_0 d I,$$

where $\preceq$ denotes the extended Bruhat order on W.

Proof. We have to prove that for $\mu \in \Lambda$ dominant and minuscule, $\mathcal{D} \cap K_0 e^\mu K_0 = \{d \in \mathcal{D},\ d \preceq e^\mu\}$. For any such weight μ, the corresponding translation can be written $e^\mu = \varpi^k w_0$ with $k \in \{0, \dots, n\}$ and $w_0 \in W_0$. By definition of the extended

Bruhat order, an element $d \in W$ satisfies $d \preceq e^\mu$ if and only if it has the form $d = \varpi^k w$ with $w \in W_0$ such that $w \preceq w_0$. So $\{d \in \mathfrak{D},\, d \preceq e^\mu\} \subset \mathfrak{D} \cap K_0 e^\mu K_0$. Let $d \in \mathfrak{D}$. Lemma 2.6 says that d can be written $d = e^\lambda w$ with $w \in W_0$ and $\lambda \in \Lambda$ a dominant weight such that $\ell(e^\lambda) = \ell(d) + \ell(w^{-1})$. If $d \in K_0 e^\mu K_0$, then $K_0 e^\lambda K_0 = K_0 e^\mu K_0$ and $\lambda = \mu$. Since ϖ has length zero, one then has $\ell(w_0 w) + \ell(w^{-1}) = \ell(w_0)$, so $w_0 w \preceq w_0$ and $d = \varpi^k w_0 w \preceq e^\mu$. □

Lemma 8.8. *For $w \in W$,*

$$\mathbf{R}(\mathrm{T}_w) = |I \backslash (K_0 \cap I w I w^{-1})|\, \mathbf{1}_{K_0 w I} = \begin{cases} \mathbf{1}_{K_0 w I} & \text{if } w \in \mathfrak{D}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition, the map (8-6) sends the characteristic function T_w of IwI onto $|I \backslash (K_0 \cap I w I w^{-1})|\, \mathbf{1}_{K_0 w I}$. We have to show that the index $|I \backslash (K_0 \cap I w I w^{-1})|$ is equal to 1 if $w \in \mathfrak{D}$ and is equal to a nontrivial power of q otherwise. If $w \in \mathfrak{D}$, then by length property, one easily checks that $K_0 \cap I w I w^{-1} = I$. Suppose now that w is not an element of $\mathfrak{D}$, that is, that it is not the minimal length element in $W_0 w$: there exists $s \in S_0$ such that $IwI = IsIswI$. Hence $IwIw^{-1} \cap K_0$ contains $IsIs$, which has q right cosets modulo I. □

Lemma 8.9. *For any dominant weight $\lambda \in \Lambda$, the following holds in $\mathbf{H} \otimes_{\mathbb{Z}[q]} k$:*

$$\mathbf{E}(\lambda) = \sum_{w \in W,\, w \preceq e^\lambda} \mathrm{T}_w.$$

Proof. Let us show that for any $x \in W$, one has

$$\mathrm{T}^*_{x^{-1}} = \sum_{w \in W,\, w \preceq x} \mathrm{T}_w \in \mathbf{H} \otimes_{\mathbb{Z}[q]} k.$$

This proves the lemma because $\mathbf{E}(\lambda) = \mathrm{T}^*_{e^{-\lambda}}$ for a dominant weight λ. It is enough to show the equality for $x \in W_{\mathrm{aff}}$, and we do it by induction on $\ell(x)$. If $x = s \in S_{\mathrm{aff}}$, then $\mathrm{T}^*_{x^{-1}} = \mathrm{T}^*_s = \mathrm{T}_s + 1 - q = \mathrm{T}_s + 1$ in $\mathbf{H} \otimes_{\mathbb{Z}[q]} k$. Now suppose $x \in W_{\mathrm{aff}}$ and $s \in S$ is such that $\ell(sx) = \ell(x) + 1$. In $\mathbf{H} \otimes_{\mathbb{Z}[q]} k$, one has by induction

$$\mathrm{T}^*_{(sx)^{-1}} = \mathrm{T}^*_s \mathrm{T}^*_{x^{-1}} = (\mathrm{T}_s + 1) \sum_{y \preceq x} \mathrm{T}_y = \sum_{y \preceq x} \mathrm{T}_s \mathrm{T}_y + \sum_{y \preceq x} \mathrm{T}_y.$$

Let $y \preceq x$. If $\ell(sy) = \ell(y) + 1$, then $\mathrm{T}_s \mathrm{T}_y = \mathrm{T}_{sy}$ and $sy \preceq sx$. Otherwise $\mathrm{T}_s \mathrm{T}_y = -\mathrm{T}_y$, so $\mathrm{T}^*_{(sx)^{-1}} = \sum_{sy' \preceq y' \preceq sx} \mathrm{T}_{y'} + \sum_{y \preceq sx,\, y \preceq sy} \mathrm{T}_y = \sum_{y \preceq sx} \mathrm{T}_y$. □

Lemma 8.10. *If μ is minuscule and not dominant, then* $\mathbf{R}(\mathbf{E}(\mu)) = 0$.

Proof. Let λ be the unique antidominant weight in the orbit of μ and $d \in W_0$ with minimal length in $W_0(\lambda) d$ such that $\mu = {}^{d^{-1}}\lambda$. Lemma 4.4 says that $\mathbf{E}(\mu) = \mathrm{T}_{e^\mu d^{-1}} \mathrm{T}^*_{d^{-1}}$. For any $w_0 \in W_0$, we have $\ell(e^\mu d^{-1}) + \ell(w_0) = \ell(e^\mu d^{-1} w_0)$, which can be seen by applying 2B2 and recalling that for any $\check{\alpha} \in \check{\Phi}^+$, if $\langle \lambda, \check{\alpha} \rangle = 0$

then $d^{-1}\check{\alpha} \in \check{\Phi}^+$. This implies that the elements of the Iwahori–Matsumoto basis appearing in the decomposition of $\mathbf{E}(\mu)$ have the form $\tau_{e^\mu d^{-1} w_0}$, with $w_0 \in W_0$. In particular, if μ is not dominant, then $e^\mu d^{-1} w_0$ is not an element of $\mathscr{D}$, by Lemma 2.6, and $\mathbf{E}(\mu)$ is sent by $\mathbf{R}$ on zero, by Lemma 8.8. □

Proof of Proposition 8.6. Let $\mu \in \Lambda$ be a minuscule weight. If it is not dominant, Lemma 8.10 says that its image by $\mathbf{R}$ is zero. If it is dominant, Lemmas 8.8 and 8.9 together say that $\mathbf{R}(\mathbf{E}(\mu))$ is the sum of the characteristic functions of $K_0 w I$, where $w \in \mathscr{D}$, $w \preceq e^\mu$, which, by Lemma 8.7, is the characteristic function of $K_0 e^\mu K_0$. □

8E. *On Barthel–Livné's unramified representations for* $\mathbf{GL}_n$. For $i \in \{1, \dots, n\}$, choose $\alpha_i \in k$ with $\alpha_n \neq 0$. Set $\alpha_0 = 1$. Define χ_0 to be the k-character of $\mathbf{A}$ with dominant support given by $\mathbf{E}(\mu_{\{n-i+1,\dots,n\}}) \mapsto \alpha_i$ for $i \in \{1, \dots, n\}$.

Define the associated character of $\mathscr{H}(G, K_0)_{\mathbb{Z}[q]} \otimes k$ by $T_i \mapsto \alpha_i$ for $i \in \{1, \dots, n\}$ and denote by

$$\frac{\operatorname{ind}_{K_0}^G \mathbf{1}_k}{\sum_i (T_i - \alpha_i)}$$

the quotient of the universal representation $\operatorname{ind}_{K_0}^G \mathbf{1}_k$ by $\sum_i (T_i - \alpha_i) \operatorname{ind}_{K_0}^G \mathbf{1}_k$.

By the results of 8D, the G-equivariant surjective morphism

$$\begin{aligned} \operatorname{ind}_I^G \mathbf{1}_k \to \frac{\operatorname{ind}_{K_0}^G \mathbf{1}_k}{\sum_i (T_i - \alpha_i)}, \\ f \mapsto \mathbf{R}(f) \mod \sum_i (T_i - \alpha_i) \end{aligned} \tag{8-8}$$

factors into a surjective G-equivariant morphism

$$\chi_0 \otimes_{\mathbf{A}} \operatorname{ind}_I^G \mathbf{1}_k \to \frac{\operatorname{ind}_{K_0}^G \mathbf{1}_k}{\sum_i (T_i - \alpha_i)}. \tag{8-9}$$

Example 8.11. Suppose that one of the α_i, $i \in \{1, \dots, n-1\}$ is nonzero. The unique character of $\mathbf{A}$ with antidominant support in the W_0-orbit of χ_0 satisfies Hypothesis $(\star)$ of Section 5D if and only if $\alpha_{i-1}\alpha_i\alpha_{i+1} \neq 0$ implies ${\alpha_i}^2 \neq \alpha_{i-1}\alpha_{i+1}$ for any $i \in \{1, \dots, n-1\}$.

Under this hypothesis and if none of the elements α_i is zero, then, by the results of [Ollivier 2006a], the representation $\chi_0 \otimes_{\mathbf{A}} \operatorname{ind}_I^G \mathbf{1}_k$ is irreducible and isomorphic to the principal series induced by the unramified character

$$T \to k^*, \quad \mu_i \mapsto \alpha_i^{-1},$$

and (8-9) is an isomorphism.

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References

[Barthel and Livné 1995] L. Barthel and R. Livné, "Modular representations of GL_2 of a local field: the ordinary, unramified case", *J. Number Theory* **55**:1 (1995), 1–27. MR 96m:22036 Zbl 0841.11026

[Bourbaki 1961] N. Bourbaki, *Algèbre commutative, I–II: Modules plats, localisation*, Actualités Scientifiques et Industrielles **1290**, Herman, Paris, 1961. MR 36 #146 Zbl 0108.04002

[Bourbaki 1968] N. Bourbaki, *Groupes et algèbres de Lie, IV–VI: Groupes de Coxeter et systèmes de Tits, groupes engendrés par des réflexions, systèmes de racines*, Actualités Scientifiques et Industrielles **1337**, Hermann, Paris, 1968. MR 39 #1590 Zbl 0186.33001

[Breuil and Paskunas 2007] C. Breuil and V. Paskunas, "Towards a modulo p Langlands correspondence for GL_2", preprint, Institut des Hautes Études Scientifiques, 2007, Available at http://inc.web.ihes.fr/prepub/PREPRINTS/2007/M/M-07-25.pdf.

[Bushnell and Kutzko 1998] C. J. Bushnell and P. C. Kutzko, "Smooth representations of reductive p-adic groups: structure theory via types", *Proc. London Math. Soc.* (3) **77**:3 (1998), 582–634. MR 2000c:22014 Zbl 0911.22014

[Cabanes and Enguehard 2004] M. Cabanes and M. Enguehard, *Representation theory of finite reductive groups*, New Mathematical Monographs **1**, Cambridge University Press, 2004. MR 2005g:20067 Zbl 1069.20032

[Carter 1985] R. W. Carter, *Finite groups of Lie type*, Wiley, New York, 1985. MR 87d:20060 Zbl 0567.20023

[Dat 1999] J.-F. Dat, "Types et inductions pour les représentations modulaires des groupes p-adiques", *Ann. Sci. École Norm. Sup.* (4) **32**:1 (1999), 1–38. MR 99m:22018 Zbl 0935.22013

[Grosse-Klönne 2009] E. Grosse-Klönne, "On special representations of p-adic reductive groups", preprint, The Erwin Schrödinger International Institute for Mathematical Physics, 2009.

[Herzig 2010] F. Herzig, "The classification of irreducible admissible mod p representations of a p-adic GL_n", preprint, 2010. arXiv 1005.1713v1

[Iwahori and Matsumoto 1965] N. Iwahori and H. Matsumoto, "On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups", *Inst. Hautes Études Sci. Publ. Math.* 25 (1965), 5–48. MR 32 #2486 Zbl 0228.20015

[Lusztig 1989] G. Lusztig, "Affine Hecke algebras and their graded version", *J. Amer. Math. Soc.* **2**:3 (1989), 599–635. MR 90e:16049 Zbl 0715.22020

[Ollivier 2006a] R. Ollivier, "Critère d'irréductibilité pour les séries principales de $\mathrm{GL}_n(F)$ en caractéristique p", *J. Algebra* **304**:1 (2006), 39–72. MR 2007k:22017 Zbl 1108.22012

[Ollivier 2006b] R. Ollivier, "Modules simples en caractéristique p de l'algèbre de Hecke du pro-p-Iwahori de $\mathrm{GL}_3(F)$", *J. Algebra* **304**:1 (2006), 1–38. MR 2007j:22030 Zbl 1107.22010

[Ollivier 2006c] R. Ollivier, "Pro-p-invariants des séries principales de $\mathrm{GL}_n(F)$ en caractéristique p", preprint, 2006.

[Ollivier 2007] R. Ollivier, "Platitude du pro-p-module universel de $\mathrm{GL}_2(F)$ en caractéristique p", *Compos. Math.* **143**:3 (2007), 703–720. MR 2008e:22017 Zbl 1170.22007

[Ollivier 2009] R. Ollivier, "Le foncteur des invariants sous l'action du pro-p-Iwahori de $\mathrm{GL}_2(F)$", *J. Reine Angew. Math.* **635** (2009), 149–185. MR 2572257 Zbl pre05633739

[Rogawski 1985] J. D. Rogawski, "On modules over the Hecke algebra of a p-adic group", *Invent. Math.* **79**:3 (1985), 443–465. MR 86j:22028 Zbl 0579.20037

[Schneider and Stuhler 1991] P. Schneider and U. Stuhler, "The cohomology of p-adic symmetric spaces", *Invent. Math.* **105**:1 (1991), 47–122. MR 92k:11057 Zbl 0751.14016

[Schneider and Teitelbaum 2006] P. Schneider and J. Teitelbaum, "Banach–Hecke algebras and p-adic Galois representations", *Doc. Math.* Extra Vol. (2006), 631–684. MR 2008b:11126 Zbl 1140.11026

[Vignéras 1998] M.-F. Vignéras, "Induced R-representations of p-adic reductive groups", *Selecta Math.* (*N.S.*) **4**:4 (1998), 549–623. MR 99k:22026 Zbl 0943.22017

[Vignéras 2004] M.-F. Vignéras, "Representations modulo p of the p-adic group $\mathrm{GL}(2, F)$", *Compos. Math.* **140**:2 (2004), 333–358. MR 2004m:22028 Zbl 1049.22010

[Vignéras 2005] M.-F. Vignéras, "Pro-p-Iwahori Hecke ring and supersingular $\overline{\mathbf{F}}_p$-representations", *Math. Ann.* **331**:3 (2005), 523–556. Erratum in **333**:2, 699–701. MR 2005m:22020 Zbl 1107.22011

[Vignéras 2006] M.-F. Vignéras, "Algèbres de Hecke affines génériques", *Represent. Theory* **10** (2006), 1–20. MR 2006i:20005 Zbl 1134.22014

[Vignéras 2007] M.-F. Vignéras, "Représentations irréductibles de $\mathrm{GL}(2, F)$ modulo p", pp. 548–563 in *L-functions and Galois representations*, edited by D. Burns et al., London Math. Soc. Lecture Note Ser. **320**, Cambridge University Press, 2007. MR 2009h:11084 Zbl 1172.11017

[Vignéras 2008] M.-F. Vignéras, "Série principale modulo p de groupes réductifs p-adiques", *Geom. Funct. Anal.* **17**:6 (2008), 2090–2112. MR 2009a:22015 Zbl 05275302

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