Geometry of quiver Grassmannians of Kronecker
type and applications to cluster algebras

Giovanni Cerulli Irelli and Francesco Esposito

Geometry of quiver Grassmannians of Kronecker type and applications to cluster algebras

Giovanni Cerulli Irelli and Francesco Esposito

We study quiver Grassmannians associated with indecomposable representations (of finite dimension) of the Kronecker quiver. We find a cellular decomposition for them and we compute their Betti numbers. As an application, we find a geometric realization for the atomic basis of cluster algebras of type $A_1^{(1)}$ found by Sherman and Zelevinsky (who called it the canonical basis) and those of type $A_2^{(1)}$ found in an earlier paper of the first author.

1. Introduction

Cluster algebras are commutative $\mathbb{Z}$-subalgebras of the field of rational functions in a finite number of indeterminates which have been introduced and studied in [Fomin and Zelevinsky 2002; 2003a; 2007; Berenstein et al. 2005]. To every quiver Q without loops and 2-cycles it is associated a coefficient-free cluster algebra $\mathcal{A}_Q$. In [Caldero and Chapoton 2006; Caldero and Keller 2006; 2008; Derksen et al. 2010] the authors describe the cluster variables of $\mathcal{A}_Q$ via a map, called the Caldero–Chapoton map, between the representations of Q and the field of rational functions in n variables (we address the reader to the survey [Keller 2010]). This map is defined in terms of the Euler–Poincaré characteristic of some complex projective varieties attached to every finite-dimensional representation M of Q and called quiver Grassmannians. By definition, the quiver Grassmannian $\mathrm{Gr}_{\boldsymbol{e}}(M)$ consists of all subrepresentations of M of dimension vector $\boldsymbol{e}$. These varieties have been considered in, for example, [Caldero and Keller 2008; Caldero and Keller 2006; Caldero and Reineke 2008; Hernandez and Leclerc 2010, §12.3; Nakajima 2011]. In this paper we obtain more geometric information about them in the case of the Kronecker quiver. In [Caldero and Zelevinsky 2006] the authors compute the Euler–Poincaré characteristic of quiver Grassmannians associated with the Kronecker

quiver and they conjecture the existence of a cellular decomposition which we find here. In [Cerulli Irelli 2011b] (see also [Poettering 2010]) a torus action on some quiver Grassmannians has been found and this allows to produce a cellular decomposition of them in the case they are smooth.

In order to get more concrete results, such as an explicit computation of Poincaré polynomials, we restrict our attention to the case of the Kronecker quiver. So in this paper we study quiver Grassmannians associated with finite-dimensional representations of the Kronecker quiver. A finite-dimensional representation of the Kronecker quiver (called from now on a *Q-representation*) is a quadruple $M = (M_1, M_2, m_a, m_b)$, where M_1 and M_2 are finite-dimensional complex vector spaces and $m_a, m_b : M_1 \to M_2$ are two linear maps between them. Given two nonnegative integers e_1 and e_2, the variety $\mathrm{Gr}_{(e_1, e_2)}(M)$ is defined as the set

$$\{(N_1, N_2) \in \mathrm{Gr}_{e_1}(M_1) \times \mathrm{Gr}_{e_2}(M_2) : m_a(N_1) \subset N_2,\ m_b(N_1) \subset N_2\},$$

where $\mathrm{Gr}_e(V)$ denotes the Grassmannian of e-dimensional vector subspaces in a vector space V. This is a projective variety which is in general not smooth. When $M_1 = M_2 = \mathbb{C}^n$ and $m_a = \mathrm{Id}$ is the identity matrix and $m_b = J_n(0)$ is an indecomposable nilpotent Jordan block, the representation M is regular indecomposable and we denote it by R_n. The corresponding quiver Grassmannians $X = \mathrm{Gr}_{(e_1, e_2)}(R_n)$ are the main subjects of this paper. We concentrate on R_n because it is an indecomposable (finite-dimensional) nonrigid Q-representation; all the other indecomposable (finite-dimensional) Q-representations have either the same quiver Grassmannians as R_n or they are rigid and hence their quiver Grassmannians are smooth and our results are already known for them (see Section 2.1). In Section 2.2 we find that a one-dimensional torus T acts on X. We provide a stratification of X (see Section 2.5)

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_s \tag{1}$$

for $s = \min(e_1, n - e_2)$ into closed subvarieties $X_k \simeq \mathrm{Gr}_{(e_1-k, e_2-k)}(R_{n-2k})$. Moreover X_{k+1} is the singular locus of X_k and the difference $X_k \setminus X_{k+1}$ is a smooth quasiprojective variety which is not complete. As a consequence of the stratification (1) we get that X is smooth if and only if $s = 0$, i.e., either $e_1 = 0$ or $e_2 = n$, in which cases the quiver Grassmannian specializes to an usual Grassmannian of vector subspaces.

In Section 2.6 we prove that Białynicki-Birula's theorem on cellular decomposition of smooth projective varieties applies to $X_k \setminus X_{k+1}$; hence we can prove that X has a decomposition

$$X = \bigcup_{L \in X^T} X_L$$

into attracting sets

$$X_L := \{N \in X : \lim_{\lambda \to 0} t_\lambda N = L\}$$

of T-fixed points L, and theses sets are affine spaces. In Section 2.7 we describe the cell X_L: if L is indecomposable then X_L is the orbit of L by the action of a unipotent group; if $L = L' \oplus L''$, with L' of "lower weight" than L'', we have

$$\dim X_{L' \oplus L''} = \dim X_{L'} + \dim X_{L'} - \langle \mathbf{dim}\, L', \mathbf{dim}\, L'' \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euler form associated with the Kronecker quiver. As a consequence of this formula we are able to compute the Poincaré polynomials of the quiver Grassmannians associated with every finite-dimensional indecomposable representation of the Kronecker quiver: for $n \geq 0$ let P_n (resp. I_n) be the indecomposable preprojective of dimension vector $(n, n+1)$ (resp. $(n+1, n)$). As usual we denote by $P_X(t) := \sum_i \dim H^i(X) t^i$ the Poincaré polynomial of the complex projective variety X. Denoting by $\mathrm{Gr}_t(s)$ the Grassmannian of t-dimensional vector subspaces of an s-dimensional vector space, we have (see Section 2.8)

$$\begin{aligned}
P_{\mathrm{Gr}_{\boldsymbol{e}}(R_n)}(t) &= P_{\mathrm{Gr}_{(e_2-e_1)}(e_2)}(t) P_{\mathrm{Gr}_{(e_2-e_1)}(n-e_1)}(t), \\
P_{\mathrm{Gr}_{\boldsymbol{e}}(P_n)}(t) &= P_{\mathrm{Gr}_{e_1}(e_2-1)}(t) P_{\mathrm{Gr}_{(e_2-e_1)}(n+1-e_1)}(t), \\
P_{\mathrm{Gr}_{\boldsymbol{e}}(I_n)}(t) &= P_{\mathrm{Gr}_{e_1}(e_2+1)}(t) P_{\mathrm{Gr}_{(e_2-e_1)}(n-e_1)}(t),
\end{aligned} \qquad (2)$$

where $\boldsymbol{e} := (e_1, e_2)$ denotes a dimension vector.

Szanto [2011] computed the polynomials which count the number of $\mathbb{F}_q$-points of the quiver Grassmannians associated with indecomposable (finite-dimensional) Q-representations. The polynomials he finds are precisely the ones we find: the Poincaré polynomials. This result is expected and should follow by standard technique in ℓ-adic cohomology of schemes. It would deserve to be analyzed in more detail. In particular it would be interesting to know if the cellular decomposition continues to hold in positive characteristic.

In Section 3 we consider the (coefficient-free) cluster algebra $\mathscr{A}$ of type $A_1^{(1)}$. The cluster variables of $\mathscr{A}$ are the rational functions x_m, $m \in \mathbb{Z}$, of the field $\mathscr{F} = \mathbb{Q}(x_1, x_2)$ recursively generated by the relation

$$x_k x_{k+2} = x_{k+1}^2 + 1.$$

It is not hard to see that every pair $\{x_k, x_{k+1}\}$ of consecutive cluster variables is a free-generating system for the field $\mathscr{F}$ and hence $\mathscr{F} = \mathbb{Q}(x_k, x_{k+1})$ and every cluster variable can be expressed as a rational function in every such pair (which are called the clusters of $\mathscr{A}$). Every cluster variable is a Laurent polynomial in every cluster of $\mathscr{A}$; this is the Laurent phenomenon proved in [Fomin and Zelevinsky 2002]. In this case the Caldero–Chapoton map $M \mapsto CC(M)$ associates to a Q-representation M of dimension vector (d_1, d_2) the Laurent polynomial

$$CC(M) := \frac{\sum_e \chi(\mathrm{Gr}_e(M)) x_1^{2(d_2-e_2)} x_2^{2e_1}}{x_1^{d_1} x_2^{d_2}}. \tag{3}$$

Caldero and Keller [2008] proved that the map $M \mapsto CC(M)$ restricts to a bijection between the indecomposable rigid Q-representations M (i.e., $\mathrm{Ext}^1(M, M) = 0$) and the cluster variables of $\mathscr{A}$ different from x_1 and x_2. Moreover it has the property that $CC(M \oplus N) = CC(M)CC(N)$, under which cluster monomials not divisible by x_1 or x_2, i.e., monomials of the form $x_k^a x_{k+1}^b$ for $k \in \mathbb{Z} \setminus \{1, 2\}$ and $a, b \geq 0$, are in bijection with rigid Q-representations.

Sherman and Zelevinsky [2004] introduced distinguished elements $\{z_n : n \geq 0\}$ of $\mathscr{F}$ recursively defined by

$$z_0 = 2, \qquad z_1 = x_0 x_3 - x_1 x_2, \qquad z_{n+1} = z_1 z_n - z_{n-1} \quad \text{for } n \geq 1, \tag{4}$$

and proved that the set

$$\boldsymbol{B} := \{\text{cluster monomials}\} \cup \{z_n : n \geq 1\}$$

is a $\mathbb{Z}$-basis of $\mathscr{A}$ such that positive linear combinations of its elements coincide with the set of all *positive* elements of $\mathscr{A}$, that is, elements that are positive Laurent polynomials in every cluster of $\mathscr{A}$. Such a basis is now called an *atomic basis* of $\mathscr{A}$, though Sherman and Zelevinsky called it a canonical basis. The terminology changed after the relationship between this basis and Lusztig's canonical basis was better understood; see [Lampe 2011; Geiß et al. 2012]. We give a new geometric realization of $\boldsymbol{B}$ by using the Caldero–Chapoton map: cluster monomials are images of rigid representations and quiver Grassmannians associated with rigid quiver representations are smooth [Caldero and Reineke 2008]. With this in mind we prove in Section 3 that

$$z_n = \frac{\sum_e \chi(\mathrm{Gr}_e(R_n)^{\mathrm{sm}}) x_1^{2(n-e_2)} x_2^{2e_1}}{x_1^n x_2^n}$$

for every $n \geq 1$, where $\mathrm{Gr}_e(R_n)^{\mathrm{sm}} := X_0 \setminus X_1$ denotes the smooth part of $\mathrm{Gr}_e(R_n)$.

A similar construction can be made in a cluster algebra $\mathscr{A}_2$ of type $A_2^{(1)}$. These cluster algebras are studied in [Cerulli Irelli 2009] and some results are recalled in Section 3.2: the atomic basis of $\mathscr{A}_2$ consists of cluster monomials together with elements $\{u_n w^k, u_n z^k : k \geq 0, n \geq 1\}$ where w and z are two cluster variables and the u_n are defined similarly to (4) as follows:

$$u_0 = 2, \qquad u_1 = zw - 2, \qquad u_{n+1} = u_1 u_n - u_{n-1} \quad \text{for } n \geq 1. \tag{5}$$

We prove that u_n is obtained by evaluating the Caldero–Chapoton map at the *smooth part* of the regular indecomposable representation of a quiver of type $A_2^{(1)}$ of dimension vector (n, n, n) that lies in a homogeneous tube.

2. Geometric structure of quiver Grassmannians

Let

$$Q := 1 \underset{b}{\overset{a}{\rightrightarrows}} 2$$

be the Kronecker quiver. As usual, we denote a complex Q-representation $M = (M_1, M_2, m_a, m_b)$ as follows:

$$M = M_1 \underset{m_b}{\overset{m_a}{\rightrightarrows}} M_2 \ .$$

A subrepresentation N of M consists of vector subspaces N_1 and N_2 of M_1 and M_2 respectively such that $m_a(N_1) \subset N_2$ and $m_b(N_1) \subset N_2$. We call

$$\mathbf{dim}(M) := (\dim M_1, \dim M_2)$$

the dimension vector of M. A morphism $g : M \to M'$ from a Q-representation M to a Q-representation M' is a pair (g_1, g_2) of linear maps $g_1 : M_1 \to M'_1$ and $g_2 : M_2 \to M'_2$ such that $m'_a \circ g_1 = g_2 \circ m_a$ and $m'_b \circ g_1 = g_2 \circ m_b$. The set of Q-representations form a category which is an abelian Krull–Schmidt category via the natural notions of direct sums, kernel and cokernel (see [Assem et al. 2006], for example). The classification of finite-dimensional Q-representations which are *indecomposable* — that is, not direct sums of nontrivial subrepresentations — goes back to [Kronecker 1890]. Here is their complete list.

There are the *indecomposable preprojectives*

$$P_n = k^n \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} k^{n+1},$$

for $n \geq 0$, where $k = \mathbb{C}$ denotes the field of complex numbers, k^n and k^{n+1} denote complex vector spaces endowed respectively with linear bases $\{v_1^{(1)}, \ldots, v_n^{(1)}\}$ and $\{v_1^{(2)}, \ldots, v_{n+1}^{(2)}\}$, and φ_1, φ_2 are the immersions in the vector subspace of k^{n+1} spanned by $\{v_1^{(2)}, \ldots, v_n^{(2)}\}$ and $\{v_2^{(2)}, \ldots, v_{n+1}^{(2)}\}$, respectively.

There are the *indecomposable regulars*

$$R_n(\lambda) = k^n \underset{J_n(\lambda)}{\overset{\mathrm{Id}}{\rightrightarrows}} k^n \ , \quad R_n(\infty) = k^n \underset{\mathrm{Id}}{\overset{J_n(0)}{\rightrightarrows}} k^n \ ,$$

where $\lambda \in k$, $J_n(\lambda)$ denotes the n-th indecomposable Jordan block of eigenvalue λ and Id is the identity matrix (in the introduction we used the notation $R_n := R_n(0)$).

Finally, there are the *indecomposable preinjectives*

$$I_n = k^{n+1} \underset{\varphi_1^t}{\overset{\varphi_2^t}{\rightrightarrows}} k^n$$

for $n \geq 0$, where φ_1^t and φ_2^t are transposes of the matrices φ_1 and φ_2 defined above.

For an indecomposable Q-representation M we denote by $\mathcal{B}_i = \{v_k^{(i)}\}$, where $i = 1, 2$, the basis of M_i with respect to which M has the previous presentation. All other finite-dimensional Q-representations are direct sums of these ones. Direct sums of indecomposable preprojectives (resp. regulars, preinjectives) are called preprojective (resp. regular, preinjective) Q-representations.

Given nonnegative integers e_1 and e_2 and a Q-representation M we consider the variety

$$\mathrm{Gr}_e(M) := \{N \leq_Q M : \mathbf{dim}(N) = (e_1, e_2)\},$$

called the $e = (e_1, e_2)$-quiver Grassmannian of M (here $N \leq_Q M$ means that N is a subrepresentation of M). This is closed inside the product $\mathrm{Gr}_{e_1}(M_1) \times \mathrm{Gr}_{e_2}(M_2)$ of usual Grassmannians of vector subspaces and so is a complex projective variety. In [Caldero and Reineke 2008] it is shown that the tangent space $T_N(\mathrm{Gr}_e(M))$ at a point N of $\mathrm{Gr}_e(M)$ equals:

$$T_N(\mathrm{Gr}_e(M)) = \mathrm{Hom}(N, M/N). \tag{6}$$

The following inequalities hold for $Z := \mathrm{Gr}_e(M)$:

$$\langle e, d - e \rangle \leq \dim Z \leq \dim T_N(Z) \leq \langle e, d - e \rangle + \dim \mathrm{Ext}^1(M, M), \tag{7}$$

where $\langle (a, b)^t, (c, d)^t \rangle := ac + bd - 2ad$ is the Euler form of Q and $d := \mathbf{dim}\, M$, so that $\mathbf{dim}(M/N) = d - e$. In particular, if M is *rigid* ($\mathrm{Ext}^1(M, M) = 0$), then all the quiver Grassmannians associated with it are smooth (see [Derksen et al. 2010, Proposition 3.5] for a generalization of this result) and they have dimension $\dim \mathrm{Gr}_e(M) = \langle e, d - e \rangle$. It is known that the rigid Q-representations are

$$P_n^{\oplus a} \oplus P_{n+1}^{\oplus b}, \quad I_n^{\oplus a} \oplus I_{n+1}^{\oplus b}$$

for all $n \geq 0$ and $a, b \geq 0$.

2.1. *Action of a group on quiver Grassmannians.*

Let M be a Q-representation of dimension vector $d = (d_1, d_2)$. We consider the group

$$\mathcal{N}(M) := \left\{ (A, B) \in \prod_{i=1}^{2} \mathrm{GL}_{d_i}(M_i) : m_a A = B m_a, \ m_b A = \lambda B m_b \text{ for some } \lambda \in \mathbb{C}^* \right\}.$$

Note that the automorphism group of M is a closed subgroup of $\mathcal{N}(M)$ (for $\lambda = 1$). The group $\mathcal{N}(M)$ acts on $\mathrm{Gr}_e(M)$ as follows $(A, B) \cdot (N_1, N_2) := (AN_1, BN_2)$.

Lemma 1. (1) *For every $n \geq 0$, $\mathcal{N}(P_n) \simeq \mathbb{C}^* \times \mathbb{C}^*$.*

(2) *For every $n \geq 0$, $\mathcal{N}(I_n) \simeq \mathbb{C}^* \times \mathbb{C}^*$.*

Proof. It follows easily from the definition that $\mathcal{N}(P_n)$ consists of diagonal matrices (A, B) of the form

$$A = \mathrm{diag}(a, a\lambda, \ldots, a\lambda^{n-1}), \quad B = \mathrm{diag}(a, a\lambda, \ldots, a\lambda^n)$$

for $a, \lambda \in \mathbb{C}^*$. Similarly for $\mathcal{N}(I_n)$. $\qquad\square$

Proposition 2. *Let M be a rigid Q-representation. Then every quiver Grassmannian $\mathrm{Gr}_e(M)$ associated with M has a cellular decomposition.*

Proof. The variety $\mathrm{Gr}_e(M)$ is smooth and $\mathcal{N}(M) \supset \mathbb{C}^* =: T$. The torus T acts with finitely many fixed points. It follows hence by Białynicki–Birula results [Białynicki-Birula 1973] (see also [Chriss and Ginzburg 1997, §2.4]) that it has a cellular decomposition into attracting sets of its T-fixed points. $\qquad\square$

In the rest of the paper we mainly concentrate on the quiver Grassmannians associated with indecomposable regular Q-representations.

2.2. *The variety $X = \mathrm{Gr}_e(R_n)$.* From now on we will focus on quiver Grassmannians associated with indecomposable regular Q-representations. It is not hard to show that $\mathrm{Gr}_e(R_n(\lambda)) = \mathrm{Gr}_e(R_n(\mu))$ for every $\lambda, \mu \in k \cup \{\infty\}$ (see [Cerulli Irelli 2011b], for example) and hence we consider the variety $X := \mathrm{Gr}_e(R_n)$ (recall that our convention is $R_n := R_n(0)$). It follows from the definition that

$$X = \{N_1 \subset N_2 \subset k^n : J_n(0)N_1 \subset N_2, \ \dim N_i = e_i, \ i = 1, 2\}$$

and hence X is a closed subvariety of a partial flag variety. In this paper we use the convention that

$$J = J_n(0) \text{ is a lower triangular matrix.} \tag{8}$$

The group $\mathcal{N} := \mathcal{N}(R_n)$ is given by

$$\mathcal{N} := \{A \in \mathrm{GL}_n(\mathbb{C}) : AJA^{-1} = \lambda J, \text{ for some } \lambda \in \mathbb{C}^*\},$$

where $J := J_n(0)$ and it acts on X as $A \cdot (N_1, N_2) = (AN_1, AN_2)$.

Lemma 3. *The group $\mathcal{N}$ is the semidirect product*

$$\mathcal{N} = \mathcal{U} \rtimes T \times Z,$$

where $\mathcal{U}$ is the unipotent radical of $\mathcal{N}$ of unipotent lower triangular Toeplitz matrices, given by

$$\mathcal{U} := \left\{ \mathbf{1}_n + \sum_{i=1}^{n-1} a_i J_n^i(0) : a_i \in \mathbb{C} \right\},$$

T is the one-dimensional torus $T = \{t_\lambda : \lambda \in \mathbb{C}^*\}$, where t_λ is the diagonal matrix $t_\lambda := \mathrm{diag}(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})$, and Z consists of central elements $\{a_0 \mathbf{1}_n : a_0 \in \mathbb{C}^*\}$.

Proof. Every element of $\mathcal{U}TZ$ belongs to $\mathcal{N}$. Conversely, let $A \in \mathcal{N}$. Then it is easy to see that the columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of A satisfy the relation $\mathbf{a}_{k+1} = J\mathbf{a}_k$ and hence $A \in \mathcal{U}TZ$. It is now easy to see that $\mathcal{U}$ is normal in $\mathcal{U}TZ$. $\qquad\square$

For example for $n = 5$, an element A of the group $\mathcal{U}$ and an element t_λ of T have the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 \\ a_2 & a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 \\ a_4 & a_3 & a_2 & a_1 & 1 \end{pmatrix} \quad t_\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & \lambda^3 & 0 \\ 0 & 0 & 0 & 0 & \lambda^4 \end{pmatrix} \tag{9}$$

for $a_1, a_2, a_3, a_4 \in \mathbb{C}$ and $\lambda \in \mathbb{C}^*$.

Let $X^T := \{N \in X : tN = N \text{ for all } t \in T\}$. Clearly $L \in X^T$ if and only if L is a coordinate subrepresentation of R_n, i.e., both L_1 and L_2 are coordinate subspaces of k^n. In the next section we will encode this information in a combinatorial tool which is called the coefficient quiver of R_n.

We conclude this section by pointing out the useful isomorphism

$$\varphi_n : \mathrm{Gr}_e \to \mathrm{Gr}_{e^*}(R_n), \quad \varphi_n(N) = N^0, \tag{10}$$

where $(e_1, e_2)^* := (n - e_2, n - e_1)$ and $N^0 := \{f \in R_n^* : f(v) = 0 \text{ for all } v \in N\}$ by using the identification $R_n \simeq R_n^*$. The torus T acts on R_n^* with contragredient action and this gives an action on $\mathrm{Gr}_{e^*}(R_n^*)$. Under the identification $R_n \simeq R_n^*$ these two actions differ by a character and so the identification $\mathrm{Gr}_e(R_n^*) \simeq \mathrm{Gr}_e(R_n)$ is T-equivariant. It is an easy check that isomorphism (10) is T-equivariant and involutive.

2.3. *The coefficient quiver of R_n.* Following [Ringel 1998] we associate to a Q-representation M with linear basis $\mathcal{B}$ a quiver $\Gamma(M, \mathcal{B})$ called the coefficient quiver of M in the basis $\mathcal{B}$. By definition, $\Gamma(M, \mathcal{B})$ has the elements of $\mathcal{B}$ as vertices and there is an arrow labeled by a (resp. b) between two vertices v and v' if the coefficient of v' in $m_a(v)$ (resp. $m_b(v)$) is nonzero. We call $\Gamma(M, \mathcal{B})$ the coefficient quiver of M in the basis $\mathcal{B}$. When the basis $\mathcal{B}$ of M is clear from the definition of M we simply write $\Gamma(M)$ for $\Gamma(M, \mathcal{B})$. Here are the coefficient quivers of some indecomposable Q-representations for $n = 4$ in the basis defined in Section 2:

$$\Gamma(P_4) = \quad \begin{array}{c} 1^{(1)} \quad\quad 2^{(1)} \quad\quad 3^{(1)} \quad\quad 4^{(1)} \\ {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \\ 1^{(2)} \quad\quad 2^{(2)} \quad\quad 3^{(2)} \quad\quad 4^{(2)} \quad\quad 5^{(2)} \end{array}$$

$$\Gamma(R_5) = \quad \begin{array}{c} 1^{(1)} \quad\quad 2^{(1)} \quad\quad 3^{(1)} \quad\quad 4^{(1)} \quad\quad 5^{(1)} \\ {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \\ 1^{(2)} \quad\quad 2^{(2)} \quad\quad 3^{(2)} \quad\quad 4^{(2)} \quad\quad 5^{(2)} \end{array}$$

$$\Gamma(I_4) = \quad \begin{array}{c} 1^{(1)} \quad\quad 2^{(1)} \quad\quad 3^{(1)} \quad\quad 4^{(1)} \quad\quad 5^{(1)} \\ \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \;\; \searrow^{b} \;\; {}^{a}\swarrow \\ 1^{(2)} \quad\quad 2^{(2)} \quad\quad 3^{(2)} \quad\quad 4^{(2)} \end{array}$$
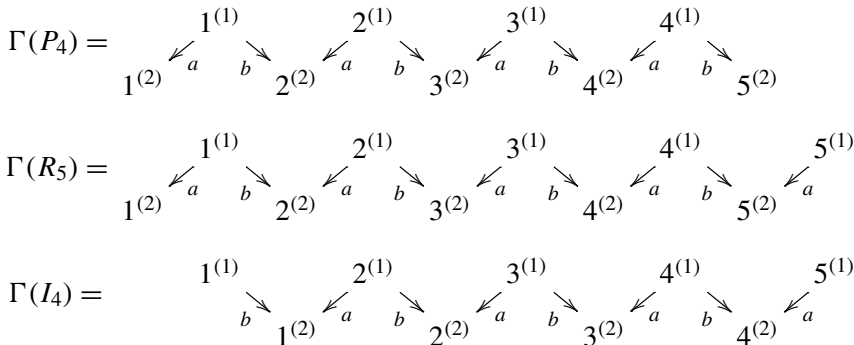
Here $k^{(i)}$, for $i = 1, 2$, denotes the vertex corresponding to $v_k^{(i)}$. Note that we use the convention (8).

In all these cases a one-dimensional torus T acts on all the associated quiver Grassmannians and the fixed points of this action are in bijection with successor closed subquivers of the corresponding coefficient quiver (i.e., subquivers γ such that if v is a vertex of γ and $\alpha : v \to v'$ is an arrow with source v then α is an arrow of γ).

Let us consider $\Gamma(R_n)$. In Lemma 3 we have seen that the torus T acts on X by $\lambda \cdot v_k^{(i)} = \lambda^{k-1} v_k^{(i)}$ ($i = 1, 2$, $k \in [1, n]$). For every $r \geq 1$ there exists a unique regular subrepresentation of R_n isomorphic to R_r and it has the property that,

$$\text{if } N \in X \text{ is such that } \lim_{\lambda \to 0} t_\lambda N = R_r, \text{ then } N = R_r. \tag{11}$$

Indeed this subrepresentation is coordinate and lies in the extreme right-hand side of $\Gamma(R_n)$. In particular the basis elements that generate R_r have maximal weights.

Roughly speaking, the flow for $\lambda \to 0$ goes from right to left in $\Gamma(R_n)$. For example the line $\langle v_1^{(2)} + v_2^{(2)} \rangle$ generated by the vector $v_1^{(2)} + v_2^{(2)}$ goes to the line generated by $v_1^{(2)}$ as follows:

$$\lim_{\lambda \to 0} \lambda \cdot (\langle v_1^{(2)} + v_2^{(2)} \rangle) = \lim_{\lambda \to 0} (\langle v_1^{(2)} + \lambda v_2^{(2)} \rangle) = \langle v_1^{(2)} \rangle.$$

2.4. *Action of the torus on Hom-spaces.* Let L and L' be indecomposable Q-representations. As we have seen in the previous sections, the torus $T = \{t_\lambda : \lambda \in \mathbb{C}^*\}$ acts on the quiver Grassmannians associated with them. The action of T naturally extends to the vector space $\text{Hom}(L, L')$ as follows: for $f \in \text{Hom}(L, L')$, $(t_\lambda f)(l) := t_\lambda f(t_{\lambda^{-1}} l)$. Following [Crawley-Boevey 1989] we endow the vector space $\text{Hom}(L, L')$ with a distinguished basis. Since $\text{Hom}(\cdot, \cdot)$ is additive, we assume that both L and L' are indecomposable. Let $\Gamma(L)$ and $\Gamma(L')$ denote the coefficient quiver of L and L' respectively. We consider the set $\mathscr{CB}(L, L')$ of triples $(\gamma, \alpha, \gamma')$ such that γ is a connected *predecessor closed* subquiver of $\Gamma(L)$ (meaning that for every vertex v of γ, every arrow $c : v' \to v$ with target v belongs to γ), γ' is a successor closed subquiver of $\Gamma(L')$, and $\alpha : \gamma_0 \to \gamma_0'$ is a bijection from the set γ_0 of vertices of γ to the set γ_0' of vertices of γ', such that for every arrow

$$v_k^{(1)} \xrightarrow{\;a\;} v_k^{(2)}$$

of γ there is an arrow

$$\alpha(v_k^{(1)}) \xrightarrow{\;a\;} \alpha(v_k^{(2)})$$

of γ' with the same label, and likewise with a replaced by b.

The next proposition is a special case of [Crawley-Boevey 1989], so we omit its proof (which is quite simple in this case).

Proposition 4. *Consider the map* $B : \mathscr{CB}(L, L') \to \mathrm{Hom}(L, L')$ *that associates to* $(\gamma, \alpha, \gamma')$ *the homomorphism*

$$f_{\gamma\gamma'}(v) = \begin{cases} \alpha(v) & \text{if } v \in \gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

The image of B *is a basis of* $\mathrm{Hom}(L, L')$ *which we call* CB-*basis (for Crawley-Boevey).*

The proposition is illustrated in the figure below, which shows an element of the CB-basis of $\mathrm{Hom}(R_3, R_3)$. The circles (resp. bullets) highlight a predecessor (successor) closed subquiver γ (resp. γ') of $\Gamma(R_3)$. The dotted arrows show the corresponding $f_{\gamma\gamma'}$.
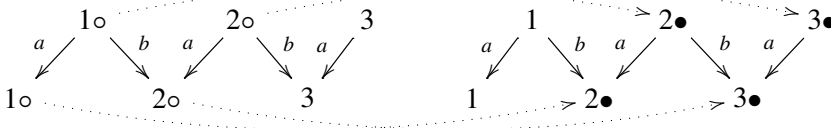
The torus T acts diagonally on the elements of the CB-basis of $\mathrm{Hom}(L, L')$ as follows: If $(\gamma, \alpha, \gamma') \in \mathscr{CB}(L, L')$ and the vertices of γ have consecutive weights $k, k+1, \dots$ and the vertices of γ' have consecutive weights $k', k'+1, \dots$ then

$$t_\lambda f_{\gamma\gamma'} = \lambda^{k'-k} f_{\gamma\gamma'}$$

and we say that $f_{\gamma\gamma'}$ has weight $k' - k$. For example, the morphism illustrated above has weight 1. We denote by $\mathrm{Hom}(L, L')^+$ the vector subspace of $\mathrm{Hom}(L, L')$ spanned by CB-basis elements with positive weight.

As an application of Proposition 4 we compute the dimension of the Hom-spaces between indecomposable Q-representations. It is known (and not difficult to prove by using Proposition 4) that $\mathrm{Hom}(R_s, P_l) = \mathrm{Hom}(I_s, P_l) = \mathrm{Hom}(I_s, R_l) = 0$ for all $s, l \geq 0$. Hence we consider the remaining cases.

Lemma 5. *For every* $l, s \geq 0$ *we have*

$$\dim \mathrm{Hom}(P_s, P_l) = [l - s + 1]_+, \tag{12}$$

$$\dim \mathrm{Hom}(P_s, R_l) = l, \tag{13}$$

$$\dim \mathrm{Hom}(P_s, I_l) = l + s, \tag{14}$$

$$\dim \mathrm{Hom}(R_s, R_l) = \min(s, l), \tag{15}$$

$$\dim \mathrm{Hom}(R_s, I_l) = s, \tag{16}$$

$$\dim \mathrm{Hom}(I_s, I_l) = [s - l + 1]_+, \tag{17}$$

where $[b]_+ := max(b, 0)$.

We conclude this section by pointing out that the action of the torus T on $\mathrm{Hom}(L, L')$ induces an action of the torus on the space $\mathrm{Ext}^1(L, L')$. With respect to this action long exact sequences in cohomology are T-equivariant.

2.5. Stratification of X. Every subrepresentation N of R_n is of the form $N = P \oplus R_r$ where P is preprojective and R_r, for $r \geq 0$, is either zero or an indecomposable regular Q-representation. Similarly the quotient $R_n/N = R_{r'} \oplus I$ where $R_{r'}$, for $r' \geq 0$, is either zero or regular indecomposable and I is preinjective. This allows us to give the following definition.

Definition 6. Let $N \in X$ with $N = P \oplus R_r$ and $R_n/N = R_{r'} \oplus I$ with P preprojective, I preinjective and some $r, r' \geq 0$. We define the integer

$$K_N = K_N(X) := \min(r, r')$$

It is easy to see that $K_N = \dim \mathrm{Ext}^1(N, R_n/N)$. Indeed

$$\dim \mathrm{Ext}^1(N, R_n/N) = \dim \mathrm{Ext}^1(R_r, R_{r'}) = \dim \mathrm{Hom}(R_{r'}, R_r) = \min(r, r')$$

where in the first equality we use the well-known fact that

$$\mathrm{Ext}^1(R, I) = \mathrm{Ext}^1(P, R) = \mathrm{Ext}^1(P, I) = 0$$

for every preprojective P, regular R and preinjective I Q-representations; in the second equality we use the AR-formula (see [Assem et al. 2006], for example); in the last equality we use (15). In particular it is known that

$$\langle \mathbf{dim}\, N, \mathbf{dim}\, N' \rangle = \dim \mathrm{Hom}(N, N') - \dim \mathrm{Ext}^1(N, N')$$

and in view of (6) we get that for every $N \in X$ the dimension of the tangent space $T_N(X)$ at N equals

$$\dim T_N(X) = \langle \mathbf{e}, n\delta - \mathbf{e} \rangle + K_N. \tag{18}$$

where $\delta := (1, 1)^t$. Equation (18) implies that a point $N \in X$ is smooth if and only if either N or R_n/N do not have a regular direct summand.

The next theorem provides a stratification of X and it is essential for our proof of the existence of a cellular decomposition of X. Let us define the strata.

Definition 7. For every integer $k \geq 0$ define the set

$$X_k = X_k(X) := \{N \in X : K_N \geq k\}$$

where K_N is given in Definition 6.

Theorem 8. (1) *The set X_{k+1} is a closed T-stable subvariety of X_k. Moreover there is a T-equivariant isomorphism*

$$X_k \simeq \mathrm{Gr}_{(e_1-k, e_2-k)}(R_{n-2k}). \tag{19}$$

(2) *The subvarieties X_k's provide a stratification of X*

$$X = X_0 \supset X_1 \supset \cdots \supset X_s \qquad (20)$$

where $s = min(e_1, n - e_2)$.

(3) *The variety $X_k \setminus X_{k+1}$ is smooth (inside X_k) and*

$$\lim_{\lambda \to 0} t_\lambda N \in X_k \setminus X_{k+1} \quad \text{for all } N \in X_k \setminus X_{k+1}.$$

Proof. We consider two subvarieties of X:

$$X_k'(X) := \{N \in \mathrm{Gr}_e(R_n) : R_k \text{ is a subrepresentation of } N\} \qquad (21)$$

for $k \in [0, e_1]$, and

$$X_k''(X) := \{N \in \mathrm{Gr}_e(R_n) : R_k \text{ is a quotient of } R_n/N\} \qquad (22)$$

for $k \in [0, n - e_2]$. It follows from the definitions that

$$X_k = X_k' \cap X_k''. \qquad (23)$$

We now collect some properties of X_k' and X_k''.

Lemma 9. (1) *The isomorphism φ_n defined in (10) induces an isomorphism*

$$X_k'(\mathrm{Gr}_e(R_n)) \simeq X_k''(\mathrm{Gr}_{e^*}(R_n)),$$

and hence also

$$X_k''(\mathrm{Gr}_e(R_n)) \simeq X_k'(\mathrm{Gr}_{e^*}(R_n)).$$

(2) *For every $k \in [0, e_1]$, $X_k'(X)$ is a T-stable subvariety of X. For every $N \in X_k'(X)$, $\lim_{\lambda \to 0} t_\lambda N \in X_k'(X)$. There is a T-equivariant isomorphism*

$$X_k'(X) \simeq \mathrm{Gr}_{e-k\delta}(R_{n-k}). \qquad (24)$$

(3) *For every $k \in [0, n - e_2]$, $X_k''(X)$ is a T-stable subvariety of X. For every $N \in X_k''(X)$, $\lim_{\lambda \to 0} t_\lambda N \in X_k''(X)$. There is a T-equivariant isomorphism*

$$X_k''(X) \simeq \mathrm{Gr}_e(R_{n-k}). \qquad (25)$$

Proof. Part (1) is a straightforward check. Part (3) follows from part (1) by using the isomorphism (10). Part (2) follows from property (11) of R_k. It remains to check (24). We consider the map

$$\Phi : X_k'(X) \to \mathrm{Gr}_{e-k\delta}(R_{n-k}) \quad \text{given by } \Phi(N) = (N/R_k).$$

Since there is a unique exact sequence of regular Q-representations

$$0 \longrightarrow R_k \overset{\iota_k}{\longrightarrow} R_n \overset{\pi_{n-k}}{\longrightarrow} R_{n-k} \longrightarrow 0 \qquad (26)$$

the map Φ is well defined and bijective. $\qquad \square$

We now conclude the proof of Theorem 8. Everything follows from Lemma 9 by (23) except the smoothness of $X_k \setminus X_{k+1}$. To prove smoothness notice that it is sufficient to prove smoothness of $X_0 \setminus X_1$ by (19), since $K_N(X_k) = K_N(X) - k$. By (18) $X_0 \setminus X_1$ is the smooth locus of X. $\qquad\square$

Corollary 10. *The variety $X = \mathrm{Gr}_e(R_n)$ is smooth if and only if $e_1 = 0$ (in which case $X \simeq \mathrm{Gr}_{e_2}(k^n)$) or $e_2 = n$ (in which case $X \simeq \mathrm{Gr}_{e_1}(k^n)$).*

Proof. By Theorem 8, X is smooth if and only if $X = X_0$ if and only if $s = \min(e_1, n - e_2) = 0$. $\qquad\square$

Corollary 11. *The quiver Grassmannian $\mathrm{Gr}_{(e_1,e_2)}(R_n)$ has dimension*

$$\dim \mathrm{Gr}_{(e_1,e_2)}(R_n) = \langle e, n\delta - e \rangle = (e_2 - e_1)(n - (e_2 - e_1))$$

Proof. In $X = \mathrm{Gr}_e(R_n)$ one can always find a smooth point, i.e., a subrepresentation N of R_n such that either N or R_n/N do not have a regular direct summand. It follows that $X_0 \setminus X_1$ is nonempty and hence the result follows from (18). $\qquad\square$

2.6. *Cellular decomposition of X.* In this section we provide a cellular decomposition of $X = \mathrm{Gr}_e(R_n)$. Following [Białynicki-Birula 1973] (see also [Chriss and Ginzburg 1997]), for every fixed point $L \in X^T$ we consider its *attracting set*, defined as

$$X_L := \{N \in X : \lim_{\lambda \to 0} t_\lambda N = L\}. \qquad (27)$$

In particular, $L \in X_L$ for every $L \in X^T$.

Theorem 12. *For every $L \in X^T$ the corresponding attracting set X_L is an affine space and $X_L \simeq T_L(X_L) \simeq T_L^+(X) := \mathrm{Hom}(L, R_n/L)^+$. Moreover*

$$X = \bigcup_{L \in X^T} X_L.$$

Proof. The results from [Białynicki-Birula 1973] on cellular decomposition of a projective variety X continue to hold if the variety is smooth but only quasiprojective, provided that the action of the torus is such that $\lim_{\lambda \to 0} t_\lambda N$ belongs to X for every $N \in X$. This is an easy consequence of Hironaka's resolution of singularities. In view of Theorem 8 we apply this to $X_0 \setminus X_1$ and we get the result. $\qquad\square$

2.7. *Description of the cells.* In this section we describe the cell X_L associated with every $L \in X^T$ (see Theorem 12). Given integers $r \in [0, n-1]$ and $k \in [1, n-r]$, we denote by $_k(P_r)$ the indecomposable preprojective subrepresentation of R_n of dimension vector $(r, r+1)$ generated by $v_k^{(1)}, v_{k+1}^{(1)}, \ldots, v_{k+r-1}^{(1)}$ and $v_k^{(2)}, v_{k+1}^{(2)}, \ldots, v_{k+r}^{(2)}$ if $r \geq 1$ and by $v_k^{(2)}$ if $r = 0$. For example, the following figure shows the subrepre-

sentation $_2(P_1)$ of R_5:
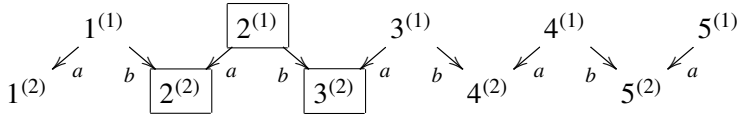
Theorem 13. (1) *If $L \in X^T$ is indecomposable then $X_L = \mathcal{U}L$. In particular if $L = {}_k(P_{e_1})$ then* $\dim X_L = n - k$.

(2) *If $L = L' \oplus L''$ with $\mathrm{Hom}(L', L'')^+ = \mathrm{Hom}(L', L'')$ then*

$$\dim X_L = \dim X_{L'} + \dim X_{L''} - \langle \boldsymbol{dim}\, L', \boldsymbol{dim}\, L'' \rangle. \tag{28}$$

Proof. (1) The unipotent group $\mathcal{U}$ defined in Lemma 3 is a subgroup of dimension $n - 1$ of the group of $n \times n$ unipotent lower triangular matrices. Let L be an indecomposable subrepresentation of R_n of dimension vector $\boldsymbol{e}$. If $L = R_r$ then $\boldsymbol{e} = (r, r)$ and $L_1 = L_2$ is the vector subspace of k^n spanned by the last r basis vectors of R_n. In particular, $AL = L$ for every $A \in \mathcal{U}$. On the other hand we have already noticed that regular subrepresentations have property (11) and hence $X_L = \{L\}$ and we get $X_L = \mathcal{U}L$ if L is regular. Let L be an indecomposable preprojective subrepresentation of R_n. Then $e_2 = e_1 + 1$ and we assume that $L = {}_k(P_{e_1})$ for some $k \in [1, n - e_1]$. It is easy to see that $\dim \mathcal{U}L = n - k$. Indeed the stabilizer of L under the action of $\mathcal{U}$ equals the stabilizer of the line generated by $v_k^{(2)}$ and hence it has dimension $k - 1$. For example in (9) the stabilizer of the third basis vector is generated by a_3 and a_4 and has dimension two. We now prove that $\dim X_L = n - k$ and since $\mathcal{U}L \subset X_L$ and they are both affine spaces, we get the equality. By Theorem 12 we have to compute $\dim T_L^+(X) = \dim \mathrm{Hom}(L, R_n/L)^+$. Since $L = {}_k(P_{e_1})$ is indecomposable the quotient R_n/L is the direct sum of at most two indecomposables as follows

$$R_n/L = R_{k-1} \oplus I_t$$

where $t = n - k - e_1$. We have

$$\mathrm{Hom}(L, R_n/L)^+ = \mathrm{Hom}(P_{e_1}, I_t).$$

In view of (14), $\dim \mathrm{Hom}(L, R_n/L)^+ = e_1 + t = n - k$ and we are done.

(2) There are short exact sequences

$$0 \longrightarrow L' \longrightarrow R_n/L'' \longrightarrow R_n/(L' \oplus L'') \longrightarrow 0$$

and

$$0 \longrightarrow L'' \longrightarrow R_n/L' \longrightarrow R_n/(L' \oplus L'') \longrightarrow 0$$

We apply the functors $\mathrm{Hom}(L'', -)$ and $\mathrm{Hom}(L', -)$ to the previous short exact sequences, then take the positive part (the part spanned by CB-basis elements with positive weight) and we get the exact sequences (see Theorem 12)

$$0 \longrightarrow 0 \longrightarrow T_{L''}(X_{L''}) \longrightarrow \mathrm{Hom}(L'', R_n/L) \longrightarrow 0$$

and

$$0 \twoheadrightarrow \mathrm{Hom}(L', L'') \twoheadrightarrow T_{L'}(X_{L'}) \twoheadrightarrow \mathrm{Hom}(L', R_n/L) \twoheadrightarrow \mathrm{Ext}^1(L', L'')^+ \twoheadrightarrow 0.$$

Indeed, we have

$$\mathrm{Ext}^1(L'', L')^+ = 0 = \mathrm{Ext}^1(L', R_n/L')^+, \tag{29}$$

$$\mathrm{Ext}^1(L', L'')^+ = \mathrm{Ext}^1(L', L''). \tag{30}$$

This is because, if $\mathrm{Hom}(M, L)^+ = 0$ then $\mathrm{Ext}^1(M, L)^+ = 0$; indeed one can always take a minimal T-equivariant injective resolution of L and apply the functor $\mathrm{Hom}(M, -)$. By summing up we get the short exact sequence

$$0 \twoheadrightarrow \mathrm{Hom}(L', L'') \twoheadrightarrow T_{L'}(X_{L'}) \oplus T_{L''}(X_{L''}) \twoheadrightarrow T_L(X_L) \twoheadrightarrow \mathrm{Ext}^1(L', L'') \twoheadrightarrow 0$$

Since for every fixed point L, $\dim X_L = \dim T_L(X_L)$ we get (28).   □

2.8. Betti numbers. We now use the results of the previous sections in order to compute the Betti numbers of $X = \mathrm{Gr}_e(R_n)$. Since X has a cellular decomposition (Theorem 12) the odd cohomology spaces of X are zero and the $2i$-th Betti number $b_{2i} = b_{2i}(X) := \dim H^{2i}(X)$ equals the number of cells of dimension i.

Before stating the main result of this section we start with the special case $e_2 = e_1 + 1$.

Theorem 14. *Let* $X = \mathrm{Gr}_{(e_1, e_1+1)}(R_n)$. *The even Betti numbers of X are*

$$b_{2i} = \begin{cases} i + 1 & \text{if } 0 \leq i \leq s, \\ s + 1 & \text{if } s \leq i \leq n - 1 - s, \\ n - i & \text{if } n - 1 - s \leq i \leq n - 1, \end{cases} \tag{31}$$

where $s = \min(e_1, n - 1 - e_1)$ *(see Theorem 8(2)).*

The Poincaré polynomial $P_X(t) := \sum_{i=0}^{\dim X} b_i t^i$ *of X equals*

$$P_X(t^{1/2}) = \left(\frac{t^{e_1+1} - 1}{t - 1} \right) \left(\frac{t^{n-e_1} - 1}{t - 1} \right) \tag{32}$$

Proof. For $k \in [0, e_1]$, consider the variety X'_k defined in (21) and the difference

$$Y'_k = Y'_k(X) := X'_k(X) \setminus X'_{k+1}(X). \tag{33}$$

By definition, $N \in Y_k$ if $N = R_k \oplus P$ for some (indecomposable) preprojective P. Since X is union of the Y_k' and each Y_k' is a union of cells (Theorem 12) we get

$$\dim H^i(X) = \sum_{k=0}^{e_1} \dim H_c^i(Y_k'), \qquad (34)$$

where $H_c^i(Y_k')$ is the i-th cohomology space of Y_k' with compact support and its dimension equals the number of cells of dimension $i/2$. In particular $\dim H_c^i(Y_k') = 0$ for i odd. We prove that

$$\dim H_c^{2i}(Y_k') = \begin{cases} 1 & \text{if } i \in [e_1 - k, n - k - 1], \\ 0 & \text{otherwise.} \end{cases} \qquad (35)$$

and hence (31) follows from (34). From the definition it follows that

$$Y_k'(X) \simeq Y_0'(\mathrm{Gr}_{(e_1-k,\,e_1+1-k)}(R_{n-k})). \qquad (36)$$

The elements of $Y_0'(\mathrm{Gr}_{(f_1,\,f_1+1)}(R_m))$ are all the indecomposable preprojective sub-representations of R_m of dimension vector $(f_1, f_1 + 1)$ and hence they are all isomorphic to P_{f_1}. For $i \in [1, m - f_1]$, Theorem 13(1) gives $\dim X_{i(P_{f_1})} = m - i$, which is in $[f_1, m - 1]$; hence for every $\ell \in [f_1, m - 1]$ there is a unique cell of dimension ℓ. For $m = n - k$ and $f_1 = e_1 - k$ we get (35). $\qquad \square$

Notice that (32) can be written as $P_X(t) = P_{\mathrm{Gr}_1(e_2)}(t) P_{\mathrm{Gr}_1(n-e_1)}(t)$, where $e_2 = e_1 + 1$. Surprisingly, this turns out to be a general fact, which we state as the main result of this section:

Theorem 15. *Let $X = \mathrm{Gr}_{(e_1,e_2)}(R_n)$. The Poincaré polynomial $P_X(t)$ of X equals*

$$P_X(t) = P_{\mathrm{Gr}_{(e_2-e_1)}(e_2)}(t) P_{\mathrm{Gr}_{(e_2-e_1)}(n-e_1)}(t). \qquad (37)$$

Proof. We proceed by induction on $n \geq e_1 \geq 0$. For $e_1 = 0$, $X \simeq \mathrm{Gr}_{e_2}(n)$ and (37) follows. Let $1 \leq e_1 \leq e_2$. The variety X can be decomposed as

$$X = Y_0'(X) \cup X_1'(X)$$

where $Y_0' = Y_0'(X)$ consists of all preprojective subrepresentations of R_n in X and X_1' consists of subrepresentations of R_n in X having a nonzero regular subrepresentation (see (21) and (33)). Moreover, by (24), $X_1'(X) \simeq \mathrm{Gr}_{(e_1-1,e_2-1)}(R_{n-1})$ and hence we have

$$P_X(t) = P_{Y_0'}(t) + P_{\mathrm{Gr}_{(e_1-1,e_2-1)}(R_{n-1})}(t). \qquad (38)$$

By the inductive hypothesis (37) holds if and only if

$$P_{Y_0'}(t) = \left(P_{\mathrm{Gr}_{(e_2-e_1)}(e_2)}(t) - P_{\mathrm{Gr}_{(e_2-e_1)}(e_2-1)}(t) \right) P_{\mathrm{Gr}_{(e_2-e_1)}(n-e_1)}(t). \qquad (39)$$

Hence we prove (39).

We make the following choice: we fix a linear basis $\{v_1, \ldots, v_s\}$ of a vector space of dimension s and we let the torus act on $\mathrm{Gr}_t(s)$ by $t_\lambda v_i = \lambda^i v_i$. We consider the vector subspace of k^{e_2} generated by $v_2, \ldots, v_{e_2}$ and the corresponding embedding $\mathrm{Gr}_{(e_2-e_1)}(e_2 - 1) \subset \mathrm{Gr}_{(e_2-e_1)}(e_2)$. With this choice the difference $\mathrm{Gr}_{(e_2-e_1)}(e_2) \setminus \mathrm{Gr}_{(e_2-e_1)}(e_2 - 1)$ is T-stable and for every point W of it, $\lim_{\lambda \to 0} t_\lambda W$ still belongs to it. The right-hand side of (39) is the Poincaré polynomial (with respect to the cohomology with compact support) of the smooth projective variety

$$G := \big(\mathrm{Gr}_{(e_2-e_1)}(e_2) \setminus \mathrm{Gr}_{(e_2-e_1)}(e_2 - 1)\big) \times \mathrm{Gr}_{(e_2-e_1)}(n - e_1).$$

The one-dimensional torus T acts on G and the attracting sets of the T-fixed points form a cellular decomposition of G. We prove that there exists a bijection between the cells of Y'_0 of dimension k and the cells of G of dimension k.

A point of Y'_0 is a direct sum of precisely $(e_2 - e_1)$ indecomposable preprojective subrepresentations of R_n (this follows by considering their dimension vectors). The T-fixed points have the form

$$L := {}_{k_1}(P_{r_1}) \oplus {}_{k_2}(P_{r_2}) \oplus \cdots \oplus {}_{k_{(e_2-e_1)}}(P_{r_{(e_2-e_1)}}),$$

where $r_1 + \cdots + r_{e_2-e_1} = e_1$, $r_i \geq 0$ and ${}_k(P_r)$ has the same meaning as in Section 2.7. In view of Theorem 13 and of (12) the dimension of the attracting cell of L equals

$$\dim X_L = n(e_2 - e_1) - \sum_{i=1}^{e_2-e_1} k_i - (e_2 - e_1)^2 + \sum_{i=1}^{e_2-e_1} \sum_{j=1}^{i} (r_j - r_i + 1).$$

We consider the set $\alpha(e, k, n)$ which parametrizes the T-fixed points of Y'_0 whose attracting set has dimension k, i.e.,

$$\alpha(e, k, n) := \big\{(k_1, k_2, \ldots, k_{e_2-e_1}, r_1, r_2, \ldots, r_{e_2-e_1}) :$$
$$1 \leq k_1 \leq k_1 + r_1 < k_2 \leq k_2 + r_2 < \cdots < k_{e_2-e_1} \leq k_{e_2-e_1} + r_{e_2-e_1} \leq n,$$
$$r_1 + r_2 + \cdots + r_{e_2-e_1} = e_1, \ r_i \geq 0,$$
$$n(e_2-e_1) - \textstyle\sum_{i=1}^{e_2-e_1} k_i - (e_2-e_1)^2 + \sum_{i=1}^{e_2-e_1} \sum_{j=1}^{i}(r_j-r_i+1) = k\big\}.$$

Now consider the T-fixed points of G and their attracting sets. The T-fixed points of $\mathrm{Gr}_t(s)$ consist of coordinate vector subspaces of dimension t and they are naturally parametrized by tuples $(a_1, \ldots, a_t)$ of integers $1 \leq a_1 < \cdots < a_t \leq s$. The corresponding cell $\mathbb{O}_{(a_1,\ldots,a_t)}$ has dimension

$$s - a_1 - (t-1) + s - a_2 - (t-2) + \cdots s - a_t = ts - \sum_{i=1}^{t} a_i - \sum_{i=1}^{t-1} i.$$

The T-fixed points of $\mathrm{Gr}_{(e_2-e_1)}(e_2) \setminus \mathrm{Gr}_{(e_2-e_1)}(e_2 - 1)$ are the coordinate vector subspaces of k^{e_2} containing v_1. Hence the following set parametrizes the cells of G

of dimension k:

$$\beta(\boldsymbol{e}, k, n) := \big\{(a_1, a_2, \ldots, a_{e_2-e_1}, b_2, b_3, \ldots, b_{e_2-e_1}) :$$
$$1 \le a_1 < a_2 < \cdots < a_{e_2-e_1} \le n - e_1, \quad 2 \le b_2 < \cdots < b_{e_2-e_1} \le e_2,$$
$$n(e_2 - e_1) - \sum_{i=1}^{e_2-e_1} a_i - \sum_{i=2}^{e_2-e_1} b_i + (e_2 - e_1 - 1) = k\big\}.$$

We consider the map $\beta(\boldsymbol{e}, k, n) \to \alpha(\boldsymbol{e}, k, n)$ defined by

$$a_1 \mapsto k_1, \quad a_i \mapsto k_i - r_1 - r_2 - \cdots - r_{i-1}, \quad b_i \mapsto \sum_{j=0}^{i-2} r_{(e_2-e_1-j)} + i,$$

for $i \in [2, e_2 - e_1]$. It is straightforward to verify that this map is a bijection between $\beta(\boldsymbol{e}, k, n)$ and $\alpha(\boldsymbol{e}, k, n)$. It follows that Y_0' and G have the same Betti numbers and hence (39) follows. $\qquad\square$

Corollary 16. *The Poincaré polynomial of a quiver Grassmannian associated with the indecomposable preprojective P_n and the indecomposable preinjective I_n, where $n \ge 0$, are given by*

$$P_{\mathrm{Gr}_{\boldsymbol{e}}(P_n)}(t) = P_{\mathrm{Gr}_{e_1}(e_2-1)}(t) P_{\mathrm{Gr}_{(e_2-e_1)}(n+1-e_1)}(t), \tag{40}$$

$$P_{\mathrm{Gr}_{\boldsymbol{e}}(I_n)}(t) = P_{\mathrm{Gr}_{e_1}(e_2+1)}(t) P_{\mathrm{Gr}_{(e_2-e_1)}(n-e_1)}(t). \tag{41}$$

Proof. Equality (41) follows from (40) by the isomorphism

$$\mathrm{Gr}_{(e_1, e_2)}(I_n) \simeq \mathrm{Gr}_{(n-e_2, n+1-e_1)}(P_n).$$

Hence we prove (40). As in the proof of Theorem 15, let $Y_0' = Y_0'(\mathrm{Gr}_{\boldsymbol{e}}(R_{n+1}))$ be the subvariety of $\mathrm{Gr}_{\boldsymbol{e}}(R_{n+1})$ of all preprojective subrepresentations of R_{n+1} of dimension vector $\boldsymbol{e}$ and let

$$G' := \mathrm{Gr}_{(e_2-e_1)}(e_2) \setminus \mathrm{Gr}_{(e_2-e_1)}(e_2 - 1)$$

with the convention that $\mathrm{Gr}_{(e_2-e_1)}(e_2-1)$ consists of all the elements of $\mathrm{Gr}_{(e_2-e_1)}(e_2)$ not containing the first basis vector (as in the proof of Theorem 15). In view of (39) it is sufficient to prove the equalities

$$P_{G'}(t) = t^{2e_1} P_{\mathrm{Gr}_{(e_2-e_1-1)}(e_2-1)}(t), \tag{42}$$

$$P_{Y_0'}(t) = t^{2e_1} P_{\mathrm{Gr}_{\boldsymbol{e}}(P_n)}(t). \tag{43}$$

The proof of (42) is similar to the proof of Theorem 15: there is an obvious bijection between the cells of G' of dimension k and the cells of $\mathrm{Gr}_{(e_2-e_1)}(e_2-1)$ of dimension $k - e_1$. Let us prove (43). Let $L \in Y_0'$. Then L is a sum of preprojective subrepresentations of R_{n+1} and L is a subrepresentation of $P_n \le_Q R_{n+1}$. By looking at the quotients R_{n+1}/L and P_n/L and using Lemma 5 one gets

$$\dim \mathrm{Hom}(L, R_{n+1}/L)^+ = \dim \mathrm{Hom}(L, P_n/L)^+ + e_1.$$

Hence the cells of Y_0' of dimension k are in bijection with the cells of $\mathrm{Gr}_e(P_n)$ of dimension $k - e_1$ and (43) holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since $\chi(X) = P_X(1)$ for a projective variety X and $\chi(\mathrm{Gr}_t(s)) = \binom{s}{t}$, we recover the following result, of which alternative proofs have also appeared in [Cerulli Irelli 2011b; Zelevinsky 2007; Szanto 2011; Poettering 2010].

Corollary 17 [Caldero and Zelevinsky 2006].

$$\chi(\mathrm{Gr}_e(R_n)) = \binom{e_2}{e_1}\binom{n-e_1}{e_2-e_1}, \tag{44}$$

$$\chi(\mathrm{Gr}_e(P_n)) = \binom{e_2-1}{e_1}\binom{n+1-e_1}{e_2-e_1}, \tag{45}$$

$$\chi(\mathrm{Gr}_e(I_n)) = \binom{e_2+1}{e_1}\binom{n-e_1}{e_2-e_1}. \tag{46}$$

3. Applications to cluster algebras

To a finite quiver Q without loops and 2-cycles is associated a (coefficient-free) cluster algebra $\mathscr{A}_Q$ ([Fomin and Zelevinsky 2002]; see also [Fomin and Zelevinsky 2003b] and [Keller 2010] for excellent surveys). This is a $\mathbb{Z}$-subalgebra of the field $\mathscr{F}$ of rational functions in n variables (n being the number of vertices of Q), generated by its *cluster variables*. The cluster variables are grouped into free-generating sets of $\mathscr{F}$ called *clusters*. The *cluster monomials* are monomials in cluster variables belonging to the same cluster. An *atomic basis* of $\mathscr{A}_Q$ is a $\mathbb{Z}$-basis $\boldsymbol{B}$ of it such that the positive linear combinations of elements of $\boldsymbol{B}$ coincide with the semiring of positive elements of $\mathscr{A}_Q$ (elements that are positive Laurent polynomials in every cluster of $\mathscr{A}_Q$); see [Sherman and Zelevinsky 2004]. The existence of such a basis has been proved only in a few cases:

- If Q is of type ADE, cluster monomials form an atomic basis of $\mathscr{A}_Q$ [Cerulli Irelli 2011a; Cerulli Irelli and Labardini-Fragoso 2011].

- When Q is the Kronecker quiver, atomic bases exist coefficients and consist of cluster monomials together with extra elements $\{z_n : n \geq 1\}$ [Sherman and Zelevinsky 2004].

- when Q is of type $A_2^{(1)}$, the atomic bases of $\mathscr{A}_Q$ have been found in [Cerulli Irelli 2009] (for every choice of the coefficients) and consist of cluster monomials together with some extra elements $\{u_n : n \geq 1\}$ possibly multiplied by particular cluster variables.

We notice that under the Caldero–Keller bijection, cluster monomials correspond, via the Caldero–Chapoton map, to rigid Q-representations and the quiver Grassmannians associated with rigid representations are smooth. In the next two sections

we propose a "truncation" of the Caldero–Chapoton map that gives a geometric realization of the extra elements $\{z_n\}$, $\{u_n\}$.

3.1. Type $A_1^{(1)}$. Let $\mathscr{F} = Q(x_1, x_2)$ be the field of rational functions in two independent variables x_1 and x_2 with rational coefficients. We recursively define elements $\{x_k : k \in \mathbb{Z}\}$ of $\mathscr{F}$ by

$$x_k x_{k+2} = x_{k+1}^2 + 1, \quad k \in \mathbb{Z}.$$

Let $\mathscr{A}$ be the $\mathbb{Z}$-subalgebra of $\mathscr{F}$ generated by all the x_k's ($k \in \mathbb{Z}$). By [Sherman and Zelevinsky 2004] the algebra $\mathscr{A}$ is the coefficient–free cluster algebra associated with the Kronecker quiver. The pairs $\{x_k, x_{k+1}\}$, $k \in \mathbb{Z}$, are free generating sets of $\mathscr{F}$ and form the *clusters* of $\mathscr{A}$. Monomials $x_k^a x_{k+1}^b$, $a, b \geq 0$, $k \in \mathbb{Z}$, are called the *cluster monomials* of $\mathscr{A}$.

Caldero and Zelevinsky [2006] defined the rational function

$$s_n := CC(R_n)$$

for every $n \geq 1$, where R_n is a regular indecomposable Q-representation of dimension (n, n). They have proved that the set $\mathscr{S} := \{\text{cluster monomials}\} \cup \{s_n : n \geq 1\}$ is a $\mathbb{Z}$-basis of $\mathscr{A}$.

The atomic basis $\boldsymbol{B}$ defined in the introduction and the basis $\mathscr{S}$ are related by (see [Caldero and Zelevinsky 2006])

$$z_n = s_n - s_{n-2} \tag{47}$$

for $n \geq 1$ and the convention that $s_{-k} = 0$ for $k > 0$, $s_0 := 1$.

Theorem 18. *The element z_n has the Laurent expansion*

$$z_n = \frac{\sum_e \chi(\mathrm{Gr}_e(R_n)^{Sm}) x_1^{2(n-e_2)} x_2^{2e_1}}{x_1^n x_2^n} \tag{48}$$

where $\mathrm{Gr}_e(R_n)^{Sm}$ denotes the smooth part of $\mathrm{Gr}_e(R_n)$.

Proof. In view of Theorem 8, $\mathrm{Gr}_e(R_n)^{Sm} = X_0 \setminus X_1$ where $X_0 = \mathrm{Gr}_e(R_n)$ and $X_1 = \mathrm{Gr}_{(e_1-1, e_2-1)}(R_{n-2})$ and hence

$$\chi(\mathrm{Gr}_e(R_n)^{Sm}) = \chi(\mathrm{Gr}_e(R_n)) - \chi(\mathrm{Gr}_{(e_1-1, e_2-1)}(R_{n-2})).$$

It is now easy to check that the right-hand side of (48) satisfies (47). □

3.2. Type $A_2^{(1)}$. We now briefly recall the construction of the atomic basis of cluster algebras of type $A_2^{(1)}$ from [Cerulli Irelli 2009]. Let $\mathscr{F} = \mathbb{Q}(x_1, x_2, x_3)$ be the field of rational functions in three (commuting) independent variables x_1, x_2 and x_3 with rational coefficients. Recursively define elements $x_m \in \mathscr{F}$ for $m \in \mathbb{Z}$ by the relation

$$x_m x_{m+3} = x_{m+1} x_{m+2} + 1. \tag{49}$$

Define also the elements $w, z \in \mathcal{F}$ by

$$w = \frac{x_1 + x_3}{x_2}, \quad z = \frac{x_1 x_2 + x_2 x_3 + 1}{x_1 x_3}. \tag{50}$$

The (coefficient–free) cluster algebra $\mathcal{A}$ of type $A_2^{(1)}$ is the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by all the x_m's, w and z (see also [Fomin and Zelevinsky 2002, Example 7.8]). This is the cluster algebra $\mathcal{A} = \mathcal{A}_{Q_2}$ associated with the affine quiver

$$Q_2: \quad 1 \longrightarrow 3 \qquad \overset{2}{\underset{}{\nearrow \searrow}} \tag{51}$$

of type $A_2^{(1)}$. The elements x_m, $m \in \mathbb{Z}$, w and z are the cluster variables of $\mathcal{A}$. The sets $\{x_m, x_{m+1}, x_{m+2}\}$, $\{x_{2m}, z, x_{2m+2}\}$ and $\{x_{2m-1}, w, x_{2m+1}\}$, $m \in \mathbb{Z}$, are the clusters of $\mathcal{A}$. The cluster monomials are monomials in cluster variables belonging to the same cluster. The exchange graph of $\mathcal{A}$ is the brick wall shown here:
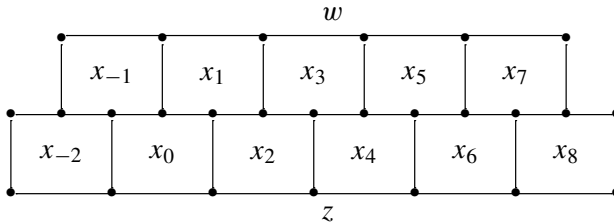
It has clusters as vertices and an edge between two vertices if the corresponding clusters share precisely two cluster variables. In this figure the cluster variables of a cluster $\mathcal{C}$ label the regions surrounding the vertex corresponding to $\mathcal{C}$.

Define elements u_n, $n \geq 0$, of $\mathcal{F}$ by the recursion

$$u_0 = 2, \quad u_1 = zw - 2, \quad u_{n+1} = u_1 u_n - u_{n-1} \quad \text{for } n \geq 1. \tag{52}$$

In [Cerulli Irelli 2009] it is shown that the set

$$\boldsymbol{B} = \{\text{cluster monomials}\} \cup \{u_n w^k, \, u_n z^k : n \geq 1, \, k \geq 0\} \tag{53}$$

is an atomic basis of $\mathcal{A}$. We now realize the elements u_n as images of the Caldero–Chapoton map. Recall that for a representation M of the quiver Q_2 of (51) the Caldero–Chapoton map $CC(M)$ is the following (see [Caldero and Zelevinsky 2006]):

$$CC(M) = \frac{\sum_e \chi(\mathrm{Gr}_e(M)) x_1^{d_2+d_3-e_2-e_3} x_2^{d_3-e_3+e_1} x_3^{e_1+e_2}}{x_1^{d_1} x_2^{d_2} x_3^{d_3}}$$

where (d_1, d_2, d_3) is the dimension vector of M. For every $n \geq 1$ let $R_{n,2}$ be the indecomposable regular Q_2-representation in an homogeneous tube, i.e.,

$$R_{n,2} = \quad k^n \xrightarrow[J_n(0)]{} k^n$$

where $J_n(0)$ denotes the $n \times n$ indecomposable nilpotent Jordan block.

Theorem 19. *For every $n \geq 1$,*

$$u_n = \frac{\sum_e \chi(\mathrm{Gr}_e(R_{n,2})^{Sm}) x_1^{2n-e_2-e_3} x_2^{n-e_3+e_1} x_3^{e_1+e_2}}{x_1^n x_2^n x_3^n} \tag{54}$$

where $\mathrm{Gr}_e(R_{n,2})^{Sm}$ denotes the smooth part of the quiver Grassmannian $\mathrm{Gr}_e(R_{n,2})$, $e := (e_1, e_2, e_3)$.

Proof. Let u'_n denote the right-hand side of (54). We prove that $\{u'_n\}$ satisfies (52). We consider the fibration of projective varieties

$$f : \mathrm{Gr}_{(e_1,e_2,e_3)}(R_{n,2}) \to \mathrm{Gr}_{(e_1,e_3)}(R_n)$$

taking (N_1, N_2, N_3) to (N_1, N_3); its fiber is $\mathrm{Gr}_{(e_2-e_1)}(e_3 - e_1)$. In particular

$$\mathrm{Gr}_{(e_1,e_2,e_3)}(R_{n,2})^{Sm} = f^{-1}\left(\mathrm{Gr}_{(e_1,e_3)}(R_n)^{Sm}\right).$$

Using this it is straightforward to check that

$$u'_n = z_n(x_1 w^{-1/2}, x_3 w^{-1/2}) \tag{55}$$

where the right-hand side means that (48) should be computed with the substitutions $x_1 \mapsto x_1 w^{-1/2}$ and $x_2 \mapsto x_3 w^{-1/2}$, and $w^{-1/2}$ is a formal variable whose square is w^{-1}. The defining equations (4) yield

$$u'_1 = z_1(x_1 w^{-1/2}, x_3 w^{-1/2}) = \frac{x_1^2 + x_3^2 + w}{x_1 x_2} = zw - 2$$

and $u'_{n+1} = u'_1 u'_n - u'_{n-1}$, so $u'_n = u_n$ for every $n \geq 1$. $\qquad\square$

4. Conclusions

The study of quiver Grassmannians from a geometric point of view has provided interesting information about cluster algebras. After this study it is natural to define the following slight modification of the Caldero–Chapoton map: for $k \geq 0$,

$$CC^{(k)}(M) := \frac{\sum_e \chi(\mathrm{Gr}_e(M)^{(k)}) x_1^{2(d_2-e_2)} x_2^{2e_1}}{x_1^{d_1} x_2^{d_2}}, \tag{56}$$

where

$$\mathrm{Gr}_e(M)^{(k)} := \{N \in \mathrm{Gr}_e(M) : \dim \mathrm{Ext}^1(N, M/N) = k\}.$$

In the case of the Kronecker quiver the map $CC^{(0)}(M)$ is defined by the smooth part of the quiver Grassmannians associated with M. This is not a general fact and it is studied in [Cerulli Irelli et al. 2012]. In that paper we study the relationship between the maps $CC^{(k)}$ and the transverse quiver Grassmannian introduced in [Dupont 2010].

We notice that for a rigid representation M we have $CC^{(0)}(M) = CC(M)$. Moreover it is easy to check that if $\mathrm{Ext}^1(M, N) = \mathrm{Ext}^1(N, M) = 0$ then $CC^0(M \oplus N) = CC^0(M)CC^0(N)$. In particular the elements $u_n w$ and $u_n z$ of the basis (53) are $CC^0(R_{n,2} \oplus W)$ and $CC^0(R_{n,2} \oplus Z)$ where W and Z are the indecomposable regular rigid Q_2-representations. We trust that this approach can help in the determination of atomic bases of cluster algebras of affine type of higher rank.

Acknowledgements

References

[Assem et al. 2006] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras*, vol. 1: Techniques of representation theory, London Math. Soc. Student Texts **65**, Cambridge University Press, Cambridge, 2006. MR 2006j:16020 Zbl 1092.16001

[Berenstein et al. 2005] A. Berenstein, S. Fomin, and A. Zelevinsky, "Cluster algebras, III: Upper bounds and double Bruhat cells", *Duke Math. J.* **126**:1 (2005), 1–52. MR 2005i:16065

[Białynicki-Birula 1973] A. Białynicki-Birula, "Some theorems on actions of algebraic groups", *Ann. of Math.* (2) **98** (1973), 480–497. MR 51 #3186 Zbl 0275.14007

[Caldero and Chapoton 2006] P. Caldero and F. Chapoton, "Cluster algebras as Hall algebras of quiver representations", *Comment. Math. Helv.* **81**:3 (2006), 595–616. MR 2008b:16015 Zbl 1119.16013

[Caldero and Keller 2006] P. Caldero and B. Keller, "From triangulated categories to cluster algebras, II", *Ann. Sci. École Norm. Sup.* (4) **39**:6 (2006), 983–1009. MR 2008m:16031 Zbl 1115.18301

[Caldero and Keller 2008] P. Caldero and B. Keller, "From triangulated categories to cluster algebras", *Invent. Math.* **172**:1 (2008), 169–211. MR 2009f:16027 Zbl 1141.18012

[Caldero and Reineke 2008] P. Caldero and M. Reineke, "On the quiver Grassmannian in the acyclic case", *J. Pure Appl. Algebra* **212**:11 (2008), 2369–2380. MR 2009f:14102 Zbl 1153.14032

[Caldero and Zelevinsky 2006] P. Caldero and A. Zelevinsky, "Laurent expansions in cluster algebras via quiver representations", *Mosc. Math. J.* **6**:3 (2006), 411–429. MR 2008j:16045 Zbl 1133.16012

[Cerulli Irelli 2009] G. Cerulli Irelli, "Canonically positive basis of cluster algebras of type $a_2^{(1)}$", preprint, 2009. arXiv 0904.2543

[Cerulli Irelli 2011a] G. Cerulli Irelli, "Positivity in skew-symmetric cluster algebras of finite type", preprint, 2011. arXiv 1102.3050

[Cerulli Irelli 2011b] G. Cerulli Irelli, "Quiver grassmannians associated with string modules", *J. Algebraic Combin.* **33**:2 (2011), 259–276. MR 2765325 Zbl 05852822

[Cerulli Irelli and Labardini-Fragoso 2011] G. Cerulli Irelli and D. Labardini-Fragoso, "Quivers with potentials associated to triangulated surfaces, III: tagged triangulations and cluster monomials", (2011). arXiv 1108.1774

[Cerulli Irelli et al. 2012] G. Cerulli Irelli, G. Dupont, and F. Esposito, "A homological interpretation of the transverse quiver grassmannians", *Algebr. Represent. Theory* (2012).

[Chriss and Ginzburg 1997] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, Boston, 1997. MR 98i:22021 Zbl 0879.22001

[Crawley-Boevey 1989] W. W. Crawley-Boevey, "Maps between representations of zero-relation algebras", *J. Algebra* **126**:2 (1989), 259–263. MR 90k:16035 Zbl 0685.16018

[Derksen et al. 2010] H. Derksen, J. Weyman, and A. Zelevinsky, "Quivers with potentials and their representations, II: Applications to cluster algebras", *J. Amer. Math. Soc.* **23**:3 (2010), 749–790. MR 2629987

[Dupont 2010] G. Dupont, "Transverse quiver grassmannians and bases in affine cluster algebras", *Algebra Number Theory* **4**:5 (2010), 599–624. MR 2011i:13027 Zbl 05779836

[Fomin and Zelevinsky 2002] S. Fomin and A. Zelevinsky, "Cluster algebras, I: Foundations", *J. Amer. Math. Soc.* **15**:2 (2002), 497–529. MR 2003f:16050

[Fomin and Zelevinsky 2003a] S. Fomin and A. Zelevinsky, "Cluster algebras. II. Finite type classification", *Invent. Math.* **154**:1 (2003), 63–121. MR 2004m:17011

[Fomin and Zelevinsky 2003b] S. Fomin and A. Zelevinsky, "Cluster algebras: notes for the CDM-03 conference", pp. 1–34 in *Current developments in mathematics,* 2003, edited by B. Mazur et al., Int. Press, Somerville, MA, 2003. MR 2005m:05235 Zbl 1119.05108

[Fomin and Zelevinsky 2007] S. Fomin and A. Zelevinsky, "Cluster algebras, IV: Coefficients", *Compos. Math.* **143**:1 (2007), 112–164. MR 2008d:16049

[Geiß et al. 2012] C. Geiß, B. Leclerc, and J. Schröer, "Generic bases for cluster algebras and the chamber ansatz", *J. Amer. Math. Soc.* **25**:1 (2012), 21–76. MR 2833478

[Hernandez and Leclerc 2010] D. Hernandez and B. Leclerc, "Cluster algebras and quantum affine algebras", *Duke Math. J.* **154**:2 (2010), 265–341. MR 2011g:17027 Zbl 05788166

[Keller 2010] B. Keller, "Cluster algebras, quiver representations and triangulated categories", pp. 76–160 in *Triangulated categories*, edited by T. Holm et al., London Math. Soc. Lecture Note Ser. **375**, Cambridge Univ. Press, Cambridge, 2010. MR 2011h:13033 Zbl 1215.16012

[Kronecker 1890] L. Kronecker, "Algebraische Reduction der Schaaren bilinearer Formen", *Sitzungsber. Akad. Berlin* (1890), 1225–1237. Reprinted as pp. 139–156 in his *Werke*, vol. 3, Teubner, Leipzig, 1931. JFM 22.0169.01

[Lampe 2011] P. Lampe, "A quantum cluster algebra of Kronecker type and the dual canonical basis", *Int. Math. Res. Not.* **2011**:13 (2011), 2970–3005. MR 2817684 Zbl 05931742

[Nakajima 2011] H. Nakajima, "Quiver varieties and cluster algebras", *Kyoto J. Math.* **51**:1 (2011), 71–126. MR 2784748 Zbl 1223.13013

[Poettering 2010] N. Poettering, "Euler characteristic of quiver Grassmannians and Ringel–Hall algebras of string algebras", preprint, 2010. To appear in Algebr. Represent. Theory. arXiv 1002.3040

[Ringel 1998] C. M. Ringel, "Exceptional modules are tree modules", *Linear Algebra Appl.* **275/276** (1998), 471–493. MR 2000c:16020 Zbl 0964.16014

[Sherman and Zelevinsky 2004] P. Sherman and A. Zelevinsky, "Positivity and canonical bases in rank 2 cluster algebras of finite and affine types", *Mosc. Math. J.* **4**:4 (2004), 947–974. MR 2006c: 16052 Zbl 1103.16018

[Szanto 2011] C. Szanto, "On the cardinalities of Kronecker quiver grassmannians", *Math. Z.* **269**:3-4 (2011), 833–846.

[Zelevinsky 2007] A. Zelevinsky, "Semicanonical basis generators of the cluster algebra of type $A_1^{(1)}$", *Electron. J. Combin.* **14**:1 (2007), Note 4, 5 pp. MR 2008j:05375

cerulli@mat.uniroma1.it   *Università degli studi di Padova, Dipartimento di Matematica Pura ed Applicata, Via Trieste 63, I-35121 Padova, Italy*

Current address: *Università di Roma "La Sapienza", Dipartimento di Matematica, Piazzale Aldo Moro 5, I-00185 Roma, Italy*

esposito@math.unipd.it   *Università degli studi di Padova, Dipartimento di Matematica Pura ed Applicata, Via Trieste 63, I-35121 Padova, Italy*

Algebra & Number Theory

msp.berkeley.edu/ant

Algebra & Number Theory

Volume 5   No. 6   2011