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We study cuspidal automorphic representations of unitary groups of $2n$ variables with ϵ-factor -1 and their central L-derivatives by constructing their arithmetic theta liftings, which are Chow cycles of codimension n on Shimura varieties of dimension $2n - 1$ of certain unitary groups. We give a precise conjecture for the arithmetic inner product formula, originated by Kudla, which relates the height pairing of these arithmetic theta liftings and the central L-derivatives of certain automorphic representations. We also prove an identity relating the archimedean local height pairing and derivatives of archimedean Whittaker functions of certain Eisenstein series, which we call an arithmetic local Siegel–Weil formula for archimedean places. This provides some evidence toward the conjectural arithmetic inner product formula.

1. Introduction

Rallis [1982] developed a formula, called the Rallis inner product formula, to determine whether a certain theta lifting is vanishing. It is used to calculate the Petersson inner product of two automorphic forms on an orthogonal group lifted from those on a symplectic group through the Weil representation. It turns out, using the Siegel–Weil formula, that the inner product is related to a diagonal integral on the doubling symplectic group of the original automorphic forms with certain Eisenstein series. This doubling method was generalized to other cases by Gelbart et al. [1987]. If we assume that the automorphic forms we lift are

cuspidal, this diagonal integral is in fact Eulerian, and decomposes into so-called local zeta integrals, which are closely related to the L-factors of the corresponding representations. In fact, Gelbart et al. prove in many cases that when everything is unramified, the local zeta integral is just the local Langlands L-factor divided by a product of certain Tate L-factors. Li [1992] extended this result for unitary groups.

Since, at that time, the key ingredient, the Siegel–Weil formula, was only known "above the convergence line", which means that the group we lift to should have a certain larger size than the group we lift from, the inner product formula could only regard the values of global L-functions at points far to the right of the central point. For example, if we lift forms from $\mathsf{Sp}(n)$ (rank $2n$ matrices) to $\mathsf{O}(2m)$ then $m > 2n + 1$. In fact, using the nonvanishing result of these L-values, Li [1992] was able to prove some nonvanishing results for the cohomology of certain arithmetic quotients, which is an important and well-known application of the inner product formula. Kudla and Rallis [1994] extended the Siegel–Weil formula with great generality for symplectic-orthogonal pairs and Ichino [2004; 2007] did that for unitary pairs using the similar idea of Kudla and Rallis. Now we can extend Rallis' original inner product formula "below the convergence line" (after regularization if necessary) which enables us to say some words about the global L-values at other points, especially the central point $\frac{1}{2}$.

Now let us stick to a special case where the dual pair are unitary groups with the same even rank, hence the related L-value is the central value. Suppose that E/F is a quadratic imaginary extension of a totally real field with τ the nontrivial Galois involution. Let us denote by $H' \cong \mathsf{U}(n, n)_F$ the unique quasisplit unitary group of rank $2n$ (with respect to τ) and by H another unitary group of the same rank. Let π be an irreducible cuspidal automorphic representation of H' and let f be a nonzero form inside it. Choosing an auxiliary Schwartz function and using the Weil representation, we get an automorphic form on H called the (regularized, if necessary) theta lifting of f. If the global epsilon factor $\epsilon(\frac{1}{2}, \pi) = 1$, then among all pure inner forms of H, the theta lifting of forms inside π should always vanish except for one possible $H = H(\pi)$. For this unitary group, the only obstruction to some theta lifting being nonvanishing is that $L(\frac{1}{2}, \pi) = 0$.

The theory is not complete since we miss another half, ones whose $\epsilon(\frac{1}{2}, \pi) = -1$. If this is the case, then $L(\frac{1}{2}, \pi)$ is automatically 0 and all theta lifting to all possible unitary groups of the same rank should vanish. A great observation of Kudla [1997; 2002; 2003; Kudla et al. 2006] was that (in the symplectic-orthogonal case) there should exist some "arithmetic theta lifting" which is a cycle on certain (integral models of a) Shimura variety and an "arithmetic Siegel–Weil formula". This arithmetic Siegel–Weil formula should be related to the central derivative $L'(\frac{1}{2}, \pi)$ of the global L-function instead of the central value $L(\frac{1}{2}, \pi)$ via an arithmetic analogue of Rallis' inner product formula (see [Kudla 2003, Section 11]). For this

direction, a particular form of the arithmetic inner product formula has already been obtained, for holomorphic cuspidal newforms of $\mathsf{PGL}(2)_{\mathbb{Q}}$ of weight 2 and level $\Gamma_0(N)$ for N square-free with epsilon factor -1, in [Kudla et al. 2006, Theorem 9.2.4], based on a lot of previous work.

In this paper, we will set up a general formulation and a more explicit formulation extending the above line. We will establish the general conjecture of an explicit form of the arithmetic inner product formula assuming some well-accepted properties of Arthur packets, the existence of Beilinson–Bloch height pairing when $F \neq \mathbb{Q}$, and some other auxiliary conjectures when $F = \mathbb{Q}$, all of which can be proved when $n = 1$. We also prove some partial results toward the general arithmetic inner product formula, namely the modularity theorem on the (noncompactified) generating series and the arithmetic analogue of the local Siegel–Weil formula at archimedean places. In the second part of this paper [Liu 2011], we will give a full proof of the arithmetic inner product formula for $n = 1$.

Before we state the main results, we would like to remark that the L-function appearing here is the so-called doubling L-series defined by Piatetski-Shapiro and Rallis (see [Harris et al. 1996] for a detailed definition for the unitary group case). This L-function is conjectured to coincide with the Langlands L-function of the standard base change $\mathsf{BC}(\pi)$ which is an irreducible automorphic representation of $\mathsf{GL}_{2n,E}$. Hence the set of central L-derivatives which can be computed by the arithmetic inner product formula at least contains those $L'(\frac{1}{2}, \Pi)$, where Π is an irreducible cuspidal automorphic representation of $\mathsf{GL}_{2n,E}$ such that $\Pi^{\vee} \cong \Pi^{\tau}$, $\Pi \otimes \epsilon_{E/F}$ is $\mathsf{GL}_{2n,F}$-distinguished, and Π_{ι} is the base change of the trivial representation of $\mathsf{U}(2n, 0)_{\mathbb{R}}$ for any archimedean place ι, where $\epsilon_{E/F}$ is the associated quadratic character by class field theory. In particular, when $n = 1$, this set is exactly the same as the one of central L-derivatives appearing in the (complete version of the) Gross–Zagier formula recently proved by X. Yuan, S.-W. Zhang and W. Zhang [Yuan et al. 2011].

More precisely, let E/F, τ, and $\epsilon_{E/F}$ be as above and ψ be an additive character of $F \backslash \mathbb{A}_F$, standard at archimedean places, which is used to define Weil representations and Fourier coefficients. For $n \geq 1$, let H_n be the unitary group over F such that for any F-algebra R, $H_n(R) = \{h \in \mathsf{GL}_{2n}(E \otimes_F R) \mid {}^t h^{\tau} w_n h = w_n\}$ where

$$w_n = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}.$$

The center of H_n is the F-torus $E^{\times,1} = \ker[\mathsf{Nm} : E^{\times} \to F^{\times}]$. Let π be an irreducible cuspidal automorphic representation of H_n and $\pi^{\vee}$ its contragredient. Let χ be a character of $\mathbb{A}_E^{\times}$ which is trivial on $E^{\times} \mathbb{A}_F^{\times}$. We can associate with χ a sequence of integers $\mathfrak{k}^{\chi} = (\mathfrak{k}_{\iota}^{\chi})_{\iota}$ for each archimedean place ι of F whose definition is in Section 3A. In particular, they are all even integers for this χ.

By the theta dichotomy proved in [Paul 1998; Gong and Grenié 2011], we get a factor $\epsilon(\pi, \chi)$ (see Section 2D for a precise definition) which is the product of the local factors $\epsilon(\pi_v, \chi_v)$ for each place v of F, such that $\epsilon(\pi_v, \chi_v) \in \{\pm 1\}$ and $\epsilon(\pi_v, \chi_v) = 1$ for almost all v. Although it is conjectured that this $\epsilon(\pi_v, \chi_v)$ is related to the local ϵ-factor in representation theory (see [Harris et al. 1996]), it is not by our definition. From these local factors, we can construct a hermitian space $\mathbb{V}(\pi, \chi)$ over $\mathbb{A}_E$ of rank $2n$ which is coherent (resp. incoherent) if $\epsilon(\pi, \chi) = 1$ (resp. -1) (for this terminology, see Section 2A). When $\epsilon(\pi, \chi) = 1$, we get the usual (extended) Rallis inner product formula. Moreover, we prove in Section 2 that when $\epsilon(\pi, \chi) = -1$, $L(\frac{1}{2}, \pi, \chi) = 0$.

Now let us assume $\epsilon(\pi, \chi) = -1$ and further assume that π_∞ is a *discrete series of weight* $(n - \mathfrak{k}^\chi/2, n + \mathfrak{k}^\chi/2)$. Careful readers may find that this is not standard terminology. We will now explain this in a more general situation. We say that a discrete series representation π of $\mathsf{U}(r, r)_\mathbb{R}$ ($r \geq 1$) is of weight (a, b) for some integers a, b such that $a + b$ is positive if the minimal type of the maximal compact subgroup $\mathsf{U}(r)_\mathbb{R} \times \mathsf{U}(r)_\mathbb{R} \subset \mathsf{U}(r, r)_\mathbb{R}$, for which we choose the standard embedding elaborated at the beginning of Section 4A (although we only write it for r even there, but it is the same for all r), is $\det_1^a \boxtimes \det_2^{-b}$, where $\det_i$ is the determinant on the i-th $\mathsf{U}(n)_\mathbb{R}$. One can prove that it is the theta correspondence (under certain Weil representation) of the trivial representation from $\mathsf{U}(a + b, 0)_\mathbb{R}$ to $\mathsf{U}(r, r)_\mathbb{R}$. Finally, the first sentence in the paragraph means that for each ι, π_ι is a discrete series of weight $(n - \mathfrak{k}_\iota^\chi/2, n + \mathfrak{k}_\iota^\chi/2)$. For π as above, the corresponding $\mathbb{V}(\pi, \chi)$ is incoherent and totally positive definite.

Moreover, for any hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ which is incoherent and totally positive-definite of rank $m \geq 2$, let $\mathbb{H} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathsf{U}(\mathbb{V})$ be the corresponding unitary group. Then we can construct a projective system of unitary Shimura varieties $(\mathrm{Sh}_K(\mathbb{H}))_K$ of dimension $m - 1$, smooth and quasiprojective over E where K is a sufficiently small open compact subgroup of $\mathbb{H}(\mathbb{A}_f)$ (for the construction, see Section 3A). Let χ be a character of $E^\times \backslash \mathbb{A}_E^\times$ such that $\chi|_{\mathbb{A}_F^\times} = \epsilon_{E/F}^m$ and $1 \leq r < m$ another integer. For any Schwartz function $\phi \in \mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_\infty K}$ (see Section 3A for notation), we can define Kudla's generating series $Z_\phi(g)$ for any $g \in H_r(\mathbb{A}_F)$ which takes values as formal sums in $\mathrm{CH}^r(\mathrm{Sh}(\mathbb{H}))_\mathbb{C}$: the inductive limit of Chow groups of codimension r cycles with complex coefficients on the Shimura varieties. For any linear functional ℓ of $\mathrm{CH}^r(\mathrm{Sh}(\mathbb{H}))_\mathbb{C}$, we can evaluate it on the generating series and hence obtain a smooth function $\ell(Z_\phi)(g)$ on $H_r(\mathbb{A}_F)$ provided that it is absolutely convergent. We prove in Section 3B the following theorem on the *modularity of the generating series*:

Theorem 3.5. (1) *If $\ell(Z_\phi)(g)$ is absolutely convergent, then it is an automorphic form of $H_r(\mathbb{A}_F)$. Moreover, $\ell(Z_\phi)_\infty$ is in a discrete series representation of weight $((m + \mathfrak{k}^\chi)/2, (m - \mathfrak{k}^\chi)/2)$.*

(2) *If $r = 1$, then $\ell(Z_\phi)(g)$ is absolutely convergent for any ℓ.*

There is also a version in the case of symplectic-orthogonal pairs which is proved in [Yuan et al. 2009]. The proof in the unitary case is similar to that of [Yuan et al. 2009] using the induction process on the codimension. Actually, the proof of the case $r = 1$ uses the result in the symplectic-orthogonal case. We will also state another version for the compactified generating series in Section 3C if the Shimura varieties are not proper, which happens in particular when $F = \mathbb{Q}$ and $m > 2$, but so far we are not able to prove it.

For simplicity, let us assume $F \neq \mathbb{Q}$ in the following discussion; then $\mathsf{Sh}_K(\mathbb{H})$ is projective. Let $m = 2n$. Similarly to [Kudla 2003; Kudla et al. 2006], for any $f \in \pi$ and Schwartz function $\phi \in \mathscr{S}(\mathbb{V}^n)^{\cup_\infty K}$, we construct a cycle Θ_ϕ^f, called the arithmetic theta lifting, which is a cycle on $\mathsf{Sh}_K(\mathbb{H})$ of codimension n. On the contragredient side, we also have $\Theta_{\phi^\vee}^{f^\vee}$ for $f^\vee \in \pi^\vee$. The definition of Θ_ϕ^f is basically the integration of f with the generating series, that is,

$$\Theta_\phi^f = \int_{H_n(F) \backslash H_n(\mathbb{A}_F)} f(g) Z_\phi(g) \, dg,$$

which is a formal sum in $\mathsf{CH}^n(\mathsf{Sh}(\mathbb{H}))_{\mathbb{C}}$ but whose (Betti) cohomology class is well-defined. We show in Section 3D that it is cohomologically trivial assuming certain properties of Arthur packets. Hence we can consider the (conjectural if $n > 1$) Beilinson–Bloch height pairing (see [Bloch 1984; Beĭlinson 1987]) $\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{BB}}$. Analogous to the coherent case, when $\mathbb{V} \neq \mathbb{V}(\pi, \chi)$, one easily shows that $\Theta_\phi^f = 0$. If $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$, we conjecture the following:

Conjecture 3.11 (arithmetic inner product formula). *Let π, χ be as above (in particular, $\epsilon(\pi, \chi) = -1$) and $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$. Then, for any $f \in \pi$, $f^\vee \in \pi^\vee$ and any $\phi, \phi^\vee \in \mathscr{S}(\mathbb{V}^n)^{\cup_\infty}$ decomposable, we have*

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{BB}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee),$$

where Z^* are normalized local zeta integrals (see Section 2C) of which almost all are 1.

We remark that this conjectural arithmetic inner product formula is different from that of Kudla (see, for example, [Kudla 2003, Section 11]) in the sense that our arithmetic theta lifting Θ_ϕ^f is canonically defined on the Shimura variety, not on any integral model. More important, it is cohomologically trivial, at least when the Shimura variety is proper, hence we can talk about its canonical height through the conjectural Beilinson–Bloch height pairing.

As we do in [Liu 2011], to prove the arithmetic inner product formula, we introduce analytic kernel functions and geometric kernel functions which carry over

all cusp forms simultaneously. The former is the derivative of certain Eisenstein series on the doubling group which deals with derivatives of L-functions, while the latter is the height pairing of the generating series which deals with that of the arithmetic theta lifting. Both kernel functions can be essentially decomposed as a sum of local terms for each place v of F. Hence we should compare them place by place. At each archimedean place of F, it turns out that we need to compare the derivatives of certain Whittaker functions with the local height pairing of special subdomains of the hermitian symmetric domain D_{m-1} of $\mathsf{U}(m-1, 1)_{\mathbb{R}}$. Let V be the complex hermitian space of signature $(m, 0)$ and V' that of signature $(m - 1, 1)$. For any nonzero $x \in V'$, we can associate a hermitian symmetric subdomain $D_x \subset D_{m-1}$ and a Green's function $\xi(x)$ to it (see Section 4B). It is of codimension 1 if the inner product of x is positive and empty if not. The Green's function, originally constructed in [Kudla 1997], is related to the Kudla–Millson form [1986] and very closely related to derivatives of Whittaker functions. But this is not the admissible Green's function used in the Beilinson–Bloch height pairing (at an archimedean place); instead, they have a certain relation which will be elaborated in [Liu 2011] when $n = 1$ and we expect that they relate for general n. For any $\vec{x} = (x_1, \ldots, x_m) \in V'^m$ such that $x_1, \ldots, x_m$ are linearly independent, we get an intersection number $H(\vec{x})_\infty$ (with respect to the Green's functions $\xi(x_i)$). It is clear that the moment matrix $T(\vec{x})$ is in $\mathrm{Her}_m(\mathbb{C})$: the set of complex hermitian matrices of rank m, and whose signature is $(m - 1, 1)$. The intersection number $H(\vec{x})_\infty$ only depends on $T = T(\vec{x})$; hence it makes sense to write it as $H(T)_\infty$. In Section 4, we prove the following *local arithmetic Siegel–Weil formula* at an archimedean place:

Proposition 4.5, Theorem 4.17. *Let $T \in \mathrm{Her}_m(\mathbb{C})$ be nonsingular with $\mathrm{sign}(T) = (p, q)$.*

(1) $\mathrm{ord}_{s=0} W_T(s, e, \Phi^0) \geq q$.

(2) *If T is positive definite, that is, $q = 0$, we have*

$$W_T(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \operatorname{tr} T}.$$

(3) *If T is of signature $(m - 1, 1)$, we have*

$$W'_T(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \operatorname{tr} T} H(T)_\infty.$$

Here, $\Phi^0 \in \mathscr{S}(V^m)$ is the standard Gaussian; $W_T(0, e, \Phi^0)$ is the T-th Whittaker integral at $s = 0$ and $e \in \mathsf{U}(m, m)_{\mathbb{R}}$; $\gamma_V \in \{\pm 1\}$ is the Weil constant and $\Gamma_m(m)$ is a product of certain usual gamma functions (see Lemma 4.3).

The study of derivatives of L-functions starts from the celebrated paper [Gross

and Zagier 1986], which studied the relation between the central derivatives of Rankin L-series and the heights of Heegner points on modular curves thus obtaining the famous Gross–Zagier formula predicted by the Birch and Swinnerton-Dyer conjecture. Later, this was generalized to the case of Shimura curves over totally real fields in [Zhang 2001a; 2001b]. The complete version of the Gross–Zagier formula has been achieved in [Yuan et al. 2011]. Moreover, Bruinier and Yang [2009] used regularized theta lifting and related the inner product to L-derivatives to give another proof of the original Gross–Zagier formula. A certain p-adic (or rigid analytic) version of the Gross–Zagier formula has been studied in [Bertolini and Darmon 1997; 1998].

There is another approach to studying L-derivatives via doubling integrals and in general derivatives of Eisenstein series, discovered by Kudla [1997; 2002; 2003; Kudla et al. 2006]. He proposed the project of the arithmetic Siegel–Weil formula and proved a special form of the arithmetic inner product formula with Rapoport and Yang. Our work follows the second approach, establishing an explicit form of the arithmetic inner product formula and, together with [Liu 2011], proving the complete version of the arithmetic inner product formula in the case of unitary groups of two variables over totally real fields.

For applications of the arithmetic inner product formula, we are able to construct nontorsion Chow cycles instead of cohomology classes in the classical case if the central derivative is nonzero. In the case of the Gross–Zagier formula [Gross and Zagier 1986; Yuan et al. 2011] and the arithmetic triple product formula [Yuan et al. 2010], we have already seen many interesting and important applications of nontorsion cycles on certain Shimura varieties.

For the positivity of the global L-function at the central point, which is a consequence of the generalized Riemann hypothesis, it is obvious that the positivity of normalized local zeta integrals (at the point 0) will imply the positivity of the central value $L(\frac{1}{2}, \pi, \chi)$. Moreover, through the arithmetic inner product formula, the positivity of normalized local zeta integrals plus the (conjectural) positivity of the Beilinson–Bloch height pairing will imply the positivity of the central derivative $L'(\frac{1}{2}, \pi, \chi)$, which is again a consequence of the generalized Riemann hypothesis!

Now we state the outline of the paper. In Section 2, we review the classical Siegel–Weil formula; in Section 2A is its generalization of the work of Ichino, and in Section 2B the doubling integral introduced by Piatetski-Shapiro and Rallis. We introduce the definition of the L-function and its relation with the local zeta integral in Section 2C. In Section 2D, we introduce the Rallis inner product formula for the central L-value in the coherent case. In Section 2E, we derive a formula for central L-derivatives using derivatives of Eisenstein series in the incoherent case.

In Section 3, we treat the geometric part of the theory. We introduce the notion of Shimura varieties of unitary groups, Kudla's special cycle, and generating series

in Section 3A. Section 3B is devoted to proving Theorem 3.5. We introduce the canonical smooth compactification in the case of higher dimensions in Section 3C. In Section 3D, we define the arithmetic theta lifting and formulate the conjecture above and two auxiliary conjectures.

Section 4 is devoted to proving Proposition 4.5, Theorem 4.17 and hence finishing the archimedean comparison on the Shimura variety in the global setting.

In the Appendix we calculate the theta correspondence for spherical representations at a nonarchimedean place. This result is a key step in the proof of Proposition 3.9. Our calculation for the unitary case follows exactly that of the symplectic-orthogonal case which is proved in [Rallis 1984].

The following conventions hold throughout this paper.

- $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = (\varprojlim_N \mathbb{Z}/N\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles and $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ is the ring of full adèles.

- For any number field K, $\mathbb{A}_K = \mathbb{A} \otimes_{\mathbb{Q}} K$, $\mathbb{A}_{f,K} = \mathbb{A}_f \otimes_{\mathbb{Q}} K$, $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R}$, and $\Gamma_K = \mathsf{Gal}(K^{\mathsf{ac}}/K)$ is the Galois group of K.

- As usual, for a subset S of places, $-_S$ (resp. $-^S$) means the S-component (resp. component away from S) for the corresponding (decomposable) adèlic object; $-_\infty$ (resp. $-_f$) is the infinite (resp. finite) part.

- The symbols Tr and Nm mean the trace (resp. reduced trace) and norm (resp. reduced norm) if they apply to fields or rings of adèles (resp. simple algebras), and tr means the trace for matrix and linear transforms.

- $\mathbf{1}_n$ and $\mathbf{0}_n$ are the $n \times n$ identity and zero matrices; ${}^t g$ is the transpose of a matrix g.

- All (skew-)hermitian spaces and quadratic spaces are assumed to be nondegenerate.

- For a scheme X over a field K, we let $\mathsf{Pic}(X)$ be the Picard group of X over K, not the Picard scheme.

2. Doubling method

2A. *Siegel–Weil formulae.* In this section, we will review the classical Siegel–Weil formula and some generalizations to be used later.

Let F be a totally real field and E a totally imaginary quadratic extension of F. We denote by τ the nontrivial element in $\mathsf{Gal}(E/F)$ and by $\epsilon_{E/F} : \mathbb{A}_F^\times / F^\times \to \{\pm 1\}$ the associated character by class field theory. Let Σ (resp. Σ_f; resp. Σ_∞) be the set of all places (resp. finite places; resp. infinite places) of F, and Σ°, Σ_f°, and Σ_∞° those of E. We fix a nontrivial additive character ψ of $\mathbb{A}_F/F$.

For positive integer r, we denote by W_r the standard skew-hermitian space over E with respect to the involution τ, which has a skew-hermitian form $\langle \cdot, \cdot \rangle$ such

that there is an E-basis $\{e_1, \ldots, e_{2r}\}$ satisfying $\langle e_i, e_j \rangle = 0$, $\langle e_{r+i}, e_{r+j} \rangle = 0$, and $\langle e_i, e_{r+j} \rangle = \delta_{ij}$ for $1 \le i, j \le r$. Let $H_r = \mathsf{U}(W_r)$ be the unitary group of W_r which is a reductive group over F. The group $H_r(F)$, in which F can be itself or its completion at some place, is generated by the standard parabolic subgroup $P_r(F) = N_r(F)M_r(F)$ and the element w_r. More precisely,

$$N_r(F) = \left\{ n(b) = \begin{pmatrix} \mathbf{1}_r & b \\ & \mathbf{1}_r \end{pmatrix} \;\middle|\; b \in \mathsf{Her}_r(E) \right\},$$

$$M_r(F) = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{\tau,-1} \end{pmatrix} \;\middle|\; a \in \mathsf{GL}_r(E) \right\},$$

and

$$w_r = \begin{pmatrix} & \mathbf{1}_r \\ -\mathbf{1}_r & \end{pmatrix}.$$

Here $\mathsf{Her}_r(E) = \{ b \in \mathsf{Mat}_r(E) \mid b^\tau = {}^t b \}$.

Degenerate principal series and Eisenstein series. We fix a place $v \in \Sigma$ and suppress it from the notation. Thus $F = F_v$ is a local field of characteristic zero, $E = E_v$ is a quadratic extension of F which may be split, and $H_r = H_{r,v} = H_r(F_v)$ is a local reductive group. Also, we denote by $\mathcal{K}_r$ its maximal compact subgroup which is the intersection of H_r with $\mathsf{GL}_{2n}(\mathcal{O}_E)$ (resp. is isomorphic to $\mathsf{U}(r) \times \mathsf{U}(r)$) if v is finite (resp. if v is infinite). For $s \in \mathbb{C}$ and a character χ of $E^\times$, we denote by $I_r(s, \chi) = \mathsf{s\text{-}Ind}_{P_r}^{H_r}(\chi |\cdot|_E^{s+r/2})$ the degenerate principal series representation (see [Kudla and Sweet 1997]) of H_r, where $\mathsf{s\text{-}Ind}$ means the nonnormalized smooth induction. Precisely, it realizes on the space of $\mathcal{K}_r$-finite functions φ_s on H_r satisfying

$$\varphi_s(n(b)m(a)g) = \chi(\det a)|\det a|_E^{s+r/2} \varphi_s(g)$$

for all $g \in H_r$, $m(a) \in M_r$, and $n(b) \in N_r$. A (holomorphic) section φ_s of $I_r(s, \chi)$ is called *standard* if its restriction to $\mathcal{K}_r$ is independent of s. It is called *unramified* if it takes value 1 on $\mathcal{K}_r$.

Now we view F and E as number fields. For a character χ of $\mathbb{A}_E^\times$ which is trivial on $E^\times$ and $s \in \mathbb{C}$, we have an admissible representation $I_r(s, \chi) = \bigotimes' I_r(s, \chi_v)$ of $H_r(\mathbb{A}_F)$, where the restricted tensor product is taken with respect to the unramified sections. For any standard section $\varphi_s = \otimes \varphi_{s,v} \in I_r(s, \chi)$, we can define an Eisenstein series as

$$E(g, \varphi_s) = \sum_{\gamma \in P_r(F) \backslash H_r(F)} \varphi_s(\gamma g).$$

The series is absolutely convergent if $\Re(s) > r/2$ and has a meromorphic continuation to the entire complex plane which is holomorphic at $s = 0$ (see [Tan 1999, Proposition 4.1]).

Hermitian spaces, Weil representations, and theta functions. Let us have a quick review of the classification of (nondegenerate) hermitian spaces. Suppose $v \in \Sigma_f$ and E is nonsplit at v. Then, up to isometry, there are two different hermitian spaces over E_v of dimension $m \geq 1$: $V^\pm$, defined by

$$\epsilon(V^\pm) = \epsilon_{E/F}\big((-1)^{m(m-1)/2} \det V^\pm\big) = \pm 1.$$

Suppose $v \in \Sigma_f$ and E is split at v. Then, up to isometry, there is only one hermitian space V^+ over E_v of dimension m. Suppose $v \in \Sigma_\infty$. Then, up to isometry, there are $m+1$ different hermitian spaces over E_v of dimension m: V_s with signature $(s, m-s)$ where $0 \leq s \leq m$. In the latter two cases, we can still define $\epsilon(V)$ in the same way. In the global case, up to isometry, all hermitian spaces V over E of dimension m are classified by signatures at infinite places and $\det V \in F^\times / \mathrm{Nm} E^\times$; particularly, V is determined by all $V_v = V \otimes_F F_v$. In general, we will also consider a hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank m. In this case, $\mathbb{V}$ is nondegenerate if there is a basis under which the representing matrix is invertible in $\mathrm{GL}_m(\mathbb{A}_E)$. For any place $v \in \Sigma$, we let $\mathbb{V}_v = \mathbb{V} \otimes_{\mathbb{A}_F} F_v$, $\mathbb{V}_f = \mathbb{V} \otimes_{\mathbb{A}_F} \mathbb{A}_{f,F}$, and define $\Sigma(\mathbb{V}) = \{v \in \Sigma \mid \epsilon(\mathbb{V}_v) = -1\}$, which is a finite set, and $\epsilon(\mathbb{V}) = \prod \epsilon(\mathbb{V}_v)$. We say $\mathbb{V}$ is *coherent* (resp. *incoherent*) if the cardinality of $\Sigma(\mathbb{V})$ is even (resp. odd), that is, $\epsilon(\mathbb{V}) = 1$ (resp. -1). By the Hasse principle, there is a hermitian space V over E such that $\mathbb{V} \cong V \otimes_F \mathbb{A}_F$ if and only if $\mathbb{V}$ is coherent. These two terminologies are introduced in the orthogonal case in [Kudla and Rallis 1994]; see also [Kudla 1997].

We fix a place $v \in \Sigma$ and suppress it from the notation. For a hermitian space V of dimension m with hermitian form $(\cdot, \cdot)$ and a positive integer r, we can construct a symplectic space $W = \mathrm{Res}_{E/F} W_r \otimes_E V$ of dimension $4rm$ over F with the skew-symmetric form $\frac{1}{2} \mathrm{Tr}_{E/F} \langle \cdot, \cdot \rangle^\tau \otimes (\cdot, \cdot)$. We let $H = \mathrm{U}(V)$ be the unitary group of V and $\mathscr{S}(V^r)$ the space of Schwartz functions on V^r. Given a character χ of $E^\times$ satisfying $\chi\big|_{F^\times} = \epsilon_{E/F}^m$, we have a splitting homomorphism

$$\tilde{\iota}_{(\chi, \mathbf{1})} : H_r \times H \to \mathrm{Mp}(W)$$

lifting the natural map $\iota : H_r \times H \to \mathrm{Sp}(W)$ (see [Harris et al. 1996, Section 1]). We thus have a Weil representation (with respect to ψ) $\omega_\chi = \omega_{\chi,\psi}$ of $H_r \times H$ on the space $\mathscr{S}(V^r)$. Explicitly, for $\phi \in \mathscr{S}(V^r)$ and $h \in H$,

- $\omega_\chi(n(b))\phi(x) = \psi(\mathrm{tr}\, bT(x))\phi(x)$,

- $\omega_\chi(m(a))\phi(x) = |\det a|_E^{m/2} \chi(\det a)\phi(xa)$,

- $\omega_\chi(w_r)\phi(x) = \gamma_V \widehat{\phi}(x)$, and

- $\omega_\chi(h)\phi(x) = \phi(h^{-1}x)$,

where $T(x) = \frac{1}{2}\big((x_i, x_j)\big)_{1 \leq i,j \leq r}$ is the moment matrix of x, γ_V is the Weil constant associated to the underlying quadratic space of V (and also ψ), and $\widehat{\phi}$ is the Fourier

transform

$$\widehat{\phi}(x) = \int_{V^r} \phi(y) \psi\left(\tfrac{1}{2} \mathrm{Tr}_{E/F}(x, {}^t y)\right) dy$$

using the self-dual measure dy on V^r with respect to ψ. Taking the restricted tensor product over all local Weil representations, we get a global $\mathscr{S}(V^r) := \bigotimes' \mathscr{S}(V_v^r)$ as a representation of $H_r(\mathbb{A}_F) \times H(\mathbb{A}_F)$.

Now for V over E, χ a character of $\mathbb{A}_E^\times/E^\times$ such that $\chi\big|_{\mathbb{A}_F^\times} = \epsilon_{E/F}^m$, and $\phi \in \mathscr{S}(V^r)$, we define the theta function

$$\theta(g, h; \phi) = \sum_{x \in V^r(E)} \omega_\chi(g, h) \phi(x),$$

which is a smooth, slowly increasing function of $H_r(F) \backslash H_r(\mathbb{A}_F) \times H(F) \backslash H(\mathbb{A}_F)$, and consider the integral

$$I_V(g, \phi) = \int_{H(F)\backslash H(\mathbb{A}_F)} \theta(g, h; \phi) \, dh$$

if it is absolutely convergent. Here we normalize dh so that $\mathrm{vol}(H(F)\backslash H(\mathbb{A}_F)) = 1$. It is absolutely convergent for all ϕ if $m > 2r$ or V is anisotropic.

Siegel–Weil formulae. It is easy to see that $\varphi_{\phi,s}(g) = \omega_\chi(g)\phi(0)\lambda_{P_r}(g)^{s-(m-r)/2}$ is a standard section in $I_r(s, \chi)$ for any $\phi \in \mathscr{S}(V^r)$, where $\lambda_{P_r}(g) = \lambda_{P_r}(n(b)m(a)k) = |\det a|_E$ under the Iwasawa decomposition with respect to P_r. Hence we can define an Eisenstein series $E(s, g, \phi) = E(g, \varphi_{\phi,s})$ and we have:

Theorem 2.1 (Siegel–Weil formula). *Let $s_0 = (m-r)/2$,*

(1) *If $m > 2r$, $E(s_0, g, \phi)$ is absolutely convergent and*

$$E(s_0, g, \phi) = I_V(g, \phi).$$

(2) *If $r < m \le 2r$ and V is anisotropic, $E(s, g, \phi)$ is holomorphic at s_0 and*

$$E(s, g, \phi)\big|_{s=s_0} = I_V(g, \phi).$$

(3) *If $m = r$ and V is anisotropic, $E(s, g, \phi)$ is holomorphic at $s_0 = 0$ and*

$$E(s, g, \phi)\big|_{s=0} = 2 I_V(g, \phi).$$

In the above theorem, (1) is the classical Siegel–Weil formula. (2) and (3) are certain generalizations which appear in [Ichino 2007, Theorem 1.1] and [Ichino 2004, Theorem 4.2], respectively. In the following, we simply write $E(s_0, g, \phi)$ for $E(s, g, \phi)\big|_{s=s_0}$ if it is holomorphic at s_0.

Remark 2.2. In case (3), if V is isotropic, we still have a (regularized) Siegel–Weil formula. But then since the theta integral $I_V(g, \phi)$ is not necessarily convergent, a regularization process must be applied. The inner product introduced in

the next section also requires a regularization process. Since the classical inner-product formula is not the purpose of this paper, we will always assume that V is anisotropic for simplicity, or pretend that the regularization process has been applied for general V in the following discussion.

2B. *Doubling integrals.* In this section, we will review the method of doubling integrals which is first introduced in [Gelbart et al. 1987].

We now let $m = 2n$ and $r = n$ with $n \geq 1$ and suppress n from the notation, except that we will use H' instead of H_n, P' instead of P_n, N' instead of N_n, and $\mathcal{K}'$ instead of $\mathcal{K}_n$. Hence $\chi|_{\mathbb{A}_F^\times} = \mathbf{1}$. Let $\pi = \bigotimes' \pi_v$ be an irreducible cuspidal automorphic representation of $H'(\mathbb{A}_F)$ contained in $L^2(H'(F)\backslash H'(\mathbb{A}_F))$ and $\pi^\vee$ realizes on the space of complex conjugation of functions in π.

We denote by $(-W)$ the skew-hermitian space over E with the form $-\langle \cdot, \cdot \rangle$. Hence we can find a basis $\{e_1^-, \ldots, e_{2n}^-\}$ satisfying $\langle e_i^-, e_j^- \rangle = 0$, $\langle e_{r+i}^-, e_{r+j}^- \rangle = 0$, and $\langle e_i^-, e_{n+j}^- \rangle = -\delta_{ij}$ for $1 \leq i \leq n$. Let $W'' = W \oplus (-W)$ be the direct sum of two skew-hermitian spaces. There is a natural embedding $\iota : H' \times H' \hookrightarrow \mathsf{U}(W'')$ which is, under the basis $\{e_1, \ldots, e_{2n}\}$ of W and $\{e_1, \ldots, e_n; e_1^-, \ldots, e_n^-; e_{n+1}, \ldots, e_{2n}; -e_{n+1}^-, \ldots, -e_{2n}^-\}$ of W'', given by $\iota(g_1, g_2) = \iota_0(g_1, g_2^\vee)$, where

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad g^\vee = \begin{pmatrix} \mathbf{1}_n & \\ & -\mathbf{1}_n \end{pmatrix} g \begin{pmatrix} \mathbf{1}_n & \\ & -\mathbf{1}_n \end{pmatrix}^{-1},$$

and

$$\iota_0(g_1, g_2) = \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}.$$

For a complete polarization $W'' = W' \oplus \overline{W}'$, where $W' = \mathrm{span}_E\{e_1, \ldots, e_n; e_1^-, \ldots, e_n^-\}$ and $\overline{W}' = \mathrm{span}_E\{e_{n+1}, \ldots, e_{2n}; -e_{n+1}^-, \ldots, -e_{2n}^-\}$, there is a Weil representation of $\mathsf{U}(W'')$, denoted by ω_χ'' (with respect to ψ), on the space $\mathscr{S}(V^{2n})$, such that $\iota^* \omega_\chi'' \cong \omega_{\chi,\psi} \boxtimes \chi \omega_{\chi,\psi}^\vee$, which is realized on the space $\mathscr{S}(V^n) \otimes \mathscr{S}(V^n)$. Here we realize the contragredient representation $\omega_{\chi,\psi}^\vee$ on the space $\mathscr{S}(V^n)$ through the bilinear pairing

$$\langle \phi, \phi^\vee \rangle = \int_{V^n(\mathbb{A}_E)} \phi(x) \phi^\vee(x) \, dx$$

for $\phi, \phi^\vee \in \mathscr{S}(V^n)$. Then $\omega_{\chi,\psi}^\vee$ is identified with $\omega_{\chi^{-1},\psi^{-1}}$.

For $\phi \in \mathscr{S}(V^n)$ and $f \in \pi$,

$$\theta_\phi^f(h) = \int_{H'(F)\backslash H'(\mathbb{A}_F)} \theta(g, h; \phi) f(g) \, dg$$

is a well-defined, slowly increasing function on $H(F)\backslash H(\mathbb{A}_F)$, where $dg = \otimes' dg_v$ such that $\mathcal{K}'_v$ gets volume 1 for any $v \in \Sigma$. Similarly, for $\phi^\vee \in \mathcal{S}(V^n)$ and $f^\vee \in \pi^\vee$, we have $\theta_{\phi^\vee}^{f^\vee}$. One should be careful that in the contragredient case, the Weil representation used to form the theta function should also be $\omega_\chi^\vee$. We have

$$
\langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H := \int_{H(F)\backslash H(\mathbb{A}_F)} \theta_\phi^f(h) \theta_{\phi^\vee}^{f^\vee}(h)\, dh
$$

$$
= \int_{H(F)\backslash H(\mathbb{A}_F)} \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} \theta(g_1, h; \phi) f(g_1) \theta(g_2, h; \phi^\vee) \\ \times f^\vee(g_2)\, dg_1\, dg_2\, dh
$$

$$
= \int_{H(F)\backslash H(\mathbb{A}_F)} \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} \theta(\iota(g_1, g_2), h; \phi \otimes \phi^\vee) \\ \times f(g_1) f^\vee(g_2) \chi^{-1}(\det g_2)\, dg_1\, dg_2\, dh
$$

$$
= \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2) \chi^{-1}(\det g_2) \\ \times \int_{H(F)\backslash H(\mathbb{A}_F)} \theta(\iota(g_1, g_2), h; \phi \otimes \phi^\vee)\, dh\, dg_1\, dg_2. \quad (2\text{-}1)
$$

We assume that V is anisotropic; then the inside integral in the last step is absolutely convergent and by Theorem 2.1(3), we have

$$
(2\text{-}1) = \frac{1}{2} \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2) \chi^{-1}(\det g_2) E(0, \iota(g_1, g_2), \phi \otimes \phi^\vee)\, dg_1 dg_2.
$$

It should be mentioned that the Eisenstein series on $\mathsf{U}(W'')$ appearing above is formed with respect to the parabolic subgroup P fixing the subspace $\overline{W}'$ (with maximal unipotent subgroup N), that is,

$$
E(s, g, \Phi) = E(g, \varphi_{\Phi,s}) = \sum_{\gamma \in P(F)\backslash \mathsf{U}(W'')(F)} \omega_\chi''(\gamma g) \Phi(0) \lambda_P(\gamma g)^s
$$

for $g \in \mathsf{U}(W'')(\mathbb{A}_F)$, $\Phi \in \mathcal{S}(V^{2n})$, and $\Re(s) > n$. The coset $P(F)\backslash \mathsf{U}(W'')(F)$ can be canonically identified with the space of isotropic n-planes in W''. Under the right action of $H'(F) \times H'(F)$ through ι, the orbit of an n-plane Z is determined by the invariant $d = \dim(Z \cap W) = \dim(Z \cap (-W))$. Let γ_d be a representative of the corresponding double coset where $0 \le d \le n$. In particular, we take

$$
\gamma_0 = \begin{pmatrix} & & \mathbf{1}_n & \\ & \mathbf{1}_n & & \\ -\mathbf{1}_n & \mathbf{1}_n & & \\ & & \mathbf{1}_n & \mathbf{1}_n \end{pmatrix} \quad \text{and } \gamma_n = \mathbf{1}_{4n}
$$

(see [Kudla and Rallis 2005]). Let St_d be the stabilizer of $P\gamma_d\iota(H' \times H')$ in $H' \times H'$. In particular $\mathsf{St}_0 = \Delta(H')$ is the diagonal. Hence for a standard section

$\varphi_s \in I_{2n}(s, \chi)$ and $\Re(s) > n$,

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^{\vee}(g_2) \chi^{-1}(\det g_2) E(\iota(g_1, g_2), \varphi_s) \, dg_1 dg_2$$

$$= \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} (f \otimes f^{\vee} \chi^{-1})(g) \sum_{\gamma \in P(F)\backslash \mathsf{U}(W'')(F)} \varphi_s(\gamma \iota(g)) \, dg$$

$$= \sum_{d=0}^{n} \int_{\mathsf{St}_d(F)\backslash H'(\mathbb{A}_F)^2} (f \otimes f^{\vee} \chi^{-1})(g) \varphi_s(\gamma_d \iota(g)) \, dg. \qquad (2\text{-}2)$$

When $d > 0$, St_d has a nontrivial unipotent radical. Since f and $f^{\vee}$ are cuspidal, we have

$$(2\text{-}2) = \int_{\Delta(H'(F))\backslash H'(\mathbb{A}_F)^2} (f \otimes f^{\vee} \chi^{-1})(g) \varphi_s(\gamma_0 \iota(g)) \, dg$$

$$= \int_{H'(F)\backslash H'(\mathbb{A}_F)} \int_{H'(\mathbb{A}_F)} f(g_1 g_2) f^{\vee}(g_1) \chi^{-1}(\det g_1)$$
$$\times \varphi_s(\gamma_0 \iota(g_1 g_2, g_1)) \, dg_1 dg_2$$

$$= \int_{H'(F)\backslash H'(\mathbb{A}_F)} \int_{H'(\mathbb{A}_F)} \pi(g_2) f(g_1) f^{\vee}(g_1) \chi^{-1}(\det g_1)$$
$$\times \varphi_s(p(g_1) \gamma_0 \iota(g_2, 1)) \, dg_1 dg_2, \quad (2\text{-}3)$$

where $p(g_1)\gamma_0 = \gamma_0 \iota(g_1, g_1)$ having the property that under the Levi decomposition $p(g_1) = n(b)m(a) \in P(\mathbb{A}_F)$, we have $\det a = \det g_1$. Hence

$$(2\text{-}3) = \int_{H'(\mathbb{A}_F)} \int_{H'(F)\backslash H'(\mathbb{A}_F)} \pi(g_2) f(g_1) f^{\vee}(g_1) \, dg_1 \varphi_s(\gamma_0 \iota(g_2, 1)) \, dg_2$$

$$= \int_{H'(\mathbb{A}_F)} \langle \pi(g) f, f^{\vee} \rangle \varphi_s(\gamma_0 \iota(g, 1)) \, dg$$

$$= \prod_{v \in \Sigma} \int_{H'_v} \langle \pi_v(g_v) f_v, f_v^{\vee} \rangle \varphi_{s,v}(\gamma_0 \iota(g_v, 1)) \, dg_v,$$

where we assume f, $f^{\vee}$ and φ_s are all decomposable. In summary, we have:

Proposition 2.3. *Let f, $f^{\vee}$ and φ_s be as above. For $\Re(s) > n$, the integral*

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^{\vee}(g_2) \chi^{-1}(\det g_2) E(\iota(g_1, g_2), \varphi_s) \, dg_1 dg_2$$

$$= \prod_{v \in \Sigma} \int_{H'_v} \langle \pi_v(g_v) f_v, f_v^{\vee} \rangle \varphi_{s,v}(\gamma_0 \iota(g_v, 1)) \, dg_v,$$

which defines an element in

$$\mathsf{Hom}_{H'(\mathbb{A}_F) \times H'(\mathbb{A}_F)}(I_{2n}(s, \chi), \pi^{\vee} \boxtimes \chi \pi) = \bigotimes_v \mathsf{Hom}_{H'_v \times H'_v}(I_{2n}(s, \chi_v), \pi_v^{\vee} \boxtimes \chi_v \pi_v).$$

2C. *Local zeta integrals.* In this section, we will study the local functional we finally found in the last section.

Fix a finite place v of F and suppress it from the notation. For $f \in \pi$, $f^\vee \in \pi^\vee$, and holomorphic section $\varphi_s \in I_{2n}(s, \chi)$, define the local zeta integral

$$Z(\chi, f, f^\vee, \varphi_s) = \int_{H'} \langle \pi(g)f, f^\vee \rangle \varphi_s(\gamma_0 \iota(g, 1)) \, dg,$$

which is absolutely convergent when $\Re(s) > 2n$. In [Harris et al. 1996, Section 6], the family of *good sections* is introduced. For any good section, the zeta integral $Z(\chi, f, f^\vee, \varphi_s)$ is a rational function in q^{-s}, where q is the cardinality of the residue field of F. In particular, it has a meromorphic continuation to the entire complex plane. Consider the family of zeta integrals

$$\{Z(\chi, f, f^\vee, \varphi_s) \mid f \in \pi, \ f^\vee \in \pi^\vee, \ \varphi_s \text{ is good}\}$$

and the fractional ideal $\mathscr{I}$ of the ring $\mathbb{C}[q^s, q^{-s}]$ in its fraction field generated by the above family. In fact, $\mathscr{I}$ is generated by $1/P(q^{-s})$, for a unique polynomial $P(X) \in \mathbb{C}[X]$ such that $P(0) = 1$. We let

$$L\left(s + \tfrac{1}{2}, \pi, \chi\right) = \frac{1}{P(q^{-s})}$$

be the local doubling *L*-series of Piatetski-Shapiro and Rallis. The same construction can also be applied to the archimedean case.

Now suppose E/F is unramified (including split) at v and ψ, χ, and π are also unramified. Let $f_0 \in \pi^{\mathcal{K}'}$, $f_0^\vee \in \pi^{\vee, \mathcal{K}'}$, and $\langle f_0, f_0^\vee \rangle = 1$, φ_s^0 be the unramified standard section. Then the calculation in [Gelbart et al. 1987] and [Li 1992] (see Theorem 3.1 of the latter reference) shows that

$$Z(\chi, f_0, f_0^\vee, \varphi_s^0) = \frac{L(s + \tfrac{1}{2}, \pi, \chi)}{b_{2n}(s)},$$

where

$$b_m(s) = \prod_{i=0}^{m-1} L(2s + m - i, \epsilon_{E/F}^i) \tag{2-4}$$

is a product of local Tate factors. For the general case,

$$\frac{b_{2n}(s) Z(\chi, f, f^\vee, \varphi_s)}{L(s + \tfrac{1}{2}, \pi, \chi)}$$

admits a meromorphic extension to the entire complex plane which is holomorphic at $s = 0$. Moreover, the *normalized zeta integral*

$$Z^*(\chi, f, f^\vee, \varphi_s) := \left. \frac{b_{2n}(s) Z(\chi, f, f^\vee, \varphi_s)}{L(s + \tfrac{1}{2}, \pi, \chi)} \right|_{s=0} \tag{2-5}$$

defines a nonzero element in $\text{Hom}_{H' \times H'}(I_{2n}(0, \chi), \pi^\vee \boxtimes \chi\pi)$ (see [Harris et al. 1996, Proof of (1) of Theorem 4.3]).

Remark 2.4. It is conjectured (see, for example, [Harris et al. 1996]) that for all irreducible admissible representations π of H' and characters χ of $E^\times$, we have

$$L(s, \pi, \chi) = L(s, \text{BC}(\pi) \otimes \chi).$$

This is known when E/F, χ, and π are all unramified due to [Li 1992] and (the similar argument for unitary groups in) [Kudla and Rallis 2005, Section 5]. It is also known when $n = 1$ due to [Harris 1993].

For further discussion, we need to recall a result on the degenerate principal series. In the following, we will use the notation H'' instead of $\text{U}(W'')$ for short and recall our embedding $\iota : H' \times H' \hookrightarrow H''$. Let V be a hermitian space of dimension $2n$ over E. Then $\varphi_\phi(g) = \omega_\chi(g)\phi(0)$ defines an H''-intertwining map $\mathscr{S}(V^{2n}) \to I_{2n}(0, \chi)$ whose image $R(V, \chi)$ is isomorphic to $\mathscr{S}(V^{2n})_H$. Recall that we denote by $V^\pm$ the two nonisometric hermitian spaces of dimension $2n$ when v is finite nonsplit, by V^+ the only hermitian space of dimension $2n$ when v is finite split (up to isometry), and by V_s ($0 \le s \le 2n$) the $2n + 1$ nonisometric hermitian spaces of dimension $2n$ when v is infinite.

Proposition 2.5. (1) *If v is finite nonsplit, $R(V^-, \chi)$ and $R(V^-, \chi)$ are irreducible and inequivalent and $I_{2n}(0, \chi) = R(V^+, \chi) \oplus R(V^-, \chi)$.*

(2) *If v is finite split, $R(V, \chi)$ is irreducible and $I_{2n}(0, \chi) = R(V^+, \chi)$.*

(3) *If v is infinite, $R(V_s, \chi)$ are irreducible and inequivalent and $I_{2n}(0, \chi) = \bigoplus_{s=0}^{2n} R(V_s, \chi)$.*

Proof. (1) is [Kudla and Sweet 1997, Theorem 1.2], (2) is [Kudla and Sweet 1997, Theorem 1.3], and (3) is [Lee 1994, Section 6, Proposition 6.11]. □

2D. *Central special values of L-functions.* In this section, we will make a connection between the theta lifting θ_ϕ^f defined in Section 2B and the central special value of the L-function of the representation π.

Recall that we have an irreducible unitary cuspidal automorphic representation π of $H' = H_n$ and a hermitian space V over E of dimension $2n$. One key question in the theory of theta lifting is whether θ_ϕ^f is nonvanishing. A sufficient condition is to look at the local invariant functional as follows.

First, we have the following *theta dichotomy*.

Proposition 2.6. *For any nonsplit place $v \in \Sigma$, $\text{Hom}_{H'_v \times H'_v}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v)$ is nonzero for exactly one hermitian space V_v (up to isometry) over E_v of dimension $2n$, which we denote by $V(\pi_v, \chi_v)$.*

Proof. If v is (real) archimedean, this is [Paul 1998, Theorem 2.9]. If v is nonarchimedean, it is due to [Gong and Grenié 2011, Theorem 2.10] and the nonvanishing of Z^*.   □

In Proposition 2.3, if we let $\varphi_s = \varphi_{\phi \otimes \phi^\vee, s}$ and denote $Z^*(s, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee) = Z^*(\chi_v, f_v, f_v^\vee, \varphi_{\phi_v \otimes \phi_v^\vee, s})$, then both sides have meromorphic continuations to the entire complex plane that are actually holomorphic at the point $s = 0$; that is, we have

$$\langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H = \frac{L(\frac{1}{2}, \pi, \chi)}{2 \prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_{v \in S} Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee), \qquad (2\text{-}6)$$

in which the product of normalized zeta integrals can actually be taken over a finite set S by the unramified calculation. In particular, for $v \notin S$, $V_v \cong V(\pi_v, \chi_v)$, that is, $\theta_{\chi_v}(\pi_v^\vee, V_v) \neq 0$. Then one necessary condition for θ_ϕ^f to be nonvanishing for some f and ϕ is that each local (normalized) zeta integral is not identically zero, which exactly means $V_v \cong V(\pi_v, \chi_v)$ for all $v \in \Sigma$. Let $\mathbb{V}(\pi, \chi)$ be the hermitian space over $\mathbb{A}_E$ such that $\mathbb{V}(\pi, \chi)_v \cong V(\pi_v, \chi_v)$ and let $\epsilon(\pi_v, \chi_v) = \epsilon(V(\pi_v, \chi_v))$ and $\epsilon(\pi, \chi) = \prod \epsilon(\pi_v, \chi_v)$. If $\epsilon(\pi, \chi) = -1$. Then $\mathbb{V}(\pi, \chi)$ is incoherent, hence for any V, the (possibly regularized) theta lifting θ_ϕ^f is always vanishing. If $\epsilon(\pi, \chi) = 1$, then $\mathbb{V}(\pi, \chi) \cong V(\pi, \chi) \otimes_F \mathbb{A}_F$ for some $V(\pi, \chi)$ over E. Assume $V(\pi, \chi)$ is anisotropic. Then there exist some $f \in \pi$ and $\phi \in \mathcal{S}(V(\pi, \chi)^n)$ such that $\theta_\phi^f \neq 0$ if and only if $L(\frac{1}{2}, \pi, \chi) \neq 0$.

We want to give another interpretation for the formula (2-6) when $\epsilon(\pi, \chi) = 1$, which is crucial for our proof in [Liu 2011]. For this purpose, let us assume the following conjecture raised by Kudla and Rallis (see [Harris et al. 1996]):

$$\dim \operatorname{Hom}_{H_v' \times H_v'}(I_{2n}(0, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v) = 1 \qquad (2\text{-}7)$$

for all components π_v of π. This is proved in [Liu 2011, Section 6B] when $n = 1$. Let $V = V(\pi, \chi)$ and $R(V, \chi) = \bigotimes' R(V_v, \chi_v)$; the functional

$$\beta(f, f^\vee, \phi, \phi^\vee) := \langle \theta_\phi^f, \theta_{\phi^\vee}^{f^\vee} \rangle_H$$

defines an element in

$$\operatorname{Hom}_{H'(\mathbb{A}_F) \times H'(\mathbb{A}_F)}(R(V, \chi), \pi^\vee \boxtimes \chi \pi) = \bigotimes_v \operatorname{Hom}_{H_v' \times H_v'}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v).$$

On the other hand, the functional

$$\alpha(f, f^\vee, \phi, \phi^\vee) := \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$$

(when everything is decomposable, otherwise we take the linear combination) also defines an element in $\bigotimes \operatorname{Hom}_{H_v' \times H_v'}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v)$ which is nonzero. But

by our assumption (2-7), this space is of dimension one. Hence β is a constant multiple of α. This constant, by (2-6), is

$$\frac{\beta}{\alpha} = \frac{L(\frac{1}{2}, \pi, \chi)}{2 \prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)}.$$

In some sense, the vanishing of $L(\frac{1}{2}, \pi, \chi)$ is the obstruction for β to be a nontrivial global invariant functional. This kind of formulation is first observed in [Yuan et al. 2011; 2010].

2E. *Vanishing of central L-values.* In this section, we will prove that the central L-value $L(\frac{1}{2}, \pi, \chi)$ vanishes when $\epsilon(\pi, \chi) = -1$.

By Proposition 2.5, we have a decomposition of $H''(\mathbb{A}_F)$-admissible representation

$$I_{2n}(0, \chi) = \bigoplus_{\mathbb{V}} R(\mathbb{V}, \chi) = \bigoplus_{\mathbb{V}} \bigotimes_v' R(\mathbb{V}_v, \chi_v),$$

where the direct sum is taken over all (isometry classes of) hermitian spaces over $\mathbb{A}_E$ of rank $2n$ and each $R(\mathbb{V}, \chi)$ is irreducible. Recall the group $H'' = \mathrm{U}(W'')$ and its standard parabolic subgroup P fixing $\overline{W}'$ whose unipotent radical is N as in Section 2B. First, we need some lemmas for local representations.

Fix any place v and suppress it from the notation. For $T \in \mathrm{Her}_{2n}(E)$, let $\Omega_T = \{x \in V^{2n} \mid T(x) = T\}$ and define a character ψ_T of $N \cong \mathrm{Her}_{2n}(E)$ by $\psi_T(n(b)) = \psi(\mathrm{tr}\, Tb)$.

Lemma 2.7. (1) *Suppose v is finite, let $\mathscr{S}(V^{2n})_{N, \psi_T}$ (resp. $R(V, \chi)_{N, \psi_T}$) be the twisted Jacquet module of $\mathscr{S}(V^{2n})$ (resp. $R(V, \chi)$) associated to N and the character ψ_T.*

  (a) *The quotient map $\mathscr{S}(V^{2n}) \to \mathscr{S}(V^{2n})_{N, \psi_T}$ can be realized by the restriction $\mathscr{S}(V^{2n}) \to \mathscr{S}(\Omega_T)$;*

  (b) *If T is nonsingular, then*

$$\dim R(V, \chi)_{N, \psi_T} = \begin{cases} 1 & \text{if } \Omega_T \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

  (2) *Suppose v is infinite, that is, $E/F = \mathbb{C}/\mathbb{R}$ and T is nonsingular, the space of H-invariant tempered distribution T on $\mathscr{S}(V^{2n})$ such that*

$$T(\omega_\chi(X)\Phi) = d\psi_T(X)T(\Phi)$$

  for $X \in \mathfrak{n} = \mathrm{Lie}\, N$ is of dimension 1 (resp. 0) if $\Omega_T \neq \varnothing$ (resp. $\Omega_T = \varnothing$).

Proof. (1) is [Rallis 1987, Lemma 4.2], (2) is [Rallis 1987, Lemma 4.2], and [Kudla and Rallis 1994, Proposition 2.9]. $\square$

We now construct the twisted Jacquet module $R(V, \chi)_{N, \psi_T}$ or the invariant distribution explicitly if it is not trivial. For a standard section $\varphi_s \in I_{2n}(s, \chi)$, define the Whittaker integral

$$W_T(g, \varphi_s) = \int_N \varphi_s(wng) \psi_T(n)^{-1} \, dn,$$

where $w = w_{2n}$ and dn is self-dual with respect to ψ. The integral $W_T(g, \varphi_s)$ is absolutely convergent when $\Re(s) > n$. It is easy to see that $W_T(e, \cdot) : I_{2n}(s, \chi) \to \mathbb{C}_{N, \psi_T}$ is an N-intertwining map. Let $W_T(s, g, \Phi) = W_T(g, \varphi_{\Phi, s})$ for $\Phi \in \mathcal{S}(V^{2n})$. We have

Lemma 2.8. *Assume T is nonsingular.*

(1) $W_T(g, \varphi_s)$ *is entire.*

(2) *The integral* $\Phi \mapsto W_T(0, e, \Phi)$ *realizes the surjective N-intertwining map*

$$\mathcal{S}(V^{2n}) \to R(V, \chi) \to R(V, \chi)_{N, \psi_T}$$

or the invariant distribution in Lemma 2.7(2).

Proof. (1) is [Karel 1979, Corollary 3.6.1] for v finite and [Wallach 1988, Theorem 8.1] for v infinite; (2) is [Kudla and Rallis 1994, Proposition 2.7]. □

Lemma 2.9. *Suppose v is finite, E/F, ψ, and χ are all unramified, and $V = V^+$. Then for Φ^0 the characteristic function of $(\Lambda^+)^{2n}$ for a self-dual $\mathcal{O}_E$-lattice Λ^+ and $T \in \mathrm{Her}_{2n}(\mathcal{O}_F)$ with $\det T \in \mathcal{O}_F^\times$, we have*

$$W_T(s, e, \Phi^0) = b_{2n}(s)^{-1}.$$

Proof. This is [Tan 1999, Proposition 3.2]. □

Now suppose we are in the global situation. We denote by $\mathcal{A}(H'')$ the space of automorphic forms of H''. For $T \in \mathrm{Her}_{2n}(F)$, define the T-th Fourier coefficient of $f(g) \in \mathcal{A}(H'')$ as

$$W_T(g, f) = \int_{N(F) \backslash N(\mathbb{A}_F)} f(ng) \psi_T(n)^{-1} \, dn.$$

For any hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank $2n$, we have a series of linear maps

$$\mathcal{E}_s : R(\mathbb{V}, \chi) \to \mathcal{A}(H'')$$

$$\Phi \mapsto E(s, g, \Phi) = E(g, \varphi_{\Phi, s})$$

for s near 0. It is an $H''(\mathbb{A}_F)$-intertwining map exactly when $s = 0$. Then for T nonsingular (and s near 0), we have

$$E_T(s, g, \Phi) := W_T(g, \mathcal{E}_s(\Phi)) = \prod_{v \in \Sigma} W_T(s, g_v, \Phi_v). \tag{2-8}$$

Lemma 2.10. *For any $H''(\mathbb{A}_F)$-intertwining operator $\mathscr{E} : R(\mathbb{V}, \chi) \to \mathscr{A}(H'')$, if $W_T(g, \cdot) \circ \mathscr{E}$ vanishes for all nonsingular T, then $\mathscr{E} = 0$.*

Proof. Fix a finite place v; by Lemma 2.7(1), we can find a section $\Phi_0 = \Phi_{v,0}\Phi^v \in \mathscr{S}(\mathbb{V}^{2n})$ with nonzero projection in $R(\mathbb{V}, \chi)$ such that $\Phi_{v,0} \in \mathscr{S}(\mathbb{V}_v^{2n})_{\text{reg}}$, the set consisting of functions supporting in the set $\{x \in \mathbb{V}_v^{2n} \mid \det T(x) \neq 0\}$. For any $g^v \in e_v H''(\mathbb{A}_F^v)$, the functional $\Phi_v \mapsto W_T(0, g^v, \Phi_v\Phi^v)$ factors through the twisted Jacquet module $\mathscr{S}(\mathbb{V}_v^{2n})_{N_v, \psi_T}$. If T is singular, then by our choice of $\Phi_{v,0}$ and Lemma 2.7(1-a), $W_T(0, g^v, \Phi_{v,0}\Phi^v) = 0$. Similarly, $W_T(0, g, \Phi_{v,0}\Phi^v) = 0$ for all $g \in P_v H''(\mathbb{A}_F^v)$ since P_v keeps the set $\mathscr{S}(\mathbb{V}_v^{2n})_{\text{reg}}$. For T nonsingular, $W_T \equiv 0$ by the assumption. Hence $\mathscr{E}(\Phi_0)(g) = 0$ for $g \in P_v H''(\mathbb{A}_F^v)$. It follows that $\mathscr{E}(\Phi_0) = 0$ and $\mathscr{E} = 0$ by our choice of Φ_0 and the irreducibility of $R(\mathbb{V}, \chi)$. $\qquad\square$

Proposition 2.11. (1) *If $\mathbb{V}$ is incoherent, then $\mathsf{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathscr{A}(H''))$ has dimension 0.*

(2) *If $\mathbb{V}$ is coherent, then $\mathsf{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathscr{A}(H''))$ has dimension 1 and $\mathscr{E}_0$ given above is a nontrivial element.*

Proof. For (1), assume that $\mathscr{E}$ is a nontrivial intertwining map. By Lemma 2.10, there is a nonsingular $T \in \mathrm{Her}_{2n}(F)$ such that $W_T(g, \cdot) \circ \mathscr{E}$ does not vanish. By parts (1-b) and (2) of Lemma 2.7, T is representable by $\mathbb{V}_v$ for any $v \in \Sigma$; that is, $\Omega_T \neq \varnothing$. But then $\mathbb{V}$ will be coherent which is a contradiction.

For (2), assume $\mathscr{E}$ and $\mathscr{E}'$ are both nontrivial intertwining maps. By Lemma 2.10, there is a nonsingular T such that $W_T(g, \cdot) \circ \mathscr{E}$ does not vanish. By parts (1-b) and (2) of Lemma 2.7, or the general fact that the Whittaker model with respect to a generic character is unique, there exists $c \in \mathbb{C}$ such that $W_T(g, \cdot) \circ \mathscr{E}' = c W_T(g, \cdot) \circ \mathscr{E}$. Furthermore, c is independent of nonsingular T since all of those which can be represented by $\mathbb{V}$ are in a single $M(F)$-orbit under the conjugation action on $N(F)$. Then by Lemma 2.10, $\mathscr{E}' - c\mathscr{E} = 0$, that is, $\dim \mathsf{Hom}_{H''(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \mathscr{A}(H'')) \leq 1$.

For the rest, we need to prove that $\mathscr{E}_0$ is actually nontrivial. Choose a non-singular $T \in \mathrm{Her}_{2n}(F)$ such that it is representable by $\mathbb{V}$ which exists since $\mathbb{V}$ is coherent. By (2-8) and Lemmas 2.8(2) and 2.9, we can find a suitable Φ such that $W_T(0, e, \Phi) \neq 0$; hence $\mathscr{E}_0 \neq 0$. $\qquad\square$

Now we can state our main result in this section.

Theorem 2.12. *If $\epsilon(\pi, \chi) = -1$, then $L\left(\frac{1}{2}, \pi, \chi\right) = 0$.*

Proof. Let $\mathbb{V} = \mathbb{V}(\pi, \chi)$; then it is incoherent. We can choose suitable f_v, $f_v^\vee$, ϕ_v, and $\phi_v^\vee$ when one of E, ψ, χ, and π is ramified at v, such that

$$Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee) \neq 0.$$

Let $f, f^\vee, \phi$, and $\phi^\vee$ be global vectors with these subscribed local components and unramified ones at the places where E, ψ, χ, and π are unramified. Then from

Proposition 2.3 (after analytic continuation), we have

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2) \chi^{-1}(\det g_2) E(0, \iota(g_1, g_2), \phi \otimes \phi^\vee) \, dg_1 dg_2$$

$$= \frac{L(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_{v \in S} Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee).$$

But $\mathscr{E}_0$ is zero on $R(\mathbb{V}, \chi)$ by Proposition 2.11(1). We have $E(0, \iota(g_1, g_2), \phi \otimes \phi^\vee) \equiv 0$. Hence $L(\frac{1}{2}, \pi, \chi) = 0$ by our choices and the fact that the Tate *L*-values appearing here are finite. $\qquad\square$

Since $L(\frac{1}{2}, \pi, \chi) = 0$, it leads us to consider its derivative at this point. In fact, we have

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2) \chi^{-1}(\det g_2) \frac{d}{ds}\Big|_{s=0} E(s, \iota(g_1, g_2), \phi \otimes \phi^\vee) \, dg_1 dg_2$$

$$= \frac{d}{ds}\Big|_{s=0} \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2) \chi^{-1}(\det g_2)$$
$$\times E(s, \iota(g_1, g_2), \phi \otimes \phi^\vee) \, dg_1 dg_2$$

$$= \frac{d}{ds}\Big|_{s=0} \frac{L(s + \frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(2s + i, \epsilon_{E/F}^i)} \prod_{v \in S} Z^*(s, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$$

$$= \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_{v \in S} Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$$
$$+ L\left(\frac{1}{2}, \pi, \chi\right) \frac{d}{ds}\Big|_{s=0} \frac{\prod_{v \in S} Z^*(s, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)}{\prod_{i=1}^{2n} L(2s + i, \epsilon_{E/F}^i)}$$

$$= \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_{v \in S} Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee). \qquad (2\text{-}9)$$

We call $E'(0, g, \Phi) = (d/ds)\big|_{s=0} E(s, g, \Phi)$ the *analytic kernel function* associated to the *test function* $\Phi \in \mathscr{S}(\mathbb{V}^{2n})$.

Recall that for $T \in \mathrm{Her}_{2n}(F)$, we let

$$E_T(s, g, \Phi) = W_T(g, \mathscr{E}_s(\Phi))$$

for *s* near 0. If *T* is nonsingular, then

$$W_T(g, \mathscr{E}_s(\Phi)) = \prod_{v \in \Sigma} W_T(s, g_v, \Phi_v)$$

if $\Phi = \bigotimes \Phi_v$ is decomposable. Hence,

$$E(s, g, \Phi) = \sum_{T \text{ sing.}} E_T(s, g, \Phi) + \sum_{T \text{ nonsing.}} \prod_{v \in \Sigma} W_T(s, g_v, \Phi_v).$$

Taking the derivative at $s = 0$, we have

$$E'(0, g, \Phi) = \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{T \text{ nonsing.}} \sum_{v \in \Sigma} W'_T(0, g_v, \Phi_v) \prod_{v' \neq v} W_T(0, g_{v'}, \Phi_{v'})$$

$$= \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{v \in \Sigma} \sum_{T \text{ nonsing.}} W'_T(0, g_v, \Phi_v) \prod_{v' \neq v} W_T(0, g_{v'}, \Phi_{v'}).$$

But we have $\prod_{v' \neq v} W_T(0, g_v, \Phi_v) \neq 0$ only if $\mathbb{V}_{v'}$ represents T for all $v' \neq v$ by Lemma 2.7(1-b). Since $\mathbb{V}$ is incoherent, $\mathbb{V}_v$ cannot represent T. For T nonsingular, there are only finitely many $v \in \Sigma$ such that T is not represented by $\mathbb{V}_v$, that is, there does not exist $x_1, \ldots, x_{2n} \in \mathbb{V}_v$ whose moment matrix is T. We denote the set of such v by $\text{Diff}(T, \mathbb{V})$. Then

$$E'(0, g, \Phi) = \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{v \in \Sigma} E_v(0, g, \Phi),$$

where

$$E_v(0, g, \Phi) = \sum_{\text{Diff}(T, \mathbb{V}) = \{v\}} W'_T(0, g_v, \Phi_v) \prod_{v' \neq v} W_T(0, g_{v'}, \Phi_{v'}). \qquad (2\text{-}10)$$

In fact, the second sum is only taken over those v which are nonsplit in E.

3. Arithmetic theta lifting

3A. *Shimura varieties of unitary groups and special cycles.* In this section, we will recall the notion of Shimura varieties of unitary groups and Kudla's special cycles on them. We fix an additive character $\psi : F \backslash \mathbb{A}_F \to \mathbb{C}$ such that ψ_ι is the standard $t \mapsto e^{2\pi i t}$ ($t \in F_\iota = \mathbb{R}$) for any $\iota \in \Sigma_\infty$ until the end of this paper. The basic references for this section are [Kudla and Millson 1990; Kottwitz 1992; Kudla 1997].

Shimura varieties of unitary groups. Let $m \geq 2$ and $1 \leq r < m$ be integers. Let $\mathbb{V}$ be a totally positive-definite incoherent hermitian space over $\mathbb{A}_E$ of rank m. Let $\mathbb{H} = \text{Res}_{\mathbb{A}_F/\mathbb{A}} U(\mathbb{V})$ be the unitary group which is a reductive group over $\mathbb{A}$ and $\mathbb{H}^{\text{der}} = \text{Res}_{\mathbb{A}_F/\mathbb{A}} SU(\mathbb{V})$ its derived subgroup. Let $\mathbb{T} \cong \text{Res}_{\mathbb{A}_F/\mathbb{A}} \mathbb{A}_E^{\times,1}$ be the maximal abelian quotient of $\mathbb{H}$ which is also isomorphic to its center. Let $T \cong \text{Res}_{F/\mathbb{Q}} E^{\times,1}$ be the unique (up to isomorphism) $\mathbb{Q}$-torus such that $T \times_{\mathbb{Q}} \mathbb{A} \cong \mathbb{T}$. Then T has the property that $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$. For any open compact subgroup K of $\mathbb{H}(\mathbb{A}_f)$, there is a Shimura variety $\text{Sh}_K(\mathbb{H})$ of dimension $m - 1$ defined over the reflex field E. For any embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ over $\iota \in \Sigma_\infty$, we have the ι°-adic uniformization

$$\text{Sh}_K(\mathbb{H})^{\text{an}}_{\iota^\circ} \cong H^{(\iota)}(\mathbb{Q}) \backslash \left(\mathscr{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_f)/K \right).$$

We briefly explain the notation above. Let $V^{(\iota)}$ be the *nearby E-hermitian space* of $\mathbb{V}$ at ι, that is, $V^{(\iota)}$ is the unique E-hermitian space (up to isometry) such that $V_v^{(\iota)} \cong \mathbb{V}_v$ for $v \neq \iota$ but $V_\iota^{(\iota)}$ is of signature $(m-1, 1)$ and $H^{(\iota)} = \mathrm{Res}_{F/\mathbb{Q}} \mathsf{U}(V^{(\iota)})$. We identify $H^{(\iota)}(\mathbb{A}_f)$ and $\mathbb{H}(\mathbb{A}_f)$ through the corresponding hermitian spaces. Let $\mathscr{D}^{(\iota^\circ)}$ be the symmetric hermitian domain consisting of all negative $\mathbb{C}$-lines in $V_\iota^{(\iota)}$ whose complex structure is given by the action of $F_\iota \otimes_F E$, which is isomorphic to $\mathbb{C}$ via ι°. The group $H^{(\iota)}(\mathbb{Q})$ diagonally acts on $\mathscr{D}^{(\iota^\circ)}$ and $\mathbb{H}(\mathbb{A}_f)/K$ via the obvious way. In fact, $\mathscr{D}^{(\iota^\circ)}$ is canonically identified with the $H^\iota(\mathbb{R})$-conjugacy class of the Hodge map $h^{(\iota)} : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \boldsymbol{G}_{\mathsf{m},\mathbb{C}} \to H_\mathbb{R}^{(\iota)} \cong \mathsf{U}(m-1, 1)_\mathbb{R} \times \mathsf{U}(m, 0)_\mathbb{R}^{d-1}$ given by

$$h^{(\iota)}(z) = \left(\begin{pmatrix} \mathbf{1}_{m-1} & \\ & \bar{z}/z \end{pmatrix}, \mathbf{1}_m, \ldots, \mathbf{1}_m \right).$$

From now on, we assume that K *is contained in the principal congruence subgroup for $N \geq 3$.* Then $\mathsf{Sh}_K(\mathbb{H})$ is a quasiprojective nonsingular E-scheme. It is proper if and only if $F \neq \mathbb{Q}$ or $F = \mathbb{Q}$, $m = 2$, and $\Sigma(\mathbb{V}) \not\supseteq \Sigma_\infty$. The set of geometric connected components of $\mathsf{Sh}_K(\mathbb{H})$ can be identified with $T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/\det(K)$.

For any other open compact subgroup $K' \subset K$, we have an étale covering map $\pi_K^{K'} : \mathsf{Sh}_{K'}(\mathbb{H}) \to \mathsf{Sh}_K(\mathbb{H})$. Let $\mathsf{Sh}(\mathbb{H})$ be the projective system of these $\mathsf{Sh}_K(\mathbb{H})$. On each $\mathsf{Sh}_K(\mathbb{H})$, we have a Hodge bundle $\mathscr{L}_K \in \mathsf{Pic}(\mathsf{Sh}_K(\mathbb{H}))_\mathbb{Q}$ which is ample. They are compatible under pull-backs of $\pi_K^{K'}$, and hence define an element $\mathscr{L} \in \mathsf{Pic}(\mathsf{Sh}(\mathbb{H}))_\mathbb{Q} := \varinjlim_K \mathsf{Pic}(\mathsf{Sh}_K(\mathbb{H}))_\mathbb{Q}$.

Special cycles. Let V_1 be an E-subspace of $\mathbb{V}_f = \mathbb{V} \otimes_{\mathbb{A}_F} \mathbb{A}_{f,F}$. We say that V_1 is *admissible* if $(\,\cdot\,,\,\cdot\,)|_{V_1}$ takes values in E and for any nonzero $x \in V_1$, (x, x) is totally positive. We have

Lemma 3.1. V_1 *is admissible if and only if for any $\iota \in \Sigma_\infty$, there is an $h \in \mathbb{H}^{\mathrm{der}}(\mathbb{A}_f)$ such that $hV_1 \subset V^{(\iota)} \subset \mathbb{V}_f$ and is totally positive definite.*

Proof. One direction is obvious. For the other direction, let us assume that V_1 is admissible and fix any ι. Take $v_1 \in V_1$ with nonzero norm. Then $q(v_1) = \frac{1}{2}(v_1, v_1)$ is locally a norm for the hermitian form on $V^{(\iota)}$ by the definition of admissibility and the fact about signatures of $\mathbb{V}$ and $V^{(\iota)}$. Thus it is a norm for some $v \in V^{(\iota)}$ by the Hasse–Minkowski theorem. Now we apply Witt's theorem to find an element $h_1 \in \mathsf{U}(\mathbb{V}_f) = \mathbb{H}(\mathbb{A}_f)$ such that $h_1 v_1 = v$ as elements in $\mathbb{V}_f$. Choose any vector $v' \in \langle v \rangle^\perp \subset V^{(\iota)}$ with nonzero norm. Let $h' \in \mathbb{H}(\mathbb{A}_f)$ fixing $\langle v' \rangle^\perp$ and multiplying $(\det h_1)^{-1}$ in the $\mathbb{A}_{f,E}$-line spanned by v'. Then $h'h_1 v_1 = h'v = v$ for $h = h'h_1 \in \mathsf{SU}(\mathbb{V}_f) = \mathbb{H}^{\mathrm{der}}(\mathbb{A}_f)$.

Replacing V_1 by hV_1 we can assume that $v_1 = v \in \mathbb{V}_f$. Since $\dim V_1 < m$, we can use induction on r by considering the orthogonal complement of v in V_1 and $V^{(\iota)}$ to find an $h \in \mathbb{H}^{\mathrm{der}}(\mathbb{A}_f)$ such that $hV_1 \subset V^{(\iota)} \subset \mathbb{V}_f$. $\qquad \square$

For admissible V_1, let $\mathbb{V}_1$ be a totally positive-definite (incoherent) hermitian space over $\mathbb{A}_E$ such that $\mathbb{V}_{1,f} \cong V_1^\perp \subset \mathbb{V}_f$. Let $\mathbb{H}_1$ be the corresponding unitary group. We have a finite morphism between Shimura varieties

$$\varsigma^{V_1} : \mathsf{Sh}_{K_1}(\mathbb{H}_1) \longrightarrow \mathsf{Sh}_K(\mathbb{H}), \qquad (3\text{-}1)$$

where $K_1 = K \cap \mathbb{H}_1(\mathbb{A}_f)$, such that the image of the map is represented, under the uniformization at some ι, by the points $(z, h_1 h) \in \mathscr{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_f)$ where h is as in Lemma 3.1 (with respect to ι), $z \perp hV_1$, and h_1 fixes all elements in hV_1. The image defines a *Kudla's special cycle* $Z(V_1)_K \in \mathsf{CH}^r(\mathsf{Sh}_K(\mathbb{H}))_{\mathbb{Q}}$. It only depends on the class KV_1.

For $x \in \mathbb{V}_f^r$, let V_x be the E-subspace of $\mathbb{V}_f$ generated by the components of x. We define

$$Z(x)_K = \begin{cases} Z(V_x)_K c_1(\mathscr{L}_K^\vee)^{r-\dim_E V_x} & \text{if } V_x \text{ is admissible}, \\ 0 & \text{otherwise}. \end{cases}$$

Generating series. First, we need a restriction of the space $\mathscr{S}(\mathbb{V}_\iota^r)$ of Weil representation when ι is infinite. We define a subspace $\mathscr{S}(\mathbb{V}_\iota^r)^{\mathsf{U}_\iota} \subset \mathscr{S}(\mathbb{V}_\iota^r)$ which consists of functions of the form

$$P(T(x))e^{-2\pi \operatorname{tr} T(x)},$$

where P is a polynomial function on $\mathrm{Her}_r(\mathbb{C})$. It is a (Lie $H_{r,\iota}$, $\mathscr{K}_{r,\iota}$)-module generated by the Gaussian

$$\phi_\infty^0(x) = e^{-2\pi \operatorname{tr} T(x)}.$$

Let $\mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_\infty} = (\bigotimes_{\iota \in \Sigma_\infty} \mathscr{S}(\mathbb{V}_\iota^r)^{\mathsf{U}_\iota}) \otimes \mathscr{S}(\mathbb{V}_f^r)$ and $\mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_\infty K} = (\bigotimes_{\iota \in \Sigma_\infty} \mathscr{S}(\mathbb{V}_\iota^r)^{\mathsf{U}_\iota}) \otimes \mathscr{S}(\mathbb{V}_f^r)^K$ for an open compact subgroup K of $\mathbb{H}(\mathbb{A}_f)$. Recall that we have a Weil representation ω_χ of $H_r(\mathbb{A}_F)$, where $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ such that $\chi|_{\mathbb{A}_F^\times} = \epsilon_{E/F}^m$. Associated to this χ, we get a sequence $\mathfrak{k}^\chi = (\mathfrak{k}_\iota^\chi)_\iota \in \mathbb{Z}^{\Sigma_\infty}$ determined by $\chi_\iota(z) = z^{\mathfrak{k}_\iota^\chi}$ for $z \in E_\iota^{\times,1} = \mathbb{C}^{\times,1}$, hence m and $\mathfrak{k}^\chi$ have the same parity.

For $\phi \in \mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_\infty K}$, we define *Kudla's generating series* to be

$$Z_\phi(g) = \sum_{x \in K \backslash \mathbb{V}_f^r} \omega_\chi(g)\phi(T(x), x)Z(x)_K$$

as a series with values in $\mathsf{CH}^r(\mathsf{Sh}_K(\mathbb{H}))_{\mathbb{C}}$ for $g \in H_r(\mathbb{A}_F)$. Here for $\phi = \phi_\infty \phi_f$, we denote $\phi(T(x), x) = \phi_\infty(y)\phi_f(x)$ for any $y \in \mathbb{V}_\infty^r$ with $T(y) = T(x)$ which does not depend on the choice of y. This makes sense since $Z(x)_K \neq 0$ only for V_x admissible and hence $T(x)$ is totally semipositive definite. It is easy to see that $Z_\phi(g)$ is compatible under pull-backs of $\pi_K^{K'}$, hence defines a series with values in $\mathsf{CH}^r(\mathsf{Sh}(\mathbb{H}))_{\mathbb{C}} := \varinjlim_K \mathsf{CH}^r(\mathsf{Sh}_K(\mathbb{H}))_{\mathbb{C}}$.

3B. *Modularity of the generating series.* In this section, we are going to prove the modularity of the generating series. This is the only section where we use Shimura varieties of orthogonal groups.

Shimura varieties of orthogonal groups. The $\mathbb{A}_F$-module $\mathbb{V}$ is also a totally positive definite quadratic space over $\mathbb{A}_F$ of rank $2m$ with quadratic form $\frac{1}{2}\mathrm{Tr}_{\mathbb{A}_E/\mathbb{A}_F}(\,\cdot\,,\,\cdot\,)$. Then its discriminant is rational and it is incoherent. Let $\mathbb{G} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{GSpin}(\mathbb{V})$ be the special Clifford group of $\mathbb{V}$ with adjoint (quotient) group $\mathbb{G}^{\mathrm{ad}} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{SO}(\mathbb{V})$ and the derived subgroup $\mathbb{G}^{\mathrm{der}} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}}\mathrm{Spin}(\mathbb{V})$. For any open compact subgroup K' of $\mathbb{G}(\mathbb{A}_f)$, there is a Shimura variety $\mathsf{Sh}_{K'}(\mathbb{G})$ defined over the reflex field F such that, for any embedding $\iota : F \hookrightarrow \mathbb{C}$, we have the following ι-adic uniformization:

$$\mathsf{Sh}_{K'}(\mathbb{G})_{\iota}^{\mathrm{an}} \cong G^{(\iota)}(\mathbb{Q})\backslash\big(\mathscr{D}'^{(\iota)} \times \mathbb{G}(\mathbb{A}_f)/K'\big),$$

where the notation is similarly defined as in the unitary case. In particular, now the symmetric hermitian domain $\mathscr{D}'^{(\iota)}$ consists of all *oriented* negative definite 2-planes in $V_{\iota}^{(\iota)}$. We denote the corresponding Hodge bundles by $\mathscr{L}'_{K'}$, special cycles $Z'(x)_{K'}$ for $x \in \mathbb{V}_f^r$ and the generating series $Z'_{\phi}(g')$ for $\phi \in \mathscr{S}(\mathbb{V}^r)^{\mathsf{U}_{\infty}K'}$ and $g' \in G_r(\mathbb{A}_f)$ (see [Yuan et al. 2009]). Here we introduce the standard skew-symmetric F-space W'_r (comparing to the space W_r in Section 2A) which has a basis $\{e_1, \ldots, e_{2r}\}$ with symmetric form $\langle e_i, e_j\rangle = 0$, $\langle e_{r+i}, e_{r+j}\rangle = 0$, and $\langle e_i, e_{r+j}\rangle = \delta_{ij}$ for $1 \le i, j \le r$, and $G_r = \mathrm{Sp}(W'_r)$ which is an F-reductive group. Similarly, when defining the generating series, we have used the Weil representation ω (with respect to ψ) of $G_r(\mathbb{A}_F) \times \mathbb{G}(\mathbb{A})$ on $\mathscr{S}(\mathbb{V}^r)$.

Pull-back formulae. In this subsection, we will fix an embedding $\iota^{\circ} : E \hookrightarrow \mathbb{C}$ over ι and suppress the latter from the notation of nearby objects: $V = V^{(\iota)}, H = H^{(\iota)}, \mathscr{D} = \mathscr{D}^{(\iota^{\circ})}, \ldots$. Hence we have our usual notions of Shimura variety $\mathsf{Sh}_K(H, X)$ (resp. $\mathsf{Sh}_{K'}(G, X')$ with a connected component X^+ of X') which is defined (to be precise) over $\iota^{\circ}(E)$ (resp. $\iota(F)$). The neutral component is the connected Shimura variety $\mathsf{Sh}_K^{\circ}(H^{\mathrm{der}}, \overline{X})$ (resp. $\mathsf{Sh}_{K'}^{\circ}(G^{\mathrm{der}}, \overline{X}^+)$ attached to the connected Shimura datum $(H^{\mathrm{der}}, \overline{X})$ (resp. $(G^{\mathrm{der}}, \overline{X}^+)$) which is defined over E_K (resp. $E_{K'}$), a finite abelian extension of $\iota(F)$ in $\mathbb{C}$. The canonical embedding $H^{\mathrm{der}} \hookrightarrow G^{\mathrm{der}}$ (see Remark 3.3(a)) between reductive groups and the embedding $\mathscr{D} \hookrightarrow \mathscr{D}'$ by forgetting the E-action define an injective map of connected Shimura data $(H^{\mathrm{der}}, \overline{X}) \to (G^{\mathrm{der}}, \overline{X}^+)$ which hence gives an embedding $i_{K'} : \mathsf{Sh}_K^{\circ}(H^{\mathrm{der}}, \overline{X}) \hookrightarrow \mathsf{Sh}_{K'}^{\circ}(G^{\mathrm{der}}, \overline{X}^+)$ which is defined over E_K providing $K \cap H^{\mathrm{der}}(\mathbb{A}_f) = K' \cap H^{\mathrm{der}}(\mathbb{A}_f)$ and K' is sufficiently small. Let $Z(x)_K^{\circ}$ (resp. $Z'(x)_{K'}^{\circ}, Z_{\phi}(g)^{\circ}, Z'_{\phi}(g')^{\circ}$) be the restriction of $Z(x)_K$ (resp. $Z'(x)_{K'}, Z_{\phi}(g), Z'_{\phi}(g')$) to the neutral component.

Proposition 3.2. *Assume K' is small enough and $K \cap H^{\mathrm{der}}(\mathbb{A}_f) = K' \cap H^{\mathrm{der}}(\mathbb{A}_f)$. For $x \in \mathbb{V}_f$, the pull-back of the special divisor $i_{K'}^* Z'(x)_{K'}^{\circ}$ is the sum of $Z(x_1)_K^{\circ}$ indexed by the classes x_1 in $K\backslash K'x$, both considered as elements in Chow groups.*

Proof. If $x = 0$, the only class in $K \backslash K'x$ is $x_1 = 0$; the proposition follows from the compatibility of Hodge bundles under pull-backs induced by maps between (connected) Shimura data. Now we assume that $\langle x, x \rangle \in E$ which is totally positive. Suppose that $(z, h) \in \mathscr{D} \times H^{\mathrm{der}}(\mathbb{A}_f)$ represents a $\mathbb{C}$-point in the scheme-theoretic intersection $\mathsf{Sh}_K^\circ(H^{\mathrm{der}}, \overline{X}) \cap Z'(x_1)_{K'}^\circ$ for some $x_1 \in K'x$. Let $g \in G(\mathbb{A}_f)$ such that $gx_1 = x_1' \in V \subset \mathbb{V}_f$. Then $z \perp \gamma x_1'$ for some $\gamma \in G(\mathbb{Q})$ and $h \in \gamma G(\mathbb{A}_f)_{x_1'}gk'$ for some $k' \in K'$, where $G(\mathbb{A}_f)_{x_1'}$ is the subgroup of $G(\mathbb{A}_f)$ fixing x_1'. We now show that $\gamma G(\mathbb{A}_f)_{x_1'}gk' \cap H^{\mathrm{der}}(\mathbb{A}_f) = G(\mathbb{A}_f)_{\gamma x_1'}\gamma gk' \cap H^{\mathrm{der}}(\mathbb{A}_f) \neq \varnothing$, that is, $G(\mathbb{A}_f)_{\gamma x_1'} \cap H^{\mathrm{der}}(\mathbb{A}_f)k'^{-1}g^{-1}\gamma^{-1} \neq \varnothing$ which is true by Lemma 3.1. Hence (z, h) represents a $\mathbb{C}$-point in the special cycle $Z(h^{-1}E\langle \gamma gx_1 \rangle)_K^\circ$ of $\mathsf{Sh}_K^\circ(H^{\mathrm{der}}, \overline{X})$. If we write $h = g_1\gamma gk'$ with some $g_1 \in G(\mathbb{A}_f)_{\gamma x_1'}$, then

$$h^{-1}E\langle \gamma gx_1 \rangle = E\langle h^{-1}\gamma gx_1 \rangle = E\langle k'^{-1}g^{-1}\gamma^{-1}g_1^{-1}\gamma gx_1 \rangle = E\langle k'^{-1}x_1 \rangle.$$

Hence the scheme-theoretic intersection is indexed by the classes x_1 in $K \backslash K'x$. This is also true in the Chow group since the intersection is proper. $\qquad\square$

Remark 3.3. (a) The canonical embedding $H^{\mathrm{der}} \hookrightarrow G^{\mathrm{der}}$ is given in the following way: first, we have an embedding $H^{\mathrm{der}} \hookrightarrow H \hookrightarrow G^{\mathrm{ad}}$ by forgetting the E-action on $V = V^{(\iota)}$. Since H^{der} is simply connected, we have a canonical lifting $H^{\mathrm{der}} \hookrightarrow G$. Since H^{der} has no nontrivial abelian quotient, the image is in fact contained in G^{der}.

(b) In the proof of Proposition 3.2, we can still use the adèlic description of the $\mathbb{C}$-points of $\mathsf{Sh}_K^\circ(H^{\mathrm{der}}, \overline{X})$ (resp. $\mathsf{Sh}_{K'}^\circ(G^{\mathrm{der}}, \overline{X}^+)$) which is compatible with that of $\mathsf{Sh}_K(\mathbb{H})$ (resp. $\mathsf{Sh}_{K'}(\mathbb{G})$) since H^{der} (resp. G^{der}) is semisimple, of noncompact type, and simply connected.

The group G_r is canonically embedded in H_r by identifying the basis $\langle e_1, \ldots, e_{2r} \rangle$ of W_r' and W_r and hence $\omega_\chi|_{G_r} = \omega$. From Proposition 3.2, we have

Corollary 3.4. *Let $r = 1$ and K, K' as in Proposition 3.2. Then $i_{K'}^* Z_\phi'(g')^\circ = Z_\phi(g')^\circ$ for $g' \in G_1(\mathbb{A}_F)$ and $\phi \in \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K'}$.*

Modularity. For a linear functional $\ell \in \mathsf{CH}^r(\mathsf{Sh}(\mathbb{H}))_\mathbb{C}^*$, we have a complex-valued series

$$\ell(Z_\phi)(g) = \sum_{x \in K \backslash \mathbb{V}_f^r} \omega_\chi(g)\phi(T(x), x)\ell(Z(x)_K)$$

for any K such that ϕ is invariant under K (which is of course independent of such choice). Our main theorem in this section is this:

Theorem 3.5 (modularity of the generating series). (1) *If $\ell(Z_\phi)(g)$ is absolutely convergent, then it is an automorphic form of $H_r(\mathbb{A}_F)$. Moreover, $\ell(Z_\phi)_\infty$ is in a discrete series representation of weight $((m + \mathfrak{k}^\chi)/2, (m - \mathfrak{k}^\chi)/2)$.*

(2) *If $r = 1$, then $\ell(Z_\phi)(g)$ is absolutely convergent for any ℓ.*

Proof. (1) We proceed as in [Yuan et al. 2009, Section 4]. First, we can assume that $\phi = \phi_\infty^0 \otimes \phi_f$ since other cases will follow from the (Lie $H_{r,\infty}$, $\mathcal{K}_{r,\infty}$)-action. Assuming the absolute convergence of $\ell(Z_\phi)(g)$, we only need to check the automorphy, that is, the invariance under left translation of $H_r(F)$. The weight part is clear.

It is easy to check the invariance under $n(b)$ and $m(a)$. For $b \in \mathrm{Her}_r(E)$, the matrix $bT(x)$ is F-rational if $Z(x)_K \neq 0$, hence $\ell(Z_\phi)(n(b)g) = \ell(Z_\phi)(g)$ for all $g \in H_r(\mathbb{A}_F)$. For $a \in \mathrm{GL}_r(E)$, we have $Z(xa)_K = Z(x)_K$, hence

$$\ell(Z_\phi)(m(a)g) = \ell(Z_\phi)(g).$$

Since $H_r(F)$ is generated by $n(b)$, $m(a)$ and $w_{r,r-1}$, where

$$w_{r,d} = \begin{pmatrix} \mathbf{1}_d & & & \\ & & & \mathbf{1}_{r-d} \\ & & \mathbf{1}_d & \\ & -\mathbf{1}_{r-d} & & \end{pmatrix}, \quad 0 \leq d \leq r, \tag{3-2}$$

we only need to check that $\ell(Z_\phi)(w_{r,r-1}g) = \ell(Z_\phi)(g)$ for all $g \in H_r(\mathbb{A}_F)$. Assuming this for $r = 1$ (see Lemma 3.6), we now prove it for general $r > 1$, following [Yuan et al. 2009; Zhang 2009].

Recall that we have assumed that $\phi = \phi_\infty^0$ and we suppress ℓ from the notation for simplicity. Then for K sufficiently small, we have

$$Z_\phi(w_{r,r-1}g) = \sum_{x \in K \backslash \mathbb{V}_f^r} \omega_\chi(w_{r,r-1}g)\phi(T(x), x)Z(x)_K$$

$$= \sum_{x \in K \backslash \mathbb{V}_f^{r-1}} \sum_{y \in K_x \backslash \mathbb{V}_f} \omega_\chi(w_{r,r-1}g)\phi(T(x,y), (x,y))Z((x,y))_K, \tag{3-3}$$

where K_x is the stabilizer of x in K, and

$$(3\text{-}3) = \sum_{x \in K \backslash \mathbb{V}_f^{r-1}} \sum_{y_1 \in K_x \backslash \mathbb{V}_f^x} \sum_{y_2 \in V_x} \omega_\chi(w_{r,r-1}g)$$

$$\times \phi(T(x, y_1 + y_2), (x, y_1 + y_2))Z((x, y_1 + y_2))_K, \tag{3-4}$$

where $\mathbb{V}_f^x$ is the orthogonal complement of $V_x = E\langle x \rangle$ in $\mathbb{V}_f$. Recall the morphism ς^{V_x} in (3-1); we have

$$(3\text{-}4) = \sum_{x \in K \backslash \mathbb{V}_f^{r-1}} \sum_{y_1 \in K_x \backslash \mathbb{V}_f^x} \sum_{y_2 \in V_x} \omega_\chi(w_{r,r-1})(\omega_\chi(g)\phi)(T(x, y_1 + y_2), (x, y_1 + y_2))$$

$$\times \varsigma_*^{V_x} Z(y_1)_{K_x}. \tag{3-5}$$

Applying the case $r = 1$ to the special cycle $Z(V_x)_{K_x}$, we have

$$(3\text{-}5) = \sum_{x \in K \backslash \mathbb{V}_f^{r-1}} \sum_{y_1 \in K_x \backslash \mathbb{V}_f^x} \sum_{y_2 \in V_x} \omega_\chi(w_{r,r-1}) \widehat{\omega_\chi(g)\phi}^{y_1}(T(x, y_1 + y_2), (x, y_1 + y_2))$$
$$\times \varsigma_*^{V_x} Z(y_1)_{K_x}, \quad (3\text{-}6)$$

where the superscript y_1 means taking the Fourier transformation with respect to the y_1 coordinate. Applying the Poisson formula (recall that ϕ_∞ is the Gaussian), we have

$$(3\text{-}6) = \sum_{x \in K \backslash \mathbb{V}_f^{r-1}} \sum_{y_1 \in K_x \backslash \mathbb{V}_f^x} \sum_{y_2 \in V_x} \omega_\chi(w_{r,r-1}) \widehat{\omega_\chi(g)\phi}^{y_1,y_2}(T(x, y_1+y_2), (x, y_1+y_2))$$
$$\times \varsigma_*^{V_x} Z(y_1)_{K_x}$$

$$= \sum_{x \in K \backslash \mathbb{V}_f^{r-1}} \sum_{y_1 \in K_x \backslash \mathbb{V}_f^x} \sum_{y_2 \in V_x} \omega_\chi(w_{r,r-1}) \widehat{\omega_\chi(g)\phi}^{y}(T(x, y_1 + y_2), (x, y_1 + y_2))$$
$$\times \varsigma_*^{V_x} Z(y_1)_{K_x}$$

$$= \sum_{x \in K \backslash \mathbb{V}_f^{r-1}} \sum_{y_1 \in K_x \backslash \mathbb{V}_f^x} \sum_{y_2 \in V_x} \omega_\chi(g)\phi(T(x, y_1 + y_2), (x, y_1 + y_2)) \varsigma_*^{V_x} Z(y_1)_{K_x}$$

$$= \sum_{x \in K \backslash \mathbb{V}_f^r} \omega_\chi(g)\phi(T(x), x) Z(x)_K = Z_\phi(g).$$

This finishes the proof of (1).

(2) follows from the argument in Lemma 3.6, following Corollary 3.4 and [Yuan et al. 2009, Theorem 1.3], which uses the result in [Kudla and Millson 1990]. □

Lemma 3.6. *If $r = 1$, then $\ell(Z_\phi)(w_1 g) = \ell(Z_\phi)(g)$ for all $g \in H_1(\mathbb{A}_F)$.*

Proof. We suppress ℓ from the notation. Further, we fix any $\iota^\circ \in \Sigma_\infty^\circ$ over $\iota \in \Sigma_\infty$ and suppress them as in the previous subsection. It is clear that we only need to prove that $Z_\phi(w_1 g) = Z_\phi(g)$ for $g \in G_1(\mathbb{A}_f)$ since $G_1(\mathbb{A}_{\infty,F}) \mathcal{H}_{1,\infty} = H_1(\mathbb{A}_{\infty,F})$. As before, we assume that ϕ_∞ is the Gaussian and K is sufficiently small. Recall that $\pi_0(\mathrm{Sh}_K(H, X)_{\iota,\mathbb{C}}) \cong T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / \det(K)$. We have the following inclusion:

$$\mathrm{CH}^1(\mathrm{Sh}_K(H, X))_\mathbb{C} \hookrightarrow \bigoplus_{\{t\} \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / \det(K)} \mathrm{CH}^1(\mathrm{Sh}_K(H, X)_{\{t\}})_\mathbb{C}, \quad (3\text{-}7)$$

where $\mathrm{Sh}_K(H, X)_{\{t\}}$ is the (canonical model of the) corresponding (geometric) connected component. Let $h \in H(\mathbb{A}_f)$ such that $\det(h) = t$ and let T_h be the Hecke operator. Then $T_h : \mathrm{Sh}_{K^h}(H, X) \to \mathrm{Sh}_K(H, X)$ induces

$$T_h^\circ : \mathrm{Sh}_{K^h}^\circ(H^{\mathrm{der}}, \bar{X}) = \mathrm{Sh}_{K^h}(H, X)_{\{1\}} \xrightarrow{\sim} \mathrm{Sh}_K(H, X)_{\{h\}} \hookrightarrow \mathrm{Sh}_K(H, X),$$

where $K^h = hKh^{-1}$. We have $T_h^{\circ,*} Z_\phi(g) = Z_\phi(g)^\circ$ which is the image of $Z_\phi(g)$ under the projection to $\mathsf{CH}^1(\mathsf{Sh}_K(H, X)_{\{t\}})_{\mathbb{C}}$ under (3-7). Here $Z_\phi(g)^\circ$ is the generating series on $\mathsf{Sh}_{K^h}^\circ(H^{\mathsf{der}}, \overline{X})$. Now shrinking K^h if necessary such that we can apply Corollary 3.4, we have $Z_\phi(g)^\circ = i_{K'}^* Z_\phi'(g)^\circ$ for $g \in G_1(\mathbb{A}_f)$. Applying [Yuan et al. 2009, Theorem 1.2 or Theorem 1.3], we conclude that $Z_\phi(w_1 g)^\circ = Z_\phi(g)^\circ$. The lemma follows by (3-7). $\hspace{2cm} \square$

3C. *Smooth compactification of unitary Shimura varieties.* In this section, we introduce the canonical smooth compactification of the unitary Shimura varieties if they are not proper and the compactified generating series on them.

Let $m \geq 2$ be an integer, $E = \mathbb{Q}(\sqrt{-D}) \subset \mathbb{C}$ for some square-free integer $D > 0$, let $\mathbb{O}_E$ be its ring of integers, and let τ be the nontrivial Galois involution on E. Let $(V, (\,\cdot\,,\,\cdot\,))$ be a hermitian space of dimension m over E whose signature is $(m-1, 1)$. If $m = 2$, we further assume that $\det V \in \mathsf{Nm}\,E^\times$. Let $H = \mathsf{U}(V)$ be the unitary group; we have the Hodge map $h : \mathbb{S} \to H_{\mathbb{R}} \cong \mathsf{U}(m-1, 1)_{\mathbb{R}}$ given by

$$h(z) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \bar{z}/z \end{pmatrix}.$$

Then we have the notion of the Shimura variety $\mathsf{Sh}_K(H, h)$ for any open compact subgroup K of $H(\mathbb{A}_f)$. For K sufficiently small, it is smooth and quasiprojective but nonproper over E of dimension $m - 1$. Hence we need to construct a smooth compactification of $\mathsf{Sh}_K(H, h)$ such that we can do height pairing. When $m = 2$, it is trivial since we only need to add cusps. When $m = 3$ and H is quasisplit, a canonical smooth compactification (even of the integral model) has been constructed in [Larsen 1992]. In fact, the same construction works in the more general case (just for compactification of the canonical model), namely any H appearing here. We should mention that, if the signature of V is (a, b) such that $a \geq b > 1$ or V is over a totally real field but not $\mathbb{Q}$ and indefinite at any archimedean place, then we should not hope that there exists a canonical smooth compactification.

Now let us assume $m > 2$. Since we are going to use modular interpretations, we should work with the group of unitary similitude. For any $v, w \in V$,

$$(v, w)' = \mathsf{Tr}_{E/\mathbb{Q}}(\sqrt{-D}(v, w))$$

defines an alternating form of V satisfying $(ev, w) = (v, e^\tau w)$ for any $e \in E$. Let $GH = \mathsf{GU}(V)$ such that for any $\mathbb{Q}$-algebra R,

$$GH(R) = \{h \in \mathsf{GL}_m(E \otimes_{\mathbb{Q}} R) \mid (hv, hw)' = \lambda(h)(v, w)' \text{ for some } \lambda(h) \in R^\times\}$$

and the Hodge map $Gh : \mathbb{S} \to GH_{\mathbb{R}} \cong \mathsf{GU}(m-1, 1)_{\mathbb{R}}$ is given by

$$Gh(z) = \begin{pmatrix} z & & & \\ & \ddots & & \\ & & z & \\ & & & \bar{z} \end{pmatrix}.$$

For any sufficiently small open compact subgroup K of $GH(\mathbb{A}_f)$, we have the Shimura variety $\mathsf{Sh}_K(GH, Gh)$ which is smooth and quasiprojective but nonproper over E of dimension $m-1$. Although we don't have a map between Shimura data, $\mathsf{Sh}_{K \cap H(\mathbb{A}_f)}(H, h)$ and $\mathsf{Sh}_K(GH, Gh)$ have the same neutral component for sufficiently small K. Hence it is the same to give a canonical smooth compactification of $\mathsf{Sh}_K(GH, Gh)$ instead of the original one. In fact, $\mathsf{Sh}_K(GH, Gh)$ is a moduli space of abelian varieties of certain PEL type. We fix a lattice $V_{\mathbb{Z}}$ of V such that $V_{\mathbb{Z}} \subset V_{\mathbb{Z}}^{\perp}$ and let $V_{\hat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$. Then Sh_K represents the following functor: for any (S, s) where S is a connected, locally noetherian E-scheme with a geometric point s, $\mathsf{Sh}_K(GH, Gh)(S, s)$ is the isomorphism classes of quadruples $(A, \theta, i, \bar{\eta})$, where

- A is an abelian scheme over S of dimension m;

- $\theta : A \to A^{\vee}$ is a polarization;

- $i : \mathcal{O}_E \hookrightarrow \mathsf{End}(A)$ such that $\mathrm{tr}(i(e); \mathrm{Lie}_S(A)) = (m-1)e + e^{\tau}$ and $\theta \circ i(e) = i(e^{\tau})^{\vee} \circ \theta$ for all $e \in \mathcal{O}_E$;

- $\bar{\eta}$ is a $\pi_1(S, s)$-invariant K-class of $\mathcal{O}_E \otimes \hat{\mathbb{Z}}$-linear symplectic similitude $\eta : V_{\hat{\mathbb{Z}}} \to \mathsf{H}_1^{\mathrm{\acute{e}t}}(A_s, \hat{\mathbb{Z}})$, where the pairing on the latter space is the θ-Weil pairing; hence the degree of θ is $[V_{\mathbb{Z}}^{\perp} : V_{\mathbb{Z}}]$.

Here in the third condition, $1 \in \mathcal{O}_E$ goes to the identity endomorphism and we view $(m-1)e + e^{\tau}$ as a constant section of $\mathcal{O}_S$ via the structure map $E \to \mathcal{O}_S$.

In the theory of toroidal compactification (see [Ash et al. 1975]), we need to choose a rational polyhedral cone decomposition. But in our case, we only have one unique choice, namely a torus in an affine line. We claim that there is a scheme $\mathsf{Sh}_{\widetilde{K}}(GH, Gh)$ with these properties:

- $\mathsf{Sh}_{\widetilde{K}}(GH, Gh)$ is smooth and proper over E.

- $\tilde{i}_K : \mathsf{Sh}_K(GH, Gh) \hookrightarrow \mathsf{Sh}_{\widetilde{K}}(GH, Gh)$ is an open immersion and for $K' \subset K$ there is a morphism $\tilde{\pi}_K^{K'}$ such that the diagram

$$\begin{array}{ccc} \mathsf{Sh}_{K'}(GH, Gh) & \xrightarrow{\tilde{i}_{K'}} & \mathsf{Sh}_{\widetilde{K'}}(GH, Gh) \\ \pi_K^{K'} \downarrow & & \downarrow \tilde{\pi}_K^{K'} \\ \mathsf{Sh}_K(GH, Gh) & \xrightarrow{\tilde{i}_K} & \mathsf{Sh}_{\widetilde{K}}(GH, Gh) \end{array}$$

commutes.

- The boundary $GY_K = \mathsf{Sh}_{\tilde{K}}(GH, Gh) - \mathsf{Sh}_K(GH, Gh)$ is a smooth divisor defined over E and each geometric component is isomorphic to an extension of an abelian variety of dimension $m - 2$ by a finite group.

The boundary part GY_K parametrizes the degeneration of abelian varieties with the above PEL data. We consider a semiabelian variety G with $i : \mathbb{O}_E \hookrightarrow \mathsf{End}(G)$ such that $\mathrm{tr}(i(e); \mathsf{Lie}(G)) = (m - 1)e + e^\tau$; then for any $e \in \mathbb{O}_E$, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & T & \xrightarrow{t} & G & \xrightarrow{\alpha} & A & \longrightarrow & 0 \\
& & & & \downarrow{\scriptstyle i(e)} & & & & \\
0 & \longrightarrow & T & \xrightarrow{t} & G & \xrightarrow{\alpha} & A & \longrightarrow & 0.
\end{array}
$$

Then the composition $\alpha \circ i(e) \circ t$ is trivial. Thus i induces actions of $\mathbb{O}_E$ on both torus part T and abelian variety part A. Suppose $X(T) = \mathbb{Z}^r$ with $r > 0$. Then r is even since E is quadratic imaginary. Further assuming $\mathrm{tr}(i(e); \mathsf{Lie}(T)) = ae + be^\tau$, we have $a + b = r$ and $a = b$. Hence there is only one possibility, namely $a = b = 1$ and $r = 2$. Then T is of rank 2 and A is an abelian variety of dimension $m - 2$ such that $\mathbb{O}_E$ acts on A and $\mathrm{tr}(i(e); \mathsf{Lie}(A)) = (m - 2)e$. Let A_1 be an elliptic curve of CM type $(\mathbb{O}_E, e \mapsto e)$. Then A is isogenous to A_1^{m-2}. Each geometric point s of GY_K corresponds to a semiabelian variety $G_s = (T_s \hookrightarrow G_s \to A_s)$ as above with certain level structure which will be defined later. For two geometric points s, s' in the same geometric connected component, the abelian variety part $A_s \cong A_{s'}$ and the rank 1 $\mathbb{O}_E$-modules $X(T_s)$ and $X(T_{s'})$ are isomorphic. It is easy to see that if A and T are fixed, then the set of such G, up to isomorphism, is parametrized by $X(T) \otimes_{\mathbb{O}_E} A$ which is an abelian variety of dimension $m - 2$.

To include the level structure, we only stick in one geometric component since it is same to others. This means that we fix T and A with $\mathbb{O}_E$-actions but, of course, not G. Let us fix a maximal isotropic subspace W of $V_{\mathbb{Z}}$. Then W is of rank 1. We have a filtration $0 \subset W \subset W^\perp \subset V_{\mathbb{Z}}$. Let B_W be the subgroup of $H(\mathbb{A}_f)$ that preserves this filtration, $N_W \subset B_W$ that acts trivially on the associated graded modules, and $U_W \subset N_W$ that acts trivially on $W^\perp$ and $V_W = N_W / U_W$. We also fix a generator w of W. On the other hand, fix an $\mathbb{O}_E \otimes \widehat{\mathbb{Z}}$ generator w_T of $\mathsf{H}_1^{\text{ét}}(T, \widehat{\mathbb{Z}})$ and a polarization $\theta_A : A \to A^\vee$ such that there exists a symplectic similitude between $\mathsf{H}_1^{\text{ét}}(A, \widehat{\mathbb{Z}})$ and $V_W \otimes \widehat{\mathbb{Z}}$. For a sufficiently small open compact subgroup $K \subset H(\mathbb{A}_f)$, let $N_{W,K} = N_W \cap K$, $U_{W,K} = U_W \cap K$, and $V_{W,K} = N_{W,K} / U_{W,K}$. Then the level structure of $(G, T \hookrightarrow G \to A)$ with respect to K is a $V_{W,K}$-class of isomorphisms $W^\perp \otimes \widehat{\mathbb{Z}} \to \mathsf{H}_1^{\text{ét}}(G, \widehat{\mathbb{Z}})$ which sends w to w_T and induces a symplectic similitude between $V_W \otimes \widehat{\mathbb{Z}}$ and $\mathsf{H}_1^{\text{ét}}(A, \widehat{\mathbb{Z}}) = \mathsf{H}_1^{\text{ét}}(G, \widehat{\mathbb{Z}})/\mathbb{O}_E \cdot w_T$. We conclude that any geometric component of GY_K is isomorphic to (a connected component of) an extension of $X(T) \otimes_{\mathbb{O}_E} A$ by $V_W / V_{W,H}$ for some T and A as above. There is a

universal object $\pi : \mathcal{G} \to \mathsf{Sh}_K^{\sim}(GH, Gh)$ which is a semiabelian scheme of relative dimension m.

Now we come back to the Shimura variety $\mathsf{Sh}_K := \mathsf{Sh}_K(H, h)$. As we have said, the canonical smooth compactification above gives a canonical smooth compactification for Sh_K, which we denote by $\mathsf{Sh}_K^{\sim}$, $Y_K = \mathsf{Sh}_K^{\sim} - \mathsf{Sh}_K$; then they will have the same property as above. We also apply the notation above to the trivial case $m = 2$. We let $\mathscr{L}_K^{\sim}$ be the line bundle on $\mathsf{Sh}_K^{\sim}$ induced from $\bigwedge^m \pi_* \Omega_{\mathcal{G}/\mathsf{Sh}_K^{\sim}(GH, Gh)}$ on $\mathsf{Sh}_K^{\sim}(GH, Gh)$ which is an extension of the Hodge bundle $\mathscr{L}_K$ on Sh_K (see Section 3A). By the canonicality of the compactification, $(\mathscr{L}_K^{\sim})_K$ defines an element in $\mathrm{Pic}(\mathsf{Sh}^{\sim})_{\mathbb{C}} = \varinjlim_K \mathrm{Pic}(\mathsf{Sh}_K^{\sim})_{\mathbb{C}}$. We also need to extend special cycles and hence the generating series. For $1 \le r < m$ and $x \in V_f^r := V^r \otimes_{\mathbb{Q}} \mathbb{A}_f$, we define the compactified special cycle as

$$Z(x)_K^{\sim} = \begin{cases} Z(V_x)_K^{\sim} c_1(\mathscr{L}_K^{\sim,\vee})^{r - \dim_E V_x} & \text{if } V_x \text{ is admissible,} \\ 0 & \text{otherwise,} \end{cases}$$

where $Z(V_x)_K^{\sim}$ is just the Zariski closure of $Z(V_x)_K$ in $\mathsf{Sh}_K^{\sim}$. We define the compactified generating series by formal series in $\mathrm{CH}^r(\mathsf{Sh}^{\sim})_{\mathbb{C}} = \varinjlim_K \mathrm{CH}^r(\mathsf{Sh}_K^{\sim})_{\mathbb{C}}$:

$$Z_\phi^{\sim}(g) = \begin{cases} \displaystyle\sum_{x \in K \backslash V_f^r} \omega_\chi(g)\phi(T(x), x)Z(x)_K^{\sim} & \text{if } m > 2, \\ \displaystyle\sum_{x \in K \backslash V_f} \omega_\chi(g)\phi(T(x), x)Z(x)_K^{\sim} + W_0\big(\tfrac{1}{2}, g, \phi\big)c_1(\mathscr{L}_K^{\sim,\vee}) & \text{if } r = 1, m = 2, \end{cases}$$

for $g \in H_r(\mathbb{A})$ and $\phi \in \mathscr{S}(V^r)^{\mathsf{U}_\infty}$. Here, $W_0(s, g, \phi) = \prod_v W_0(s, g_v, \phi_v)$, which is holomorphic at $s = \frac{1}{2}$. Moreover, we define the following *positive partial compactified generating series* as

$$Z_{\phi,+}^{\sim}(g) = \sum_{\substack{x \in K \backslash V_f^r \\ T(x) \gg \mathbf{0}_r}} \omega_\chi(g)\phi(T(x), x)Z(x)_K^{\sim},$$
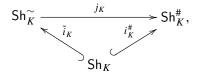
where the sum is taken over all x such that $T(x)$ is totally positive definite. We would like to propose the following conjecture on the *modularity of the compactified generating series*:

Conjecture 3.7. *Let ℓ be a linear functional on $\mathrm{CH}^r(\mathsf{Sh}^{\sim})_{\mathbb{C}}$ such that $\ell(Z_\phi^{\sim})(g)$ is absolutely convergent. Then if $1 \le r \le m - 2$, $\ell(Z_\phi^{\sim})(g)$ is a holomorphic automorphic form of $H_r(\mathbb{A}_F)$; if $r = 1, m = 2$, $\ell(Z_\phi^{\sim})(g)$ is an automorphic form of $H_1(\mathbb{A}_F)$, not necessarily holomorphic; in general, if $r = m - 1$, $\ell(Z_{\phi,+}^{\sim})(g)$ is the sum of the positive-definite Fourier coefficients of an automorphic form of $H_{m-1}(\mathbb{A}_F)$.*

The case $m = 2$ ($r = 1$) will be proved in [Liu 2011, Section 3B] and is actually not far from Theorem 3.5 as we point out there.

Fix a rational prime ℓ. There are class maps $\mathsf{cl} : \mathsf{CH}^r(\mathsf{Sh}_K^{\sim})_{\mathbb{C}} \to \mathsf{H}_{\text{ét}}^{2r}(\mathsf{Sh}_K^{\sim} \times_E E^{\mathsf{ac}}, \mathbb{Z}_\ell(r))^{\Gamma_E} \otimes_{\mathbb{Z}_\ell} \mathbb{C}$ compatible under $\tilde{\pi}_K^{K'}$, which induces $\mathsf{cl} : \mathsf{CH}^r(\mathsf{Sh}^{\sim})_{\mathbb{C}} \to \mathsf{H}_{\text{ét}}^{2r}(\mathsf{Sh}^{\sim} \times_E E^{\mathsf{ac}}, \mathbb{Z}_\ell(r))^{\Gamma_E} \otimes_{\mathbb{Z}_\ell} \mathbb{C} \subset \mathsf{H}_{\mathsf{B}}^{2r}(\mathsf{Sh}^{\sim}, \mathbb{C})$ (the Betti cohomology) where the two cohomology groups are defined as inductive limits as K varies. Let $\mathsf{H}_Y^{\bullet}(\mathsf{Sh}^{\sim}, \mathbb{C}) = \varinjlim_K \mathsf{H}_{Y_K}^{\bullet}(\mathsf{Sh}_K^{\sim}(\mathbb{C}), \mathbb{C})$ be the inductive limit of cohomology groups with support in Y_K as K varies. Then since Y is a smooth divisor, we have $\mathsf{H}_Y^{\bullet}(\mathsf{Sh}^{\sim}, \mathbb{C}) \cong \mathsf{H}_{\mathsf{B}}^{\bullet-2}(Y, \mathbb{C}) = \varinjlim_K \mathsf{H}_{\mathsf{B}}^{\bullet-2}(Y_K(\mathbb{C}), \mathbb{C})$.

On the other hand, let us denote by $\mathsf{Sh}_K^{\#}$ the Baily–Borel compactification of Sh_K. Hence we have the commutative diagram

$$\mathsf{Sh}_K^{\sim} \xrightarrow{\;\; j_K \;\;} \mathsf{Sh}_K^{\#},$$

with maps $\tilde{i}_K$, $i_K^{\#}$ to Sh_K

which is compatible when K varies, and, more importantly, Hecke equivariant. We also denote by $\mathsf{IH}^{\bullet}(\mathsf{Sh}^{\#}, \mathbb{C}) = \varinjlim_K \mathsf{IH}^{\bullet}(\mathsf{Sh}_K^{\#}(\mathbb{C}), \mathbb{C})$ the inductive limit of the intersection cohomology groups. Then by [Beĭlinson et al. 1982, Théorème 6.2.5], we have the exact sequence

$$\mathsf{H}_Y^{\bullet}(\mathsf{Sh}^{\sim}, \mathbb{C}) \longrightarrow \mathsf{H}_{\mathsf{B}}^{\bullet}(\mathsf{Sh}^{\sim}, \mathbb{C}) \xrightarrow{\;\; j_* \;\;} \mathsf{IH}^{\bullet}(\mathsf{Sh}^{\#}, \mathbb{C}) \longrightarrow 0. \qquad (3\text{-}8)$$

Let $\mathsf{H}_{\partial}^{\bullet}(\mathsf{Sh}^{\sim})$ be the image of the first map which is isomorphic to a quotient of $\mathsf{H}_{\mathsf{B}}^{\bullet-2}(Y, \mathbb{C})$.

3D. *Arithmetic theta lifting and inner product formula.* We next define the arithmetic theta lifting and prove its cohomological triviality under certain assumptions. We then formulate the conjectural arithmetic inner product formula in general.

Arithmetic theta lifting. We assume Conjecture 3.7 and the following assumptions on A-packets which are a certain part of the Langlands–Arthur conjecture (see [Arthur 1984; 1989]); they should be proved by a similar method to that in [Arthur 2012] (which handles the case of symplectic and orthogonal groups):

- A-packets are defined for all unitary groups $\mathsf{U}(m)_{/F}$. We denote by $\mathsf{AP}(\mathsf{U}(m)_{\mathbb{A}_F})$ the set of A-packets of $\mathsf{U}(m)$ and by $\mathsf{AP}(\mathsf{U}(m)_{\mathbb{A}_F})_{\mathsf{disc}} \subset \mathsf{AP}(\mathsf{U}(m)_{\mathbb{A}_F})$ the subset of discrete A-packets.

- If Π_1 and Π_2 are in $\mathsf{AP}(\mathsf{U}(m)_{\mathbb{A}_F})_{\mathsf{disc}}$ such that for almost all $v \in \Sigma$, $\Pi_{1,v}$ and $\Pi_{2,v}$ contain the same unramified representation, then $\Pi_1 = \Pi_2$.

- Let $\mathsf{U}(m)^*$ be the quasisplit unitary group. Then we have the correspondence between A-packets of inner forms: $\mathsf{JL} : \mathsf{AP}(\mathsf{U}(m)_{\mathbb{A}_F})_{\mathsf{disc}} \to \mathsf{AP}(\mathsf{U}(m)_{\mathbb{A}_F}^*)_{\mathsf{disc}}$.

Definition 3.8. Let π be an irreducible cuspidal automorphic representation of $H_r(\mathbb{A}_F)$ realized in $L^2(H_r(F)\backslash H_r(\mathbb{A}_F))$. We assume that $1 \le r \le m - 2$ or $r = 1, m = 2$. For any $\phi \in \mathscr{S}(\mathbb{V}^r)^{U_\infty}$ and any cusp form $f \in \pi$, the integral

$$
\Theta_\phi^f = \begin{cases} \int_{H_r(F)\backslash H_r(\mathbb{A}_F)} f(g) Z_\phi(g)\, dg \in \mathsf{CH}^r(\mathsf{Sh})_\mathbb{C} & \text{if } \mathsf{Sh} \text{ is proper,} \\ \int_{H_r(F)\backslash H_r(\mathbb{A}_F)} f(g) Z_{\widetilde{\phi}}(g)\, dg \in \mathsf{CH}^r(\mathsf{Sh}^\sim)_\mathbb{C} & \text{otherwise,} \end{cases}
$$

is called the *arithmetic theta lifting* of f which is a (formal sum of) codimension r cycle(s) on a certain (compactified) Shimura variety of dimension $m - 1$. Its cohomology class (restricted to Sh) is well-defined due to [Kudla and Millson 1990]. The original idea of this construction comes from Kudla; see [Kudla 2003, Section 8] or [Kudla et al. 2006, Section 9.1]. He constructed the arithmetic theta series as an Arakelov divisor on a certain integral model of a Shimura curve.

In the following discussion, let $m = 2n$ and $r = n$. Let π be an irreducible cuspidal automorphic representation of $H_n(\mathbb{A}_F)$, χ a character of $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$ such that π_∞ is a discrete series representation of weight $(n - \ell^\chi/2, n + \ell^\chi/2)$, and $\epsilon(\pi, \chi) = -1$. Then the (equal-rank) theta correspondence of π_ι (under ω_χ) is the trivial representation of $U(2n, 0)_\mathbb{R}$ for any archimedean place ι. Hence $\mathbb{V}(\pi, \chi)$ is a totally positive-definite incoherent hermitian space over $\mathbb{A}_E$. Now we fix an incoherent hermitian space $\mathbb{V}$ which is totally positive-definite of rank $2n$ and let $(\mathsf{Sh}_K)_K$ be the associated Shimura varieties.

We fix an embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ inducing $\iota : F \hookrightarrow \mathbb{C}$ if $F \ne \mathbb{Q}$. Then similarly we have the class map $\mathsf{cl} : \mathsf{CH}^\bullet(\mathsf{Sh}_K)_\mathbb{C} \to \mathsf{H}_\mathsf{B}^{2\bullet}(\mathsf{Sh}_{K,\iota^\circ}(\mathbb{C}), \mathbb{C})$. By a theorem in [Zucker 1982, Section 6] concerning the L^2-cohomology and the intersection cohomology, we have a (compatible system of) Hecke equivariant isomorphisms:

$$
\mathsf{H}_{(2)}^\bullet(\mathsf{Sh}_K) = \begin{cases} \mathsf{H}_\mathsf{B}^\bullet(\mathsf{Sh}_{K,\iota^\circ}(\mathbb{C}), \mathbb{C}) & \text{if } \mathsf{Sh}_K \text{ is proper,} \\ \mathsf{IH}^\bullet(\mathsf{Sh}_K^\#, \mathbb{C}) & \text{otherwise.} \end{cases}
$$

We let

$$
\mathsf{H}_{(2)}^\bullet(\mathsf{Sh}) = \varinjlim_K \mathsf{H}_{(2)}^\bullet(\mathsf{Sh}_K).
$$

In the nonproper case, we compose the map j_* in (3-8) to get a class map still denoted by $\mathsf{cl} : \mathsf{CH}^\bullet(\mathsf{Sh}^\sim) \to \mathsf{H}_{(2)}^{2\bullet}(\mathsf{Sh})$.

Proposition 3.9. *The class* $\mathsf{cl}(\Theta_\phi^f) = 0$ *in* $\mathsf{H}_{(2)}^{2n}(\mathsf{Sh})$, *that is, if* Sh *is proper* (*resp. nonproper*), Θ_ϕ^f *is cohomologically trivial* (*resp. such that* $\mathsf{cl}(\Theta_\phi^f) \in \mathsf{H}_\partial^{2n}(\mathsf{Sh}^\sim)$).

Proof. If Sh is nonproper, we can assume $n > 1$. By our definition of the arithmetic theta lifting, for $\phi = \phi_\infty \phi_f$ with fixed ϕ_∞, $\mathsf{cl}(\Theta_\phi^f)$ defines an element in

$$
\mathsf{Hom}_{H_n(\mathbb{A}_{f,F})}\big(\mathscr{S}(\mathbb{V}_f^n) \otimes \pi_f, \mathsf{H}_{(2)}^{2n}(\mathsf{Sh})\big),
$$

where $H_n(\mathbb{A}_{f,F})$ acts trivially on the L^2-cohomology.

Let $V^{(\iota)}$ be the nearby hermitian space of $\mathbb{V}$ at ι (see Section 3A) and $H^{(\iota)} = \mathrm{U}(V^{\iota})$. Then since $Z_{\widetilde{\omega(h)\phi}}(g) = T_h^* Z_{\widetilde{\phi}}(g)$ for all $h \in H^{(\iota)}(\mathbb{A}_{f,F})$ where T_h is the Hecke operator of h, we see that $\mathrm{cl}(\Theta_\phi^f)$ in fact defines an element

$$
\begin{aligned}
H_{\Theta,\phi_\infty} &\in \mathrm{Hom}_{H_n(\mathbb{A}_{f,F}) \times H^{(\iota)}(\mathbb{A}_{f,F})}\big(\mathscr{S}(\mathbb{V}_f^n) \otimes \pi_f, \mathsf{H}_{(2)}^{2n}(\mathrm{Sh})\big) \\
&= \mathrm{Hom}_{H_n(\mathbb{A}_{f,F}) \times H^{(\iota)}(\mathbb{A}_{f,F})}\big(\mathscr{S}(\mathbb{V}_f^n), \pi_f^\vee \otimes \mathsf{H}_{(2)}^{2n}(\mathrm{Sh})\big),
\end{aligned}
$$

where $H_n(\mathbb{A}_{f,F}) \times H^{(\iota)}(\mathbb{A}_{f,F})$ acts on $\mathscr{S}(\mathbb{V}_f^n)$ through the Weil representation ω_χ and $H^{(\iota)}(\mathbb{A}_{f,F})$ acts on $\mathsf{H}_{(2)}^{2\bullet}(\mathrm{Sh})$ through Hecke operators and on π_f trivially. As an $H^{(\iota)}(\mathbb{A}_{f,F})$-representation, we have the following well-known decomposition (see, for example, [Borel and Wallach 2000, Chapter XIV]):

$$
\mathsf{H}_{(2)}^{2n}(\mathrm{Sh}) = \bigoplus_\sigma m_{\mathsf{disc}}(\sigma)\mathsf{H}^{2n}\big(\mathrm{Lie}\, H_\infty^{(\iota)}, K_{H_\infty^{(\iota)}}; \sigma_\infty\big) \otimes \sigma_f,
$$

where the direct sum is taken over all irreducible discrete automorphic representations of $H^{(\iota)}(\mathbb{A}_F)$. If the invariant functional $H_{\Theta,\phi_\infty} \neq 0$, then some σ_f with

$$
m_{\mathsf{disc}}(\sigma)\mathsf{H}^{2n}\big(\mathrm{Lie}\, H_\infty^{(\iota)}, K_{H_\infty^{(\iota)}}; \sigma_\infty\big) \neq 0
$$

is the theta correspondence $\theta(\pi_f^\vee)$ of $\pi_f^\vee$.

We define a character $\tilde{\chi}$ of $E^{\times,1}\backslash\mathbb{A}_E^{\times,1}$ in the following way. For any $x \in \mathbb{A}_E^{\times,1}$, we can write $x = e/e^\tau$ for some $e \in \mathbb{A}_E^\times$ and define $\tilde{\chi}(x) = \chi(e)$ which is well-defined since $\chi|_{\mathbb{A}_F^\times} = 1$.

For all finite places v such that $v \nmid 2$ and ψ_v, χ_v, and π_v are unramified, we have $H_v^{(\iota)} \cong H_{n,v}$. Let $\Sigma \in \mathrm{AP}(H_{\mathbb{A}_F}^{(\iota)})_{\mathsf{disc}}$ be the A-packet containing σ and $\Pi \in \mathrm{AP}(H_{n,\mathbb{A}_F})_{\mathsf{disc}}$ containing π. Then by Corollary A.6, we have that for v as above, $\mathrm{JL}(\Sigma)_v = \mathrm{JL}(\Sigma_v) = \Sigma_v$ and $\Pi_v \otimes \tilde{\chi}_v$ contain the same unramified representation, hence $\mathrm{JL}(\Sigma)$ and $\Pi \otimes \tilde{\chi}$ coincide. In particular,

$$
\mathrm{JL}(\Sigma_\infty) = \mathrm{JL}(\Sigma)_\infty = \Pi_\infty \otimes \tilde{\chi}_\infty,
$$

which implies that Σ_∞ is a discrete series L-packet (see [Adams 2011]). This contradicts our assumption since for any discrete series representation σ_∞, we can have $\mathsf{H}^\bullet(\mathrm{Lie}\, H_\infty^{(\iota)}, K_{H_\infty^{(\iota)}}; \sigma_\infty) \neq 0$ only in the middle dimension, which is $2n - 1$, not $2n$ (see [Borel and Wallach 2000, Chapter II, Theorem 5.4]). Thus $H_{\Theta,\phi_\infty} = 0$ and we prove the proposition. $\qquad\square$

The proposition says that Θ_ϕ^f is automatically cohomologically trivial at least in the proper case.

Conjecture 3.10. *When* Sh *is nonproper,* $\mathrm{cl}(\Theta_\phi^f) \in \mathsf{H}_\partial^{2n}(\mathrm{Sh}^\thicksim)$ *is* 0 *for any cusp form* $f \in \pi$ *and* ϕ *as above.*

When $n = 1$, this is proved in [Liu 2011] just by computing the degree of the generating series which is the linear combination of an Eisenstein series and (possibly) an automorphic character (that is, one-dimensional automorphic representation) of $H_1(\mathbb{A})$. Hence $\mathsf{cl}(\Theta_\phi^f)$ is zero since f is cuspidal. For the general case, we believe that the same phenomenon will happen.

Main conjecture. Let us assume Conjecture 3.10 and the existence of Beilinson–Bloch height pairing on smooth proper schemes over number fields. Then Θ_ϕ^f is cohomologically trivial and we let

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{BB}}^K$$

be the Beilinson–Bloch height on Sh_K (resp. $\mathsf{Sh}_{\widetilde{K}}$) if it is proper (resp. nonproper) for sufficiently small K. Let $\mathsf{vol}(K)$ be the volume with respect to the Haar measure defined in the proof of Theorem 4.20. Then

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{BB}} := \mathsf{vol}(K) \langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{BB}}^K$$

is a well-defined number which is independent of K.

If $\mathbb{V} \not\cong \mathbb{V}(\pi, \chi)$, then $\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{BB}} = 0$ for any f, $f^\vee$ and ϕ, $\phi^\vee$ since otherwise, it defines a nonzero functional

$$\gamma(f, f^\vee, \phi, \phi^\vee) \in \mathsf{Hom}_{H_n(\mathbb{A}_{f,F}) \times H_n(\mathbb{A}_{f,F})}(R(\mathbb{V}_f, \chi_f), \pi_f^\vee \boxtimes \chi_f \pi_f)$$

which contradicts the fact that the latter space is zero. This will imply that, assuming the conjecture that the Beilinson–Bloch height pairing is nondegenerate, any arithmetic theta lifting $\Theta_\phi^f = 0$.

If $\mathbb{V} \cong \mathbb{V}(\pi)$, we have the following main conjecture:

Conjecture 3.11 (arithmetic inner product formula). *Let π be an irreducible cuspidal automorphic representation of $H_n(\mathbb{A}_F)$, χ a character of $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$ such that π_∞ is a discrete series representation of weight $(n - \mathfrak{k}^\chi/2, n + \mathfrak{k}^\chi/2)$, $\epsilon(\pi, \chi) = -1$, and $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$. Then, for any $f \in \pi$, $f^\vee \in \pi^\vee$, and any $\phi, \phi^\vee \in \mathscr{S}(\mathbb{V}_f^n)^{\mathsf{U}_\infty}$ decomposable, we have*

$$\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{BB}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee),$$

where in the last product almost all factors are 1.

We remark that when $n = 1$, the height pairing $\langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{BB}}$ is just the Néron–Tate height pairing, hence is defined without any assumption.

4. Comparison at infinite places

4A. *Archimedean Whittaker integrals.* In this section, we will calculate certain Whittaker integrals $W_T(s, g, \Phi)$ and their first derivatives (at $s = 0$) at an archimedean place.

Elementary reductions. We fix an archimedean place $\iota : F \hookrightarrow \mathbb{C}$ and suppress it from the notation. Hence we have $H' \cong \mathrm{U}(n, n)$ and $H'' \cong \mathrm{U}(2n, 2n)$ with parabolic subgroup P, $V \cong \mathbb{V}_\iota$ the standard positive definite $2n$-dimensional complex hermitian space, and χ a character of $\mathbb{C}^\times$ which is trivial on $\mathbb{R}^\times$. In this section, we write a^* instead of ${}^t a^\tau$ for a complex matrix. Recall that we are always writing the elements in H'' in matrix form with respect to the basis $\{e_1, \ldots, e_n; e_1^-, \ldots, e_n^-; e_{n+1}, \ldots, e_{2n}; -e_{n+1}^-, \ldots, -e_{2n}^-\}$ under which P is the standard maximal parabolic subgroup. The map

$$\mathrm{U}(2n) \times \mathrm{U}(2n) \longrightarrow H'' \cong \mathrm{U}(2n, 2n)$$

$$(k_1, k_2) \longmapsto [k_1, k_2] := \frac{1}{2} \begin{pmatrix} k_1 + k_2 & -ik_1 + ik_2 \\ ik_1 - ik_2 & k_1 + k_2 \end{pmatrix}$$

is an isomorphism onto its image which is a maximal compact subgroup of H'', denoted by $\mathcal{K}$. Let Φ^0 be the standard Gaussian in $\mathscr{S}(V^{2n})$ and $\chi(z) = z^{2\ell}$ ($|z| = 1$) for some $\ell \in \mathbb{Z}$. Then $\omega_\chi([k_1, k_2])\Phi^0 = (\det k_1)^{n+\ell}(\det k_2)^{-n+\ell}\Phi^0$. Our first goal is to analyze the integral

$$W_T(s, g, \Phi^0) = \int_{\mathrm{Her}_{2n}(\mathbb{C})} \varphi_{\Phi^0, s}(wn(u)g)\psi_T(n(u))^{-1}\, du \qquad (4\text{-}1)$$

for $T \in \mathrm{Her}_{2n}(\mathbb{C})$ and $\Re(s) > n$, where du is the self-dual measure with respect to (the standard) ψ. Write $g = n(b)m(a)[k_1, k_2]$ under the Iwasawa decomposition. Then

$$(4\text{-}1) = \int_{\mathrm{Her}_{2n}(\mathbb{C})} \omega_\chi(wn(u)n(b)m(a)[k_1, k_2])\Phi^0(0)$$
$$\times \lambda_P(wn(u)n(b)m(a)[k_1, k_2])^s \psi(-\operatorname{tr} Tu)\, du$$

$$= \psi(\operatorname{tr} Tb)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell}$$
$$\times \int_{\mathrm{Her}_{2n}(\mathbb{C})} \omega_\chi(wn(u)m(a))\Phi^0(0)\lambda_P(wn(u)m(a))^s \psi(-\operatorname{tr} Tu)\, du. \qquad (4\text{-}2)$$

Since $wn(u)m(a) = wm(a)n(a^{-1}ua^{*,-1}) = m(a^{*,-1})wn(a^{-1}ua^{*,-1})$ and after changing variable $du = |\det a|_\mathbb{C}^{2n} d(a^{-1}ua^{*,-1})$, we have

$$(4\text{-}2) = \psi(\operatorname{tr} Tb)|\det a|_\mathbb{C}^{n-s}\chi(\det a)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell}$$
$$\times \int_{\mathrm{Her}_{2n}(\mathbb{C})} \omega_\chi(wn(u))\Phi^0(0)\lambda_P(wn(u))^s \psi(-\operatorname{tr} a^*Tau)\, du$$

$$= \psi(\operatorname{tr} Tb)|\det a|_{\mathbb{C}}^{n-s} \chi(\det a)(\det k_1)^{n+\ell}(\det k_2)^{-n+\ell} W_{a^*Ta}(s, e, \Phi^0). \quad (4\text{-}3)$$

Hence we only need to consider $W_T(s, e, \Phi^0)$. From now on, we will not restrict ourselves to considering the case of dimension $2n$. Let $m \geq 1$ be an integer, V the totally positive-definite complex hermitian space of dimension m, and Φ^0 the standard Gaussian. For $T \in \operatorname{Her}_m(\mathbb{C})$, we can still consider $W_T(s, e, \Phi^0)$ which has the usual integral presentation for $\Re(s) > m/2$.

Lemma 4.1. *For $u \in \operatorname{Her}_m(\mathbb{C})$, $\omega_\chi(wn(u))\Phi^0(0) = \gamma_V \det(\mathbf{1}_m - iu)^m$, where γ_V is the Weil constant.*

Proof. By definition,

$$\gamma_V^{-1}\omega_\chi(wn(u))\Phi^0(0) = \int_{V^m} \omega_\chi(n(u))\Phi^0(x)\,dx = \int_{V^m} \psi(\operatorname{tr} uT(x))\Phi^0(x)\,dx, \quad (4\text{-}4)$$

where γ_V is the Weil constant. Write $u = k\,\mathrm{diag}[\ldots, u_j, \ldots]k^*$ with $u_j \in \mathbb{R}$ ($j = 1, \ldots, m$) and $k \in \mathrm{U}(m)$. Then

$$(4\text{-}4) = \int_{V^m} \psi(\operatorname{tr} k\,\mathrm{diag}[\ldots, u_j, \ldots]k^*T(x)kk^{-1})\Phi^0(x)\,dx$$
$$= \int_{V^m} \psi(\operatorname{tr}\mathrm{diag}[\ldots, u_j, \ldots]T(xk))e^{-2\pi\,\operatorname{tr} T(x)}\,dx. \quad (4\text{-}5)$$

Changing the variable $x \mapsto xk$ and using $\operatorname{tr} T(x) = \operatorname{tr} T(xk)$, we have

$$(4\text{-}5) = \int_{V^m} \exp(2\pi i\,\operatorname{tr}\mathrm{diag}[\ldots, u_j, \ldots]T(x) - 2\pi\,\operatorname{tr} T(x))\,dx$$
$$= \prod_{j=1}^m \int_V \exp(2\pi i u_j T(x_j) - 2\pi T(x_j))\,dx_j. \quad (4\text{-}6)$$

We identify V with $\mathbb{C}^m$ and $(\,\cdot\,,\,\cdot\,)$ with the usual hermitian form on $\mathbb{C}^m$, and then the self-dual measure dx_j on V is just the usual Lebesgue measure dx on $\mathbb{C}^m \cong \mathbb{R}^{2m}$. Hence

$$(4\text{-}6) = \prod_{j=1}^m \int_{\mathbb{R}^{2m}} e^{-\pi(1-iu_j)\|x\|^2}\,dx = \prod_{j=1}^m \left(\int_{-\infty}^\infty e^{-\pi(1-iu_j)t^2}\,dt\right)^{2m}$$
$$= \prod_{j=1}^m (1-iu_j)^{-m} = \det(\mathbf{1}_m - iu)^{-m}. \qquad \square$$

Lemma 4.2. *For $u \in \operatorname{Her}_m(\mathbb{C})$, $\lambda_P(wn(u)) = \det(\mathbf{1}_m + u^2)^{-1}$.*

Proof. We have

$$wn(u)\begin{pmatrix} i\mathbf{1}_m \\ \mathbf{1}_m \end{pmatrix} = \begin{pmatrix} & \mathbf{1}_m \\ -\mathbf{1}_m & \end{pmatrix}\begin{pmatrix} \mathbf{1}_m & u \\ & \mathbf{1}_m \end{pmatrix}\begin{pmatrix} i\mathbf{1}_m \\ \mathbf{1}_m \end{pmatrix} = \begin{pmatrix} \mathbf{1}_m \\ -i\mathbf{1}_m - u \end{pmatrix}.$$

Then,

$$1_m(-i1_m - u)^{-1} = -u(1_m + u^2)^{-1} + i(1_m + u^2)^{-1}.$$

Hence $\lambda_P(wn(u)) = \det(1_m + u^2)^{-1}$ which is a positive real number. $\square$

Combining Lemmas 4.1 and 4.2, we have for $\Re(s) > m/2$,

$$\gamma_V^{-1} W_T(s, e, \Phi^0) = \int_{\mathrm{Her}_m(\mathbb{C})} \psi(-\operatorname{tr} Tu) \det(1_m + iu)^{-s} \det(1_m - iu)^{-s-m} \, du.$$

Now we proceed as in [Shimura 1982, Case II]. First, we need to introduce some new notation which may be different from that in [Shimura 1982]. Let

$$\mathrm{Her}_m^+(\mathbb{C}) = \{x \in \mathrm{Her}_m(\mathbb{C}) \mid x > 0\},$$
$$\mathfrak{h}_m = \{x + iy \mid x \in \mathrm{Her}_m(\mathbb{C}), \, y \in \mathrm{Her}_m^+(\mathbb{C})\},$$
$$\mathfrak{h}_m' = \{x + iy \mid x \in \mathrm{Her}_m^+(\mathbb{C}), \, y \in \mathrm{Her}_m(\mathbb{C})\}.$$

Lemma 4.3 (Siegel; see [Shimura 1982, Section 1]).   (1) *For $z \in \mathfrak{h}_m'$ and $\Re(s) > m - 1$, we have*

$$\int_{\mathrm{Her}_m^+(\mathbb{C})} e^{-\operatorname{tr}(zx)} \det(x)^{s-m} \, dx = \Gamma_m(s) \det(z)^{-s},$$

where dx is induced from the self-dual measure on $\mathrm{Her}_m(\mathbb{C})$ and

$$\Gamma_m(s) = (2\pi)^{m(m-1)/2} \prod_{j=0}^{m-1} \Gamma(s - j).$$

(2) *For $x \in \mathrm{Her}_m(\mathbb{C})$, $b \in \mathrm{Her}_m^+(\mathbb{C})$ and $\Re(s) > 2m - 1$, we have*

$$\Gamma_m(s) \int_{\mathrm{Her}_m(\mathbb{C})} e^{2\pi i \operatorname{tr}(ux)} \det(b + 2\pi iu)^{-s} \, du = \begin{cases} e^{-\operatorname{tr}(xb)} \det(x)^{s-m} & \text{if } x \in \mathrm{Her}_m^+(\mathbb{C}), \\ 0 & \text{if } x \notin \mathrm{Her}_m^+(\mathbb{C}). \end{cases}$$

Now for $\Re(s) > m - 1$, by Lemma 4.3(1),

$$\gamma_V^{-1} W_T(s, e, \Phi^0) = \int_{\mathrm{Her}_m(\mathbb{C})} \psi(-\operatorname{tr} Tu) \frac{1}{\Gamma_m(s)}$$

$$\times \int_{\mathrm{Her}_m^+(\mathbb{C})} e^{-\operatorname{tr}(1_m + iu)x} \det(x)^{s-m} \det(1_m - iu)^{-s-m} \, dx \, du$$

$$= \frac{1}{\Gamma_m(s)} \int_{\mathrm{Her}_m^+(\mathbb{C})} e^{-\operatorname{tr} x} \det(x)^{s-m}$$

$$\times \int_{\mathrm{Her}_m(\mathbb{C})} e^{-i \operatorname{tr}(x + 2\pi T)u} \det(1_m - iu)^{-s-m} \, du \, dx. \quad (4\text{-}7)$$

Apply Lemma 4.3(2) with $b = 1_m$, $x = x + 2\pi T$, and $s = s + m$, and perform

the change of variable $u \mapsto -u/(2\pi)$ to obtain

$$(4\text{-}7) = \frac{1}{\Gamma_m(s)} \times \int_{x>0, x+2\pi T>0} e^{-\operatorname{tr} x} \det(x)^{s-m}$$

$$\times \frac{(2\pi)^{m^2}}{\Gamma_m(s+m)} e^{-\operatorname{tr}(x+2\pi T)} \det(x + 2\pi T)^s \, dx. \quad (4\text{-}8)$$

We change the variable $x \mapsto x/\pi + T$,

$$(4\text{-}8) = \frac{(2\pi)^{m^2} \pi^{2ms}}{\Gamma_m(s)\Gamma_m(s+m)} \int_{x>-T, x>T} e^{-\operatorname{tr} 2\pi x} \det(x+T)^s \det(x-T)^{s-m} \, dx$$

$$= \frac{(2\pi)^{m^2} \pi^{2ms}}{\Gamma_m(s)\Gamma_m(s+m)} \eta(2\pi \mathbf{1}_m, T; s+m, s), \quad (4\text{-}9)$$

where the function $\eta(g, h; \alpha, \beta)$ for $g \in \operatorname{Her}_m^+(\mathbb{C})$, $h \in \operatorname{Her}_m(\mathbb{C})$, $\Re(\alpha) \gg 0$, and $\Re(\beta) \gg 0$ is introduced in [Shimura 1982, (1.26)]. In what follows, we assume that T is nonsingular with $\operatorname{sign}(T) = (p, q)$ for $p + q = m$. We write $T = k\operatorname{diag}[t_1 \ldots, t_p, -t_{p+1}, \ldots, -t_m]k^*$ with $k \in \mathsf{U}(m)$, $t_j \in \mathbb{R}_{>0}$ and let $a = k\operatorname{diag}[\ldots, t_j^{1/2}, \ldots]$. Then $T = a\varepsilon_{p,q}a^*$ where $\varepsilon_{p,q} = \operatorname{diag}[\mathbf{1}_p, -\mathbf{1}_q]$. It is easy to see that we have

$$\eta(g, T; \alpha, \beta) = |\det T|^{\alpha+\beta-m} \eta(a^*ga, \varepsilon_{p,q}; \alpha, \beta), \quad (4\text{-}10)$$

$$\eta(g, \varepsilon_{p,q}; \alpha, \beta) = 2^{m(\alpha+\beta-m)} e^{-\operatorname{tr} g} \zeta_{p,q}(2g; \alpha, \beta). \quad (4\text{-}11)$$

We introduce $\zeta_{p,q}$ as in [Shimura 1982, (4.16)]. For $g \in \operatorname{Her}_m^+(\mathbb{C})$ and $p+q = m$, let $\varepsilon_p = \operatorname{diag}[\mathbf{1}_p, \mathbf{0}_q]$ and $\varepsilon_q' = \operatorname{diag}[\mathbf{0}_p, \mathbf{1}_q]$. Then

$$\zeta_{p,q}(g; \alpha, \beta) = \int_{X_{p,q}} e^{-\operatorname{tr}(gx)} \det(x + \varepsilon_p)^{\alpha-m} \det(x + \varepsilon_q')^{\beta-m} \, dx,$$

where $X_{p,q} := \{x \in \operatorname{Her}_m(\mathbb{C}) \mid x + \varepsilon_p > 0, x + \varepsilon_q' > 0\}$ with the measure induced from the self-dual one on $\operatorname{Her}_m(\mathbb{C})$. Then $X_{m,0} = \operatorname{Her}_m^+(\mathbb{C})$. If $q = 0$, we simply write $\zeta_m = \zeta_{m,0}$.

Analytic continuation. Following [Shimura 1982, (4.17)], we let

$$\omega_{p,q}(g; \alpha, \beta) = \Gamma_q(\alpha - p)^{-1} \Gamma_p(\beta - q)^{-1} \det^+(\varepsilon_{p,q}g)^{\beta-q/2}$$

$$\times \det^-(\varepsilon_{p,q}g)^{\alpha-p/2} \zeta_{p,q}(g; \alpha, \beta), \quad (4\text{-}12)$$

where $\det^{\pm}$ denotes the absolute value of the product of positive or negative eigenvalues (equal to 1 if there are no such eigenvalues) of a nonsingular element in $\operatorname{Her}_m(\mathbb{C})$. It is proved in [Shimura 1982, Section 4] that $\omega_{p,q}(g; \alpha, \beta)$ can be continued as a holomorphic function in (α, β) to the whole $\mathbb{C}^2$ and satisfies the

functional equation $\omega_{p,q}(g; m - \beta, m - \alpha) = \omega_{p,q}(g; \alpha, \beta)$. Also, if $q = 0$, we simply write $\omega_m = \omega_{m,0}$.

Lemma 4.4. *If $q = 0$, then $\omega_m(g; m, \beta) = \omega_m(g; \alpha, 0) = 1$.*

Proof. The integral defining $\zeta_m(g; m, \beta)$,

$$\int_{\mathrm{Her}_m^+(\mathbb{C})} e^{-\,\mathrm{tr}(gx)} \det(x)^{\beta-m} \, dx,$$

is absolutely convergent for $\Re(\beta) > m - 1$ and is equal to $\Gamma_m(\beta) \det(g)^{-\beta}$ by Lemma 4.3(1). Hence $\omega_m(g; m, \beta) \equiv 1$, which proves the lemma by the functional equation. $\qquad \square$

Proposition 4.5. *Let $T \in \mathrm{Her}_m(\mathbb{C})$ be nonsingular with $\mathrm{sign}(T) = (p, q)$.*

(1) $\mathrm{ord}_{s=0} W_T(s, e, \Phi^0) \geq q$.

(2) *If T is positive definite, that is, $q = 0$, then*

$$W_T(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \,\mathrm{tr}\, T}.$$

Proof. (1) Combining (4-9), (4-10), (4-11), and (4-12), we have

$$\gamma_V^{-1} W_T(s, e, \Phi^0) = \frac{\Gamma_q(m + s - p)\Gamma_p(s - q)}{\Gamma_m(s)\Gamma_m(s + m)} (2\pi)^{m^2 + 2ms} |\det T|^{2s} e^{-2\pi \,\mathrm{tr}(a^*a)}$$
$$\times \det{}^+(4\pi T)^{q/2-s} \det{}^-(4\pi T)^{p/2-m-s} \omega_{p,q}(4\pi a^*a; m + s, s). \quad (4\text{-}13)$$

All terms except the gamma factors are holomorphic for any $s \in \mathbb{C}$. But

$$\frac{\Gamma_q(m + s - p)\Gamma_p(s - q)}{\Gamma_m(s)\Gamma_m(s + m)} = \frac{(2\pi)^{-pq - m(m-1)/2}}{\Gamma(s) \cdots \Gamma(s - q + 1) \times \Gamma(s + m) \cdots \Gamma(s + m - p + 1)}.$$

Hence $\mathrm{ord}_{s=0} W_T(s, e, \Phi^0) \geq -\mathrm{ord}_{s=0} \Gamma(s) \cdots \Gamma(s - q + 1) = q$.

(2) If T is positive definite, then $\mathrm{tr}(a^*a) = \mathrm{tr}\, T$. By (4-13) and Lemma 4.4, we have

$$\gamma_V^{-1} W_T(0, e, \Phi^0) = \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \,\mathrm{tr}\, T} \omega_m(4\pi a^*a; m, 0) = \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \,\mathrm{tr}\, T}. \qquad \square$$

The case $q = 1$. By Proposition 4.5(1), the T-th coefficient will not contribute to the analytic kernel function $E'(0, g, \Phi)$ if $\mathrm{sign}(T) = (p, q)$ with $q \geq 2$. For this reason, we now focus on the case $q = 1$, that is, the functions $\zeta_{m-1,1}(g; \alpha, \beta)$ and $\omega_{m-1,1}(g; \alpha, \beta)$. We can assume that $g = \mathrm{diag}[a, b]$ with $a \in \mathrm{Her}_{m-1}^+(\mathbb{C})$ and $b \in \mathbb{R}_{>0}$. We write elements in X in the form

$$\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}, \quad x \in \mathrm{Her}_{m-1}(\mathbb{C}), \quad y \in \mathbb{R}, \quad z \in \mathrm{Mat}_{m-1,1}(\mathbb{C}).$$

Then (see [Shimura 1982, p. 288])

$$X = \{(x, y, z) \mid x > 0, y > 0, x + \mathbf{1}_{m-1} > zy^{-1}z^*, y + 1 > z^*x^{-1}z\}$$
$$= \{(x, y, z) \mid x + \mathbf{1}_{m-1} > 0, y + 1 > 0, x > z(y+1)^{-1}z^*, y > z^*(x+\mathbf{1}_{m-1})^{-1}z\}.$$

We have

$$\zeta_{m-1,1}(g; \alpha, \beta)$$
$$= \int_X e^{-\operatorname{tr}(ax)-by} \det \begin{pmatrix} x + \mathbf{1}_{m-1} & z \\ z^* & z \end{pmatrix}^{\alpha-m} \det \begin{pmatrix} x & z \\ z^* & y+1 \end{pmatrix}^{\beta-m} dx\, dy\, dz, \quad (4\text{-}14)$$

where we use the self-dual measure dx on $\operatorname{Her}_{m-1}(\mathbb{C})$, the Lebesgue measure dy on $\mathbb{R}$, and $dz = 2^{m-1} \times$ the Lebesgue measure on $\operatorname{Mat}_{m-1,1}(\mathbb{C})$. Now we make a change of variables as in [Shimura 1982, p. 289] as follows. Put

$$f = (x + \mathbf{1}_{m-1})^{-1/2} z (y+1)^{-1/2}.$$

Then $\mathbf{1}_{m-1} - ff^* > 0$. Put $r = (1 - f^*f)^{1/2}$ and $s = (\mathbf{1}_{m-1} - ff^*)^{1/2}$; also $w = s^{-1}f = fr^{-1}$, $u = x - ww^*$, and $v = y - w^*w$. The map $(x, y, z) \mapsto (u, v, w)$ maps X bijectively onto $Y = \operatorname{Her}_{m-1}^+(\mathbb{C}) \times \mathbb{R}_{>0} \times \operatorname{Mat}_{m-1,1}(\mathbb{C})$. Then the Jacobian

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det(\mathbf{1}_{m-1} + x)(1 + y)^{m-1}(1 + w^*w)^{-m}$$

for the measure $\partial(u, v, w)$ on Y induced from that on $\operatorname{Her}_{m-1}(\mathbb{C}) \times \mathbb{R} \times \operatorname{Mat}_{m-1,1}(\mathbb{C})$ as an open subset. We have

$$\det \begin{pmatrix} x + \mathbf{1}_{m-1} & z \\ z^* & y \end{pmatrix} = \det(u + \mathbf{1}_{m-1} + ww^*)v \det(\mathbf{1}_{m-1}ww^*)^{-1},$$

$$\det \begin{pmatrix} x & z \\ z^* & y+1 \end{pmatrix} = (v + 1 + w^*w) \det(u) \det(\mathbf{1}_{m-1}ww^*)^{-1}.$$

We obtain that

$$(4\text{-}14) = \int_Y e^{-\operatorname{tr}(au+aww^*)-(bv+bw^*w)} \det(\mathbf{1}_{m-1} + ww^*)^{m-\alpha-\beta}$$

$$\times \det(u+\mathbf{1}_{m-1}+ww^*)^{\alpha-m+1} \det(u)^{\beta-m} \times (v+1+w^*w)^{\beta-1}v^{\alpha-m}\, du\, dv\, dw$$

$$= \int_{\operatorname{Mat}_{m-1,1}(\mathbb{C})} e^{-\operatorname{tr}(aww^*)-bw^*w} \zeta_1(b(1 + w^*w); \beta, \alpha - m + 1)$$

$$\times \int_{\operatorname{Her}_{m-1}^+(\mathbb{C})} e^{-\operatorname{tr}(au)} \det(u + \mathbf{1}_{m-1} + ww^*)^{\alpha-m-1} \det(u)^{\beta-m}\, du\, dw. \quad (4\text{-}15)$$

On the other hand, again by (4-10), (4-11), and (4-12), we have

$$\gamma_V^{-1} W_T(s, e, \Phi^0) = \frac{(2\pi)^{m^2+2ms} |\det T|^{2s}}{\Gamma_m(s)\Gamma_m(s+m)} e^{-2\pi \operatorname{tr}(a^*a)} \zeta_{m-1,1}(4\pi a^*a; m+s, s). \quad (4\text{-}16)$$

Assuming $T \sim \mathrm{diag}[a_1, \ldots, a_{m-1}, -b]$ with $a_j, b \in \mathbb{R}_{>0}$, then $a^*a = \mathrm{diag}[a_1, \ldots, a_{m-1}, b]$. By (4-16) and Proposition 4.5(1),

$$\gamma_V^{-1} W_T'(0, e, \Phi^0) = \lim_{s \to 0} \frac{(2\pi)^{m^2}}{s\Gamma_m(s)\Gamma_m(m)}$$

$$\times e^{-2\pi(a_1+\cdots+a_{m-1}+b)} \zeta_{m-1,1}(4\pi \mathrm{diag}[a_1, \ldots, a_{m-1}, b]; m, s). \quad (4\text{-}17)$$

Plugging in (4-15) for $(\alpha, \beta) = (m, s)$:

$$(4\text{-}17) = \lim_{s \to 0} \frac{2^{m-1}(2\pi)^{m^2}}{s\Gamma_m(s)\Gamma_m(m)} e^{-2\pi(a_1+\cdots+a_{m-1}+b)}$$

$$\times \int_{\mathbb{C}^{m-1}} e^{-4\pi[(a_1+a_m)w_1\bar{w}_1+\cdots+(a_{m-1}+a_m)w_{m-1}\bar{w}_{m-1}]} \zeta_1(4\pi b(1 + w^*w); 0, 1)$$

$$\times \int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} e^{-4\pi \,\mathrm{tr}\,\mathrm{diag}[a_1,\ldots,a_{m-1}]u}$$

$$\times \det(u + \mathbf{1}_{m-1} + ww^*)\det(u)^{s-m} \, du\, dw_1 \cdots dw_{m-1}. \quad (4\text{-}18)$$

It is easy to see that

$$\zeta_1(4\pi b(1 + w^*w); 0, 1) = -e^{4\pi b(1+w^*w)} \mathrm{Ei}(-4\pi b(1 + w^*w)), \quad (4\text{-}19)$$

where Ei is the classical exponential integral

$$\mathrm{Ei}(z) := -\int_1^\infty \frac{e^{zt}}{t} \, dt.$$

The main difficulty is calculating the inside integral, the one over $\mathrm{Her}_{m-1}^+(\mathbb{C})$. We temporarily let $g_0 = 4\pi \mathrm{diag}[a_1, \ldots, a_{m-1}]$ and consider the integral

$$\int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} e^{-\mathrm{tr}(ug_0)} \det(u + \mathbf{1}_{m-1} + ww^*)\det(u)^{s-m} \, du. \quad (4\text{-}20)$$

We define a differential operator $\Delta = \det\left(\dfrac{\partial}{\partial g_{jk}}\right)_{j,k=1}^{m-1}$. Then

$$\Delta e^{-\mathrm{tr}(ug)} = (-1)^{m-1} \det(u)e^{-\mathrm{tr}(ug)}.$$

Hence

$$(4\text{-}20) = e^{\mathrm{tr}(\mathbf{1}_{m-1}+ww^*)g_0}$$

$$\times \int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} e^{-\mathrm{tr}(u+\mathbf{1}_{m-1}+ww^*)g_0} \det(u + \mathbf{1}_{m-1} + ww^*)\det(x)^{s-m} \, du$$

$$= (-1)^{m-1} e^{\mathrm{tr}(\mathbf{1}_{m-1}+ww^*)g_0}$$

$$\times \int_{\mathrm{Her}_{m-1}^+(\mathbb{C})} \Delta\Big|_{g=g_0} e^{-\mathrm{tr}(u+\mathbf{1}_{m-1}+ww^*)g} \det(x)^{s-m} \, du. \quad (4\text{-}21)$$

We can exchange Δ and the integration by analytic continuation; then

$$
\begin{aligned}
(4\text{-}21) &= (-1)^{m-1} e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \int_{\operatorname{Her}_{m-1}^+(\mathbb{C})} e^{-\operatorname{tr}(u+\mathbf{1}_{m-1}+ww^*)g} \det(x)^{s-m}\, du \\
&= (-1)^{m-1} e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \left(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \zeta_{m-1}(g; m-1, s-1) \right) \\
&= (-1)^{m-1} e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \left(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \det(g)^{1-s} \Gamma_{m-1}(s-1) \right) \\
&= (-1)^{m-1} \Gamma_{m-1}(s-1) e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \\
&\qquad\qquad\qquad\qquad \times \left(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \det(g)^{1-s} \right). \quad (4\text{-}22)
\end{aligned}
$$

We plug (4-19) and (4-22) in (4-18):

$$
\begin{aligned}
(4\text{-}18) &= \lim_{s\to 0} \frac{\Gamma_{m-1}(s-1)(-2)^{m-1}(2\pi)^{m^2}}{s\,\Gamma_m(s)\Gamma_m(m)} e^{-2\pi \operatorname{tr} T} \\
&\quad \times \int_{\mathbb{C}^{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})} e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \\
&\quad \times \left(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \det(g) \right)(-\operatorname{Ei})(-4\pi b(1+w^*w))\, dw_1 \cdots dw_{m-1} \\
&= \frac{(2\pi)^{m^2}(-2)^{m-1}}{\Gamma_m(m)(2\pi)^{m-1}} e^{-2\pi \operatorname{tr} T} \int_{\mathbb{C}^{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})} \\
&\quad \times e^{\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g_0} \Delta\big|_{g=g_0} \left(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g} \det(g) \right) \\
&\qquad\qquad\qquad \times (-\operatorname{Ei})(-4\pi b(1+w^*w))\, dw_1 \cdots dw_{m-1}. \quad (4\text{-}23)
\end{aligned}
$$

Now we make a change of variables. Let

$$
D_{m-1} = \{ z = (z_1, \ldots, z_{m-1}) \in \mathbb{C}^{m-1} \mid z\bar{z} := z_1 \bar{z}_1 + \cdots + z_{m-1} \bar{z}_{m-1} < 1 \}
$$

be the open unit disc in $\mathbb{C}^{m-1}$. Then the map

$$
w_j = \frac{z_j}{(1-z\bar{z})^{1/2}}, \quad j = 1, \ldots, m-1 \quad (4\text{-}24)
$$

is a C^∞-homeomorphism from $\mathbb{C}^{m-1}$ to D_{m-1}. To calculate the Jacobian, let $w_j = u_j + v_j i$ and $z_j = x_j + y_j i$ be the corresponding real and imaginary parts. Then

$$
\frac{\partial u_j}{\partial x_k} = \frac{x_j x_k}{(1-z\bar{z})^{3/2}}, \quad k \neq j; \qquad \frac{\partial u_j}{\partial x_j} = \frac{x_j^2}{(1-z\bar{z})^{3/2}} + \frac{1}{(1-z\bar{z})^{1/2}}; \qquad \frac{\partial u_j}{\partial y_k} = \frac{x_j y_k}{(1-z\bar{z})^{3/2}};
$$

$$
\frac{\partial v_j}{\partial y_k} = \frac{y_j y_k}{(1-z\bar{z})^{3/2}}, \quad k \neq j; \qquad \frac{\partial v_j}{\partial y_j} = \frac{y_j^2}{(1-z\bar{z})^{3/2}} + \frac{1}{(1-z\bar{z})^{1/2}}; \qquad \frac{\partial v_j}{\partial x_k} = \frac{y_j x_k}{(1-z\bar{z})^{3/2}}.
$$

Let $c_j = x_j$ and $c_{m-1+j} = y_j$ for $j = 1, \ldots, m-1$. Then by Lemma 4.6(2) below with $n = 2m+2$, $c = {}^t(c_1, \ldots, c_{2m-2})$, and $\epsilon = 1 - z\bar{z}$,

$$\frac{\partial(u_1, v_1, \ldots, u_{m-1}, v_{m-1})}{\partial(x_1, y_1, \ldots, x_{m-1}, y_{m-1})}$$

$$= \frac{\partial(u_1, \ldots, u_{m-1}; v_1, \ldots, v_{m-1})}{\partial(x_1, \ldots, x_{m-1}; y_1, \ldots, y_{m-1})}$$

$$= (1 - z\bar{z})^{-3(m-1)} \det((1 - z\bar{z})\mathbf{1}_{2m-2} + cc^*)$$

$$= (1 - z\bar{z})^{-3(m-1)}(1 - z\bar{z})^{2m-3}(1 - z\bar{z} + x_1^2 + \cdots + x_{m-1}^2 + y_1^2 + \cdots + y_{m-1}^2)$$

$$= (1 - z\bar{z})^{-m}. \tag{4-25}$$

Lemma 4.6. *Let $c \in \mathrm{Mat}_{n \times 1}(\mathbb{C})$. Then*

(1) $\det(\mathbf{1}_n + cc^*) = 1 + c^*c$ *and*

(2) *for $\epsilon > 0$, $\det(\epsilon\mathbf{1}_n + cc^*) = \epsilon^{n-1}(\epsilon + c^*c)$.*

Proof. (1) is [Shimura 1982, Lemma 2.2]. Since it is not difficult, we will give a proof here, following Shimura. We claim that $\det(\mathbf{1}_n + scc^*) = 1 + sc^*c$ for all $c \in \mathbb{R}$. Since they are both polynomials in s, we only need to prove this for $s < 0$. We have

$$\begin{pmatrix} \mathbf{1}_n & -\sqrt{-s}c \\ & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \sqrt{-s}c \\ \sqrt{-s}c^* & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \\ -\sqrt{-s}c^* & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n + scc^* & \\ & 1 \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{1}_n & \\ -\sqrt{-s}c^* & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \sqrt{-s}c \\ \sqrt{-s}c^* & 1 \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & -\sqrt{-s}c \\ & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & \\ & 1 + sc^*c \end{pmatrix}.$$

Hence $\det(\mathbf{1}_n + scc^*) = 1 + sc^*c$. (2) follows from (1) immediately. $\square$

Now we write the Lebesgue measure $dz_1 \cdots dz_{m-1}$ in the differential form of degree $(m-1, m-1)$ on D_{m-1} which is

$$dz_1 \cdots dz_{m-1} = \frac{1}{(-2i)^{m-1}} \Omega := \frac{1}{(-2i)^{m-1}} \bigwedge_{j=1}^{m-1} dz_j \wedge d\bar{z}_j,$$

where in the latter we view dz_j as a $(1, 0)$-form, not the Lebesgue measure anymore. By (4-25), we have

$$(4\text{-}23) = \frac{(2\pi)^{m^2}}{\Gamma_m(m)(2\pi i)^{m-1}} e^{-2\pi \, \mathrm{tr} \, T} \int_{D_{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})}(1 - z\bar{z})^{-m}$$

$$\times e^{\mathrm{tr}(\mathbf{1}_{m-1} + ww^*)g_0} \Delta\big|_{g=g_0} \big(e^{-\mathrm{tr}(\mathbf{1}_{m-1} + ww^*)g} \det(g)j\big)(-\mathrm{Ei})(-4\pi b(1 + w^*w))\Omega, \tag{4-26}$$

where w_j are as in (4-24). The final step is finished by the following lemma.

Lemma 4.7. *For* $g_0 = 4\pi \operatorname{diag}[a_1, \ldots, a_{m-1}]$,

$$\Delta\big|_{g=g_0}\big(e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g}\det(g)\big) = e^{-\operatorname{tr}(\mathbf{1}_{m-1}+ww^*)g}$$
$$\times \sum_{1 \le s_1 < \cdots < s_t \le m-1} (-4\pi)^t (m-1-t)!(a_{s_1} \cdots a_{s_t})(1 + w_{s_1}\bar{w}_{s_1} + \cdots + w_{s_t}\bar{w}_{s_t}),$$

where the sum is taken over all subsets of $\{1, \ldots, m-1\}$.

Proof. Let $u_{jk} = -(1 + w_j\bar{w}_k)$ and $g = (g_{jk})_{j,k=1}^{m-1}$ be the matrix variables. For short, we also use $|g|$ to indicate the determinant of the square matrix g. For subsets $I, J \subset \{1, \ldots, m-1\}$ of the same cardinality, we denote by $g_{J,K}$ (resp. $g^{J,K}$) the (square) matrix obtained by keeping (resp. omitting) the rows indexed in J and the columns indexed in K. Let $\mathfrak{S}_{m-1}$ be the group of $(m-1)$-permutations; for $\sigma \in \mathfrak{S}_{m-1}$ and a subset $J = \{j_1 < \cdots < j_t\} \subset \{1, \ldots, m-1\}$, let $\epsilon_J(\sigma) \in \{\pm 1\}$ be a factor which only depends on J and σ. This factor comes from the combinatorics in taking successive partial derivatives. In our application, we only need to know its value in the case σ maps J to itself. In this case, let σ_J be the restriction of σ to J. Then $\epsilon_J(\sigma) = (-1)^{|\sigma_J|}$. Now we compute

$$\frac{\partial}{\partial g_{1,\sigma(1)}} e^{\operatorname{tr}(ug)}|g| = u_{\sigma(1),1} e^{\operatorname{tr}(ug)}|g| + \epsilon_{\{1\}}(\sigma) e^{\operatorname{tr}(ug)}|g^{\{1\},\{\sigma(1)\}}|,$$

$$\frac{\partial}{\partial g_{2,\sigma(2)}} \frac{\partial}{\partial g_{1,\sigma(1)}} e^{\operatorname{tr}(ug)}|g| = u_{\sigma(2),2} u_{\sigma(1),1} e^{\operatorname{tr}(ug)}|g| + \epsilon_{\{2\}}(\sigma) u_{\sigma(1),1} e^{\operatorname{tr}(ug)}|g^{\{2\},\{\sigma(2)\}}|$$
$$+ \epsilon_{\{1\}}(\sigma) u_{\sigma(2),2} e^{\operatorname{tr}(ug)}|g^{\{1\},\{\sigma(1)\}}| + \epsilon_{\{1,2\}}(\sigma) e^{\operatorname{tr}(ug)}|g^{\{1,2\},\{\sigma(1),\sigma(2)\}}|.$$

By induction, we have

$$\frac{\partial}{\partial g_{m-1,\sigma(m-1)}} \cdots \frac{\partial}{\partial g_{1,\sigma(1)}} e^{\operatorname{tr}(ug)}|g| = \sum_{1 \le j_1 < \cdots < j_t \le m-1} \epsilon_{\{j_1,\ldots,j_t\}}(\sigma) u_{\sigma(s_{m-1-t}),s_{m-1-t}}$$
$$\cdots u_{\sigma(s_1),s_1} e^{\operatorname{tr}(ug)}|g^{\{j_1,\ldots,j_t\},\{\sigma(j_1),\ldots,\sigma(j_t)\}}|,$$

where $\{s_1 < \cdots < s_{m-1-t}\}$ is the complement of $\{j_1, \ldots, j_t\}$. Summing over σ, we have

$$\Delta\big|_{g=g_0} e^{\operatorname{tr}(ug)}|g| = e^{\operatorname{tr}(ug_0)} \sum_{\sigma \in \mathfrak{S}_{m-1}} (-1)^{|\sigma|} \sum_{1 \le j_1 < \cdots < j_t \le m-1} \epsilon_{\{j_1,\ldots,j_t\}}(\sigma) u_{\sigma(s_{m-1-t}),s_{m-1-t}}$$
$$\cdots u_{\sigma(s_1),s_1} |g_0^{\{j_1,\ldots,j_t\},\{\sigma(j_1),\ldots,\sigma(j_t)\}}|.$$

Changing the order of summation, since g_0 is diagonal, we have

$$\Delta\big|_{g=g_0} e^{\operatorname{tr}(ug)}|g|$$
$$= e^{\operatorname{tr}(ug_0)} \sum_{J=\{j_1 < \cdots < j_t\}} \sum_{\sigma(J)=J} (-1)^{|\sigma|}(-1)^{|\sigma_J|} u_{\sigma(s_{m-1-t}),s_{m-1-t}} \cdots u_{\sigma(s_1),s_1} |g_0^{J,J}|$$

$$= e^{\operatorname{tr}(ug_0)} \sum_{J=\{j_1<\cdots<j_t\}} t! |g_0^{J,J}| \sum_{\sigma':\bar{J}\to\bar{J}} (-1)^{|\sigma'|} u_{\sigma(s_{m-1-t}),s_{m-1-t}} \cdots u_{\sigma(s_1),s_1}$$

$$= e^{\operatorname{tr}(ug_0)} \sum_{J=\{j_1<\cdots<j_t\}} t! |g_0^{J,J}| \, |u^{J,J}| = e^{\operatorname{tr}(ug_0)} \sum_{J'=\{s_1<\cdots<s_t\}} (m-1-t)! |(g_0)_{J',J'}| \, |u_{J',J'}|.$$

The lemma follows by Lemma 4.6(1). □

In conclusion, using (4-26), we obtain our main result.

Proposition 4.8. *For $T \sim \operatorname{diag}[a_1, \dots, a_{m-1}, -b]$ of signature $(m-1, 1)$, we have*

$$W_T'(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)(2\pi i)^{m-1}} e^{-2\pi\operatorname{tr}T} \int_{D_{m-1}} e^{-4\pi(a_1 w_1\bar{w}_1 + \cdots + a_{m-1}w_{m-1}\bar{w}_{m-1})}$$

$$\times \sum_{1\le s_1<\cdots<s_t\le m-1} (-4\pi)^t (m-1-t)! (a_{s_1} \cdots a_{s_t})(1 + w_{s_1}\bar{w}_{s_1} + \cdots + w_{s_t}\bar{w}_{s_t})$$

$$\times (-\operatorname{Ei})(-4\pi b(1 + w^*w))(1 - z\bar{z})^{-m}\Omega,$$

where w_j are C^∞-functions in z as in (4-24) and Ω is the volume form in z.

4B. *An archimedean local height function.* In this section, we will introduce a notion of height on the symmetric domain which will finally contribute to the local height pairing at an archimedean place. We will also prove some important properties of this height. A basic reference for archimedean Green's currents and height pairing is [Soulé 1992, Chapter II].

Green's functions. We still let $m \ge 2$ be an integer and V' the complex hermitian space of signature $(m-1, 1)$. We identify V' with $\mathbb{C}^m$ equipped with the hermitian form $(z, z') = z_1\bar{z}_1' + \cdots + z_{m-1}\bar{z}_{m-1}' - z_m\bar{z}_m'$ for $z = (z_1, \dots, z_m)$ and $z' = (z_1', \dots, z_m')$ in $\mathbb{C}^m$. The hermitian domain $\mathscr{D}$ of $\mathsf{U}(V') \cong \mathsf{U}(n-1, 1)$ can be identified with the $(m-1)$-dimensional complex open unit disc D_{m-1} through

$$z = [z_1 : \cdots : z_m] \in \mathscr{D} \mapsto \left(\frac{z_1}{z_m}, \dots, \frac{z_{m-1}}{z_m}\right) \in D_{m-1}$$

and we will not distinguish them anymore. Given any $x \ne 0 \in V'^r$ ($1 \le r \le m-1$) with nonsingular moment matrix $T(x)$, let D_x be the subspace of D_{m-1} consisting of lines perpendicular to all components in x which is nonempty if and only if $T(x)$ is positive definite. Now suppose $r = 1$, for $z \in D_{m-1}$, and let $x = x_z + x^z$ be the orthogonal decomposition with respect to the line z, that is, $x_z \in z$ and $x^z \perp z$. Let $R(x, z) = -(x_z, x_z)$ which is nonnegative since z is negative definite and $R(x, z) = 0$ if and only if $x = 0$ or $z \in D_x$. Explicitly, let $x = (x_1, \dots, x_m) \in V'$ and $z = (z_1, \dots, z_{m-1}) \in D_{m-1}$; then

$$R(x, z) = \frac{(x_1\bar{z}_1 + \cdots + x_{m-1}\bar{z}_{m-1} - x_m)(\bar{x}_1 z_1 + \cdots + \bar{x}_{m-1}z_{m-1} - \bar{x}_m)}{1 - z\bar{z}},$$

where we recall that $z\bar{z} = z_1\bar{z}_1 + \cdots + z_{m-1}\bar{z}_{m-1}$. We define

$$\xi(x, z) = -\mathrm{Ei}(-2\pi R(x, z)).$$

For each $x \neq 0 \in V'$, $\xi(x, \cdot)$ is a smooth function on $D_{m-1} - D_x$ and has logarithmic growth along D_x if not empty. Hence we can view it as a current $[\xi(x, \cdot)]$ on D. On the other hand, we recall the *Kudla–Millson form* $\varphi \in [\mathscr{S}(V'^r) \otimes A^{r,r}(D_{m-1})]^{\mathrm{U}(V')}$ $(1 \leq r \leq m - 1)$ constructed in [Kudla and Millson 1986] and let $\omega(x, \cdot) = e^{2\pi \operatorname{tr} T(x)}\varphi(x, \cdot)$. Then we have

Proposition 4.9. *Let $x \neq 0 \in V'$, as currents; we have*

$$dd^c[\xi(x)] + \delta_{D_x} = [\omega(x)].$$

Proof. We will only give a proof for $m = 2$ since the proof for general m is similar but involves tedious computations.

First, we prove that $dd^c\xi(x) = \omega(x)$ holds away from D_x. Let $x = (x_1, x_2)$ and $z \in D_1 - D_x$. Sometimes we simply write R for $R(x)$. We have the formula

$$dd^c\xi(x) = \frac{1}{2\pi i}\left\{\frac{e^{-2\pi R}}{R^2}(R\partial\bar{\partial}R - \partial R \wedge \bar{\partial}R) - 2\pi\frac{e^{-2\pi R}}{R}\partial R \wedge \bar{\partial}R\right\}. \quad (4\text{-}27)$$

Computing each term, we have

$$R(x, z) = \frac{(x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)}{1 - z\bar{z}},$$

$$\bar{\partial}R = \frac{x_1(\bar{x}_1 z - \bar{x}_2)(1 - z\bar{z}) + (x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)z}{(1 - z\bar{z})^2}\,d\bar{z},$$

$$\partial R = \frac{\bar{x}_1(x_1\bar{z} - x_2)(1 - z\bar{z}) + (x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)\bar{z}}{(1 - z\bar{z})^2}\,dz,$$

$$\partial\bar{\partial}R = \left(\frac{x_1\bar{x}_1}{1 - z\bar{z}} + \frac{x_1\bar{z}(\bar{x}_1 z - \bar{x}_2) + (x_1\bar{z} - x_2)(2\bar{x}_1 z - \bar{x}_2) + 2Rz\bar{z}}{(1 - z\bar{z})^2}\right)dz \wedge d\bar{z},$$

$$\partial R \wedge \bar{\partial}R = \left(\frac{x_1\bar{x}_1 R}{1 - z\bar{z}} + \frac{\bar{x}_1 z(x_1\bar{z} - x_2)R + x_1\bar{z}(\bar{x}_1 z - \bar{x}_2)R + R^2 z\bar{z}}{(1 - z\bar{z})^2}\right)dz \wedge d\bar{z}.$$

Hence,

$$R\partial\bar{\partial}R - \partial R \wedge \bar{\partial}R = \left(\frac{(x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)R}{(1 - z\bar{z})^2} - \frac{R^2 z\bar{z}}{(1 - z\bar{z})^2}\right)dz \wedge d\bar{z} = R^2\frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2} \quad (4\text{-}28)$$

and

$$\partial R \wedge \bar{\partial} R = (x_1 \bar{x}_1 (1 - z\bar{z}) + \bar{x}_1 z (x_1 \bar{z} - x_2) + x_1 \bar{z} (\bar{x}_1 z - \bar{x}_2) + Rz\bar{z}) \frac{R \, dz \wedge d\bar{z}}{(1 - z\bar{z})^2}$$

$$= (x_1 \bar{x}_1 + \bar{x}_1 z x_2 + x_1 \bar{z} \bar{x}_2 + x_1 \bar{x}_1 z\bar{z}) \frac{R \, dz \wedge d\bar{z}}{(1 - z\bar{z})^2}$$

$$= ((x, x) + (\bar{x}_1 z - \bar{x}_2)(x_1 \bar{z} - x_2) + Rz\bar{z}) \frac{R \, dz \wedge d\bar{z}}{(1 - z\bar{z})^2} \qquad (4\text{-}29)$$

$$= (R(x, z) + (x, x)) \frac{R \, dz \wedge d\bar{z}}{(1 - z\bar{z})^2}.$$

Plugging in (4-28) and (4-29), we have

$$(4\text{-}27) = \left(1 - 2\pi (R(x, z) + (x, x))\right) e^{-2\pi R(x, z)} \frac{dz \wedge d\bar{z}}{2\pi i (1 - z\bar{z})^2} = \omega(x, z).$$

The rest is the same as the proof of Proposition 11.1 of [Kudla 1997], from Lemma 11.2 on page 606. We omit it. □

The proposition says that $\xi(x)$ is a Green's function of logarithmic type for D_x. Now we consider $x = (x_1, \dots, x_r) \in V''^r$ with nonsingular moment matrix $T(x)$. Then using the star product of the Green's current, we have a Green's current $\Xi_x := [\xi(x_1)] * \cdots * [\xi(x_r)]$ for D_x and as currents of degree (r, r) and

$$dd^c([\xi(x_1)] * \cdots * [\xi(x_r)]) + \delta_{D_x} = [\omega(x_1) \wedge \cdots \wedge \omega(x_r)] = [\omega(x)].$$

A height function. For $x = (x_1, \dots, x_m) \in V'^m$ with nonsingular moment matrix $T(x)$, we define the *height function*

$$H(x)_\infty := \langle 1, \Xi_x \rangle = \langle 1, [\xi(x_1)] * \cdots * [\xi(x_m)] \rangle.$$

Since $\xi(hx, hz) = \xi(x, z)$ for $h \in \mathsf{U}(V')$, the height function satisfies $H(hx)_\infty = H(x)_\infty$ and thus only depends on the (nonsingular) moment matrix $T(x)$. We sometimes simply write $H(T)_\infty$ for this function. Our main result is this:

Proposition 4.10 (invariance under $\mathsf{U}(m)$). *The height function $H(T)_\infty$ only depends on the eigenvalues of T, that is, for any $k \in \mathsf{U}(m)$, $H(kTk^*)_\infty = H(T)_\infty$.*

Proof. We prove this by induction on m. We will treat the case $m = 2$ in the next subsection. Now suppose $m \geq 3$ and the proposition holds for $m - 1$. Since $\mathsf{U}(m)$ is generated by the elements

$$\begin{pmatrix} k' & \\ & 1 \end{pmatrix}, \quad k' \in \mathsf{U}(m - 1)$$

and elements of the form $k = (k_{i,j})_{i,j=1}^m$ with entries $k_{i,\sigma(i)} \in \mathsf{U}(1)$ and zero for others for some $\sigma \in \mathfrak{S}_m$. We only need to prove that $H((x'k', x_m))_\infty = H((x', x_m))_\infty$

where $x = (x', x_m) \in V'^{m-1} \oplus V' = V'^m$ with $T(x) = T$. By definition,

$$H((x', x_m))_\infty = \langle 1, [\xi(x_1)] * \ldots * [\xi(x_{m-1})]|_{D_{x_m}} \rangle + \int_{D_{m-1}} \omega(x_1) \wedge \ldots \wedge \omega(x_{m-1}) \wedge \xi(x_m)$$

$$= H(x')_\infty + \int_{D_{m-1}} \omega(x') \wedge \xi(x_m).$$

By induction, $H(x'k')_\infty = H(x')_\infty$. Moreover, $\omega(x') = \omega(x'k')$, by [Kudla and Millson 1986, Theorem 3.2(ii)]. Hence $H((x'k', x_m))_\infty = H((x', x_m))_\infty$. $\square$

Invariance under $\mathsf{U}(2)$: *A calculus exercise.* Now we consider the case $m = 2$. Suppose

$$T = \begin{pmatrix} d_1 & m \\ \bar{m} & d_2 \end{pmatrix}, \quad d_1, d_2 \in \mathbb{R}.$$

Choose a complex number ϵ with norm 1 such that $\epsilon^2 m \in \mathbb{R}$. Then

$$\begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} d_1 & m \\ \bar{m} & d_2 \end{pmatrix} \begin{pmatrix} \epsilon & \\ & \epsilon^{-1} \end{pmatrix}^* = \begin{pmatrix} d_1 & \epsilon^2 m \\ \epsilon^2 m & d_2 \end{pmatrix} \in \mathsf{Sym}_2(\mathbb{R}).$$

Now we write the elements of $\mathsf{SO}(2)$ in the form

$$k_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}$$

and write $T[\theta] = k_\theta T k_\theta^*$. Since $\xi(\epsilon x) = \xi(x)$ for any $x \in V'$, we have reduced the problem to proving this:

Proposition 4.11. *For any* $T \in \mathsf{Sym}_2(\mathbb{R})$, *we have* $H(T[\theta])_\infty = H(T)_\infty$.

Proof. The proof is similar to that in [Kudla 1997, Section 13]. Here is the idea. We construct a differentiable map

$$\alpha : \mathbb{R} \times D_1 \to \mathsf{Her}_2(\mathbb{C})^{\det=0}$$

and a 2-form Ξ on the latter space such that the integration of $\alpha_{\theta_1}^*(\Xi) - \alpha_{\theta_0}^*(\Xi)$ on D_1 calculates the difference $H(T[\theta_1])_\infty - H(T[\theta_0])_\infty$, where $\alpha_\theta = \alpha(\theta, \cdot)$. Then we try to apply Stokes' theorem. The difficulty is that $\alpha^*(\Xi)$ has singularities along the real axis, hence a limit process should be taken to get the correct result. The difference between our proof and Kudla's is that we have different symmetric domains. Although they are conformal to each other, we need to take different 2-forms and limits of the integration domains.

Suppose that $T = \mathrm{diag}[a, -b]$ with $a, b > 0$. Let

$$T[\theta] = \begin{pmatrix} d_{1,\theta} & m_\theta \\ m_\theta & d_{2,\theta} \end{pmatrix} \in \mathsf{Sym}_2(\mathbb{R})$$

and let $x_0 = (\sqrt{2a}, 0) \in V'$, $y_0 = (0, \sqrt{2b}) \in V'$. For $\theta \in \mathbb{R}$, let

$$x_\theta = x_0 k_\theta = (x_{1,\theta}, x_{2,\theta}) = \cos \theta \cdot x_0 - \sin \theta \cdot y_0,$$
$$y_\theta = y_0 k_\theta = (y_{1,\theta}, y_{2,\theta}) = \sin \theta \cdot x_0 + \cos \theta \cdot y_0.$$

We have $dx_\theta/d\theta = -y_\theta$, $dy_\theta/d\theta = x_\theta$, and $H(T[\theta])_\infty = H((x_\theta, y_\theta))_\infty$. Let $z_{x,\theta} = x_{2,\theta}/x_{1,\theta}$ and $z_{y,\theta} = y_{2,\theta}/y_{1,\theta}$). Then $D_{x_\theta} = [z_{x,\theta}, 1]$ and $D_{y_\theta} = [z_{y,\theta}, 1]$, if not empty. We make the convention that if $|z| \geq 1$, then $f(z)$ is 0 for any function f.

Lemma 4.12 [Kudla 1997, Lemma 11.4]. *We have*

$$H((x_\theta, y_\theta))_\infty = \xi(x_\theta, z_{y,\theta}) + \int_{D_1} \xi(y_\theta)\omega(x_\theta)$$

$$= \xi(y_\theta, z_{x,\theta}) + \int_{D_1} \xi(x_\theta)\omega(y_\theta)$$

$$= \xi(x_\theta, z_{y,\theta}) + \xi(y_\theta, z_{x,\theta}) - \int_{D_1} d\xi(x_\theta) \wedge d^c \xi(y_\theta).$$

We now write $x = (x_1, x_2) \in V'$, $y = (y_1, y_2) \in V'$ and $R_1 = R(x)$, $R_2 = R(y)$ and consider the following integral in general:

$$I(T) = I((x, y)) := -\int_{D_1} d\xi(x) \wedge d^c \xi(y)$$

$$= -\frac{1}{4\pi i} \int_{D_1} (\partial + \bar{\partial})\xi(x) \wedge (\partial - \bar{\partial})\xi(y)$$

$$= -\frac{i}{4\pi} \int_{D_1} \partial\xi(x) \wedge \bar{\partial}\xi(y) + \partial\xi(y) \wedge \bar{\partial}\xi(x)$$

$$= -\frac{i}{4\pi} \int_{D_1} \frac{e^{-2\pi(R_1+R_2)}}{R_1 R_2}(\partial R_1 \wedge \bar{\partial} R_2 + \partial R_2 \wedge \bar{\partial} R_1).$$

For $z \in D_1$, letting $x(z) = (1 - z\bar{z})^{-1/2}(z, 1) \in V'$ and $M = (x, x(z))\overline{(y, x(z))}$, we have

Lemma 4.13. *Let $2m = (x, y)$. Then*

$$\partial R_1 \wedge \bar{\partial} R_2 + \partial R_2 \wedge \bar{\partial} R_1 = 2(R_1 R_2 + \bar{m}M + m\overline{M})\frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2},$$

$$\partial R_1 \wedge \bar{\partial} R_2 - \partial R_2 \wedge \bar{\partial} R_1 = 2(\bar{m}M - m\overline{M})\frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2}.$$

Proof. By definition,

$$R_1 = \frac{(x_1\bar{z} - x_2)(\bar{x}_1 z - \bar{x}_2)}{1 - z\bar{z}}.$$

Hence,

$$\partial R_1 = \frac{(x_1\bar{z} - x_2)\bar{x}_1 + \bar{z}R_1}{1 - z\bar{z}} dz$$

and similarly for ∂R_2, $\bar{\partial} R_1$, and $\bar{\partial} R_2$. We compute

$\partial R_1 \wedge \bar{\partial} R_2$

$= \{(x_1\bar{z}-x_2)(\bar{y}_1z-\bar{y}_2)\bar{x}_1y_1+(\bar{y}_1z-\bar{y}_2)y_1\bar{z}R_1+(x_1\bar{z}-x_2)\bar{x}_1zR_2+z\bar{z}R_1R_2\}\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}$

$= \{(x_1\bar{z}-x_2)(\bar{y}_1z-\bar{y}_2)\bar{x}_1y_1+(\bar{y}_1z-\bar{y}_2)y_2R_1+(x_1\bar{z}-x_2)\bar{x}_1zR_2+R_1R_2\}\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}$

$= \left\{(x_1\bar{z}-x_2)(\bar{y}_1z-\bar{y}_2)\left(\bar{x}_1y_1+\dfrac{y_2(\bar{x}_1z-\bar{x}_2)}{1-z\bar{z}}\right)+(x_1\bar{z}-x_2)\bar{x}_1zR_2+R_1R_2\right\}\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}$

$= \left\{(x_1\bar{z}-x_2)(\bar{y}_1z-\bar{y}_2)\dfrac{\bar{x}_1y_1-\bar{x}_2y_2-\bar{x}_1z(y_1\bar{z}-y_2)}{1-z\bar{z}}H(x_1\bar{z}-x_2)\bar{x}_1zR_2+R_1R_2\right\}$

$$\times\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}$$

$= (2\bar{m}M+R_1R_2)\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}.$

The lemma follows from a similar calculation for $\partial R_2 \wedge \bar{\partial} R_1$. $\qquad\square$

We define a morphism $\alpha: \mathbb{R} \times D_1 \longrightarrow \mathrm{Her}_2(\mathbb{C})^{\det=0}$ between two 3-dimensional real analytic spaces where

$$\alpha(\theta, z) = \begin{pmatrix} R_1 & M \\ \overline{M} & R_2 \end{pmatrix} = \begin{pmatrix} (x_\theta, x(z))\overline{(x_\theta, x(z))} & (x_\theta, x(z))\overline{(y_\theta, x(z))} \\ \overline{(x_\theta, x(z))}(y_\theta, x(z)) & (y_\theta, x(z))\overline{(y_\theta, x(z))} \end{pmatrix}$$

and $\alpha_\theta := \alpha(\theta, \cdot)$. By an easy computation, we see that

$$\dfrac{dR_1}{d\theta} = -(M+\overline{M}), \quad \dfrac{dR_2}{d\theta} = M+\overline{M}, \quad \dfrac{dM}{d\theta} = R_1+R_2. \tag{4-30}$$

Hence R_1+R_2 and $M-\overline{M}$, which are the values at $\theta=0$, are independent of θ:

$$R_1+R_2 = \dfrac{2az\bar{z}+2b}{1-z\bar{z}} = -2a+\dfrac{2(a+b)}{1-z\bar{z}}, \quad M-\overline{M} = \dfrac{2\sqrt{ab}(z-\bar{z})}{1-z\bar{z}}. \tag{4-31}$$

By Lemma 4.13 and the fact that $2m_\theta = (x_\theta, y_\theta) \in \mathbb{R}$, we have

$$I(T[\theta]) = -\dfrac{i}{2\pi}\int_{D_1}\dfrac{e^{-2\pi(R_1+R_2)}}{R_1R_2}(R_1R_2+m(M+\overline{M}))\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}$$

$$= \left(-\dfrac{i}{2\pi}\int_{D_1}e^{-2\pi(R_1+R_2)}\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}\right)$$

$$+\left(-\dfrac{i}{2\pi}\int_{D_1}\dfrac{e^{-2\pi(R_1+R_2)}}{R_1R_2}m(M+\overline{M})\dfrac{dz\wedge d\bar{z}}{(1-z\bar{z})^2}\right)$$

$$=: I'(T[\theta]) + I''(T[\theta]). \tag{4-32}$$

By (4-31), the integral $I'(T[\theta])$ is independent of θ; hence we now only consider the second one, $I''(T[\theta])$. We define a differential form of degree two on (the smooth locus of) $\mathrm{Her}_2(\mathbb{C})^{\det=0}$:

$$\Xi = -\frac{i}{4\pi} \frac{e^{-2\pi(R_1+R_2)}}{R_1 R_2} \frac{M+\overline{M}}{M-\overline{M}} \, dR_1 \wedge dR_2$$

which has singularities along $R_1 R_2 (M - \overline{M}) = 0$.

Lemma 4.14. (1) *For a fixed $\theta \in \mathbb{R}$,*

$$\alpha_\theta^*(\Xi) = -\frac{i}{2\pi} \frac{e^{-2\pi(R_1+R_2)}}{R_1 R_2} m(M+\overline{M}) \frac{dz \wedge d\bar{z}}{(1-z\bar{z})^2};$$

(2) *On* $\mathrm{Her}_2(\mathbb{C})^{\det=0}$, *we have*

$$d\Xi = \frac{i}{\pi} \frac{e^{-2\pi(R_1+R_2)}}{(M-\overline{M})^2(M+\overline{M})} \, d(M-\overline{M}) \wedge dR_1 \wedge dR_2.$$

Proof. (1) follows from Lemma 4.13. For (2), by the relation

$$(M+\overline{M})^2 - (M-\overline{M})^2 = 4R_1 R_2,$$

we have

$$\frac{d}{d(M-\overline{M})} \frac{M+\overline{M}}{M-\overline{M}} = -\frac{4R_1 R_2}{(M-\overline{M})^2(M+\overline{M})}. \qquad \square$$

Let $D_1^+ = \{z \in D_1 \mid \Im(z) \geq 0\}$. Since $\alpha_\theta^*(\Xi)/dz \wedge d\bar{z}$ is invariant under $z \mapsto \bar{z}$, by (4-32) and Lemmas 4.12 and 4.14(1), we have

$$H(T[\theta_1])_\infty - H(T[\theta_0])_\infty$$

$$= \xi(x_{\theta_1}, z_{y,\theta_1}) + \xi(y_{\theta_1}, z_{x,\theta_1}) - \xi(x_{\theta_0}, z_{y,\theta_0}) - \xi(y_{\theta_0}, z_{x,\theta_0}) + I(T[\theta]) - I(T[\theta_0])$$

$$= \xi(x_{\theta_1}, z_{y,\theta_1}) + \xi(y_{\theta_1}, z_{x,\theta_1}) - \xi(x_{\theta_0}, z_{y,\theta_0}) - \xi(y_{\theta_0}, z_{x,\theta_0}) + I''(T[\theta]) - I''(T[\theta_0])$$

$$= \xi(x_{\theta_1}, z_{y,\theta_1}) + \xi(y_{\theta_1}, z_{x,\theta_1}) - \xi(x_{\theta_0}, z_{y,\theta_0}) - \xi(y_{\theta_0}, z_{x,\theta_0}) + \int_{D_1} \alpha_{\theta_1}^*(\Xi) - \int_{D_1} \alpha_{\theta_0}^*(\Xi)$$

$$= \xi(x_{\theta_1}, z_{y,\theta_1}) + \xi(y_{\theta_1}, z_{x,\theta_1}) - \xi(x_{\theta_0}, z_{y,\theta_0}) - \xi(y_{\theta_0}, z_{x,\theta_0})$$

$$+ 2\int_{D_1^+} \alpha_{\theta_1}^*(\Xi) - 2\int_{D_1^+} \alpha_{\theta_0}^*(\Xi). \quad (4\text{-}33)$$

We see that the form $\alpha_\theta^*(\Xi)$ has (possible) singularities when $R_1 R_2 = 0$, that is, the (possible) points $z_{x,\theta}, z_{y,\theta}$. An easy calculation shows that

$$z_{x,\theta} = \frac{x_{2,\theta}}{x_{1,\theta}} = -\tan\theta \cdot \sqrt{\frac{b}{a}} \in \mathbb{R}, \qquad z_{y,\theta} = \frac{y_{2,\theta}}{y_{1,\theta}} = \cot\theta \cdot \sqrt{\frac{b}{a}} \in \mathbb{R}.$$

Now we assume that $[\theta_0, \theta_1] \subset (0, \pi/2)$. Then 0 will not be a singular point for $\theta \in [\theta_0, \theta_1]$. Our goal is to calculate the value

$$\int_{D_1^+} \alpha_{\theta_0}^*(\Xi) - \int_{D_1^+} \alpha_{\theta_1}^*(\Xi).$$

For any $\epsilon > 0$ small enough, let $B_{1,\epsilon}$ be the (oriented) path $\{z = re^{i\epsilon} \mid r \in [0, 1)\}$ from $r = 0$ to $r = 1$, $B_{2,\epsilon}$ the path $\{z = re^{i(\pi-\epsilon)} \mid r \in [0, 1)\}$ from $r = 1$ to $r = 0$, and $D_\epsilon \subset D_1^+$ the area containing points on or above the lines $B_{1,\epsilon}$ and $B_{2,\epsilon}$. By our assumption, $\alpha_\theta^*(\Xi)$ is nonsingular on D_ϵ for any $\theta \in [\theta_0, \theta_1]$. By Stokes' theorem and the fact that $e^{-2\pi(R_1+R_2)}$ decays rapidly as $|z|$ goes to 1, we have

$$\int_{[\theta_0, \theta_1] \times D_\epsilon} \alpha^*(d\Xi) = \int_{D_\epsilon} \alpha_{\theta_1}^*(\Xi) - \int_{D_\epsilon} \alpha_{\theta_0}^*(\Xi) + \int_{[\theta_0, \theta_1] \times (B_{2,\epsilon}+B_{1,\epsilon})} \alpha^*(\Xi). \quad (4\text{-}34)$$

Lemma 4.15.
$$\int_{[\theta_0, \theta_1] \times D_\epsilon} \alpha^*(d\Xi) = 0.$$

Proof. By (4-30) and (4-31), we have

$$dR_1 = \partial R_1 + \bar{\partial} R_1 - (M + \overline{M}) \, d\theta,$$
$$dR_2 = \partial R_2 + \bar{\partial} R_2 + (M + \overline{M}) \, d\theta,$$
$$d(M - \overline{M}) = 2\sqrt{ab}\left(\partial \frac{z - \bar{z}}{1 - z\bar{z}} + \bar{\partial} \frac{z - \bar{z}}{1 - z\bar{z}}\right).$$

Hence

$$\alpha^*(d(M - \overline{M}) \wedge dR_1 \wedge dR_2)$$
$$= 2\sqrt{ab}(M + \overline{M})\left(\partial \frac{z - \bar{z}}{1 - z\bar{z}} \wedge \bar{\partial}(R_1 + R_2) - \partial(R_1 + R_2) \wedge \bar{\partial} \frac{z - \bar{z}}{1 - z\bar{z}}\right)$$
$$= 4\sqrt{ab}(a + b)(M + \overline{M})\left(\partial \frac{z - \bar{z}}{1 - z\bar{z}} \wedge \bar{\partial} \frac{1}{1 - z\bar{z}} - \partial \frac{1}{1 - z\bar{z}} \wedge \bar{\partial} \frac{z - \bar{z}}{1 - z\bar{z}}\right)$$

and by Lemma 4.14(2),

$$\alpha^*(d\Xi) = \frac{4i\sqrt{ab}(a+b)}{\pi} \frac{e^{-2\pi(R_1+R_2)}}{(M - \overline{M})^2}\left(\partial \frac{z - \bar{z}}{1 - z\bar{z}} \wedge \bar{\partial} \frac{1}{1 - z\bar{z}} - \partial \frac{1}{1 - z\bar{z}} \wedge \bar{\partial} \frac{z - \bar{z}}{1 - z\bar{z}}\right)$$
$$= \frac{4i\sqrt{ab}(a+b)}{\pi} \frac{e^{-2\pi(R_1+R_2)}}{(M - \overline{M})^2} \frac{z + \bar{z}}{(1 - z\bar{z})^3} \, dz \wedge d\bar{z}.$$

Since $z \mapsto -\bar{z}$ keeps the domain $[\theta_0, \theta_1] \times D_\epsilon$ and maps $\alpha^*(d\Xi)/dz \wedge d\bar{z}$ to its negative, the integral is zero. $\qquad \square$

Hence by (4-34),

$$\int_{D_1^+} \alpha_{\theta_0}^*(\Xi) - \int_{D_1^+} \alpha_{\theta_1}^*(\Xi) = \lim_{\epsilon \to 0} \int_{D_\epsilon} \alpha_{\theta_0}^*(\Xi) - \lim_{\epsilon \to 0} \int_{D_\epsilon} \alpha_{\theta_1}^*(\Xi)$$

$$= \lim_{\epsilon \to 0} \int_{[\theta_0,\theta_1] \times (B_{1,\epsilon} + B_{2,\epsilon})} \alpha^*(\Xi). \quad (4\text{-}35)$$

A simple computation shows that on $[\theta_0, \theta_1] \times (B_{2,\epsilon} + B_{1,\epsilon})$,

$$\alpha^*(\Xi) = \frac{-i(a+b)}{\pi} \frac{e^{-2\pi(R_1+R_2)}}{R_1 R_2} \frac{(M+\overline{M})^2}{M-\overline{M}} \frac{r}{(1-r^2)^2} dr \wedge d\theta$$

$$= \frac{-i(a+b)}{\pi} \frac{e^{-2\pi(R_1+R_2)}(M-\overline{M})}{R_1 R_2} \frac{r}{(1-r^2)^2} dr \wedge d\theta$$

$$+ \frac{-4i(a+b)}{\pi} \frac{e^{-2\pi(R_1+R_2)}}{M-\overline{M}} \frac{r}{(1-r^2)^2} dr \wedge d\theta.$$

Since the integrations of the second form on the two paths cancel each other, we have

$$(4\text{-}35) = \int_{\theta_0}^{\theta_1} d\theta \cdot \frac{-i(a+b)}{\pi} \lim_{\epsilon \to 0} \int_{B_{1,\epsilon}+B_{2,\epsilon}} \frac{e^{-2\pi(R_1+R_2)}(M-\overline{M})}{R_1 R_2} \frac{r}{(1-r^2)^2} dr$$

$$= \int_{\theta_0}^{\theta_1} d\theta \cdot \frac{4\sqrt{ab}(a+b)}{\pi} \lim_{\epsilon \to 0} \sin \epsilon \int_{B_{1,\epsilon}+B_{2,\epsilon}} \frac{e^{-2\pi(R_1+R_2)}}{R_1 R_2} \frac{r^2}{(1-r^2)^3} dr.$$

$$(4\text{-}36)$$

To proceed, we need the following lemma.

Lemma 4.16. *Let $f(r)$ be a C^∞-function on $[0,1)$ which is rapidly decreasing as $r \to 1$. Then for any $c_1, c_2, d_1, d_2 > 0$,*

$$\lim_{\epsilon \to 0^+} \int_0^1 \frac{\sin \epsilon}{(c_1^2 r^2 + c_2^2 - 2c_1 c_2 r \cos \epsilon)(d_1^2 r^2 + d_2^2 + 2d_1 d_2 r \cos \epsilon)} f(r) dr$$

$$= \begin{cases} \dfrac{\pi c_1}{c_2(c_1 d_2 + c_2 d_1)^2} f\left(\dfrac{c_2}{c_1}\right), & c_1 > c_2, \\ 0, & c_1 \le c_2. \end{cases}$$

Proof. The case $c_1 \le c_2$ follows from the fact that f is rapidly decreasing. To prove the first case, we only need to prove that

$$\lim_{\epsilon \to 0^+} \int_0^1 \frac{\sin \epsilon}{c_1^2 r^2 + c_2^2 - 2c_1 c_2 r \cos \epsilon} dr = \frac{\pi}{c_1 c_2}. \quad (4\text{-}37)$$

The integral of the left-hand side of (4-37) (for small $\epsilon > 0$) equals

$$\sin\epsilon \int_0^1 \frac{1}{(c_1 r - c_2 \cos\epsilon)^2 + c_2^2(1 - \cos\epsilon)}$$

$$= \frac{\sin\epsilon}{c_1 c_2 \sqrt{1 - \cos\epsilon}} \int_{-\frac{\cos\epsilon}{\sqrt{1-\cos\epsilon}}}^{\frac{c_1 - c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}} \frac{1}{\left(\frac{c_1 r - c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}\right)^2 + 1} \, d\frac{c_1 r - c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}}$$

$$= \frac{\sin\epsilon}{c_1 c_2 \sqrt{1 - \cos\epsilon}} \left(\arctan\frac{c_1 r - c_2\cos\epsilon}{c_2\sqrt{1-\cos\epsilon}} + \arctan\frac{\cos\epsilon}{\sqrt{1-\cos\epsilon}} \right).$$

Let $\epsilon \to 0^+$, and the limit is $\pi/c_1 c_2$. $\qquad\square$

Applying the lemma, we have

$$(4\text{-}36) = \int_{\theta_0}^{\theta_1} \sqrt{ab}(a+b) \left(\frac{e^{-2\pi R_1(z_{y,\theta})}}{R_1(z_{y,\theta})} \frac{y_{1,\theta} y_{2,\theta}}{d_{2,\theta}^2} + \frac{e^{-2\pi R_2(z_{x,\theta})}}{R_2(z_{x,\theta})} \frac{x_{1,\theta} x_{2,\theta}}{d_{1,\theta}^2} \right) d\theta.$$

$$(4\text{-}38)$$

But

$$\frac{dR_1(z_{y,\theta})}{d\theta} = \frac{d}{d\theta}(R_1(z_{y,\theta}) + R_2(z_{y,\theta})) = \frac{4(a+b)r}{(1-r^2)^2}\bigg|_{r=y_{2,\theta}/y_{1,\theta}} \cdot \frac{d}{d\theta}\left(\frac{y_{2,\theta}}{y_{1,\theta}}\right)$$

$$= 2\sqrt{ab}(a+b)\frac{y_{1,\theta} y_{2,\theta}}{d_{2,\theta}^2},$$

$$\frac{dR_2(z_{x,\theta})}{d\theta} = 2\sqrt{ab}(a+b)\frac{x_{1,\theta} x_{2,\theta}}{d_{1,\theta}^2}.$$

Hence

$$(4\text{-}38) = \frac{1}{2}\left(\int_{R_1(z_{y,\theta_0})}^{R_1(z_{y,\theta_1})} \frac{e^{-2\pi R_1(z_{y,\theta})}}{R_1(z_{y,\theta})} \, dR_1(z_{y,\theta}) + \int_{R_2(z_{x,\theta_0})}^{R_2(z_{x,\theta_1})} \frac{e^{-2\pi R_2(z_{x,\theta})}}{R_2(z_{x,\theta})} \, dR_2(z_{x,\theta}) \right)$$

$$= \frac{1}{2}(\xi(x_{\theta_1}, z_{y,\theta_1}) + \xi(y_{\theta_1}, z_{x,\theta_1}) - \xi(x_{\theta_0}, z_{y,\theta_0}) - \xi(y_{\theta_0}, z_{x,\theta_0}))$$

which, by (4-33), implies that

$$H(T[\theta_1])_\infty - H(T[\theta_0])_\infty = 0$$

for $[\theta_0, \theta_1] \in (0, \pi/2)$. The same argument works for other intervals and the constancy of $H(T[\theta])_\infty$ for all $\theta \in \mathbb{R}$ follows from the continuity. This finishes the proof of Proposition 4.11. $\qquad\square$

4C. *An arithmetic local Siegel–Weil formula.* In this section, we will find a relation between derivatives of Whittaker functions and the height functions defined above. Further, we will prove a local arithmetic analogue of the Siegel–Weil formula at an archimedean place for general dimensions.

Comparison on the hermitian domain. We are going to prove a relation between $W_T'(0, e, \Phi^0)$ and $H(T)_\infty$. Now suppose $T \sim \mathrm{diag}[a_1, \ldots, a_{m-1}, -b]$ which is hermitian of signature $(m - 1, 1)$. By Proposition 4.10, $H(T)_\infty$ only depends on $a_1, \ldots, a_{m-1}, b$. Hence, if we let $x_j = (\ldots, \sqrt{2a_j}, \ldots) \in \mathbb{C}^m \cong V'$ with the j-th entry $\sqrt{2a_j}$ and all others zero for $j = 1, \ldots, m - 1$ and $x_m = (0, \ldots, 0, \sqrt{2b})$, then $H(T)_\infty = H((x_1, \ldots, x_m))_\infty$. Since $(x_m, x_m) < 0$, we have $D_{x_m} = \varnothing$ and

$$H(T)_\infty = \int_{D_{m-1}} \omega(x_1) \wedge \cdots \wedge \omega(x_{m-1}) \wedge \xi(x_m).$$

Our main result is the following *local arithmetic Siegel–Weil formula* at an archimedean place:

Theorem 4.17. *For $T \in \mathsf{Her}_m(\mathbb{C})$ of signature $(m - 1, 1)$, we have*

$$W_T'(0, e, \Phi^0) = \gamma_V \frac{(2\pi)^{m^2}}{\Gamma_m(m)} e^{-2\pi \, \mathrm{tr}\, T} H(T)_\infty.$$

Proof. By the above discussion, we can assume $T = \mathrm{diag}[a_1, \ldots, a_{m-1}, -b]$ and, by Proposition 4.8, we need to prove that

$$(2\pi i)^{m-1} \int_{D_{m-1}} \omega(x_1) \wedge \cdots \wedge \omega(x_{m-1}) \wedge \xi(x_m) = \int_{D_{m-1}} e^{-4\pi(a_1 w_1 \bar{w}_1 + \cdots + a_{m-1} w_{m-1} \bar{w}_{m-1})}$$

$$\times \sum_{1 \leq s_1 < \cdots < s_t \leq m-1} (-4\pi)^t (m - 1 - t)! (a_{s_1} \cdots a_{s_t})(1 + w_{s_1} \bar{w}_{s_1} + \cdots + w_{s_t} \bar{w}_{s_t})$$

$$\times (-\mathsf{Ei})(-4\pi b(1 + w^* w))(1 - z\bar{z})^{-m} \Omega. \quad (4\text{-}39)$$

By definition and (4-24), we have

$$R_j(z) := R(x_j, z) = \frac{2a_j z_j \bar{z}_j}{1 - z\bar{z}} = 2a_j w_j \bar{w}_j, \quad j = 1, \ldots, m - 1,$$

$$R_m(z) := R(x_m, z) = \frac{-2b}{1 - z\bar{z}} = -2b(1 + w^* w).$$

Hence $\xi(x_m) = -\mathsf{Ei}(-4\pi b(1 + w^* w))$. Now we need an explicit formula for $\omega(x_j)$. By (4-27), we need to calculate $\bar{\partial} R_j$, ∂R_j, and $\partial \bar{\partial} R_j$ for $j = 1, \ldots, m - 1$. We have

$$(1 - z\bar{z})R_j = 2a_j z_j \bar{z}_j$$

$$\implies \quad \bar{\partial}(1 - z\bar{z})R_j + (1 - z\bar{z})\bar{\partial} R_j = 2a_j z_j \, d\bar{z}_j, \quad (4\text{-}40)$$

$$\implies \quad \bar{\partial} R_j = \frac{2a_j z_j \, d\bar{z}_j + R_j \bar{\partial}(z\bar{z})}{1 - z\bar{z}}. \quad (4\text{-}41)$$

Similarly,

$$\partial R_j = \frac{2a_j \bar{z}_j \, dz_j + R_j \partial(z\bar{z})}{1 - z\bar{z}}. \quad (4\text{-}42)$$

Differentiating (4-40) again and plugging in (4-41) and (4-42), we have

$$\partial\bar\partial(1-z\bar z)R_j+\partial R_j\bar\partial(1-z\bar z)+\partial(1-z\bar z)\bar\partial R_j+(1-z\bar z)\partial\bar\partial R_j=2a_j\,dz_jd\bar z_j$$

which implies that

$$R_j=\frac{1}{(1-z\bar z)^2}\big(2a_j(1-z\bar z)\,dz_jd\bar z_j+2a_j\bar z_j\,dz_j\bar\partial(z\bar z)+2a_jz_j\partial(z\bar z)\,d\bar z_j$$
$$+2R_j\partial(z\bar z)\bar\partial(z\bar z)+R_j(1-z\bar z)\partial\bar\partial(z\bar z)\big).\quad(4\text{-}43)$$

Taking the wedge of (4-41) and (4-42), we have

$$\partial R_j\wedge\bar\partial R_j$$
$$=\frac{4a_j^2z_j\bar z_j\,dz_jd\bar z_j+2a_jR_j\bar z_j\,dz_j\bar\partial(z\bar z)+2a_jR_jz_j\partial(z\bar z)d\bar z_j+R_j^2\partial(z\bar z)\bar\partial(z\bar z)}{(1-z\bar z)^2}.\quad(4\text{-}44)$$

Combining (4-43) and (4-44), we have

$$\frac{1}{R_j^2}(R_j\partial\bar\partial R_j-\partial R_j\wedge\bar\partial R_j)=\frac{\partial(z\bar z)\bar\partial(z\bar z)}{(1-z\bar z)^2}+\frac{\partial\bar\partial(z\bar z)}{1-z\bar z}.\quad(4\text{-}45)$$

For simplicity, we make some substitutions. Let

$$\omega=\partial(z\bar z)\bar\partial(z\bar z)+(1-z\bar z)\partial\bar\partial(z\bar z),$$
$$\omega_j=(1-z\bar z)\bar z_j\,dz_jd\bar z_j+\bar z_j\,dz_j\bar\partial(z\bar z)+z_j\partial(z\bar z)d\bar z_j$$
$$+w_j\bar w_j\partial(z\bar z)\bar\partial(z\bar z),\quad j=1,\dots,m-1.$$

Then $2\pi i\omega(x_j)=-\partial\bar\partial\xi(x_j)=e^{-4\pi a_jw_j\bar w_j}(\omega-4\pi a_j\omega_j)(1-z\bar z)^2$. Hence to prove (4-39), we only need to prove the following equality between $(m-1,m-1)$-forms on D_{m-1}:

$$\bigwedge_{j=1}^{m-1}(\omega-4\pi a_j\omega_j)$$
$$=\sum_{s_1<\cdots<s_t}(-4\pi)^t(m-1-t)!(a_{s_1}\cdots a_{s_t})(1+w_{s_1}\bar w_{s_1}+\cdots+w_{s_t}\bar w_{s_t})(1-z\bar z)^{m-2}\Omega$$

which follows from the claim that for any subset $\{s_1<\cdots<s_t\}\subset\{1,\dots,m-1\}$, we have

$$\omega_{s_1}\wedge\cdots\wedge\omega_{s_t}\wedge\omega^{m-1-t}$$
$$=(m-1-t)!(1+w_{s_1}\bar w_{s_1}+\cdots+w_{s_t}\bar w_{s_t})(1-z\bar z)^{m-2}\Omega.\quad(4\text{-}46)$$

This will be proved in the next lemma where, without loss of generality, we assume that $s_j=j$. The theorem follows. □

Lemma 4.18. *Let w_j, Ω, ω, and ω_j be as above; for any integer $0 \le t \le m - 1$, we have the following equality between $(m-1, m-1)$-forms*:

$$\left(\bigwedge_{j=1}^{t} \omega_j \right) \wedge \omega^{m-1-t} = (m - 1 - t)! \left(1 + \sum_{j=1}^{t} w_j \bar{w}_j \right) (1 - z\bar{z})^{m-2} \Omega.$$

Proof. For $j = 1, \ldots, m - 1$, we let

$$\sigma_j = \bar{z}_j \, dz_j \bar{\partial}(z\bar{z}), \quad \sigma'_j = z_j \partial(z\bar{z}) \, d\bar{z}_j, \quad \delta_j = (1 - z\bar{z}) \, dz_j d\bar{z}_j.$$

Then

$$\sum_{k=1}^{m-1} \sigma_k = \sum_{k=1}^{m-1} \sigma'_k, \quad \omega = \sum_{k=1}^{m-1} (\sigma_k + \delta_k), \quad \omega_j = \delta_j + \sigma_j + \sigma'_j + w_j \bar{w}_j \sum_{k=1}^{m-1} \sigma_k,$$

and

$$\sigma_j \wedge \sigma_j = 0, \quad \sigma'_j \wedge \sigma'_j = 0, \quad \delta_j \wedge \delta_j = 0.$$

We introduce the $(m - 1) \times (m - 1)$ matrix

$$Z = \begin{pmatrix} \bar{z}_1 z_1 & \bar{z}_2 z_1 & \cdots & \bar{z}_{m-1} z_1 \\ \bar{z}_1 z_2 & \bar{z}_2 z_2 & \cdots & \bar{z}_{m-1} z_2 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{z}_1 z_{m-1} & \bar{z}_2 z_{m-1} & \cdots & \bar{z}_{m-1} z_{m-1} \end{pmatrix}$$

and recall the notation $Z_{J,K}$ (see the proof of Lemma 4.7) for subsets $J, K \subset \{1, \ldots, m - 1\}$ with $|J| = |K|$. It is easy to see that $|Z_{J,K}| \ne 0$ only if $|J| \le 1$ where $|Z_{\{j\},\{k\}}| = \bar{z}_j z_k$ and $|Z_{\varnothing,\varnothing}| = 1$.

Now we consider three subsets $I, J, K \subset \{1, \ldots, m - 1\}$ with $|I| + |J| + |K| = m - 1$. Writing

$$\sigma_I = \bigwedge_{i \in I} \sigma_i$$

and similarly for σ'_J and δ_K, we have the following equalities

$$\sigma_I \sigma'_J \delta_K := \sigma_I \wedge \sigma'_J \wedge \delta_K = \begin{cases} \epsilon_{I,J,K} |Z_{I,\overline{J \cup K}}| |Z_{\overline{I \cup K},J}| (1 - z\bar{z})^{|K|} \Omega, & (I \cup J) \cap K = \varnothing, \\ 0, & (I \cup J) \cap K \ne \varnothing, \end{cases}$$

where $\epsilon_{I,J,K} \in \{\pm 1\}$ is a factor only depending on I, J, K. This is not zero only if $|I| \le 1$ and $|J| \le 1$. Explicitly,

$$\sigma_I \sigma'_J \delta_K = \begin{cases} \bar{z}_i z_i \bar{z}_j z_j (1 - z\bar{z})^{m-3} \Omega, & i \ne j, I = \{i\}, J = \{j\}, K = \overline{I \cup J}, \\ -\bar{z}_i z_j \bar{z}_j z_i (1 - z\bar{z})^{m-3} \Omega, & i \ne j, I = J = \{i\}, K = \overline{I \cup \{j\}}, \\ \bar{z}_i z_i (1 - z\bar{z})^{m-2} \Omega, & I \cup J = \{i\}, K = \overline{\{i\}}, \\ (1 - z\bar{z})^{m-1} \Omega, & I = J = \varnothing, K = \{1, \ldots, m - 1\}. \end{cases}$$

Now we compute

$$\left(\bigwedge_{j=1}^{t}\omega_j\right)\wedge\omega^{m-1-t}$$

$$=\bigwedge_{j=1}^{t}\left(\delta_j+\sigma_j+\sigma_j'+w_j\bar{w}_j\sum_{k=1}^{m-1}\sigma_k\right)\wedge\left(\sum_{k=1}^{m-1}\sigma_k+\sum_{k=1}^{m-1}\delta_k\right)^{m-1-t}$$

$$=\left(\sum_{L\sqcup M\sqcup N\sqcup P=\{1,\dots,t\}}\delta_L\sigma_M\sigma_N'w_P\bar{w}_P\left(\sum_{k=1}^{m-1}\sigma_k\right)^{|P|}\right)$$

$$\times\left(\sum_{\substack{Q\subset 1,\dots,m-1\\|Q|\le m-1-t}}\frac{(m-1-t)!}{(m-1-t-|Q|)!}\left(\sum_{k=1}^{m-1}\sigma_k\right)^{m-1-t-|Q|}\delta_Q\right)$$

$$=\sum_{L,M,N,P,Q}\frac{(m-1-t)!}{(m-1-t-|Q|)!}\delta_{L\cup Q}\sigma_M\sigma_N'w_P\bar{w}_P\left(\sum_{k=1}^{m-1}\sigma_k\right)^{|P|+m-1-t-|Q|}$$

$$=:\sum_{L,M,N,P,Q}T_{L,M,N,P,Q}, \tag{4-47}$$

where $w_P=\prod_{p\in P}w_p$ and similarly for $\bar{w}_P$. We now classify and calculate all the terms $T_{L,M,N,P,Q}$ which are not zero. It is easy to see that $|Q|\ge m-2-t$ if $T_{L,M,N,P,Q}\ne 0$. We now list all cases where $T_{L,M,N,P,Q}$ may not be zero.

Case I: $|Q|=m-1-t$. Then $|P|\le 1$:

Case I-1: $|P|=0$. Then $Q=\{t+1,\dots,m-1\}$ and $|M|\le 1$, $|N|\le 1$:

Case I-1a: $M=\{m\}$ and $N=\{n\}$ for $m\ne n\in\{1,\dots,t\}$. Then the sum of corresponding terms is

$$\sum T_{L,M,N,P,Q}=(m-1-t)!\sum_{\substack{m,n=1\\m\ne n}}^{t}z_m\bar{z}_m z_n\bar{z}_n(1-z\bar{z})^{m-3}\Omega. \tag{4-48}$$

Case I-1b: $M\cup N=\{m\}$ for $1\le m\le t$. Then the sum of corresponding terms is

$$\sum T_{L,M,N,P,Q}=2(m-1-t)!\sum_{m=1}^{t}z_m\bar{z}_m(1-z\bar{z})^{m-2}\Omega. \tag{4-49}$$

Case I-1c: $M=N=\varnothing$. The corresponding term is

$$T_{L,M,N,P,Q}=T_{\{1,\dots,t\},\varnothing,\varnothing,\varnothing,\{t+1,\dots,m-1\}}=(m-1-t)!(1-z\bar{z})^{m-1}\Omega. \tag{4-50}$$

<u>Case I-2:</u> $|P| = 1$. Then $M = N = \varnothing$. Suppose $P = \{p\}$ for $1 \le p \le t$. Then $Q = \{p, t+1, \ldots, m-1\} - \{q\}$ for some q inside. The sum of the corresponding terms is

$$\sum T_{L,M,N,P,Q} = (m-1-t)! \sum_{p=1}^{t} w_p \bar{w}_p \Big(z_p \bar{z}_p + \sum_{q=t+1}^{m-1} z_q \bar{z}_q \Big)(1-z\bar{z})^{m-2}\Omega. \quad (4\text{-}51)$$

<u>Case II:</u> $|Q| = m-2-t$. Then $M = N = P = \varnothing$ and $|Q| = \{t+1, \ldots, m-1\} - \{q\}$ for some q inside. The sum of the corresponding terms is

$$\sum T_{L,M,N,P,Q} = (m-1-t)! \sum_{q=t+1}^{m-1} z_q \bar{z}_q (1-z\bar{z})^{m-2}\Omega. \quad (4\text{-}52)$$

Taking the sum from (4-48) to (4-52), we have

$$(4\text{-}47) = (m-1-t)!(1-z\bar{z})^{m-2}\Omega \Bigg\{ \sum_{p=1}^{t} w_p \bar{w}_p \Big(z_p \bar{z}_p + \sum_{q=t+1}^{m-1} z_q \bar{z}_q \Big)$$

$$+ \frac{\sum_{\substack{m,n=1 \\ m \ne n}}^{t} z_m \bar{z}_m z_n \bar{z}_n}{1-z\bar{z}} + 2\sum_{m=1}^{t} z_m \bar{z}_m + \sum_{q=t+1}^{m-1} z_q \bar{z}_q + (1-z\bar{z}) \Bigg\}$$

$$= (m-1-t)!(1-z\bar{z})^{m-2}\Omega \Bigg\{ 1 + \sum_{m=1}^{t} z_m \bar{z}_m$$

$$+ \frac{\sum_{\substack{m,n=1 \\ m \ne n}}^{t} z_m \bar{z}_m z_n \bar{z}_n + \sum_{p=1}^{t} z_p \bar{z}_p \big(z_p \bar{z}_p + \sum_{q=t+1}^{m-1} z_q \bar{z}_q \big)}{1-z\bar{z}} \Bigg\}$$

$$= (m-1-t)!(1-z\bar{z})^{m-2}\Omega \frac{\sum_{m=1}^{t} z_m \bar{z}_m}{1-z\bar{z}}$$

$$= (m-1-t)!(1 + \sum_{j=1}^{t} w_j \bar{w}_j)(1-z\bar{z})^{m-2}\Omega.$$

This finishes the proof of the lemma.   □

Remark 4.19. W. Zhang has proved Theorem 4.17 independently (unpublished) using a similar method, assuming invariance under $\mathsf{U}(2)$ (Proposition 4.11).

Comparison on the Shimura variety. Now we use the previous results to compute the archimedean local height pairing on the unitary Shimura varieties with respect to suitable Green's currents. Let $n \ge 1$ be a positive integer. We recall the notation for groups in Section 2B and the notation for Shimura varieties in Section 3A with $m = 2n$ and $r = n$. We let M_K be the variety $\mathsf{Sh}_K(\mathbb{H})$ for simplicity. For decomposable $\phi_i = \phi_\infty^0 \phi_{i,f}$ with the Gaussian ϕ_∞^0 at infinite places and $\phi_{i,f} \in$

$\mathscr{S}(\mathbb{V}_f^n)^K$ $(i = 1, 2)$ such that $\phi_{1,v} \otimes \phi_{2,v} \in \mathscr{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ for some $v \in \Sigma_f$, the generating series $Z_{\phi_1}(g_1)$ and $Z_{\phi_2}(g_2)$, defined by the Weil representation $\omega_{\chi,\psi}$ with standard ψ_∞, will not intersect on M_K providing $g_i \in P_v' H'(\mathbb{A}_F^v)$. For any infinite place $\iota^\circ \in \Sigma_\infty^\circ$ over $\iota \in \Sigma_\infty$, we will attach a Green's current $\Xi_{\phi_i}(g_i)_{\iota^\circ}$ to $Z_{\phi_i}(g_i) \in \mathrm{CH}^n(M_K)$ and consider the local height pairing

$$\mathrm{vol}(K)\langle (Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota^\circ}), (Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota^\circ}) \rangle_{M_K}.$$

Our main theorem is the following:

Theorem 4.20. *Let ϕ_i, g_i $(i = 1, 2)$ and ι°, ι be as above. Then there is a unique Haar measure on $\mathbb{H}(\mathbb{A}_f)$ which only depends on ψ_f such that*

$$E_\iota(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) = -2\mathrm{vol}(K)\langle (Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota^\circ}), (Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota^\circ}) \rangle_{M_K}$$

where E_ι is given in (2-10) and $\mathrm{vol}(K)$ is the volume of K determined by this measure.

Proof. We can assume that $K = \prod K_v$ is decomposable and sufficiently small. To do this, we consider the uniformization of M_K at ι° and suppress the superscript (ι) for the nearby objects such as H, $\mathscr{D}$, and V. We have

$$(M_K)_{\iota^\circ}^{\mathrm{an}} \cong H(\mathbb{Q})\backslash(\mathscr{D} \times H(\mathbb{A}_f)/K) = \coprod_{\{h\}} M_{K,\{h\}},$$

and $M_{K,\{h\}} = \Gamma_{\{h\}}\backslash\mathscr{D}$ is a geometric connected component with $\Gamma_{\{h\}} := H(\mathbb{Q}) \cap hKh^{-1}$ viewed as a subgroup of $H(\mathbb{Q})$, where $\{h\}$ goes through the double cosets $H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K$.

By our assumption on ϕ_i and g_i, we may write the generating series in the following way:

$$Z_{\phi_i}(g_i) = \sum_{T_i \in \mathrm{Her}_n^+(E)} \sum_{h_i \in H_{x_{T_i}}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(g_i)\phi_i(T_i, h_i^{-1}x_{T_i})Z(h_i^{-1}x_{T_i})_K$$

for $i = 1, 2$, where $\mathrm{Her}_n^+(E)$ is the set of totally positive-definite hermitian matrices in $\mathrm{Her}_n(E)$ and $x_{T_i} \in V^n$ is any element (if it exists) such that $T(x_{T_i}) = T_i$ since $H(\mathbb{Q})$ acts transitively on $\Omega_T := \{x \in V^n \mid T(x) = T\}$ for $T \in \mathrm{Her}_n^+(E)$. By definition,

$$Z(h_i^{-1}x_{T_i})_K = \sum_h Z_{x_{T_i},h},$$

where $Z_{x_{T_i},h}$ is a cycle in $\mathrm{CH}^n(M_{K,\{h\}})$ represented by the points (z, h) with $z \in D_{x_{T_i}}$ and the sum takes over a set of representatives h in the double coset

$$H_{x_{T_i}}(\mathbb{Q})\backslash H_{x_{T_i}}(\mathbb{A}_f)h_i K/K.$$

Then we have

$$Z_{\phi_i}(g_i) = \sum_{T_i \in \mathsf{Her}_n^+(E)} \sum_{h_i \in H_{x_{T_i}}(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K} \omega_\chi(g_i)\phi_i(T_i, h_i^{-1}x_{T_i})Z_{x_{T_i}, h_i}.$$

Writing $g_{i,\iota} = n(b_i)m(a_i)[k_{i,1}, k_{i,2}]$ for the Iwasawa decomposition as in Section 4A, let $\Xi_{x_i a_i, h_i}$ be the $(n-1, n-1)$ Green's current of $D_{x_i a_i}$ on the hermitian symmetric domain $(\mathcal{D}, h) \subset \mathcal{D} \times H(\mathbb{A}_f)/K$. We define a current

$$\Xi_{\phi_i}(g_i)_{\iota^\circ} = \sum_{x_i} \sum_{h_i \in H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K} \omega_\chi(g_i)\phi_i(T(x_i), h_i^{-1}x_i)\Xi_{x_i a_i, h_i}$$

on $\mathcal{D} \times H(\mathbb{A}_f)/K$, where x_i is taken over all elements in V^n whose components are linearly independent. It It projects to a current on $H(\mathbb{Q}) \backslash \mathcal{D} \times H(\mathbb{A}_f)/K$, which is a Green's current for $Z_{\phi_i}(g_i)$. Then we have

$$\langle (Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota^\circ}), (Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota^\circ}) \rangle_{M_K}$$

$$= \sum_{x_1, x_2} \sum_{h \in H(\mathbb{Q}) \backslash H(\mathbb{A}_f)/K} \omega_\chi''(\iota(g_1, g_2^\vee))\phi_1 \otimes \phi_2\big(T(x_1) \oplus T(x_2), (h^{-1}x_1, h^{-1}x_2)\big)$$

$$\times \int_{H(\mathbb{Q}) \backslash (\mathcal{D} \times H(\mathbb{A}_f)/K)} \Xi_{x_1 a_1, h} * \Xi_{x_2 a_2, h}$$

$$= \sum_T \sum_{h \in H(\mathbb{A}_f)/K} \omega_\chi''(\iota(g_1, g_2^\vee))\phi_1 \otimes \phi_2(T, h^{-1}x_T)H(a^*Ta)_\infty$$

$$= \sum_T H(a^*Ta)_\infty \prod_{v \in \Sigma_\infty} \omega_{\chi_v}''(\iota(g_{1,v}, g_{2,v}^\vee))\Phi_v^0(T)$$

$$\times \prod_{v \in \Sigma_f} \sum_{h_v \in H_v/K_v} \omega_{\chi_v}''(\iota(g_{1,v}, g_{2,v}^\vee))\phi_{1,v} \otimes \phi_{2,v}(h_v^{-1}x_T), \quad \text{(4-53)}$$

where the sum is taken over all nonsingular $T \in \mathsf{Her}_{2n}(E)$ that are moment matrices of some $x_T \in V^{2n}$ and $a = a_1 \oplus a_2 \in \mathsf{GL}_{2n}(\mathbb{C})$. We compute for each v.

Case I: $v = \iota$, by (4-3) and Theorem 4.17 for the coefficient a^*Ta, we have

$$H(a^*Ta)_\infty \omega_{\chi_\iota}''(\iota(g_{1,\iota}, g_{2,\iota}^\vee))\Phi_\iota^0(T) = \gamma_{\mathbb{V}_\iota}^{-1} \frac{\Gamma_{2n}(2n)}{(2\pi)^{4n^2}} W_T'(0, \iota(g_{1,\iota}, g_{2,\iota}^\vee), \Phi_\iota^0).$$

Recalling the local Tate factors (2-4), we have

$$H(a^*Ta)_\infty \omega_{\chi_\iota}''(\iota(g_{1,\iota}, g_{2,\iota}^\vee))\Phi_\iota^0(T) = \gamma_{\mathbb{V}_\iota}^{-1} b_{2n,\iota}(0) W_T'(0, \iota(g_{1,\iota}, g_{2,\iota}^\vee), \Phi_\iota^0). \quad \text{(4-54)}$$

Case II: $v \in \Sigma_\infty$, $v \neq \iota$. By (4-3) and Proposition 4.5(2), we have

$$\omega_{\chi_v}''(\iota(g_{1,v}, g_{2,v}^\vee))\Phi_v^0(T) = \gamma_{\mathbb{V}_v}^{-1} b_{2n,v}(0) W_T(0, \iota(g_{1,v}, g_{2,v}^\vee), \Phi_v^0). \quad \text{(4-55)}$$

Case III: $v \in \Sigma_f$. Recalling the set Ω_T defined in Section 2E, it is easy to see that $\Omega_T \neq \varnothing$ is a single orbit of the left translation by H_v whose stabilizer at

any point is trivial. Hence any Haar measure $d'h_v$ on H_v induces a measure $d'x$ on Ω_T. We have

$$\sum_{h_v \in H_v/K_v} \omega''_{\chi_v}(\iota(g_{1,v}, g^{\vee}_{2,v}))\phi_{1,v} \otimes \phi_{2,v}(h_v^{-1}x_T)$$

$$= \text{vol}'(K_v)^{-1} \int_{H_v} \omega''_{\chi_v}(\iota(g_{1,v}, g^{\vee}_{2,v}))\phi_{1,v} \otimes \phi_{2,v}(h_v^{-1}x_T) \, d'h_v,$$

where $\text{vol}'(K_v)$ is the volume of K_v under the measure $d'h_v$. By [Rallis 1987, Lemma 4.2], we can choose a unique measure $d'h_v$ such that

$$W_T(0, \iota(g_{1,v}, g^{\vee}_{2,v}), \phi_{1,v} \otimes \phi_{2,v}) = W_T\left(0, e, \omega''_{\chi_v}(\iota(g_{1,v}, g^{\vee}_{2,v}))\phi_{1,v} \otimes \phi_{2,v}\right)$$

$$= \gamma_{\mathbb{V}_v} b_{2n,v}(0)^{-1} \int_{H_v} \omega''_{\chi_v}(\iota(g_{1,v}, g^{\vee}_{2,v}))\phi_{1,v} \otimes \phi_{2,v}(h_v^{-1}x_T) \, d'h_v. \quad (4\text{-}56)$$

By Lemma 2.9, for almost all v, we have $\text{vol}'(K_v) = 1$.

Now taking the product of (4-54), (4-55), and (4-56), we have

$$(4\text{-}53) = -b_{2n}(0)\text{vol}'(K)^{-1}E_{\iota}(0, \iota(g_1, g^{\vee}_2), \phi_1 \otimes \phi_2).$$

Now we take the modified measure $(2b_{2n}(0))^{-1} \prod_{v \in \Sigma_f} d'h_v$ under which we have the desired identity in Theorem 4.20. $\qquad\square$

Appendix: Theta correspondence of spherical representations

In this appendix, we consider the theta correspondence of spherical representations for unitary groups since we cannot find literature in this case. We follow [Rallis 1984] where the symplectic-orthogonal case was discussed.

Let $F/\mathbb{Q}_p$ be a finite field extension with $p \neq 2$ and E/F an unramified quadratic field extension with $\text{Gal}(E/F) = \{1, \tau\}$. Let $\mathbb{O}_F$ (resp. $\mathbb{O}_E$) be the ring of integers of F (resp. E), ϖ a uniformizer of $\mathbb{O}_F$, and q the cardinality of $\mathbb{O}_F/\varpi\mathbb{O}_F$. Let ψ be the unramified additive character of F which determines an additive character of E by composing with $\frac{1}{2}\text{Tr}_{E/F}$. Let dx be the Haar measure of E which is self-dual with respect to $\psi \circ \frac{1}{2}\text{Tr}_{E/F}$ and $d^{\times}x = dx/|x|_E$ the Haar measure of $E^{\times}$, normalized such that $|\varpi|_E = q^{-2}$.

Let $n, m \geq 1$ be two integers and $r = \min\{m, n\}$. Let $(W_n, \langle \cdot, \cdot \rangle)$ be a skew hermitian space over E whose skew hermitian form is given by

$$\begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}$$

under the basis $\{e_1, \ldots, e_n; e_1^*, \ldots, e_n^*\}$ and $(V_m, (\,\cdot\,,\,\cdot\,))$ a hermitian space over E whose hermitian form is given by

$$\begin{pmatrix} & \mathbf{1}_m \\ \mathbf{1}_m & \end{pmatrix}$$

under the basis $\{f_1, \ldots, f_m; f_1^*, \ldots, f_m^*\}$. Let $H_n' = \mathsf{U}(W_n)$, $H_m = \mathsf{U}(V_m)$ and $K_n' = \mathsf{U}(W_n) \cap \mathsf{GL}_{2n}(\mathbb{O}_E)$, $K_m = \mathsf{U}(V_m) \cap \mathsf{GL}_{2m}(\mathbb{O}_E)$ be hyperspecial maximal compact subgroups. We have a Weil representation $\omega = \omega_{\mathbf{1},\psi}$ of $H_n' \times H_m$ on the space of Schwartz functions $\mathscr{S}(V_m^n)$ defined as

- $\omega\left(\begin{pmatrix} A & \\ & {}^t A^{\tau,-1} \end{pmatrix}\right)\phi(x) = |\det A|_E^m \phi(xA),$

- $\omega\left(\begin{pmatrix} \mathbf{1}_n & B \\ & \mathbf{1}_n \end{pmatrix}\right)\phi(x) = \psi(\operatorname{tr} BT(x))\phi(x),$

- $\omega\left(\begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}\right)\phi(x) = \widehat{\phi}(x),$

- $\omega(h)\phi(x) = \phi(h^{-1}x),$

where $\phi \in \mathscr{S}(V_m^n)$, $A \in \mathsf{GL}_n(E)$, $B \in \operatorname{Her}_n(E)$, $h \in H_m$, and $\widehat{\phi}$ is the Fourier transform with respect to $\psi \circ \frac{1}{2}\operatorname{Tr}_{E/F}$ and dx.

Let $W_{n,i}^* = \operatorname{span}_E\{e_{i+1}^*, \ldots, e_n^*\}$ for $0 \le i \le n$ and $V_{m,j}^* = \operatorname{span}_E\{f_{j+1}^*, \ldots, f_m^*\}$ for $0 \le j \le m$. Then we have filtration of the maximal isotropic subspaces $W_{n,0}^*$ and $V_{m,0}^*$:

$$W_{n,0}^* \supset W_{n,1}^* \supset \cdots \supset W_{n,n}^* = \{0\}, \quad V_{m,0}^* \supset V_{m,1}^* \supset \cdots \supset V_{m,m}^* = \{0\}.$$

Then, up to conjugacy, the maximal parabolic subgroups of $H_n' \times H_m$ are precisely those subgroups $P_{n,i}' \times P_{m,j}$ consisting of elements (h', h) stabilizing the subspace $W_{n,i}^* \otimes V_{m,j}^*$. Let $N_{n,i}' \times N_{m,j}$ be its unipotent radical. Also the Levi factor of $P_{n,i}' \times P_{m,j}$ is isomorphic to $(\mathsf{GL}_{n-i}(E) \times H_i') \times (\mathsf{GL}_{m-j}(E) \times H_j)$. We also define the algebraic closed subsets Σ_t of V_m^n for $0 \le t \le n$ to be

$$\Sigma_t = \{x = (x_1, \ldots, x_n) \in V_m^n \mid (x_i, x_j) = 0 \text{ for } t+1 \le j \le n\}.$$

We say that a function $\phi \in \mathscr{S}(V_m^n)$ is *spherical* if it is invariant under the action of $K_n' \times K_m$. Then we have

Lemma A.1. *Let ϕ be a spherical function in $\mathscr{S}(V_m^n)$ such that for any $h' \in H_n'$, $\omega(h')\phi$ vanishes on the subset Σ_0, then $\omega(h')\phi$ vanishes identically.*

Proof. The proof follows exactly that of [Rallis 1984, Proposition 2.2]. □

Now we identify V_m^n with $\operatorname{Mat}_{2m \times n}(E)$ via the basis $\{f_1, \ldots, f_m; f_1^*, \ldots, f_m^*\}$. Then as a $\mathsf{GL}_n(E) \times H_m$-module, the action is given by $(A, h).X = hXA^{-1}$. We have the following version of [Rallis 1984, Lemma 3.1]:

Lemma A.2. *Let* $\Sigma_0^{(i)} = \{X \in \Sigma_0 \mid \text{rank}(X) = i\}$. *Then* $\Sigma_0^{(i)}$ *(if nonempty) is an orbit under* $\text{GL}_n(E) \times H_m$ *and* Σ_0 *is a disjoint union of orbits of the form* $\Sigma_0^{(i)}$ *for* $i = 0, 1, \ldots, r$ *where* $\Sigma_0^{(r)}$ *is the unique open one.*

Let us review some facts about spherical representations of $H'_n \times H_m$. Consider the minimal parabolic subgroup $B'_n \times B_{m,r}$ of $H'_n \times H_m$ defined as follows. Let

$$B'_n = \left\{ \begin{pmatrix} A & \\ & {}^t A^{\tau,-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & B \\ & \mathbf{1}_n \end{pmatrix} \;\middle|\; A \text{ is lower triangular and } B \text{ is hermitian} \right\},$$

which has a decomposition $B'_n = T'_n \cdot U'_n$, with

$$T'_n = \{\text{diag}[t_1, \ldots, t_n, t_1^{\tau,-1}, \ldots, t_n^{\tau,-1}] \mid t_i \in E^\times\}$$

and U'_n the unipotent radical of B'_n. Let

$$B_{m,r} = \left\{ \begin{pmatrix} A & \\ & {}^t A^{\tau,-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_m & B \\ & \mathbf{1}_m \end{pmatrix} \;\middle|\; A = \begin{pmatrix} A_1 & A_2 \\ & A_3 \end{pmatrix} \right\},$$

where $A_1 \in \text{Mat}_{r \times r}(E)$ is lower triangular, $A_3 \in \text{Mat}_{(m-r) \times (m-r)}(E)$ is upper triangular, $A_2 \in \text{Mat}_{r \times (m-r)}(E)$, and B is skew-hermitian. We have a decomposition $B_{m,r} = T_m \cdot U_{m,r}$ where $T_m = T'_m$ and $U_{m,r}$ is the unipotent radical of $B_{m,r}$.

For $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{C}^n$, we define the space $I'(\nu)$ consisting of all locally constant functions $\varphi : H'_n \to \mathbb{C}$ satisfying

$$\varphi(h' \cdot t' \cdot u') = \delta'^{-1/2}_n(t') \prod_{i=1}^n |t'_i|_E^{\nu_i} \varphi(h')$$

for all $h' \in H'_n$, $t \in T'_n$ and $u' \in U'_n$ where δ'_n is the modulus function of B'_n. We have $\delta'_n(t') = \prod_{i=1}^n |t'_i|_E^{2i-1}$. These $I'(\nu)$ give all spherical principal series of H'_n. Let $\mathscr{S}(H'_n /\!/ K'_n)$ be the spherical Hecke algebra of H'_n. Then we have the Fourier–Satake isomorphism $\text{FS} : \mathscr{S}(H'_n /\!/ K'_n) \to \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(H'_n)}$ such that for any $f' \in \mathscr{S}(H'_n /\!/ K'_n)$,

$$\text{FS}(f')(q^{2\nu_1}, q^{-2\nu_1}, \ldots, q^{2\nu_n}, q^{-2\nu_n}) = \text{trace}_{I'(\nu)}(f').$$

For $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{C}^m$, we define the space $I(\mu)$ consisting of all locally constant functions $\varphi : H_m \to \mathbb{C}$ satisfying

$$\varphi(h \cdot t \cdot u) = \delta_{m,r}^{-1/2}(t) \prod_{j=1}^m |t_j|_E^{\mu_j} \varphi(h)$$

for all $h \in H_m$, $t \in T_m$, and $u \in U_{m,r}$, where $\delta_{m,r}$ is the modulus function of $B_{m,r}$. We have $\delta_{m,r}(t) = \prod_{j=1}^r |t_j|_E^{2m-2r+2j-1} \prod_{j=r+1}^m |t_j|_E^{2m-2j+1}$. These $I(\mu)$ give all spherical principal series of H_m. Let $\mathscr{S}(H_m /\!/ K_m)$ be the spherical Hecke

algebra of H_m. Then we have the Fourier–Satake isomorphism $\mathrm{FS} : \mathscr{S}(H_m // K_m) \to \mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(H_m)}$ such that for any $f \in \mathscr{S}(H_m // K_m)$,

$$\mathrm{FS}(f)(q^{2\mu_1}, q^{-2\mu_1}, \ldots, q^{2\mu_m}, q^{-2\mu_m}) = \mathrm{trace}_{I(\mu)}(f).$$

Now we are going to construct a certain explicit intertwining operator from $\mathscr{S}(V_m^n)$ to $I'(\nu) \otimes I(\mu)$. To do this, we introduce the subgroup $Y_r = A_r \cdot L_r$ of $\mathrm{GL}_r(E)$, where

$$A_r = \{\mathrm{diag}[a_1, \ldots, a_r] \mid a_i \in E^\times\}, \quad L_r = \left\{ \begin{pmatrix} 1 & & \\ & \ddots & \\ l_{ij} & & 1 \end{pmatrix} \,\middle|\, l_{ij} \in E \right\}.$$

It has a right invariant measure given by

$$dy_r = \prod_{i=1}^{r} |a_i|_E^{2i-(r+1)} d^\times a_i \prod_{1 \leq j < i \leq r} \tilde{d} l_{i,j},$$

where $\tilde{d} l_{i,j}$ is a certain measure on L_r normalized as in [Rallis 1984, p. 490]. For $\sigma = (\sigma_1, \ldots, \sigma_r) \in \mathbb{C}^r$ such that $\Re(\sigma_i) \gg 0$, the integral

$$Z_\sigma(\phi) = \int_{Y_r} \phi\left(\begin{pmatrix} y_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right) \prod_{i=1}^{r} |a_i|_E^\sigma \, dy_r$$

is absolutely convergent. We define a functional $\mathscr{Z}_\sigma$ sending ϕ to the function $(h', h) \mapsto Z_\sigma(\omega(h'^{-1}, h^{-1})\phi)$. It is a nonzero $H_n' \times H_m$-intertwining map from $\mathscr{S}(V_m^n)$ to $\mathscr{S}(H_n' \times H_m)$; moreover:

Lemma A.3. *For $\Re(\sigma_i) \gg 0$, the image of the above intertwining map $\mathscr{Z}_\sigma$ lies in $I'(\nu) \otimes I(\mu)$ where*

$$\nu = \left(2 + \sigma_1 - m - \tfrac{3}{2}, \ldots, 2r + \sigma_r - m - \tfrac{3}{2}, (r+1) - m - \tfrac{1}{2}, \ldots, n - m - \tfrac{1}{2} \right),$$
$$\mu = \left(-2 - \sigma_1 + m + \tfrac{3}{2}, \ldots, -2r + \sigma_r + m + \tfrac{3}{2}, -(r+1) + m + \tfrac{1}{2}, \ldots, \tfrac{1}{2} \right).$$

Proof. We have

$$\mathscr{Z}_\sigma(\phi)(h't'u', htu) = \int_{Y_r} \omega(u'^{-1}t'^{-1}h'^{-1}, u^{-1}t^{-1}h^{-1})\phi\left(\begin{pmatrix} y_r & \\ & \end{pmatrix} \right) \prod_{i=1}^{r} |a_i|_E^\sigma \, dy_r$$

$$= \int_{Y_r} \omega(t'^{-1}h'^{-1}, t^{-1}h^{-1})\phi\left(\begin{pmatrix} y_r & \\ & \end{pmatrix} \right) \prod_{i=1}^{r} |a_i|_E^\sigma \, dy_r$$

$$= \int_{Y_r} |\det t'|_E^{-m} \omega(h'^{-1}, h^{-1})\phi\left(t\begin{pmatrix} y_r & \\ & \end{pmatrix} t'^{-1} \right) \prod_{i=1}^{r} |a_i|_E^\sigma dy_r. \quad \text{(A.1)}$$

Changing the variable $y_r \mapsto y_r'' = \mathrm{diag}[t_1, \ldots, t_r]y_r$, we have

$$\prod_{i=1}^{r}|a_i|_E^\sigma \, dy_r = \prod_{i=1}^{r}|t_i|_E^{r-3i-\sigma_i+2}\prod_{i=1}^{r}|t_i a_i|_E^\sigma \, dy_r''.$$

Changing the variable $y_r \mapsto y_r' = \mathrm{diag}[t_1'^{-1}, \ldots, t_r'^{-1}]$, then

$$\prod_{i=1}^{r}|a_i|_E^\sigma \, dy_r = \prod_{i=1}^{r}|t_i|_E^{i+\sigma_i-1}\prod_{i=1}^{r}|t_i'^{-1} a_i|_E^\sigma \, dy_r'.$$

Hence

$$(\mathrm{A.1}) = \prod_{i=1}^{r}|t_i'|_E^{i+\sigma_i-m-1}\prod_{i=r+1}^{n}|t_i'|_E^{-m}\prod_{j=1}^{r}|t_j|_E^{r-3j-\sigma_j+2}$$

$$\times \int_{Y_r}\omega(h'^{-1},h^{-1})\phi\left(\begin{pmatrix} & y_r \\ & \end{pmatrix}\right)\prod_{i=1}^{r}|a_i|_E^\sigma \, dy_r$$

$$= \prod_{i=1}^{r}|t_i'|_E^{i+\sigma_i-m-1}\prod_{i=r+1}^{n}|t_i'|_E^{-m}\prod_{j=1}^{r}|t_j|_E^{r-3j-\sigma_j+2}\mathscr{L}_\sigma(\phi)(h',h). \qquad \square$$

From this lemma, it is easy to see the following. If $m \ge n = r$, there is a surjective homomorphism $\Phi_{m,n} : \mathscr{S}(H_m /\!\!/ K_m) \to \mathscr{S}(H_n' /\!\!/ K_n')$ which has the property

$$\mathscr{L}_\sigma \circ (\Phi_{m,n}(f) - f) = 0$$

for all $f \in \mathscr{S}(H_m /\!\!/ K_m)$ and $\Re(\sigma_i) \gg 0$. Using the Fourier–Satake isomorphism, the map $\Phi_{m,n}$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(H_m)} \longrightarrow \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(H_n')},$$

where

$$\begin{aligned} \log_q X_j &\mapsto \log_q X_j, & j &= 1, \ldots, n, \\ \log_q X_j &\mapsto 2m - 2j + 1, & j &= n+1, \ldots, m. \end{aligned}$$

In particular, when $m = n$, $\Phi_{m,n}$ is the identity map.

If $n > m = r$, similarly there is a surjective homomorphism $\Phi_{n,m}' : \mathscr{S}(H_n' /\!\!/ K_n') \to \mathscr{S}(H_m /\!\!/ K_m)$ which has the property

$$\mathscr{L}_\sigma \circ (f' - \Phi_{n,m}'(f')) = 0$$

for all $f' \in \mathscr{S}(H_n' /\!\!/ K_n')$ and $\Re(\sigma_i) \gg 0$. Using the Fourier–Satake isomorphism, the map $\Phi_{n,m}'$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(H_n')} \longrightarrow \mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(H_m)},$$

where

$$\begin{aligned}
\log_q X_i &\mapsto \log_q X_i, & i &= 1, \ldots, m, \\
\log_q X_i &\mapsto 2m - 2i + 1, & i &= m + 1, \ldots, n.
\end{aligned}$$

In fact, we have

Lemma A.4. *Let $\phi \in \mathcal{S}(V_m^n)$ be spherical and $Z_\sigma(\phi) \equiv 0$ for all $\sigma \in \mathbb{C}^r$ with $\Re(\sigma_i) \gg 0$. Then $\omega(h')\phi$ vanishes on Σ_0 for all $h' \in H_n'$.*

Proof. It suffices to show that $\omega(h')\phi$ vanishes on $\Sigma_0^{(r)}$ since it is dense open in Σ_0. Since $\Sigma_0^{(r)}$ is a $\mathsf{GL}_n(E) \times H_m$-orbit, we only need to show that

$$\omega(h', h)\phi\left(\begin{pmatrix} \mathbf{1}_r \\ & \end{pmatrix}\right) \equiv 0$$

for all (h', h). But we can write $h' = b'k'$ with $b' \in B_n'$, $k' \in K_n'$, and $h = bk$ with $b \in B_{m,r}'$ and $k \in K_m$. Then since ϕ is spherical, we have

$$\omega(h', h)\phi\left(\begin{pmatrix} \mathbf{1}_r \\ & \end{pmatrix}\right) = \omega(b', b)\phi\left(\begin{pmatrix} \mathbf{1}_r \\ & \end{pmatrix}\right) = \phi\left(\begin{pmatrix} X \\ & \end{pmatrix}\right)$$

with $X \in \mathrm{Mat}_{r \times r}(E)$. Hence the lemma follows from [Rallis 1984, Lemma 5.2] for $k = E$. $\qquad\square$

Combining Lemmas A.1, A.3, and A.4, we have

Proposition A.5. *The ideal*

$$\mathcal{I}_{n,m} = \{f \in \mathcal{S}(H_n'/\!/K_n') \otimes \mathcal{S}(H_m/\!/K_m) \mid \omega(f) \equiv 0\}$$

is generated by

$$\{\Phi_{m,n}(f) - f \mid f \in \mathcal{S}(H_m/\!/K_m)\} \quad (\textit{resp. } \{f' - \Phi_{n,m}'(f') \mid f' \in \mathcal{S}(H_n'/\!/K_n')\})$$

if $m \geq n$ (resp. $m < n$).

We have a similar result for the Weil representation of $\mathsf{GL}_n(F) \times \mathsf{GL}_m(F)$ on $\mathcal{S}(\mathrm{Mat}_{m \times n}(F))$ given by $\omega(g', g)\phi(x) = \phi(g^{-1}xg')$ (see [Rallis 1984, Section 6]). Without lost of generality, we assume that $n \geq m$; then the ideal

$$\mathcal{I}_{n,m} = \{f \in \mathcal{S}(\mathsf{GL}_n(F)/\!/\mathsf{GL}_n(\mathbb{O}_F)) \otimes \mathcal{S}(\mathsf{GL}_m(F)/\!/\mathsf{GL}_m(\mathbb{O}_F)) \mid \omega(f) \equiv 0\}$$

is generated by

$$\{f - \Psi_{n,m}(f) \mid f \in \mathcal{S}(\mathsf{GL}_n(F)/\!/\mathsf{GL}_n(\mathbb{O}_F))\}.$$

In terms of the Fourier–Satake isomorphism, the surjective homomorphism $\Psi_{n,m}$ is given by

$$\mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]^{W(\mathsf{GL}_n(F))} \longrightarrow \mathbb{C}[X_1, X_1^{-1}, \ldots, X_m, X_m^{-1}]^{W(\mathsf{GL}_m(F))},$$

where

$$\log_q X_i \mapsto -\log_q X_i + \frac{n-m}{2}, \qquad i = 1, \ldots, m,$$

$$\log_q X_i \mapsto -i + \frac{n+1}{2}, \qquad i = m+1, \ldots, n.$$

Corollary A.6. (1) *If π is an unramified irreducible admissible representation of H'_n, then the theta correspondence of π to $H_n \cong H'_n$ is nontrivial and isomorphic to π, that is, $\theta_1(\pi, V_n) = \pi$.*

(2) *If π is an unramified irreducible admissible representation of $\mathsf{GL}_n(F)$ and χ an unramified character of $F^\times$, then the theta correspondence of π to $\mathsf{GL}_n(F)$ through the Weil representation ω_χ, where $\omega_\chi(g', g)\phi(x) = \chi(\det g')\phi(g^{-1}xg')$ for $\phi \in \mathscr{S}(\mathsf{Mat}_{n \times n}(F))$, is nontrivial and isomorphic to $\pi^\vee \otimes \chi$.*

Acknowledgements

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