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We prove the arithmetic inner product formula conjectured in the first paper of this series for $n = 1$, that is, for the group $\mathrm{U}(1,1)_F$ unconditionally. The formula relates central L-derivatives of weight-2 holomorphic cuspidal automorphic representations of $\mathrm{U}(1,1)_F$ with ϵ-factor -1 with the Néron–Tate height pairing of special cycles on Shimura curves of unitary groups. In particular, we treat all kinds of ramification in a uniform way. This generalizes the arithmetic inner product formula obtained by Kudla, Rapoport, and Yang, which holds for certain cusp eigenforms of $\mathrm{PGL}(2)_{\mathbb{Q}}$ of square-free level.

1. Introduction

The Birch–Swinnerton-Dyer conjecture predicts a deep relation between rational points on rational elliptic curves and the associated analytic object called the Hasse–Weil zeta function or L-function. This conjecture has also been generalized to higher dimensions and to more general varieties and motives by Beilinson, Bloch and others. Gross and Zagier [1986] studied the relation between the central derivative of the L-function of a rational elliptic curve and the height pairing of Heegner points on it, through the arithmetic theory of modular curves and Rankin L-series. After elaborate computations, they obtained the famous Gross–Zagier formula, which is exactly predicted by the Birch–Swinnerton-Dyer conjecture. This was

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generalized by S.-W. Zhang [2001a; 2001b] to the case of Shimura curves over totally real fields. The complete version of the Gross–Zagier formula was recently achieved by X. Yuan, S.-W. Zhang and W. Zhang [Yuan et al. 2011]. Bruinier and Yang [2009] used regularized theta lifting and relate the inner product to L-derivatives to give another proof of the original Gross–Zagier formula. Certain p-adic (or rigid analytic) versions of the Gross–Zagier formula have been studied in [Bertolini and Darmon 1997; 1998].

Kudla [1997; 2002; 2003; Kudla et al. 2006] found another way to study L-derivatives, or more generally, derivatives of Siegel Eisenstein series. It was his great discovery that the theory of doubling integrals established in [Gelbart et al. 1987] can be used instead of the classical Rankin–Selberg convolution and that derivatives of (Siegel) Eisenstein series are also related to the height pairing of certain arithmetic objects. His project on the arithmetic Siegel–Weil formula sheds new light on this area. More importantly, the idea should work for higher dimensions and in both symplectic-orthogonal and unitary cases. Kudla, Rapoport, and Yang [Kudla et al. 2006] have also proved a special form of the arithmetic inner product formula for quaternion Shimura curves over $\mathbb{Q}$ of minimal level.

Extending that work, we set up in [Liu 2011] a general, explicit formulation of arithmetic theta lifting. Conjecture 3.11 of that paper gave an arithmetic inner product formula for unitary groups; we also proved the modularity theorem for the generating series (Theorem 3.5) and an archimedean arithmetic Siegel–Weil formula for any dimension (Theorems 4.17 and 4.20) predicted by this formulation. In this second paper, we prove the complete version of the arithmetic inner product formula for unitary groups of two variables over totally real fields.

The following is a detailed introduction. Let F be a totally real field, E/F a quadratic imaginary extension, τ the nontrivial Galois involution, $\epsilon_{E/F}$ the associated quadratic character by class field theory, and ψ an additive character of $F\backslash\mathbb{A}_F$, standard at archimedean places. For $n \geq 1$, let H_n be the unitary group over F such that for any F-algebra R, $H_n(R) = \{h \in \mathrm{GL}_{2n}(E \otimes_F R) \mid {}^t h^\tau w_n h = w_n\}$ where

$$w_n = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix}.$$

The center of H_n is the F-torus $E^{\times,1} = \ker[\mathrm{Nm}: E^\times \to F^\times]$. Let π be an irreducible cuspidal automorphic representation of H_n and $\pi^\vee$ its contragredient. Let χ be a character of $\mathbb{A}_E^\times$ which is trivial on $E^\times \mathbb{A}_F^\times$.

By the theta dichotomy proved in [Paul 1998; Gong and Grenié 2011], we get a factor $\epsilon(\pi, \chi)$ (see Section 2A for a precise definition) which is the product of local ones $\epsilon(\pi_v, \chi_v)$ for each place v of F, such that $\epsilon(\pi_v, \chi_v) \in \{\pm 1\}$ and $\epsilon(\pi_v, \chi_v) = 1$ for almost all v. Although it is conjectured that this $\epsilon(\pi_v, \chi_v)$ is related to the local ϵ-factor in representation theory (see [Harris et al. 1996]), it is

not the same, according to our definition. From these local factors, we can construct a hermitian space $\mathbb{V}(\pi, \chi)$ over $\mathbb{A}_E$ of rank $2n$ which is coherent (resp. incoherent) if $\epsilon(\pi, \chi) = 1$ (resp. -1). When $\epsilon(\pi, \chi) = 1$, we get the usual (generalized) Rallis inner product formula (see [Kudla and Rallis 1994; Ichino 2004; 2007], also [Liu 2011, Section 2] in our setting).

Now let us assume $n = 1$ and $\epsilon(\pi, \chi) = -1$. Then the central L-value $L(\frac{1}{2}, \pi, \chi)$ vanishes where the global L-function $L(s, \pi, \chi) = \prod_v (s, \pi_v, \chi_v)$ is the product of local ones, which are essentially defined as the common denominators of local zeta integrals by Piatetski-Shapiro and Rallis (see [Gelbart et al. 1987; Harris et al. 1996]; this will be recalled in Section 2A). It is natural to ask the value of $L'(\frac{1}{2}, \pi, \chi)$. For this purpose, we further assume that for any archimedean place ι of F, π_ι is a discrete series representation of weight 2 such that its central character $\omega_{\pi_\iota} = \chi_\iota^{-1}$. Then the corresponding $\mathbb{V}(\pi, \chi)$ is incoherent and totally positive definite of rank 2. Now for any hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ which is incoherent and totally positive definite of rank 2, let $\mathbb{H} = \mathrm{Res}_{\mathbb{A}_F/\mathbb{A}} \mathsf{U}(\mathbb{V})$ be the corresponding unitary group. Then we can construct a projective system of unitary Shimura curves $(\mathrm{Sh}_K(\mathbb{H}))_K$, smooth and quasiprojective over E, where K is a sufficiently small open compact subgroup of $\mathbb{H}(\mathbb{A}_f)$. These curves are nonproper if and only if $F = \mathbb{Q}$ and $\epsilon(\pi_v, \chi_v) = 1$ for all finite places v of F. In any case, we denote by $(M_K)_K$ the (compactified, if necessary) system of unitary Shimura curves for simplicity. For any $f \in \pi$ and Schwartz function $\phi \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty}$ (see Section 3B), we construct a cycle Θ^f_ϕ, called the arithmetic theta lifting, which is a divisor on M_K of degree 0 for any K fixing ϕ, through the Weil representation ω_χ. On the contragredient side, we also have $\Theta^{f^\vee}_{\phi^\vee}$ for $f^\vee \in \pi^\vee$ (but through $\omega^\vee_\chi$). We prove the following *arithmetic inner product formula for* $\mathsf{U}(1,1)_F$:

Theorem 1.1. *Let π, χ be as above and let $\mathbb{V}$ be any totally positive-definite incoherent hermitian space over $\mathbb{A}_E$ of rank 2. Then*

(1) *If $\mathbb{V} \ncong \mathbb{V}(\pi, \chi)$, then the arithmetic theta lifting Θ^f_ϕ is a torsion class for any $f \in \pi$ and $\phi \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty}$.*

(2) *If $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$, then for any $f \in \pi$, $f^\vee \in \pi^\vee$ and any $\phi, \phi^\vee \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty}$ decomposable, we have*

$$\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee} \rangle_{\mathrm{NT}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{L_F(2) L(1, \epsilon_{E/F})} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee),$$

where we take the Néron–Tate height pairing on some M_K (same as Beilinson–Bloch pairing on curves) such that ϕ and $\phi^\vee$ are invariant under K and we normalize it by a volume factor such that the resulting pairing is independent of the K we choose. The terms Z^* in the product are normalized local zeta integrals defined in Section 2A, of which almost all are 1.

We remark that the L-function $L(s, \pi, \chi)$ defined by Piatetski-Shapiro and Rallis (see [Harris et al. 1996] for a detailed definition for the unitary group case) coincides with $L(s, \mathrm{BC}(\pi) \otimes \chi)$ when $n = 1$; this is conjectured to be true for any n. In particular, the set of L-derivatives appearing in the above main theorem is exactly the same as in the Gross–Zagier formula in full generality, recently proved in [Yuan et al. 2011].

Our basic idea is similar to that of [Kudla 1997; Kudla et al. 2006]. The difference is that those works consider a certain integral model of the Shimura curve associated to a $\mathbb{Q}$-quaternion algebra and view the generating series and hence the arithmetic theta lifting as Arakelov divisors on that integral model. It has a canonical integral model in their minimal level case. But for general-level structures and even higher dimensional Shimura varieties, it is not all known. Instead, we work over canonical models of (unitary) Shimura varieties over reflex fields and define the generating series and the arithmetic theta lifting as usual Chow (co)cycles. In this way, we can formulate a precise version of the arithmetic inner product formula assuming that the Beilinson–Bloch height pairing, which is just the Néron–Tate pairing in the case of curves, is well-defined. At least in the case of $\mathrm{U}(1, 1)_F$, everything is well-defined.

For the proof, we use theories of Siegel Eisenstein series, Arakelov geometry, local heights, and p-divisible groups. The geometry part of this method actually goes back to [Gross and Zagier 1986]. Instead of explicit place-by-place computation (which is possible in the minimal level case) as in [Kudla et al. 2006], we greatly use the theory of theta lifting, certain multiplicity one results, modularity of the generating series, and various techniques for choosing test functions to avoid explicit computations at bad places which are almost impossible in the case of general levels. This allows us to prove the result for all kinds of ramification, from both representations and geometry, in a uniform way. This new idea was first proposed by Yuan, Zhang, and Zhang, and was used in their recent work on the general Gross-Zagier formula and the arithmetic triple product formula [Yuan et al. 2010; 2011].

The paper is organized as follows. In Section 2, we start by reviewing the method of doubling integrals, especially the integral presentation of L-functions and L-derivatives for unitary groups. In particular, we recall the analytic kernel function $E'(0, g, \Phi)$. Usually, it is extremely difficult to calculate its Fourier coefficients explicitly. But we prove later in the section that for a certain "nice" choice of test functions, we can kill all irregular Fourier coefficients and even arbitrary finitely many derivatives of regular ones. This nice choice is quite delicate and hence not easy to describe at this point. Finally, we have the following decomposition for nice Φ and g in a subgroup of $H_{2n}(\mathbb{A}_F)$ which is dense in $H_{2n}(F)\backslash H_{2n}(\mathbb{A}_F)$:

$$E'(0, g, \Phi) = \sum_{v \notin S} E_v(0, g, \Phi),$$

where S is a certain finite set of finite places of F which are "bad". The term $E_v(0, g, \Phi)$ is a sum of products of local Whittaker functions away from v and their derivatives at v; it is 0 if v is split in E.

In Section 3, we review the definition of Néron–Tate and Beilinson–Bloch height pairing on curves over number fields. Using this, we have a parallel construction of the kernel function for the height-pairing side when $n = 1$, namely the geometric kernel function $\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2)$ which is essentially the height pairing of two generating series $Z_{\phi_1}(g_1)$ and $Z_{\phi_2}(g_2)$. Thanks to the theorem of modularity of the generating series proved in [Liu 2011], it is not difficult to see that $\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2)$ is an automorphic form of $H_1 \times H_1$. Analogous to the analytic side, for nice choice of $\phi_1 \otimes \phi_2$, we have a decomposition for g_i inside a subgroup of $H_1(\mathbb{A}_F)$ which is dense in $H_1(F)\backslash H_1(\mathbb{A}_F)$:

$$\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) = -\mathrm{vol}(K) \sum_{v^\circ \in \Sigma^\circ} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{v^\circ} + \text{Eisenstein series and (possibly) automorphic characters,}$$

where v° takes over all places of E and the local height pairing is taken over a certain integral model of M_K. The terms of automorphic characters appear only in the case where the original Shimura curve is nonproper due to the nonvanishing of a certain intertwining operator.

Section 4 is dedicated to comparing the corresponding terms in two kernel functions for good finite places, namely the analytic side $E_v(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ and the geometric side $\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{v^\circ}$ with $v^\circ | v$.

Section 5 is dedicated to treating bad places appearing only on the geometric side. We prove that, for nicely chosen $\phi_1 \otimes \phi_2$, these (finitely many nonzero) height pairings are Eisenstein series and theta series.

We reach the final stage of the proof in Section 6. First, we introduce the notion of holomorphic projection and compute that for the analytic kernel function. By the comparison theorem at infinite places proved in [Liu 2011, Section 4], it turns out that after doing holomorphic projection, we will get the correct Green's function. Second, the difference between the (holomorphic projection of the) analytic kernel function and the geometric kernel function

$$\mathrm{Pr}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) - \mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2)$$

is now a linear combination of Eisenstein series, automorphic characters (that is, one-dimensional automorphic representations), and theta series for (g_1, g_2) inside a subgroup of $H_1(\mathbb{A}_F) \times H_1(\mathbb{A}_F)$ which is dense in $H_1(F)\backslash H_1(\mathbb{A}_F) \times H_1(F)\backslash H_1(\mathbb{A}_F)$.

But the key thing is that they are both automorphic forms; hence they really differ by a linear combination of Eisenstein series, automorphic characters, and theta series. Now we integrate automorphic forms $f \in \pi$ and $f^\vee \in \pi^\vee$ with this difference and get zero since π is cuspidal and $\epsilon(\pi, \chi) = -1$! This has already implied the arithmetic inner product formula but only for nicely chosen $\phi \otimes \phi^\vee$. To obtain the full formula, we need to use the multiplicity-one result proved in Section 6B. We introduce functionals

$$\alpha(f, f^\vee, \phi, \phi^\vee) := \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee),$$
$$\gamma(f, f^\vee, \phi, \phi^\vee) := \langle \Theta_\phi^f, \Theta_{\phi^\vee}^{f^\vee} \rangle_{\mathsf{NT}},$$

which are obviously inside $\mathsf{Hom}_{H_1(\mathbb{A}_F) \times H_1(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \pi^\vee \boxtimes \chi\pi)$, whose dimension is 1 when $\mathbb{V} = \mathbb{V}(\pi, \chi)$. Moreover, by [Harris et al. 1996] we know that as a functional, $\alpha \neq 0$. Hence γ is a constant multiple of α. To calculate this constant, we only need to plug in certain $f, f^\vee, \phi, \phi^\vee$ such that $\alpha(f, f^\vee, \phi, \phi^\vee) \neq 0$. By the density result proved in Section 2D, we can choose nice $\phi \otimes \phi^\vee$ and $f, f^\vee$ such that $\alpha(f, f^\vee, \phi, \phi^\vee) \neq 0$ where the constant has already been computed. As a consequence, we obtain the arithmetic inner product formula for any $f, f^\vee, \phi, \phi^\vee$.

The following conventions hold throughout this paper.

- $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = (\varprojlim_N \mathbb{Z}/N\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles, $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ is the ring of full adèles.
- For any number field K, $\mathbb{A}_K = \mathbb{A} \otimes_{\mathbb{Q}} K$, $\mathbb{A}_{f,K} = \mathbb{A}_f \otimes_{\mathbb{Q}} K$, $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R}$, and $\Gamma_K = \mathsf{Gal}(K^{\mathsf{ac}}/K)$ is the Galois group of K.
- As usual, for a subset S of places, $-_S$ (resp. $-^S$) means the S-component (resp. component away from S) for the corresponding (decomposable) adèlic object; $-_\infty$ (resp. $-_f$) is the infinite (resp. finite) part.
- The symbols Tr and Nm mean the trace (resp. reduced trace) and norm (resp. reduced norm) if they apply to fields or rings of adèles (resp. simple algebras), and tr means the trace for matrices and linear transformations.
- $\mathbf{1}_n$ and $\mathbf{0}_n$ are the $n \times n$ identity and zero matrices; ${}^t g$ is the transpose of a matrix g.
- All (skew-)hermitian spaces and quadratic spaces are assumed to be nondegenerate.
- For a ring R, sometimes R also stands for its spectrum $\mathsf{Spec}\, R$ or $\mathsf{Spf}\, R$ (if it causes no confusion) according to the context.
- For a scheme X over a field K, we let $\mathsf{Pic}(X)$ be the Picard group of X over K, not the Picard scheme.

2. Analytic kernel functions

2A. ***Doubling method.*** We briefly recall results mainly from [Gelbart et al. 1987; Li 1992; Harris et al. 1996] with the setups and notation of [Liu 2011, Section 2].

Let F be a totally real field and E a totally imaginary quadratic extension of F. We denote by τ the nontrivial element in $\mathrm{Gal}(E/F)$ and $\epsilon_{E/F} : \mathbb{A}_F^\times / F^\times \to \{\pm 1\}$ the associated character by class field theory. Let Σ (resp. Σ_f; resp. Σ_∞) be the set of all places (resp. finite places; resp. infinite places) of F, and Σ°, Σ_f°, and Σ_∞° those of E. We fix a nontrivial additive character ψ of $\mathbb{A}_F/F$.

For a positive integer r, we denote by W_r the standard skew-hermitian space over E with respect to the involution τ, which has a skew-hermitian form $\langle \cdot\,, \cdot \rangle$ such that there is an E-basis $\{e_1, \ldots, e_{2r}\}$ satisfying $\langle e_i, e_j \rangle = 0$, $\langle e_{r+i}, e_{r+j} \rangle = 0$ and $\langle e_i, e_{r+j} \rangle = \delta_{ij}$ for $1 \le i, j \le r$. Let $H_r = \mathrm{U}(W_r)$ be the unitary group of W_r which is a reductive group over F. The group $H_r(F)$, in which F can be itself or its completion at some place, is generated by the standard parabolic subgroup $P_r(F) = N_r(F) M_r(F)$ and the element w_r. Precisely,

$$N_r(F) = \left\{ n(b) = \begin{pmatrix} \mathbf{1}_r & b \\ & \mathbf{1}_r \end{pmatrix} \;\middle|\; b \in \mathrm{Her}_r(E) \right\},$$

$$M_r(F) = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{\tau,-1} \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_r(E) \right\},$$

and

$$w_r = \begin{pmatrix} & \mathbf{1}_r \\ -\mathbf{1}_r & \end{pmatrix},$$

where $\mathrm{Her}_r(E) = \{ b \in \mathrm{Mat}_r(E) \mid b^\tau = {}^t b \}$.

We fix a place $v \in \Sigma$ and suppress it from the notation. Thus $F = F_v$ is a local field of characteristic zero, $E = E_v$ is a quadratic extension of F which may be split and $H_r = H_{r,v} = H_r(F_v)$ is a local reductive group. Also, we denote by $\mathcal{K}_r$ the maximal compact subgroup of H_r which is the intersection of H_r with $\mathrm{GL}_{2n}(\mathcal{O}_E)$ (resp. is isomorphic to $\mathrm{U}(r) \times \mathrm{U}(r)$) if v is finite (resp. if v is infinite). For $s \in \mathbb{C}$ and a character χ of $E^\times$, we denote by $I_r(s, \chi) = \mathrm{s\text{-}Ind}_{P_r}^{H_r}(\chi |\cdot|_E^{s+r/2})$ the degenerate principal series representation (see [Kudla and Sweet 1997]) of H_r, where s-Ind means the unnormalized smooth induction. Precisely, it realizes on the space of $\mathcal{K}_r$-finite functions φ_s on H_r satisfying

$$\varphi_s(n(b)m(a)g) = \chi(\det a) |\det a|_E^{s+r/2} \varphi_s(g)$$

for all $g \in H_r$, $m(a) \in M_r$, and $n(b) \in N_r$. A (holomorphic) section φ_s of $I_r(s, \chi)$ is called *standard* if its restriction to $\mathcal{K}_r$ is independent of s. It is called *unramified* if it takes value 1 on $\mathcal{K}_r$. Now we view F and E as number fields. For a character χ of $\mathbb{A}_E^\times$ which is trivial on $E^\times$ and $s \in \mathbb{C}$, we have an admissible representation

$I_r(s,\chi)=\bigotimes' I_r(s,\chi_v)$ of $H_r(\mathbb{A}_F)$, where the restricted tensor product is taken with respect to the unramified sections.

Let us have a quick review of the classification of (nondegenerate) hermitian spaces. If $v\in\Sigma_f$ and E is nonsplit at v, there are, up to isometry, two different hermitian spaces over E_v of dimension $m\geq 1$: $V^{\pm}$ defined by

$$\epsilon(V^{\pm})=\epsilon_{E/F}\big((-1)^{m(m-1)/2}\det V^{\pm}\big)=\pm 1.$$

If $v\in\Sigma_f$ and E is split at v, there is, up to isometry, only one hermitian space V^+ over E_v of dimension m. If $v\in\Sigma_\infty$, there are, up to isometry, $m+1$ different hermitian spaces over E_v of dimension m: V_s with signature $(s,m-s)$ where $0\leq s\leq m$. In the later two cases, we can still define $\epsilon(V)$ in the same way. In the global case, up to isometry, all hermitian spaces V over E of dimension m are classified by signatures at infinite places and $\det V\in F^\times/\mathsf{N}_{E/F}E^\times$; particularly, V is determined by all $V_v=V\otimes_F F_v$. In general, we will also consider a hermitian space $\mathbb{V}$ over $\mathbb{A}_E$ of rank m. In this case, $\mathbb{V}$ is nondegenerate if there is a basis under which the representing matrix is invertible in $\mathsf{GL}_m(\mathbb{A}_E)$. For any place $v\in\Sigma$, we let $\mathbb{V}_v=\mathbb{V}\otimes_{\mathbb{A}_F}F_v$, $\mathbb{V}_f=\mathbb{V}\otimes_{\mathbb{A}_F}\mathbb{A}_{f,F}$, and define $\Sigma(\mathbb{V})=\{v\in\Sigma\mid\epsilon(\mathbb{V}_v)=-1\}$, which is a finite set, and $\epsilon(\mathbb{V})=\prod\epsilon(\mathbb{V}_v)$. We say $\mathbb{V}$ is *coherent* (resp. *incoherent*) if the cardinality of $\Sigma(\mathbb{V})$ is even (resp. odd), that is, $\epsilon(\mathbb{V})=1$ (resp. -1). By the Hasse principle, there is a hermitian space V over E such that $\mathbb{V}\cong V\otimes_F\mathbb{A}_F$ if and only if $\mathbb{V}$ is coherent. These two terminologies are introduced in the orthogonal case in [Kudla and Rallis 1994]; see also [Kudla 1997].

We fix a place $v\in\Sigma$ and suppress it from the notation. For a hermitian space V of dimension m with hermitian form $(\,\cdot\,,\cdot\,)$ and a positive integer r, we can construct a symplectic space $\boldsymbol{W}=\mathsf{Res}_{E/F}W_r\otimes_E V$ of dimension $4rm$ over F with the skew-symmetric form $\frac{1}{2}\mathsf{Tr}_{E/F}\langle\,\cdot\,,\cdot\,\rangle^\tau\otimes(\,\cdot\,,\cdot\,)$. We let $H=\mathsf{U}(V)$ be the unitary group of V and $\mathcal{S}(V^r)$ the space of Schwartz functions on V^r. Given a character χ of $E^\times$ satisfying $\chi\big|_{F^\times}=\epsilon_{E/F}^m$, we have a splitting homomorphism

$$\tilde{\iota}_{(\chi,\mathbf{1})}:H_r\times H\to\mathsf{Mp}(\boldsymbol{W})$$

lifting the natural map $\iota:H_r\times H\to\mathsf{Sp}(\boldsymbol{W})$ (see [Harris et al. 1996, Section 1]). We thus have a Weil representation (with respect to ψ) $\omega_\chi=\omega_{\chi,\psi}$ of $H_r\times H$ on the space $\mathcal{S}(V^r)$. Explicitly, for $\phi\in\mathcal{S}(V^r)$ and $h\in H$, we have:

- $\omega_\chi(n(b))\phi(x)=\psi(\mathrm{tr}\,bT(x))\phi(x)$.
- $\omega_\chi(m(a))\phi(x)=|\det a|_E^{m/2}\chi(\det a)\phi(xa)$.
- $\omega_\chi(w_r)\phi(x)=\gamma_V\widehat{\phi}(x)$.
- $\omega_\chi(h)\phi(x)=\phi(h^{-1}x)$,

where $T(x)=\frac{1}{2}\left((x_i,x_j)\right)_{1\le i,j\le r}$ is the moment matrix of x, γ_V is the Weil constant associated to the underlying quadratic space of V (and also ψ), and $\widehat{\phi}$ is the Fourier transform

$$\widehat{\phi}(x)=\int_{V^r}\phi(y)\psi\left(\tfrac{1}{2}\mathrm{Tr}_{E/F}(x,{}^t y)\right)dy,$$

using the self-dual measure dy on V^r with respect to ψ. Taking the restricted tensor product over all local Weil representations, we get a global $\mathcal{S}(V^r):=\bigotimes'\mathcal{S}(V_v^r)$ as a representation of $H_r(\mathbb{A}_F)\times H(\mathbb{A}_F)$.

We now let $m=2n$ and $r=n$ with $n\ge 1$ and suppress n from our notation, except that we will use H' instead of H_n, P' instead of P_n, N' instead of N_n, and $\mathcal{H}'$ instead of $\mathcal{H}_n$. Hence $\chi\big|_{\mathbb{A}_F^\times}=\mathbf{1}$. Let $\pi=\bigotimes'\pi_v$ be an irreducible cuspidal automorphic representation of $H'(\mathbb{A}_F)$ contained in $L^2(H'(F)\backslash H'(\mathbb{A}_F))$ and $\pi^\vee$ realizes on the space of complex conjugation of functions in π.

We denote by $(-W)$ (recall that $W=W_n$) the skew-hermitian space over E with the form $-\langle\,\cdot\,,\cdot\,\rangle$. Hence we can find a basis $\{e_1^-,\dots,e_{2n}^-\}$ satisfying $\langle e_i^-,e_j^-\rangle=0$, $\langle e_{r+i}^-,e_{r+j}^-\rangle=0$ and $\langle e_i^-,e_{n+j}^-\rangle=-\delta_{ij}$ for $1\le i\le n$. Let $W''=W\oplus(-W)$ be the direct sum of two skew-hermitian spaces. There is a natural embedding $\iota: H'\times H'\hookrightarrow H'':=\mathrm{U}(W'')$ given, under the basis $\{e_1,\dots,e_{2n}\}$ of W and $\{e_1,\dots,e_n;e_1^-,\dots,e_n^-;e_{n+1},\dots,e_{2n};-e_{n+1}^-,\dots,-e_{2n}^-\}$ of W'', by $\iota(g_1,g_2)=\iota_0(g_1,g_2^\vee)$, where

$$g_1=\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix},\quad g_2=\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix},\quad g^\vee=\begin{pmatrix}\mathbf{1}_n & \\ & -\mathbf{1}_n\end{pmatrix}g\begin{pmatrix}\mathbf{1}_n & \\ & -\mathbf{1}_n\end{pmatrix}^{-1},$$

and

$$\iota_0(g_1,g_2)=\begin{pmatrix}a_1 & & b_1 & \\ & a_2 & & b_2\\ c_1 & & d_1 & \\ & c_2 & & d_2\end{pmatrix}.$$

For a complete polarization $W''=W'\oplus\overline{W}'$, where $W'=\mathrm{span}_E\{e_1,\dots,e_n;e_1^-,\dots,e_n^-\}$ and $\overline{W}'=\mathrm{span}_E\{e_{n+1},\dots,e_{2n};-e_{n+1}^-,\dots,-e_{2n}^-\}$, there is a Weil representation of H'', denoted by ω_χ'' (with respect to ψ), on the space $\mathcal{S}(V^{2n})$, such that $\iota^*\omega_\chi''\cong\omega_{\chi,\psi}\boxtimes\chi\omega_{\chi,\psi}^\vee$, which is realized on the space $\mathcal{S}(V^n)\otimes\mathcal{S}(V^n)$. Let P be the parabolic subgroup of H'' fixing the subspace $\overline{W}'$ whose maximal unipotent subgroup is denoted by N.

Let $\mathbb{V}$ be a hermitian space over $\mathbb{A}_E$ of rank $2n$. We have a linear map

$$\mathcal{S}(\mathbb{V}^{2n})\to I_{2n}(s,\chi)$$

given by $\varphi_{\Phi,s}(g)=\omega_\chi''(g)\Phi(0)\lambda_P(g)^s$. When $s=0$, it is $H''(\mathbb{A}_F)$-equivariant and we denote by $R(\mathbb{V},\chi)=\bigotimes_v' R(\mathbb{V}_v,\chi)$ the image of this map. We define the

Eisenstein series

$$E(g,\varphi_s)=\sum_{\gamma\in P(F)\backslash H''(F)}\varphi_s(\gamma g)$$

for any standard section φ_s and $E(s,g,\Phi)=E(g,\varphi_{\Phi,s})$. It is absolutely convergent when $\Re(s)>n$ and has a meromorphic continuation to the entire complex plane, holomorphic at $s=0$ (see [Tan 1999, Proposition 4.1]).

By [Gelbart et al. 1987; Li 1992] (see also [Liu 2011, Proposition 2.3]), for any $f\in\pi$, $f^\vee\in\pi^\vee$, and standard $\varphi_s\in I_{2n}(s,\chi)$ which are decomposable, we have for $\Re(s)>n$

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1)f^\vee(g_2)\chi^{-1}(\det g_2)E(\iota(g_1,g_2),\varphi_s)\,dg_1\,dg_2 = \prod_{v\in\Sigma} Z(\chi_v,f_v,f_v^\vee,\varphi_{s,v}),$$

where

$$Z(\chi_v,f_v,f_v^\vee,\varphi_{s,v})=\int_{H'_v}\langle\pi_v(g_v)f_v,f_v^\vee\rangle\varphi_{s,v}(\gamma_0\iota(g_v,1))\,dg_v$$

is the *local zeta integral*, which has a meromorphic continuation to the entire complex plane. Here,

$$\gamma_0=\begin{pmatrix} & & \mathbf{1}_n & \\ & \mathbf{1}_n & & \\ -\mathbf{1}_n & \mathbf{1}_n & & \\ & & \mathbf{1}_n & \mathbf{1}_n\end{pmatrix}.$$

Let us temporarily suppress v in the following. As in [Harris et al. 1996] (see also [Liu 2011, Section 2C]), we define the local L-factors $L(s,\pi,\chi)$ through these local zeta integrals and define the normalized one to be

$$Z^*(\chi,f,f^\vee,\varphi_s):=\frac{b_{2n}(s)Z(\chi,f,f^\vee,\varphi_s)}{L(s+\frac{1}{2},\pi,\chi)}\bigg|_{s=0}, \tag{2-1}$$

which is a nonzero element in $\mathrm{Hom}_{H'\times H'}(I_{2n}(0,\chi),\pi^\vee\boxtimes\chi\pi)$ (see [Harris et al. 1996, Proof of (1), Theorem 4.3]), where

$$b_m(s)=\prod_{i=0}^{m-1}L(2s+m-i,\epsilon_{E/F}^i). \tag{2-2}$$

We let $Z^*(s,\chi,f,f^\vee,\Phi)=Z^*(\chi,f,f^\vee,\varphi_{\Phi,s})$.

When everything is unramified, $Z^*(s,\chi,f,f^\vee,\Phi)=1$, by [Li 1992]. Hence, globally (and assuming everything is decomposable; otherwise we take a linear

combination), the assignment

$$\alpha(f, f^\vee, \Phi) := \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \Phi_v)$$

defines an element in

$$\mathrm{Hom}_{H'(\mathbb{A}_F)\times H'(\mathbb{A}_F)}(R(\mathbb{V}, \chi), \pi^\vee \boxtimes \chi\pi) = \bigotimes_v \mathrm{Hom}_{H'_v\times H'_v}(R(\mathbb{V}_v, \chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v).$$

By analytic continuation, we have

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2) E(s, \iota(g_1, g_2), \Phi)\, dg_1\, dg_2 = \frac{L(s+\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(2s+i, \epsilon_{E/F}^i)} \prod_v Z^*(s, \chi_v, f_v, f_v^\vee, \Phi_v)$$

for any $s \in \mathbb{C}$, where in the last product almost all factors are 1.

By the theta dichotomy proved in [Paul 1998] in the archimedean case and [Gong and Grenié 2011] in the nonarchimedean case (see [Liu 2011, Proposition 2.6] for our statement), we have $\mathrm{Hom}_{H'_v\times H'_v}(R(V_v, \chi_v), \pi_v^\vee \boxtimes \chi_v\pi_v) \neq 0$ for exactly one V_v (up to isometry) over E_v of dimension $2n$. We denote this hermitian space by $V(\pi_v, \chi_v)$ and $\epsilon(\pi_v, \chi_v) = \epsilon(V(\pi_v, \chi_v))$. Let $\mathbb{V}(\pi, \chi)$ be the hermitian space over $\mathbb{A}_E$, unique up to isometry, such that $\mathbb{V}(\pi, \chi)_v \cong V(\pi_v, \chi_v)$ for any $v \in \Sigma$ and $\epsilon(\pi, \chi) = \prod \epsilon(\pi_v, \chi_v)$. Hence we can choose $f \in \pi$, $f^\vee \in \pi^\vee$, and $\Phi \in \mathcal{S}(\mathbb{V}(\pi, \chi)^{2n})$ such that $\alpha(f, f^\vee, \Phi) \neq 0$. If $\epsilon(\pi, \chi) = -1$, then $\mathbb{V}(\pi, \chi)$ is incoherent and $E(0, g, \Phi) = 0$ by [Liu 2011, Proposition 2.11(1)]. Then $L(\frac{1}{2}, \pi, \chi) = 0$ and we have

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2) E'(0, \iota(g_1, g_2), \Phi)\, dg_1\, dg_2 = \frac{L'(\frac{1}{2}, \pi, \chi)}{\prod_{i=1}^{2n} L(i, \epsilon_{E/F}^i)} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \Phi_v). \quad (2\text{-}3)$$

We call $E'(0, g, \Phi) = d/ds|_{s=0} E(s, g, \Phi)$ the *analytic kernel function* associated to the *test function* $\Phi \in \mathcal{S}(\mathbb{V}^{2n})$.

For any $T \in \mathrm{Her}_{2n}(E)$, we have the T-th Fourier coefficient

$$E_T(s, g, \Phi) = \int_{N(F)\backslash N(\mathbb{A}_F)} E(s, ng, \Phi)\psi_T(n)^{-1}\, dn,$$

where $\psi_T(n(b)) = \psi(\mathrm{tr}\, Tb)$ and locally dn_v is the self-dual measure with respect to ψ_v. It turns out that for nonsingular T, the Fourier coefficient has a decomposition as

$$E_T(s, g, \Phi) = W_T(s, g, \Phi) := \prod_{v\in\Sigma} W_T(s, g_v, \Phi_v) \quad (2\text{-}4)$$

if $\Phi = \bigotimes \Phi_v$ is decomposable. Here, locally for any standard section $\varphi_{s,v}$ in $I_{2n}(s, \chi_v)$, we define the Whittaker integral as

$$W_T(g_v, \varphi_{s,v}) = \int_{N_v} \varphi_{s,v}(wn_v g_v)\psi_T(n_v)^{-1}\, dn_v,$$

where $w = w_{2n}$ and $W_T(s, g_v, \Phi_v) := W_T(g, \varphi_{\Phi_v,s})$ for any $T \in \mathrm{Her}_{2n}(E_v)$.

Hence we have

$$E(s, g, \Phi) = \sum_{T \text{ sing.}} E_T(s, g, \Phi) + \sum_{T \text{ nonsing.}} \prod_{v\in\Sigma} W_T(s, g_v, \Phi_v).$$

Taking the derivative at $s = 0$, we have

$$\begin{aligned} E'(0, g, \Phi) &= \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{T \text{ nonsing.}} \sum_{v\in\Sigma} W'_T(0, g_v, \Phi_v) \prod_{v'\neq v} W_T(0, g_{v'}, \Phi_{v'}) \\ &= \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{v\in\Sigma} \sum_{T \text{ nonsing.}} W'_T(0, g_v, \Phi_v) \prod_{v'\neq v} W_T(0, g_{v'}, \Phi_{v'}). \end{aligned}$$

But $\prod_{v'\neq v} W_T(0, g_v, \Phi_v) \neq 0$ only if $\mathbb{V}_{v'}$ represents T for all $v' \neq v$. Since $\mathbb{V}$ is incoherent, $\mathbb{V}_v$ cannot represent T. For T nonsingular, there are only finitely many $v \in \Sigma$ such that T is not represented by $\mathbb{V}_v$, that is, there does not exist $x_1, \ldots, x_{2n} \in \mathbb{V}_v$ whose moment matrix is T. We denote the set of such v by $\mathrm{Diff}(T, \mathbb{V})$. Then

$$E'(0, g, \Phi) = \sum_{T \text{ sing.}} E'_T(0, g, \Phi) + \sum_{v\in\Sigma} E_v(0, g, \Phi),$$

where

$$E_v(0, g, \Phi) = \sum_{\mathrm{Diff}(T,\mathbb{V})=\{v\}} W'_T(0, g_v, \Phi_v) \prod_{v'\neq v} W_T(0, g_{v'}, \Phi_{v'}). \tag{2-5}$$

In fact, the second sum is only taken over those v which are nonsplit in E.

2B. ***Regular test functions.*** In this section, we will prove that the summation of $E'_T(0, g, \Phi)$ over singular T's vanishes for a certain choice of Φ and g in a suitable subset of $H''(\mathbb{A}_F)$. We follow the ideas in [Yuan et al. 2010].

For a finite place v, recall the definition

$$\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}} = \{\Phi_v \in \mathcal{S}(\mathbb{V}_v^{2n}) \mid \Phi_v(x) = 0 \text{ if } \det T(x) = 0\}.$$

We call the elements in this set *regular test functions*.

Fix a finite subset $S \subset \Sigma_f$ with $|S| = k > 0$ and let $\mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}} = \bigotimes_{v\in S} \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$. Our main result in this section is:

Proposition 2.1. *For $\Phi = \Phi_S \Phi^S \in \mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}} \otimes \mathcal{S}(\mathbb{V}^{S,2n})$, $\mathrm{ord}_{s=0} E_T(s, g, \Phi) \geq k$ for T singular and $g \in P(\mathbb{A}_{F,S})H''(\mathbb{A}_F^S)$.*

We can assume that $\Phi = \bigotimes \Phi_v$ is decomposable with $\Phi_v \in \mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ for $v \in S$ and $\mathrm{rank}\, T = 2n - r < 2n$. Choose $a \in \mathrm{GL}_{2n}(E)$ such that

$$aT\,{}^t a^\tau = \begin{pmatrix} \mathbf{0}_r & \\ & \widetilde{T} \end{pmatrix}, \tag{2-6}$$

with $\widetilde{T} \in \mathrm{Her}_{2n-r}(E)$. Then

$$E_T(s, g, \Phi) = E_{aT\,{}^t a^\tau}(s, m(a)g, \Phi).$$

Hence we can assume that T is of the form (2-6).

First, we need a more explicit formula for the singular coefficient E_T. By definition, for $\Re(s) > n$

$$\begin{aligned} E_T(s, g, \Phi) &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash H''(F)} \varphi_{\Phi,s}(\gamma n g)\psi_T(n)^{-1}\, dn \\ &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash H''(F)} r(g)\varphi_{\Phi,s}(\gamma n)\psi_T(n)^{-1}\, dn, \end{aligned} \tag{2-7}$$

where $r(g)$ means the H'' action on $I_{2n}(s, \chi)$ by right translation. We need to unfold this summation. Let

$$w_{2n,d} = \begin{pmatrix} \mathbf{1}_d & & & \\ & & & \mathbf{1}_{2n-d} \\ & & \mathbf{1}_d & \\ & -\mathbf{1}_{2n-d} & & \end{pmatrix} \tag{2-8}$$

for $0 \le d \le 2n$ be a set of representatives of the double coset $W_{P/N}\backslash W_{H''}/W_{P/N}$ of Weyl groups, thus $w_{2n,0} = w_{2n} = w$. We have a Bruhat decomposition

$$H''(F) = \coprod_{d=0}^{2n} P(F) w_{2n,d} P(F),$$

where F can be the global field or its local completions.

Lemma 2.2. *If $v \in S$ and $g_v \in P_v$, the support of $r(g_v)\varphi_{\Phi_v,s}$ is contained in $P(F_v)wN(F_v)$.*

Proof. It suffices to prove that $\varphi_{\Phi_v,s}$ vanishes on $P(F_v)w_{2n,d}P(F_v)$ for $d > 0$ since $g_v \in P(F_v)$. For $g = n(b_1)m(a_1)w_{2n,d}n(b_2)m(a_2) \in P(F_v)w_{2n,d}P(F_v)$, we have

$$\begin{aligned} \varphi_{\Phi_v,s}(g) &= \omega_{\chi_v}(g)\Phi(0)\lambda(g)^s \\ &= \chi_v(\det a_1 a_2)|\det a_1 a_2|_{E_v}^n \lambda(g)^s \int_{\mathbb{V}_v^{2n-d}} \psi_{b_2}(T(x))\Phi_v(x a_2)\, dx, \end{aligned}$$

where $\mathbb{V}_v^{2n-d}$ is viewed, via $(x_1, \ldots, x_{2n-d}) \mapsto (0, \ldots, 0, x_1, \ldots, x_{2n-d})$, as a subset of $\mathbb{V}_v^{2n}$. Since Φ_v is regular and $d > 0$, $\Phi_v(x a_2) = 0$ for $x \in \mathbb{V}_v^{2n-d}$. □

By the lemma, we have for $g \in P(\mathbb{A}_{F,S})H''(\mathbb{A}_F^S)$,

$$\begin{aligned}(2\text{-}7) &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in P(F)\backslash P(F)wP(F)} r(g)\varphi_{\Phi,s}(\gamma n)\psi_T(n)^{-1}\,dn \\ &= \int_{N(F)\backslash N(\mathbb{A}_F)} \sum_{\gamma \in wN(F)} r(g)\varphi_{\Phi,s}(\gamma n)\psi_T(n)^{-1}\,dn \\ &= \int_{N(\mathbb{A}_F)} r(g)\varphi_{\Phi,s}(wn)\psi_T(n)^{-1}\,dn \\ &= \int_{N(\mathbb{A}_F)} \varphi_s(wn)\psi_T(n)^{-1}\,dn \\ &= \prod_{v\in\Sigma}\int_{N_v} \varphi_{v,s}(wn_v)\psi_T(n_v)^{-1}\,dn_v, \end{aligned} \tag{2-9}$$

where we write φ_s instead of $r(g)\varphi_{\Phi,s}$ for simplicity. Let $S' \subset \Sigma$ be the finite subset containing all infinite places and ramified places, away from which χ_v and ψ_v are unramified; $\varphi_{v,s}$ is the (unique) unramified section in $I_{2n}(s, \chi_v)$ (hence $S' \supset S$) and $\det \widetilde{T} \in \mathbb{O}_{F_v}^\times$. Then

$$(2\text{-}9) = \left(\prod_{v\in S'} W_T(e, \varphi_{v,s})\right) W_T(e, \varphi_s^{S'}). \tag{2-10}$$

By [Kudla and Rallis 1994, p. 36] and [Tan 1999, Proposition 3.2], we have

$$W_T(e, \varphi_s^{S'}) = \frac{a_{2n}^{S'}(s)}{a_{2n-r}^{S'}(s-\frac{1}{2}r)b_{2n}^{S'}(s)},$$

where

$$a_{m,v}(s) = \prod_{i=0}^{m-1} L_v(2s+i-m+1, \epsilon_{E/F}^i) \quad \text{and} \quad b_{m,v}(s) = \prod_{i=0}^{m-1} L_v(2s+m-i, \epsilon_{E/F}^i),$$

as in (2-2). Hence $W_T(e, \varphi_s^{S'})$ has a meromorphic continuation to the entire complex plane. For $v \in S'$, we normalize the Whittaker functional to be

$$W_T^*(e, \varphi_{v,s}) = \frac{a_{2n-r,v}(s-\frac{1}{2}r)b_{2n,v}(s)}{a_{2n,v}(s)} W_T(e, \varphi_{v,s}).$$

Using the argument and notation of [Kudla and Rallis 1994, p. 35], we have

$$W_T(e, \varphi_{v,s}) = W_{\widetilde{T}}(e, i^* \circ U_{r,v}(s)\varphi_{v,s}).$$

By [Piatetski-Shapiro and Rallis 1987, Section 4], the (local) intertwining operator $U_{r,v}(s)$ has a meromorphic extension to the entire complex plane. By [Liu 2011, Lemma 2.8(1)], which combines results from [Karel 1979] and [Wallach 1988],

$W_T(e,\varphi_{v,s})$ and hence $W_T^*(e,\varphi_{v,s})$ have meromorphic continuations to the entire complex plane. Together with the meromorphic continuation of W_T away from S' and W_T^* in S', (2-10) has a meromorphic continuation which equals

$$\frac{a_{2n}(s)}{a_{2n-r}(s-\frac{1}{2}r)b_{2n}(s)}\prod_{v\in S'}W_T^*(e,\varphi_{v,s}).$$

Proof of Proposition 2.1. Consider the point $s=0$, $b_{2n}(0)=\prod_{i=1}^{2n}L(i,\epsilon_{E/F}^i)\in\mathbb{C}^\times$, and

$$\frac{a_{2n}(0)}{a_{2n-r}(-\frac{1}{2}r)}=\prod_{i=0}^{r-1}L(-i,\epsilon_{E/F}^{i+1})\in\mathbb{C}^\times.$$

Let $\kappa_v=\operatorname{ord}_{s=0}W_T^*(e,\cdot)$ be the order of the functional at $s=0$ for $v\in S'$ and $\kappa_v'=\operatorname{ord}_{s=0}W_T^*(s,e,\cdot)|_{\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}}$ for $v\in S$. Since $E_T(e,\varphi_\Phi)=0$ if $\Phi=\bigotimes\Phi_v$ for at least one Φ_v regular, by (2-10) and the proof of [Liu 2011, Lemma 2.10], we have $\kappa_{v_0}'+\sum_{v_0\neq v\in S'}\kappa_v\geq 1$ for any v_0 in S. Also by the definition of W_T, we see that

$$\varphi_{v,0}\mapsto s^{-\kappa_v}W_T^*(e,\varphi_{v,s})\big|_{s=0}$$

is a nontrivial N-intertwining map from $I_{2n}(0,\chi)$ to $\mathbb{C}_{N,\psi_T}$. Now if $v\in S$, our $\varphi_{v,0}=\varphi_{\Phi_v,0}$ for a regular test function $\Phi_v\in\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$. By [Rallis 1987, Lemma 4.2] stated as [Liu 2011, Lemma 2.7(1-a)], $\varphi_{v,0}$ goes to 0 under the above map, that is, $\kappa_v'\geq\kappa_v+1$ for $v\in S$. Hence

$$\operatorname{ord}_{s=0}\prod_{v\in S'}W_T^*(e,\varphi_{v,s})\geq\sum_{v\in S}\kappa_v'+\sum_{v\in S'-S}\kappa_v\geq k-1+\kappa_{v_0}'+\sum_{v_0\neq v\in S'}\kappa_v\geq k.\quad\square$$

In conclusion, if we choose S such that $|S|\geq 2$ and a decomposable test function

$$\Phi=\Phi_S\Phi^S\in\mathcal{S}(\mathbb{V}_S^{2n})_{\mathrm{reg}}\otimes\mathcal{S}(\mathbb{V}^{S,2n}),$$

then for $g\in P(\mathbb{A}_{F,S})H''(\mathbb{A}_F^S)$,

$$E'(0,g,\Phi)=\sum_{v\in\Sigma}E_v(0,g,\Phi).\tag{2-11}$$

2C. ***Test functions of higher discriminant.*** In this section, we will show that if we have a better choice of Φ_v for $v\in S$, we can even make $W_T'(0,e,\Phi_v)=0$ for any nonsingular T which is not representable by $\mathbb{V}_v$. We follow the ideas in [Yuan et al. 2010].

Since the argument is local, we fix one $v\in S$ which is nonsplit and suppress it from the notation in this section. Let V be one of $V^\pm$ and V' the other one which

is not isometric to V. For $d \in \mathbb{Z}$, let

$$\begin{aligned}
\mathrm{Her}_{2n}^0(E) &= \{T \in \mathrm{Her}_{2n}(E) \mid \det T \neq 0\},\\
\mathscr{H} &= \{b \in \mathrm{Her}_{2n}^0(E) \mid b = T(x) \text{ for some } x \in V^{2n}\},\\
\mathscr{H}' &= \{b' \in \mathrm{Her}_{2n}^0(E) \mid b' = T(x') \text{ for some } x' \in V'^{2n}\},\\
\mathscr{H}'_d &= \{b' + b'' \mid b' \in \mathscr{H}' \text{ and } b'' \in \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})\} \cap \mathrm{Her}_{2n}^0(E),
\end{aligned}$$

where $\mathfrak{p}_E$ is the maximal ideal of $\mathbb{O}_E$. Then $\mathrm{Her}_{2n}^0(E) = \mathscr{H} \amalg \mathscr{H}'$,

$$\cdots \subsetneqq \mathscr{H}'_{-1} \subsetneqq \mathscr{H}'_0 \subsetneqq \mathscr{H}'_1 \subsetneqq \cdots,$$

$\bigcap_d \mathscr{H}'_d = \mathscr{H}'$, and $\bigcup_d \mathscr{H}'_d = \mathrm{Her}_{2n}^0(E)$. We say that a test function $\Phi \in \mathcal{S}(V^{2n})$ is *of discriminant d* if

$$\{T(x) \mid x \in \mathrm{Supp}(\Phi)\} \cap \mathscr{H}'_d = \varnothing.$$

We denote by $\mathcal{S}(V^{2n})_d$ the space of such functions, and set

$$\mathcal{S}(V^{2n})_{\mathrm{reg},d} = \mathcal{S}(V^{2n})_{\mathrm{reg}} \cap \mathcal{S}(V^{2n})_d.$$

Lemma 2.3. *For any $d \in \mathbb{Z}$, $\mathcal{S}(V^{2n})_{\mathrm{reg},d}$ is not empty.*

Proof. Fix any d; in fact, we only need to prove that there exists $T \notin \mathscr{H}'_d$ such that $\det T \neq 0$. Then $(T + \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})) \cap \mathscr{H}' = \varnothing$. Any test function with support whose elements have moment matrices contained in $(T + \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})) \cap \mathrm{Her}_{2n}^0(E)$, which is open, will be in $\mathcal{S}(V^{2n})_{\mathrm{reg},d}$. Now we want to find such a T. Take any $T_1 \in \mathscr{H}$ with $\det T_1 \neq 0$. Since $\mathscr{H}$ is open, we can find a neighborhood $T_1 + \mathrm{Her}_{2n}(\mathfrak{p}_E^{\nu}) \subset \mathscr{H}$ for a $\nu \in \mathbb{Z}$. If $\nu \leq -d$, then we are done. Otherwise, let ϖ be the uniformizer of F. Then $\varpi^{-\nu-d}(T_1 + \mathrm{Her}_{2n}(\mathfrak{p}_E^{\nu})) \subset \mathscr{H}$. But

$$\varpi^{-\nu-d}(T_1 + \mathrm{Her}_{2n}(\mathfrak{p}_E^{\nu})) = (\varpi^{-\nu-d} T_1 + \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})).$$

Hence $T = \varpi^{-\nu-d} T_1$ will serve for our purpose. □

Since ψ is nontrivial, we can define its *discriminant* d_ψ to be the largest integer d such that the character ψ_T is trivial on $N(\mathbb{O}_F) \cong \mathrm{Her}_{2n}(\mathbb{O}_E)$ for all $T \in \mathrm{Her}_{2n}(\mathfrak{p}_E^{-d})$. We need to mention that this is not the conductor of a p-adic additive character. But the difference between them only depends on n and the ramification of E/F. The main result of this section is:

Proposition 2.4. *Let $d \geq d_\psi$ be an integer. Given $\Phi \in \mathcal{S}(V^{2n})_{\mathrm{reg},d}$, we have $W_T(s, e, \Phi) \equiv 0$ for $T \in \mathscr{H}'$ nonsingular.*

Proof. For $\Re(s) > n$,

$$W_T(s, e, \Phi) = \int_N \omega_\chi(wn)\Phi(0)\lambda(wn)^s \psi_T(n)^{-1}\, dn$$

is absolutely convergent. Hence it equals

$$\begin{aligned}\int_{\mathrm{Her}_{2n}(E)}\left(\int_{V^{2n}}\psi(\operatorname{tr} bT(x))\Phi(x)\,dx\right)\lambda(wn(b))^s\psi(-\operatorname{tr} Tb)\,db&\\ =\int_{V^{2n}}\Phi(x)\,dx\int_{\mathrm{Her}_{2n}(E)}\lambda(wn(b))^s\psi\big(\operatorname{tr}(T(x)-T)b\big)\,db&\\ =\int_{V^{2n}}\Phi(x)\,dx\int_{\mathrm{Her}_{2n}(E)}\lambda(wn(b))^s\psi_{T(x)-T}(n(b))\,db.&\qquad(2\text{-}12)\end{aligned}$$

Since $\lambda(wn(b)n(b_1))=\lambda(wn(b))$ for $b_1\in\mathrm{Her}_{2n}(\mathbb{O}_E)$,

$$\begin{aligned}(2\text{-}12)=\int_{V^{2n}}\Phi(x)\,dx\int_{\mathrm{Her}_{2n}(E)/\mathrm{Her}_{2n}(\mathbb{O}_E)}\lambda(wn(b))^s\psi_{T(x)-T}(n(b))\,db&\\ \times\int_{\mathrm{Her}_{2n}(\mathbb{O}_E)}\psi_{T(x)-T}(n(b_1))\,db_1,&\end{aligned}$$

in which the last integral is zero for all $x\in\mathrm{Supp}(\Phi)$ by our assumption on Φ. Hence $W_T(s,e,\Phi)\equiv 0$ after continuation. In particular, $W_T'(0,e,\Phi)=0$. □

In conclusion, if S is a finite subset of Σ_f with $|S|\geq 2$, $\Phi=\bigotimes\Phi_v\in\mathcal{S}(\mathbb{V}^{2n})$ with $\Phi_v\in\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg}}$ for $v\in S$, and $\Phi_v\in\mathcal{S}(\mathbb{V}_v^{2n})_{\mathrm{reg},d_v}$ for $v\in S$ nonsplit with $d_v\geq d_{\psi_v}$, then

$$E'(0,g,\Phi)=\sum_{v\notin S}E_v(0,g,\Phi)\qquad(2\text{-}13)$$

for $g\in e_S H''(\mathbb{A}_F^S)$.

2D. ***Density of test functions.*** In the previous two sections, we have made particular choices of test functions to simplify the formula of the analytic kernel functions. But for our proof of the main theorem, arbitrary choices will not suffice. In this section, we will show that there are "sufficiently many" test functions satisfying these particular choices we have made in the sense of Proposition 2.8. We follow the ideas in [Yuan et al. 2011].

We keep the notation from the previous two sections. In particular, v will be a place in S and will be suppressed from the notation. Recall that we have an H''-intertwining map $\mathcal{S}(V^{2n})\twoheadrightarrow\mathcal{S}(V^{2n})_H\cong R(V,\chi)\hookrightarrow I_{2n}(0,\chi)$ through the Weil representation ω_χ''. Hence we have an $H'\times H'$ admissible representation on $\mathcal{S}(V^{2n})$ through the embedding ι defined in Section 2A.

Lemma 2.5. *If v is nonsplit, then for any $d\in\mathbb{Z}$ we have*

$$\mathcal{S}(V^{2n})_{\mathrm{reg}}=\omega_\chi''(m(F^\times\mathbf{1}_{2n}))\mathcal{S}(V^{2n})_{\mathrm{reg},d}.$$

Proof. Fix $d \in \mathbb{Z}$. For any function $\Phi \in \mathcal{S}(V^{2n})_{\mathrm{reg}}$, $\mathrm{Supp}(\Phi)$ is a compact subset of $\mathcal{H}$. Since $\mathrm{Her}^0_{2n}(E)\backslash\mathcal{H}'_d$ is open and

$$\bigcup_d (\mathrm{Her}^0_{2n}(E)\backslash\mathcal{H}'_d) = \mathrm{Her}^0_{2n}(E)\backslash\bigcap_d \mathcal{H}'_d = \mathrm{Her}^0_{2n}(E)\backslash\mathcal{H}' = \mathcal{H},$$

$(\mathrm{Her}^0_{2n}(E)\backslash\mathcal{H}'_d)_{d\in\mathbb{Z}}$ is an open covering of $\mathrm{Supp}(\Phi)$, and hence has a finite subcover. Then there exists $d_0 \in \mathbb{Z}$ such that $\mathrm{Supp}(\Phi) \cap \mathcal{H}'_{d_0} = \varnothing$. If $d_0 \geq d$, then we are done. Otherwise, consider $\Phi' = \omega''_\chi(m(\varpi^{d_0-d}\mathbf{1}_{2n}))\Phi$; then $\mathrm{Supp}(\Phi') \cap \mathcal{H}'_d = \varnothing$. The lemma follows. □

In the rest of this section, let $n = 1$. Then $H' = \mathrm{U}(W_1)$.

Lemma 2.6. *Let π be an irreducible admissible representation of H' which is not of dimension 1 and $A : \mathcal{S}(V) \to \pi$ a surjective H'-intertwining map, where H' acts on $\mathcal{S}(V)$ through a Weil representation ω. Then for any ϕ with $A(\phi) \neq 0$, there is $\phi' \in \mathcal{S}(V)_{\mathrm{reg}}$ such that $A(\phi') \neq 0$ and $\mathrm{Supp}(\phi') \subset \mathrm{Supp}(\phi)$.*

Proof. Let $f = A(\phi)$. If there exists $n \in N$ such that $\pi(n)f \neq f$, then

$$A(\omega(n)\phi - \phi) = \pi(n)f - f \neq 0$$

but

$$\omega(n)\phi(x) - \phi(x) = \big(\psi(bT(x)) - 1\big)\phi(x),$$

where $n = n(b)$. We see that $\phi' = \omega(n)\phi - \phi \in \mathcal{S}(V)_{\mathrm{reg}}$ and $\mathrm{Supp}(\phi') \subset \mathrm{Supp}(\phi)$. If $\pi(n)f = f$ for any $n \in N$, then f will be fixed by an open subgroup of H' containing N since π is smooth. But any such subgroup will contain $\mathrm{SU}(W_1)$, hence π factors through $H'/\mathrm{SU}(W_1) = \mathrm{U}(W_1)/\mathrm{SU}(W_1) \cong E^{\times,1}$, which contradicts the assumption on π. □

Lemma 2.7. *If π_1 and π_2 are two irreducible admissible representations of H' which are not of dimension 1, then for any surjective $H' \times H'$-intertwining map $B : \mathcal{S}(V) \otimes \mathcal{S}(V) = \mathcal{S}(V^2) \to \pi_1 \boxtimes \pi_2$ where $H' \times H'$ acts on $\mathcal{S}(V) \otimes \mathcal{S}(V)$ by a pair of Weil representations $\omega_1 \boxtimes \omega_2$, there is an element $\Phi = \phi_1 \otimes \phi_2 \in \mathcal{S}(V^2)_{\mathrm{reg}}$ such that $B(\Phi) \neq 0$.*

Proof. Let $\Phi' \in \mathcal{S}(V^2)$ be such that $B(\Phi') \neq 0$. Write $\Phi' = \sum \phi_{i,1} \otimes \phi_{i,2}$ as an element in $\mathcal{S}(V) \otimes \mathcal{S}(V)$. Hence we can assume that there is $\phi_1 \otimes \phi_2$ such that $B(\phi_1 \otimes \phi_2) \neq 0$. By Lemma 2.6, we can also assume that $\phi_1 \in \mathcal{S}(V)_{\mathrm{reg}}$. For $x \in \mathrm{Supp}(\phi_1)$, let V_x be the subspace of V generated by x and V^x its orthogonal complement. Both V_x and V^x are nondegenerate hermitian spaces of dimension 1. As an H'-representation, $\mathcal{S}(V) = \mathcal{S}(V_x) \otimes \mathcal{S}(V^x)$. Now write $\phi_2 = \sum \phi_{i,x} \otimes \phi_i^x$ according to this decomposition. We can assume there is a $\phi_x \otimes \phi^x$ such that $B(\phi_1 \otimes (\phi_x \otimes \phi^x)) \neq 0$, since as an H'-representation, $\mathcal{S}(V^x)$ is generated by the

subspace $\mathcal{S}(V^x)_{\mathrm{reg}}$. We can then write

$$\phi_x \otimes \phi^x = \sum \omega_2(g_j)(\omega_2^{-1}(g_j)\phi_x \otimes \phi_j^x),$$

with $\phi_j^x \in \mathcal{S}(V^x)_{\mathrm{reg}}$. So we can further assume that $B(\phi_1 \otimes (\phi_x \otimes \phi^x)) \neq 0$ with $\phi^x \in \mathcal{S}(V^x)_{\mathrm{reg}}$, that is, $\mathrm{Supp}(\phi_x \otimes \phi^x) \cap V_x = \varnothing$. Applying Lemma 2.6 again, we can further assume there exists $\phi_2^{(x)} \in \mathcal{S}(V)_{\mathrm{reg}}$ such that $\mathrm{Supp}(\phi_2^{(x)}) \subset \mathrm{Supp}(\phi_x \otimes \phi^x)$ and $B(\phi_1 \otimes \phi_2^{(x)}) \neq 0$. The condition that $\mathrm{Supp}(\phi_2) \cap V^x = \varnothing$ is open for x. Hence we can find a neighborhood U_x of x such that $\phi_1|_{U_x} \otimes \phi_2^{(x)} \in \mathcal{S}(V^2)_{\mathrm{reg}}$. Since $\mathrm{Supp}(\phi_1)$ is compact, we can find Φ of this kind such that $B(\Phi) \neq 0$. □

Recall the zeta integrals introduced in Section 2A. For $\Phi \in \mathcal{S}(V^{2n})$, we write $Z^*(s, \chi, f, f^\vee, \Phi) = Z^*(\chi, f, f^\vee, \varphi_{\Phi,s})$. Combining Lemmas 2.5 and 2.7, we have:

Proposition 2.8. *Let $n = 1$, $v \in \Sigma_f$, π be an irreducible cuspidal automorphic representation of H', and $V_v = V(\pi_v, \chi_v)$. For any $d \in \mathbb{Z}$, we can find $f_v \in \pi_v$, $f_v^\vee \in \pi_v^\vee$, and $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(V_v^2)_{\mathrm{reg},d}$ (resp. $\mathcal{S}(V_v^2)_{\mathrm{reg}}$) if v is nonsplit (resp. split) in E, such that the (normalized) zeta integral $Z^*(0, \chi_v, f_v, f_v^\vee, \phi_{1,v} \otimes \phi_{2,v}) \neq 0$.*

3. Geometric kernel functions

3A. ***Néron–Tate height pairing on curves.*** In this section, we will review the general theory of the Néron–Tate height pairing on curves over number fields and some related facts.

Height pairing of cohomologically trivial cycles. Let E be any number field, not necessarily CM, and let M be a connected smooth projective curve over E, not necessarily geometrically connected. Let $\mathrm{CH}^1(M)^0_{\mathbb{C}}$ be the group of *cohomologically trivial cycles* which is the kernel of the map

$$\deg : \mathrm{CH}^1(M)_{\mathbb{C}} \longrightarrow \mathrm{H}^2_{\text{ét}}(M_{E^{\mathrm{ac}}}, \mathbb{Z}_\ell(1))^{\Gamma_E} \otimes_{\mathbb{Z}_\ell} \mathbb{C} \cong \mathbb{C}$$

for any fixed rational prime number ℓ. Let $\mathcal{M}$ be a regular model of M, that is, a regular scheme, flat and projective over $\mathrm{Spec}\, \mathcal{O}_E$ with generic fiber $\mathcal{M}_E \cong M$.

Recall that an *arithmetic $\mathbb{C}$-divisor* is a datum $(\mathcal{Z}, g_{\iota^\circ})$, where $\mathcal{Z} \in \mathrm{Z}^1(\mathcal{M})_{\mathbb{C}}$ is a usual divisor and g_{ι° is a Green's function (that is, a Green's (0,0)-form of logarithmic type [Soulé 1992, II.2]) for the divisor $\mathcal{Z}_{\iota^\circ}(\mathbb{C})$ on $\mathcal{M}_{\iota^\circ}(\mathbb{C})$ for each $\iota^\circ : E \hookrightarrow \mathbb{C}$. We denote by $\widehat{\mathrm{Z}}^1_{\mathbb{C}}(\mathcal{M})$ the group of arithmetic $\mathbb{C}$-divisors. For a nonzero rational function f on $\mathcal{M}$, we define the associated *principal arithmetic divisor* to be

$$\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), -\log|f_{\iota^\circ,\mathbb{C}}|^2).$$

The quotient of $\widehat{\mathrm{Z}}^1_{\mathbb{C}}(\mathcal{M})$ divided by the $\mathbb{C}$-subspace generated by the principal arithmetic divisors is the *arithmetic Chow group*, denoted by $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$. Inside $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$, there is a subspace $\mathrm{CH}^1_{\mathrm{fin}}(\mathcal{M})_{\mathbb{C}}$ which is $\mathbb{C}$-generated by $(\mathcal{Z}, 0)$ with $\mathcal{Z}$ supported

on special fibers. Let $\mathrm{CH}^1_{\mathrm{fin}}(\mathcal{M})^{\perp}_{\mathbb{C}} \subset \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$ be the orthogonal complement under the $\mathbb{C}$-bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{GS}} : \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M}) \times \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M}) \to \mathbb{C}.$$

Recall that an arithmetic divisor $(\mathscr{Z}, g_{\iota^\circ})$ is *flat* if we have the following equality in the space $\mathrm{D}^{1,1}(M_{\iota^\circ}(\mathbb{C}))$ of $(1,1)$-currents:

$$dd^c[g_{\iota^\circ}] + \delta_{\mathscr{Z}_{\iota^\circ}(\mathbb{C})} = 0$$

for any ι°, where $d^c = (4\pi i)^{-1}(\partial - \bar{\partial})$, $[-]$ is the associated current, and δ is the Dirac current. Flatness is well-defined in $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$. Now we introduce the subgroup $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})^0$ of $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})$ consisting of elements (represented by) $(\mathscr{Z}, g_{\iota^\circ})$ such that $(\mathscr{Z}, g_{\iota^\circ}) \in \mathrm{CH}^1_{\mathrm{fin}}(\mathcal{M})^{\perp}_{\mathbb{C}}$, $\mathscr{Z}_E \in \mathrm{CH}^1(M)^0_{\mathbb{C}}$, and $(\mathscr{Z}, g_\iota)$ is flat. Hence we have a natural map

$$\begin{aligned} p_{\mathcal{M}} : \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M})^0 &\to \mathrm{CH}^1(M)^0_{\mathbb{C}} \\ (\mathscr{Z}, g_\iota) &\mapsto \mathscr{Z}_E \end{aligned} \tag{3-1}$$

which is surjective. Now we can define the *Néron–Tate height pairing*:

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathrm{NT}} : \mathrm{CH}^1(M)^0_{\mathbb{C}} \times \mathrm{CH}^1(M)^0_{\mathbb{C}} &\to \mathbb{C} \\ (Z_1, Z_2) &\mapsto \langle (\mathscr{Z}_1, g_{1,\iota^\circ}), (\mathscr{Z}_2, g_{2,\iota^\circ}) \rangle_{\mathrm{GS}}, \end{aligned} \tag{3-2}$$

where $(\mathscr{Z}_i, g_{i,\iota^\circ})$ $(i = 1, 2)$ is any preimage of Z_i under $p_{\mathcal{M}}$. It is easy to see that this is independent of the choices of preimages and also the regular model $\mathcal{M}$.

Modification of the height pairing. Practically, the cycles we are interested are not automatically cohomologically trivial. We need to make some modifications with respect to some auxiliary data. This is quite easy if we are working over a curve. Let $\widehat{\mathrm{Pic}}(M)$ be the abelian group of isomorphism classes of *hermitian line bundles* on M. Recall that a hermitian line bundle is $\overline{\mathscr{L}} = (\mathscr{L}, \|\cdot\|_{\iota^\circ})$, where $\mathscr{L} \in \mathrm{Pic}(M)$ and $\|\cdot\|_{\iota^\circ}$ is a (smooth) hermitian metric on the holomorphic line bundle $\mathscr{L}_{\iota^\circ,\mathbb{C}}$. We assume that $\deg c_1(\mathscr{L}) \neq 0$. For any $Z \in \mathrm{CH}^1(M)$, the divisor

$$Z^0_{\mathscr{L}} = Z - \frac{\deg Z}{\deg c_1(\mathscr{L})} c_1(\mathscr{L}) \in \mathrm{CH}^1(M)^0_{\mathbb{C}}.$$

Now we define the modified height pairing with respect to $\overline{\mathscr{L}}$:

$$\langle Z_1, Z_2 \rangle_{\overline{\mathscr{L}}} := \langle Z^0_{1,\mathscr{L}}, Z^0_{2,\mathscr{L}} \rangle_{\mathrm{NT}}$$

for any $Z_i \in \mathrm{CH}^1(M)_{\mathbb{C}}$ $(i = 1, 2)$. In particular, we need to choose a suitable Green's function on Z_i when computing via (3-2). We say that the Green's function g_{ι° of Z is *$\overline{\mathscr{L}}$-admissible* if the following equalities between $(1,1)$-currents hold:

$$dd^c[g_{\iota^\circ}] + \delta_{Z_{\iota^\circ}(\mathbb{C})} = \frac{\deg Z}{\deg c_1(\mathscr{L})} [c_1(\mathscr{L}_{\iota^\circ,\mathbb{C}}, \|\cdot\|_{\iota^\circ})],$$

$$\int_{M_{\iota^\circ}(\mathbb{C})} g_{\iota^\circ} \cdot c_1(\mathscr{L}_{\iota^\circ,\mathbb{C}}, \|\cdot\|_{\iota^\circ}) = 0,$$

where $c_1(\mathscr{L}_{\iota^\circ,\mathbb{C}}, \|\cdot\|_{\iota^\circ}) \in \mathsf{A}^{1,1}(M_{\iota^\circ}(\mathbb{C}))$ is the *Chern form* associated to the hermitian holomorphic line bundle $(\mathscr{L}_{\iota^\circ,\mathbb{C}}, \|\cdot\|_{\iota^\circ})$ which is a $(1,1)$-form.

3B. ***Decomposition of geometric kernel functions.*** In this section, we will apply the above section to the special case of Shimura curves of unitary groups. We fix until the end of this paper an additive character $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}$ such that ψ_ι is the standard character $\phi^0_\infty : t \mapsto e^{2\pi i t}$ $(t \in F_\iota = \mathbb{R})$ for any $\iota \in \Sigma_\infty$.

Shimura curves of unitary groups. We review the setup of [Liu 2011, Section 3A] in the particular case $m = 2$ and $r = 1$. Hence $\mathbb{V}$ is a totally positive-definite hermitian space over $\mathbb{A}_E$ of rank 2. Let $\mathbb{H} = \mathsf{Res}_{\mathbb{A}_F/\mathbb{A}}\mathsf{U}(\mathbb{V})$ be the unitary group which is a reductive group over $\mathbb{A}$ and $\mathbb{H}^{\mathsf{der}} = \mathsf{Res}_{\mathbb{A}_F/\mathbb{A}}\mathsf{SU}(\mathbb{V})$ its derived subgroup. Let $\mathbb{T} \cong \mathsf{Res}_{\mathbb{A}_F/\mathbb{A}}\mathbb{A}_E^{\times,1}$ be the maximal abelian quotient of $\mathbb{H}$ which is also isomorphic to its center. Let $T \cong \mathsf{Res}_{F/\mathbb{Q}}E^{\times,1}$ be the unique (up to isomorphism) $\mathbb{Q}$-torus such that $T \times_{\mathbb{Q}} \mathbb{A} \cong \mathbb{T}$. Then T has the property that $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$. For any open compact subgroup K of $\mathbb{H}(\mathbb{A}_f)$, which we always assume *to be contained in the principal congruence subgroup for* $N \geq 3$, there is a Shimura curve $\mathsf{Sh}_K(\mathbb{H})$ that is smooth over the reflex field E. For any embedding $\iota^\circ : E \hookrightarrow \mathbb{C}$ over $\iota \in \Sigma_\infty$, we have the following ι°-adic uniformization:

$$\mathsf{Sh}_K(\mathbb{H})^{\mathsf{an}}_{\iota^\circ} \cong H^{(\iota)}(\mathbb{Q})\backslash(\mathscr{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_f)/K).$$

We briefly explain the notation above. Let $V^{(\iota)}$ be the *nearby E-hermitian space* of $\mathbb{V}$ at ι, that is, $V^{(\iota)}$ is the unique E-hermitian space (up to isometry) such that $V^{(\iota)}_v \cong \mathbb{V}_v$ for $v \neq \iota$ but $V^{(\iota)}_\iota$ is of signature $(1,1)$ and $H^{(\iota)} = \mathsf{Res}_{F/\mathbb{Q}}\mathsf{U}(V^{(\iota)})$. We identify $H^{(\iota)}(\mathbb{A}_f)$ and $\mathbb{H}(\mathbb{A}_f)$ through the corresponding hermitian spaces. Let $\mathscr{D}^{(\iota^\circ)}$ be the hermitian domain consisting of all negative $\mathbb{C}$-lines in $V^{(\iota)}_\iota$ whose complex structure is given by the action of $F_\iota \otimes_F E$, which is isomorphic to $\mathbb{C}$ via ι°. The group $H^{(\iota)}(\mathbb{Q})$ diagonally acts on $\mathscr{D}^{(\iota^\circ)}$ and $\mathbb{H}(\mathbb{A}_f)/K$ via the obvious way. In fact, $\mathscr{D}^{(\iota^\circ)}$ is canonically identified with the $H^\iota(\mathbb{R})$-conjugacy class of the Hodge map $h^{(\iota)} : \mathbb{S} = \mathsf{Res}_{\mathbb{C}/\mathbb{R}}\boldsymbol{G}_{\mathsf{m},\mathbb{C}} \to H^{(\iota)}_{\mathbb{R}} \cong \mathsf{U}(1,1)_{\mathbb{R}} \times \mathsf{U}(2,0)^{d-1}_{\mathbb{R}}$ given by

$$h^{(\iota)}(z) = \left(\begin{pmatrix} 1 & \\ & \bar{z}/z \end{pmatrix}, \mathbf{1}_2, \ldots, \mathbf{1}_2\right).$$

The Shimura curve $\mathsf{Sh}_K(\mathbb{H})$ is nonproper if and only if $F = \mathbb{Q}$ and $\Sigma(\mathbb{V}) = \{\infty\}$. In this case, we can add cusps to make it proper. We denote by M_K the compactified (resp. original) Shimura curve if $\mathsf{Sh}_K(\mathbb{H})$ is nonproper (resp. proper) and by M the projective system of $(M_K)_K$ with respect to the projection $\pi^{K'}_K : M_{K'} \to M_K$. On each M_K, we have a Hodge bundle $\mathscr{L}_K \in \mathsf{Pic}(M_K)_{\mathbb{Q}}$ which is ample. They are

compatible under pull-backs of $\pi_K^{K'}$, hence define an element

$$\mathscr{L} \in \mathrm{Pic}(M)_{\mathbb{Q}} := \varinjlim_K \mathrm{Pic}(M_K)_{\mathbb{Q}}.$$

Now we briefly recall the construction of *Kudla's special cycles* $Z(x)_K$ and *generating series* $Z_\phi(g)$ (see, for example, [Kudla 1997]). Here, we will make a consistent formulation as in [Liu 2011, Section 3A]. We say $x \in \mathbb{V}$ is *admissible* if $(x,x) \in E$ is totally positive definite. For x admissible, we define $Z(x)_K$ in the following manner: under the uniformization at some ι, it is represented by the points $(z, h_1 h) \in \mathscr{D}^{(\iota^\circ)} \times \mathbb{H}(\mathbb{A}_f)$ where $h \in \mathbb{H}^{\mathrm{der}}(\mathbb{A}_f)$ as in [Liu 2011, Lemma 3.1] (with respect to ι), $z \perp hx$, and h_1 fixes hx. The cycle (actually a divisor) is in fact defined over E and hence independent of ι. We define $Z(x)_K = c_1(\mathscr{L}_K^\vee)$ when $x = 0$ and 0 for all other x. In any case, $Z(x)_K$ defines an element in $\mathrm{CH}^1(M_K)_{\mathbb{Q}}$ which only depends on the class $K\langle x\rangle$.

As in [Liu 2011, Section 3A], we define a subspace $\mathscr{S}(\mathbb{V}_\iota)^{\mathsf{U}_\iota} \subset \mathscr{S}(\mathbb{V}_\iota)$ which consists of functions of the form

$$P(T(x))e^{-2\pi T(x)},$$

where P is a polynomial function on $\mathrm{Her}_1(\mathbb{C}) = \mathbb{R}$. It is a $(\mathrm{Lie}H_{r,\iota}, \mathscr{K}_{r,\iota})$-module which is generated by the Gaussian

$$\phi_\infty^0(x) = e^{-2\pi T(x)}.$$

Let

$$\mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty} = \Big(\bigotimes_{\iota\in\Sigma_\infty} \mathscr{S}(\mathbb{V}_\iota)^{\mathsf{U}_\iota}\Big) \otimes \mathscr{S}(\mathbb{V}_f), \quad \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K} = \Big(\bigotimes_{\iota\in\Sigma_\infty} \mathscr{S}(\mathbb{V}_\iota)^{\mathsf{U}_\iota}\Big) \otimes \mathscr{S}(\mathbb{V}_f)^K$$

for an open compact subgroup K of $\mathbb{H}(\mathbb{A}_f)$. Recall that we have the Weil representation $\omega_\chi = \omega_{\chi,\psi}$ of $H'(\mathbb{A}_F)$ on $\mathscr{S}(\mathbb{V})$ (see Section 2A) with $\chi : E^\times\backslash\mathbb{A}_E^\times$ such that $\chi\big|_{\mathbb{A}_F^\times} = 1$. Associated to this χ, we define a sequence $\mathfrak{k}^\chi = (\mathfrak{k}_\iota^\chi)_\iota \in \mathbb{Z}^{\Sigma_\infty}$ determined by $\chi_\iota(z) = z^{\mathfrak{k}_\iota^\chi}$ for $z \in E_\iota^{1,\times} = \mathbb{C}^{\times,1}$. In particular, $\mathfrak{k}_\iota^\chi$ are all even.

Let us recall the definition of Kudla's generating series and define the (modified) compactified one as in [Liu 2011, Section 3C]. They are

$$Z_\phi(g) = \sum_{x\in K\backslash\mathbb{V}_f} \omega_\chi(g)\phi(T(x),x)Z(x)_K,$$

$$Z_\phi^\sim(g) = \begin{cases} Z_\phi(g) & \text{if } \mathrm{Sh}(\mathbb{H})_K \text{ is proper,} \\ Z_\phi(g) + W_0\big(\frac{1}{2}, g, \phi\big)c_1(\mathscr{L}_K^\vee) & \text{if not,} \end{cases}$$

as series with values in $\mathrm{CH}^1(M_K)_{\mathbb{C}}$ for $\phi \in \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K}$ and $g \in H'(\mathbb{A}_F)$, where $W_0(s,g,\phi) = \prod_v W_0(s, g_v, \phi_v)$ which is holomorphic at $s = \frac{1}{2}$. Here for $\phi = \phi_\infty\phi_f$, we denote $\phi(T(x),x) = \phi_\infty(y)\phi_f(x)$ for any $y \in \mathbb{V}_\infty$ with $T(y) = T(x)$ which does not depend on the choice of y. This makes sense since $Z(x)_K \neq 0$ only for x

admissible or equal to 0 and hence $T(x)$ is totally positive definite or 0. It is easy to see that $Z_\phi(g)$ and $Z_\phi^\sim(g)$ are compatible under pull-backs of $\pi_K^{K'}$, and hence define series with values in $\mathrm{CH}^1(M)_\mathbb{C} := \varinjlim_K \mathrm{CH}^1(M_K)_\mathbb{C}$. Readers may view the modification in the nonproper case as an analogy of the classical Eisenstein series $G_2(\tau)$ (which is not a modular form!). It becomes modular if we add a term $-\pi/\Im\tau$ at the price of being nonholomorphic (see, for example, [Diamond and Shurman 2005, p. 18]).

Now we apply the argument of the previous section to the curve M_K. The cycles whose heights we want to compute are the generating series $Z_\phi^\sim(g)$ which are not necessarily cohomologically trivial. We use the dual of the Hodge bundle $\mathscr{L} = (\mathscr{L}_K)_K \in \mathrm{Pic}(M)$ to modify as in the above section. The metric of $\mathscr{L}_{\iota^\circ,\mathbb{C}}$ for some $\iota^\circ \in \Sigma_\infty^\circ$ over $\iota \in \Sigma_\infty$ is the one descended from the H'_ι-invariant metric

$$\|v\|_{\iota^\circ} = \tfrac{1}{2}(v, v)_\iota$$

for $v \in V_\iota^{(\iota)}$ and the hermitian form $(\,\cdot\,,\cdot\,)_\iota$ of $V^{(\iota)}$ at ι. We denote by $\overline{\mathscr{L}} = (\overline{\mathscr{L}}_K)_K \in \widehat{\mathrm{Pic}}(M)$ the corresponding metrized line bundle. Since $\mathscr{L}$ is ample, $\deg c_1(\mathscr{L}_K) \neq 0$. For $\phi \in \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K}$ and $g \in H'(\mathbb{A}_F)$, we define *the arithmetic theta series* as

$$\Theta_\phi(g) = Z_\phi^\sim(g) - \frac{\deg Z_\phi^\sim(g)}{\deg c_1(\mathscr{L}_{K'}^\vee)} c_1(\mathscr{L}_{K'}^\vee)$$

on any curve $M_{K'}$ with $K' \subset K$. The ratio

$$D(g, \phi) := \frac{\deg Z_\phi^\sim(g)}{\deg c_1(\mathscr{L}_{K'}^\vee)}$$

is independent of the choice of K'.

Definition 3.1. The series $\Theta_\phi(g)$ is called the *arithmetic theta series*. It is a $\mathrm{CH}^1(M)^0_\mathbb{C}$-valued automorphic form of $H'(\mathbb{A}_F)$ by Corollary 3.3.

Degree of the generating series. In this subsection, we will compute the degree function $D(g, \phi)$. From $\phi \in \mathscr{S}(\mathbb{V})^{\mathsf{U}_\infty K}$, which is decomposable, we can form an Eisenstein series

$$E(s, g, \phi) = \sum_{\gamma \in P'(F)\backslash H'(F)} \omega_\chi(\gamma g)\phi(0)\lambda_{P'}(\gamma g)^{s-1/2}$$

on $H'(\mathbb{A}_F)$, which is absolutely convergent if $\Re(s) > \frac{1}{2}$ and has a meromorphic continuation to the entire complex plane. We take Tamagawa measures (with respect to ψ) dh on $\mathbb{H}(\mathbb{A})$, $d\bar{h}$ on $\mathbb{A}_E^{\times,1} = \mathbb{H}/\mathbb{H}^{\mathrm{der}}(\mathbb{A})$, and dh_x on $\mathbb{H}(\mathbb{A})_x$ which is the stabilizer of $x \in \mathbb{V}$. Now for any $v \in \Sigma$, let $b \in F_v^\times$ such that $\Omega_b := \{x \in \mathbb{V}_v \mid T(x) = b\} \neq \varnothing$. Then the local Whittaker integral $W_b(s, e, \phi_v)$ has a holomorphic continuation to

the entire complex plane and $W_b\big(\frac{1}{2}, e, \phi_v\big)$ is not identically zero. Hence we have an N_v-intertwining map

$$\mathcal{S}(\mathbb{V}_v) \to \mathbb{C}_{N_v,\phi_b}, \quad \phi_v \mapsto W_b\big(\tfrac{1}{2}, e, \phi_v\big).$$

On the other hand, by [Rallis 1987, Lemma 4.2] for v finite and [Rallis 1987, Lemma 4.2] and [Kudla and Rallis 1994, Proposition 2.10] for v infinite (see also [Ichino 2004, Proposition 6.2]), we have

$$W_b\big(\tfrac{1}{2}, e, \phi_v\big) = \gamma_{\mathbb{V}_v} \int_{\Omega_b} \phi_v(x)\, d\mu_{v,b}(x) \tag{3-3}$$

for the quotient measure $d\mu_{v,b} = dh_v/dh_{v,x}$ on Ω_b for any $x \in \Omega_b$.

Proposition 3.2. *The Eisenstein series $E(s, g, \phi)$ is holomorphic at $s = \frac{1}{2}$ and $D(g, \phi) = E(s, g, \phi)\big|_{s=\frac{1}{2}}$.*

Proof. We can assume that ϕ is decomposable. For $b \in F^\times$, let

$$D_b(g, \phi) = \frac{1}{\deg c_1(\mathscr{L}_{K'}^\vee)} \sum_{\substack{x \in K' \backslash \mathbb{V}_f \\ T(x)=b}} \omega_\chi(g)\phi(b, x) \deg Z(x)_{K'}$$

be the b-th Fourier coefficient of $D(g, \phi)$. Now we compute the degree of $Z(x)_{K'}$ when $T(x) = b$ is totally positive. Without lost of generality, let us assume x is contained in the image of some (rational) nearby hermitian space $V^{(\iota)} \hookrightarrow \mathbb{V}_f$ and K' is sufficiently small. The isomorphism $\det : H_x^{(\iota)} \to E^{\times,1}$ induces a surjective map $H_x^{(\iota)} \backslash \mathbb{H}(\mathbb{A}_f)_x / (K' \cap \mathbb{H}(\mathbb{A}_f)_x) \to E^{\times,1} \backslash \mathbb{A}_{E,f}^{\times,1} / \det K'$. Hence

$$\deg Z(x)_{K'} = \left| \frac{\det K'}{K' \cap \mathbb{H}(\mathbb{A}_f)_x} \right| = \frac{\mathrm{vol}(\det K', d\bar{h}_f)}{\mathrm{vol}(K' \cap \mathbb{H}(\mathbb{A}_f)_x, dh_{f,x})}.$$

When $b \neq 0$ and is not totally positive, $\deg Z(x)_{K'} = 0$ by definition. Hence for b totally positive,

$$\begin{aligned} D_b(g, \phi) &= \frac{1}{\deg c_1(\mathscr{L}_{K'}^\vee)} \sum_{\substack{x \in K' \backslash \mathbb{V}_f \\ T(x)=b}} \omega_\chi(g)\phi(b, x) \frac{\mathrm{vol}(\det K')}{\mathrm{vol}(K' \cap \mathbb{H}(\mathbb{A}_f)_x)} \\ &= \frac{\omega_\chi(g_\infty)\phi_\infty(b)\, \mathrm{vol}(\det K')}{\deg c_1(\mathscr{L}_{K'}^\vee)\, \mathrm{vol}(K')} \int_{\substack{x \in \mathbb{V}_f \\ T(x)=b}} \omega_\chi(g)\phi(x)\, d\mu_b(x) \\ &= \frac{\omega_\chi(g_\infty)\phi_\infty(b)\, \mathrm{vol}(\det K')}{\deg c_1(\mathscr{L}_{K'}^\vee)\, \mathrm{vol}(K')} \prod_{v \in \Sigma_f} \int_{\Omega_b} \omega_\chi(g_v)\phi_v(x)\, d\mu_{v,b}(x) \end{aligned} \tag{3-4}$$

and $D_b(g, \phi) = 0$ otherwise.

On the other hand, $E_b(s, g, \phi)$ is holomorphic at $s = \frac{1}{2}$ for $b \neq 0$. For b not totally positive, $E_b(s, g, \phi)\big|_{s=\frac{1}{2}} = 0$; otherwise,

$$E_b(s, g, \phi)\big|_{s=\frac{1}{2}} = W_b\big(\tfrac{1}{2}, g, \phi\big) = \prod_{v \in \Sigma} W_b\big(\tfrac{1}{2}, g_v, \phi_v\big)$$

and so, by (3-3),

$$\begin{aligned} E_b(s, g, \phi)\big|_{s=\frac{1}{2}} &= \prod_{v \in \Sigma} \gamma_{\mathbb{V}_v} \int_{\Omega_b} \omega_\chi(g_v)\phi_v(x)\, d\mu_{v,b}(x) \\ &= -\mathrm{vol}(\Omega_\infty)\omega_\chi(g_\infty)\phi_\infty(b) \prod_{v \in \Sigma_f} \int_{\Omega_b} \omega_\chi(g_v)\phi_v(x)\, d\mu_{v,b}(x), \quad (3\text{-}5) \end{aligned}$$

where $\mathrm{vol}(\Omega_\infty) = \mathrm{vol}(\Omega_{\infty,b})$ for any totally positive b. Let

$$D = \frac{\mathrm{vol}(\det K')}{\mathrm{vol}(\Omega_\infty) \deg c_1(\mathscr{L}_{K'}) \mathrm{vol}(K')}.$$

Now we compute the constant term

$$D_0(g, \phi) = \omega_\chi(g)\phi(0) + W_0\big(\tfrac{1}{2}, g, \phi\big).$$

On the other hand, the constant term of $E\big(\frac{1}{2}, g, \phi\big)$ is

$$E_0\big(\tfrac{1}{2}, g, \phi\big) = \omega_\chi(g)\phi(0) + W_0\big(\tfrac{1}{2}, g, \phi\big).$$

Here the intertwining term $W_0(\frac{1}{2}, g, \phi)$ is nonzero only if $\mathrm{Sh}_K(\mathbb{H})$ is nonproper, that is, $|\Sigma(\mathbb{V})| = 1$. Now, if $\mathrm{Sh}_K(\mathbb{H})$ is proper, then we can apply the theorem of modularity of the generating series (see [Liu 2011, Theorem 3.5]) to see that $D(g, \phi)$ is already an automorphic form. Comparing the ratio of the constant term and nonconstant terms, we find that $D = 1$. Second, if $\mathrm{Sh}_K(\mathbb{H})$ is not proper, we calculate the degree of the Hodge bundle in the classical way on modular curves and find that $D = 1$.

Let $E(g, \phi) = E(s, g, \phi)\big|_{s=\frac{1}{2}} - W_0(\frac{1}{2}, g, \phi)$; then

$$\Theta_\phi(g) = Z_\phi(g) - E(g, \phi)c_1(\mathscr{L}_K^\vee).$$

If $|\Sigma(\mathbb{V})| > 1$, $W_0(\frac{1}{2}, g, \phi) = 0$; otherwise, it equals $C(\tilde{\chi} \circ \det)$, where C is a constant and $\tilde{\chi}$ is the descent of χ to $\mathbb{A}_E^{\times,1}$. Precisely, $\tilde{\chi}(x) = \chi(e_x)$ for any $e_x \in \mathbb{A}_E^\times$ such that $x = e_x/e_x^\tau$. In any case, $E(g, \phi)$ is a linear combination of an Eisenstein series and an automorphic character for g in $P'_\infty H'(\mathbb{A}_{f,F})$. □

From this computation, we have the following corollary of the *modularity of the generating series in the compactified case*:

Corollary 3.3. *For any linear functional $\ell \in \mathrm{CH}^1(M)^*_{\mathbb{C}}$, $\ell(Z_\phi^\sim)(g)$ (and hence $\ell(\Theta_\phi)(g)$ as well) is absolutely convergent and an automorphic form of $H'(\mathbb{A}_F)$.*

Proof. We only need to prove that $Z^\sim_\phi(\gamma g) = Z^\sim_\phi(g) \in \mathrm{Pic}(M)_\mathbb{C}$ for any $\gamma \in H'(\mathbb{Q})$. But by [Liu 2011, Theorem 3.5] and the fact that the Hodge bundle is supported on the cusps, $Z^\sim_\phi(\gamma g) = Z^\sim_\phi(g)$ in $\mathrm{CH}^1(\mathrm{Sh}_K(\mathbb{H}))_\mathbb{C}$, so their difference must be supported on the set of cusps. By a theorem due to Manin [1972] and Drinfeld [1973], which posits that any two cusps are the same in $\mathrm{CH}^1(M_K)_\mathbb{C}$, we have an exact sequence

$$\mathbb{C} \to \mathrm{CH}^1(M_K)_\mathbb{C} \to \mathrm{CH}^1(\mathrm{Sh}_K(\mathbb{H}))_\mathbb{C} \to 0.$$

Hence we only need to prove that $\deg Z^\sim_\phi(\gamma g) = \deg Z^\sim_\phi(g)$, which is true by the above proposition. □

Definition 3.4. Let π be an irreducible cuspidal automorphic representation of $H'(\mathbb{A}_F)$. For any cusp form $f \in \pi$ and $\phi \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty}$, the integral

$$\Theta^f_\phi = \int_{H'(F)\backslash H'(\mathbb{A}_F)} f(g)\Theta_\phi(g)\, dg \in \mathrm{CH}^1(M)^0_\mathbb{C}$$

is called the *arithmetic theta lifting* of f which is a divisor on (compactified) Shimura curves. The original idea of this construction comes from Kudla [2003, Section 8; 2006, Section 9.1]. He constructed the arithmetic theta series as an Arakelov divisor on a certain integral model of a Shimura curve.

Geometric kernel functions. For $\Phi = \sum \phi_{i,1} \otimes \phi_{i,2}$ with $\phi_{i,j} \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty K}$, we define the *geometric kernel function* associated to the test function Φ to be

$$\mathbb{E}(g_1, g_2; \Phi) := \mathrm{vol}(K') \sum \langle \Theta_{\phi_{i,1}}(g_1), \Theta_{\phi_{i,2}}(g_2) \rangle^{K'}_{\mathsf{NT}},$$

where the measure giving $\mathrm{vol}(K')$ is defined in [Liu 2011, Theorem 4.20], and the superscript K' means that we are taking the Néron–Tate height pairing on the curve $M_{K'}$ for some $K' \subset K$. The function is independent of what K' we choose. By Corollary 3.3, $\mathbb{E}(g_1, g_2; \Phi) \in \mathcal{A}(H' \times H')$, the space of automorphic forms of the group $H'(\mathbb{A}_F) \times H'(\mathbb{A}_F)$. Now let us just work over M_K and choose a regular model $\mathcal{M}_K$ of it. We fix an arithmetic line bundle $\widehat{\omega}_K$ to extend $\overline{\mathcal{L}}^\vee_K$. Of course, the metric on $\widehat{\omega}_K$ is same as that on $\overline{\mathcal{L}}^\vee_K$.

Now since the map $p_{\mathcal{M}_K}$ (see (3-1)) is surjective, we can fix an inverse linear map $p^{-1}_{\mathcal{M}_K}$ and write

$$\widehat{\Theta}_\phi(g) := p^{-1}_{\mathcal{M}_K}(\Theta_\phi(g)) = ([Z_\phi(g)]^{\mathsf{Zar}}, g_{\iota^\circ}) + (\mathcal{V}_\phi(g), 0) - E(g, \phi)\widehat{\omega}_K,$$

where g_{ι° is an $\overline{\mathcal{L}}_K$-admissible Green's function of $Z_\phi(g)$ and $\mathcal{V}_\phi(g)$ is the sum of (finitely many) vertical components supported on special fibers. We also write simply

$$\widehat{Z}_\phi(g) = ([Z_\phi(g)]^{\mathsf{Zar}}, g_{\iota^\circ}) + (\mathcal{V}_\phi(g), 0).$$

Then we have for $\phi_i \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty K}$ $(i = 1, 2)$,

$$\begin{aligned}
&\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2)\\
&\quad = \mathrm{vol}(K)\langle \Theta_{\phi_1}(g_1), \Theta_{\phi_2}(g_2)\rangle^K_{\mathsf{NT}}\\
&\quad = -\mathrm{vol}(K)\langle \widehat{\Theta}_{\phi_1}(g_1), \widehat{\Theta}_{\phi_2}(g_2)\rangle_{\mathsf{GS}}\\
&\quad = -\mathrm{vol}(K)\langle \widehat{Z}_{\phi_1}(g_1) - E(g_1, \phi_1)\widehat{\omega}_K, \widehat{Z}_{\phi_2}(g_2) - E(g_2, \phi_2)\widehat{\omega}_K\rangle_{\mathsf{GS}}\\
&\quad = -\mathrm{vol}(K)\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathsf{GS}} + E(g_1, \phi_1)\,\mathrm{vol}(K)\langle \widehat{\omega}_K, \widehat{\Theta}_{\phi_2}(g_2)\rangle_{\mathsf{GS}}\\
&\qquad\qquad + E(g_2, \phi_2)\,\mathrm{vol}(K)\langle \widehat{\Theta}_{\phi_1}(g_1), \widehat{\omega}_K\rangle_{\mathsf{GS}}\\
&\qquad\qquad + E(g_1, \phi_1)E(g_2, \phi_2)\,\mathrm{vol}(K)\langle \widehat{\omega}_K, \widehat{\omega}_K\rangle_{\mathsf{GS}}, \qquad (3\text{-}6)
\end{aligned}$$

where the Gillet–Soulé pairing are taken on the model $\mathcal{M}_K$. By Corollary 3.3, $A(g, \phi) := \mathrm{vol}(K)\langle \widehat{\omega}_K, \widehat{\Theta}_\phi(g)\rangle_{\mathsf{GS}}$ is an automorphic form of H' which may depend on K, and also on the model $\mathcal{M}_K$ since we do not require any canonicality of $p^{-1}_{\mathcal{M}_K}$. Let $C = \mathrm{vol}(K)\langle \widehat{\omega}_K, \widehat{\omega}_K\rangle_{\mathsf{GS}}$. Then

$$\begin{aligned}
(3\text{-}6) = -\mathrm{vol}(K)\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathsf{GS}} + E(g_1, \phi_1)A(g_2, \phi_2)\\
+ A(g_1, \phi_1)E(g_2, \phi_2) + CE(g_1, \phi_1)E(g_2, \phi_2). \qquad (3\text{-}7)
\end{aligned}$$

Now we assume that ϕ_1 and ϕ_2 are decomposable and $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathsf{reg}}$ for some $v \in \Sigma_f$. Then $Z_{\phi_1}(g_1)$ and $Z_{\phi_2}(g_2)$ will not intersect on the generic fiber if $g_i \in P'_v H'(\mathbb{A}_F^v)$ $(i = 1, 2)$. Then

$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathsf{GS}} = \sum_{v^\circ \in \Sigma^\circ} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}, \qquad (3\text{-}8)$$

where the intersection $\langle \cdot\,, \cdot \rangle_{v^\circ}$ is taken on the local model $\mathcal{M}_{K;\mathfrak{p}^\circ} := \mathcal{M}_K \times_{\mathbb{O}_E} \mathbb{O}_{E_{\mathfrak{p}^\circ}}$ if $v^\circ = \mathfrak{p}^\circ$ is finite and $M_{K,\iota^\circ}(\mathbb{C})$ if $v^\circ = \iota^\circ$ is infinite. Combining (3-7) and (3-8), we have for such ϕ_i and g_i $(i = 1, 2)$,

$$\begin{aligned}
\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) = -\mathrm{vol}(K) \sum_{v^\circ \in \Sigma^\circ} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ} + E(g_1, \phi_1)A(g_2, \phi_2)\\
+ A(g_1, \phi_1)E(g_2, \phi_2) + CE(g_1, \phi_1)E(g_2, \phi_2). \qquad (3\text{-}9)
\end{aligned}$$

4. Comparison at finite places: good reduction

4A. ***Nonarchimedean Whittaker integrals.*** In this section, we calculate certain Whittaker integrals $W_T(s, g, \Phi)$ and their derivatives (at $s = 0$) at a nonarchimedean place when T has rank 2.

Let $F/\mathbb{Q}_p$ be a finite extension and E/F a quadratic extension with $\mathsf{Gal}(E/F) = \{1, \tau\}$. We fix a uniformizer ϖ of F and let q be the cardinality of the residue field of F. Let V^+ (resp. V^-) be the two-dimensional E-hermitian space with $\epsilon(V^+) = 1$ (resp. $\epsilon(V^-) = -1$); it is unique up to isometry. Set $H^\pm = \mathsf{U}(V^\pm)$, and let $\Lambda^\pm$

be a maximal $\mathcal{O}_E$-lattice in $V^{\pm}$ where the hermitian form takes values in $\mathcal{O}_E$. Let $\phi^{0\pm} \in \mathcal{S}(V^{\pm})$ (resp. $\Phi^{0\pm} \in \mathcal{S}((V^{\pm})^2)$) be the characteristic function of $\Lambda^{\pm}$ (resp. $(\Lambda^{\pm})^2$), and let $K_0^{\pm}$ be the stabilizer of $\Lambda^{\pm}$ in $H^{\pm}$ which is a maximal compact subgroup. Recall that we have local reductive groups $H' \cong H_1$, $H'' \cong H_2$, $P, \ldots$.

Now, we assume that E/F is unramified and $p > 2$. Let ψ be the unramified character of F. For nonsingular $T \in \mathrm{Her}_2(E)$, we consider the Whittaker integral,

$$W_T(s, g, \Phi^{0+}) = \int_{\mathrm{Her}_2(E)} \varphi_{\Phi^{0+},s}(wn(u)g)\psi_T(n(u))^{-1}\, du, \tag{4-1}$$

for $\Re(s) > 1$, where du is the self-dual measure with respect to ψ. We write $g = n(b)m(a)k$ under the Iwasawa decomposition of H''. Then

$$\begin{aligned}(4\text{-}1) &= \int_{\mathrm{Her}_2(E)} \omega''_{\mathbf{1}}(wn(u)n(b)m(a)k)\Phi^{0+}(0)\lambda_P(wn(u)n(b)m(a)k)^s\psi(-\operatorname{tr} Tu)\, du \\ &= \psi(\operatorname{tr} Tb)\int_{\mathrm{Her}_2(E)} \lambda_P(wn(u)m(a))^s\psi(-\operatorname{tr} Tu)\, du \\ &= \psi(\operatorname{tr} Tb)|\det a|_E^{1-s} W_{{}^t a^\tau T a}(s, e, \Phi^{0+}).\end{aligned}$$

Hence we only need to consider the integral $W_T(s, e, \Phi^{0+})$. If $T \notin \mathrm{Her}_2(\mathcal{O}_E)$, then $W_T(s, e, \Phi^{0+})$ is identically 0. For $T \in \mathrm{Her}_2(\mathcal{O}_E)$, it is known (see [Kudla 1997, Appendix], for example) that for $r \in \mathbb{Z}$ and $r > 1$, $W_T(r, e, \Phi^{0+}) = \gamma_{V^+}\alpha_F(\mathbf{1}_{2+r}, T)$ where γ_{V^+} is the Weil constant and α_F is the classical representation density (for hermitian matrices). From [Hironaka 1999], we see that for $r \geq 0$

$$\alpha_F(\mathbf{1}_{2+r}, T) = P_F(\mathbf{1}_2, T; (-q)^{-r})$$

for a polynomial $P_F(\mathbf{1}_2, T; X) \in \mathbb{Q}[X]$. By analytic continuation, we see that

$$W_T(s, e, \Phi^{0+}) = \gamma_{V^+}P_F(\mathbf{1}_2, T; (-q)^{-s}).$$

If $\mathrm{ord}(\det T)$ is odd, that is, if T cannot be represented by V^+, then we know that $W_T(0, e, \Phi^{0+}) = P_F(\mathbf{1}_2, T; 1) = 0$. Taking the derivative at $s = 0$, we have

$$W'_T(0, e, \Phi^{0+}) = -\gamma_{V^+}\log q \cdot \frac{d}{dX}P_F(\mathbf{1}_2, T; X)\Big|_{X=1}.$$

Proposition 4.1 [Hironaka 1999]. *If T is $\mathrm{GL}_2(\mathcal{O}_E)$-equivalent to $\mathrm{diag}[\varpi^a, \varpi^b]$ with $0 \leq a < b$, then*

$$P_F(\mathbf{1}_2, T; X) = (1 + q^{-1}X)(1 - q^{-2}X)\sum_{l=0}^{a}(qX)^l\left(\sum_{k=0}^{a+b-2l}(-X)^k\right).$$

Corollary 4.2. *If $a + b$ is odd, then*

$$W'_T(0, e, \Phi^{0+}) = \gamma_{V^+}b_2(0)^{-1}\log q \cdot \frac{1}{2}\sum_{l=0}^{a} q^l(a + b - 2l + 1).$$

4B. ***Integral models.*** We now introduce the integral model of the Shimura curves M_K with full-level structure at $\mathfrak{p}$ and integral special subschemes. First, we fix some notation for Sections 4 and 5.

- For any rational prime p, we fix an isomorphism $\iota_{(p)} : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ once for all.
- Let ι_i $(i = 1, \dots, d)$ be all the embeddings of F into $\mathbb{C}$, and let ι_i° and $\iota_i^\bullet$ be the embeddings of E into $\mathbb{C}$ that induce ι_i.
- For $\mathfrak{p}$ a finite place of F, let $\mathfrak{p}^\circ$ be that of E over $\mathfrak{p}$ if $\mathfrak{p}$ is nonsplit and $\mathfrak{p}^\circ$ and $\mathfrak{p}^\bullet$ be those of E if $\mathfrak{p}$ is split. We fix a uniformizer ϖ of $F_\mathfrak{p}$.
- For a number field F, $T_F = \mathrm{Res}_{F/\mathbb{Q}} \boldsymbol{G}_{\mathsf{m},F}$ and $T^1_{F'} = \mathrm{Res}_{F/\mathbb{Q}} F'^{\times,1}$ for any quadratic extension F'/F where $F'^{\times,1} = \ker[\mathrm{Nm} : F'^\times \to F^\times]$. If F is totally real, then F^+ is the set of all totally positive elements.
- For any finite extension $L/\mathbb{Q}_p$ of local fields with ring of integers $\mathbb{O}_L$ and maximal ideal $\mathfrak{q} \subset \mathbb{O}_L$, let U_L^s be the subgroup of $\mathbb{O}_L^\times$ congruent to 1 mod $\mathfrak{q}^s$. Denote by L^{nr} the maximal unramified extension of L and $\hat{L}$ its completion whose ring of integers is $\mathbb{O}_{\hat{L}}$. Let L^s be the finite extension of $L^{\mathrm{nr}} = L^0$ corresponding to U_L^s through local class field theory and $\hat{L}^s$ its completion.
- Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$.

If the Shimura curve $\mathrm{Sh}_K(\mathbb{H})$ is nonproper (then $F = \mathbb{Q}$), we add cusps to make it proper — this holds also for integral models (see Remark 4.10 and [Katz and Mazur 1985] for details). In what follows, we will not pay any attention to these cusps since they will not affect our later computations. In the first two subsections, we recall some results in [Carayol 1986] which are useful for us.

Change of Shimura data. Let $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ $(1 \le r \le d)$ be all places of F dividing p and $\mathfrak{p}^\circ$ the place of E above $\mathfrak{p}$. We assume that the embedding

$$\iota_{(p)} \circ \iota_1^\circ : E \hookrightarrow \mathbb{C}_p$$

induces the place $\mathfrak{p}^\circ$. As before, we suppress ι_1 for the nearby objects. Hence, we have the hermitian space V over E of dimension two, whose signature is $(1, 1)$ at ι_1 and $(2, 0)$ elsewhere, the unitary group H, and the Shimura curve $M_K = \mathrm{Sh}_K(H, X)$ for a sufficiently small open compact subgroup $K \subset H(\mathbb{A}_f)$, which is a smooth projective curve defined over $\iota_1^\circ(E)$. Here X is the conjugacy class of the Hodge map $h : \mathbb{S} \to H_\mathbb{R}$ defined by

$$z = x + iy \mapsto \left[\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)^{-1} \times z, 1, \dots, 1\right] \in H(\mathbb{R}) \subset (\mathrm{GL}_2(\mathbb{R}) \times_{\mathbb{R}^\times} \mathbb{C}^\times) \times (\mathbb{H}^\times \times_{\mathbb{R}^\times} \mathbb{C}^\times)^{d-1},$$

where we identify $T_E(\mathbb{R})$ with $(\mathbb{C}^\times)^d$ via $(\iota_1^\circ, \dots, \iota_d^\circ)$. We denote by $\nu : H \to T_E^1$ the determinant map. We have the zero-dimensional Shimura variety

$$L_K = \mathrm{Sh}_{\nu(K)}(T_E^1, \nu(X))$$

and a smooth morphism (also denoted by) $\nu : M_K \to L_K$ between $\iota_1^\circ(E)$-schemes such that the fiber of each geometric point is connected.

Now let us define a subgroup $K_{\mathfrak{p},n}$ of $\mathrm{U}(V_\mathfrak{p})$. Recalling the notation in the last section and assuming that $V_\mathfrak{p} \cong V^\pm$, we have the lattice $\Lambda^\pm$ (if $\mathfrak{p}$ is split, we only have the positive one). For any integer $n \geq 0$, we define $K_{\mathfrak{p},n}$ to be the subgroup of $K_0^\pm$ whose elements act on $\Lambda^\pm/\varpi^n\Lambda^\pm$ trivially. Then $K_{\mathfrak{p},0} = K_0^\pm$ is a maximal compact subgroup. For $K = K_{\mathfrak{p},n} \times K^\mathfrak{p}$, we write $M_{n,K^\mathfrak{p}}$ (resp. $L_{n,K^\mathfrak{p}}$) for M_K (resp. L_K).

Since the Shimura datum (H, X) is not of PEL type when $F \neq \mathbb{Q}$ (even when $F = \mathbb{Q}$, we still need another PEL datum for later computation, see Remark 4.10), we need a variation in order to obtain the moduli interpretation, and hence the integral model. This is analogous to the case considered in [Carayol 1986; Zhang 2001a; 2001b; Yuan et al. 2011] and we refer for the detailed proof of various facts to [Carayol 1986]. We choose a negative number $\lambda \in \mathbb{Q}$ such that the extension $\mathbb{Q}(\sqrt{\lambda})$ is split at p and the CM extension $F^\dagger = F(\sqrt{\lambda})/F$ with $\mathrm{Gal}(F^\dagger/F) = \{1, \tau^\dagger\}$ is not isomorphic to E/F. We fix a square root of λ in $\mathbb{C}$ with positive imaginary part, say λ', and a square root of λ in $\mathbb{Q}_p$, λ_p. Let ι_i^1 (resp. ι_i^2) be the embeddings of $F^\dagger$ into $\mathbb{C}$ above ι_i $(i = 1, \ldots, d)$ which sends $\sqrt{\lambda}$ to λ' (resp. $-\lambda'$). Since p is split in $\mathbb{Q}(\sqrt{\lambda})$, each $\mathfrak{p}_i$ $(i = 1, \ldots, r)$ is split in $F^\dagger$; we denote by $\mathfrak{p}_i^1$ (resp. $\mathfrak{p}_i^2$) the place above $\mathfrak{p}_i$ that sends $\sqrt{\lambda}$ to λ_p (resp. $-\lambda_p$) and assume that $\iota_{(p)} \circ \iota_1^1$ induces $\mathfrak{p}_1^1$.

By the Hasse principle, we see that there is a unique quaternion algebra B over F, up to isomorphism, such that B, as an F-quadratic space (of dimension 4), is isometric to V viewed as an F-quadratic space with the quadratic form $\frac{1}{2}\mathrm{Tr}_{E/F}(\,\cdot\,,\cdot\,)$ where $(\,\cdot\,,\cdot\,)$ is the hermitian form on V. More precisely, for v finite, $B_v = B \otimes_F F_v$ is nonsplit if and only if v is nonsplit and $V_v \cong V^-$. Also, $B_{\iota_1}(\mathbb{R}) \cong \mathrm{Mat}_2(\mathbb{R})$ and $B_{\iota_i}(\mathbb{R}) \cong \mathbb{H}$ for $i > 1$. We identify the two quadratic spaces B and V through a fixed isometry; hence V has both left and right multiplication by B. We fix an embedding $E \hookrightarrow B$ through which the action of E induced from the left multiplication of B coincides with the E-vector space structure of V. Let $G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$ with center $T \cong T_F$ and

$$G^\dagger := G \times_T T_{F^\dagger} \xrightarrow{\nu^\dagger} T \times T^1_{F^\dagger},$$

where $\nu^\dagger$ sends $(g \times z)$ to $(\mathrm{N}mg \cdot zz^{\tau^\dagger}, z/z^{\tau^\dagger})$. Consider the subtorus $T^\dagger = \boldsymbol{G}_{\mathrm{m},\mathbb{Q}} \times T^1_{F^\dagger}$ and let $H^\dagger$ be the preimage of $T^\dagger$ under $\nu^\dagger$. We define the Hodge map $h^\dagger : \mathbb{S} \to H^\dagger_\mathbb{R} \subset G_\mathbb{R} \times_{T_\mathbb{R}} T_{F^\dagger,\mathbb{R}}$ by

$$z = x + iy \mapsto \left[\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)^{-1} \times 1, \mathbf{1}_2 \times z^{-1}, \ldots, \mathbf{1}_2 \times z^{-1}\right]$$

and let $X^\dagger$ be the $H^\dagger(\mathbb{R})$ conjugacy class of $h^\dagger$, where we identify $T_{F^\dagger}(\mathbb{R})$ with $(\mathbb{C}^\times)^d$ through $(\iota_1^1, \ldots, \iota_d^1)$. So we have the Shimura curve $M^\dagger_{K^\dagger} := \mathrm{Sh}_{K^\dagger}(H^\dagger, X^\dagger)$

which is defined over $\iota_1^1(F^\dagger)$ for an open compact subgroup $K^\dagger$ of $H^\dagger(\mathbb{A}_f)$. Similarly, we have the smooth morphism $\nu^\dagger : M^\dagger_{K^\dagger} \to L^\dagger_{K^\dagger}$. Moreover $h^\dagger(i)$ defines a complex structure on V_{ι_1}; hence V_{ι_1} becomes a complex hermitian space of dimension 2 which is isometric to its original complex hermitian space structure from the E-hermitian space V. Then $X^\dagger$ can be identified with the set of negative definite complex lines in V_{ι_1}; and hence $X^\dagger$ is isomorphic to X as hermitian symmetric domains.

As in [Carayol 1986, Section 2.2], we can view $H^\dagger$ as a group of symplectic similitude. In fact, let $B^\dagger = B \otimes_F F^\dagger$ and let $b \mapsto \bar{b}$ be the involution of the second kind on $B^\dagger$ which is the tensor product of the canonical involution of B and the conjugation of $F^\dagger$. Consider the $\mathbb{Q}$-vector space $V^\dagger$ underlying $B^\dagger$. We define a symplectic form by

$$\psi^\dagger(v, w) := \mathrm{Tr}_{F^\dagger/\mathbb{Q}}(\sqrt{\lambda}\mathrm{Tr}_{B^\dagger/F^\dagger}(v\bar{w}))$$

for $v, w \in B^\dagger$. Then $H^\dagger$ can be identified with the group of $B^\dagger$-linear symplectic similitude of $(V^\dagger, \psi^\dagger)$ with the left action given by $h.v = vh^{-1}$ and hence $H^\dagger(\mathbb{Q}_p)$ can be identified with the group $\mathbb{Q}_p^\times \times \prod_{i=1}^r B_{\mathfrak{p}_i}^\times$. For any open compact subgroups $K_p^{\dagger,\mathfrak{p}}$ of $\prod_{i=2}^r B_{\mathfrak{p}_i}^\times$ and $K^{\dagger,p}$ of $H^\dagger(\mathbb{A}_f^p)$, we simply write $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ for $M^\dagger_{K^\dagger}$ where $K^\dagger = \mathbb{Z}_p^\times \times \mathcal{O}_{B_\mathfrak{p}}^\times \times K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p}$ and similarly for $L^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$.

Through $\iota_{(p)}$, we can take the base change of the curve M_K to $E_{\mathfrak{p}^\circ}$ and the curve $M^\dagger_{K^\dagger}$ to $F^\dagger_{\mathfrak{p}_1^\circ} \cong F_\mathfrak{p}$ embedded in $E_{\mathfrak{p}^\circ}$, which are denoted by $M_{K;\mathfrak{p}^\circ}$ and $M^\dagger_{K^\dagger;\mathfrak{p}}$, and similarly for L and $L^\dagger$. Since H and $H^\dagger$ have the same derived subgroup, which is also the derived subgroup of G, we have

Proposition 4.3 [Carayol 1986, Section 4]. *Let $K^\mathfrak{p} \subset H_f^\mathfrak{p} := \mathsf{U}(V \otimes_F \mathbb{A}_{f,F}^\mathfrak{p})$ be an open compact subgroup which is decomposable and sufficiently small. Then there is an open compact subgroup $K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p} \subset \prod_{i=2}^r B_{\mathfrak{p}_i}^\times \times H^\dagger(\mathbb{A}_f^p)$, such that the geometric neutral components $M^\circ_{0,K^\mathfrak{p};\mathfrak{p}^\circ}$ and $M^{\dagger,\circ}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}$ are defined and isomorphic over $E^{\mathrm{nr}}_{\mathfrak{p}^\circ}$.*

Moduli interpretations and integral models. From the Hodge map defined above, we have a Hodge filtration $0 \subset \mathrm{Fil}^0(V_\mathbb{C}^\dagger) = (V_\mathbb{C}^\dagger)^{0,-1} \subset V_\mathbb{C}^\dagger$ and define $t^\dagger(b) = \mathrm{tr}(b; V_\mathbb{C}^\dagger/\mathrm{Fil}^0(V_\mathbb{C}^\dagger)) \in \iota_1^1(F^\dagger)$ for $b \in B^\dagger$. For sufficiently small $K^\dagger$, the curve $M^\dagger_{K^\dagger}$ represents the following moduli functor (see [Kottwitz 1992]) on the category of locally noetherian schemes over $\iota_1^1(F^\dagger)$: for such a scheme S, $M^\dagger_{K^\dagger}(S)$ is the set of equivalence classes of quadruples $(A, \theta, i, \bar{\eta})$, where

- A is an abelian scheme over S of dimension $4d$;
- $\theta : A \to A^\vee$ is a polarization;
- $i : B^\dagger \hookrightarrow \mathrm{End}^0(A)$ is a monomorphism of $\mathbb{Q}$-algebras such that, for all $b \in B^\dagger$, we have $\mathrm{tr}(i(b); \mathrm{Lie}_S(A)) = t^\dagger(b)$ and $\theta \circ i(b) = i(\bar{b})^\vee \circ \theta$;

- $\bar\eta$ is a $K^\dagger$-level structure; that is, for a chosen geometric point s on each connected component of S, $\bar\eta$ is a $\pi_1(S,s)$-invariant $K^\dagger$-orbit of $B^\dagger\otimes\mathbb{A}_f$-linear symplectic similitude $\eta: V^\dagger\otimes\mathbb{A}_f\to \mathrm{H}_1^{\text{ét}}(A_s,\mathbb{A}_f)$, where the pairing on the latter space is the θ-Weil pairing.

Here in the third condition, we view $t^\dagger(b)$ as a constant section of $\mathcal{O}_S$ via the structure map $\iota_1^1(F^\dagger)\to\mathcal{O}_S$. This convention is also applied to other trace conditions appearing later. The two quadruples $(A,\theta,i,\bar\eta)$ and $(A',\theta',i',\bar\eta')$ are equivalent if there is an isogeny $A\to A'$ which takes θ to a $\mathbb{Q}^\times$-multiple of θ', i to i', and $\bar\eta$ to $\bar\eta'$.

Now taking the base change through $\iota_{(p)}$, we obtain the functor $M^\dagger_{K^\dagger;\mathfrak{p}}$ over the completion $[\iota_1^1(F^\dagger)]^\wedge_{\iota_{(p)}}\cong F_{\mathfrak{p}}$. For any $(A,\theta,i,\bar\eta)\in M^\dagger_{K^\dagger;\mathfrak{p}}(S,s)$, $\mathsf{Lie}_S(A)$ is a $B_p^\dagger=B^\dagger\otimes_{\mathbb{Q}}\mathbb{Q}_p$-module. Since the algebra $B_p^\dagger=B\otimes_F(F^\dagger\otimes\mathbb{Q}_p)$ decomposes as

$$B_p^\dagger=B_1^1\oplus B_2^1\oplus\cdots\oplus B_r^1\oplus B_1^2\oplus B_2^2\oplus\cdots\oplus B_r^2,$$

where $B_i^j=B^\dagger\otimes_F F^\dagger_{\mathfrak{p}_i^j}$ is isomorphic to $B_{\mathfrak{p}_i}$ as $F_{\mathfrak{p}_i}$-algebra, the $B_p^\dagger$-module $\mathsf{Lie}_S(A)$ decomposes as

$$\mathsf{Lie}_S(A)\ =\ \bigoplus_{i=1}^r\mathsf{Lie}_i^1(A)\ \oplus\ \bigoplus_{i=1}^r\mathsf{Lie}_i^2(A),$$

while

$$A_{p^\infty}\ =\ \bigoplus_{i=1}^r(A_{p^\infty})_i^1\ \oplus\ \bigoplus_{i=1}^r(A_{p^\infty})_i^2,$$

for the associated p-divisible group. Since the involution $b\mapsto\bar b$ on $B_p^\dagger$ changes the factors B_i^1 and B_i^2, by computing the trace we see that the condition that $\mathrm{tr}(i(b);\mathsf{Lie}_S(A))=t^\dagger(b)$ is equivalent to

$$\mathrm{tr}(b\in B_1^2;\mathsf{Lie}_1^2(A))=\mathrm{Tr}_{B_1^2/F_{\mathfrak{p}}}(b)\text{ and }\mathsf{Lie}_i^2(A)=0\text{ for }i\geq 2. \tag{4-2}$$

Fix a maximal order $\Lambda_i^2=\mathcal{O}_{B_{\mathfrak{p}_i}}$ of B_i^2 for each $i=1,\dots,r$ and let Λ_i^1 be the dual of Λ_i^2. Then

$$\Lambda_p\ =\ \bigoplus_{i=1}^r\Lambda_i^1\ \oplus\ \bigoplus_{i=1}^r\Lambda_i^2\ \subset\ \bigoplus_{i=1}^r(V_p^\dagger)_i^1\ \oplus\ \bigoplus_{i=1}^r(V_p^\dagger)_i^2\ =\ V_p^\dagger\ =\ V^\dagger\otimes\mathbb{Q}_p$$

is a $\mathbb{Z}_p$-lattice in $V_p^\dagger$ and self-dual under $\psi^\dagger$. There is a unique maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}^\dagger\subset B^\dagger$ such that $\bar{\mathcal{O}}^\dagger=\mathcal{O}^\dagger$ and $\mathcal{O}^\dagger_{\mathfrak{p}_i^2}=\mathcal{O}_{B_{\mathfrak{p}_i}}$ acting on Λ_i^2 where $\mathcal{O}^\dagger_{\mathfrak{p}_i^2}$ is the image of $\mathcal{O}^\dagger\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p\subset B^\dagger\otimes_{\mathbb{Q}}\mathbb{Q}_p=B_p^\dagger$ in the B_i^2 component. Then the functor $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}$ is isomorphic to the following moduli functor in the category of locally noetherian schemes over $F_{\mathfrak{p}}$: for such a scheme S, $M^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}(S)$ is the set of equivalence classes of quintuples $(A,\theta,i,\bar\eta^p,\bar\eta_p^{\mathfrak{p}})$ where

- A is an abelian scheme over S of dimension $4d$;
- $\theta : A \to A^\vee$ is a prime-to-p polarization;
- $i : \mathcal{O}^\dagger \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is a monomorphism such that (4-2) is satisfied and $\theta \circ i(b) = i(\bar{b})^\vee \circ \theta$ for all $b \in \mathcal{O}^\dagger$;
- $\bar{\eta}^p$ is a $K^{\dagger,p}$-level structure, that is, a $\pi_1(S,s)$-invariant $K^{\dagger,p}$-class of $B^\dagger \otimes \mathbb{A}_f^p$-linear symplectic similitudes $\eta^p : V^\dagger \otimes \mathbb{A}_f^p \to \mathsf{H}_1^{\text{ét}}(A_s, \mathbb{A}_f^p)$;
- $\bar{\eta}_p^{\mathfrak{p}}$ is a $K_p^{\dagger,p}$-level structure, that is, a $\pi_1(S,s)$-invariant $K_p^{\dagger,\mathfrak{p}}$-class of isomorphisms of $\mathcal{O}^\dagger$-modules $\eta_p^{\mathfrak{p}} : \bigoplus_{i=2}^r \Lambda_i^2 \to \bigoplus_{i=2}^r \mathsf{H}_1^{\text{ét}}(A_s, \mathbb{Z}_p)_i^2$.

The two quintuples $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ and $(A', \theta', i', (\bar{\eta}^p)', (\bar{\eta}_p^{\mathfrak{p}})')$ are equivalent if there is a prime-to-p isogeny $A \to A'$ such that it carries θ to a $\mathbb{Z}_{(p)}^\times$-multiple of θ', i to i', $\bar{\eta}^p$ to $(\bar{\eta}^p)'$, and $\bar{\eta}_p^{\mathfrak{p}}$ to $(\bar{\eta}_p^{\mathfrak{p}})'$.

We are going to extend this moduli functor to $\mathcal{O}_{F_{\mathfrak{p}}}$ to get an integral model of $M_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p};\mathfrak{p}}^\dagger$. Now let us consider an abelian scheme (A, θ, i) which is a part of the datum defined just above, but for A over an $\mathcal{O}_{F_{\mathfrak{p}}}$-scheme S. Through θ, we see that $(A_{p^\infty})_i^1$ and $(A_{p^\infty})_i^2$ are Cartier dual to each other. We replace (4-2) by

$$\mathrm{tr}(b \in \mathcal{O}_{B_{\mathfrak{p}}} \subset B_1^2; \mathrm{Lie}_1^2(A)) = \mathrm{Tr}_{B_1^2/F_{\mathfrak{p}}}(b) \in \mathcal{O}_{F_{\mathfrak{p}}} \text{ and } \mathrm{Lie}_i^2(A) = 0 \text{ for } i \geq 2. \quad (4\text{-}3)$$

This means that the p-divisible group $(A_{p^\infty})_i^2$ is étale for $i \geq 2$. We let $T_pA = \varprojlim_n A[p^n](S)$. Then $(T_pA)_i^2$ is isomorphic to Λ_i^2 as $\mathcal{O}^\dagger$-modules if S is simply connected.

Now we define a moduli functor $\mathcal{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^\dagger$ on the category of locally noetherian schemes over $\mathcal{O}_{F_{\mathfrak{p}}}$: for such a scheme S, $\mathcal{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^\dagger(S)$ is the set of equivalence classes of quintuple $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ where

- (A, θ, i) is as in the last moduli problem but satisfies (4-3);
- $\bar{\eta}^p$ is a $K^{\dagger,p}$-level structure, that is, a $\pi_1(S,s)$-invariant $K^{\dagger,p}$-class of $B^\dagger \otimes \mathbb{A}_f^p$-linear symplectic similitudes $\eta^p : V^\dagger \otimes \mathbb{A}_f^p \to \mathsf{H}_1^{\text{ét}}(A_s, \mathbb{A}_f^p)$;
- $\bar{\eta}_p^{\mathfrak{p}}$ is a $K_p^{\dagger,p}$-level structure, that is, a $\pi_1(S,s)$-invariant $K_p^{\dagger,\mathfrak{p}}$-class of isomorphisms of $\mathcal{O}^\dagger$-modules $\eta_p^{\mathfrak{p}} : \bigoplus_{i=2}^r \Lambda_i^2 \to \bigoplus_{i=2}^r (T_pA_s)_i^2$.

The two quintuples $(A, \theta, i, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ and $(A', \theta', i', (\bar{\eta}^p)', (\bar{\eta}_p^{\mathfrak{p}})')$ are equivalent if there exists a prime-to-p isogeny $A \to A'$ satisfying the same requirements as in the last moduli problem. For sufficiently small $K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p}$, this moduli functor is represented by a regular scheme (also denoted by) $\mathcal{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^\dagger$, which is flat and projective over $\mathcal{O}_{F_{\mathfrak{p}}}$. Using Proposition 4.3, we get a regular scheme $\mathcal{M}_{0,K^{\mathfrak{p}}}$ flat and projective over $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$ whose generic fiber is $M_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. Here, we also need to use the fact that $\mathcal{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^\dagger$ is stable for $K^{\dagger,p}$ small and the results in [Deligne and Mumford 1969, Section 1] to make the descent argument. By construction, the neutral components of $\mathcal{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^\dagger \times_{\mathcal{O}_{F_{\mathfrak{p}}}} \mathcal{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}}$ and $\mathcal{M}_{0,K^{\mathfrak{p}}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{E_{\mathfrak{p}^\circ}^{\mathrm{nr}}}$ are isomorphic.

We shall write $(\mathscr{A}, \theta, i)$ for (a part of the datum of) the universal object over $\mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$. We also denote by $\mathscr{X}^{\dagger} = (\mathscr{A}_{p^\infty})^2_1 \to \mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ the universal p-divisible group with $\mathbb{O}_{B_{\mathfrak{p}}}$-action and an action by $\prod_{i=2}^{r} B_{\mathfrak{p}_i}^{\times} \times H^{\dagger}(\mathbb{A}_f^p)$ compatible with that on the underlying scheme $\mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$. We also have a p-divisible group $\mathscr{X} \to \mathscr{M}_{0,K^{\mathfrak{p}}}$ with an action by $H_f^{\mathfrak{p}}$ compatible with that on $\mathscr{M}_{0,K^{\mathfrak{p}}}$.

Remark 4.4. In fact, when $\mathfrak{p}|2$ and $B_{\mathfrak{p}}$ is division, the condition (4-3) is not enough. One needs to pose that $(A_{p^\infty})^2_1$ is special (see [Boutot and Carayol 1991, Section II.2]) at all geometric points of characteristic p.

Now consider the case where $\epsilon(\mathbb{V}_{\mathfrak{p}}) = 1$, that is, $V_{\mathfrak{p}} \cong V^+$, and $B_{\mathfrak{p}} \cong \mathsf{Mat}_2(F_{\mathfrak{p}})$ or $\mathsf{U}(V_{\mathfrak{p}})$ is quasisplit. At the outset, we introduce some notation for the Morita equivalence. Let R be a (commutative) ring (with 1) and M a (left) R-module (or p-divisible group according to the context). Let $m > 0$ be any integer. We denote by $M^{\sharp} = M^m$ (arranged in a column) the left $\mathsf{Mat}_m(R)$-module in the natural way. Conversely, for any left $\mathsf{Mat}_m(R)$-module N, we denote by $N^{\flat} = eN$ the (left) R-module, where $e = \mathsf{diag}[1, 0, \dots, 0] \in \mathsf{Mat}_m(R)$ and the action is given by $r.(en) = (e \times \mathsf{diag}[r, \dots, r]).n$ for $r \in R$ and $n \in N$. It is easy to see that the functors $(-)^{\sharp}$ and $(-)^{\flat}$ are a pair of equivalences between corresponding categories.

We identify $\Lambda^2_1 = \mathbb{O}_{B_{\mathfrak{p}}}$ with $\mathsf{Mat}_2(\mathbb{O}_{F_{\mathfrak{p}}})$ and hence $\mathbb{O}^{\times}_{B_{\mathfrak{p}}}$ with $\mathsf{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}})$. Using Morita equivalence, we easily see that in the moduli problem $\mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$, we can replace the first condition in (4-3) by the following:

$$\mathrm{tr}(b \in \mathbb{O}_{F_{\mathfrak{p}}}; \mathrm{Lie}^2_1(A)^{\flat}) = b. \tag{4-4}$$

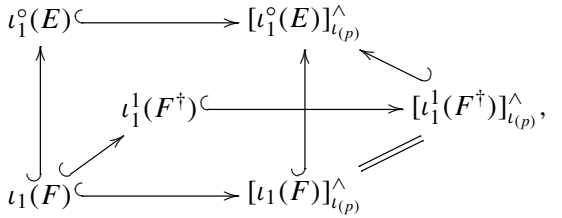
Consider a geometric point $s : \mathrm{Spec}\, \mathbb{F} \to \mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ of characteristic p and let $\hat{\mathbb{O}}_{(s)}$ be the completion of the henselization of the local ring at s. By the Serre–Tate theorem, it is the universal deformation ring of $(\mathscr{A}_s, \theta_s, i_s)$ which is the same as that of $(\mathscr{A}_{s,p^\infty}, \theta_s, i_s)$. By the conditions in the moduli problem and the Morita equivalence, we see that this is the same deformation ring of the p-divisible group $\mathscr{X}_s^{\dagger,\flat} = (\mathscr{A}_{s,p^\infty})^{2,\flat}_1$ which is an $\mathbb{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2. Hence $\hat{\mathbb{O}}_{(s)} \cong \mathbb{O}_{\hat{F}_{\mathfrak{p}}}[\![t]\!]$. We have

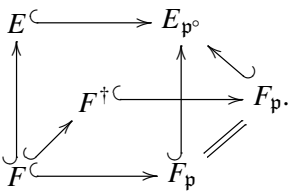
Proposition 4.5 [Carayol 1986, Section 6]. *The scheme $\mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ (resp. $\mathscr{M}_{0,K^{\mathfrak{p}}}$) is smooth and projective over $\mathbb{O}_{F_{\mathfrak{p}}}$ (resp. $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$).*

For a geometric point s of characteristic p on $\mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ (resp. $\mathscr{M}_{0,K^{\mathfrak{p}}}$), there are two cases. We say s is *ordinary* if the formal part of $\mathscr{X}^{\dagger}_s$ (resp. $\mathscr{X}_s$) is of height 1; *supersingular* if $\mathscr{X}^{\dagger}_s$ (resp. $\mathscr{X}_s$) is formal. We denote by $[\mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_{\mathrm{s.s.}}$, (resp. $[\mathscr{M}_{0,K^{\mathfrak{p}}}]_{\mathrm{s.s.}}$) the supersingular locus of the scheme $\mathscr{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ (resp. $\mathscr{M}_{0,K^{\mathfrak{p}}}$).

A basic abelian scheme. To give the moduli interpretation of the special cycles, we need first to construct a basic abelian scheme. We fix an imaginary element μ in E, that is, $\mu^{\tau} = -\mu$ and $\mu \neq 0$. Since we only care about the place $\mathfrak{p}$, we

identify the following two isomorphic commutative diagrams to ease our notation:





where $[-]^\wedge_{\iota_{(p)}}$ means the completion in $\mathbb{C}_p$ through $\iota_{(p)}$.

Now let $E^\dagger = E \otimes_F F^\dagger$, which is a CM field of degree $4d$ and a subalgebra of $B^\dagger$ extending the fixed embedding $E \hookrightarrow B$. We also denote by $e \mapsto \bar{e}$ the canonical involution of the second kind of $E^\dagger$ such that the subfield fixed by this involution is totally real. The maps $\iota_i^\circ \otimes \iota_i^j : E \otimes_F F^\dagger \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ and $\iota_i^\bullet \otimes \iota_i^j : E \otimes_F F^\dagger \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ $(i = 1, \dots, d,\ j = 1, 2)$ give $4d$ different complex embeddings of $E^\dagger$ into $\mathbb{C}$, where $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ is the usual multiplication. We choose a CM type $\Phi = \{\iota_1^\circ \otimes \iota_1^1, \iota_1^\circ \otimes \iota_1^2; \iota_i^\circ \otimes \iota_i^1, \iota_i^\bullet \otimes \iota_i^1 \mid i = 2, \dots, d\}$ of $E^\dagger$. Then Φ determines a Hodge map $h^\ddagger : \mathbb{S} \to T^\ddagger_{\mathbb{R}}$ where $T^\ddagger$ is the subtorus of $\mathrm{Res}_{E^\dagger/\mathbb{Q}} \boldsymbol{G}_{\mathrm{m},E^\dagger}$ consisting of elements e such that $e\bar{e} \in \boldsymbol{G}_{\mathrm{m},\mathbb{Q}}$. The Shimura varieties $M^\ddagger_{K^\ddagger} = \mathrm{Sh}_{K^\ddagger}(T^\ddagger, \{h^\ddagger\})$ basically parametrize abelian varieties over $E^{\dagger,\Phi}$ with CM by $E^\dagger$ of type Φ. It is finite and projective over $\mathrm{Spec}\, E^{\dagger,\Phi}$ where $E^{\dagger,\Phi}$ is the reflex field of $(E^\dagger, \Phi)$.

To make this more precise, let $V^\ddagger$ be the $\mathbb{Q}$-vector space underlying $E^\dagger$. Define a symplectic form

$$\psi^\ddagger(v, w) := \mathrm{Tr}_{F^\dagger/\mathbb{Q}}(\sqrt{\lambda}\mathrm{Tr}_{E^\dagger/F^\dagger}(v\bar{w}))$$

for $v, w \in E^\dagger$. Then $T^\ddagger$ can be identified with the group of $E^\dagger$-linear symplectic similitude of $(V^\ddagger, \psi^\ddagger)$ and $T^\ddagger(\mathbb{Q}_p)$ can be identified with $\mathbb{Q}_p^\times \times \prod_{i=1}^r E^\times_{\mathfrak{p}_i}$. The Hodge map $h^\ddagger$ induces a filtration $0 \subset \mathrm{Fil}^0(V^\ddagger_{\mathbb{C}}) \subset V^\ddagger_{\mathbb{C}}$ such that $t^\ddagger(e) = \mathrm{tr}(e; V^\ddagger_{\mathbb{C}}/\mathrm{Fil}^0(V^\ddagger_{\mathbb{C}})) = \sum_{\iota \in \Phi} \iota(e)$ for $e \in E^\dagger$. Since we have identified E (resp. $F^\dagger$) with its embedding through ι_1° (resp. ι_1^1), we now identify $E^\dagger$ with its embedding through $\iota_1^\circ \otimes \iota_1^1$, that is, with $\iota_1^\circ(E).\iota_1^1(F^\dagger) \subset \mathbb{C}$.

Lemma 4.6. *The reflex field $E^{\dagger,\Phi}$ is $E^\dagger$.*

Proof. By definition, $E^{\dagger,\Phi}$ is the field generated by the numbers $t^\ddagger(e)$ for all $e \in E^\dagger$. Let $e = (x + y\mu) \times (x' + y'\lambda')$ be an element in $E^\dagger$ with $x, y, x', y' \in F$. Then

$$\begin{aligned} t^\ddagger(e) &= (x + y\mu)(2x') + \sum_{i=2}^{d} 2\iota_i(x)(\iota_i(x') + \iota_i(y')\lambda') \\ &= 2\mathrm{Tr}_{F/\mathbb{Q}}(xx') + 2\mathrm{Tr}_{F/\mathbb{Q}}(xy')\lambda' + 2yx'\mu - 2xy'\lambda'. \end{aligned}$$

Hence $E^{\dagger,\Phi} = E^\dagger$. □

As before, the algebra $E_p^\dagger = E^\dagger \otimes_{\mathbb{Q}} \mathbb{Q}_p$ has a decomposition

$$E_p^\dagger = \bigoplus_{i=1}^r E_i^1 \oplus \bigoplus_{i=1}^r E_i^2 \cong \bigoplus_{i=1}^r E_{\mathfrak{p}_i} \oplus \bigoplus_{i=1}^r E_{\mathfrak{p}_i},$$

which is also true for its modules. Let π_1^1 be the projection of $E_p^\dagger$ to the first factor E_1^1. The additive map $t^\ddagger$ extends to a map $t_p^\ddagger : E_p^\dagger \to E_p^\dagger$. From the calculation in the above lemma, we find that, for $(e_i^j) = (e_1^1, \dots, e_r^1; e_1^2, \dots, e_r^2) \in E_p^\dagger$,

$$\pi_1^1 \circ t_p^\ddagger((e_i^j)) = \sum_{i=1}^r \mathrm{Tr}_{E_{\mathfrak{p}_i}/\mathbb{Q}_p}(e_i^1) + e_1^1 + e_1^2 - \mathrm{Tr}_{E_{\mathfrak{p}_1}/F_{\mathfrak{p}_1}}(e_1^1). \tag{4-5}$$

Let $\mathcal{O}^\ddagger \subset \mathcal{O}^\dagger$ be the unique maximal $\mathbb{Z}_{(p)}$-order in $E^\dagger$ such that $\bar{\mathcal{O}}^\ddagger = \mathcal{O}^\ddagger$ and the ring of integers is $\mathcal{O}^\ddagger_{\mathfrak{p}_i^2} = \mathcal{O}_{E_{\mathfrak{p}_i}}$, where $\mathcal{O}^\ddagger_{\mathfrak{p}_i^2}$ is the image of $\mathcal{O}^\ddagger \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ in the E_i^2 component. For any abelian variety A over an $E_{\mathfrak{p}^\circ}$-scheme S with an action by $\mathcal{O}^\ddagger$, $\mathrm{Lie}_S(A)$ is an $E_p^\dagger$-module, and hence decomposes as the direct sum of $\mathrm{Lie}_i^j(A)$ $(i = 1, \dots, r,\ j = 1, 2)$. In view of (4-3) and (4-5), we pose the trace condition as

$$\mathrm{tr}(e \in \mathcal{O}_{E_{\mathfrak{p}}} \subset E_1^2; \mathrm{Lie}_1^2(A)) = e_{\mathfrak{p}^\circ} \in E_{\mathfrak{p}^\circ} \text{ and } \mathrm{Lie}_i^2(A) = 0 \text{ for } i \geq 2, \tag{4-6}$$

where $e_{\mathfrak{p}^\circ}$ is just e if $E_{\mathfrak{p}} = E_{\mathfrak{p}^\circ}$ is a field or the component in $E_{\mathfrak{p}^\circ}$ if $E_{\mathfrak{p}} = E_{\mathfrak{p}^\circ} \oplus E_{\mathfrak{p}^\bullet}$ is split.

Let

$$K^\ddagger = \mathbb{Z}_p^\times \times \prod_{i=1}^r \mathcal{O}_{E_{\mathfrak{p}_i}}^\times \times K^{\ddagger,p}$$

be an open compact subgroup of $T^\ddagger(\mathbb{A}_f)$. Set $M^\ddagger_{00,K^{\ddagger,p}} = M^\ddagger_{K^\ddagger}$, and let $M^\ddagger_{00,K^{\ddagger,p};\mathfrak{p}^\circ}$ be the base change under $\iota_{(p)} \circ (\iota_1^\circ \otimes \iota_1^1) : E^\dagger \hookrightarrow E_{\mathfrak{p}^\circ}$. Then for sufficiently small $K^{\ddagger,p}$, $M^\ddagger_{00,K^{\ddagger,p}}$ represents the following moduli functor (due to (4-6)) on the category of locally noetherian schemes over $E_{\mathfrak{p}^\circ}$: for such a scheme S, $M^\ddagger_{00,K^{\ddagger,p}}(S)$ is the set of equivalence classes of quadruples $(A, \vartheta, j, \bar{\eta}^p)$ where

- A is an abelian scheme over S of dimension $2d$;
- $\vartheta : A \to A^\vee$ is a prime-to-p polarization;
- $j : \mathcal{O}^\ddagger \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is a monomorphism of $\mathbb{Z}_{(p)}$-algebras such that (4-6) is satisfied and $\vartheta \circ j(e) = j(\bar{e})^\vee \circ \vartheta$ for all $e \in \mathcal{O}^\ddagger$;
- $\bar{\eta}^p$ is a $K^{\ddagger,p}$-structure, that is, a $\pi_1(S, s)$-invariant $K^{\ddagger,p}$-class of $E^\dagger \otimes \mathbb{A}_f^p$-linear symplectic similitude $\eta^p : V^\ddagger \otimes \mathbb{A}_f^p \to \mathrm{H}_1^{\text{ét}}(A_s, \mathbb{A}_f^p)$.

The notion of equivalence is similarly defined as before. Moreover, we can extend this moduli functor to be over $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$. We omit the detailed definition. One can similarly prove that the extended one, say $\mathcal{M}^\ddagger_{00,K^{\ddagger,p}}$, is finite, projective, smooth over $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$, and connected, and hence is isomorphic to $\mathrm{Spec}\ \mathcal{O}_{E^\natural}$ for some finite

unramified extension of local fields $E^\natural/E_{\mathfrak{p}^\circ}$. Fix an embedding $\iota^\natural : E^\natural \hookrightarrow E^{\mathrm{nr}}_{\mathfrak{p}^\circ}$ and let $(\mathscr{E}, \vartheta, j)$ be the universal object over $\mathcal{M}^{\ddagger}_{00,K^{\ddagger,p}} \times_{\mathbb{O}_{E^\natural},\iota^\natural} \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}} \cong \mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$ and $\mathscr{Y} = (\mathscr{E}_{p^\infty})^2_1$. Fix a geometric point $s : \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}} \hookrightarrow \mathbb{C}$ of characteristic zero and an $\mathbb{O}^\ddagger$-generator $\boldsymbol{x}$ of $\mathsf{H}^{\mathsf{B}}_1(\mathscr{E}_s, \mathbb{Z}_{(p)})$ where $\mathsf{H}^{\mathsf{B}}_1$ is the first Betti homology. We call the quadruple $(\mathscr{E}, \vartheta, j; \boldsymbol{x})$ a *basic unitary datum*.

For any $x \in V$ which is positive definite, we have a subscheme $Z(x)_K$ on M_K which is a special cycle as in Section 3B. Let us consider the curve $M_{0,K^{\mathfrak{p}};\mathfrak{p}^\circ}$ and its subscheme $Z(x)_{0,\mathfrak{p}^\circ}$, which is the base change of $Z(x)_{K_{\mathfrak{p},0}K^{\mathfrak{p}}}$ in $M_{0,K^{\mathfrak{p}}}$. We denote by $Z(x)^\circ_{0,\mathfrak{p}^\circ}$ the neutral component of $Z(x)_{0,\mathfrak{p}^\circ} \times_{E^\circ_{\mathfrak{p}}} E^{\mathrm{nr}}_{\mathfrak{p}^\circ}$. By Proposition 4.3, we have an irreducible closed subscheme $Z(x)^\dagger_{0,\mathfrak{p}}$ on $M^{\dagger,\circ}_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p};\mathfrak{p}}$ by passing between the neutral components, which is an element in $\mathsf{CH}^1(M^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p};\mathfrak{p}} \times_{F_{\mathfrak{p}}} E^{\mathrm{nr}}_{\mathfrak{p}^\circ})$. For $K^{\mathfrak{p}}$ sufficiently small (which is independent of x), $Z(x)^\dagger_{0,\mathfrak{p},o}(\mathbb{C})$ is represented by (z, e) for any complex geometric point $o : E^{\mathrm{nr}}_{\mathfrak{p}^\circ} \hookrightarrow \mathbb{C}$ where $z \perp x$ and $e \in H^\dagger(\mathbb{A}_f)$ is the identity element. Actually, $Z(x)^\dagger_{0,\mathfrak{p}}$ is defined for any $x \in V^\dagger - \{0\}$ if we view $x = x \otimes 1 \in V^\dagger$ and extend by $Z(ax)^\dagger_{0,\mathfrak{p}} = Z(x)^\dagger_{0,\mathfrak{p}}$ for $a \in F^{\dagger,\times}$. Let $[Z(x)^\dagger_{0,\mathfrak{p}}]^{\mathsf{Zar}}$ be the Zariski closure of $Z(x)^\dagger_{0,\mathfrak{p}}$ in $\mathcal{M}^\ddagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \times_{\mathbb{O}_{F_{\mathfrak{p}}}} \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$.

We fix a basic unitary datum $(\mathscr{E}, \vartheta, j; \boldsymbol{x})$. Since $\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$ is simply connected, we can extend $\boldsymbol{x}$ to a section $\boldsymbol{x}^p$ of the lisse $\mathbb{A}^p_f$-sheaf $\mathsf{H}^{\text{ét}}_1(\mathscr{E}, \mathbb{A}^p_f)$ over $\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$, hence a section $\boldsymbol{x}^p_S$ of $\mathsf{H}^{\text{ét}}_1(\mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S, \mathbb{A}^p_f)$ for any $\mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$-scheme S. If S is an $E^{\mathrm{nr}}_{\mathfrak{p}^\circ}$-scheme, we have a section $\boldsymbol{x}_S$ of $\mathsf{H}^{\text{ét}}_1(\mathscr{E} \times_{\mathrm{Spec}\, E^{\mathrm{nr}}_{\mathfrak{p}^\circ}} S, \mathbb{A}_f)$. In particular,

$$(\boldsymbol{x}_{\mathfrak{p},S}, \boldsymbol{x}^{\mathfrak{p}}_{p,S}) \in (T_p(\mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S))^2_1 \oplus \bigoplus_{i=2}^{r} (T_p(\mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S))^2_i.$$

Let $\boldsymbol{E} = \mathscr{E} \times_{\mathbb{O}^{\mathrm{nr}}_{\mathfrak{p}^\circ}} \mathbb{F}$ be the special fiber. Since $(\boldsymbol{E}_{p^\infty})^2_i$ is étale for $i \geq 2$, $(T_p \boldsymbol{E})^2_i$ is canonically isomorphic to $(\mathsf{H}^{\mathsf{B}}_1(\mathscr{E}_s, \mathbb{Z}_p))^2_i$. Hence $\boldsymbol{x}$ canonically determines an element $\boldsymbol{x}^{\mathfrak{p}}_p \in \bigoplus_{i=2}^r (T_p \boldsymbol{E})^2_i$. For any $\mathbb{F}$-scheme S, we have element $\boldsymbol{x}^{\mathfrak{p}}_{p,S}$.

Proposition 4.7. *Let $x \in V^\dagger \cap \Lambda^2_1$ such that $Z(x)^\dagger_{0,\mathfrak{p}}$ is nonempty and let S be an $\mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$-scheme. For any morphism $S \to [Z(x)^\dagger_{0,\mathfrak{p}}]^{\mathsf{Zar}} \hookrightarrow \mathcal{M}^\dagger_{0,K^{\dagger,\mathfrak{p}}_p,K^{\dagger,p}} \times_{\mathbb{O}_{F_{\mathfrak{p}}}} \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$ inducing the quintuple $(A, \theta, i_A, \bar{\eta}^p, \bar{\eta}^{\mathfrak{p}}_p)$, there is a quasihomomorphism $\varrho_A : \mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S \to A$ satisfying the following conditions*:

- *For any $e \in \mathbb{O}^\ddagger$, the following diagram commutes*:

$$\begin{array}{ccc} \mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S & \xrightarrow{\varrho_A} & A \\ {\scriptstyle j(e)}\downarrow & & \downarrow{\scriptstyle i_A(e)} \\ \mathscr{E} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S & \xrightarrow{\varrho_A} & A \end{array} \tag{4-7}$$

- *ϱ_A induces a homomorphism from $\mathscr{Y} \times_{\mathrm{Spec}\, \mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}} S$ to $(A_{p^\infty})^2_1$.*
- *For any geometric point $s \in S$, the map $\varrho_{A_s,*} : \mathsf{H}^{\text{ét}}_1(\mathscr{E}_s, \mathbb{A}^p_f) \to \mathsf{H}^{\text{ét}}_1(A_s, \mathbb{A}^p_f)$ sends $\boldsymbol{x}^p_s$ into $\bar{\eta}^p(x)$.*

- *For any geometric point $s \in S$, the map*

$$\varrho_{A_s,*} : \bigoplus_{i=2}^{r} (T_p \mathscr{C}_s)_i^2 \to \bigoplus_{i=2}^{r} (T_p A_s)_i^2 \otimes_{\mathbb{O}_{F_{\mathfrak{p}_i}}} F_{\mathfrak{p}_i}$$

sends $\boldsymbol{x}_{p,s}^{\mathfrak{p}}$ into $\bar{\eta}_p^{\mathfrak{p}}(x)$.

Proof. Let $o : E_{\mathfrak{p}^\circ}^{\mathsf{nr}} \hookrightarrow \mathbb{C}$ be any embedding such that it gives a complex geometric point of $[Z(x)_{0,\mathfrak{p}}^{\dagger}]^{\mathsf{Zar}}$ which corresponds to the quintuple $(A_o, \theta_o, i_{A_o}, (\bar{\eta}^p)_o, (\bar{\eta}_p^{\mathfrak{p}})_o)$, and also to the point represented by (z, e). This means that we can find a symplectic similitude $\gamma : \mathsf{H}_1^{\mathsf{B}}(A_o, \mathbb{Q}) \to V^{\dagger}$ such that $\gamma^{-1}(x) \in \mathsf{Lie}(A_o)$ is an i-eigenvector of $i_{A_o}(h^{\dagger}(i))$ (see [Kottwitz 1992, Section 8] for the complex points of Shimura varieties of PEL type). Consider the operator $i_{A_o}(h^{\dagger}(i)) - i$ acting on $\mathsf{Lie}(A_o)$ which is by zero on the $\mathbb{Z}_{(p)}$-sublattice $\Upsilon_{(p)} = i_{A_o}(\mathbb{O}^{\ddagger})(\gamma^{-1}(x))$ of rank $4d$, and hence is also by zero on the $\mathbb{R}$-subspace it generates, which is $\Upsilon_{\mathbb{R}} = i_{A_o}(E^{\dagger} \otimes_{\mathbb{Q}} \mathbb{R})(\gamma^{-1}(x))$. But since, on this space, multiplying i is the same as $i_{A_o}(h^{\dagger}(i))$, we see that $\Upsilon_{\mathbb{R}}/\Upsilon_{(p)}$ defines a complex subtorus (up to prime-to-p isogeny) E of dimension $2d$ and a quasihomomorphism $\varrho'_{A_o} : E \to A_o$, hence E is a complex abelian variety. It is easy to see that E is isogenous to $\mathscr{C} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathsf{nr}}}, o} \mathbb{C}$, hence we can find a unique quasihomomorphism $\varrho_E : \mathscr{C} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathsf{nr}}}, o} \mathbb{C} \to E$ sending $\boldsymbol{x}$ to $\gamma^{-1}x \in \Upsilon_{(p)} = \mathsf{H}_1^{\mathsf{B}}(E, \mathbb{Z}_{(p)})$, and $\varrho_{A_o} = \varrho_E \circ \varrho'_{A_o}$ satisfies the properties in the proposition. Since o is arbitrary, it is not difficult to see that ϱ_{A_o} descends to a quasihomomorphism $\varrho_A : \mathscr{C} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathsf{nr}}}} E_{\mathfrak{p}^\circ}^{\mathsf{nr}} \to A$ satisfying all properties but where $(A, \theta, i_A, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ is the quintuple on $Z(x)_{0,\mathfrak{p}}^{\dagger}$. If we denote by A' the corresponding abelian scheme over $[Z(x)_{0,\mathfrak{p}}^{\dagger}]^{\mathsf{Zar}}$, the quasihomomorphism ϱ_A uniquely extends to $\varrho_{A'} : \mathscr{C} \to A'$, satisfying (4-7). The other properties follow from the comparison theorems (for homology) and the assumption that $x \in \Lambda_1^2$. For general S, we only need to pull back $\varrho_{A'}$. □

Integral special subschemes in the quasisplit case. We now assume that $\mathfrak{p}$ is nonsplit in E. For any element $t \in F^{\dagger,\times} \cap \mathbb{O}_{F_{\mathfrak{p}}}$, we define a functor $\mathscr{Z}(t)_{0,\mathfrak{p}}^{\dagger}$ in the following way. For any $\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathsf{nr}}}$-scheme S, $\mathscr{Z}(t)_{0,\mathfrak{p}}^{\dagger}(S)$ is a sextuple $(A, \theta, i_A, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}}; \varrho_A)$ where $(A, \theta, i_A, \bar{\eta}^p, \bar{\eta}_p^{\mathfrak{p}})$ is an element in $\mathscr{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^{\dagger}(S)$ and $\varrho_A : \mathscr{C} \times_{\mathsf{Spec}\,\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathsf{nr}}}} S \to A$ is a nontrivial quasihomomorphism such that

- we have a commutative diagram like (4-7);
- ϱ induces a homomorphism from $\mathscr{Y} \times_{\mathsf{Spec}\,\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathsf{nr}}}} S$ to $(A_{p^\infty})_1^2$; and
- the nontrivial quasihomomorphism $\vartheta^{-1} \circ \varrho_A^{\vee} \circ \theta \circ \varrho_A$ induces $j(t) \times_{\mathsf{Spec}\,\mathbb{F}} S_{\mathsf{spe}} : \boldsymbol{E} \times_{\mathsf{Spec}\,\mathbb{F}} S_{\mathsf{spe}} \to \boldsymbol{E} \times_{\mathsf{Spec}\,\mathbb{F}} S_{\mathsf{spe}}$ where $S_{\mathsf{spe}} = S \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}^{\mathsf{nr}}}} \mathbb{F}$.

These properties cut out a subscheme, which is still denoted by $\mathscr{Z}(t)_{0,\mathfrak{p}}^{\dagger}$, of $\mathscr{M}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}^{\dagger}$. By the positivity property of the Rosati involution, one easily sees that it is nonempty if and only if t is totally positive. Let o be an $E_{\mathfrak{p}^\circ}^{\mathsf{nr}}$-point of the

generic fiber $\mathscr{Z}(t)^\dagger_{0,\mathfrak{p}}$ such that it is equal to some $Z(x)^\dagger_{0,\mathfrak{p}}$. We write $T(o)=t$ to indicate their relation. If $\mathscr{Z}(o)^\dagger_{0,\mathfrak{p}}$ is the unique irreducible component of $\mathscr{Z}(t)^\dagger_{0,\mathfrak{p}}$ containing o, then it is a closed subscheme of $\mathscr{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}\times_{\mathbb{O}_{F_\mathfrak{p}}}\mathbb{O}_{E^{\mathrm{nr}}_{\mathfrak{p}^\circ}}$. By construction, o corresponds to a special cycle $Z(x)^\dagger_{0,\mathfrak{p}}$. Hence we have the following identity between sets:

$$\bigcup_{t\in F^{\dagger,+}\cap\mathbb{O}_{F_\mathfrak{p}}}\left\{\text{generic point of }\mathscr{Z}(o)^\dagger_{0,\mathfrak{p}}\;\middle|\;T(o)=t\right\}=\left\{Z(x)^\dagger_{0,\mathfrak{p}}\;\middle|\;x\in V^\dagger\cap\Lambda_1^2-\{0\}\right\}$$
$$=\left\{Z(x)^\circ_{0,\mathfrak{p}^\circ}\;\middle|\;x\in V\cap\Lambda_\mathfrak{p}-\{0\}\right\},$$

where $\Lambda_\mathfrak{p}=\Lambda^\pm$ is the self-dual lattice of $V_\mathfrak{p}$. Hence, again by Proposition 4.3, we obtain an *integral special subscheme* $\mathscr{Z}(x)^\circ_{0,\mathfrak{p}^\circ}$ of $\mathscr{M}^\circ_{0,K^\mathfrak{p}}$ such that its generic fiber is $Z(x)^\circ_{0,\mathfrak{p}^\circ}$.

From now on, we assume further that $\epsilon(\mathbb{V}_p)=1$.

Proposition 4.8. *The special fiber* $[\mathscr{Z}(o)^\dagger_{0,\mathfrak{p}}]_{\mathsf{spe}}$ *(resp.,* $[\mathscr{Z}(x)^\circ_{0,\mathfrak{p}^\circ}]_{\mathsf{spe}}$*) of* $\mathscr{Z}(o)^\dagger_{0,\mathfrak{p}}$ *(resp.* $\mathscr{Z}(x)^\circ_{0,\mathfrak{p}^\circ}$*) lies in the supersingular locus* $[\mathscr{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_{\mathsf{s.s.}}$ *(resp.* $[\mathscr{M}_{0,K^\mathfrak{p}}]_{\mathsf{s.s.}}$*).*

Proof. We only need to prove this for $[\mathscr{Z}(o)^\dagger_{0,\mathfrak{p}}]_{\mathsf{spe}}$. Let $s=(A,\theta,i_A,\bar\eta^p,\bar\eta^\mathfrak{p}_p;\varrho_A)$ be an $\mathbb{F}$-point of $\mathscr{Z}(o)^\dagger_{0,\mathfrak{p}}$. Then we have a nontrivial homomorphism between $\mathbb{O}_{F_\mathfrak{p}}$-modules $\varrho_{A,*}:\boldsymbol{Y}\to(A_{p^\infty})^2_1=(\mathscr{X}^\dagger_s)\cong(\mathscr{X}^{\dagger,\flat}_s)^{\oplus 2}$. Hence there is at least one projection $(\mathscr{X}^{\dagger,\flat}_s)^{\oplus 2}\to\mathscr{X}^{\dagger,\flat}_s$ whose composition with $\varrho_{A,*}$ is nonzero. We know that both $\boldsymbol{Y}$ and $\mathscr{X}^{\dagger,\flat}_s$ are $\mathbb{O}_{F_\mathfrak{p}}$-modules of dimension 1 and height 2. But since we have assumed that $\mathfrak{p}$ is nonsplit in E, $\boldsymbol{Y}$ is formal, which implies that $\mathscr{X}^\dagger_s$ is also formal; that is, s is located in the supersingular locus. □

We need to study the supersingular loci of $\mathscr{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ and $\mathscr{M}_{0,K^\mathfrak{p}}$. We fix an integral special subscheme $\mathscr{Z}(o)^\dagger_{0,\mathfrak{p}}$ with $T(o)=1$ and an $\mathbb{F}$-point s of it. We set $\boldsymbol{A}=\mathscr{A}_s$ and $\boldsymbol{X}=\mathscr{X}^\dagger_s$, which is a formal $\mathbb{O}_{F_\mathfrak{p}}$-module of dimension 2 and height 4 over $\mathbb{F}$ with an action by $\mathsf{GL}_2(\mathbb{O}_{F_\mathfrak{p}})$. In fact, the isomorphism class of $\boldsymbol{X}$ is independent of the choice of o and s. We denote by $(\boldsymbol{A}^0,\theta^0,i_{\boldsymbol{A}^0})$ the (unique) isogeny class of the abelian variety with polarization and endomorphism $(\boldsymbol{A},\theta,i_{\boldsymbol{A}})$. Let $\breve B$ be the division algebra over F obtained from B by changing Hasse invariants at ι_1 and $\mathfrak{p}$, hence both B_{ι_1} and $B_\mathfrak{p}$ are division algebras. Let $\breve B^\dagger=\breve B\otimes_F F^\dagger$, $\breve G=\mathsf{Res}_{F/\mathbb{Q}}\breve B^\times$ with center T, and

$$\breve G^\dagger:=\breve G\times_T T_{F^\dagger}\xrightarrow{\breve\nu^\dagger}T\times T_{F^\dagger},$$

where $\breve\nu^\dagger(\breve g\times z)=(\mathsf{N}\breve g\cdot zz^{\tau^\dagger},z/z^{\tau^\dagger})$. Let $\breve H^\dagger$ be the preimage of $T^\dagger$ under $\breve\nu^\dagger$. Then we have $\mathsf{End}(\boldsymbol{A}^0,i_{\boldsymbol{A}^0})\cong\breve B^\dagger$ as an $F^\dagger$-algebra and $\mathsf{Aut}(\boldsymbol{A}^0,\theta^0,i_{\boldsymbol{A}^0})\cong\breve H^\dagger(\mathbb{Q})$ (see [Carayol 1986, Section 11]). We can also choose the isomorphism such that the involution of the second kind on $\breve B^\dagger$ induced by θ is the tensor product of the canonical involution of $\breve B$ and the conjugation of $F^\dagger$. In what follows, we identity

$\mathrm{End}(\boldsymbol{A}^0, i_{\boldsymbol{A}^0})$ with $\breve{B}^\dagger$ and $\mathrm{Aut}(\boldsymbol{A}^0, \theta^0, i_{\boldsymbol{A}^0})$ with $\breve{H}^\dagger(\mathbb{Q})$. We also identify $\breve{H}^\dagger(\mathbb{A}_f^p)$ with $H^\dagger(\mathbb{A}_f^p)$ through the level structure $(\bar\eta^p, \bar\eta_p^{\mathfrak{p}})$ of $\boldsymbol{A}$.

Let $S^\dagger = [\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_{\mathsf{s.s.}}(\mathbb{F})$ be the set of supersingular points in the special fiber together with a p-divisible group $\mathscr{X}^\dagger\big|_{S^\dagger}$ on it. The group $H^\dagger(\mathbb{A}_f)$ acts on $\mathscr{X}^\dagger\big|_{S^\dagger}$ which is compatible with its action on $S^\dagger$. It is easy to see that the action factors through $\mathbb{Z}_p^\times \times \mathsf{SL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$. Hence its normalizer $\mathsf{SL}_2(F_{\mathfrak{p}})$ acts trivially on $S^\dagger$ and the action factors through $F_{\mathfrak{p}}^\times \times \prod_{i=2}^r B_{\mathfrak{p}_i}^\times \times H^\dagger(\mathbb{A}_f^p)$.

By Honda–Tate theory, it is proved in [Carayol 1986, Section 11.3] that

$$F_{\mathfrak{p}}^\times \times \prod_{i=2}^r B_{\mathfrak{p}_i}^\times \times H^\dagger(\mathbb{A}_f^p)$$

acts transitively on $S^\dagger$ and its stabilizer at s is $\breve{H}^\dagger(\mathbb{Q}) \times (\mathcal{O}_{F_{\mathfrak{p}}}^\times \times K_p^{\dagger,\mathfrak{p}} \times K^{\dagger,p})$. Hence we have

$$S^\dagger \cong \breve{H}^\dagger(\mathbb{Q}) \backslash \left(\mathbb{Z} \times \prod_{i=2}^r B_{\mathfrak{p}_i}^\times / K_p^{\dagger,\mathfrak{p}} \times \breve{H}^\dagger(\mathbb{A}_f^p)/K^{\dagger,p} \right).$$

Similarly for $S = [\mathcal{M}_{0,K^{\mathfrak{p}}}]_{\mathsf{s.s.}}(\mathbb{F})$, we have a p-divisible group $\mathscr{X}\big|_S$ with an action by $H(\mathbb{A}_f)$ which is compatible with that on S. The action factors through $\mathsf{SL}_2(\mathcal{O}_{F_{\mathfrak{p}}}) \subset H(\mathbb{A}_f)$. Hence its normalizer $\mathsf{SL}_2(F_{\mathfrak{p}})$ acts trivially on S and the action factors through $E_{\mathfrak{p}^\circ}^{\times,1} \times H_f^{\mathfrak{p}}$. Let Z be the center of H. Then the stabilizer at s is $\breve{H}(\mathbb{Q}) \times (E_{\mathfrak{p}^\circ}^{\times,1} \times K^{\mathfrak{p}})$, where $\breve{H} = Z \cdot \breve{H}^{\dagger,\mathsf{der}}$. Hence we have

$$S \cong \breve{H}(\mathbb{Q}) \backslash \breve{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}},$$

where $\breve{H}_f^{\mathfrak{p}} = H_f^{\mathfrak{p}}$.

Moreover, if we denote by $[\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_s^\wedge$ (resp. $[\mathcal{M}_{0,K^{\mathfrak{p}}}]_s^\wedge$) the formal completion at the point s, then we have

$$[\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]_s^\wedge \cong \mathcal{N}, \qquad [\mathcal{M}_{0,K^{\mathfrak{p}}}]_s^\wedge \cong \mathcal{N}',$$

where $\mathcal{N} = \mathrm{Spf}\, R_{F_{\mathfrak{p}},2}$ with $R_{F_{\mathfrak{p}},2} = \mathcal{O}_{\hat F_{\mathfrak{p}}}[\![t]\!]$ and $\mathcal{N}' = \mathcal{N} \times_{\mathcal{O}_{\hat F_{\mathfrak{p}}}} \mathcal{O}_{\hat E_{\mathfrak{p}^\circ}}$. Hence we have the following p-adic uniformization of the formal completion at the supersingular locus:

$$[\mathcal{M}^\dagger_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}]^\wedge_{\mathsf{s.s.}} \times_{\mathcal{O}_{F_{\mathfrak{p}}}} \mathcal{O}_{\hat F_{\mathfrak{p}}} \cong \breve{H}^\dagger(\mathbb{Q}) \backslash \left(\mathcal{N} \times \mathbb{Z} \times \prod_{i=2}^r B_{\mathfrak{p}_i}^\times / K_p^{\dagger,\mathfrak{p}} \times \breve{H}^\dagger(\mathbb{A}_f^p)/K^{\dagger,p} \right)$$

and

$$[\mathcal{M}_{0,K^{\mathfrak{p}}}]^\wedge_{\mathsf{s.s.}} \times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}} \mathcal{O}_{\hat E_{\mathfrak{p}^\circ}} \cong \breve{H}(\mathbb{Q}) \backslash \mathcal{N}' \times \breve{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}},$$

where $\breve{H}^\dagger(\mathbb{Q})$ (resp. $\breve{H}(\mathbb{Q})$) acts on $\mathcal{N}$ (resp. $\mathcal{N}'$) through the $\mathfrak{p}$-component, which is trivial on the center. Such uniformization is a special case of that considered in [Rapoport and Zink 1996].

Next we must determine the formal completion $[\mathscr{Z}(o)^{\dagger}_{0,\mathfrak{p}}]^{\wedge}_{\mathsf{spe}}$ and $[\mathscr{Z}(x)^{\circ}_{0,\mathfrak{p}^{\circ}}]^{\wedge}_{\mathsf{spe}}$ of the integral special subschemes. We consider the $\mathbb{Z}_{(p)}$-module

$$\breve{\Lambda} = \mathsf{End}((\boldsymbol{E}, j), (\boldsymbol{A}, i_{\boldsymbol{A}})) := \{\varrho : \boldsymbol{E} \to \boldsymbol{A} \mid \varrho \circ j(e) = i_{\boldsymbol{A}}(e) \circ \varrho \text{ for all } e \in \mathbb{O}^{\ddagger}\},$$

which has an action by $\mathbb{O}^{\ddagger}$ through $e.\breve{x} = \breve{x} \circ j(e)$ for $e \in \mathbb{O}^{\ddagger}$ and $\breve{x} \in \breve{\Lambda}$. Hence $\breve{\Lambda}_{\mathbb{Q}} = \breve{\Lambda} \otimes \mathbb{Q}$ has an $E^{\dagger}$-vector space structure which is of dimension 2. The group $\breve{H}^{\dagger}(\mathbb{Q})$ acts on $\breve{\Lambda}_{\mathbb{Q}}$ through $\breve{h}.\breve{x} = \breve{h} \circ \breve{x}$. Hence we have an action of $\breve{H}(\mathbb{Q})$ on $\breve{\Lambda}_{\mathbb{Q}}$. Let $\breve{V}$ be the $\mathbb{Q}$-vector subspace of $\breve{\Lambda}_{\mathbb{Q}}$ generated by $\breve{H}(\mathbb{Q}).\breve{x}_0$ which is an E-vector space of dimension 2, where $\breve{x}_0 = \varrho_{\boldsymbol{A}}$. We define an E-hermitian form $(\,\cdot\,,\,\cdot\,)'$ on it in the following way:

$$(\breve{x}, \breve{y})' = j^{-1} \circ \vartheta^{-1} \circ \breve{y}^{\vee} \circ \theta \circ \breve{x} \in E, \qquad \breve{x}, \breve{y} \in \breve{V} \subset \breve{\Lambda}_{\mathbb{Q}}. \tag{4-8}$$

By the following lemma, $\breve{H}$ is the unitary group of $(\breve{V}, (\,\cdot\,,\,\cdot\,)')$ (and the hermitian form really takes values in E). In particular $\breve{V}$ is totally positive definite and is isometric to the nearby hermitian space $V^{(\mathfrak{p})}$ of $\mathbb{V}$.

Lemma 4.9. *For any $\breve{b} \in \breve{B}^{\times,1} = \breve{H}^{\dagger,\mathsf{der}}(\mathbb{Q})$, we have $(\breve{b}.\breve{x}_0, \breve{b}.\breve{x}_0)' = 1$, where $\breve{B}^{\times,1}$ is the subgroup of all norm-1 elements.*

Proof. By definition,

$$\begin{aligned}(\breve{b}.\breve{x}_0, \breve{b}.\breve{x}_0)' &= j^{-1} \circ \vartheta^{-1} \circ (\breve{b} \circ \breve{x}_0)^{\vee} \circ \theta \circ (\breve{b}.\breve{x}_0) = j^{-1} \circ \vartheta^{-1} \circ \breve{x}_0^{\vee} \circ (\breve{b}^{\vee} \circ \theta \circ \breve{b}) \circ \breve{x}_0 \\ &= j^{-1} \circ \vartheta^{-1} \circ \breve{x}_0^{\vee} \circ (\bar{\breve{b}}^{\vee,-1} \circ \theta \circ \breve{b}) \circ \breve{x}_0 \\ &= j^{-1} \circ \vartheta^{-1} \circ \breve{x}_0^{\vee} \circ \theta \circ \breve{x}_0 = (\breve{x}_0, \breve{x}_0)' = 1. \end{aligned}$$

□

The level structure $(\bar{\eta}^{p}, \bar{\eta}^{\mathfrak{p}}_{p})$ of $\boldsymbol{A}$ gives a $K^{\mathfrak{p}}$-class of isometries

$$V \otimes_F \mathbb{A}^{\mathfrak{p}}_{f,F} \to \breve{V} \otimes_F \mathbb{A}^{\mathfrak{p}}_{f,F}$$

by sending x to $\breve{x} \in \breve{V} \otimes_F \mathbb{A}^{\mathfrak{p}}_{f,F}$ such that $\breve{x}_*(\boldsymbol{x}^{p}_{s}) \in \bar{\eta}^{p}(x)$ and $\breve{x}_*(\boldsymbol{x}^{\mathfrak{p}}_{p,s}) \in \bar{\eta}^{\mathfrak{p}}_{p}(x)$. We identify these two spaces through any isometry in this class. For the place $\mathfrak{p}$, we let $\breve{\Lambda}_{\mathfrak{p}} = \mathsf{End}((\boldsymbol{Y}, j), (\boldsymbol{X}, i_{\boldsymbol{X}}))$. Then $\breve{\Lambda}_{\mathfrak{p}}$ is a maximal lattice in $\breve{V}_{\mathfrak{p}}$ such that the hermitian form restricted to it takes values in $\mathbb{O}_{E_{\mathfrak{p}^{\circ}}}$.

Let $\pi : \mathscr{N}' \to [\mathcal{M}^{\dagger}_{0,K^{\dagger,\mathfrak{p}}_{p},K^{\dagger,p}}]^{\wedge}_{\mathsf{s.s.}} \times_{\mathbb{O}_{F_{\mathfrak{p}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^{\circ}}}$ be the natural projection map. Then the base change $\pi^{-1}([\mathscr{Z}(o)^{\dagger}_{0,\mathfrak{p}}]^{\wedge}_{s})$ is $\mathscr{Z}(\breve{x}_0) \times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^{\circ}}}$. Here, $\mathscr{Z}(\breve{x}_0)$ is a cycle on $\mathscr{N}$ where the endomorphism $\breve{x}_0 \in \breve{\Lambda}_{\mathfrak{p}}$ deforms. In the next section, we will define and discuss in detail the cycle $\mathscr{Z}(\breve{x})$ for any $\breve{x} \in \breve{\Lambda}_{\mathfrak{p}} - \{0\}$.

For any $\breve{h} \in \breve{H}^{\mathfrak{p}}_{f}/K^{\mathfrak{p}}$, we denote by $\mathscr{Z}(\breve{x}, \breve{h})$ the cycle of $[\mathcal{M}_{0,K^{\mathfrak{p}}}]^{\wedge}_{\mathsf{s.s.}} \times_{\mathbb{O}_{E_{\mathfrak{p}^{\circ}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^{\circ}}}$ represented by $(\mathscr{Z}(\breve{x}) \times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^{\circ}}}, \breve{h})$. For any $h \in H^{\mathfrak{p}}_{f}/K^{\mathfrak{p}}$, we denote by $\mathscr{Z}(x, h)$ the cycle of $\mathcal{M}_{0,K^{\mathfrak{p}}} \times_{\mathbb{O}_{E_{\mathfrak{p}^{\circ}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^{\circ}}}$ which is the translation of $\mathscr{Z}(x)^{\circ}_{0,\mathfrak{p}^{\circ}}$ by the Hecke operator of h. Since $\breve{\Lambda}_{\mathbb{Q}} \cap \breve{\Lambda}_{\mathfrak{p}} = (\breve{H}^{\dagger}(\mathbb{Q}) \cap \mathsf{GL}(\breve{\Lambda}_{\mathfrak{p}})).(\breve{V} \cap \breve{\Lambda}_{\mathfrak{p}})$, we have the following

identity between sets:

$$\begin{aligned}\left\{\mathscr{Z}(\check{x},\check{h}) \;\middle|\; \check{x}\in\check{H}(\mathbb{Q})\backslash(\check{V}\cap\check{\Lambda}_{\mathfrak{p}}-\{0\}),\check{h}\in\check{H}_{\check{x}}(\mathbb{Q})\backslash\check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}\right\}\\ =\left\{[\mathscr{Z}(x,h)]^{\wedge}_{\mathsf{spe}} \;\middle|\; x\in H(\mathbb{Q})\backslash(V\cap\Lambda_{\mathfrak{p}}-\{0\}),h\in H_x(\mathbb{Q})\backslash H_f^{\mathfrak{p}}/K^{\mathfrak{p}}\right\}. \quad (4\text{-}9)\end{aligned}$$

Remark 4.10. In the case $F=\mathbb{Q}$, the Shimura datum (H, X) is of PEL type. But for our later computation, we still want to change the datum. Recall that from the hermitian space V, we get a unique quaternion algebra B over $\mathbb{Q}$ which is indefinite. Now the group $H^\dagger = B^\times$ and $M^\dagger_{K^\dagger} := \mathsf{Sh}_{K^\dagger}(H^\dagger, X^\dagger)$ is just the usual Shimura curve over $\mathbb{Q}$ (the modular curve if B is the matrix algebra). In this case, we don't introduce the auxiliary imaginary field $\mathbb{Q}(\sqrt{\lambda})$ anymore. We still have Proposition 4.3.

For moduli interpretations, it is well known that $M^\dagger_{K^\dagger}$ parametrizes abelian surfaces with $\mathcal{O}_B$ action and $K^\dagger$-level structure. But notice that for local decomposition, we only have one term which is $B_p = B_1^2$. If B_p is a matrix algebra for some rational prime p we still have the object $(\mathscr{A}_{p^\infty})_1^{2,\flat}$, which is just $(\mathscr{A}_{p^\infty})^\flat$, and Proposition 4.5 holds. In particular, when $M^\dagger_{K^\dagger}$ is the modular curve, we use the results in [Katz and Mazur 1985] to construct the compactified integral models.

The basic abelian scheme in this case is just the elliptic curve over $\mathcal{O}_{E^{\mathrm{nr}}_{p^\circ}}$ with generic fiber of CM type $(E, \iota^\bullet)$. Moreover, we have a similar but simpler version of Propositions 4.7 and 4.8. For various kinds of special cycles, we can define similar notions and their relation (4-9) still holds.

4C. ***Local intersection numbers.*** In this section, we study the formal scheme $\mathcal{N}$ and its formal special subschemes $\mathscr{Z}(\check{x})$. Then we compute certain intersection numbers of these formal special subschemes. In fact, the case we consider is essentially the same one as in [Kudla and Rapoport 2011] with the signature $(1, 1)$, only with mild modifications. We keep assuming that $\epsilon(\mathbb{V}_{\mathfrak{p}}) = 1$ and $\mathfrak{p}$ is nonsplit in E.

Formal special subschemes. Let $(\boldsymbol{X}, i_{\boldsymbol{X}})$ be as in the last section. Then $\boldsymbol{X}^\flat$ is a formal $\mathcal{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2. We define a moduli functor $\mathcal{N}$ on $\mathfrak{Nilp}_{\mathcal{O}_{\hat{F}_{\mathfrak{p}}}}$, the category of $\mathcal{O}_{\hat{F}_{\mathfrak{p}}}$-schemes where ϖ is locally nilpotent. For any $S\in\mathsf{Obj}\,\mathfrak{Nilp}_{\mathcal{O}_{\hat{F}_{\mathfrak{p}}}}$, $\mathcal{N}(S)$ is the set of equivalence classes of the couples (G, ρ_G) where

- G is an $\mathcal{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2 over S, and
- $\rho_G : G\times_S S_{\mathsf{spe}} \to \boldsymbol{X}^\flat\times_{\mathbb{F}} S_{\mathsf{spe}}$ is a quasiisogeny of height 0 (isomorphism actually).

Two couples (G, ρ_G), $(G', \rho_{G'})$ are equivalent if there is an isomorphism $G'\to G$ sending ρ_G to $\rho_{G'}$. Then $\mathcal{N}$ is represented by a formal scheme of finite type over $\mathsf{Spf}\,\mathcal{O}_{\hat{F}_{\mathfrak{p}}}$ of relative dimension 1 which is just $\mathsf{Spf}\,\mathcal{O}_{\hat{F}_{\mathfrak{p}}}[\![t]\!]$.

Recall that we have a two-dimensional $E_{\mathfrak{p}^\circ}$-hermitian space $\check{V}_{\mathfrak{p}} = \mathrm{Hom}((\boldsymbol{Y}, j), (\boldsymbol{X}, i_{\boldsymbol{X}})) \otimes \mathbb{Q} \cong V^-$. For any $\check{x} \in \check{\Lambda}_{\mathfrak{p}} - \{0\}$, we define a subfunctor $\mathscr{Z}(\check{x})$ of $\mathcal{N}$ as follows: for any $S \in \mathrm{Obj}\, \mathfrak{Nilp}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}$, $\mathscr{Z}(\check{x})(S)$ is the set of equivalence classes of (G, ρ_G) such that the composed homomorphism

$$\mathcal{Y} \times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} S_{\mathsf{spe}} = \boldsymbol{Y} \times_{\mathbb{F}} S_{\mathsf{spe}} \xrightarrow{\check{x}} \boldsymbol{X} \times_{\mathbb{F}} S_{\mathsf{spe}} \xrightarrow{(\rho_G^\sharp)^{-1}} G^\sharp \times_S S_{\mathsf{spe}}$$

extends to a homomorphism $\mathcal{Y} \times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} S \to G^\sharp$. Then $\mathscr{Z}(\check{x})$ is represented by a closed formal subscheme of $\mathcal{N}$ (in fact, one can show that it is a relative divisor as in [Kudla and Rapoport 2011, Proposition 3.5]). For linearly independent $\vec{x} = (\check{x}_1, \check{x}_2) \in (\check{\Lambda}_{\mathfrak{p}} - \{0\})^2$, two formal special subschemes $\mathscr{Z}(\check{x}_1)$ and $\mathscr{Z}(\check{x}_2)$ intersect properly at the unique closed point in $\mathcal{N}_{\mathsf{red}}$. Assuming that $\vec{y} = \vec{x}g$ for some $g \in \mathrm{GL}_2(\mathbb{O}_{E_{\mathfrak{p}^\circ}})$ such that $\vec{y} = (\check{y}_1, \check{y}_2)$ has the moment matrix $T(\vec{y}) = \mathrm{diag}[\varpi^a, \varpi^b]$, then of $a, b \geq 0$, one is odd and the other is even.

Intersection numbers. Now we assume that $\mathfrak{p} \nmid 2$ *is unramified in* E. Let $\overline{\boldsymbol{Y}}$ be the unique (up to isomorphism) formal $\mathbb{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2 over $\mathbb{F}$ but with action by $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$ such that the induced character of $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$ on $\mathrm{Lie}(\overline{\boldsymbol{Y}})$ is the one twisted by τ from that on $\mathrm{Lie}(\boldsymbol{Y})$. By [Kudla and Rapoport 2011, Lemma 4.2], there is an isomorphism

$$\rho_{\boldsymbol{X}} : \boldsymbol{Y} \times \overline{\boldsymbol{Y}} \to \boldsymbol{X}$$

in $\mathrm{Hom}_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}(\boldsymbol{Y} \times \overline{\boldsymbol{Y}}, \boldsymbol{X})$ such that, as elements of $\mathrm{Hom}_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}(\boldsymbol{Y}, \boldsymbol{Y} \times \overline{\boldsymbol{Y}})$,

$$\rho_{\boldsymbol{X}}^{-1} \circ \check{y}_i = \begin{cases} \mathsf{inc}_i \circ \Pi^a, & i = 1, \\ \mathsf{inc}_i \circ \Pi^b, & i = 2, \end{cases}$$

where inc_i denotes the inclusion into the i-th factor of the product and Π is a fixed uniformizer of $\mathrm{End}(\boldsymbol{Y})$. We identify $\boldsymbol{Y} \times \overline{\boldsymbol{Y}}$ with $\boldsymbol{X}$, and hence $\check{\Lambda}_{\mathfrak{p}}$ with $\mathrm{Hom}_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}(\boldsymbol{Y}, \boldsymbol{Y} \times \overline{\boldsymbol{Y}})$. If we denote by $\mathrm{Def}(\boldsymbol{X}^\flat; \vec{x})$ the subring of $\mathrm{Def}(\boldsymbol{X}^\flat) = \mathbb{O}_{\hat{F}_{\mathfrak{p}}}[\![t]\!]$ deforming $\vec{x}$, then

$$\mathscr{Z}(\check{x}_1) \cdot \mathscr{Z}(\check{x}_2) = \mathsf{length}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathrm{Def}(\boldsymbol{X}^\flat; \vec{x}) = \mathsf{length}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathrm{Def}(\boldsymbol{X}^\flat; \vec{y}) = \mathscr{Z}(\check{y}_1) \cdot \mathscr{Z}(\check{y}_2).$$

Let F_s be a quasicanonical lifting of level s which is an $\mathbb{O}_{F_{\mathfrak{p}}}$-module over $\mathbb{O}_{\hat{F}^s_{\mathfrak{p}}}$, unique up to the Galois action (see [Gross 1986]). Hence it defines a morphism $\mathrm{Spf}\, \mathbb{O}_{\hat{F}^s_{\mathfrak{p}}} \to \mathcal{N}$ which is a closed immersion. Let $\mathscr{Z}_s$ be the divisor on $\mathcal{N}$ defined by the image, which is independent of the choice of F_s. We have the following proposition generalizing [Kudla and Rapoport 2011, Proposition 8.1].

Proposition 4.11. *As divisors on* $\mathcal{N}$,

$$\mathscr{Z}(\check{y}_1) = \sum_{\substack{s=0 \\ \text{even}}}^{a} \mathscr{Z}_s, \qquad \mathscr{Z}(\check{y}_2) = \sum_{\substack{s=1 \\ \text{odd}}}^{b} \mathscr{Z}_s.$$

Proof. The original proof of [Kudla and Rapoport 2011, Proposition 8.1] again works for one direction:

$$\sum_{\substack{s=0\\ \text{even}}}^{a} \mathscr{Z}_s \le \mathscr{Z}(\check{y}_1), \qquad \sum_{\substack{s=1\\ \text{odd}}}^{b} \mathscr{Z}_s \le \mathscr{Z}(\check{y}_2).$$

To prove the other side, we only need to prove that the intersection multiplicities of both sides with the special fiber $\mathcal{N}_{\mathsf{spe}} = \mathsf{Spf}\, \mathbb{F}[\![t]\!]$ are the same. For the left side we have

$$\sum_{\substack{s=0\\ \text{even}}}^{a} \mathscr{Z}_s \cdot \mathcal{N}_{\mathsf{spe}} = \sum_{\substack{s=0\\ \text{even}}}^{a} [\mathbb{O}_{F_{\mathfrak{p}}}^{\times} : U_{F_{\mathfrak{p}}}^{s}] = \frac{q^{a+1}-1}{q-1},$$

$$\sum_{\substack{s=1\\ \text{odd}}}^{b} \mathscr{Z}_s \cdot \mathcal{N}_{\mathsf{spe}} = \sum_{\substack{s=1\\ \text{odd}}}^{b} [\mathbb{O}_{F_{\mathfrak{p}}}^{\times} : U_{F_{\mathfrak{p}}}^{s}] = \frac{q^{b+1}-1}{q-1},$$

where q is the cardinality of the residue field of $F_{\mathfrak{p}}$. The assertion follows from the following proposition, which is a generalization of [Kudla and Rapoport 2011, Proposition 8.2]. □

Proposition 4.12. *For* $\check{y} \in \mathsf{Hom}_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}}(\boldsymbol{Y}, \boldsymbol{X})$, *the intersection multiplicity*

$$\mathscr{Z}(\check{y}) \cdot \mathcal{N}_{\mathsf{spe}} = \frac{q^{v+1}-1}{q-1},$$

where $v \geqslant 0$ *is the valuation of* $(\check{y}, \check{y})'$.

Proof. We generalize the proof of [Kudla and Rapoport 2011, Proposition 8.2] to the case $F_{\mathfrak{p}} \neq \mathbb{Q}_p$ again by using the theory of windows and displays of p-divisible groups [Zink 2001; 2002]. In the proof, we simply write $F = F_{\mathfrak{p}}$, $E = E_{\mathfrak{p}^\circ}$ and let e and f be the ramification index and extension degree of residue fields of $F/\mathbb{Q}_p$, respectively. Then $q = p^f$. Let $R = \mathbb{F}[\![t]\!]$ and $A = W[\![t]\!]$ where $W = W(\mathbb{F})$. We extend the Frobenius automorphism σ on W to A by setting $\sigma(t) = t^p$. For any $s \geq 1$, we set $R_s = R/t^s$ and $A_s = A/t^s$. Then A (resp. A_s) is a frame of R (resp. R_s). The category of formal p-divisible groups over R is equivalent to the category of pairs (M, α), consisting of a free A-module of finite rank and an A-linear injective homomorphism $\alpha : M \to M^{(\sigma)} := A \otimes_{A,\sigma} M$, such that $\mathsf{coker}(\alpha)$ is a free R-module.

First, we treat the case $f = 1$. Consider the p-divisible group $\boldsymbol{Y}$ over $\mathbb{F}$ of dimension 1 and (absolute) height $2e$ with action by $\mathbb{O}_E$. It corresponds to the pair (N, β), where N is the $\mathbb{Z}/2$-graded free $\mathbb{O}_{\hat{F}} = \mathbb{O}_F \otimes_{\mathbb{Z}_p} W$-module of rank 2 (which is a free W-module of rank $2e$) with $N_i = \mathbb{O}_{\hat{F}} \cdot n_i$ $(i = 1, 2)$ and $\beta(n_0) = \varpi \otimes n_1$, $\beta(n_1) = 1 \otimes n_0$. We extend $\mathbb{O}_F$-linearly the Frobenius automorphism on W to $\mathbb{O}_{\hat{F}}$.

Similarly as in the proof of [Kudla and Rapoport 2011, Proposition 8.2], the p-divisible group $\boldsymbol{X}$ over $\mathbb{F}$ corresponds to (M, α) there and its universal deformation is (M, α_t). The only difference is that we should replace p by ϖ. The proof there works exactly in this case.

Now we treat the general case and assume that $f \geq 2$. Consider the p-divisible group $\boldsymbol{Y}$ over $\mathbb{F}$. It corresponds to the pair (N, β), where N is a $\mathbb{Z}/2$-graded free $\mathbb{O}_F \otimes_{\mathbb{Z}_p} W$-module of rank 2. Since

$$\mathbb{O}_F \otimes_{\mathbb{Z}_p} W = \bigoplus_{j=0}^{f-1} \mathbb{O}_F \otimes_{W(k),\sigma^j} W =: \bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)},$$

where k is the residue field of F, we can write

$$N = \left(\bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)} e_{0,j}\right) \oplus \left(\bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)} e_{1,j}\right)$$

and $\beta(e_{i,j}) = e_{i,j+1}$ for $i = 1, 2$ and $0 \leqslant j < f-1$; $\beta(e_{0,f-1}) = e_{1,0}$ and $\beta(e_{1,f-1}) = \varpi e_{0,0}$. Similarly, the p-divisible group $\overline{\boldsymbol{Y}}$ corresponds to $(\overline{N}, \bar{\beta})$ where we write

$$\overline{N} = \left(\bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)} \bar{e}_{0,j}\right) \oplus \left(\bigoplus_{j=0}^{f-1} \mathbb{O}_{\hat{F}}^{(\sigma^j)} \bar{e}_{1,j}\right)$$

and $\bar{\beta}(\bar{e}_{i,j}) = \bar{e}_{i,j+1}$ for $i = 0, 1$ and $0 \leqslant j < f-1$; $\bar{\beta}(\bar{e}_{1,f-1}) = \bar{e}_{0,0}$ and $\bar{\beta}(\bar{e}_{0,f-1}) = \varpi \bar{e}_{1,0}$. Then we extend them to $\mathbb{F}[\![t]\!]$ by scalars, still denoted by N and $\overline{N}$.

The p-divisible group $\boldsymbol{X}$ corresponds the direct sum $(M, \alpha) := (N, \beta) \oplus (\overline{N}, \bar{\beta})$. Under the basis $\{e_{0,0}, \bar{e}_{1,0}, \ldots, e_{0,f-1}, \bar{e}_{1,f-1}; e_{1,0}, \bar{e}_{0,0}, \ldots, e_{1,f-1}, \bar{e}_{0,f-1}\}$, the matrix of α is

$$\alpha = \begin{pmatrix} & & & & & & & \varpi & \\ & & & & & & & & \varpi \\ 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & & & & & & & \end{pmatrix}.$$

Let (M, α_t) corresponds to the universal deformation of (X, i_X) over $\mathbb{F}[\![t]\!]$. We can write in the same basis

$$\alpha_t = \begin{pmatrix} 1 & & & & t & & \\ & 1 & & -t & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix} \cdot \alpha.$$

Explicitly, we have

$$\begin{aligned} \alpha_t(e_{i,j}) &= e_{i,j+1}, \quad i = 0, 1 \text{ and } j = 0, \dots, f-2, \\ \alpha_t(e_{0,f-1}) &= e_{1,0} - t\bar{e}_{1,0}, \qquad \alpha_t(e_{1,f-1}) = \varpi e_{0,0}, \\ \alpha_t(\bar{e}_{i,j}) &= \bar{e}_{i,j+1}, \quad i = 0, 1 \text{ and } j = 0, \dots, f-2, \\ \alpha_t(\bar{e}_{1,f-1}) &= \bar{e}_{0,0} + te_{0,0}, \qquad \alpha_t(\bar{e}_{0,f-1}) = \varpi \bar{e}_{1,0}. \end{aligned}$$

Now we denote by $\sigma^k(\alpha) : M^{(\sigma^k)} \to M^{(\sigma^{k+1})}$ the induced homomorphism for $k \geqslant 0$. Then, formally, we have

$$\begin{aligned} \sigma^k(\alpha)^{-1}(e_{i,j}) &= e_{i,j-1}, \quad i = 1, 2 \text{ and } j = 1, \dots, f-1, \\ \sigma^k(\alpha)^{-1}(e_{0,0}) &= \frac{1}{\varpi} e_{1,f-1}, \qquad \sigma^k(\alpha)^{-1}(e_{1,0}) = e_{0,f-1} + \frac{t^{p^k}}{\varpi} \bar{e}_{0,f-1}, \\ \sigma^k(\alpha)^{-1}(\bar{e}_{i,j}) &= \bar{e}_{i,j-1}, \quad i = 1, 2 \text{ and } j = 1, \dots, f-1, \\ \sigma^k(\alpha)^{-1}(\bar{e}_{1,0}) &= \frac{1}{\varpi} \bar{e}_{0,f-1}, \qquad \sigma^k(\alpha)^{-1}(\bar{e}_{0,0}) = \bar{e}_{1,f-1} - \frac{t^{p^k}}{\varpi} e_{1,f-1}. \end{aligned}$$

Now let $\check{y}$ correspond to the graded A_1-linear homomorphism $\gamma : N \otimes_A A_1 \to M$. Then the length $\ell = \mathscr{Z}(\check{y}) \cdot \mathscr{N}_{\mathsf{spe}}$ of the deformation space of γ is the maximal number a such that there exists a diagram of the form

$$\begin{array}{ccc} N & \xrightarrow{\beta} & N^{(\sigma)} \\ {\scriptstyle \tilde{\gamma}}\downarrow & & \downarrow{\scriptstyle \tilde{\gamma}^{(\sigma)}} \\ M & \xrightarrow{\alpha_t} & M^{(\sigma)} \end{array}$$

that commutes modulo t^a, where $\tilde{\gamma}$ lifts γ.

Case i: $v = 2r$ *is even.* We may assume that $\gamma = \varpi^r \mathsf{inc}_1$, represented by the $4f \times 2f$ matrix

$$X(0) = \begin{pmatrix} \varpi^r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \varpi^r & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varpi^r \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If $r = 0$, in order to lift $\gamma \bmod t^p$, we search for a $4f \times 2f$ matrix $X(1)$ with entries in A_p such that $X(1) \equiv X(0)$ in A_1 and satisfies

$$\alpha_t \circ X(1) = \sigma(X(1)) \circ \beta.$$

But $\sigma(X(1)) = \sigma(X(0)) = X(0)$. Hence we need to find the largest $a \leqslant p$ such that $\alpha_t^{-1} \circ X(0) \circ \beta$ has integral entries $\bmod t^a$. But the entry at the place $(e_{0,f-1}, \bar{e}_{0,f-1})$ is t/ϖ, so the largest a is just 1. Hence when $v = r = 0$, the proposition holds.

If $r > 0$, we first show that we can lift $\gamma \bmod t^{q^{2r}}$. By induction, we introduce $X(k)$ for $k \geq 1$ by requiring that $X(k+1) \equiv X(k)$ in A_{p^k} and $\alpha_t \circ X(k+1) = \sigma(X(k+1)) \circ \beta$. But $\sigma(X(k+1)) = \sigma(X(k))$; hence formally we should have $X(k+1) = \alpha_t^{-1} \circ \sigma(X(k)) \circ \beta$. We need to show that

$$X(2rf) = \alpha_t^{-1} \circ \sigma(\alpha_t)^{-1} \circ \cdots \circ \sigma^{2rf-1}(\alpha_t)^{-1} \circ X(0) \circ \beta^{2rf}$$

has integral entries. Let $x_{i,j;i',j'}$ (resp. $\bar{x}_{i,j;i',j'}$) be the entry of $X(2rf) \bmod \varpi$ at the place $(e_{i,j}, e_{i',j'})$ (resp. $(e_{i,j}, \bar{e}_{i',j'})$). Then among all these terms, the only nonzero terms are

$$\bar{x}_{0,j;0,j} = (-1)^{r-1} t^{p^{f-1-j}(q^{2r-2}+q^{2r-3}+\cdots+1)}, \qquad j = 0, \ldots, f-1,$$
$$x_{1,j;1,j} = (-1)^{r} t^{p^{f-1-j}(q^{2r-1}+q^{2r-2}+\cdots+1)}, \qquad j = 0, \ldots, f-1,$$

which shows that we can lift $\gamma \bmod t^{q^{2r}}$. Next, we consider the lift of $\gamma \bmod t^{pq^{2r}}$, that is, the matrix

$$X(2rf+1) = \alpha_t^{-1} \circ \sigma(X(2rf)) \circ \beta.$$

It has exactly one entry which is not integral: the place of $(e_{0,f-1}, \bar{e}_{0,f-1})$ whose nonintegral part is

$$\frac{t}{\varpi}(-1)^r t^{p \cdot p^{f-1}(q^{2r-1}+q^{2r-2}+\cdots+1)} = \frac{(-1)^r}{\varpi} t^{q^{2r}+q^{2r-1}+\cdots+1}.$$

It turns out that the length $\ell = \mathscr{Z}(\check{y}) \cdot \mathscr{N}_{\mathsf{spe}}$ is exactly

$$\frac{q^{2r+1}-1}{q-1} = \frac{q^{v+1}-1}{q-1}.$$

Case ii: $v = 2s+1$ *is even.* We may assume that We may assume that $\gamma = \varpi^s \mathsf{inc}_2 \circ \Pi$, where Π is the endomorphism of $\boldsymbol{Y}$ determined by $\Pi(e_{0,j}) = e_{0,j}$ and $\Pi(e_{1,j}) = \varpi e_{0,j}$ for $j = 0, \dots, f-1$. Then γ is represented by the $4f \times 2f$ matrix

$$\begin{pmatrix} & & & 0 & \cdots & 0 \\ & & & \varpi^{s+1} & \cdots & 0 \\ & & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & 0 \\ & & & 0 & \cdots & \varpi^{s+1} \\ 0 & \cdots & 0 & & & \\ \varpi^s & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & & \\ 0 & \cdots & 0 & & & \\ 0 & \cdots & \varpi^s & & & \end{pmatrix}.$$

Similarly, we introduce the notation $Y(k)$ for $k \geq 0$. We first show that γ lifts mod $t^{q^{2s+1}}$, that is, the matrix

$$Y((2s+1)f) = \alpha_t^{-1} \circ \sigma(\alpha_t)^{-1} \circ \cdots \circ \sigma^{(2s+1)f-1}(\alpha_t)^{-1} \circ Y(0) \circ \beta^{(2s+1)f}$$

has integral entries. Let $y_{i,j;i',j'}$ (resp. $\bar{y}_{i,j;i',j'}$) be the entry of $Y((2s+1)f)$ mod ϖ at the place $(e_{i,j}, e_{i',j'})$ (resp. $(e_{i,j}, \bar{e}_{i',j'})$). Then among all these terms, the only nonzero terms are

$$\begin{aligned} \bar{y}_{0,j;0,j} &= (-1)^s t^{p^{f-1-j}(q^{2s-1}+q^{2s-2}+\cdots+1)}, \qquad j = 0, \dots, f-1, \\ y_{1,j;1,j} &= (-1)^{s+1} t^{p^{f-1-j}(q^{2s}+q^{2s-1}+\cdots+1)}, \qquad j = 0, \dots, f-1, \end{aligned}$$

which shows that we can lift γ mod $t^{q^{2s+1}}$. Next we consider the lift of γ mod $t^{pq^{2s+1}}$, that is, the matrix

$$Y((2s+1)f+1) = \alpha_t^{-1} \circ \sigma\big(Y((2s+1)f)\big) \circ \beta.$$

It has exactly one entry which is not integral: the place of $(e_{0,f-1}, \bar{e}_{0,f-1})$ whose nonintegral part is

$$\frac{t}{\varpi}(-1)^{s+1} t^{p \cdot p^{f-1}(q^{2s}+q^{2s-1}+\cdots+1)} = \frac{(-1)^{s+1}}{\varpi} t^{q^{2s+1}+q^{2s}+\cdots+1}.$$

It turns out that the length $\ell = \mathscr{L}(\check{y}) \cdot \mathscr{N}_{\mathsf{spe}}$ is exactly

$$\frac{q^{2s+2}-1}{q-1} = \frac{q^{v+1}-1}{q-1}.$$

This proves the proposition. □

The results in [Görtz and Rapoport 2007] used in the proof of [Kudla and Rapoport 2011, Proposition 8.4] also work for $F_\mathfrak{p}$, not just $\mathbb{Q}_p$. For $0 < s \le b$ odd we have

$$\mathscr{Z}(\check{y}_1)\cdot\mathscr{Z}_s = \begin{cases} \dfrac{q^{a+1}-1}{q-1}, & a < s, \\ \dfrac{q^s-1}{q-1} + \dfrac{1}{2}(a+1-s)[\mathbb{O}_{F_\mathfrak{p}}^\times : U_{F_\mathfrak{p}}^s], & a \ge s. \end{cases}$$

By summing over s, we get the following *local arithmetic Siegel–Weil formula* at a good finite place:

Theorem 4.13. *Let $\check{x}_i \in \check{\Lambda}_\mathfrak{p} - \{0\}$, for $i = 1, 2$, be linearly independent. Then the intersection multiplicity $\mathscr{Z}(\check{x}_1)\cdot\mathscr{Z}(\check{x}_2)$ only depends on the $\mathrm{GL}_2(\mathbb{O}_{E_{\mathfrak{p}^\circ}})$-equivalence class of the moment matrix $T = T(\vec{x})$. Moreover, if we set $T \sim \mathrm{diag}[\varpi^a, \varpi^b]$ with $0 \le a < b$, then*

$$H_\mathfrak{p}(T) := \mathscr{Z}(\check{x}_1)\cdot\mathscr{Z}(\check{x}_2) = \frac{1}{2}\sum_{l=0}^{a} q^l(a+b+1-2l),$$

where q is the cardinality of the residue field of $F_\mathfrak{p}$.

4D. ***Comparison at good places.*** In this section, we will consider the local height pairing $\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}$ at a finite place v° of E which is good. Recall that we have a Shimura curve M_K and we assume that K is sufficiently small and decomposable. We also assume that ϕ_i $(i = 1, 2)$ are decomposable.

Let $\mathcal{S} \subset \Sigma_f$ be a finite subset with $|\mathcal{S}| \ge 2$ such that for any finite place $\mathfrak{p} \notin \mathcal{S}$, we have

- $\mathfrak{p} \nmid 2$, $\mathfrak{p}$ is unramified or split in E;
- $\epsilon(\mathbb{V}_\mathfrak{p}) = 1$;
- $\phi_{i,\mathfrak{p}} = \phi_\mathfrak{p}^0$ $(i = 1, 2)$ are the characteristic functions of a self-dual lattice $\Lambda_\mathfrak{p} = \Lambda^+ \subset \mathbb{V}_\mathfrak{p}$;
- $K_\mathfrak{p}$ is the subgroup of $\mathrm{U}(\mathbb{V}_\mathfrak{p})$ stabilizing $\Lambda_\mathfrak{p}$, that is, $K_\mathfrak{p}$ is a hyperspecial maximal compact subgroup;
- χ and ψ are unramified at $\mathfrak{p}$.

Fix a place $\mathfrak{p} \notin \mathcal{S}$ such that $\mathfrak{p}$ is nonsplit in E. We have the generating series

$$\begin{aligned} Z_{\phi_i}(g_i) &= \sum_{x_i \in K\backslash\mathbb{V}} \omega_\chi(g_i)\phi_i(x_i)Z(x_i)_K \\ &= \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(g_i)\phi_i(h_i^{-1}x_i)Z(h_i^{-1}x_i)_K, \end{aligned} \tag{4-10}$$

where $V = V^{(\iota_1)}$ is the nearby (coherent) hermitian space as in Section 4B. Write $g_{i,\mathfrak{p}} = n(b_{i,\mathfrak{p}})m(a_{i,\mathfrak{p}})k_{i,\mathfrak{p}}$ in the Iwasawa decomposition and choose any number $e_i \in E^\times$ such that $\mathrm{val}_{\mathfrak{p}}(e_i) = \mathrm{val}_{\mathfrak{p}}(a_{i,\mathfrak{p}})$. Let $\tilde{g}_i = m(e_i)^{-1} g_i$; then

$$\begin{aligned}
(4\text{-}10) &= \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(\tilde{g}_i)\phi_i(h_i^{-1}x_i e_i) Z(h_i^{-1}x_i)_K \\
&= \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(\tilde{g}_i)\phi_i(h_i^{-1}x_i e_i) Z(h_i^{-1}x_i e_i)_K \\
&= \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \omega_\chi(\tilde{g}_i)\phi_i(h_i^{-1}x_i) Z(h_i^{-1}x_i)_K \\
&= \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{A}_f)\backslash H(\mathbb{A}_f)/K} \psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}} T(x_i))\phi_{\mathfrak{p}}^0 \otimes (\omega_\chi(\tilde{g}_i^{\mathfrak{p}})\phi_i^{\mathfrak{p}})(h_i^{-1}x_i) Z(h_i^{-1}x_i)_K
\end{aligned}$$

since in the Iwasawa decomposition $\tilde{g}_{i,\mathfrak{p}} = n(\tilde{b}_{i,\mathfrak{p}})m(\tilde{a}_{i,\mathfrak{p}})k_{i,\mathfrak{p}}$, $\tilde{a}_{i,\mathfrak{p}} \in \mathbb{O}_{E_{\mathfrak{p}^\circ}}$. In what follows, we assume that $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for at least one $v \in \mathcal{S}$ and $g_i \in P_v' H'(\mathbb{A}_F^v)$. Then if $Z(h_i^{-1}x_i)_K$ appears in the generating series $Z_{\phi_i}(g_i)$, we must have $x_i \in V - \{0\}$ and $h_{i,\mathfrak{p}}^{-1}x_i \in \Lambda_{\mathfrak{p}}$. Hence by the last part of Section 4B, we can extend $Z(h_i^{-1}x_i)_K$ to a union of integral special subschemes $\mathcal{Z}(x_i, h_i)$ on the smooth model $\mathcal{M}_{0,K^{\mathfrak{p}}}$. We define

$$\begin{aligned}
&\mathcal{Z}_{\phi_i}(g_i) \\
&= \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} \psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}} T(x_i))\phi_{\mathfrak{p}}^0 \otimes (\omega_\chi(\tilde{g}_i^{\mathfrak{p}})\phi_i^{\mathfrak{p}})(h_i^{-1}x_i)\mathcal{Z}(x_i, h_i),
\end{aligned}$$

which is a cycle of $\mathcal{M}_{0,K^{\mathfrak{p}}}$ extending the generating series $Z_{\phi_i}(g_i)$.

If $\mathfrak{p} \notin \mathcal{S}$ is split in E, we just take $\mathcal{Z}_{\phi_i}(g_i)$ to be the Zariski closure of $Z_{\phi_i}(g_i)$ in $\mathcal{M}_{0,K^{\mathfrak{p}}}$.

We now state the main theorem of this section. Here vol is the same volume as in [Liu 2011, Theorem 4.20].

Theorem 4.14. *Suppose that $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for at least one $v \in \mathcal{S}$ and $g_i \in P_v' H'(\mathbb{A}_F^v)$. Let $\mathfrak{p}$ be a finite place not in $\mathcal{S}$, let $\mathcal{M}_{K;\mathfrak{p}^\circ} = \mathcal{M}_{0,K^{\mathfrak{p}}}$ be the smooth local model introduced in Section 4B, and let $\mathcal{Z}_{\phi_i}(g_i)$ be the cycle introduced above.*

(1) *For $\mathfrak{p}$ nonsplit in E,*

$$E_{\mathfrak{p}}(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) = -\mathrm{vol}(K)\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ},$$

where, by definition,

$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ} = \log q^2 (\mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{Z}_{\phi_2}(g_2))$$

(2) $\mathcal{Z}_{\phi_1}(g_1) \cdot \mathcal{Z}_{\phi_2}(g_2) = 0$ *for $\mathfrak{p}$ split in E.*

Combining this with [Liu 2011, Theorem 4.20], we have:

Corollary 4.15. *Assume that $\phi_i = \phi_\infty^0 \phi_{i,f}$ satify $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for all $v \in \mathcal{S}$ and $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg},d_v}$ with $d_v \geq d_{\psi_v}$ for nonsplit $v \in \mathcal{S}$* (see Section 2C for the notation). *Assume further that $g_i \in e_{\mathcal{S}} H'(\mathbb{A}_F^{\mathcal{S}})$ $(i = 1, 2)$, and that the local model $\mathcal{M}_{K;\mathfrak{p}^\circ}$ is $\mathcal{M}_{0,K^{\mathfrak{p}}}$ for all finite places $\mathfrak{p}^\circ|\mathfrak{p} \notin \mathcal{S}$. Then*

$$E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) = -\mathrm{vol}(K) \sum_{\substack{v^\circ | v \\ v \notin \mathcal{S}}} \langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{v^\circ}.$$

(Here the Green's functions used in archimedean places are those defined in [Liu 2011, Theorem 4.20], not the admissible Green's functions defined in Section 3B.)

Proof of Theorem 4.14. (1) Since the special fiber of $\mathscr{Z}_{\phi_i}(g_i)$ locates in the supersingular locus, we have

$$\mathscr{Z}_{\phi_1}(g_1) \cdot \mathscr{Z}_{\phi_2}(g_2) = [\mathscr{Z}_{\phi_1}(g_1)]^\wedge_{\mathsf{spe}} \cdot [\mathscr{Z}_{\phi_2}(g_2)]^\wedge_{\mathsf{spe}}. \tag{4-11}$$

But

$$[\mathscr{Z}_{\phi_i}(g_i)]^\wedge_{\mathsf{spe}} = \sum_{\check{x}_i \in \check{H}(\mathbb{Q}) \backslash \check{V}} \sum_{\check{h}_i \in \check{H}_{\check{x}_i}(\mathbb{Q}) \backslash \check{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}}} \psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}} T(\check{x}_i)) \check{\phi}_{\mathfrak{p}}^0 \otimes \omega_\chi(\tilde{g}_i^{\mathfrak{p}}) \phi_i^{\mathfrak{p}}(\check{h}_i^{-1} \check{x}_i) \mathscr{Z}(\check{x}_i, \check{h}_i), \tag{4-12}$$

where $\check{\phi}_{\mathfrak{p}}^0$ is the characteristic function of $\check{\Lambda}_{\mathfrak{p}}$. For any $t_i \in F \cap \mathbb{O}_{F_{\mathfrak{p}}}$ which is totally positive, we fix an element $\check{x}_{t_i} \in \check{V} \cap \check{\Lambda}_{\mathfrak{p}}$ with $T(\check{x}_{t_i}) = t_i$. Then

$$\begin{aligned} (4\text{-}12) &= \sum_{t_i} \psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}} t_i) \sum_{\check{h}_i \in \check{H}_{\check{x}_i}(\mathbb{Q}) \backslash \check{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}}} \omega_\chi(\tilde{g}_i^{\mathfrak{p}}) \phi_i^{\mathfrak{p}}(\check{h}_i^{-1} \check{x}_i) \mathscr{Z}(\check{x}_i, \check{h}_i) \\ &= \sum_{\check{x}_i \in \check{V} \cap \check{\Lambda}_{\mathfrak{p}} - \{0\}} \psi_{\mathfrak{p}}(\tilde{b}_{i,\mathfrak{p}} T(\check{x}_i)) \sum_{\check{h}_i \in \check{H}(\mathbb{Q}) \backslash \check{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}}} \omega_\chi(\tilde{g}_i^{\mathfrak{p}}) \phi_i^{\mathfrak{p}}(\check{h}_i^{-1} \check{x}_i) \mathscr{Z}(\check{x}_i, \check{h}_i). \end{aligned}$$

Two formal cycles $\mathscr{Z}(\check{x}_1, \check{h}_1)$ and $\mathscr{Z}(\check{x}_2, \check{h}_2)$ intersect only if $\check{h}_1$ and $\check{h}_2$ are in the same double coset of $\check{H}(\mathbb{Q}) \backslash \check{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}}$. Hence

$$\begin{aligned} (4\text{-}11) &= \sum_{\vec{x} = (x_1, x_2)} \psi_{\mathfrak{p}}(\tilde{b}_{\mathfrak{p}} T(\vec{x})) \sum_{\check{h} \in \check{H}(\mathbb{Q}) \backslash \check{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}}} \omega''_\chi(\iota(\tilde{g}_1^{\mathfrak{p}}, \tilde{g}_2^{\mathfrak{p},\vee})) \phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\check{h}^{-1} \vec{x}) \mathscr{Z}(\check{x}_1) \cdot \mathscr{Z}(\check{x}_2) \\ &= \sum_{T \in \mathrm{GL}_2(E_{\mathfrak{p}^\circ}) \cap \mathrm{Her}_2(\mathbb{O}_{E_{\mathfrak{p}^\circ}})} \psi_{\mathfrak{p}}(\tilde{b}_{\mathfrak{p}} T(\vec{x})) \\ &\qquad \times \sum_{\check{h} \in \check{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}}} \omega''_\chi(\iota(\tilde{g}_1^{\mathfrak{p}}, \tilde{g}_2^{\mathfrak{p},\vee})) \phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\check{h}^{-1} \vec{x}) H_{\mathfrak{p}}(T), \end{aligned} \tag{4-13}$$

where $\tilde{b}_{\mathfrak{p}} = \mathrm{diag}[\tilde{b}_{1,\mathfrak{p}}, \tilde{b}_{2,\mathfrak{p}}]$. By Theorem 4.13, Corollary 4.2 and following the same steps in the proof of [Liu 2011, Theorem 4.20], we get

$$-\mathrm{vol}(K)\log q^2(\mathscr{Z}_{\phi_1}(g_1)\cdot\mathscr{Z}_{\phi_2}(g_2)) = E_{\mathfrak{p}}(0, \iota(\tilde{g}_1, \tilde{g}_2^{\vee}), \phi_1\otimes\phi_2). \tag{4-14}$$

By definition,

(4-14)

$$= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W_T'(0, \iota(\tilde{g}_{1,\mathfrak{p}}, \tilde{g}_{2,\mathfrak{p}}^{\vee}), \phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}) \prod_{v\neq\mathfrak{p}} W_T(0, \iota(\tilde{g}_{1,v}, \tilde{g}_{2,v}^{\vee}), \phi_{1,v}\otimes\phi_{2,v})$$

$$= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W_{e^{\tau}Te}'(0, \iota(g_{1,\mathfrak{p}}, g_{2,\mathfrak{p}}^{\vee}), \phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}) \times \prod_{v\neq\mathfrak{p}} W_{e^{\tau}Te}(0, \iota(g_{1,v}, g_{2,v}^{\vee}), \phi_{1,v}\otimes\phi_{2,v})$$

$$= \sum_{\mathrm{Diff}(T,\mathbb{V})=\{\mathfrak{p}\}} W_T'(0, \iota(g_{1,\mathfrak{p}}, g_{2,\mathfrak{p}}^{\vee}), \phi_{1,\mathfrak{p}}\otimes\phi_{2,\mathfrak{p}}) \prod_{v\neq\mathfrak{p}} W_T(0, \iota(g_{1,v}, g_{2,v}^{\vee}), \phi_{1,v}\otimes\phi_{2,v})$$

$$= E_{\mathfrak{p}}(0, \iota(g_1, g_2^{\vee}), \phi_1\otimes\phi_2),$$

where $e = \mathrm{diag}[e_1, e_2] \in \mathsf{GL}_2(E)$.

(2) We will prove this in a more general case in Lemma 5.1. □

5. Comparison at finite places: bad reduction

In this section, we discuss the contribution of the local height pairing at a finite place in $\mathcal{S}$. There are three cases we need to consider: the split case (that is, $\mathsf{U}(\mathbb{V}_{\mathfrak{p}})$ is split), the quasisplit case (that is, $\mathsf{U}(\mathbb{V}_{\mathfrak{p}})$ is quasisplit but not split) and the nonsplit case (that is, $\mathsf{U}(\mathbb{V}_{\mathfrak{p}})$ is not quasisplit).

5A. ***Split case.*** We first discuss the contribution of the local height pairing at a split (finite) place in $\mathcal{S}$.

We fix a prime $\mathfrak{p}\in\mathcal{S}$ which is split in E and any $\mathfrak{p}^{\circ}\in\Sigma_f^{\circ}$ over $\mathfrak{p}$. What we want to consider is the height pairing $\langle\widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^{\circ}}$ on a certain model $\mathscr{M}_{K;\mathfrak{p}^{\circ}}$, for $\phi_{1,\mathcal{S}}\otimes\phi_{2,\mathcal{S}}\in\mathcal{S}(\mathbb{V}_{\mathcal{S}}^2)_{\mathrm{reg}}$ and $g_i\in e_{\mathcal{S}}H'(\mathbb{A}_F^{\mathcal{S}})$. We assume that $K = K_{\mathfrak{p}}K^{\mathfrak{p}}$ with $K^{\mathfrak{p}}$ sufficiently small and $K_{\mathfrak{p}} = K_{\mathfrak{p},n}$ for $n\geq 0$, hence $M_K = M_{n,K^{\mathfrak{p}}}$. In Section 4B, we constructed a smooth integral model $\mathscr{M}_{0,K^{\mathfrak{p}}}$ for $M_{0,K^{\mathfrak{p}};\mathfrak{p}^{\circ}}$, a p-divisible group $\mathscr{X}\to\mathscr{M}_{0,K^{\mathfrak{p}}}$, and hence $\mathscr{X}^{\flat}\to\mathscr{M}_{0,K^{\mathfrak{p}}}$, which is an $\mathcal{O}_{F_{\mathfrak{p}}}$-module of dimension 1 and height 2. A Drinfeld ϖ^n-structure for an $\mathcal{O}_{F_{\mathfrak{p}}}$-module X of height 2 over an $\mathcal{O}_{F_{\mathfrak{p}}}$-scheme S is an $\mathcal{O}_{F_{\mathfrak{p}}}$-homomorphism,

$$\alpha_n : (\mathcal{O}_{F_{\mathfrak{p}}}/\varpi^n\mathcal{O}_{F_{\mathfrak{p}}})^2 \to X[\varpi^n](S),$$

such that the image forms a full set of sections of $X[\varpi^n]$ in the sense of [Katz and Mazur 1985, Section 1.8]. Let $\mathscr{M}_{n,K^{\mathfrak{p}}} = \mathscr{M}_{0,K^{\mathfrak{p}}}(n)$ be the universal scheme

over $\mathcal{M}_{0,K^{\mathfrak{p}}}$ of the Drinfeld ϖ^n-structure α_n (see [Harris and Taylor 2001, Lemma II.2.1]). Then $\mathcal{M}_{n,K^{\mathfrak{p}}}$ is regular, finite over $\mathcal{M}_{0,K^{\mathfrak{p}}}$, and its generic fiber is $M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. We compute the intersection number after a base change $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ} = M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ} \times_{F_{\mathfrak{p}}} F_{\mathfrak{p}}^n$. Then $\mathcal{M}'_{n,K^{\mathfrak{p}}}$, the normalization of $\mathcal{M}_{n,K^{\mathfrak{p}}} \times_{\mathbb{O}_{F_{\mathfrak{p}}}} \mathbb{O}_{F_{\mathfrak{p}}^n}$, is still regular and its generic fiber is $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. We denote by $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{ord}}$ the ordinary locus of the special fiber $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{spe}} = \mathcal{M}'_{n,K^{\mathfrak{p}}} \times_{\mathbb{O}_{F_{\mathfrak{p}}^n}} \mathbb{F}$ which is also the smooth locus. The set of connected components of $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{spe}}$ canonically corresponds to the set of geometric connected component of $M_{n,K^{\mathfrak{p}}}$, and hence to $E^{\times,1}\backslash\mathbb{A}_E^{\times,1}/\nu(K)$. The set of irreducible components on each connected component of $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{spe}}$, that is, the *Igusa curves*, corresponds to the set $\boldsymbol{P}(V_{\mathfrak{p}})/K_{\mathfrak{p},n}$, where $\boldsymbol{P}(V_{\mathfrak{p}})$ is the set of all $E_{\mathfrak{p}} \cong F_{\mathfrak{p}} \oplus F_{\mathfrak{p}}$-lines in $V_{\mathfrak{p}}$ where $\mathrm{U}(V_{\mathfrak{p}})$ acts from right by $l.h = h^{-1}l$ for $l \in \boldsymbol{P}(V_{\mathfrak{p}})$ and $h \in \mathrm{U}(V_{\mathfrak{p}})$. Together, the set of irreducible components of $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{spe}}$ is

$$\mathrm{Ig}_{n,K^{\mathfrak{p}}} := \boldsymbol{P}(V_{\mathfrak{p}})/K_{\mathfrak{p},n} \times (E^{\times,1}\backslash\mathbb{A}_E^{\times,1}/\nu(K)).$$

Now we consider the special cycles. We use the same notation for the base change of special cycles $Z(x)_K$ and the generating series $Z_{\phi_i}(g_i)$ on $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. As before, we denote by $\mathscr{Z}(x)_K$ (resp. $\mathscr{Z}_{\phi_i}(g_i)$) the Zariski closure of $Z(x)_K$ (resp. $Z_{\phi_i}(g_i)$) in $\mathcal{M}'_{n,K^{\mathfrak{p}}}$. Since $\mathfrak{p}$ is split in E, the special fiber $[\mathscr{Z}_{\phi_i}(g_i)]_{\mathrm{spe}} \subset [\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{ord}}$. Let $\boldsymbol{P}(V)$ be the set of E-lines in V. Then the set of geometric special points of $M_{n,K^{\mathfrak{p}}}$ (also of $M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$, $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$) is

$$\mathrm{Sp}_K := H(\mathbb{Q})\backslash\boldsymbol{P}(V) \times H(\mathbb{A}_f)/K = \coprod_{l\in H(\mathbb{Q})\backslash\boldsymbol{P}(V)} H_l(\mathbb{Q})\backslash H(\mathbb{A}_f)/K$$

and the set $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{ord}}(\mathbb{F})$ is

$$\coprod_{l\in H(\mathbb{Q})\backslash\boldsymbol{P}(V)} H_l(\mathbb{Q})\backslash\big((N_l\backslash\mathrm{U}(V_{\mathfrak{p}})/K_{\mathfrak{p},n}) \times H_f^{\mathfrak{p}}/K^{\mathfrak{p}}\big),$$

where $N_l \subset \mathrm{U}(V_{\mathfrak{p}})$ is the unipotent subgroup of the parabolic subgroup fixing l. The reduction map

$$\mathrm{Sp}_K \to [\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{ord}}(\mathbb{F}) \to \mathrm{Ig}_{n,K^{\mathfrak{p}}} \tag{5-1}$$

is given by $(l, h) \mapsto (l, h_{\mathfrak{p}}, h^{\mathfrak{p}}) \mapsto (h_{\mathfrak{p}}^{-1}l, \nu(h_{\mathfrak{p}}h^{\mathfrak{p}}))$ (see [Zhang 2001b, Section 5.4] for a discussion).

We compute the local height pairing on the model $\mathcal{M}'_{n,K^{\mathfrak{p}}}$. We write $\widehat{Z}_{\phi_i}(g_i) = \mathscr{Z}_{\phi_i}(g_i) + \mathscr{V}_{\phi_i}(g_i)$ for some cycle $\mathscr{V}_{\phi_i}(g_i)$ supported on the special fiber as in

Section 3B. Let $\omega_{\mathfrak{p}^\circ}$ be the base change of ω_K to $\mathcal{M}'_{n,K^{\mathfrak{p}}}$. We have

$$\begin{aligned}
&(\log q)^{-1}\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2)\rangle_{\mathfrak{p}^\circ}\\
&\quad=(\mathscr{Z}_{\phi_1}(g_1)+\mathscr{V}_{\phi_1}(g_1))\cdot(\mathscr{Z}_{\phi_2}(g_2)+\mathscr{V}_{\phi_2}(g_2)-E(g_2,\phi_2)\omega_{\mathfrak{p}^\circ}+E(g_2,\phi_2)\omega_{\mathfrak{p}^\circ})\\
&\quad=\mathscr{Z}_{\phi_1}(g_1)\cdot(\mathscr{Z}_{\phi_2}(g_2)+\mathscr{V}_{\phi_2}(g_2)-E(g_2,\phi_2)\omega_{\mathfrak{p}^\circ})\\
&\qquad\qquad +E(g_2,\phi_2)(\mathscr{Z}_{\phi_1}(g_1)+\mathscr{V}_{\phi_1}(g_1))\cdot\omega_{\mathfrak{p}^\circ}\\
&\quad=\mathscr{Z}_{\phi_1}(g_1)\cdot\mathscr{Z}_{\phi_2}(g_2)+\mathscr{Z}_{\phi_1}(g_1)\cdot\mathscr{V}_{\phi_2}(g_2)+E(g_2,\phi_2)\mathscr{V}_{\phi_1}(g_1)\cdot\omega_{\mathfrak{p}^\circ},
\end{aligned}\tag{5-2}$$

where q is the cardinality of the residue field of $F_{\mathfrak{p}}$.

Now we discuss the above three height pairings respectively. First, we have

Lemma 5.1. *Under the weaker hypotheses that $\phi_{1,v}\otimes\phi_{2,v}\in\mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ and $g_i\in P'_vH'(\mathbb{A}_F^v)$ for some finite place v other than $\mathfrak{p}$, $\mathscr{Z}_{\phi_1}(g_1)$ and $\mathscr{Z}_{\phi_2}(g_2)$ do not intersect.*

Proof. This is clear from the first arrow of the reduction map (5-1). □

Second, we define a function $\nu(\,\cdot\,;\phi_2,g_2)$ on $\mathbb{V}_{\mathfrak{p}}-\{0\}$ in the following way. For any $x\in\mathbb{V}_{\mathfrak{p}}-\{0\}$, write $\bar{x}$ for the line in $\boldsymbol{P}(\mathbb{V}_{\mathfrak{p}})$ containing x. Then $\nu(x;\phi_2,g_2)$ is the coefficient of the geometric irreducible component represented by $(\bar{x},1)$ in $\mathrm{lg}_{n,K^{\mathfrak{p}}}$ appearing in $\mathscr{V}_{\phi_2}(g_2)$. It is a locally constant function and

$$\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)=\frac{\mathrm{vol}(\det K)}{\mathrm{vol}(K)}\phi_{1,\mathfrak{p}}\otimes\nu(\,\cdot\,;\phi_2,g_2)$$

extends to a function in $\mathcal{S}(\mathbb{V}_{\mathfrak{p}})$ such that $\nu(0,\phi_{1,\mathfrak{p}};\phi_2,g_2)=0$ since $\phi_{1,\mathfrak{p}}(0)=0$. Then the intersection number

$$\begin{aligned}
\mathscr{Z}_{\phi_1}(g_1)\cdot\mathscr{V}_{\phi_2}(g_2)&=\sum_{x\in K\backslash\mathbb{V}_f}\omega_\chi(g_1)\phi_1(x)\mathscr{Z}(x)_K\cdot\mathscr{V}_{\phi_2}(g_2)\\
&=\sum_{\substack{x\in K\backslash\mathbb{V}_f\\ T(x)\in F^+}}\frac{\mathrm{vol}(K)}{\mathrm{vol}(K\cap\mathbb{H}(\mathbb{A}_f)_x)}\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes(\omega_\chi(g_1^{\mathfrak{p}})\phi_1^{\mathfrak{p}})(x)
\end{aligned}\tag{5-3}$$

since $g_1\in e_{\mathfrak{p}}H'(\mathbb{A}^{\mathfrak{p}})$.

On the other hand, we let

$$\begin{aligned}
&E(s,g,\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}})\\
&\qquad=\sum_{\gamma\in P'(F)\backslash H'(F)}\omega_\chi(\gamma g)(\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}})(0)\lambda_{P'}(\gamma g)^{s-\frac{1}{2}}
\end{aligned}$$

be an Eisenstein series which is holomorphic at $s=\frac{1}{2}$. Then we have

$$(5\text{-}3)=E(s,g,\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}})\big|_{s=\frac{1}{2}}-W_0\big(\tfrac{1}{2},g,\nu(\,\cdot\,,\phi_{1,\mathfrak{p}};\phi_2,g_2)\otimes\phi_1^{\mathfrak{p}}\big)$$

by the standard Siegel–Weil argument and the argument in Proposition 3.2. For simplicity, we let

$$E_{(\mathfrak{p}^\circ)}(g, \phi_1; \phi_2, g_2)$$
$$= \log q \big[E(s, g, \nu(\cdot, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \phi_1^{\mathfrak{p}})\big|_{s=\frac{1}{2}} - W_0\big(\tfrac{1}{2}, g, \nu(\cdot, \phi_{1,\mathfrak{p}}; \phi_2, g_2) \otimes \phi_1^{\mathfrak{p}}\big)\big].$$

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1, \phi_1) = \log q \mathscr{V}_{\phi_1}(g_1) \cdot \omega_{\mathfrak{p}^\circ}.$$

Then in summary, we have

Proposition 5.2. *For $\phi_{1,\mathcal{S}} \otimes \phi_{2,\mathcal{S}} \in \mathcal{S}(\mathbb{V}_{\mathcal{S}}^2)_{\mathrm{reg}}$ and $g_i \in e_{\mathcal{S}} H'(\mathbb{A}_F^{\mathcal{S}})$ $(i = 1, 2)$,*

$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{\mathfrak{p}^\circ} = E_{(\mathfrak{p}^\circ)}(g_1, \phi_1; g_2, \phi_2) + A_{(\mathfrak{p}^\circ)}(g_1, \phi_1) E(g_2, \phi_2).$$

5B. ***Quasisplit case.*** In this section, we discuss the contribution of the local height pairing at a nonsplit (finite) place $\mathfrak{p}$ in $\mathcal{S}$ such that $\epsilon(\mathbb{V}_{\mathfrak{p}}) = 1$.

We fix such a $\mathfrak{p}$ and denote by $\mathfrak{p}^\circ$ the unique place of E over $\mathfrak{p}$ as usual. As before, we need to consider the height pairing $\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{\mathfrak{p}^\circ}$ on a certain model $\mathcal{M}_{K;\mathfrak{p}^\circ}$, for $\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}} \in \mathcal{S}(\mathbb{V}_{\mathfrak{p}}^2)_{\mathrm{reg}}$ and $g_i \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$. We assume that $K = K_{\mathfrak{p}} K^{\mathfrak{p}}$ with $K^{\mathfrak{p}}$ sufficiently small and $K_{\mathfrak{p}} = K_{\mathfrak{p},n}$ for $n \geq 0$. In Section 4B, we have fixed an isometry between $\mathbb{V}_{\mathfrak{p}}$ and $\mathsf{Mat}_2(F_{\mathfrak{p}})$ sending $\Lambda_{\mathfrak{p}}$ to $\mathsf{Mat}_2(\mathbb{O}_{F_{\mathfrak{p}}})$. Hence $\mathbb{V}_{\mathfrak{p}}$ has an action by $\mathsf{GL}_2(F_{\mathfrak{p}})$ by both left and right multiplication where the latter is $E_{\mathfrak{p}^\circ}$-linear. We write $\mathbb{V}_{\mathfrak{p}}^\flat$ with respect to the left multiplication and $\mathsf{GL}_2(F_{\mathfrak{p}})$ acts on $\mathcal{S}(\mathbb{V}_{\mathfrak{p}})$ via $(g.\phi)(x) = \phi(xg)$. By enlarging n (to be an least 1), we assume that $\phi_{i,\mathfrak{p}}$ is not only invariant under $K_{\mathfrak{p},n}$ but also $\mathbf{1}_2 + \varpi^n \mathsf{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}})$.

We let $\mathcal{M}_{n,K^{\mathfrak{p}}}$ be the normalization of $\mathcal{M}_{0,K^{\mathfrak{p}}}$ in $M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$ which is regular and finite over $\mathcal{M}_{0,K^{\mathfrak{p}}}$. We take a base change $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ} = M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ} \times_{E_{\mathfrak{p}^\circ}} E_{\mathfrak{p}^\circ}^{(n)}$, where $E_{\mathfrak{p}^\circ}^{(n)} = E_{\mathfrak{p}^\circ} F_{\mathfrak{p}}^n$ and $\hat{E}_{\mathfrak{p}^\circ}^{(n)} = E_{\mathfrak{p}^\circ} \hat{F}_{\mathfrak{p}}^n$. Let $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ be the normalization of $\mathcal{M}_{n,K^{\mathfrak{p}}} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}} \mathbb{O}_{E_{\mathfrak{p}^\circ}^{(n)}}$, which in turn is a regular model of $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. Then the set of supersingular points is

$$S_n = [\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathrm{s.s.}}(\mathbb{F}) \cong \breve{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1} / \nu(K_{\mathfrak{p},n}) \times \breve{H}_f^{\mathfrak{p}} / K^{\mathfrak{p}}),$$

where $\breve{H}(\mathbb{Q})$ acts on the first factor by multiplying the determinant. For any point $s \in S_n$, the completion $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_s^\wedge$ at the point s is isomorphic to a formal scheme $\mathcal{N}'_n$ over $\mathsf{Spf}\, \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{(n)}}$. It can be constructed in the following way. We have a p-divisible group $\mathscr{X}^{\mathrm{univ}} \to \mathcal{N} \cong \mathsf{Spf}\, R_{F_{\mathfrak{p}},2}$. Let $R_{F_{\mathfrak{p}},2,n}$ (see [Harris and Taylor 2001, Lemma II.2.2]) be such that $\mathsf{Spec}\, R_{F_{\mathfrak{p}},2,n} = (\mathsf{Spec}\, R_{F_{\mathfrak{p}},2})(n)$ is the universal scheme of the Drinfeld ϖ^n-structure for $\mathscr{X}^{\mathrm{univ},\flat}$ which is even defined over $\mathsf{Spec}\, R_{F_{\mathfrak{p}},2}$. Let $\mathsf{Spec}\, R'_n$ be the normalization of $\mathsf{Spec}\, R_{F_{\mathfrak{p}},2} \otimes_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{(n)}}$ in any connected component of $\mathsf{Spec}\, R_{F_{\mathfrak{p}},2,n} \otimes_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}} \hat{E}_{\mathfrak{p}^\circ}^{(n)}$. Then $\mathcal{N}'_n \cong \mathsf{Spf}\, R'_n$ and is finite over $\mathcal{N}' \times_{\mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{\mathrm{nr}}}} \mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{(n)}}$. The generic fiber $\mathcal{N}'_{n,\eta} := \mathsf{Spec}\, R'_n \otimes \hat{E}_{\mathfrak{p}^\circ}^{(n)}$ is Galois over $\mathcal{N}'_\eta := \mathsf{Spec}\, R_{F_{\mathfrak{p}},2} \otimes \hat{E}_{\mathfrak{p}^\circ}^{(n)}$

with Galois group $\mathsf{SL}_2(\mathbb{O}_{F_\mathfrak{p}}/\varpi^n\mathbb{O}_{F_\mathfrak{p}})$. Moreover, it inherits a universal p-divisible $\mathscr{X}' \to \mathsf{Spec}\, R'_n$, a universal Drinfeld ϖ^n-structure

$$\alpha'_{n,\eta} : \Lambda_\mathfrak{p}^\flat/\varpi^n\Lambda_\mathfrak{p}^\flat \to \mathscr{X}'^\flat_\eta[\varpi^n](\mathcal{N}'_{n,\eta})$$

for $\mathscr{X}'^\flat_\eta$. In particular, we have the uniformization

$$[\mathcal{M}'_{n,K^\mathfrak{p}}]^\wedge_{\mathsf{s.s.}} \cong \breve{H}(\mathbb{Q})\backslash(\mathcal{N}'_n \times E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \breve{H}_f^\mathfrak{p}/K^\mathfrak{p}).$$

In Section 4B, we construct the (irreducible) integral special subscheme $\mathscr{Z}(x, h')$ for $x \in V \cap \Lambda_\mathfrak{p} - \{0\}$, $h' \in H_f^\mathfrak{p}/K^\mathfrak{p}$. We still write $\mathscr{Z}(x, h')$ for its base change under the map $\mathcal{M}'_{n,K^\mathfrak{p}} \to \mathcal{M}_{0,K^\mathfrak{p}} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}} \mathbb{O}_{E_{\mathfrak{p}^\circ}^{(n)}}$. But now it is not irreducible anymore. We write $\mathscr{Z}(x, h)$ with $h \in K_{\mathfrak{p},0}/K_{\mathfrak{p},n} \times H_f^\mathfrak{p}/K^\mathfrak{p}$ and $h^\mathfrak{p} = h'$, for all its irreducible components such that its complex geometric fiber (point) is represented by (z, h) with $z \perp x$. Each $\mathscr{Z}(x, h)$ is defined over $\mathbb{O}_{E_{\mathfrak{p}^\circ}^{(n)}}$, is geometrically irreducible, and $[\mathscr{Z}(x, h)]_{\mathsf{spe}} \subset [\mathcal{M}'_{n,K^\mathfrak{p}}]_{\mathsf{s.s.}}$.

We have a p-divisible group $\mathscr{X}'\big|_{\mathscr{Z}(\check{x})} \to \mathscr{Z}(\check{x})$ where we use the same notation for the pull-back of $\mathscr{Z}(\check{x})$ from $\mathsf{Spf}\, R_{F_\mathfrak{p},2}$ to $\mathcal{N}'_n$. Consider $\mathscr{X}'_\eta\big|_{\mathscr{Z}(\check{x})_\eta} \to \mathscr{Z}(\check{x})_\eta$ and

$$\alpha'_{n,\eta} : \Lambda_\mathfrak{p}^\flat/\varpi^n\Lambda_\mathfrak{p}^\flat \xrightarrow{\sim} \mathscr{X}'^\flat_\eta[\varpi^n](\mathscr{Z}(\check{x})_{\eta,0}),$$

where $\mathscr{Z}(\check{x})_{\eta,0}$ is some connected component of $\mathscr{Z}(\check{x})_\eta$. By the definition of $\mathscr{Z}(\check{x})$, the element $\check{x} \in \mathsf{Hom}((\boldsymbol{Y}, j), (\boldsymbol{X}, i_X))$ canonically induces a homomorphism $\varrho_{\check{x}} : {}^{\mathcal{Y}}\boldsymbol{y} \times_{\mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}^{\mathsf{nr}}}} \mathscr{Z}(\check{x}) \to \mathscr{X}'\big|_{\mathscr{Z}(\check{x})}$, hence $\varrho_{\check{x}} : {}^{\mathcal{Y}}\boldsymbol{y}_\eta \times_{\hat{E}_{\mathfrak{p}^\circ}^{\mathsf{nr}}} \mathscr{Z}(\check{x})_\eta \to \mathscr{X}'_\eta\big|_{\mathscr{Z}(\check{x})_\eta}$. In particular, we have an element

$$\varrho_{\check{x},*}(\boldsymbol{x}_\mathfrak{p}) \in T_\mathfrak{p}\big(\mathscr{X}'_\eta\big|_{\mathscr{Z}(\check{x})_\eta}\big) = \varprojlim_n \mathscr{X}'_\eta[\varpi^n](\mathscr{Z}(\check{x})_\eta) = \big(\varprojlim_n \mathscr{X}'^\flat_\eta[\varpi^n](\mathscr{Z}(\check{x})_\eta)\big)^\sharp.$$

For each connected component $\mathscr{Z}(\check{x})_{\eta,0}$, $\alpha'_{n,\eta}$ extends to a $(\mathbf{1}_2 + \varpi^n\mathsf{GL}_2(\mathbb{O}_{F_\mathfrak{p}}))$-class of isomorphisms $\bar{\eta}_\mathfrak{p} : \Lambda_\mathfrak{p} \xrightarrow{\sim} T_\mathfrak{p}(\mathscr{X}'_{\eta,0})$, where $\mathscr{X}'_{\eta,0}$ is the restriction of $\mathscr{X}'_\eta$ to $\mathscr{Z}(\check{x})_{\eta,0}$. Let $x = \bar{\eta}_\mathfrak{p}^{-1}(\varrho_{\check{x},*}(\boldsymbol{x}_\mathfrak{p}))$, which is well-defined in $\Lambda_\mathfrak{p}/(\mathbf{1}_2 + \varpi^n\mathsf{GL}_2(\mathbb{O}_{F_\mathfrak{p}}))$. By construction, we have the following property:

$$(x, x)/(\check{x}, \check{x})' \in 1 + \varpi^n\mathbb{O}_{F_\mathfrak{p}} \text{ and } x \in \varpi^m\Lambda_\mathfrak{p} \Leftrightarrow \check{x} \in \varpi^m\check{\Lambda}_\mathfrak{p} \text{ for all } m \geq 0. \quad (5\text{-}4)$$

We denote by $\mathscr{Z}(\check{x}, x)$ the union of all irreducible components of $\mathscr{Z}(\check{x})$ containing $\mathscr{Z}(\check{x})_{\eta,0}$ whose $\bar{\eta}_\mathfrak{p}^{-1}(\varrho_{\check{x},*}(\boldsymbol{x}_\mathfrak{p})) = x$. It is nonempty only when (5-4) is satisfied. Hence for a fixed $\check{x}$, the number of x such that $\mathscr{Z}(\check{x}, x)$ is nonempty is at most $|\mathsf{SL}_2(\mathbb{O}_{F_\mathfrak{p}}/\varpi^n\mathbb{O}_{F_\mathfrak{p}})|$. Now for any $\check{h} \in E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^\mathfrak{p}/K^\mathfrak{p}$, we let $\mathscr{Z}(\check{x}, x, \check{h})$ be the cycle of $[\mathcal{M}'_{n,K^\mathfrak{p}}]^\wedge_{\mathsf{s.s.}}$ represented by $(\mathscr{Z}(\check{x}, x), \check{h})$. Then we have the following identity between sets:

$$\begin{aligned}&\big\{\mathscr{Z}(\check{x}, x, \check{h}) \,\big|\, \check{x} \in \check{H}(\mathbb{Q})\backslash(\check{\Lambda}_\mathfrak{p} - \{0\}), \check{h} \in \check{H}_{\check{x}}(\mathbb{Q})\backslash(E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^\mathfrak{p}/K^\mathfrak{p})\big\}\\ =&\big\{[\mathscr{Z}(x, h)]^\wedge_{\mathsf{spe}} \,\big|\, x \in H(\mathbb{Q})\backslash(V \cap \Lambda_\mathfrak{p} - \{0\}), h \in H_x(\mathbb{Q})\backslash(K_{\mathfrak{p},0}/K_{\mathfrak{p},n} \times H_f^\mathfrak{p}/K^\mathfrak{p})\big\}. \quad (5\text{-}5)\end{aligned}$$

Now we can consider the height pairing. Pick an element $e \in E^\times$ such that $-\mathrm{val}_{\mathfrak{p}}(e)$ is sufficiently large. Then

$$Z_{\phi_i}(g_i) = \sum_{x_i \in K\backslash \mathbb{V}_f} \omega_\chi(g_i)\phi_i(x_i)Z(x_i)_K = \sum_{x_i \in K\backslash \mathbb{V}_f} \omega_\chi(g_i)\phi_i(x_i e)Z(x_i e)_K$$
$$= \sum_{x_i \in K\backslash \mathbb{V}_f} \omega_\chi(g_i)\phi_i(x_i e)Z(x_i)_K.$$

Hence we can assume that $\phi_{i,\mathfrak{p}}$ is supported on $\Lambda_{\mathfrak{p}}$. Recall that we assume $\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}} \in \mathcal{S}(\mathbb{V}_{\mathfrak{p}}^2)^{K_{\mathfrak{p},n}}_{\mathrm{reg}}$, $g_i \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$, and $\phi_{i,\mathfrak{p}}$ is also invariant under $\mathbf{1}_2 + \varpi^n \mathrm{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}})$. We define

$$\mathscr{Z}_{\phi_i}(g_i) = \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} \phi_{\mathfrak{p}} \otimes (\omega_\chi(g_i)\phi_i^{\mathfrak{p}})(h_i^{-1}x_i)\mathscr{Z}(x_i, h_i)$$

which is a cycle of $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ extending the generating series $Z_{\phi_i}(g_i)$ on $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. The special fiber $[\mathscr{Z}_{\phi_i}(g_i)]_{\mathsf{spe}} \subset [\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{s.s.}}$ and

$$[\mathscr{Z}_{\phi_i}(g_i)]^\wedge_{\mathsf{spe}} = \sum_{\check{x}_i \in \check{V} \cap \check{\Lambda}_{\mathfrak{p}} - \{0\}, x_i} \sum_{\check{h}_i \in \check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{i,\mathfrak{p}}(x_i)$$
$$\times (\omega_\chi(g_i)\phi_i^{\mathfrak{p}})(\check{h}_i^{-1}\check{x}_i)\mathscr{Z}(\check{x}_i, x_i, \check{h}_i).$$

We have a similar decomposition as in (5-2) but the first term is not zero anymore. First, we consider

$$\mathscr{Z}_{\phi_1}(g_1) \cdot \mathscr{Z}_{\phi_2}(g_2) = [\mathscr{Z}_{\phi_1}(g_1)]^\wedge_{\mathsf{spe}} \cdot [\mathscr{Z}_{\phi_2}(g_2)]^\wedge_{\mathsf{spe}}$$
$$= \sum_{\check{x}_1, \check{x}_2 \in \check{V} \cap \check{\Lambda}_{\mathfrak{p}} - \{0\}} \sum_{x_1, x_2} \sum_{\check{h} \in \check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2)$$
$$\big(\omega''_\chi(\iota(g_1, g_2^\vee))\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\check{h}^{-1}(\check{x}_1, \check{x}_2))\big)\mathscr{Z}(\check{x}_1, x_1) \cdot \mathscr{Z}(\check{x}_2, x_2). \quad (5\text{-}6)$$

Now the key point is to analyze the last intersection number. We have:

Lemma 5.3. *We can extend*

$$\mu(\check{x}_1, \check{x}_2; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}) := \sum_{x_1, x_2 \in \Lambda_{\mathfrak{p}}/(\mathbf{1}_2 + \varpi^n \mathrm{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}}))} \phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2)\mathscr{Z}(\check{x}_1, x_1) \cdot \mathscr{Z}(\check{x}_2, x_2)$$

to a function in $\mathcal{S}(\check{V}_{\mathfrak{p}}^2)$.

Proof. First we note that $\mathscr{Z}(\check{x}_1, x_1)$ and $\mathscr{Z}(\check{x}_2, x_2)$ intersect properly unless $\check{x}_1$ and $\check{x}_2$ are $E_{\mathfrak{p}^\circ}$-colinear and x_1 and x_2 are also $E_{\mathfrak{p}^\circ}$-colinear. But in this case, $\phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2) = 0$ by our regularity assumption. For $(\check{x}_1, \check{x}_2) \notin (\check{\Lambda}_{\mathfrak{p}} - \{0\})^2$, we let $\mu(\check{x}_1, \check{x}_2; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}) = 0$. Hence $\mu(\check{x}_1, \check{x}_2; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}})$ is now a function on $\check{V}_{\mathfrak{p}}^2$ which is compactly supported. We only need to prove that it is locally constant. We have several cases.

- If one $\check{x}_i$, say $\check{x}_1$, is not in $\check{\Lambda}_{\mathfrak{p}}$, then μ is locally zero at $(\check{x}_1, \check{x}_2)$.
- If, say, $\check{x}_1 = 0$, then since $\phi_{1,\mathfrak{p}}$ vanishes on a neighborhood of 0 and (5-4), μ is also locally zero.
- If both $\check{x}_i$ are in $\Lambda_{\mathfrak{p}} - \{0\}$, but are not $E_{\mathfrak{p}^\circ}$-colinear, choose a neighborhood U such that any $(x_1', x_2') \in U$ is still not $E_{\mathfrak{p}^\circ}$-colinear; $(x_i', x_i')'/(\check{x}_i, \check{x}_i)' \in 1 + \varpi^n \mathbb{O}_{F_{\mathfrak{p}}}$; (x_1', x_2') and $(\check{x}_1, \check{x}_2)$ span the same $\mathbb{O}_{E_{\mathfrak{p}^\circ}}$-sublattice in $\check{\Lambda}_{\mathfrak{p}}$. Then μ is locally constant on U.
- If both $\check{x}_i$ are in $\Lambda_{\mathfrak{p}} - \{0\}$ and $E_{\mathfrak{p}^\circ}$-colinear, we choose U as above. Then for x_1, x_2 not $E_{\mathfrak{p}^\circ}$-colinear, $\mathscr{Z}(\check{x}_1, x_1) \cdot \mathscr{Z}(\check{x}_2, x_2)$ is locally constant on U, and hence μ is also. □

By the lemma,

$$(5\text{-}6) = \sum_{\vec{x} \in \check{V}^2} \sum_{\check{h} \in \check{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \big(\omega_\chi''(\iota(g_1, g_2^\vee))\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\check{h}^{-1}\vec{x})\big)\mu(\vec{x}; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}).$$

Since the set $\check{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})$ is finite, we let

$$\theta_{(\mathfrak{p}^\circ)}^{\mathrm{hor}}(\,\cdot\,; \phi_1, \phi_2) = \log q \sum_{\vec{x} \in \check{V}^2} \omega_\chi''(\,\cdot\,)\Phi^{\mathrm{hor}}(\vec{x})$$

be the theta series for the Schwartz function

$$\Phi^{\mathrm{hor}} = \sum_{\check{h} \in \check{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \mu(\,\cdot\,,\cdot\,; \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}) \otimes (\omega_\chi''(\check{h})\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}).$$

Then we have

Lemma 5.4. *For $g_i \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\log q \, \mathscr{Z}_{\phi_1}(g_1) \cdot \mathscr{Z}_{\phi_2}(g_2) = \theta_{(\mathfrak{p}^\circ)}^{\mathrm{hor}}(\iota(g_1, g_2^\vee); \phi_1, \phi_2),$$

where q is the cardinality of the residue field of $E_{\mathfrak{p}^\circ}$.

Now we consider the second term, $\mathscr{Z}_{\phi_1}(g_1) \cdot \mathscr{V}_{\phi_2}(g_2)$. For any

$$\check{h}_1 \in \check{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}),$$

we write $s(\check{h}_1)$ for the corresponding supersingular point in S_n. Then

$$\begin{aligned}\mathscr{Z}_{\phi_1}(g_1) \cdot \mathscr{V}_{\phi_2}(g_2) &= [\mathscr{Z}_{\phi_1}(g_1)]_{\mathsf{spe}}^{\wedge} \cdot [\mathscr{V}_{\phi_2}(g_2)]_{\mathsf{s.s.}}^{\wedge} \\ &= \sum_{\check{x}_1 \in \check{V} \cap \check{\Lambda}_{\mathfrak{p}} - \{0\}, x_1} \sum_{\check{h}_1 \in \check{H}(\mathbb{Q}) \backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{1,\mathfrak{p}}(x_1)(\omega_\chi(g_1)\phi_1^{\mathfrak{p}})(\check{h}_1^{-1}\check{x}_1) \\ &\qquad \mathscr{Z}(\check{x}_1, x_1, \check{h}_1) \cdot [\mathscr{V}_{\phi_2}(g_2)]_{s(\check{h}_1)}^{\wedge}. \quad (5\text{-}7)\end{aligned}$$

Lemma 5.5. *For any $\check{h}_1$, we can extend*

$$\nu(\check{x}_1, \phi_{1,\mathfrak{p}}, \check{h}_1; \phi_2, g_2) := \sum_{x_1 \in \Lambda_{\mathfrak{p}}/(\mathbf{1}_2 + \varpi^n \mathrm{GL}_2(\mathbb{O}_{F_{\mathfrak{p}}}))} \phi_{1,\mathfrak{p}}(x_1) \mathscr{Z}(\check{x}_1, x_1, \check{h}_1) \cdot [\mathscr{V}_{\phi_2}(g_2)]^{\wedge}_{s(\check{h}_1)}$$

to a function in $\mathscr{S}(\check{V}_{\mathfrak{p}})$.

Proof. The proof is similar to that of Lemma 5.3. In fact, for $\check{x}_1 \in \check{\Lambda}_{\mathfrak{p}} - \{0\}$, let U be a neighborhood such that for any $x_1' \in U$, $(x_i', x_i')'/(\check{x}_i, \check{x}_i)' \in 1 + \varpi^n \mathbb{O}_{F_{\mathfrak{p}}}$. Then $\mathscr{Z}(\,\cdot\,, x_1, \check{h}_1) \cdot [\mathscr{V}_{\phi_2}(g_2)]^{\wedge}_{s(\check{h}_1)}$ is locally constant on U. □

The lemma implies that

(5-7) =

$$\sum_{\check{x}_1 \in \check{V}} \sum_{\check{h}_1 \in \check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{1,\mathfrak{p}}(x_1)(\omega_\chi(g_1)\phi_1^{\mathfrak{p}})(\check{h}_1^{-1}\check{x}_1)\nu(\check{x}_1, \phi_{1,\mathfrak{p}}, \check{h}_1; \phi_2, g_2).$$

Let $\theta^{\mathsf{ver}}_{(\mathfrak{p}^\circ)}(\,\cdot\,, \phi_1; g_2, \phi_2) = \sum_{\check{x}_1 \in \check{V}} \omega_\chi(\,\cdot\,)\phi^{\mathsf{ver}}(\check{x}_1)$ be the theta series for the Schwartz function

$$\phi^{\mathsf{ver}} = \sum_{\check{h}_1 \in \check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \nu(\,\cdot\,, \phi_{1,\mathfrak{p}}, \check{h}_1; \phi_2, g_2) \otimes \omega_\chi(\check{h}_1)\phi_1^{\mathfrak{p}}.$$

Then we have:

Lemma 5.6. *For $g_i \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\log q \, \mathscr{Z}_{\phi_1}(g_1) \cdot \mathscr{V}_{\phi_2}(g_2) = \theta^{\mathsf{ver}}_{(\mathfrak{p}^\circ)}(g_1, \phi_1; g_2, \phi_2)$$

is a theta series for $g_1 \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$.

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1, \phi_1) = \log q \, \mathscr{V}_{\phi_1}(g_1) \cdot \omega_{\mathfrak{p}^\circ}.$$

Then, in summary, we have:

Proposition 5.7. *For $\phi_{1,\mathcal{S}} \otimes \phi_{2,\mathcal{S}} \in \mathscr{S}(\mathbb{V}_{\mathcal{S}}^2)_{\mathsf{reg}}$ and $g_i \in e_{\mathcal{S}} H'(\mathbb{A}_F^{\mathcal{S}})$ $(i = 1, 2)$,*

$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{\mathfrak{p}^\circ} = \theta^{\mathsf{hor}}_{(\mathfrak{p}^\circ)}(\iota(g_1, g_2^\vee); \phi_1, \phi_2) + \theta^{\mathsf{ver}}_{(\mathfrak{p}^\circ)}(g_1, \phi_1; g_2, \phi_2) + A_{(\mathfrak{p}^\circ)}(g_1, \phi_1)E(g_2, \phi_2).$$

5C. ***Nonsplit case.*** In this section, we discuss the contribution of the local height pairing at a nonsplit (finite) place $\mathfrak{p}$ in $\mathcal{S}$ such that $\epsilon(\mathbb{V}_{\mathfrak{p}}) = -1$.

We fix such a $\mathfrak{p}$ and denote by $\mathfrak{p}^\circ$ the unique place of E over $\mathfrak{p}$ as usual. As before, we need to consider the height pairing $\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{\mathfrak{p}^\circ}$ on a certain model $\mathcal{M}_{K;\mathfrak{p}^\circ}$, for $\phi_{1,\mathfrak{p}} \otimes \phi_{2,\mathfrak{p}} \in \mathscr{S}(\mathbb{V}_{\mathfrak{p}}^2)_{\mathsf{reg}}$ and $g_i \in e_{\mathfrak{p}} H'(\mathbb{A}_F^{\mathfrak{p}})$. We assume that $K = K_{\mathfrak{p}} K^{\mathfrak{p}}$ with $K^{\mathfrak{p}}$ sufficiently small and $K_{\mathfrak{p}} = K_{\mathfrak{p},n}$ for $n \geq 0$. In Section 4B, we

have fixed an isometry between $\mathbb{V}_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$, the division quaternion algebra over $F_{\mathfrak{p}}$, sending $\Lambda_{\mathfrak{p}}$ to $\mathcal{O}_{B_{\mathfrak{p}}}$, the maximal order. Hence $\mathbb{V}_{\mathfrak{p}}$ has an action of $B_{\mathfrak{p}}$ by both left and right multiplication where the latter is $E_{\mathfrak{p}^\circ}$-linear. Also, $B_{\mathfrak{p}}$ acts on $\mathcal{S}(\mathbb{V}_{\mathfrak{p}})$ via $(g.\phi)(x)=\phi(xg)$. By enlarging n (to be an least 1), we assume that $\phi_{i,\mathfrak{p}}$ is not only invariant under $K_{\mathfrak{p},n}$ but also $\mathbf{1}_2+\varpi^n\mathcal{O}_{B_{\mathfrak{p}}}$. Moreover, we assume that $\phi_{i,\mathfrak{p}}$ is supported on $\Lambda_{\mathfrak{p}}$.

We need to choose some model for $M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. If $n=0$, we have already constructed a regular model $\mathcal{M}_{0,K^{\mathfrak{p}}}$ which is flat and projective, but not smooth over $\mathcal{O}_{E_{\mathfrak{p}^\circ}}$. Similar to the quasisplit case in Section 4B, we let $\breve{B}$ be the quaternion algebra over F by changing the Hasse invariant at ι_1 and $\mathfrak{p}$ from which we construct $\breve{B}^\dagger$ and an algebraic group $\breve{H}^\dagger$ over $\mathbb{Q}$. We let $\breve{H}=Z\cdot\breve{H}^{\dagger,\mathrm{der}}$. Then we have the following variant of the *Cherednik–Drinfeld uniformization theorem* (see [Boutot and Carayol 1991, Chapitre III]):

$$\begin{array}{ccc}
[\mathscr{X}]^\wedge_{\mathsf{spe}}\times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}} & \xrightarrow{\sim} & \breve{H}(\mathbb{Q})\backslash\mathscr{X}'\times\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}\\
\downarrow & & \downarrow\\
[\mathcal{M}_{0,K^{\mathfrak{p}}}]^\wedge_{\mathsf{spe}}\times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}} & \xrightarrow{\sim} & \breve{H}(\mathbb{Q})\backslash\Omega'\times\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}},
\end{array}$$

where $\Omega'=\Omega\times_{\mathcal{O}_{F_{\mathfrak{p}}}}\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}}$ (resp. $\mathscr{X}'=\mathscr{X}^{\mathrm{univ}}\times_{\mathcal{O}_{F_{\mathfrak{p}}}}\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}}$) and Ω (resp. $\mathscr{X}^{\mathrm{univ}}$) is the *formal Drinfeld upper half plane* over $\mathcal{O}_{F_{\mathfrak{p}}}$ (resp. universal $\mathcal{O}_{B_{\mathfrak{p}}}$-module over Ω).

For general $n\geq 1$, we construct an integral model of the base change $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}=M_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}\times_{E_{\mathfrak{p}^\circ}}E^{(n)}_{\mathfrak{p}^\circ}$ as follows. Let $\Omega^{\mathrm{rig}}_n=\mathscr{X}^{\mathrm{univ,rig}}[\varpi^n]-\mathscr{X}^{\mathrm{univ,rig}}[\varpi^{n-1}]$ be the étale covering over Ω^{rig} with Galois group $(\mathcal{O}_{B_{\mathfrak{p}}}/\varpi^n\mathcal{O}_{B_{\mathfrak{p}}})^\times$. Consider $\Omega^{\mathrm{rig}}_n\times_{F_{\mathfrak{p}}}\hat{E}^{(n)}_{\mathfrak{p}^\circ}$, it has $(\mathcal{O}_{F_{\mathfrak{p}}}/\varpi^n\mathcal{O}_{F_{\mathfrak{p}}})^\times$ connected components. Pick any connected component Σ_n which is étale over $\Omega^{\mathrm{rig}}\times_{F_{\mathfrak{p}}}\hat{E}^{(n)}_{\mathfrak{p}^\circ}$ with Galois group $(\mathcal{O}_{B_{\mathfrak{p}}}/\varpi^n\mathcal{O}_{B_{\mathfrak{p}}})^{\times,1}$. Then it is easy to see that

$$M'^{\mathrm{rig}}_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}\cong\breve{H}(\mathbb{Q})\backslash(\Sigma_n\times E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}),$$

where $\breve{H}(\mathbb{Q})$ acts on Σ_n through the $\mathfrak{p}$-component modulo center and acts on $E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})$ via the determinant map.

Let Ω_n be the normalization of $\Omega'\times_{\mathcal{O}_{\hat{E}_{\mathfrak{p}^\circ}}}\mathcal{O}_{\hat{E}^{(n)}_{\mathfrak{p}^\circ}}$ in Σ_n. It is not regular but has double points; we blow up these points to get a regular formal scheme Ω'_n and (for sufficiently small $K^{\mathfrak{p}}$)

$$\breve{H}(\mathbb{Q})\backslash(\Omega'_n\times E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n})\times\breve{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})$$

is regular, flat, and projective over $\mathcal{O}_{\hat{E}^{(n)}_{\mathfrak{p}^\circ}}$, where $\breve{H}(\mathbb{Q})$ acts on Ω'_n by the universal property of normalization and blowing-up of double points. By Grothendieck's existence theorem, we have a regular scheme $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ that is flat and projective over $\mathrm{Spec}\,\mathcal{O}_{E^{(n)}_{\mathfrak{p}^\circ}}$, and a morphism $\pi_n:\mathcal{M}'_{n,K^{\mathfrak{p}}}\to\mathcal{M}'_{0,K^{\mathfrak{p}}}\times_{\mathcal{O}_{E_{\mathfrak{p}^\circ}}}\mathcal{O}_{E^{(n)}_{\mathfrak{p}^\circ}}$ such that the following

diagram commutes:

$$\begin{array}{ccc} [\mathcal{M}'_{n,K^{\mathfrak{p}}}]^{\wedge}_{\mathsf{spe}} & \xrightarrow{\sim} & \breve{H}(\mathbb{Q})\backslash(\Omega'_n \times E^{\times,1}_{\mathfrak{p}^\circ}/\nu(K_{\mathfrak{p},n}) \times \breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}}) \\ \downarrow{\scriptstyle \pi_n^{\wedge}} & & \downarrow \\ [\mathcal{M}'_{0,K^{\mathfrak{p}}} \times_{\mathbb{O}_{E_{\mathfrak{p}^\circ}}} \mathbb{O}_{E^{(n)}_{\mathfrak{p}^\circ}}]^{\wedge}_{\mathsf{spe}} & \xrightarrow{\sim} & (\breve{H}(\mathbb{Q})\backslash\Omega' \times \breve{H}^{\mathfrak{p}}_f/K^{\mathfrak{p}}) \times_{\mathbb{O}_{\hat{E}_{\mathfrak{p}^\circ}}} \mathbb{O}_{\hat{E}^{(n)}_{\mathfrak{p}^\circ}}. \end{array}$$

Now let us define the integral special subschemes on these models. We recall the integral special subschemes $\mathscr{Z}(o)^{\dagger}_{\mathfrak{p}}$ (resp. $\mathscr{Z}(x)^{\circ}_{0,\mathfrak{p}^\circ}$) on $\mathcal{M}^{\dagger}_{0,K_p^{\dagger,\mathfrak{p}},K^{\dagger,p}}$ (resp. $\mathcal{M}_{0,K^{\mathfrak{p}}}$) defined in Section 4B. Similar to the quasisplit case, we fix an integral special subscheme $\mathscr{Z}(o)^{\dagger}_{\mathfrak{p}}$ with $T(o)=1$. Let s be the unique geometric point in the Zariski closure of the generic fiber of $\mathscr{Z}(o)^{\dagger}_{\mathfrak{p}}$. We set $\boldsymbol{A}=\mathcal{A}_s$ and $\boldsymbol{X}=\mathcal{X}^{\dagger}_s$, a special formal $\mathbb{O}_{B_{\mathfrak{p}}}$-module of height 4. The isogeny class of $\boldsymbol{A}$ is independent of o. We denote by $(\boldsymbol{A}^0,\theta^0,i_{\boldsymbol{A}^0})$ the corresponding abelian variety up to isogeny. Then we have $\mathsf{End}(\boldsymbol{A}^0,i_{\boldsymbol{A}^0})\cong\breve{B}^{\dagger}$ as an $F^{\dagger}$-algebra and $\mathsf{Aut}(\boldsymbol{A}^0,\theta^0,i_{\boldsymbol{A}^0})\cong\breve{H}^{\dagger}(\mathbb{Q})$. We define $\breve{\Lambda}=\mathsf{Hom}((\boldsymbol{E},j),(\boldsymbol{A},i_{\boldsymbol{A}}))$ and $\breve{\Lambda}_{\mathbb{Q}}=\breve{\Lambda}\otimes\mathbb{Q}$. Let $\breve{V}\subset\breve{\Lambda}_{\mathbb{Q}}$ be the sub-E-vector space generated by $\breve{H}(\mathbb{Q}).x_0$ where $x_0=\varrho_{\boldsymbol{A}}$. One can define a hermitian form $(\,\cdot\,,\cdot\,)'$ on $\breve{V}$ as in (4-8) such that $(\breve{V},(\,\cdot\,,\cdot\,)')$ is isometric to the nearby hermitian space $V^{(\mathfrak{p})}$ of $\mathbb{V}$ and has the unitary group $\breve{H}$. The level structure $(\bar{\eta}^p,\bar{\eta}^{\mathfrak{p}}_p)$ of $\boldsymbol{A}$ gives a $K^{\mathfrak{p}}$-class of isometries $V\otimes_F\mathbb{A}^{\mathfrak{p}}_{f,F}\to\breve{V}\otimes_F\mathbb{A}^{\mathfrak{p}}_{f,F}$. We identify $V\otimes_F\mathbb{A}^{\mathfrak{p}}_{f,F}$ with $\breve{V}\otimes_F\mathbb{A}^{\mathfrak{p}}_{f,F}$ via a fixed isometry in this class. For the place $\mathfrak{p}$, we let $\breve{\Lambda}_{\mathfrak{p}}=\mathsf{Hom}((\boldsymbol{Y},j),(\boldsymbol{X},i_{\boldsymbol{X}}))$, which is a self-dual lattice in $\breve{V}_{\mathfrak{p}}$. We are going to define a formal special subscheme $\mathscr{Z}(\breve{x})$ on $\hat{\Omega}:=\Omega\times_{\mathbb{O}_{F_{\mathfrak{p}}}}\mathbb{O}_{\hat{F}_{\mathfrak{p}}}$.

Let us first recall the moduli problem represented by $\hat{\Omega}$. For any element $S\in\mathsf{Obj}\,\mathfrak{Nilp}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}$, $\hat{\Omega}(S)$ is the set of equivalence classes of couples (Φ,ρ_Φ) where

- Φ is a special formal $\mathbb{O}_{B_{\mathfrak{p}}}$-module of height 4 over S and
- $\rho_\Phi:\Phi\times_S S_{\mathsf{spe}}\to\boldsymbol{X}\times_{\mathbb{F}}S_{\mathsf{spe}}$ is a quasiisogeny of height 0.

Two couples (Φ,ρ_Φ) and $(\Phi',\rho_{\Phi'})$ are equivalent if there is an isomorphism $\Phi'\to\Phi$ sending ρ_Φ to $\rho_{\Phi'}$. For any $\breve{x}\in\breve{\Lambda}_{\mathfrak{p}}$, we define a subfunctor $\mathscr{Z}(\breve{x})$ as follows: for any $S\in\mathsf{Obj}\,\mathfrak{Nilp}_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}$, $\mathscr{Z}(\breve{x})(S)$ is the set of equivalence classes (Φ,ρ_Φ) such that the composed quasihomomorphism

$$\mathcal{Y}\times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}}}}S_{\mathsf{spe}}=\boldsymbol{Y}\times_{\mathbb{F}}S_{\mathsf{spe}}\xrightarrow{\breve{x}}\boldsymbol{X}\times_{\mathbb{F}}S_{\mathsf{spe}}\xrightarrow{\rho_\Phi^{-1}}\Phi\times_S S_{\mathsf{spe}}$$

extends to a homomorphism $\mathcal{Y}\times_{\mathbb{O}_{\hat{F}_{\mathfrak{p}^\circ}}}S\to\Phi$.

Now we proceed exactly as in Section 5B. We use the same notation for the pull-back of $\mathscr{Z}(x)^{\circ}_{0,\mathfrak{p}^\circ}$ to the scheme $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ and define $\mathscr{Z}(x,h)$ for $x\in V\cap\Lambda_{\mathfrak{p}}-\{0\}$, $h\in K_{\mathfrak{p},0}/K_{\mathfrak{p},n}\times H^{\mathfrak{p}}_f/K^{\mathfrak{p}}$. We also have the formal special subscheme $\mathscr{Z}(\breve{x},x,h)$

for $\check{x} \in \check{\Lambda}_{\mathfrak{p}} - \{0\}$, $\check{h} \in E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}$, and x satisfying a similar relation as in (5-4). The identity of sets (5-5) still holds. There is only one difference: when $n \geq 1$, we only keep the irreducible component which is not supported on the special fiber when defining $\mathscr{Z}(\check{x}, x, h)$.

Now we can consider the height pairing. We define

$$\mathscr{Z}_{\phi_i}(g_i) = \sum_{x_i \in H(\mathbb{Q})\backslash V} \sum_{h_i \in H_{x_i}(\mathbb{Q})\backslash H(\mathbb{A}_f)/K} \phi_{\mathfrak{p}} \otimes (\omega_\chi(g_i)\phi_i^{\mathfrak{p}})(h_i^{-1}x_i)\mathscr{Z}(x_i, h_i),$$

which is a cycle of $\mathcal{M}'_{n,K^{\mathfrak{p}}}$ extending the generating series $Z_{\phi_i}(g_i)$ on $M'_{n,K^{\mathfrak{p}};\mathfrak{p}^\circ}$. We have

$$[\mathscr{Z}_{\phi_i}(g_i)]^\wedge_{\mathsf{spe}} = \sum_{\check{x}_i \in \check{V} \cap \check{\Lambda}_{\mathfrak{p}} - \{0\}, x_i} \sum_{\check{h}_i \in \check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{i,\mathfrak{p}}(x_i)(\omega_\chi(g_i)\phi_i^{\mathfrak{p}})(\check{h}_i^{-1}\check{x}_i)\mathscr{Z}(\check{x}_i, x_i, \check{h}_i).$$

We have a similar decomposition as in (5-2) but the first term is not zero anymore. First, we consider

$$\begin{aligned}\mathscr{Z}_{\phi_1}(g_1) \cdot \mathscr{Z}_{\phi_2}(g_2) &= [\mathscr{Z}_{\phi_1}(g_1)]^\wedge_{\mathsf{spe}} \cdot [\mathscr{Z}_{\phi_2}(g_2)]^\wedge_{\mathsf{spe}} \\ &= \sum_{\check{x}_1, \check{x}_2 \in \check{V} \cap \check{\Lambda}_{\mathfrak{p}} - \{0\}} \sum_{x_1, x_2} \sum_{\check{h} \in \check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2) \\ &\quad \times \big(\omega''_\chi(\iota(g_1, g_2^\vee))\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\check{h}^{-1}(\check{x}_1, \check{x}_2))\big)\mathscr{Z}(\check{x}_1, x_1, \check{h}) \cdot \mathscr{Z}(\check{x}_2, x_2, \check{h}). \quad (5\text{-}8)\end{aligned}$$

We have the following lemma, whose proof is similar to that of Lemma 5.3.

Lemma 5.8. *We can extend*

$$\mu(\check{x}_1, \check{x}_2; \check{h}, \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}) := \sum_{x_1, x_2 \in \Lambda_{\mathfrak{p}}/(\mathbf{1}_2 + \varpi^n \mathsf{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}}))} \phi_{1,\mathfrak{p}}(x_1)\phi_{2,\mathfrak{p}}(x_2)\mathscr{Z}(\check{x}_1, x_1, \check{h}) \cdot \mathscr{Z}(\check{x}_2, x_2, \check{h})$$

to a function in $\mathcal{S}(\check{V}_{\mathfrak{p}}^2)$.

By the lemma,

$$(5\text{-}6) = \sum_{\vec{x} \in \check{V}^2} \sum_{\check{h} \in \check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \big(\omega''_\chi(\iota(g_1, g_2^\vee))\phi_1^{\mathfrak{p}} \otimes \phi_2^{\mathfrak{p}}(\check{h}^{-1}\vec{x})\big)\mu(\vec{x}; \check{h}, \phi_{1,\mathfrak{p}}, \phi_{2,\mathfrak{p}}).$$

Since the set $\check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n}) \times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})$ is finite, we let

$$\theta^{\mathsf{hor}}_{(\mathfrak{p}^\circ)}(\,\cdot\,; \phi_1, \phi_2) = \log q \sum_{\vec{x} \in \check{V}^2} \omega''_\chi(\,\cdot\,)\Phi^{\mathsf{hor}}(\vec{x})$$

be the theta series for the Schwartz function

$$\Phi^{\mathrm{hor}} = \sum_{\check{h}\in \check{H}(\mathbb{Q})\backslash (E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n})\times \check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \mu(\,\cdot\,,\cdot\,;\check{h},\phi_{1,\mathfrak{p}},\phi_{2,\mathfrak{p}})\otimes(\omega''_\chi(\check{h})\phi_1^{\mathfrak{p}}\otimes\phi_2^{\mathfrak{p}}).$$

Lemma 5.9. *For $g_i \in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\log q\,\mathscr{Z}_{\phi_1}(g_1)\cdot\mathscr{Z}_{\phi_2}(g_2) = \theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\iota(g_1,g_2^\vee);\phi_1,\phi_2),$$

where q is the cardinality of the residue field of $E_{\mathfrak{p}^\circ}$.

Now we consider the second term, $\mathscr{Z}_{\phi_1}(g_1)\cdot\mathscr{V}_{\phi_2}(g_2)$. For any

$$\check{h}_1 \in \check{H}(\mathbb{Q})\backslash(E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n})\times\check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}}),$$

we write $s(\check{h}_1)$ for the corresponding connected component of $[\mathcal{M}'_{n,K^{\mathfrak{p}}}]_{\mathsf{spe}}$. Then

$$\begin{aligned}\mathscr{Z}_{\phi_1}(g_1)\cdot\mathscr{V}_{\phi_2}(g_2) &= [\mathscr{Z}_{\phi_1}(g_1)]^\wedge_{\mathsf{spe}}\cdot[\mathscr{V}_{\phi_2}(g_2)]^\wedge_{\mathsf{spe}}\\ &= \sum_{\check{x}_1\in\check{V}\cap\check{\Lambda}_{\mathfrak{p}}-\{0\},x_1}\ \sum_{\check{h}_1\in\check{H}(\mathbb{Q})\backslash(E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n})\times\check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})} \phi_{1,\mathfrak{p}}(x_1)(\omega_\chi(g_1)\phi_1^{\mathfrak{p}})(\check{h}_1^{-1}\check{x}_1)\\ &\qquad \mathscr{Z}(\check{x}_1,x_1,\check{h}_1)\cdot[\mathscr{V}_{\phi_2}(g_2)]^\wedge_{s(\check{h}_1)}. \quad (5\text{-}9)\end{aligned}$$

Lemma 5.10. *For any $\check{h}_1$, we can extend*

$$\nu(\check{x}_1,\phi_{1,\mathfrak{p}},\check{h}_1;\phi_2,g_2) := \sum_{x_1\in\Lambda_{\mathfrak{p}}/(\mathbf{1}_2+\varpi^n\mathsf{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}}))}\phi_{1,\mathfrak{p}}(x_1)\mathscr{Z}(\check{x}_1,x_1,\check{h}_1)\cdot[\mathscr{V}_{\phi_2}(g_2)]^\wedge_{s(\check{h}_1)}$$

to a function in $\mathcal{S}(\check{V}_{\mathfrak{p}})$.

This lemma implies that (5-9) equals

$$\sum_{\check{x}_1\in\check{V}}\sum_{\check{h}_1\in\check{H}(\mathbb{Q})\backslash(E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n})\times\check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})}\phi_{1,\mathfrak{p}}(x_1)\times(\omega_\chi(g_1)\phi_1^{\mathfrak{p}})(\check{h}_1^{-1}\check{x}_1)\nu(\check{x}_1,\phi_{1,\mathfrak{p}},\check{h}_1;\phi_2,g_2).$$

Let

$$\theta^{\mathsf{ver}}_{(\mathfrak{p}^\circ)}(\,\cdot\,,\phi_1;g_2,\phi_2) = \sum_{\check{x}_1\in\check{V}}\omega_\chi(\,\cdot\,)\phi^{\mathsf{ver}}(\check{x}_1)$$

be the theta series for the Schwartz function

$$\phi^{\mathsf{ver}} = \sum_{\check{h}_1\in\check{H}(\mathbb{Q})\backslash(E_{\mathfrak{p}^\circ}^{\times,1}/\nu(K_{\mathfrak{p},n})\times\check{H}_f^{\mathfrak{p}}/K^{\mathfrak{p}})}\nu(\,\cdot\,,\phi_{1,\mathfrak{p}},\check{h}_1;\phi_2,g_2)\otimes\omega_\chi(\check{h})\phi_1^{\mathfrak{p}}.$$

Lemma 5.11. *For $g_i \in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$,*

$$\log q\,\mathscr{Z}_{\phi_1}(g_1)\cdot\mathscr{V}_{\phi_2}(g_2) = \theta^{\mathsf{ver}}_{(\mathfrak{p}^\circ)}(g_1,\phi_1;g_2,\phi_2)$$

is a theta series for $g_1\in e_{\mathfrak{p}}H'(\mathbb{A}_F^{\mathfrak{p}})$.

Finally, we let

$$A_{(\mathfrak{p}^\circ)}(g_1, \phi_1) = \log q^{\mathcal{V}_{\phi_1}(g_1)} \cdot \omega_{\mathfrak{p}^\circ}.$$

Then, in summary, we have:

Proposition 5.12. *For $\phi_{1,\mathcal{S}} \otimes \phi_{2,\mathcal{S}} \in \mathcal{S}(\mathbb{V}_{\mathcal{S}}^2)_{\mathrm{reg}}$ and $g_i \in e_{\mathcal{S}} H'(\mathbb{A}_F^{\mathcal{S}})$ $(i = 1, 2)$,*

$$\langle \widehat{Z}_{\phi_1}(g_1), \widehat{Z}_{\phi_2}(g_2) \rangle_{\mathfrak{p}^\circ} = \theta^{\mathrm{hor}}_{(\mathfrak{p}^\circ)}(\iota(g_1, g_2^\vee); \phi_1, \phi_2) + \theta^{\mathrm{ver}}_{(\mathfrak{p}^\circ)}(g_1, \phi_1; g_2, \phi_2) + A_{(\mathfrak{p}^\circ)}(g_1, \phi_1) E(g_2, \phi_2).$$

6. An arithmetic inner product formula

6A. ***Holomorphic projection.*** In this section, we calculate the holomorphic projection of the analytic kernel function $E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ and its relation with the geometric kernel function when $n = 1$. We follow the general theory for the GL_2 case in [Gross and Zagier 1986; Zhang 2001a; 2001b; Yuan et al. 2011].

Holomorphic projection in general. Let $\mathfrak{k} = (\mathfrak{k}_\iota)_\iota \in \mathbb{Z}^{\Sigma_\infty}$ be a sequence of integers. We denote by $\mathscr{A}_0(H') \subset \mathscr{A}(H')$ the subspace of cuspidal automorphic forms of $H' = \mathrm{U}(W_1)$ and by $\mathscr{A}_0^{\mathfrak{k}}(H') \subset \mathscr{A}_0(H')$ those ones whose archimedean component is in a discrete series representation of weight $(1+\mathfrak{k}, 1-\mathfrak{k})$. Let Z' be the center of H', and hence isomorphic to $E^{\times,1}$ as an F-torus. From $\mathfrak{k}$, we define a character $\zeta^{\mathfrak{k}}$ of Z'_∞ by $\zeta_\iota^{\mathfrak{k}}(z) = z^{2\mathfrak{k}_\iota}$. Let $\mathscr{A}(H', \zeta^{\mathfrak{k}})$ be the subspace of $\mathscr{A}(H')$ consisting of all forms which have the archimedean central character $\zeta^{\mathfrak{k}}$. It is obvious that $\mathscr{A}_0^{\mathfrak{k}}(H') \subset \mathscr{A}(H', \zeta^{\mathfrak{k}})$. For any element in $\mathscr{A}_0^{\mathfrak{k}}(H')$ and any $t \in F^+$, the t-th archimedean Whittaker function (with respect to the standard ψ_∞^0) is $W_t^{\mathfrak{k}}$, where

$$W_t^{\mathfrak{k}}(n(b)m(a)[k_1, k_2]) = \prod_{\iota \in \Sigma_\infty} e^{2\pi i t(b_\iota + i a_\iota \bar{a}_\iota)} (a_\iota \bar{a}_\iota) k_{1,\iota}^{1+\mathfrak{k}_\iota} k_{2,\iota}^{1-\mathfrak{k}_\iota}$$

for all $a = (a_\iota) \in E_\infty^\times$, $b = (b_\iota) \in F_\infty$ and $[k_1, k_2] = ([k_{1,\iota}, k_{2,\iota}])$ in the standard maximal compact subgroup $\mathscr{K}'_\infty$.

We let $\mathscr{A}_0^{\mathfrak{k}}(H' \times H')$ (resp. $\mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$) to be the subspace of $\mathscr{A}(H' \times H')$ consisting of functions F such that $F(\cdot, g_2)$ and $F(g_1, \cdot)$ are both in $\mathscr{A}_0^{\mathfrak{k}}(H')$ (resp. $\mathscr{A}(H', \zeta^{\mathfrak{k}})$) for all $g_1, g_2 \in H'(\mathbb{A}_F)$. For any form $F \in \mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$ and any $f_1 \otimes f_2 \in \mathscr{A}_0^{\mathfrak{k}}(H' \times H')$, we can define the usual Petersson inner product as

$$(F, f_1 \otimes f_2)_{H'} = \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} F(g_1, g_2) \overline{f_1(g_1) f_2(g_2)}\, dg_1\, dg_2.$$

Definition 6.1. The *holomorphic projection* Pr is a linear map from $\mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$ to $\mathscr{A}_0^{\mathfrak{k}}(H' \times H')$, such that $\mathrm{Pr}(F)$ is the unique form in $\mathscr{A}_0^{\mathfrak{k}}(H' \times H')$ satisfying $(\mathrm{Pr}(F), f_1 \otimes f_2)_{H'} = (F, f_1 \otimes f_2)_{H'}$ for any $f_1 \otimes f_2 \in \mathscr{A}_0^{\mathfrak{k}}(H' \times H')$.

For any automorphic form $F \in \mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$, we define the Whittaker function for a nontrivial character ψ of $F\backslash\mathbb{A}_F$ to be

$$F_{\psi,s}(g_1, g_2) = (4\pi)^{2d} W^{\mathfrak{k}}(g_{1,\infty}) W^{\mathfrak{k}}(g_{2,\infty}) \int_{[Z'(F_\infty)N'(F_\infty)\backslash H'(F_\infty)]^2} \lambda_{P'}(h_1)^s \lambda_{P'}(h_2)^s \times F_\psi(g_{1,f}h_1, g_{2,f}h_2) \overline{W^{\mathfrak{k}}(h_1) W^{\mathfrak{k}}(h_2)}\, dh_1\, dh_2, \quad (6\text{-}1)$$

where $W^{\mathfrak{k}} = W_t^{\mathfrak{k}}$ such that $\psi_\infty(x) = \psi_\infty^0(tx)$ and $d = [F : \mathbb{Q}]$ as before.

Proposition 6.2. *Let $F \in \mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$ be a form with asymptotic behavior*

$$F(m(a_1)g_1, m(a_2)g_2) = O_{g_1,g_2}(|a_1 a_2|_{\mathbb{A}_E}^{1-\epsilon})$$

as $a_i \in \mathbb{A}_E^\times$ and $|a_1 a_2|_{\mathbb{A}_E} \to \infty$ for some $\epsilon > 0$. Then the holomorphic projection $\mathrm{Pr}(F)$ has the Whittaker function

$$\mathrm{Pr}(F)_\psi(g_1, g_2) = \lim_{s \to 0} F_{\psi,s}(g_1, g_2).$$

Proof. First, we can decompose $F = \sum_s F_{\zeta_{s,1},\zeta_{s,2}}$ as a finite sum of element $F_{\zeta_{s,1},\zeta_{s,2}} \in \mathscr{A}(H' \times H', \zeta^{\mathfrak{k}})$ which has central character $\zeta_{s,1} \boxtimes \zeta_{s,2}$ such that $(\zeta_{s,1}, \zeta_{s,2})$ are distinct pairs. One can easily show that if F satisfies the asymptotic behavior in the proposition, so does each $F_{\zeta_{s,1},\zeta_{s,2}}$.

Now consider any Whittaker function $W^i(g_i) = W^{\mathfrak{k}}(g_{i,\infty}) W_f^i(g_{i,f})$ $(i = 1, 2)$ of $H'(\mathbb{A}_F)$ with central character ζ_i such that $W_f^i(g_{i,f})$ is compactly supported modulo $Z'(\mathbb{A}_{f,F})N'(\mathbb{A}_{f,F})$, we define the Poincaré series as

$$P_{W^i}(g_i) = \lim_{s \to 0^+} \sum_{\gamma \in Z'(F)N'(F)\backslash H'(F)} W^i(\gamma g_i) \lambda_{P'}(\gamma_\infty g_{i,\infty})^s.$$

If (ζ_1, ζ_2) doesn't appear in $\{(\zeta_{s,1}, \zeta_{s,2})\}$, then the Petersson inner product

$$(F, P_{W^1} \otimes P_{W^2})_{H'}$$

is automatically zero; hence we only need to consider the case where it appears. Then, assuming that F has the asymptotic behavior as in the proposition, we have, after choosing suitable quotient measures dg_1 and dg_2,

$$\begin{aligned}
&(F, P_{W^1} \otimes P_{W^2})_{H'} \\
&\quad = \int F_{\zeta_1,\zeta_2}(g_1, g_2) \overline{P_{W^1}(g_1) P_{W^2}(g_2)}\, dg_1\, dg_2 \\
&\quad = \lim_{s \to 0^+} \int F_{\zeta_1,\zeta_2}(g_1, g_2) \overline{W^1(g_1) W^2(g_2)} \lambda_{P'}(g_{1,\infty})^s \lambda_{P'}(g_{2,\infty})^s\, dg_1\, dg_2 \\
&\quad = \lim_{s \to 0^+} \int (F_{\zeta_1,\zeta_2})_\psi(g_1, g_2) \overline{W^1(g_1) W^2(g_2)} \lambda_{P'}(g_{1,\infty})^s \lambda_{P'}(g_{2,\infty})^s\, dg_1\, dg_2,
\end{aligned}$$

where the first two integrals are taken over $[Z'(\mathbb{A}_F)H'(F)\backslash H'(\mathbb{A}_F)]^2$ and that last over $[Z'(\mathbb{A}_F)N'(\mathbb{A}_F)\backslash H'(\mathbb{A}_F)]^2$.

Since $(\mathrm{Pr}(F), P_{W^1}\otimes P_{W^2})_{H'}$ is equal to $(F, P_{W^1}\otimes P_{W^2})_{H'}$, its value is

$$\int_{[Z'(F_\infty)N'(F_\infty)\backslash H'(F_\infty)]^2} W^{\natural}(g_1)W^{\natural}(g_2)\overline{W^{\natural}(g_1)W^{\natural}(g_2)}\,dg_1\,dg_2\times$$
$$\int_{[Z'(\mathbb{A}_{f,F})N'(\mathbb{A}_{f,F})\backslash H'(\mathbb{A}_{f,F})]^2} (\mathrm{Pr}(F)_{\zeta_1,\zeta_2})_\psi(g_{1,f}, g_{2,f})\overline{W^1_f(g_{1,f})W^2_f(g_{2,f})}\,dg_{1,f}dg_{2,f}.$$

The first factor equals $(4\pi)^{-2d}$. Therefore

$$(\mathrm{Pr}(F), P_{W^1}\otimes P_{W^2})_{H'} = (4\pi)^{-2d}\times$$
$$\int_{[Z'(\mathbb{A}_{f,F})N'(\mathbb{A}_{f,F})\backslash H'(\mathbb{A}_{f,F})]^2} (\mathrm{Pr}(F)_{\zeta_1,\zeta_2})_\psi(g_{1,f}, g_{2,f})\overline{W^1_f(g_{1,f})W^2_f(g_{2,f})}\,dg_{1,f}dg_{2,f}.$$

Since this holds for all possible W^1_f, W^2_f, and (ζ_1, ζ_2), we conclude that

$$\mathrm{Pr}(F)_\psi(g_1, g_2) = \lim_{s\to 0} F_{\psi,s}(g_1, g_2). \qquad \square$$

Now suppose F does not satisfy an asymptotic behavior as in Proposition 6.2.

Definition 6.3. For any $F\in\mathscr{A}(H'\times H', \zeta^{\natural})$, we let

$$\widetilde{\mathrm{Pr}}(F)_\psi(g_1, g_2) = \operatorname*{const}_{s\to 0} F_{\psi,s}(g_1, g_2),$$

where $\operatorname{const}_{s\to 0}$ denotes the constant term at $s=0$ after the meromorphic continuation (around 0). We define the *quasiholomorphic projection* of F to be

$$\widetilde{\mathrm{Pr}}(F)(g_1, g_2) = \sum_\psi \widetilde{\mathrm{Pr}}(F)_\psi(g_1, g_2),$$

where the sum is taken over all nontrivial characters of $F\backslash\mathbb{A}_F$. The above proposition just says that for F satisfying that asymptotic behavior, we have $\widetilde{\mathrm{Pr}}(F) = \mathrm{Pr}(F)$. In fact, the definition can apply to more general functions just in

$$L^2(N'(F)\backslash H'(\mathbb{A}_F), \zeta^{\natural})\otimes L^2(N'(F)\backslash H'(\mathbb{A}_F), \zeta^{\natural}).$$

Holomorphic projection of the analytic kernel function. Now we want to apply the above theory to the particular form $E'(0, \iota(g_1, g_2^\vee), \phi_1\otimes\phi_2))\in\mathscr{A}(H'\times H', \chi^\circ)$ for $\phi_\infty = \phi^\vee_\infty = \phi^0_\infty$, where $\chi^\circ := \chi\big|_{Z'_\infty} = \zeta^{\natural\chi}/2$. Unfortunately, this form does not have the asymptotic behavior stated in Proposition 6.2. To find its holomorphic projection, we introduce the following function:

$$F(s; g_1, g_2; \phi_1, \phi_2) = E\big(s+\tfrac{1}{2}, g_1, \phi_1\big)E\big(s+\tfrac{1}{2}, g_2, \phi_2\big)\in\mathscr{A}(H'\times H', \chi^\circ),$$

where we use the Weil representation $\omega_{\chi,\psi}$ in both Eisenstein series on $H'(\mathbb{A}_F)$. This function is holomorphic at $s=0$. We claim that

Proposition 6.4. *The difference* $E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)$ *has the asymptotic behavior stated in Proposition 6.2.*

Proof. Since it is symmetric in g_1 and g_2, without lost of generality, we prove the asymptotic behavior for g_1. Consider the Fourier expansion

$$E\big(s, \iota(m(a_1)g_1, g_2^\vee), \phi_1 \otimes \phi_2\big) = \sum_{T \in \mathrm{Her}_2(E)} E_T\big(s, \iota(m(a_1)g_1, g_2^\vee), \phi_1 \otimes \phi_2\big)$$

in which all terms except those with

$$T = \begin{pmatrix} 0 & 0 \\ 0 & d_2 \end{pmatrix}$$

in the summation are bounded as $|a_1|_{\mathbb{A}_E} \to \infty$. Hence we only need to consider those T. Before we compute these terms, we recall some matrices representing elements in Weyl groups:

$$w_2 = \begin{pmatrix} & \mathbf{1}_2 \\ -\mathbf{1}_2 & \end{pmatrix}, \qquad w_{2,1} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}, \qquad w_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Case I: $d_2 \in F^\times$. We have

$$\begin{aligned} &E_T(s, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) \\ &= \int_{\mathrm{Her}_2(\mathbb{A}_E)} \omega_\chi(w_2 n(b)\iota(g_1, g_2^\vee))\phi_1 \otimes \phi_2(0)\lambda_P(w_2 n(b)\iota(g_1, g_2^\vee))^s \psi(\mathrm{tr}\, Tb)^{-1} db \\ &\quad + \int_{\mathrm{Her}_2(\mathbb{A}_E)} \omega_\chi(w_{2,1} n(b)\iota(g_1, g_2^\vee))\phi_1 \otimes \phi_2(0)\lambda_P(w_{2,1} n(b)\iota(g_1, g_2^\vee))^s \psi(\mathrm{tr}\, Tb)^{-1} db \\ &\qquad\qquad + \text{terms that are bounded,} \end{aligned}$$

where the first term is just $W_T(s, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ and the second term equals

$$\begin{aligned} W_{1,d_2}(s; g_1, g_2^\vee; \phi_1, \phi_2) &:= \int_{\mathbb{A}_F} \omega_\chi\left(w_{2,1} n \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix} \iota(g_1, g_2^\vee)\right)\phi_1 \otimes \phi_2(0)\lambda_{P'}(g_1)^s \\ &\qquad \times \lambda_{P'}(w_1 n(b_2) g_2)^s \psi(d_2 b_2)\, db_2 \\ &= \omega_\chi(g_1)\phi_1(0)\lambda_{P'}(g_1)^s \times W_{-d_2}\big(s + \tfrac{1}{2}, g_2, \phi_2\big). \end{aligned}$$

Case II: $d_2 = 0$. We have, apart from the terms $W_T(s, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ and $W_{1,0}(s; g_1, g_2^\vee; \phi_1, \phi_2)$, another term,

$$W_{2,0}(s; g_1, g_2^\vee; \phi_1, \phi_2) := \omega_\chi(g_1)\phi_1(0)\lambda_{P'}(g_1)^s \times \omega_\chi(g_2^\vee)\phi_2(0)\lambda_{P'}(g_2)^s.$$

Now the term $W_T(s, \iota(m(a_1)g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ has asymptotic behavior $O_{g_1,g_2}(1)$ as $|a_1|_{\mathbb{A}_E} \to \infty$, hence we don't need to consider it. What is left is

$$E_{00}(s, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) := W_{2,0}(s; g_1, g_2^\vee; \phi_1, \phi_2) + \sum_{d_2 \in F} W_{1,d_2}(s; g_1, g_2^\vee; \phi_1, \phi_2)$$
$$= \omega_\chi(g_1)\phi_1(0)\lambda_{P'}(g_1)^s \times E\big(s + \tfrac{1}{2}, g_2, \phi_2\big).$$

It turns out that our form $F(s; g_1, g_2; \phi_1, \phi_2)$ is just the Eisenstein series (in g_1) of the section $E_{00}(s, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$, namely

$$F(s; g_1, g_2; \phi_1, \phi_2) = \sum_{\gamma \in P'(F)\backslash H'(F)} E_{00}(s, \iota(\gamma g_1, g_2^\vee), \phi_1 \otimes \phi_2).$$

Viewing it as a function in g_1, we have the Fourier expansion

$$F(s; g_1, g_2; \phi_1, \phi_2) = \sum_{d_1 \in F} E_{d_1}\big(s + \tfrac{1}{2}, g_1, \phi_1\big) \times E\big(s + \tfrac{1}{2}, g_2, \phi_2\big).$$

For all $d_1 \in F^\times$, the term $E_{d_1}\big(s + \frac{1}{2}, m(a_1)g_1, \phi_1\big)$ decays exponentially as $|a_1|_{\mathbb{A}_E}$ goes to infinity. Hence we only need to consider the term

$$\begin{aligned} &F_0(s; g_1, g_2; \phi_1, \phi_2) \\ &\quad = E_0\big(s + \tfrac{1}{2}, g_1, \phi_1\big) E\big(s + \tfrac{1}{2}, g_2, \phi_2\big) \\ &\quad = \big(\omega_\chi(g_1)\phi_1(0)\lambda_{P'}(g_1)^s + W_0\big(s + \tfrac{1}{2}, g_1, \phi_1\big)\big) E\big(s + \tfrac{1}{2}, g_2, \phi_2\big) \\ &\quad = E_{00}(s, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) + W_0\big(s + \tfrac{1}{2}, g_1, \phi_1\big) E\big(s + \tfrac{1}{2}, g_2, \phi_2\big). \end{aligned}$$

Now the proposition is equivalent to showing that

$$\begin{aligned} &E'_{00}(0, \iota(\gamma g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'_0(0; g_1, g_2; \phi_1, \phi_2) \\ &\quad = -\frac{d}{ds}\Big|_{s=0} \big(W_0\big(s + \tfrac{1}{2}, g_1, \phi_1\big) E\big(s + \tfrac{1}{2}, g_2, \phi_2\big)\big) \\ &\quad = -W_0\big(\tfrac{1}{2}, g_1, \phi_1\big) E'\big(\tfrac{1}{2}, g_2, \phi_2\big) - W'_0\big(\tfrac{1}{2}, g_1, \phi_1\big) E\big(\tfrac{1}{2}, g_2, \phi_2\big) \end{aligned}$$

has the asymptotic behavior (in g_1). This is true since

$$W_0\big(\tfrac{1}{2}, m(a_1)g_1, \phi_1\big) = O_{g_1,g_2}(1), \quad W'_0\big(\tfrac{1}{2}, m(a_1)g_1, \phi_1\big) = O_{g_1,g_2}(\log|a_1|_{\mathbb{A}_E}). \quad \square$$

By the proposition, we have

$$\begin{aligned} &\mathsf{Pr}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) \qquad (6\text{-}2) \\ &= \mathsf{Pr}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)\big) + \mathsf{Pr}(F'(0; g_1, g_2; \phi_1, \phi_2)) \\ &= \widetilde{\mathsf{Pr}}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)\big) + \mathsf{Pr}(F'(0; g_1, g_2; \phi_1, \phi_2)). \end{aligned}$$

Since

$$F'(0; g_1, g_2; \phi_1, \phi_2) = E'\big(\tfrac{1}{2}, g_1, \phi_1\big) E\big(\tfrac{1}{2}, g_2, \phi_2\big) + E\big(\tfrac{1}{2}, g_1, \phi_1\big) E'\big(\tfrac{1}{2}, g_2, \phi_2\big),$$

its holomorphic projection $\mathrm{Pr}(F'(0; g_1, g_2; \phi_1, \phi_2)) = 0$. Then

$$\begin{aligned}(6\text{-}2) &= \widetilde{\mathrm{Pr}}(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) - F'(0; g_1, g_2; \phi_1, \phi_2)) \\ &= \widetilde{\mathrm{Pr}}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) - \widetilde{\mathrm{Pr}}(F'(0; g_1, g_2; \phi_1, \phi_2)) \\ &= \widetilde{\mathrm{Pr}}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) - \widetilde{\mathrm{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)E\big(\tfrac{1}{2}, g_2, \phi_2\big)\big) \\ &\qquad - \widetilde{\mathrm{Pr}}\big(E\big(\tfrac{1}{2}, g_1, \phi_1\big)E'\big(\tfrac{1}{2}, g_2, \phi_2\big)\big).\end{aligned}$$

It is easy to see that

$$\widetilde{\mathrm{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)E\big(\tfrac{1}{2}, g_2, \phi_2\big)\big) = \widetilde{\mathrm{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)\big)E_*\big(\tfrac{1}{2}, g_2, \phi_2\big),$$

where

$$E_*\big(\tfrac{1}{2}, g_i, \phi_i\big) = \sum_{d_i \in F^\times} W_{d_i}\big(\tfrac{1}{2}, g_i, \phi_i\big).$$

In particular, if $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ for at least one finite place v, then given g_1 and g_2 in $P_v' H'(\mathbb{A}_F^v)$, each $E_*(\frac{1}{2}, g_i, \phi_i)$, $i = 1, 2$, is a linear combination of an Eisenstein series and an automorphic character. In summary:

Proposition 6.5. *The holomorphic projection of the analytic kernel function is*

$$\begin{aligned}\mathrm{Pr}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) &= \widetilde{\mathrm{Pr}}\big(E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)\big) \\ &\quad - \widetilde{\mathrm{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)\big)E_*\big(\tfrac{1}{2}, g_2, \phi_2\big) - E_*\big(\tfrac{1}{2}, g_1, \phi_1\big)\widetilde{\mathrm{Pr}}\big(E'\big(\tfrac{1}{2}, g_2, \phi_2\big)\big).\end{aligned}$$

Quasiholomorphic projection of the analytic kernel function. Now we are going to compute the quasiholomorphic projection of $E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ under the following assumption:

(REG) $\phi_i = \phi_\infty^0 \phi_{i,f}$ *with* $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}}$ *for all* $v \in \mathcal{S}$ *and* $\phi_{1,v} \otimes \phi_{2,v} \in \mathcal{S}(\mathbb{V}_v^2)_{\mathrm{reg}, d_v}$ *for* $v \in \mathcal{S}$ *nonsplit with* $d_v \geq d_{\psi_v}$*;* $g_i \in e_{\mathcal{S}} H'(\mathbb{A}_F^{\mathcal{S}})$.

Recall from (2-13) that

$$E'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2) = \sum_{v \notin \mathcal{S}} E_v'(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2).$$

It is clear that when we apply $\widetilde{\mathrm{Pr}}$ to the above expression, nothing will change except the terms $E_\iota(0, \iota(g_1, g_2^\vee), \phi_1 \otimes \phi_2)$ for $\iota \in \Sigma_\infty$. Now we just fix one $\iota \in \Sigma_\infty$ and consider, by [Liu 2011, Theorem 4.20],

$$-2\,\mathrm{vol}(K)\langle (Z_{\phi_1}(g_1), \Xi_{\phi_1}(g_1)_{\iota^\circ}), (Z_{\phi_2}(g_2), \Xi_{\phi_2}(g_2)_{\iota^\circ})\rangle_{M_K},$$

which is (after forgetting the constant $-2\,\mathrm{vol}(K)$), by definition, the integration over the (complex) Shimura curve of

$$\Bigg(\sum_{x_1\neq 0}\sum_{h_1\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K}\omega_\chi(g_1)\phi_i(T(x_1),h_1^{-1}x_1)\Xi_{x_1a_1,h_1}\Bigg)$$
$$*\Bigg(\sum_{x_2\neq 0}\sum_{h_2\in H(\mathbb{Q})\backslash H(\mathbb{A}_f)/K}\omega_\chi(g_2)\phi_2(T(x_2),h_2^{-2}x_2)\Xi_{x_2a_2,h_2}\Bigg).$$

See [Liu 2011, Section 4C] for the notation. Since this expression and the process of taking $\widetilde{\mathsf{Pr}}$ is symmetric in g_1 and g_2, let us just do the first variable and hence omit the subscript 1 in the following calculation. Of course we only need to do this for the neutral component, hence we consider the current

$$\sum_{x\neq 0}\omega_\chi(g)\phi(T(x),x)\Xi_{xa}.$$

It is clear that for $T(x)=t\notin F^+$,

$$\widetilde{\mathsf{Pr}}(\omega_\chi(-)\phi(t,x)\Xi_{xa})_{\psi_{t'}}(g)=0$$

for any $t'\in F^+$ (which is just the set of all totally positive numbers in F), hence these terms vanish after applying $\widetilde{\mathsf{Pr}}$. For those x such that $T(x)=t\in F^+$, the corresponding term will contribute to the t-th Fourier coefficient in the quasiholomorphic projection. Namely,

$$\widetilde{\mathsf{Pr}}\Bigg(\sum_{x\neq 0}\omega_\chi(g)\phi(T(x),x)\Xi_{xa}\Bigg)=\sum_{t\in F^+}\widetilde{\mathsf{Pr}}\Bigg(\sum_{T(x)=t}\omega_\chi(-)\phi(t,x)\Xi_{xa}\Bigg)_{\psi_t}(g),$$

where, similar to (6-1),

$$\widetilde{\mathsf{Pr}}\Bigg(\sum_{T(x)=t}\omega_\chi(-)\phi(T(x),x)\Xi_{xa}\Bigg)_{\psi_t}(g)=\operatorname*{const}_{s\to 0}(4\pi t)W_t^{\mathfrak{k}_\chi/2}(g_\iota)$$
$$\times\sum_{T(x)=t}\int_{Z'(\mathbb{R})N'(\mathbb{R})\backslash H'(\mathbb{R})}\lambda_{P'}(h)^s\omega_\chi(g^\iota h)\phi(t,x)\Xi_{xa}\,dh,\quad(6\text{-}3)$$

where we identify F_ι with $\mathbb{R}$ in the domain of the integral and a is such that $h=n(b)m(a)k$ of h in the Iwasawa decomposition. Making the substitution $y=a\bar{a}$, we have

$$(6\text{-}3)=\operatorname*{const}_{s\to 0}(4\pi t)W_t^{\mathfrak{k}_\chi/2}(g_\iota)\sum_{T(x)=t}\omega_\chi(g^\iota)\phi^\iota(t,x)\int_0^\infty\Xi_{\sqrt{y}\cdot x}y^se^{-4\pi ty}\,dy$$
$$=\operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)\int_0^\infty\Xi_{\sqrt{y}\cdot x}y^se^{-4\pi ty}\,dy.\quad(6\text{-}4)$$

If we let $\delta_x(z) = R(x,z)/2t = -(x_z, x_z)/(x,x)$, then

$$\begin{aligned}
(6\text{-}4) &= \operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)\int_0^\infty\left(\int_1^\infty \frac{e^{-4\pi t y u\delta_x(z)}}{u}\,du\right)y^s e^{-4\pi t y}\,dy\\
&= \operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)t^{-1-s}\int_0^\infty\int_1^\infty \frac{e^{-4\pi y u\delta_x(z)}}{u}y^s e^{-4\pi t y}\,du\,dy\\
&= \operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)t^{-1-s}\int_1^\infty \frac{1}{u}\left(\int_0^\infty e^{-4\pi y(1+u\delta_x(z))}\right)du\\
&= \operatorname*{const}_{s\to 0}(4\pi t)\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)\frac{\Gamma(1+s)}{(4\pi t)^{1+s}}\int_1^\infty \frac{du}{u(1+u\delta_x(z))^{1+s}}\\
&= \operatorname*{const}_{s\to 1}\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)\int_1^\infty \frac{du}{u(1+u\delta_x(z))^{s}}. \qquad (6\text{-}5)
\end{aligned}$$

Admissible Green's function. As in [Gross and Zagier 1986], we introduce the Legendre function of the second type:

$$Q_{s-1}(t) = \int_0^\infty (t+\sqrt{t^2-1}\cosh u)^{-s}\,du, \quad t>1,\ s>0.$$

Then the admissible Green's function attached to the divisor $\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)Z_x$ (on the neutral component) is

$$\Xi_\phi^{\mathrm{adm}}(g)_t = \operatorname*{const}_{s\to 1} 2\sum_{T(x)=t}\omega_\chi(g)\phi(t,x)Q_{s-1}(1+2\delta_x(z)).$$

By a result of Gross and Zagier, we have

$$\int_1^\infty \frac{du}{u(1+uc)^s} = 2Q_{s-1}(1+2c) + O(c^{-s-1}), \quad c\to+\infty.$$

Combining (6-5), Corollary 4.15, and Proposition 6.5, we have:

Proposition 6.6. *Under the assumptions (REG) for $\phi_1\otimes\phi_2$ and g_i, we have*

$$\begin{aligned}
\mathrm{Pr}\big(E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)\big) &= -\mathrm{vol}(K)\sum_{\substack{v^\circ\mid v\\ v\notin\mathcal{S}}}\langle\widehat{Z}_{\phi_1}(g_1),\widehat{Z}_{\phi_2}(g_2)\rangle_{v^\circ}\\
&\quad-\widetilde{\mathrm{Pr}}\big(E'(\tfrac12,g_1,\phi_1)\big)E_*(\tfrac12,g_2,\phi_2) - E_*(\tfrac12,g_1,\phi_1)\widetilde{\mathrm{Pr}}\big(E'(\tfrac12,g_2,\phi_2)\big),
\end{aligned}$$

where at the archimedean places we are using admissible Green's functions.

6B. ***Uniqueness of local invariant functionals.*** We now fix a place $v\in\Sigma$ and suppress it from the notation. We prove that the space $\mathrm{Hom}_{H'\times H'}(I_2(0,\chi),\pi^\vee\boxtimes\chi\pi)$ is of dimension 1, following [Harris et al. 1996].

From χ, we have defined an automorphic character π_χ of H' in the following way. Given $g \in H'$, $\det g \in E^{\times,1}$; hence we can write $\det g = e_g/e_g^\tau$ for some $e_g \in E^\times$, by Hilbert's Theorem 90. Define $\pi_\chi(g) = \chi(e_g)$, which is well-defined since $\chi|_{F^\times} = 1$.

Proposition 6.7. *For an irreducible admissible representation $\pi \ncong \pi_\chi^{-1}$ of H', we have* $\dim \mathrm{Hom}_{H'\times H'}(I_2(0,\chi), \pi^\vee \boxtimes \chi\pi) = 1$ *and $L(s,\pi,\chi)$ is holomorphic at $s = \frac{1}{2}$.*

First, we have a double coset decomposition

$$H'' = P\gamma_0\iota(H'\times H') \sqcup P\iota(H'\times H') =: \Omega_0 \sqcup \Omega_1$$

with Ω_0 open and Ω_1 closed. Hence we have a filtration $I_2(0,\chi) \supset I_2^{(0)}(0,\chi)$ where

$$I_2^{(0)}(0,\chi) = \{\varphi \in I_2(0,\chi) \mid \mathrm{Supp}\varphi \subset \Omega_0\},$$

which is invariant under the action of $H'\times H'$ by right translation through ι. As $H'\times H'$ representations, we have $Q_2^{(0)}(0,\chi) = I_2^{(0)}(0,\chi)$ and $Q_2^{(1)}(0,\chi) = I_2(0,\chi)/I_2^{(0)}(0,\chi)$. We have an $H'\times H'$ intertwining operator

$$\begin{aligned} Q_2^{(0)}(0,\chi) &\xrightarrow{\sim} \mathcal{S}(H')(1\otimes\chi) \\ \varphi &\longmapsto \Psi(g) = \varphi(\gamma_0\iota(g,\mathbf{1}_2)), \end{aligned}$$

where $\mathcal{S}(H')$ is the space of Schwartz functions on H', since

$$\varphi(\gamma_0\iota(g,\mathbf{1}_2)\iota(g_1,g_2)) = \varphi(\gamma_0\iota(g_2,g_2)\iota(g_2^{-1}gg_1,\mathbf{1}_2)) = \chi(\det g_2)\varphi(\gamma_0\iota(g_2^{-1}gg_1,\mathbf{1}_2)).$$

There is on $\mathcal{S}(H')\otimes(\pi\boxtimes\pi^\vee)$ a unique $H'\times H'$-invariant functional (up to a constant) given by

$$\Psi\otimes(f\otimes f^\vee) \mapsto \int_{H'} \langle\pi(g)f, f^\vee\rangle\Psi(g)\,dg.$$

But,

$$\begin{aligned} \mathrm{Hom}_{H'\times H'}(\mathcal{S}(H')\otimes(\pi\boxtimes\pi^\vee),\mathbb{C}) &= \mathrm{Hom}_{H'\times H'}(\mathcal{S}(H'),\pi^\vee\boxtimes\pi) \\ &= \mathrm{Hom}_{H'\times H'}(\mathcal{S}(H')\otimes(1\boxtimes\chi),\pi^\vee\boxtimes\chi\pi) \\ &= \mathrm{Hom}_{H'\times H'}(Q_2^{(0)}(0,\chi),\pi^\vee\boxtimes\chi\pi). \end{aligned}$$

For the representation $Q_2^{(1)}(0,\chi)$, we have:

Lemma 6.8. *If $\pi \ncong \pi_\chi^{-1}$, then* $\mathrm{Hom}_{H'\times H'}(Q_2^{(1)}(0,\chi),\pi^\vee\boxtimes\chi\pi) = 0$.

Proof. We have an intertwining isomorphism $Q_2^{(1)}(0,\chi) \to I_1(\frac{1}{2},\chi)\boxtimes I_1(\frac{1}{2},\chi)$ given by

$$\bar\varphi \mapsto \big((g_1,g_2)\mapsto \bar\varphi(\iota(g_1,g_2))\big),$$

since $\bar\varphi(\iota(p_1g_1,p_2g_2))=\bar\varphi(\iota(p_1,p_2)\iota(g_1,g_2))=\chi(a_1a_2)|a_1a_2|_E\bar\varphi(\iota(g_1,g_2))$ where $p_1=n(b_1)m(a_1)k_1$ and $p_2=n(b_2)m(a_2)k_2$. Hence

$$\begin{aligned}&\mathrm{Hom}_{H'\times H'}(Q_2^{(1)}(0,\chi),\pi^\vee\boxtimes\chi\pi)\\ &\qquad=\mathrm{Hom}_{H'\times H'}\big(I_1(\tfrac{1}{2},\chi)\boxtimes I_1(\tfrac{1}{2},\chi),\pi^\vee\boxtimes\chi\pi\big)\\ &\qquad=\mathrm{Hom}_{H'\times H'}\big(\pi\boxtimes\chi^{-1}\pi^\vee, I_1(-\tfrac{1}{2},\chi^{-1})\boxtimes I_1(-\tfrac{1}{2},\chi^{-1})\big).\end{aligned}$$

By [Kudla and Sweet 1997, Theorem 1.2] for v finite nonsplit, [Kudla and Sweet 1997, Theorem 1.3] for v finite split, and [Lee 1994, Theorem 6.10 (1-b)] for v infinite, the only irreducible H'-submodule contained in $I_1(-\frac{1}{2},\chi^{-1})$ is isomorphic to π_χ^{-1}. The lemma follows by our assumption on π. □

Proof of Proposition 6.7. The normalized zeta integral (2-1) has already defined a nonzero element in $\mathrm{Hom}_{H'\times H'}(I_2(0,\chi),\pi^\vee\boxtimes\chi\pi)$, so the dimension is at least 1. If it is higher than one, we can find a nonzero element in $\mathrm{Hom}_{H'\times H'}(I_2(0,\chi),\pi^\vee\boxtimes\chi\pi)$ whose restriction to $I_2^{(0)}(0,\chi)$ is zero since $\dim\mathrm{Hom}_{H'\times H'}(Q_2^{(0)}(0,\chi),\pi^\vee\boxtimes\chi\pi)=1$. Then it defines a nonzero element in $\mathrm{Hom}_{H'\times H'}(Q_2^{(1)}(0,\chi),\pi^\vee\boxtimes\chi\pi)$ which is 0 by the above lemma. Hence $\dim\mathrm{Hom}_{H'\times H'}(I_2(0,\chi),\pi^\vee\boxtimes\chi\pi)=1$.

For the L-factor part, the restriction of the normalized zeta integral to $I_2^{(0)}(0,\chi)$ is nonzero. But the original zeta integral has already been absolutely convergent at $s=0$ if $\varphi\in I_2^{(0)}(0,\chi)$, hence $L(s,\pi,\chi)$ cannot have a pole at $s=\frac{1}{2}$ since $b_2(s)$ is holomorphic and nonzero at $s=0$. □

Remark 6.9. Proposition 6.7 is conjectured to be true for any n, s, χ, and irreducible admissible representation π. This is proved in [Harris et al. 1996] for π supercuspidal — more precisely, for π *not occurring in the boundary at the point* s, which exactly equivalent to the assumption of Lemma 6.8 when $n=1$.

6C. ***Final proof.*** In this section, we prove the main theorem by combining all the results we have obtained.

We need to compare the (holomorphic projection of the) analytic kernel function and the geometric kernel function defined in Section 3B. First, we still assume (REG) on $\phi_1\otimes\phi_2$ and g_i. Let

$$\mathfrak{E}(g_1,g_2;\phi_1\otimes\phi_2)=\mathrm{Pr}\big(E'(0,\iota(g_1,g_2^\vee),\phi_1\otimes\phi_2)\big)-\mathbb{E}(g_1,g_2;\phi_1\otimes\phi_2),$$

which is in $\mathscr{A}(H'\times H',\chi^\circ)$. By (3-9), and Propositions 5.2, 5.7, 5.12, and 6.6, we have that the restriction of $\mathfrak{E}(g_1,g_2;\phi_1\otimes\phi_2)$ to the subset $[e_S H'(\mathbb{A}_F^S)]^2$ is equal

to the sum of the following terms:

$$\mathfrak{E}_{\mathrm{I}}(g_1, g_2; \phi_1 \otimes \phi_2) = -E(g_1, \phi_1)A(g_2, \phi_2) - A(g_1, \phi_1)E(g_2, \phi_2) - CE(g_1, \phi_1)E(g_2, \phi_2);$$

$$\mathfrak{E}_{\mathrm{II}}(g_1, g_2; \phi_1 \otimes \phi_2) = \sum_{\substack{\mathfrak{p}^\circ|\mathfrak{p} \\ \mathfrak{p}\in\mathcal{S}}} A_{(\mathfrak{p}^\circ)}(g_1, \phi_1)E(g_2, \phi_2);$$

$$\mathfrak{E}_{\mathrm{III}}(g_1, g_2; \phi_1 \otimes \phi_2) = \sum_{\substack{\mathfrak{p}^\circ|\mathfrak{p} \text{ split} \\ \mathfrak{p}\in\mathcal{S}}} E_{(\mathfrak{p}^\circ)}(g_1, \phi_1; g_2, \phi_2);$$

$$\mathfrak{E}_{\mathrm{IV}}(g_1, g_2; \phi_1 \otimes \phi_2) = \sum_{\substack{\mathfrak{p}^\circ|\mathfrak{p} \text{ nonsplit} \\ \mathfrak{p}\in\mathcal{S}}} \theta^{\mathsf{hor}}_{(\mathfrak{p}^\circ)}(\iota(g_1, g_2^\vee); \phi_1, \phi_2) + \theta^{\mathsf{ver}}_{(\mathfrak{p}^\circ)}(g_1, \phi_1; g_2, \phi_2);$$

$$\mathfrak{E}_{\mathrm{V}}(g_1, g_2; \phi_1, \phi_2) = -\widetilde{\mathsf{Pr}}\big(E'\big(\tfrac{1}{2}, g_1, \phi_1\big)\big)E_*\big(\tfrac{1}{2}, g_2, \phi_2\big) - E_*\big(\tfrac{1}{2}, g_1, \phi_1\big)\widetilde{\mathsf{Pr}}\big(E'\big(\tfrac{1}{2}, g_2, \phi_2\big)\big).$$

Now given any cuspidal automorphic representation π of H' such that π_∞ is a discrete series of weight $(1 - \mathfrak{k}^\chi/2, 1 + \mathfrak{k}^\chi/2)$ and $\epsilon(\pi, \chi) = -1$, for any $f \in \pi$ and $f^\vee \in \pi^\vee$, the integral

$$\int_{[P'_\infty e_{\mathcal{S}} H'(\mathbb{A}^{\mathcal{S}}_{f,F})]^2} f(g_1) f^\vee(g_2^\vee)\chi^{-1}(\det g_2)\mathfrak{E}_?(g_1, g_2; \phi_1 \otimes \phi_2) = 0$$

for $? = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}$, since each term involves either Eisenstein series, automorphic characters, or theta series when restricted to $e_{\mathcal{S}} H'(\mathbb{A}_F^{\mathcal{S}})$ which is dense in $H'(F)\backslash H'(\mathbb{A}_F)$! Hence we have

$$\int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2)E'(0, \iota(g_1, g_2), \phi_1 \otimes \phi_2) = \int_{[H'(F)\backslash H'(\mathbb{A}_F)]^2} f(g_1) f^\vee(g_2)\chi^{-1}(\det g_2)\mathbb{E}(g_1, g_2^\vee; \phi_1 \otimes \phi_2). \quad (6\text{-}6)$$

Recall our definition of the geometric kernel function, which is

$$\mathbb{E}(g_1, g_2; \phi_1 \otimes \phi_2) = \mathsf{vol}(K) \sum \langle \Theta_{\phi_1}(g_1), \Theta_{\phi_2}(g_2)\rangle^K_{\mathsf{NT}},$$

and we are using the Weil representation ω_χ in the formation of both $\Theta_{\phi_i}(g_i)$ $(i = 1, 2)$. If we now use $\omega_\chi^\vee$ to form the second and, to be consistent with the previous convention, write $\phi = \phi_1$ and $\phi^\vee = \phi_2$, then $\Theta_{\phi_2}(g_2^\vee) = \Theta_{\phi^\vee}(g_2)\chi(\det g_2)$.

Recall our definition of arithmetic theta lifting (with respect to the Weil representation ω_χ) in Section 3B:

$$\Theta^f_\phi = \int_{H'(F)\backslash H(\mathbb{A}_F)} f(g)\Theta_\phi(g)\, dg, \quad (6\text{-}7)$$

which is an element in $\mathrm{CH}^1(M)^0_{\mathbb{C}}$ and also $\Theta^{f^\vee}_{\phi^\vee}$ with respect to $\omega^\vee_\chi$, where $M = (M_K)_K$ is the projective system of (compactified) Shimura curves. For any K' under which ϕ and $\phi^\vee$ are invariant, the height pairing $\mathrm{vol}(K')\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle^{K'}_{\mathrm{NT}}$ calculated on $M_{K'}$ is independent of K', where $\mathrm{vol}(K')$ is defined as before. Hence we can use $\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}}$ to denote this number.

Recall that we have a totally positive-definite incoherent hermitian space $\mathbb{V}(\pi, \chi)$. We now prove our main theorem:

Theorem 6.10 (Arithmetic inner product formula). *Let π and χ be as above and $\mathbb{V}$ any totally positive-definite incoherent hermitian space over $\mathbb{A}_E$ of rank 2. Then*

(1) *If $\mathbb{V} \ncong \mathbb{V}(\pi, \chi)$, then the arithmetic theta lifting $\Theta^f_\phi = 0$ for any $f \in \pi$ and $\phi \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty}$;*

(2) *If $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$, then for any $f \in \pi$, $f^\vee \in \pi^\vee$, and any $\phi, \phi^\vee \in \mathcal{S}(\mathbb{V})^{\mathsf{U}_\infty}$ decomposable, we have*

$$\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{L_F(2)L(1, \epsilon_{E/F})} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee),$$

where almost all normalized zeta integrals (see Section 2A) appearing in the product are 1.

Proof. We first prove (2). Recall that in Section 2A, we defined the functional

$$\alpha(f, f^\vee, \phi, \phi^\vee) = \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$$

in the space $\bigotimes \mathrm{Hom}_{H'_v \times H'_v}(R(\mathbb{V}_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v)$ which is nonzero since $\mathbb{V} \cong \mathbb{V}(\pi, \chi)$. By Proposition 2.8 for $v \in \mathcal{S}$ and the fact that π_∞ is a discrete series representation of weight $(1 - \mathfrak{k}^\chi/2, 1 + \mathfrak{k}^\chi/2)$, we can choose local components f_v and $f_v^\vee$ for all $v \in \Sigma$ and ϕ_v and $\phi_v^\vee$ for $v \in \Sigma_f$ such that $\phi \otimes \phi^\vee$ satisfies the assumption (REG) and $\alpha(f, f^\vee, \phi, \phi^\vee) \neq 0$. On the other hand, the functional

$$\gamma(f, f^\vee, \phi, \phi^\vee) := \langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}} = \mathrm{vol}(K)\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle^K_{\mathrm{NT}}$$

is also in $\bigotimes_v \mathrm{Hom}_{H'_v \times H'_v}(R(\mathbb{V}_v, \chi_v), \pi_v^\vee \boxtimes \chi_v \pi_v)$ whose dimension is 1 according to Proposition 6.7. Hence we know that the ratio γ/α is a constant. By our special choice of f, ϕ, and $\phi^\vee$ and by (6-7), (6-6), and (2-3), we have

$$\frac{\gamma}{\alpha} = \frac{L'(\frac{1}{2}, \pi, \chi)}{L_F(2)L(1, \epsilon_{E/F})}.$$

Hence

$$\langle \Theta^f_\phi, \Theta^{f^\vee}_{\phi^\vee}\rangle_{\mathrm{NT}} = \frac{L'(\frac{1}{2}, \pi, \chi)}{L_F(2)L(1, \epsilon_{E/F})} \prod_v Z^*(0, \chi_v, f_v, f_v^\vee, \phi_v \otimes \phi_v^\vee)$$

for *any* $f \in \pi$, $f^\vee \in \pi^\vee$, and $\phi, \phi^\vee \in \mathcal{S}(\mathbb{V}_f)^{\mathsf{U}_\infty}$.

For (1), the functional γ is zero since $\mathbb{V} \ncong \mathbb{V}(\pi, \chi)$. If we take $\phi^\vee = \bar{\phi}$ and $f^\vee = \bar{f}$, then $\Theta_\phi^f = 0$ since the Néron–Tate height pairing on curves is positive definite. □

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