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If F^+ is a totally real field, if n is an odd integer and if Π is a regular, algebraic, essentially self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F^+})$, then we calculate the image of any complex conjugation under the l-adic representations $r_{l,\iota}(\Pi)$ associated to Π.

Introduction

Let F^+ denote a totally real number field and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. It is known that to a regular, algebraic, essentially self-dual, cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_{F^+})$ one can associate a continuous semisimple Galois representation

$$r_{l,\iota}(\Pi) : \mathrm{Gal}(\bar{F}^+/F^+) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l).$$

(For the definition of "regular, algebraic, essentially self-dual, cuspidal" see the start of Section 1.) This representation is known to be de Rham and its Hodge–Tate numbers are known. (They can be simply calculated from the infinitesimal character of π_∞.) For all finite places v of F^+ not dividing l one can calculate the Frobenius semisimplification of the restriction of $r_{l,\iota}(\Pi)$ to a decomposition group above v in terms of π_v via the local Langlands correspondence. This uniquely (in fact, over) determines $r_{l,\iota}(\Pi)$. (See [Shin 2011; Clozel et al. 2011; Caraiani 2010; Chenevier and Harris 2011].) The representation $r_{l,\iota}(\Pi)$ is conjectured to be irreducible. This is known if Π is discrete series at some finite place [Taylor and Yoshida 2007]. Moreover $r_{l,\iota}(\Pi)^\vee \cong r_{l,\iota}(\Pi) \otimes \mu$ for some character μ of $\mathrm{Gal}(\bar{F}^+/F^+)$ which is either totally odd (takes the value -1 on all complex conjugations) or totally even (takes the value $+1$ on all complex conjugations).

Frank Calegari raised the question as to whether, for an infinite place v of F^+ one can calculate the conjugacy class of $r_{l,\iota}(\Pi)(c_v)$, where $c_v \in \mathrm{Gal}(\bar{F}^+/F^+)$ is a

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complex conjugation for v. This conjugacy class has order two, so it is semisimple with eigenvalues ± 1. The problem is to determine how many $+1$'s and how many -1's occur. Because Π was assumed to be regular, we expect that the number of $+1$'s and -1's differ by at most one:

$$|\operatorname{tr} r_{l,\iota}(\Pi)(c_v)| \leq 1.$$

As we know the determinant of $r_{l,\iota}(\Pi)$ this would completely determine the conjugacy class of $r_{l,\iota}(\Pi)(c_v)$.

If μ is totally odd then [Bellaïche and Chenevier 2011] shows that n is even and that $r_{l,\iota}(\Pi)$ preserves an alternating pairing up to multiplier μ. In this case, because $\mathrm{GSp}_n(\overline{\mathbb{Q}}_l)$ has a unique conjugacy class of elements of order two and multiplier -1, we see that $\operatorname{tr} r_{l,\iota}(\Pi)(c_v) = 0$ for all $v|\infty$. So the problem lies in the case that μ is totally even, i.e., that $r_{l,\iota}(\Pi)$ preserves an orthogonal pairing up to multiplier μ.

In this paper we will prove this conjecture in the case n is odd:

Proposition 1. *Suppose that F^+ is a totally real field, that n is an odd positive integer and that Π a regular, algebraic, essentially self-dual, cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_{F^+})$. Suppose also that $r_{l,\iota}(\Pi)$ is irreducible. If*

$$c \in \mathrm{Gal}(\overline{F}^+/F^+)$$

is a complex conjugation (for some embedding $\overline{F}^+ \hookrightarrow \mathbb{C}$) then

$$|\operatorname{tr} r_{l,\iota}(\Pi)(c)| \leq 1.$$

We believe that essentially the same method works if n is even and Π is discrete series at a finite place, though we haven't taken the trouble to write the argument down in this case. (One would work with the construction of $r_{l,\iota}(\Pi)$ given in [Harris and Taylor 2001] rather than that given in [Shin 2011].) However we do not see how to treat the general case when n is even. When $r_{l,\iota}(\Pi)$ is reducible one can calculate the trace of $r(c)$ for some representation of r of $\mathrm{Gal}(\overline{F}^+/F^+)$ with the same restriction to $\mathrm{Gal}(\overline{F}^+/F)$, but this does not seem to be very helpful.

The construction of $r_{l,\iota}(\Pi)$ is via piecing together twists of representations of $\mathrm{Gal}(\overline{F}^+/F)$ which arise in the cohomology of unitary group Shimura varieties, as F runs over certain imaginary CM fields. For none of these twisted restrictions does complex conjugation make sense. For an infinite place of F one can assign a natural sign to the representations of $\mathrm{Gal}(\overline{F}^+/F)$ that arise in the cohomology of these Shimura varieties, because they are essentially conjugate self-dual. (See [Clozel et al. 2008] or [Bellaïche and Chenevier 2011].) As Calegari has stressed this sign is not related to the image of complex conjugation in our representation of $\mathrm{Gal}(\overline{F}^+/F^+)$. This latter image only makes sense for the Galois representations coming from certain automorphic forms on the unitary groups, namely those that arise from an automorphic form on $\mathrm{GL}_n(\mathbb{A}_{F^+})$ by some functoriality.

In the case that n is odd the unitary groups employed by Shin [2011] have rank n and we are able to use the moduli theoretic interpretation of its Shimura variety to write descent data to the maximal totally real subfield of F. This descent data does not commute with the action of the finite adelic points of the unitary group. However in the special case of an automorphic representation π which arises by functoriality from an automorphic form on GL_n over a totally real field we are able to show that, up to twist, this descent data preserves the π^∞ isotypical component of the cohomology, and hence gives a geometric realization of $r_{l,\iota}(\Pi)(c_v)$. Because of its geometric construction, $r_{l,\iota}(\Pi)(c_v)$ also makes sense in the world of variations of Hodge structures. Finally we can appeal to the fact that the Hodge structure corresponding to $r_{l,\iota}(\Pi)$ is regular (i.e., each $h^{p,q} \leq 1$) to show that $|\operatorname{tr} r_{l,\iota}(\Pi)(c_v)| \leq 1$.

In the case that n is even and Π is not discrete series at any finite place, [Shin 2011] realizes twists of $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\bar{F}^+/F)}$ in the cohomology of the Shimura varieties for unitary groups of rank $n+1$. One takes the π^∞ isotypic component of the cohomology for an unstable automorphic representation π of the unitary group, which one constructs from Π using the theory of endoscopy. In this case our descent data relates the π^∞ isotypic component of the cohomology, not to itself, but to a twist of the $(\pi')^\infty$ isotypic component for a second unstable automorphic representation π' of the unitary group also arising from Π. (This π' is not even nearly equivalent to a twist of π.) This does not seem to be helpful.

Notation. Let us establish some notation that we will use throughout the paper.

If ρ is a representation κ_ρ will denote its central character.

If F is a p-adic field with valuation v then F^{nr} will denote its maximal unramified extension and $\mathrm{Frob}_v \in \mathrm{Gal}(F^{\mathrm{nr}}/F)$ will denote geometric Frobenius. Moreover $\mathrm{Art}_F : F^\times \to \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$ will denote the Artin map (normalized to take uniformizers to geometric Frobenius elements). Suppose that $V/\overline{\mathbb{Q}}_l$ is a finite-dimensional vector space and that

$$r : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}(V)$$

is a continuous homomorphism. If either $l \neq p$ or $l = p$ and V is de Rham (i.e., $\dim_{\overline{\mathbb{Q}}_l} (V \otimes_{\tau,F} B_{\mathrm{DR}})^{\mathrm{Gal}(\bar{F}/F)} = \dim_{\overline{\mathbb{Q}}_l} V$ for all continuous embeddings $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$) then we may associate to r a Weil–Deligne representation $\mathrm{WD}(r)$ of the Weil group W_K of K over $\overline{\mathbb{Q}}_l$. In the case $l \neq p$ the Weil–Deligne representation $\mathrm{WD}(r)$ determines r up to equivalence. (See for instance [Taylor and Yoshida 2007, Section 1] for details.) If (r, N) is a Weil–Deligne representation of W_K then we will let $(r, N)^{\mathrm{F\text{-}ss}} = (r^{\mathrm{ss}}, N)$ denote the Frobenius semisimplification of (r, N). We will write rec_F for the local Langlands correspondence — a bijection from irreducible smooth representations of $\mathrm{GL}_n(F)$ over $\mathbb{C}$ to n-dimensional Frobenius semisimple Weil–Deligne representations of the Weil group W_F of F. (See the Introduction or

Section VII.2 of [Harris and Taylor 2001].) Recall that if χ is a character of $F^\times$ then $\mathrm{rec}(\chi)=\chi\circ\mathrm{Art}_F^{-1}$.)

If $F=\mathbb{R}$ or $\mathbb{C}$ we will write $\mathrm{Art}_F : F^\times \twoheadrightarrow \mathrm{Gal}(\bar{F}/F)$. If $F=\mathbb{R}$ then we will denote by c the nontrivial element of $\mathrm{Gal}(\bar{F}/F)$ and denote by sgn the unique surjection $F^\times \twoheadrightarrow \{\pm 1\}$.

If F is a number field then

$$\mathrm{Art}_F=\prod_v \mathrm{Art}_{F_v} : \mathbb{A}_F^\times/\overline{F^\times(F_\infty^\times)^0} \xrightarrow{\sim} \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$$

will denote the Artin map. If v is a real place of F then we will let c_v denote the image of $c\in\mathrm{Gal}(\bar{F}_v/F_v)$ in $\mathrm{Gal}(\bar{F}/F)$. Thus c_v is well defined up to conjugacy. Suppose that

$$\chi : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$$

is a continuous character for which there exists $a\in\mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})}$ such that

$$\chi|_{(F_\infty^\times)^0} : x \;\mapsto \prod_{\tau\in\mathrm{Hom}(F,\mathbb{C})} (\tau x)^{a_\tau}$$

(i.e., an algebraic grossencharacter). Suppose also that $\iota:\overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Then we define

$$r_{l,\iota}(\chi) : \mathrm{Gal}(\bar{F}/F) \to \overline{\mathbb{Q}}_l^\times$$

to be the continuous character such that

$$\iota\Biggl((r_{l,\iota}(\chi)\circ\mathrm{Art}_F)(x)\prod_{\tau\in\mathrm{Hom}(F,\mathbb{C})}(\iota^{-1}\tau)(x_l)^{-a_\tau}\Biggr)=\chi(x)\prod_{\tau\in\mathrm{Hom}(F,\mathbb{C})}(\tau x)^{-a_\tau}.$$

1. Statement of the main result

Now let F^+ be a totally real field. By a *RAESDC* (regular, algebraic, essentially self dual, cuspidal) automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_{F^+})$ we mean a cuspidal automorphic representation such that

- $\pi^\vee\cong\pi\otimes(\chi\circ\det)$ for some continuous character $\chi:\mathbb{A}_{F^+}^\times/(F^+)^\times\to\mathbb{C}^\times$ with $\chi_v(-1)$ independent of $v|\infty$, and
- π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F^+ to $\mathbb{Q}$ of GL_n.

Note that χ is necessarily algebraic. Also, if n is odd and $\pi^\vee\cong\pi\otimes(\chi\circ\det)$, then $\chi_v(-1)$ is necessarily independent of $v|\infty$, in fact it is necessarily 1 for all such v.

If F^+ is totally real we will write $(\mathbb{Z}^n)^{\mathrm{Hom}(F^+,\mathbb{C}),+}$ for the set of $a=(a_{\tau,i})\in(\mathbb{Z}^n)^{\mathrm{Hom}(F^+,\mathbb{C})}$ satisfying

$$a_{\tau,1}\geq\cdots\geq a_{\tau,n}.$$

If $F^{+\prime}/F^+$ is a finite totally real extension we define $a_{F^{+\prime}} \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^{+\prime},\mathbb{C}),+}$ by

$$(a_{F^{+\prime}})_{\tau,i} = a_{\tau|_{F^+},i}.$$

If $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^+,\mathbb{C}),+}$, let Ξ_a denote the irreducible algebraic representation of $\mathrm{GL}_n^{\mathrm{Hom}(F^+,\mathbb{C})}$ which is the tensor product over τ of the irreducible representations of GL_n with highest weights a_τ. We will say that a RAESDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_{F^+})$ has *weight a* if π_∞ has the same infinitesimal character as $\Xi_a^\vee$.

Fix once and for all an isomorphism $\iota : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. The following theorem is proved in [Shin 2011] (see also [Clozel et al. 2011]). (This is not explicitly stated in [Shin 2011], but see Remark 7.6 of that reference. For the last sentence see [Taylor and Yoshida 2007].)

Theorem 1.1. *Let F_0^+ be a totally real field and let n be an odd positive integer. Let $a \in (\mathbb{Z}^n)^{\mathrm{Hom}(F_0^+,\mathbb{C}),+}$. Suppose further that Π is a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F_0^+})$ of weight a. Specifically suppose that $\Pi^\vee \cong \Pi\chi$ where $\chi : \mathbb{A}_{F_0^+}^\times/(F_0^+)^\times \to \mathbb{C}^\times$ and $\chi_v(-1)$ is independent of $v|\infty$. Then there is a continuous semisimple representation*

$$r_{l,\iota}(\Pi) : \mathrm{Gal}(\bar{F}_0^+/F_0^+) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

with the following properties.

(1) *For every prime $v \nmid l$ of F_0^+ we have*

$$\mathrm{WD}(r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\bar{F}_{0,v}^+/F_{0,v}^+)})^{\text{F-ss}} = r_l(\iota^{-1}\,\mathrm{rec}(\Pi_v \otimes |\det|_v^{(1-n)/2}).$$

(2) $r_{l,\iota}(\Pi)^\vee = r_{l,\iota}(\Pi)\epsilon^{n-1} r_{l,\iota}(\chi)$.

(3) $\det r_{l,\iota}(\Pi) = r_{l,\iota}(\kappa_\Pi)\epsilon_l^{n(1-n)/2}$.

(4) *If $v|l$ is a prime of F_0^+ then the restriction $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\bar{F}_{0,v}^+/F_{0,v}^+)}$ is de Rham. Moreover, if Π_v is unramified, if $(F_{0,v}^+)^0$ denotes the maximal unramified subextension of $F_{0,v}^+/\mathbb{Q}_l$ and if $\tau : (F_{0,v}^+)^0 \hookrightarrow \overline{\mathbb{Q}}_l$ then $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\bar{F}_{0,v}^+/F_{0,v}^+)}$ is crystalline and the characteristic polynomial of $\phi^{[(F_{0,v}^+)^0:\mathbb{Q}_l]}$ on*

$$(r_{l,\iota}(\Pi) \otimes_{\tau,(F_{0,v}^+)^0} B_{\mathrm{cris}})^{\mathrm{Gal}(\bar{F}_{0,v}^+/F_{0,v}^+)}$$

equals the characteristic polynomial of

$$\iota^{-1}\,\mathrm{rec}_{F_{0,v}^+}(\Pi_v \otimes |\det|_v^{(1-n)/2})(\mathrm{Frob}_v).$$

(5) *If $v|l$ is a prime of F_0^+ and if $\tau : F_0^+ \hookrightarrow \overline{\mathbb{Q}}_l$ lies above v then*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i (r_{l,\iota}(\Pi) \otimes_{\tau,F_{0,v}} B_{\mathrm{DR}})^{\mathrm{Gal}(\bar{F}_{0,v}^+/F_{0,v}^+)} = 0$$

unless $i = a_{\iota\tau,j} + n - j$ *for some* $j = 1, \ldots, n$ *in which case*

$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i (r_{l,\iota}(\Pi) \otimes_{\tau, F^+_{0,v}} B_{\mathrm{DR}})^{\mathrm{Gal}(\bar{F}^+_{0,v}/F^+_{0,v})} = 1.$$

(6) *If* Π *is discrete series at some finite place then* $r_{l,\iota}(\Pi)$ *is irreducible.*

The purpose of this paper is to calculate $r_{l,\iota}(\Pi)(c_v)$ for any infinite place v of F_0^+.

Proposition 1.2. *Keep the notation and assumptions of the above theorem and suppose that* $r_{l,\iota}(\Pi)$ *is irreducible.* (*In particular we are assuming that n is odd.*) *Let* v *denote an infinite place of* F_0^+. *Then*

$$r_{l,\iota}(\Pi)(c_v)$$

is semisimple with eigenvalues 1 *of multiplicity* $(n + \kappa_{\Pi,v}(-1))/2$ *and* -1 *with multiplicity* $(n - \kappa_{\Pi,v}(-1))/2$.

2. A geometric realization of complex conjugation

We must recall some of the construction of $r_{l,\iota}(\Pi)$ and explain how the action of complex conjugation can be constructed geometrically.

The basic set-up. There is a constant $\alpha \in \mathbb{Z}$ such that $a_{\tau,j} + a_{\tau,n+1-j} = \alpha$ for all $j = 1, \ldots, n$ and all $\tau : F_0^+ \hookrightarrow \mathbb{C}$. Thus

$$\chi|_{((F^+_{0,\infty})^\times)^0} = \mathbf{N}^\alpha_{F_0^+/\mathbb{Q}}.$$

Shin shows that one can choose

- a soluble Galois totally real extension F^+/F_0^+,
- an imaginary quadratic field E in which l splits,
- an embedding $\tau_0 : F = F^+E \hookrightarrow \mathbb{C}$,
- a continuous character

$$\phi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times,$$

- a continuous character

$$\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times,$$

with the following properties.

- $[F^+ : \mathbb{Q}]$ is even and > 2.
- If Ram denotes the set of (finite) rational primes above which any of F, Π, ϕ, or ψ ramifies, then every prime of F^+ above a prime of Ram splits in F.
- $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\bar{F}/F)}$ remains irreducible.

- $\phi\phi^c = \chi_F$ and $\phi|_{F_\infty^\times} = \prod_\tau \tau^{-\beta_\tau}$ where $\beta_\tau + \beta_{\tau c} = -\alpha$.
- $\psi^c/\psi = (\kappa_\Pi|_{\mathbb{A}^\times}^{[F^+:F_0^+]} \circ \mathbf{N}_{E/\mathbb{Q}})\phi|_{\mathbb{A}_E^\times}^n$.
- $\psi_\infty = \tau_0^{-\epsilon}(\tau_0 \circ c)^{-\epsilon'}$ with $\epsilon, \epsilon' \in \mathbb{Z}$.
- ψ is unramified at the prime of E above l corresponding to $\iota^{-1} \circ \tau_0$.

Let $V = F^n$ and let

$$\langle\ ,\ \rangle : V \times V \to \mathbb{Q}$$

be a nondegenerate alternating bilinear form such that

$$\langle xv, w\rangle = \langle v, {}^c x w\rangle$$

for all $x \in F$ and $v, w \in V$. Let G be the reductive subgroup of $\mathrm{GL}(V/F)$ consisting of elements which preserve $\langle\ ,\ \rangle$ up to a $\mathbb{G}_m$-multiple and let $\nu : G \to \mathbb{G}_m$ denote the multiplier character. We may, and do, suppose that V is chosen so that

- G is quasisplit at all finite places;
- if $\tau : F \hookrightarrow \mathbb{C}$ satisfies $\tau|_E = \tau_0|_E$ then the Hermitian form on $V \otimes_{F,\tau} \mathbb{C}$ defined by

  $$(v, w) \mapsto \langle v, iw\rangle$$

  has a maximal positive definite subspace of dimension 0 if $\tau \neq \tau_0$ and 1 if $\tau = \tau_0$.

(See [Shin 2011, Lemma 5.1].) There is an identification of $G \times_{\mathbb{Q}} E$ with the product of GL_1 and the restriction of scalars from F to E of GL_n. The map sends g to the product of its multiplier and its action on the direct summand $V \otimes_{E,1} E$ of $V \otimes_{\mathbb{Q}} E = V \otimes_{E,1} E \oplus V \otimes_{E,c} E$.

The group G. Letting $\ker^1(\mathbb{Q}, G)$ denote the kernel of

$$H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G),$$

using the fact that n is odd, we see from [Kottwitz 1992, Section 8] that there is an identification

$$\ker^1(\mathbb{Q}, G) \cong ((F^+)^\times \cap (\mathbb{A}^\times \mathbf{N}_{F/F^+}\mathbb{A}_F^\times))/\mathbb{Q}^\times(\mathbf{N}_{F/F^+}F^\times).$$

As F/F^+ is unramified at all finite primes we see that $\mathbf{N}_{F/F^+}\mathbb{A}_F^\times \supset \widehat{\mathbb{Z}}^\times \mathbb{R}_{>0}^\times$ so that $\mathbb{A}^\times \mathbf{N}_{F/F^+}\mathbb{A}_F^\times = \mathbb{Q}^\times \mathbf{N}_{F/F^+}\mathbb{A}_F^\times$. Because $(F^+)^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times = \mathbf{N}_{F/F^+}F^\times$ we conclude that

$$\ker^1(\mathbb{Q}, G) \cong \mathbb{Q}^\times((F^+)^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times)/\mathbb{Q}^\times(\mathbf{N}_{F/F^+}F^\times) = \{1\}.$$

It follows from the proof of Lemma 3.1 of [Shin 2011] that the Tamagawa number $\tau(G) = 2$.

Let T denote the quotient of G by its derived subgroup. Then we may identify T by

$$T(R) = \{(x, y) \in R^\times \times (R \otimes_{\mathbb{Q}} F)^\times : \; x^n = y^c y\}$$

for any $\mathbb{Q}$-algebra R. The quotient map $d : G \to T$ sends g to $(\nu(g), \det g)$. Also let Z denote the centre of G so that

$$Z(R) = \{(x, y) \in R^\times \times (R \otimes_{\mathbb{Q}} F)^\times : \; x = y^c y\}$$

for any $\mathbb{Q}$-algebra R. The map $d|_Z$ sends (x, y) to (x, y^n) and the map $\nu|_Z$ sends (x, y) to x. Note that $Z \times E$ can be identified with the product of $\mathbb{G}_m$ with the restriction of scalars from F to E of $\mathbb{G}_m$ and the norm map sends (a, b) to $(a^c a, {}^c ab/{}^c b)$. Then

$$\nu : Z(\mathbb{A})/Z(\mathbb{Q})(\mathbf{N}_{E/\mathbb{Q}} Z(\mathbb{A}_E)) \xrightarrow{\sim} \mathbb{A}^\times/\mathbb{Q}^\times(\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times) \cong \mathrm{Gal}(E/\mathbb{Q}).$$

[To see this note that the left hand side is

$$\{y \in \mathbb{A}_F^\times : \; y^c y \in \mathbb{A}^\times\}/\mathbb{A}_E^\times\{y \in F^\times : \; y^c y \in \mathbb{Q}^\times\}\{y/{}^c y : \; y \in \mathbb{A}_F^\times\}.$$

As $\{y/{}^c y : \; y \in \mathbb{A}_F^\times\} = \mathbb{A}_F^{\mathbf{N}_{F/F^+}=1}$ we see that the group in the previous displayed equations maps isomorphically under $\nu = \mathbf{N}_{F/F^+}$ to

$$\begin{aligned} &(\mathbb{A}^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times)/(\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)(\mathbb{Q}^\times \cap \mathbf{N}_{F/F^+}F^\times) \\ \cong \; &(\mathbb{A}^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times)/((\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)\mathbb{Q}^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times). \end{aligned}$$

There is a natural injection from here to $\mathbb{A}^\times/(\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)\mathbb{Q}^\times$. It only remains to see that this map is surjective, i.e., that

$$\mathbb{A}^\times/\mathbb{Q}^\times(\mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times)(\mathbb{A}^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times) = \{1\}.$$

However as F/F^+ is everywhere unramified we have that

$$(\mathbb{A}^\times \cap \mathbf{N}_{F/F^+}\mathbb{A}_F^\times) \supset \widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}^\times,$$

while $\mathbb{A}^\times = \mathbb{Q}^\times\widehat{\mathbb{Z}}^\times\mathbb{R}_{>0}^\times$.]

The involution I. We can choose a $\mathbb{Q}$-linear map $I : V \to V$ such that

- $I(xv) = {}^c x I(v)$ for all $x \in F$ and $v \in V$;
- $\langle Iv, Iw\rangle = -\langle v, w\rangle$ for all $v, w \in V$;
- $I^2 = 1$.

[To see this note that with respect to a suitable basis we have

$$\langle v, w\rangle = \mathrm{tr}_{F/\mathbb{Q}}({}^t v D^c w)$$

for some diagonal matrix D with ${}^c D = -D$. With respect to such a basis we can take I to simply be complex conjugation on coordinates.] The choice of I gives rise to an automorphism # of G of order two:

$$g^\# = IgI.$$

Note that

$$\nu \circ \# = \nu$$

and that

$$\det g^\# = {}^c \det g.$$

If we identify $G \times E$ with the product of $\mathbb{G}_m$ and the restriction of scalars from F to E of GL_n then # differs by composition with an inner automorphism from the automorphism:

$$(x, g) \mapsto (x, x\,{}^t g^{-1}).$$

Base change from $G(\mathbb{A}^\infty)$ to $(\mathbb{A}_E^\infty)^\times \times \mathrm{GL}_n(\mathbb{A}_F^\infty)$. As in [Harris and Taylor 2001, Section VI.2] we can define the base change $\mathrm{BC}(\tilde{\pi})$ of an irreducible admissible representation $\tilde{\pi}$ of $G(\mathbb{A}^\infty)$ which is unramified at a place v of $\mathbb{Q}$, unless all primes of F^+ above v split in F. The base change lift, $\mathrm{BC}(\tilde{\pi})$, is an irreducible admissible representation of $(\mathbb{A}_E^\infty)^\times \times \mathrm{GL}_n(\mathbb{A}_F^\infty)$. Note that if $\delta_{E/\mathbb{Q}}$ denotes the nontrivial character of $\mathbb{A}^\times/\mathbb{Q}^\times \mathbf{N}_{E/\mathbb{Q}}\mathbb{A}_E^\times$ then

$$\mathrm{BC}(\tilde{\pi}) = \mathrm{BC}(\tilde{\pi} \otimes (\delta_{E/\mathbb{Q}} \circ \nu)).$$

Also note that $\tilde{\pi}$ and $\tilde{\pi} \otimes (\delta_{E/\mathbb{Q}} \circ \nu)$ have different central characters and so can not be isomorphic. (Recall that

$$\nu : Z(\mathbb{A}^\infty) \twoheadrightarrow (\mathbb{A}^\infty)^\times \cap \mathbf{N}_{F/F^+}(\mathbb{A}_F^\infty)^\times \supset \widehat{\mathbb{Z}}^\times,$$

and that $\delta_{E/\mathbb{Q}}$ is ramified at some finite prime.) We have that

$$\kappa_{\mathrm{BC}(\tilde{\pi})} = \kappa_{\tilde{\pi}} \circ \mathbf{N},$$

where $\mathbf{N}$ denotes the norm map $Z(\mathbb{A}_E^\infty) \to Z(\mathbb{A}^\infty)$. If

$$\mathrm{BC}(\tilde{\pi}) = (\tilde{\phi}, \widetilde{\Pi})$$

then

$$\mathrm{BC}(\tilde{\pi}^\#) = (\tilde{\phi}\kappa_{\widetilde{\Pi}}|_{(\mathbb{A}_E^\infty)^\times}, \widetilde{\Pi}^\vee)$$

and

$$\kappa_{\tilde{\pi}^\#} = \kappa_{\tilde{\pi}}\kappa_{\widetilde{\Pi}}^c|_{Z(\mathbb{A}^\infty)},$$

where we think of $Z(\mathbb{A}^\infty) \subset (\mathbb{A}_F^\infty)^\times$.

Define

$$\begin{array}{rcl}\omega : T(\mathbb{A})/T(\mathbb{Q}) & \to & \mathbb{C}^\times \\ (x, y) & \mapsto & \phi^c(y)^{-1}\kappa_{\Pi, F^+}(x)^{-1}.\end{array}$$

Note that

$$\omega^\# \omega = 1.$$

With the functorialities of the previous paragraph the next lemma is easy to verify.

Lemma 2.1. *Suppose that $\tilde{\pi}$ is as in the previous paragraph and that*

$$\mathrm{BC}(\tilde{\pi}) = (\psi^\infty, \Pi_F \phi).$$

Then

(1) $\kappa_{\tilde{\pi}^\# \otimes (\omega^\infty \circ d)} = \kappa_{\tilde{\pi}}$;

(2) $\mathrm{BC}(\tilde{\pi}^\# \otimes (\omega^\infty \circ d)) = \mathrm{BC}(\tilde{\pi})$;

(3) *and there exists an automorphism $A_{\tilde{\pi}}$ of the underlying space of $\tilde{\pi}$ such that*

$$A_{\tilde{\pi}} \tilde{\pi}(g) = \tilde{\pi}(g^\#) \omega(d(g)) A_{\tilde{\pi}}$$

for all $g \in G(\mathbb{A}^\infty)$ and $A_{\tilde{\pi}}^2 = 1$. Moreover $A_{\tilde{\pi}}$ is unique up to sign.

Weights. We identify $G \times_{\mathbb{Q}} \mathbb{C}$ with

$$\mathbb{G}_m \times \prod_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})} \mathrm{GL}(V \otimes_{F,\tau} \mathbb{C}),$$

where $\mathrm{Hom}_{E,\tau_0}(F, \mathbb{C})$ denotes the set of embeddings $\tau : F \hookrightarrow \mathbb{C}$ with $\tau|_E = \tau_0|_E$. The identification sends g to its multiplier and its push forward to each $\mathrm{GL}(V \otimes_{F,\tau} \mathbb{C})$. Let ξ denote the irreducible representations of $G \times_{\mathbb{Q}} \mathbb{C}$ with highest weights $(b_0; b_{\tau,i})_{\tau|_E = \tau_0|_E}$, where

- $b_0 = \epsilon$;
- $b_{\tau,i} = a_{\tau|_{F_0^+},i} + \beta_\tau$.

Then $\xi^\#$ has highest weights

$$\Big(b_0 + \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C}), i} b_{\tau,i}; -b_{\tau,n+1-i}\Big)_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C});\ i=1,\dots,n}.$$

Also let ζ be the irreducible representation with highest weights

$$\Big(-n\Big([F^+ : \mathbb{Q}]\alpha/2 + \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})} \beta_\tau\Big); \alpha + 2\beta_\tau\Big)_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C});\ i=1,\dots,n}.$$

Then

- ζ is one-dimensional;
- $\xi^\# \otimes \zeta \cong \xi$;
- $\zeta^\# \cong \zeta^\vee$;
- and $\omega|_{T(\mathbb{R})} = \zeta^{-1}$.

Shimura varieties. Let U denote an open compact subgroup of $G(\mathbb{A}^\infty)$. Consider the functor $\mathfrak{X}_U$ from connected, locally noetherian F-schemes with a specified geometric point to sets, which sends a pair $(S, \bar{s})$ to the set of equivalence classes of 4-tuples

$$(A, i, \lambda, \bar{\eta})$$

where

(1) A/S is an abelian scheme of relative dimension n;

(2) $i : F \hookrightarrow \mathrm{End}^0(A/S)$ is such that for all $x \in F$ we have
$$\mathrm{tr}(x|_{\mathrm{Lie}\, A}) = x - {}^c x + n\, \mathrm{tr}_{F/E}\, {}^c x;$$

(3) $\lambda : A \to A^\vee$ is a polarization such that $i(x)^\vee \circ \lambda = \lambda \circ i({}^c x)$ for all $x \in F$;

(4) $\bar{\eta}$ is a $\pi_1(S, \bar{s})$-invariant U-orbit of $\mathbb{A}_F^\infty$-isomorphisms $\eta : V \otimes \mathbb{A}^\infty \xrightarrow{\sim} VA_{\bar{s}}$ such that for some isomorphism $\eta_0 : \mathbb{A}^\infty \xrightarrow{\sim} \mathbb{A}^\infty(1)$ and for all $v, w \in V \otimes \mathbb{A}^\infty$ we have
$$\langle \eta v, \eta w \rangle_\lambda = \eta_0 \langle v, w \rangle,$$
where $\langle\ ,\ \rangle_\lambda$ denotes the λ-Weil pairing.

Two 4-tuples $(A, i, \lambda, \bar{\eta})$ and $(A', i', \lambda', \bar{\eta}')$ are considered equivalent if there is an isogeny

$$\gamma : A \to A'$$

such that

(1) $\gamma i(x) = i'(x)\gamma$ for all $x \in F$,

(2) $\gamma^\vee \lambda' \gamma \in \mathbb{Q}^\times \lambda$,

(3) and $(V\gamma_{\bar{s}}) \circ \bar{\eta} = \bar{\eta}'$.

This functor is canonically independent of the choice of base point $\bar{s}$ and so can be considered as a functor from connected, locally noetherian F-schemes to sets. It can be extended to all locally noetherian F-schemes by setting

$$\mathfrak{X}_U(S_1 \amalg S_2) = \mathfrak{X}_U(S_1) \times \mathfrak{X}_U(S_2).$$

(See for instance [Harris and Taylor 2001, Section III.1] for more details. We are using $\mathrm{End}^0(A/S)$ to denote $\mathrm{End}(A/S) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $VA_{\bar{s}}$ for $(\lim_{\leftarrow N} A[N](k(\bar{s}))) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $k(\bar{s})$ denotes the residue field of $\bar{s}$.)

If U is sufficiently small then $\mathfrak{X}_U$ is represented by an abelian scheme

$$\mathscr{A}_U / X_U / \operatorname{Spec} F.$$

If $V \subset U$ is an open subgroup there is a natural map $X_V \to X_U$ such that $\mathscr{A}_U$ pulls back to $\mathscr{A}_V$. The inverse system of the X_U's carries a natural action of $G(\mathbb{A}^\infty)$, as does the inverse system of the $\mathscr{A}_U$'s. If V is a normal open subgroup of U

then U acts on X_V and induces an isomorphism between U/V and $\mathrm{Gal}(X_V/X_U)$. Thus $\iota^{-1}\xi$ gives a representation of U and hence a lisse $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{L}_\xi$ on X_U. The $\overline{\mathbb{Q}}_l$-vector space

$$H^i(X, \mathcal{L}_\xi) = \lim_{\to U} H^i(X_U \times \bar{F}, \mathcal{L}_\xi)$$

has an action of $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\bar{F}/F)$. It is admissible and semisimple as a $G(\mathbb{A}^\infty)$-module. If U is an open, compact subgroup of $G(\mathbb{A}^\infty)$ then

$$H^i(X, \mathcal{L}_\xi)^U = H^i(X_U \times \bar{F}, \mathcal{L}_\xi)$$

is a continuous representation of $\mathrm{Gal}(\bar{F}/F)$ on a finite-dimensional $\overline{\mathbb{Q}}_l$-vector space.

The pull back $X_U \times_{F,c} F$ represents the functor $\mathfrak{X}'_U$ defined exactly as $\mathfrak{X}_U$ except that the condition

$$\mathrm{tr}(x|_{\mathrm{Lie}\,A}) = x - {}^c x + n\,\mathrm{tr}_{F/E}\,{}^c x$$

is replaced by the condition

$$\mathrm{tr}(x|_{\mathrm{Lie}\,A}) = {}^c x - x + n\,\mathrm{tr}_{F/E}\,x.$$

There is a map of functors $\mathfrak{X}_U \to \mathfrak{X}'_U$ which sends $(A, i, \lambda, \bar{\eta})$ to $(A, i \circ c, \lambda, \overline{\eta \circ I})$. This induces an F-linear map $X_U \to X_U \times_{F,c} F$ and hence a c-linear map, which we will also denote I,

$$\begin{array}{ccc} X_U & \xrightarrow{I} & X_U \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,F & \xrightarrow{c} & \mathrm{Spec}\,F. \end{array}$$

We have

- $I^2 = 1$;
- $IgI = g^\#$ for $g \in G(\mathbb{A}^\infty)$;
- and a natural isomorphism $I^*\mathcal{L}_\xi \otimes \mathcal{L}_\zeta \cong \mathcal{L}_\xi$, i.e.,

$$I^*\mathcal{L}_\xi \cong \mathcal{L}_{\xi^\#}. \tag{2-1}$$

Thus I provides a way to descend the system of the X_U to F^+; however this descended system of varieties no longer has an action of $G(\mathbb{A}^\infty)$ defined over F^+.

Complex points and connected components. We will need to consider the complex uniformization of $X_U \times_{F,\tau} \mathbb{C}$ for every homomorphism $\tau : F \hookrightarrow \mathbb{C}$. So suppose $\tau : F \hookrightarrow \mathbb{C}$. There is a nondegenerate alternating form

$$\langle\ ,\ \rangle_\tau : V \times V \to \mathbb{Q}$$

such that

$$\langle xv, w\rangle_\tau = \langle v, {}^c x w\rangle_\tau$$

for all $x \in F$ and $v, w \in V$ and such that

- there is an isomorphism $j_\tau : (V \otimes_{\mathbb{Q}} \mathbb{A}^\infty, \langle\ ,\ \rangle) \xrightarrow{\sim} (V \otimes_{\mathbb{Q}} \mathbb{A}^\infty, \langle\ ,\ \rangle_\tau)$ as $\mathbb{A}_F^\infty$-modules with alternating $\mathbb{A}^\infty$-bilinear pairing;
- if $\tau' : F \hookrightarrow \mathbb{C}$ satisfies $\tau'|_E = \tau|_E$ then the Hermitian form on $V \otimes_{F,\tau'} \mathbb{C}$ defined by
$$(v, w) \mapsto \langle v, iw\rangle_\tau$$
has a maximal positive definite subspace of dimension 0 if $\tau' \neq \tau$ and 1 if $\tau' = \tau$.

Let G_τ denote the group of symplectic F-linear similitudes for $(V, \langle\ ,\ \rangle_\tau)$ and $G_{\tau,1}$ the kernel of the multiplier character $G_\tau \to \mathbb{G}_m$. Note that $G_\tau \times_{\mathbb{Q}} \mathbb{A}^\infty \cong G \times_{\mathbb{Q}} \mathbb{A}^\infty$ and that $G_\tau / G_{\tau,1} \xrightarrow{\sim} T$. Choose a $\mathbb{Q}$-linear map $I_\tau : V \to V$ such that

- $I_\tau(xv) = {}^c x I_\tau(v)$ for all $x \in F$ and $v \in V$;
- $\langle I_\tau v, I_\tau w\rangle = -\langle v, w\rangle$ for all $v, w \in V$;
- $I_\tau^2 = 1$.

We may, and shall, take $\langle\ ,\ \rangle_{\tau_0} = \langle\ ,\ \rangle$ and $I_{\tau_0} = I$.

Let Ω_τ denote the set of homomorphisms
$$h : \mathbb{C} \to \mathrm{End}_{F \otimes_{\mathbb{Q}} \mathbb{R}}(V \otimes_{\mathbb{Q}} \mathbb{R})$$
such that

- $\langle h(z)v, w\rangle_\tau = \langle v, h({}^c z)w\rangle_\tau$ for all $z \in \mathbb{C}$ and $v, w \in V \otimes \mathbb{R}$,
- $\langle v, h(i)v\rangle_\tau \geq 0$ for all $v \in V$.

Then Ω_τ forms a single conjugacy class for $G_{\tau,1}(\mathbb{R})$ [Kottwitz 1992, Lemma 4.3]. This gives Ω_τ a topology (the quotient topology) and, as the group $G_{\tau,1}(\mathbb{R})$ is connected, we see that Ω_τ is connected. There are $G(\mathbb{A}^\infty)$-equivariant homeomorphisms (see [Kottwitz 1992, Section 8], for example)
$$G_\tau(\mathbb{Q}) \backslash (G(\mathbb{A}^\infty)/U \times \Omega_\tau) \xrightarrow{\sim} (X_U \times_{F,\tau} \mathbb{C})(\mathbb{C}).$$

Let Λ be a $\mathbb{Z}$-lattice in V. The map sends (g, h) to a the equivalence class of a four-tuple $(A, i, \lambda, \bar{\eta})$, which is determined as follows. The abelian variety A is characterized by the complex uniformization $A(\mathbb{C}) = (V \otimes_{\mathbb{Q}} \mathbb{R})/\Lambda$ with the complex structure coming from h. The map i arises from the natural action of F on $V \otimes_{\mathbb{Q}} \mathbb{R}$ and the (quasi)polarization λ corresponds to the Riemann form $\langle\ ,\ \rangle_\tau$. Note that VA is naturally identified with $V \otimes_{\mathbb{Q}} \mathbb{A}^\infty$. The level structure $\bar{\eta}$ is the class of $j_\tau \circ g$. Under $I \times c_\tau$ this is taken to $({}^c A, i \circ c, \lambda, \bar{\eta} \circ I)$, which has analytic uniformization as $(V \otimes_{\mathbb{Q}} \mathbb{R})/\Lambda$ but with the complex structure coming from $h \circ c$. The F action is the complex conjugate of the usual one. The Riemann form is sent to its negative and the level structure is $j_\tau \circ g \circ I$. The map $I \otimes 1_{\mathbb{R}}$ shows that this is isomorphic to the abelian variety with additional structure corresponding to $((j_\tau^{-1} I_\tau j_\tau I) g^\#, I_\tau h I_\tau) \in G(\mathbb{A}^\infty) \times \Omega_\tau$. Set $s_\tau = j_\tau^{-1} I_\tau j_\tau I \in G(\mathbb{Q})$ and note that $s_\tau^\# s_\tau = 1$.

We conclude that there is a bijection ς_τ :

$$\pi_0(X_U \times_F \bar{F}) \cong \pi_0(X_U \times_{F,\tau} \mathbb{C})(\mathbb{C}) \cong G_\tau(\mathbb{Q}) \backslash G_\tau(\mathbb{A}^\infty)/U \xrightarrow{\sim} T(\mathbb{Q}) \backslash T(\mathbb{A}^\infty)/d(U).$$

(For the bijectivity of the third map, which is given by d, see [Milne 2005, Theorem 5.17] and the discussion following it.) Write ς for ς_{τ_0}. The map ς_τ is $G(\mathbb{A}^\infty)$-equivariant. It is also $I \times c_\tau$ equivariant if we let $I \times c_\tau$ act on $T(\mathbb{Q}) \backslash T(\mathbb{A}^\infty)/d(U)$ via $t \mapsto d(s_\tau) t^\#$. Note that because of the $G(\mathbb{A}^\infty)$ equivariance we must have $\varsigma_\tau = u_\tau \varsigma$ for some $u_\tau \in T(\mathbb{A})$. Thus we see that

- $\varsigma(Cg) = d(g)\varsigma(C)$ for all $C \in \pi_0(X_U \times_F \bar{F})$ and all $g \in G(\mathbb{A}^\infty)$,
- and for any infinite place v of $\bar{F}$ there is an $s_v \in T(\mathbb{A})$ such that $\varsigma((I \times c_v)x) = s_v \varsigma(x)^\#$ and $s_v s_v^\# = 1$.

(If $v|_F$ arises from $\tau : F \hookrightarrow \mathbb{C}$ then $s_v = d(s_\tau) u_\tau^\# u_\tau^{-1}$.)

We wish to also know the $\mathrm{Gal}(\bar{F}/F)$-equivariance of ς. Note that the X_U are the canonical models for the Shimura varieties $\mathrm{Sh}_U(G, [h^{-1}])$. (See [Kottwitz 1992, Section 8] and note that $\ker^1(\mathbb{Q}, G) = (0)$.) Define a map

$$r : \mathbb{A}_F^\times \to T(\mathbb{A}_E) \xrightarrow{\mathbf{N}_{E/\mathbb{Q}}} T(\mathbb{A})$$

where the first map sends

$$x \mapsto (\mathbf{N}_{F/E} x, x)^{-1}.$$

Note that $r \circ \mathrm{Art}_F^{-1}$ is a well defined map

$$(r \circ \mathrm{Art}_F^{-1}) : \mathrm{Gal}(\bar{F}/F) \to T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R}).$$

Then according to [Milne 2005, Section 13] we have

$$\varsigma(\sigma x) = (r \circ \mathrm{Art}_F^{-1})(\sigma)\varsigma(x)$$

for all $x \in \pi_0(X_U \times_F \bar{F})$ and all $\sigma \in \mathrm{Gal}(\bar{F}/F)$.

H^0 of sheaves on our Shimura varieties. Let $\tilde{\xi}$ be the irreducible representation of $G \times \mathbb{C}$ which has highest weight $(\tilde{b}_0, \tilde{b}_{\tau,i})_{\tau|_E = \tau_0|_E}$. The description of the previous section allows us to calculate $H^0(X_U \times \bar{F}, \mathcal{L}_{\tilde{\xi}})$. It will be (0) unless $\tilde{b}_{\tau,i} = \tilde{b}_\tau$ is independent of i. In this case $\tilde{\xi}$ factors through a map $T \times \mathbb{C} \to \mathbb{G}_m$ which we will also denote $\tilde{\xi}$. We can then identify $H^0(X_U \times \bar{F}, \mathcal{L}_{\tilde{\xi}})$ with the space of functions

$$f : T(\mathbb{A})/T(\mathbb{R})T(\mathbb{Q}) \to \overline{\mathbb{Q}}_l$$

such that

$$f(tu) = (\iota^{-1}\tilde{\xi})(u_l)^{-1} f(t)$$

for all $t \in T(\mathbb{A})$ and all $u \in d(U)$. The action of $G(\mathbb{A}^\infty)$ is via

$$(gf)(t) = (\iota^{-1}\tilde{\xi})(g_l) f(td(g))$$

and the action of $\mathrm{Gal}(\bar{F}/F)$ is via

$$(\sigma f)(t) = f((r \circ \mathrm{Art}_F^{-1})(\sigma)t).$$

The map that sends f to $\tilde{f}$ defined by

$$\tilde{f}(t) = (\iota^{-1} \circ \tilde{\xi})(t_\infty)^{-1}(\iota^{-1}\tilde{\xi})(t_l) f(t),$$

establishes an isomorphism between $H^0(X_U \times \bar{F}, \mathcal{L}_{\tilde{\xi}})$ and the space of functions $\tilde{f} : T(\mathbb{A})/T(\mathbb{Q})d(U) \to \overline{\mathbb{Q}}_l$ such that

$$\tilde{f}(tu_\infty) = (\iota^{-1} \circ \tilde{\xi})(u_\infty)^{-1} \tilde{f}(t)$$

for all $t \in T(\mathbb{A})$ and $u_\infty \in T(\mathbb{R})$. Now the action of $G(\mathbb{A}^\infty)$ is via right translation $((g\tilde{f})(t) = \tilde{f}(td(g)))$ and the action of $\mathrm{Gal}(\bar{F}/F)$ is via

$$(\sigma \tilde{f})(t) = (\iota^{-1} \circ \tilde{\xi})(s_\infty)(\iota^{-1}\tilde{\xi})(s_l)^{-1} \tilde{f}(st)$$

where s is a lift of $(r \circ \mathrm{Art}_F^{-1})(\sigma)$ to $T(\mathbb{A})$. From this it follows that we can write

$$H^0(X, \mathcal{L}_{\tilde{\xi}}) = \bigoplus_{\widetilde{\omega}} \overline{\mathbb{Q}}_l v_{\widetilde{\omega}}$$

where $\widetilde{\omega}$ runs over continuous characters

$$T(\mathbb{A})/T(\mathbb{Q}) \to \mathbb{C}^\times$$

such that $\widetilde{\omega}|_{T(\mathbb{R})} = \tilde{\xi}^{-1}$, and where:

- the action of $G(\mathbb{A}^\infty)$ on $v_{\widetilde{\omega}}$ is via $\iota^{-1} \circ \widetilde{\omega} \circ d$;
- the action of $\mathrm{Gal}(\bar{F}/F)$ on $v_{\widetilde{\omega}}$ is via $r_{l,\iota}(\widetilde{\omega} \circ r)$;
- and, if v is an infinite place of $\bar{F}$, then $(I \times c_v)v_{\widetilde{\omega}} \in \overline{\mathbb{Q}}_l v_{\widetilde{\omega}^\#}$.

In particular cupping with $v_{\delta_{E/\mathbb{Q}} \circ \nu} \in H^0(X, \overline{\mathbb{Q}}_l)$ we see that

$$\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^i(X, \mathcal{L}_\xi)) \cong \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}(\pi \otimes (\delta_{E/\mathbb{Q}} \circ \nu)), H^i(X, \mathcal{L}_\xi)).$$

If v is a place of $\bar{F}$ above infinity then $I \times c_v$ defines a map $X_U \times_F \bar{F} \to X_U \times_F \bar{F}$, which in turn induces a map

$$H^i(X, \mathcal{L}_\xi) \to H^i(X, \mathcal{L}_{\xi^\#}).$$

Composing this with the cup product with $\omega(s_v)^{-1/2}\upsilon_\omega \in H^0(X, \mathcal{L}_\zeta)$, we get a map

$$I_v : H^i(X, \mathcal{L}_\xi) \to H^i(X, \mathcal{L}_\xi),$$

such that

- $I_v g I_v = g^\#(\iota^{-1} \circ \omega \circ d)(g)$ for $g \in G(\mathbb{A}^\infty)$;
- and $I_v \sigma I_v = (c_v \sigma c_v) r_{l,\iota}((\psi_F \phi)^c/(\psi_F \phi))(\sigma)$ for $\sigma \in \mathrm{Gal}(\bar{F}/F)$.

Galois representations. Shin shows that

- $\bigoplus_{\mathrm{BC}(\tilde{\pi})=(\psi^\infty, \Pi_F^\infty \otimes \phi^\infty)} \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\tilde{\pi}, H^i(X, \mathcal{L}_\xi)) \neq (0)$ if and only if $i = n-1$;
- $$\bigoplus_{\mathrm{BC}(\tilde{\pi})=(\psi^\infty, \Pi_F^\infty \otimes \phi^\infty)} \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\tilde{\pi}, H^{n-1}(X, \mathcal{L}_\xi))^{\mathrm{ss}} \cong r_{l,\iota}(\Pi)|^\vee_{\mathrm{Gal}(\bar{F}/F)} \otimes r_{l,\iota}((\psi_F^{-1}\phi^{-1})^2).$$

(See in particular Theorem 6.4, Corollary 6.5 and the proof of Lemma 3.1 of [Shin 2011]. The sums run over $\tilde{\pi}$ which only ramify above rational primes v, such that all places of F^+ above v split in F.) From the irreducibility of $r_{l,\iota}(\Pi)|_{\mathrm{Gal}(\bar{F}/F)}$ we see that at most two $\tilde{\pi}$'s can contribute to the latter sum. On the other hand if $\tilde{\pi}$ contributes so does $\tilde{\pi} \otimes (\delta_{E/\mathbb{Q}} \circ \nu)$, because one can cup with $\upsilon_{\delta_{E/\mathbb{Q}} \circ \nu}$. Thus exactly two $\tilde{\pi}$'s contribute. Choose one of them and from now on reserve the notation π for this one. Thus we have the following.

- Suppose that $\tilde{\pi}$ is an irreducible representation of $G(\mathbb{A}^\infty)$ and $j \in \mathbb{Z}_{\geq 0}$ such that
  - if $\tilde{\pi}$ is ramified above a rational prime v, then all places of F^+ above v split in F;
  - $\mathrm{BC}(\tilde{\pi}) = (\psi^\infty, \Pi_F^\infty \otimes \phi^\infty)$;
  - and $\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\tilde{\pi}, H^j(X, \mathcal{L}_\xi)) \neq (0)$.

  Then $j = n-1$ and $\tilde{\pi} \cong \pi$ or $\pi \otimes (\delta_{E/\mathbb{Q}} \circ \nu)$.
- $\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^{n-1}(X, \mathcal{L}_\xi)) \otimes r_{l,\iota}(\psi_F \phi) \cong r_{l,\iota}(\Pi)|^\vee_{\mathrm{Gal}(\bar{F}/F)}$.
- $\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}(\pi \otimes (\delta_{E/\mathbb{Q}} \circ \nu)), H^{n-1}(X, \mathcal{L}_\xi)) \otimes r_{l,\iota}(\psi_F \phi) \cong r_{l,\iota}(\Pi)|^\vee_{\mathrm{Gal}(\bar{F}/F)}$.

If v is an infinite place of $\bar{F}$ then the map

$$f \mapsto I_v \circ f \circ A_\pi$$

induces a map $\tilde{c}_v$ on

$$\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^{n-1}(X, \mathcal{L}_\xi)) \otimes r_{l,\iota}(\psi_F \phi)$$

such that

$$\tilde{c}_v \circ \sigma \circ \tilde{c}_v = (c_v \sigma c_v)$$

for all $\sigma \in \mathrm{Gal}(\bar{F}/F)$. Because $r_{l,\iota}(\Pi)|^{\vee}_{\mathrm{Gal}(\bar{F}/F)}$ is irreducible, we conclude that $\tilde{c}_v$ corresponds to a scalar multiple of $r_{l,\iota}(\Pi)^{\vee}(c_v)$. We can, and shall, replace $\tilde{c}_v$ by a scalar multiple so that $\tilde{c}_v^2 = 1$, so that $\tilde{c}_v = \pm r_{l,\iota}(\Pi)^{\vee}(c_v)$. We finally have our geometric realization of $r_{l,\iota}(\Pi)(c_v)$. To prove our proposition it suffices to check that the trace of $\tilde{c}_v$ on

$$\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^{n-1}(X, \mathcal{L}_\xi))$$

is ± 1. This we will do in the next section by working with the variations of Hodge structure analogue of our l-adic sheaves.

3. Calculation of the trace of $\tilde{c}_v$

We must recall an alternative construction of the sheaves $\mathcal{L}_\xi$, $\mathcal{L}_{\xi^\#}$ and $\mathcal{L}_\zeta$, which will make sense also for variations of Hodge structures. First we recall the theory of Young symmetrizers.

Young symmetrizers. Let k denote a field of characteristic 0 and let $\mathcal{C}$ denote a Tannakian category over k in the terminology of [Deligne 1990]. Suppose that $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$ satisfies $e_1 \ge e_2 \ge \cdots \ge e_n \ge 0$. Let S_e denote the symmetric group on the set $\mathcal{T}_e$ of pairs of integers (i, j) with $1 \le i \le n$ and $1 \le j \le e_i$. Let S_e^+ denote the subgroup of S_e consisting of elements σ with $\sigma(i, j) = (i, j')$ some j' and let S_e^- denote the subgroup of S_e consisting of elements σ with $\sigma(i, j) = (i', j)$ for some i'. Further we set

$$A_e^{\pm} = \sum_{\sigma \in S_e^{\pm}} (\pm)^{\sigma} \sigma \in \mathbb{Q}[S_e],$$

where $(+)^\sigma = 1$ and $(-)^\sigma$ denotes the sign of σ. Note that $(A_e^{\pm})^2 = (\# S_e^{\pm}) A_e^{\pm}$ and $(A_e^+ A_e^-)^2 = m(e)(A_e^+ A_e^-)$ and $(A_e^- A_e^+)^2 = m(e)(A_e^- A_e^+)$ for some nonzero integer $m(e)$ [Fulton and Harris 1991, Theorem 4.3]. If W is an object of $\mathcal{C}$ we define

$$\mathcal{S}_e(W) = W^{\otimes \mathcal{T}_e} A_e^+ A_e^-,$$

where S_e acts on $W^{\otimes \mathcal{T}_e}$ from the right by

$$(\otimes_{t \in \mathcal{T}_e} w_t) h = \otimes_{t \in \mathcal{T}_e} w_{ht}.$$

Then $\mathcal{S}_e$ is a functor from $\mathcal{C}$ to itself. Note that $\mathcal{S}_{(1,\dots,1)}(W) = \bigwedge^n W$. Right multiplication by A_e^+ defines an isomorphism

$$\mathcal{S}_e(W) \xrightarrow{\sim} W^{\otimes \mathcal{T}_e} A_e^- A_e^+,$$

with inverse given by right multiplication by $m(e)^{-1}A_e^-$. Thus we get natural isomorphisms

$$\mathcal{S}_e(W)^\vee = (W^{\otimes \mathcal{T}_e} A_e^+ A_e^-)^\vee \xrightarrow{\sim} (W^\vee)^{\otimes \mathcal{T}_e} A_e^- A_e^+ \xrightarrow{\sim} \mathcal{S}_e(W^\vee).$$

Let $e' = (e_1 + 1, \dots, e_n + 1)$. Let

$$\iota : \mathcal{T}_{e'} \xrightarrow{\sim} \mathcal{T}_{(1,\dots,1)} \amalg \mathcal{T}_e$$

be the bijection which sends $(i, 1)$ to $(i, 1)$ in the first part and, if $j > 1$, sends (i, j) to $(i, j-1)$ in the second part. Then ι induces an isomorphism

$$\iota^* : W^{\otimes n} \otimes W^{\otimes \mathcal{T}_e} \to W^{\otimes \mathcal{T}_{e'}}.$$

Note that

$$A_{e'}^+ \circ \iota^* \circ (A_{(1,\dots,1)}^- \otimes A_e^- A_e^+) = (\# S_e^+)(A_{e'}^- A_{e'}^+) \circ \iota^*$$

so that we get a natural surjection

$$\left(\bigwedge^n W\right) \otimes \mathcal{S}_e(W) \xrightarrow{\sim} W^{\otimes n} A_{(1,\dots,1)}^- \otimes W^{\otimes \mathcal{T}_e} A_e^- A_e^+ \twoheadrightarrow W^{\otimes \mathcal{T}_{e'}} A_{e'}^- A_{e'}^+ \xrightarrow{\sim} \mathcal{S}'_{e'}(W),$$

where the middle map is $A_{e'}^+ \circ \iota^*$. If W has rank n then this map is an isomorphism. (This can be checked after applying a fibre functor where one can either count dimension, or use the fact that the map is $\mathrm{GL}(W)$ equivariant and $\left(\bigwedge^n W\right) \otimes \mathcal{S}_e(W)$ is an irreducible $\mathrm{GL}(W)$-module.) Thus for any $e = (e_1, \dots, e_n) \in (\mathbb{Z}^n)^+$ and any W of rank n we can define

$$\mathcal{S}_e(W) = \mathcal{S}_{e'}(W) \otimes \left(\bigwedge^n W\right)^{\otimes -f}$$

where $f \in \mathbb{Z}$ satisfies $f \geq -e_n$ and where $e' = (e_1 + f, \dots, e_n + f)$. We see that up to natural isomorphism this does not depend on the choice of f.

Lemma 3.1. *If $e \in (\mathbb{Z}^n)^+$ equals $(e_1, \dots, e_n)$ set $e^* = (-e_n, \dots, -e_1) \in (\mathbb{Z}^n)^+$. If W has rank n then there are natural isomorphisms*

$$\mathcal{S}_{e+(f,f,\dots,f)}(W) \cong \mathcal{S}_e(W) \otimes \mathcal{S}_{(f,f,\dots,f)}(W)$$

and

$$\mathcal{S}_e(W) \cong \mathcal{S}_{e^*}(W^\vee).$$

Proof. The first assertion has already been proved so we turn to the second. We may reduce to the case $e_n \geq 0$ and we may choose $f \in \mathbb{Z}_{\geq e_1}$. Set $e' = (f - e_n, \dots, f - e_1)$. Then it will suffice to show that

$$\mathcal{S}_e(W) \cong \mathcal{S}_{e'}(W)^\vee \otimes \left(\bigwedge^n W\right)^{\otimes f}.$$

It even suffices to find a nontrivial natural map

$$\mathcal{S}_e(W) \otimes \mathcal{S}_{e'}(W) \to \left(\bigwedge^n W\right)^{\otimes f} = (W^{\otimes \mathcal{T}_{(f,\dots,f)}}) A_{(f,\dots,f)}^-.$$

(For this then gives a nontrivial natural map $\mathcal{S}_e(W) \to \mathcal{S}_{e'}(W)^\vee \otimes \left(\bigwedge^n W\right)^{\otimes f}$, which we can check is an isomorphism after applying a fibre functor, in which case the left and right hand sides become irreducible $\mathrm{GL}(W)$-modules.) To this end let ι denote the bijection

$$\iota : \mathcal{T}_{(f,\ldots,f)} \xrightarrow{\sim} \mathcal{T}_e \amalg \mathcal{T}_{e'}$$

which sends (i, j) to (i, j) if $j \le e_i$ and to $(n+1-i,\, f+1-i)$ if $j > e_i$, and let ι^* denote the induced map

$$W^{\otimes \mathcal{T}_e} \otimes W^{\bigotimes \mathcal{T}_{e'}} \xrightarrow{\sim} W^{\bigotimes \mathcal{T}_{(f,\ldots,f)}}.$$

Then we consider the map

$$A^-_{(f,\ldots,f)} \circ \iota^* : \mathcal{S}_e(W) \otimes \mathcal{S}_{e'}(W) \to \mathcal{S}_{(f,\ldots,f)}(W).$$

We must show that if W has rank n then this map is nontrivial. We can reduce this to the case of $\overline{\mathbb{Q}}$-vector spaces by applying a fibre functor. In this case let $w_1, \ldots, w_n$ be a basis of W. Consider the element

$$x = (\otimes_{\mathcal{T}_e} u_t) A^-_e \otimes (\otimes_{\mathcal{T}_{e'}} v_t) A^-_{e'} \in W^{\otimes \mathcal{T}_e} \otimes W^{\otimes \mathcal{T}_{e'}}$$

where $u_{(i,j)} = w_i$ and $v_{(i,j)} = w_{n+1-i}$. Then

$$\begin{aligned}(\iota^* x) A^-_{(f,\ldots,f)} = {} & \left(\prod_{i=1}^f (\#\{j :\ e_j < i\})!(\#\{j :\ e_j \ge i\})!\right) (\otimes_{\mathcal{T}_{(f,\ldots,f)}} x_t) A^-_{(f,\ldots,f)} \\ \ne {} & 0,\end{aligned}$$

where $x_{(i,j)} = w_i$. The lemma follows. □

The relative cohomology of $\mathcal{A}/X_U$. If ϖ denotes the projection map from the universal abelian variety $\mathcal{A}$ to X_U then we decompose

$$R^1 \varpi_* \overline{\mathbb{Q}}_l = \bigoplus_{\tau \in \mathrm{Hom}(F,\mathbb{C})} \mathcal{L}_\tau$$

where $\mathcal{L}_\tau$ is the subsheaf of $R^1 \varpi_* \overline{\mathbb{Q}}_l$ where the action of F coming from the endomorphisms of the universal abelian variety is via $\iota^{-1}\tau$. The sheaves $\mathcal{L}_\tau$ on the inverse system of the X_U's carry a natural action of $G(\mathbb{A}^\infty)$ (coming from the action of $G(\mathbb{A}^\infty)$ on the inverse system of the $\mathcal{A}/X_U$. Let Std_τ denote the representation of $G \times_{\mathbb{Q}} \mathbb{C}$ on $V \otimes_{F,\tau} \mathbb{C}$, so that $\mathrm{Std}_{\tau c} \cong \nu\, \mathrm{Std}_\tau^\vee$. Then $\mathcal{L}_\tau \cong \mathcal{L}_{\mathrm{Std}_\tau^\vee}$ with the $G(\mathbb{A}^\infty)$-actions. We also define an action of $G(\mathbb{A}^\infty)$ on the sheaves $\overline{\mathbb{Q}}_l(1)$ by letting $g : g^* \overline{\mathbb{Q}}_l(1) \to \overline{\mathbb{Q}}_l(1)$ be $\nu(g_l)^{-1}$ times the canonical map. Then $\mathcal{L}_{\nu^m} \cong \overline{\mathbb{Q}}_l(m)$ with the $G(\mathbb{A}^\infty)$-actions. Moreover the Weil pairing gives $G(\mathbb{A}^\infty)$-equivariant isomorphisms

$$\mathcal{L}_\tau \cong \mathcal{L}_{\tau c}^\vee \otimes \overline{\mathbb{Q}}_l(-1)$$

corresponding to $\mathcal{L}_{\mathrm{Std}_\tau^\vee} \cong \mathcal{L}_{\mathrm{Std}_{\tau c}} \otimes \mathcal{L}_{\nu^{-1}}$.

Suppose that $\tilde{\xi}$ is an irreducible representation of $G \times_{\mathbb{Q}} \mathbb{C}$ with highest weight $(\tilde{b}_0, \tilde{b}_{\tau,i})_{\tau|_E=\tau_0|_E}$. Then we see that

$$\mathcal{L}_{\tilde{\xi}} \cong \Big(\bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\dots,\tilde{b}_{\tau,n})}(\mathcal{L}_\tau^\vee) \Big) \otimes \overline{\mathbb{Q}}_l(\tilde{b}_0),$$

with their $G(\mathbb{A}^\infty)$-actions.

Note that there are natural isomorphisms $I^*\mathcal{L}_\tau \cong \mathcal{L}_{\tau c}$ and hence, by Lemma 3.1, natural isomorphisms

$$\begin{aligned} I^*\Big(\bigotimes_{\tau|_E=\tau_0|_E} &\mathcal{S}_{(\tilde{b}_{\tau,1},\dots,\tilde{b}_{\tau,n})}(\mathcal{L}_\tau^\vee) \Big) \otimes \overline{\mathbb{Q}}_l(\tilde{b}_0) \\ &\cong \Big(\bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\dots,\tilde{b}_{\tau,n})}(\mathcal{L}_{\tau c}^\vee) \Big) \otimes \overline{\mathbb{Q}}_l(\tilde{b}_0) \\ &\cong \Big(\bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\dots,\tilde{b}_{\tau,n})}(\mathcal{L}_{\tau}(1)) \Big) \otimes \overline{\mathbb{Q}}_l(\tilde{b}_0) \\ &\cong \Big(\bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(-\tilde{b}_{\tau,n},\dots,-\tilde{b}_{\tau,1})}(\mathcal{L}_\tau^\vee) \Big) \otimes \overline{\mathbb{Q}}_l\Big(\tilde{b}_0 + \sum_{\tau|_E=\tau_0|_E} \sum_i b_{\tau,i}\Big). \end{aligned}$$

This isomorphism coincides up to scalar multiples with our previous isomorphism $I^*\mathcal{L}_{\tilde{\xi}} \cong \mathcal{L}_{\tilde{\xi}^\#}$ of (2-1).

Betti realizations. Fix $\sigma : \overline{F} \hookrightarrow \mathbb{C}$ which gives rise to our infinite place v of $\overline{F}$ and suppose that $\sigma|_E = \tau_0|_E$. Set $X_{U,\sigma}(\mathbb{C})$ to be the complex manifold $(X_U \times_{F,\sigma} \mathbb{C})(\mathbb{C})$. If $\tau : F \hookrightarrow \mathbb{C}$ let L_τ denote the maximal subsheaf of $R^1\varpi_*\mathbb{C}$ on $X_{U,\sigma}(\mathbb{C})$ where the action of F from endomorphisms of the universal abelian variety is via τ. The system of locally constant sheaves L_τ have a natural action of $G(\mathbb{A}^\infty)$. Also let $\mathbb{C}(1)$ denote the constant sheaf and endow the system of sheaves $\mathbb{C}(1)/X_{U,\sigma}(\mathbb{C})$ with an action of $G(\mathbb{A}^\infty)$ by letting $g : g^*\mathbb{C}(1) \to \mathbb{C}(1)$ be $|\nu(g)|^{-1}$ times the natural map. Then

$$L_\tau \cong L_{\tau c}^\vee \otimes \mathbb{C}(-1).$$

If $\tilde{\xi}$ is the irreducible representation of $G \times_{\mathbb{Q}} \mathbb{C}$ with highest weight $(\tilde{b}_0, \tilde{b}_{\tau,i})_{\tau|_E=\tau_0|_E}$, then we define a locally constant sheaf of finite-dimensional $\mathbb{C}$-vector spaces $L_{\tilde{\xi}}$ on $X_{U,\sigma}(\mathbb{C})$ as

$$\Big(\bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\dots,\tilde{b}_{\tau,n})}(L_\tau^\vee) \Big) \otimes \mathbb{C}(\tilde{b}_0).$$

Then $L_{\tilde{\xi}}$ is the locally constant sheaf associated to the pull back of $\mathcal{L}_{\tilde{\xi}}$ to $X_U \times_{F,\sigma} \mathbb{C}$, thought of as a sheaf of $\mathbb{C}$-vector spaces via ι^{-1}. This correspondence is $G(\mathbb{A}^\infty)$-equivariant. Note that by Lemma 3.1 if $\tilde{\xi}'$ is one-dimensional then

$$L_{\tilde{\xi}} \otimes L_{\tilde{\xi}'} \xrightarrow{\sim} L_{\tilde{\xi}\otimes\tilde{\xi}'}.$$

Let ${}^c X_{U,\sigma}(\mathbb{C})$ denote the complex conjugate complex manifold of $X_{U,\sigma}(\mathbb{C})$, that is, the same topological space but with complex conjugate charts. Then $I \times c$ induces an isomorphism

$$I \times c : X_{U,\sigma}(\mathbb{C}) \xrightarrow{\sim} {}^c X_{U,\sigma}(\mathbb{C}).$$

As we described above in the l-adic setting, Lemma 3.1 together with the isomorphisms $L_\tau \cong L_{\tau c}^\vee \otimes \mathbb{C}(-1)$ gives rise to an isomorphism

$$(I \times c)^* L_{\tilde{\xi}} \cong L_{\tilde{\xi}^\#}$$

compatible with the corresponding isomorphism in the l-adic setting ($I^* \mathcal{L}_{\tilde{\xi}} \cong \mathcal{L}_{\tilde{\xi}^\#}$).

We set

$$H^i(X_\sigma(\mathbb{C}), L_{\tilde{\xi}}) = \lim_{\to U} H^i(X_{U,\sigma}(\mathbb{C}), L_{\tilde{\xi}})$$

which is naturally a $G(\mathbb{A}^\infty)$-module and which satisfies

$$H^i(X_\sigma(\mathbb{C}), L_{\tilde{\xi}}) \cong H^i(X, \mathcal{L}_{\tilde{\xi}}) \otimes_{\overline{\mathbb{Q}}_l, \iota} \mathbb{C}$$

as $\mathbb{C}[G(\mathbb{A}^\infty)]$-modules. Again as in the l-adic setting we have a decomposition

$$H^0(X_\sigma(\mathbb{C}), L_\zeta) = \bigoplus_{\widetilde{\omega}} \mathbb{C} v_{\widetilde{\omega}, B},$$

where $\widetilde{\omega}$ runs over continuous characters

$$T(\mathbb{A})/T(\mathbb{Q}) \to \mathbb{C}^\times$$

with $\widetilde{\omega}|_{T(\mathbb{R})} = \zeta^{-1}$, and where $G(\mathbb{A}^\infty)$ acts on $v_{\widetilde{\omega},B}$ via $\widetilde{\omega} \circ d$. If we define

$$I_{v,B} : H^i(X_\sigma(\mathbb{C}), L_\xi) \to H^i(X_\sigma(\mathbb{C}), L_\xi)$$

to be the composite

$$H^i(X_\sigma(\mathbb{C}), L_\xi) \xrightarrow{I \times c} H^i(X_\sigma(\mathbb{C}), L_{\xi^\#}) \xrightarrow{\cup v_{\omega,B}} H^i(X_\sigma(\mathbb{C}), L_\xi).$$

Then under the isomorphism $H^i(X_\sigma(\mathbb{C}), L_\xi) \cong H^i(X, \mathcal{L}_\xi) \otimes_{\overline{\mathbb{Q}}_l, \iota} \mathbb{C}$, this map $I_{v,B}$ corresponds to a scalar multiple of the previous map $I_v \otimes 1$.

Again we can define a map $\tilde{c}_{v,B}$ on

$$\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), L_\xi)) \cong \mathbb{C}^n$$

to be the map which sends

$$f \mapsto I_{v,B} \circ f \circ A_\pi.$$

Then $\tilde{c}_{v,B}$ corresponds to a scalar multiple of the map $\tilde{c}_v$ previously defined on $\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\iota^{-1}\pi, H^{n-1}(X, \mathcal{L}_\xi))$. Rescaling $\tilde{c}_{v,B}$ we may, and shall, suppose that $\tilde{c}_{v,B}^2 = 1$, in which case it corresponds to $\pm\tilde{c}_v$. Then it suffices to show that the trace of $\tilde{c}_{v,B}$ is ± 1.

Variation of Hodge structures I: generalities. We begin with a rather lengthy reminder about variations of pure Hodge structures on complex manifolds. We do this because we have not found a single clear reference for all the material we need, although it is all standard.

Recall that a (pure) $\mathbb{R}$-Hodge structure of weight w is a finite-dimensional $\mathbb{R}$-vector space H together with a decreasing, exhaustive and separated filtration Fil^i on the $\mathbb{C}$-vector space $H \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$H \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Fil}^i(H \otimes_{\mathbb{R}} \mathbb{C}) \oplus (1 \otimes c)\,\mathrm{Fil}^{w-1-i}(H \otimes_{\mathbb{R}} \mathbb{C})$$

for all i. In this case $H \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_i H^{i,w-i}$, where

$$H^{i,w-i} = (\mathrm{Fil}^i\, H \otimes_{\mathbb{R}} \mathbb{C}) \cap (1 \otimes c)(\mathrm{Fil}^{w-i}\, H \otimes_{\mathbb{R}} \mathbb{C}).$$

By a polarization on $(H, \{\mathrm{Fil}^i\})$ we mean a perfect bilinear pairing

$$\langle\ ,\ \rangle : H \times H \to \mathbb{R}$$

such that the $\langle\ ,\ \rangle$-orthogonal complement of $\mathrm{Fil}^i\, H \otimes_{\mathbb{R}} \mathbb{C}$ is $\mathrm{Fil}^{w-1-i}\, H \otimes_{\mathbb{R}} \mathbb{C}$ and such that the following property holds. Define a sesquilinear pairing $(\ ,\ )$ on $H \otimes_{\mathbb{R}} \mathbb{C}$ by extending $\langle\ ,\ \rangle$ to a $\mathbb{C}$-bilinear pairing on $H \otimes \mathbb{C}$ and defining

$$(x, y) = \sqrt{-1}^{\,-w} \langle x, (1 \otimes c)y \rangle.$$

Note that $(\ ,\ )$ restricts to a perfect sesquilinear pairing on each $H^{i,w-i}$. We require that $(\ ,\ )$ is Hermitian (i.e., $(y, x) = c(x, y)$) and that the restriction of $(-1)^i(\ ,\ )$ to $H^{i,w-i}$ is positive definite. If $\phi : (H_1, \{\mathrm{Fil}_1^i\}) \to (H_2, \{\mathrm{Fil}_2^i\})$ is a map of $\mathbb{R}$-Hodge structures (i.e., a linear map $\phi : H_1 \to H_2$ such that $\phi \otimes 1$ maps $\mathrm{Fil}^i\, H_1 \otimes_{\mathbb{R}} \mathbb{C}$ to $\mathrm{Fil}^i\, H_2 \otimes_{\mathbb{R}} \mathbb{C}$ for all i) then

$$(\phi \otimes 1)(\mathrm{Fil}^i\, H_1 \otimes_{\mathbb{R}} \mathbb{C}) = (\mathrm{Fil}^i\, H_2 \otimes_{\mathbb{R}} \mathbb{C}) \cap (\phi(H_1) \otimes_{\mathbb{R}} \mathbb{C})$$

for all i. It follows that the category of $\mathbb{R}$-Hodge structures of weight w is an abelian category. The restriction of a polarization to a subobject is again a polarization and the orthogonal complement of the subobject is again a subobject. It follows that the full subcategory of polarizable pure Hodge structures is also (semisimple) abelian. The direct sums of over all integers w of the abelian category of $\mathbb{R}$-Hodge structures of weight w and of the abelian category of polarizable $\mathbb{R}$-Hodge structures of weight w are Tannakian. We will refer to them as the categories of pure $\mathbb{R}$-Hodge structures and of pure polarizable $\mathbb{R}$-Hodge structures; although strictly speaking their objects are not pure, but direct sums of pure objects.

A (pure) $\mathbb{C}$-Hodge structure of weight w is a $\mathbb{C}$-vector space H together with two decreasing, exhaustive and separated filtrations Fil^i and $\overline{\mathrm{Fil}}^i$ on H such that $H = \mathrm{Fil}^i H \oplus \overline{\mathrm{Fil}}^{w-1-i} H$ for all i. If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a $\mathbb{C}$-Hodge structure of weight w then we define the underlying $\mathbb{R}$-Hodge structure to be

$$(H, \{\mathrm{Fil}^i H \oplus \overline{\mathrm{Fil}}^i H\}),$$

where

$$H \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} H \oplus H \supset \mathrm{Fil}^i H \oplus \overline{\mathrm{Fil}}^i H$$

is given by $x \otimes a \mapsto (ax, ({}^c a)x)$. This establishes an equivalence of categories between $\mathbb{C}$-Hodge structures of weight w and $\mathbb{R}$-Hodge structures of weight w with an action of $\mathbb{C}$. If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a $\mathbb{C}$-Hodge structure of weight $\mathbb{R}$ then $H = \bigoplus H^{i,w-i}$, where $H^{i,w-i} = \mathrm{Fil}^i H \cap \overline{\mathrm{Fil}}^{w-i} H$. By a polarization on $\mathbb{H}$ we mean a perfect Hermitian pairing

$$(\ ,\ ) : H \times H \to \mathbb{C},$$

such that for all i the orthogonal complement of $\mathrm{Fil}^i H$ is $\overline{\mathrm{Fil}}^{w-1-i} H$ and the restriction of $(-1)^i(\ ,\ )$ to $H^{i,w-i}$ is positive definite. This is equivalent to a polarization $\langle\ ,\ \rangle$ of the underlying $\mathbb{R}$-Hodge structure such that

$$\langle ax, y\rangle = \langle x, ({}^c a)y\rangle$$

for all $a \in \mathbb{C}$ and $x, y \in H$. The equivalence is given by

$$\langle x, y\rangle = \mathrm{Re}\, \sqrt{-1}^{\,w}(x, y).$$

The category of polarizable $\mathbb{C}$-Hodge structures of weight w is the full subcategory of the category of $\mathbb{C}$-Hodge structures of weight w whose objects are those that admit a polarization. It is closed under taking subobjects and quotients. By the category of (polarizable) pure $\mathbb{C}$-Hodge structures we mean the direct sum over w of the categories of (polarizable) $\mathbb{C}$-Hodge structures of weight w. They are Tannakian categories. (Again objects of these categories are not strictly speaking pure, but the direct sum of pure objects of different weights.)

If $(H, \{\mathrm{Fil}^i\})$ is an $\mathbb{R}$-Hodge structure of weight w then we define

$$(H, \{\mathrm{Fil}^i\}) \otimes \mathbb{C} = (H \otimes_{\mathbb{R}} \mathbb{C}, \{\mathrm{Fil}^i\}, \{(1 \otimes c)\,\mathrm{Fil}^i\}),$$

a $\mathbb{C}$-Hodge structure of weight w. If $(H, \{\mathrm{Fil}^i\})$ is polarizable, so is $(H, \{\mathrm{Fil}^i\}) \otimes \mathbb{C}$. (Define $(x \otimes a, y \otimes b) = \sqrt{-1}^{\,-w} a({}^c b)\langle x, y\rangle$.)

If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a $\mathbb{C}$-Hodge structure we define its complex conjugate ${}^c\mathbb{H} = (H, \{\overline{\mathrm{Fil}}^i\}, \{\mathrm{Fil}^i\})$.

Recall also that a variation of $\mathbb{R}$-Hodge structures $\mathbb{H}$ of weight w on a complex manifold Y is a pair $(H, \{\mathrm{Fil}^i\})$, where H is a locally constant sheaf of finite-dimensional $\mathbb{R}$-vector spaces, where $\{\mathrm{Fil}^i\}$ is an exhaustive, separated, decreasing filtration of $H \otimes_{\mathbb{R}} \mathbb{O}_Y$ by local $\mathbb{O}_Y$-direct summands, such that

- the pull back of $\mathbb{H}$ to any point of Y is a pure $\mathbb{C}$-Hodge structure of weight w,
- and $1 \otimes d : \mathrm{Fil}^i(H \otimes_{\mathbb{R}} \mathbb{O}_Y) \to (\mathrm{Fil}^{i-1}(H \otimes_{\mathbb{R}} \mathbb{O}_Y)) \otimes_{\mathbb{O}_Y} \Omega^1_Y$.

If $\phi : \mathbb{H}_1 \to \mathbb{H}_2$ is a morphism of variation of $\mathbb{R}$-Hodge structures of weight w on Y then $(\phi \otimes 1)\,\mathrm{Fil}^i(H_1 \otimes_{\mathbb{R}} \mathbb{O}_Y) = ((\phi H_1) \otimes_{\mathbb{R}} \mathbb{O}_Y) \cap \mathrm{Fil}^i(H_2 \otimes_{\mathbb{R}} \mathbb{O}_Y)$. It follows that the category of variations of $\mathbb{R}$-Hodge structures of weight w on Y is abelian. By a polarization on $\mathbb{H}$ we mean a perfect bilinear pairing

$$\langle\ ,\ \rangle : H \times H \to \mathbb{R}$$

whose pull-back to any point of Y is a polarization. The full subcategory of the category of variations of $\mathbb{R}$-Hodge structures of weight w on Y consisting of polarizable objects is a semisimple abelian subcategory closed under taking subobjects and quotients. By the category of (polarizable) pure variations of $\mathbb{R}$-Hodge structures on Y we mean the direct sum over w of the categories of (polarizable) variations of $\mathbb{R}$-Hodge structures of weight w on Y. They are Tannakian categories. (Again objects of these categories are not strictly speaking pure, but the direct sum of pure objects of different weights.)

The pull back of a (polarizable) variation of $\mathbb{R}$-Hodge structures of weight w by any morphism is clearly again a (polarizable) variation of $\mathbb{R}$-Hodge structures of weight w. If Y is a compact Kähler manifold and $\mathbb{H}$ is a polarizable variation of $\mathbb{R}$-Hodge structures of weight w on Y then $H^i(Y, H)$ has a natural structure of a polarizable $\mathbb{R}$-Hodge structure of weight $i + w$ [Zucker 1979, Theorem (2.9)]. More precisely, we define $\Omega^\bullet(\mathbb{H})$ to be the complex

$$H \otimes_{\mathbb{R}} \mathbb{O}_Y \to H \otimes_{\mathbb{R}} \Omega^1_Y \to H \otimes_{\mathbb{R}} \Omega^2_Y \to \cdots,$$

and filter it by setting $\mathrm{Fil}^i\,\Omega^\bullet(\mathbb{H})$ to be the subcomplex

$$\mathrm{Fil}^i(H \otimes_{\mathbb{R}} \mathbb{O}_Y) \to \mathrm{Fil}^{i-1}(H \otimes_{\mathbb{R}} \mathbb{O}_Y) \otimes_{\mathbb{O}_Y} \Omega^1_Y \to \mathrm{Fil}^{i-2}(H \otimes_{\mathbb{R}} \mathbb{O}_Y) \otimes_{\mathbb{O}_Y} \Omega^2_Y \to \cdots.$$

Then the spectral sequence

$$E_1^{i,j} = \mathbb{H}^{i+j}(Y, \mathrm{gr}^i\,\Omega^\bullet(\mathbb{H})) \Rightarrow \mathbb{H}^{i+j}(Y, \Omega^\bullet(\mathbb{H})) \cong H^{i+j}(Y, H) \otimes_{\mathbb{R}} \mathbb{C}$$

degenerates at E_1 and defines the (Hodge) filtration on $H^i(Y, H) \otimes_{\mathbb{R}} \mathbb{C}$.

If $f : X \to Y$ is a smooth family of compact Kähler manifolds over a complex manifold Y then $R^i f_* \mathbb{R}$ is naturally a polarizable variation of $\mathbb{R}$-Hodge structures of weight i. (See the Introduction and first two sections of [Zucker 1979].) More precisely, let $\Omega^\bullet_{X/Y}$ denote the complex

$$\mathcal{O}_X \to \Omega^1_{X/Y} \to \Omega^2_{X/Y} \to \cdots$$

and let $\mathrm{Fil}^i\,\Omega^\bullet_{X/Y}$ denote the subcomplex

$$\Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y} \to \cdots$$

Then the filtration on $(R^i f_*\mathbb{R}) \otimes \mathcal{O}_Y \cong \mathbb{R}^i f_*\Omega^\bullet_{X/Y}$ is the one induced by the spectral sequence

$$E_1^{i,j} = R^j f_*\Omega^i_{X/Y} \Rightarrow \mathbb{R}^{i+j} f_*\Omega^\bullet_{X/Y} \cong R^{i+j} f_*\mathbb{R} \otimes_{\mathbb{R}} \mathcal{O}_Y.$$

If moreover Y is a compact Kähler manifold then the Leray spectral sequence

$$E_2^{i,j} = H^i(Y, R^j f_*\mathbb{R}) \Rightarrow H^{i+j}(X, \mathbb{R})$$

degenerates at E_2 and the $\mathbb{R}$-Hodge structure on $H^i(Y, R^j f_*\mathbb{R})$ is compatible with the $\mathbb{R}$-Hodge structure on $H^{i+j}(X, \mathbb{R})$ [Zucker 1979, Proposition (2.16)].

By a variation of $\mathbb{C}$-Hodge structures $\mathbb{H}$ of weight w on a complex manifold Y we mean a triple $(H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$, where H is a locally constant sheaf of finite-dimensional $\mathbb{C}$-vector spaces, $\{\mathrm{Fil}^i\}$ is an exhaustive, separated, decreasing filtration of $H \otimes_{\mathbb{C}} \mathcal{O}_Y$ by local $\mathcal{O}_Y$-direct summands, and $\{\overline{\mathrm{Fil}}^i\}$ is an exhaustive, separated, decreasing filtration of $H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}$ by local $\mathcal{O}_{{}^cY}$-direct summands such that

- the pull back of $\mathbb{H}$ to any point of Y is a pure $\mathbb{C}$-Hodge structure of weight w,
- $1 \otimes d : \mathrm{Fil}^i(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \to (\mathrm{Fil}^{i-1}(H \otimes_{\mathbb{C}} \mathcal{O}_Y)) \otimes_{\mathcal{O}_Y} \Omega^1_Y$,
- and $1 \otimes d : \overline{\mathrm{Fil}}^i(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}) \to (\overline{\mathrm{Fil}}^{i-1}(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY})) \otimes_{\mathcal{O}_{{}^cY}} \Omega^1_{{}^cY}$.

(Recall that ${}^cY$ denote the same underlying topological space as Y but with complex conjugate charts.) If $\mathbb{H}$ is a variation of $\mathbb{C}$-Hodge structures of weight w on Y then $(H, \{\mathrm{Fil}^i \oplus (1 \otimes c)\overline{\mathrm{Fil}}^i\})$ is a variation of $\mathbb{R}$-Hodge structures of weight w on Y, where we think of $\mathrm{Fil}^i \oplus (1 \otimes c)\overline{\mathrm{Fil}}^i$ contained in

$$(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \oplus (1 \otimes c)(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}) = (H \otimes_{\mathbb{C}} \mathcal{O}_Y) \oplus (H \otimes_{\mathbb{C},c} \mathcal{O}_Y) = H \otimes_{\mathbb{R}} \mathcal{O}_Y.$$

This establishes an equivalence of categories between variations of $\mathbb{C}$-Hodge structures of weight w on Y and variations of $\mathbb{R}$-Hodge structures of weight w on Y together with an action of $\mathbb{C}$. Thus the category of variations of $\mathbb{C}$-Hodge structures of weight w on Y is abelian. By the category of pure variations of $\mathbb{C}$-Hodge structures of weight w on Y we mean the direct sum over w of the categories of variations of $\mathbb{C}$-Hodge structures of weight w. It is a Tannakian category. (Again the objects are not strictly speaking pure, but the direct sum of pure objects of different weights.)

By a polarization of a variation of $\mathbb{C}$-Hodge structures of weight w on Y we mean a perfect Hermitian pairing

$$(\ , \ ) : H \times H \to \mathbb{C}$$

such that the pull back to any point of Y is a polarization. The category of polarizable $\mathbb{C}$-Hodge structures of weight w on Y is equivalent to the category of $\mathbb{R}$-Hodge structures of weight w on Y together with an action of $\mathbb{C}$, which admit a polarization for which the adjoint of any $a \in \mathbb{C}$ is ${}^c a$. Thus the category of polarizable variations of $\mathbb{C}$-Hodge structures of weight w on Y is a full abelian subcategory of the category of variations of $\mathbb{C}$-Hodge structures of weight w on Y and is closed under subobjects and quotients. By the category of pure polarizable variations of $\mathbb{C}$-Hodge structures of weight w on Y we mean the direct sum over w of the categories of variations of $\mathbb{C}$-Hodge structures of weight w. It is again a Tannakian category. (And again the objects are not strictly speaking pure, but the direct sum of pure objects of different weights.)

If $(H, \{\mathrm{Fil}^i\})$ is a variation $\mathbb{R}$-Hodge structures of weight w on Y then we define

$$(H, \{\mathrm{Fil}^i\}) \otimes \mathbb{C} = (H \otimes_{\mathbb{R}} \mathbb{C}, \{\mathrm{Fil}^i\}, \{(1 \otimes c)\,\mathrm{Fil}^i\}),$$

a variation of $\mathbb{C}$-Hodge structures of weight w on Y. If $(H, \{\mathrm{Fil}^i\})$ is polarizable then so is $(H, \{\mathrm{Fil}^i\}) \otimes \mathbb{C}$. (Define $(x \otimes a, y \otimes b) = \sqrt{-1}^{\,-w} a({}^c b)\langle x, y\rangle$.)

If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a variation of $\mathbb{C}$-Hodge structures of weight w on Y we define its complex conjugate ${}^c\mathbb{H} = (H, \{\overline{\mathrm{Fil}}^i\}, \{\mathrm{Fil}^i\})$.

The pull back of a (polarizable) variation of $\mathbb{C}$-Hodge structures of weight w by any morphism is clearly again a (polarizable) variation of $\mathbb{C}$-Hodge structures of weight w. If Y is a compact Kähler manifold and $\mathbb{H}$ is a polarizable variation of $\mathbb{C}$-Hodge structures of weight w on Y then $H^i(Y, H)$ has a natural structure of a polarizable $\mathbb{C}$-Hodge structure of weight $i + w$). More precisely, define $\Omega^\bullet_Y(\mathbb{H})$ to be the complex

$$H \otimes_{\mathbb{C}} \mathcal{O}_Y \to H \otimes_{\mathbb{C}} \Omega^1_Y \to H \otimes_{\mathbb{C}} \Omega^2_Y \to \cdots$$

filtered by setting $\mathrm{Fil}^i\,\Omega^\bullet_Y(\mathbb{H})$ to be the subcomplex

$$\mathrm{Fil}^i(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \to \mathrm{Fil}^{i-1}(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \Omega^1_Y \to \mathrm{Fil}^{i-2}(H \otimes_{\mathbb{C}} \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \Omega^2_Y \to \cdots ;$$

similarly $\Omega^\bullet_{{}^cY}(\mathbb{H})$ is the complex

$$H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY} \to H \otimes_{\mathbb{C}} \Omega^1_{{}^cY} \to H \otimes_{\mathbb{C}} \Omega^2_{c}Y \to \cdots$$

with $\mathrm{Fil}^i\,\Omega^\bullet_{{}^cY}(\mathbb{H}))$ the subcomplex

$$\overline{\mathrm{Fil}}^i(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}) \to \overline{\mathrm{Fil}}^{i-1}(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}) \otimes_{\mathcal{O}_{{}^cY}} \Omega^1_{{}^cY} \to \overline{\mathrm{Fil}}^{i-2}(H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}) \otimes_{\mathcal{O}_{{}^cY}} \Omega^2_{{}^cY} \ldots .$$

Then the spectral sequences

$$E_1^{i,j} = \mathbb{H}^{i+j}(Y, \mathrm{gr}^i\, \Omega_Y^\bullet(\mathbb{H})) \Rightarrow \mathbb{H}^{i+j}(Y, \Omega_Y^\bullet(\mathbb{H})) \cong H^{i+j}(Y, H)$$

and

$$\bar{E}_1^{i,j} = \mathbb{H}^{i+j}({}^cY, \mathrm{gr}^i\, \Omega_{{}^cY}^\bullet(\mathbb{H})) \Rightarrow \mathbb{H}^{i+j}(Y, \Omega_{{}^cY}^\bullet(\mathbb{H})) \cong H^{i+j}(Y, H)$$

degenerate at E_1 and define the (Hodge) filtrations on $H^i(Y, H)$. (This can be easily deduced from the corresponding facts for variations of $\mathbb{R}$-Hodge structures.)

If $f : X \to Y$ is a smooth family of compact Kähler manifolds over a complex manifold Y then $R^i f_*\mathbb{C}$ is naturally a polarizable variation of $\mathbb{C}$-Hodge structures of weight i. More precisely, the filtrations on $(R^i f_*\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_Y \cong \mathbb{R}^i f_*\Omega_{X/Y}^\bullet$ and $(R^i f_*\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY} \cong \mathbb{R}^i f_*\Omega_{{}^cX/{}^cY}^\bullet$ are the ones induced by the spectral sequences

$$E_1^{i,j} = R^j f_*\Omega_{X/Y}^i \Rightarrow \mathbb{R}^{i+j} f_*\Omega_{X/Y}^\bullet \cong R^{i+j} f_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_Y$$

and

$$\bar{E}_1^{i,j} = R^j f_*\Omega_{{}^cX/{}^cY}^i \Rightarrow \mathbb{R}^{i+j} f_*\Omega_{{}^cX/{}^cY}^\bullet \cong R^{i+j} f_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}.$$

If moreover Y is a compact Kähler manifold then the Leray spectral sequence

$$E_2^{i,j} = H^i(Y, R^j f_*\mathbb{C}) \Rightarrow H^{i+j}(X, \mathbb{C})$$

degenerates at E_2 and the $\mathbb{C}$-Hodge structure on $H^i(Y, R^j f_*\mathbb{C})$ is compatible with the $\mathbb{C}$-Hodge structure on $H^{i+j}(X, \mathbb{C})$. (Again this is all easily deduced from the case of $\mathbb{R}$-Hodge structures.)

For example $\mathbb{C}(m)$ is the variation of pure $\mathbb{C}$-Hodge structures of weight $-2m$ with underlying locally constant sheaf $\mathbb{C}$ and with $\mathrm{Fil}^i = (0)$ and $\overline{\mathrm{Fil}}^i = (0)$ for $i > -m$, but with Fil^i and $\overline{\mathrm{Fil}}^i$ everything for $i \le m$.

If $\mathbb{H} = (H, \{\mathrm{Fil}^i\}, \{\overline{\mathrm{Fil}}^i\})$ is a variation of pure $\mathbb{C}$-Hodge structures of weight w on Y we define a variation pure $\mathbb{C}$-Hodge structures $\mathbb{H}\{j_1, j_2\}$ of weight $w + j_1 + j_2$ on Y by setting $H\{j_1, j_2\} = H$ and

$$\mathrm{Fil}^i\, H\{j_1, j_2\} \otimes_{\mathbb{C}} \mathcal{O}_Y = \mathrm{Fil}^{i-j_1}\, H \otimes_{\mathbb{C}} \mathcal{O}_Y,$$
$$\overline{\mathrm{Fil}}^i H\{j_1, j_2\} \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY} = \overline{\mathrm{Fil}}^{i-j_2} H \otimes_{\mathbb{C}} \mathcal{O}_{{}^cY}.$$

Thus $\mathbb{C}(j) = \mathbb{C}(0)\{-j, -j\}$.

Variation of Hodge structures II. We will give $\mathbb{C}(j)$ (the constant variation of pure $\mathbb{C}$-Hodge structures of weight $-2j$ on $X_{U,\sigma}(\mathbb{C})$) an action of $G(\mathbb{A}^\infty)$ by letting $g : g^*\mathbb{C}(j) \to \mathbb{C}(j)$ be $|\nu(g)^{-j}|$ times the natural map. If $\mathbb{H}/X_{U,\sigma}(\mathbb{C})$ is a collection of variations of pure $\mathbb{C}$-Hodge structures with an action of $G(\mathbb{A}^\infty)$ we will give $\mathbb{H}\{j_1, j_2\}$ the action induced from the one on $\mathbb{H}$. Thus the actions of $G(\mathbb{A}^\infty)$ on $\mathbb{C}(j)$ and $\mathbb{C}(0)\{-j, -j\}$ are different.

$R^1\varpi_*\mathbb{C}$ is a variation of pure $\mathbb{C}$-Hodge structures of weight 1 on $X_{U,\sigma}(\mathbb{C})$ and we can decompose

$$R^1\varpi_*\mathbb{C} = \bigoplus_{\tau\in\mathrm{Hom}(F,\mathbb{C})} \mathbb{L}_\tau$$

where $\mathbb{L}_\tau$ is a variation of pure $\mathbb{C}$-Hodge structures of weight 1 extending L_τ. The projective system of variations of pure $\mathbb{C}$-Hodge structures $\mathbb{L}_\tau/X_{U,\sigma}(\mathbb{C})$ as U varies has an action of $G(\mathbb{A}^\infty)$. We have $G(\mathbb{A}^\infty)$-equivariant isomorphisms

$$\mathbb{L}_\tau \cong \mathbb{L}_{\tau c}^\vee \otimes \mathbb{C}(-1).$$

Also, if $\sigma, \tau \in \mathrm{Hom}_{E,\tau_0}(F, \mathbb{C})$ then

$$\left(\bigwedge^n \mathbb{L}_\tau\right)/X_{U,\sigma}(\mathbb{C})$$

is noncanonically isomorphic to $\mathbb{C}\{0, n\}$ if $\sigma \neq \tau$ and to $\mathbb{C}\{1, n-1\}$ if $\sigma = \tau$. This identification is not $G(\mathbb{A}^\infty)$-equivariant.

For $\tilde{\xi}$ an irreducible representation of $G \times_{\mathbb{Q}} \mathbb{C}$ with highest weight $(\tilde{b}_0, \tilde{b}_{\tau,i})$, we can then define a variation of pure $\mathbb{C}$-Hodge structures $\mathbb{L}_{\tilde{\xi}}$ of weight

$$-2\tilde{b}_0 - \sum_{\tau|_E=\tau_0|_E} \sum_i \tilde{b}_{\tau,i}$$

extending $L_{\tilde{\xi}}$ by

$$\mathbb{L}_{\tilde{\xi}} = \left(\bigotimes_{\tau|_E=\tau_0|_E} \mathcal{S}_{(\tilde{b}_{\tau,1},\dots,\tilde{b}_{\tau,n})}(\mathbb{L}_\tau^\vee)\right) \otimes \mathbb{C}(\tilde{b}_0).$$

Again the system $\mathbb{L}_{\tilde{\xi}}/X_{U,\sigma}(\mathbb{C})$ has an action of $G(\mathbb{A}^\infty)$. Again by Lemma 3.1 we see that if $\tilde{\xi}'$ is one-dimensional then there is a natural isomorphism

$$\mathbb{L}_{\tilde{\xi}} \otimes \mathbb{L}_{\tilde{\xi}'} \xrightarrow{\sim} \mathbb{L}_{\tilde{\xi}\otimes\tilde{\xi}'}.$$

We set

$$H^i(X_\sigma(\mathbb{C}), \mathbb{L}_{\tilde{\xi}}) = \lim_{\to U} H^i(X_{U,\sigma}(\mathbb{C}), \mathbb{L}_{\tilde{\xi}}).$$

It is a direct limit of pure $\mathbb{C}$-Hodge structures with an action of $G(\mathbb{A}^\infty)$, such that the fixed subspace of any open subgroup of $G(\mathbb{A}^\infty)$ is a (finite-dimensional) pure $\mathbb{C}$-Hodge structure of weight $w = i - 2\tilde{b}_0 - \left(\sum_{\tau|_E=\tau_0|_E} \sum_j \tilde{b}_{\tau,j}\right)$.

If $\tilde{b}_{\tau,j} = \tilde{b}_\tau$ is independent of j for all $\tau \in \mathrm{Hom}_{E,\tau_0}(F, \mathbb{C})$ and if $\sigma|_E = \tau_0|_E$ then

$$\mathbb{L}_{\tilde{\xi}} \cong \mathbb{C}(0)\{-\tilde{b}_\sigma - \tilde{b}_0, \tilde{b}_\sigma - \tilde{b}_0 - n \sum_{\tau\in\mathrm{Hom}_{E,\tau_0}(E,\mathbb{C})} \tilde{b}_\tau\}$$

noncanonically on $X_{U,\sigma}(\mathbb{C})$. If

$$\widetilde{\omega} : T(\mathbb{A})/T(\mathbb{Q}) \longrightarrow \mathbb{C}^\times$$

is a continuous character with $\widetilde{\omega}|_{T(\mathbb{R})} = \tilde{\xi}^{-1}$ then $\upsilon_{\widetilde{\omega},B}$ spans a sub pure $\mathbb{C}$-Hodge structure of $H^0(X_\sigma(\mathbb{C}), \mathbb{L}_{\tilde{\xi}})$ isomorphic to

$$\mathbb{C}(0)\{-\tilde{b}_\sigma - \tilde{b}_0, \tilde{b}_\sigma - \tilde{b}_0 - n \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(E,\mathbb{C})} \tilde{b}_\tau\}.$$

The choice of $\widetilde{\omega}$ fixes an equivariant isomorphism

$$\mathbb{L}_{\tilde{\xi}} \cong \mathbb{C}(0)\{-\tilde{b}_\sigma - \tilde{b}_0, \tilde{b}_\sigma - \tilde{b}_0 - n \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(E,\mathbb{C})} \tilde{b}_\tau\}(\widetilde{\omega} \circ d).$$

The map $(I \times c) : X_{U,\sigma}(\mathbb{C}) \to {}^cX_{U,\sigma}(\mathbb{C})$ lifts to a map $\mathscr{A}_\sigma(\mathbb{C}) \to {}^c\mathscr{A}_\sigma(\mathbb{C})$. We deduce that there is a natural isomorphism

$$(I \times c)^*\mathbb{L}_\tau \cong {}^c\mathbb{L}_{\tau c},$$

and hence applying Lemma 3.1 and the isomorphism $\mathbb{L}_\tau \cong \mathbb{L}_{\tau c}^\vee \otimes \mathbb{C}(-1)$ we get natural isomorphisms

$$(I \times c)^*\mathbb{L}_{\tilde{\xi}} \cong {}^c\mathbb{L}_{\tilde{\xi}^\#}$$

extending our previous isomorphism $(I \times c)^*L_{\tilde{\xi}} \cong L_{\tilde{\xi}^\#}$. Thus we get maps

$$H^i(X_\sigma(\mathbb{C}), \mathbb{L}_{\tilde{\xi}}) \to H^i({}^cX_\sigma(\mathbb{C}), {}^c\mathbb{L}_{\tilde{\xi}^\#}) \cong {}^cH^i(X_\sigma(\mathbb{C}), \mathbb{L}_{\tilde{\xi}^\#}).$$

Now suppose that $\sigma|_E = \tau_0|_E$. The line $\mathbb{C}\upsilon_{\omega,B}$ is a subpure $\mathbb{C}$-Hodge structure of $H^0({}^cX_\sigma(\mathbb{C}), {}^c\mathbb{L}_\zeta)$ isomorphic to $\mathbb{C}\{\gamma, -\gamma\}$ with

$$\gamma = \alpha + 2\beta_\sigma - n \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})} (\beta_\tau + \alpha/2).$$

Thus the cup product map

$$\cup \upsilon_{\omega,B} : {}^c\mathbb{L}_{\xi^\#} \to ({}^c\mathbb{L}_\xi)\{-\gamma, \gamma\}$$

is a map of variations of pure $\mathbb{C}$-Hodge structures. Thus the map

$$I_{v,B} : H^i(X_\sigma(\mathbb{C}), L_\xi) \to H^i(X_\sigma(\mathbb{C}), L_\xi)$$

extends to a map of pure $\mathbb{C}$-Hodge structures

$$I_{v,B} : H^i(X_\sigma(\mathbb{C}), \mathbb{L}_\xi) \to ({}^cH^i(X_\sigma(\mathbb{C}), \mathbb{L}_\xi))\{-\gamma, \gamma\},$$

or to a map of pure $\mathbb{C}$-Hodge structures

$$I_{v,B} : H^i(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)\{\epsilon + \beta_\sigma, \epsilon' - \alpha - \beta_\sigma\} \to {}^c(H^i(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)\{\epsilon + \beta_\sigma, \epsilon' - \alpha - \beta_\sigma\}).$$

$\big($Note that $\epsilon' - \alpha - \beta_\sigma - (\epsilon + \beta_\sigma) = -\alpha - 2\beta_\sigma + n \sum_{\tau \in \mathrm{Hom}_{E,\tau_0}(F,\mathbb{C})} (\beta_\tau + \alpha/2) = -\gamma.\big)$

If we set

$$\mathbb{H} = \mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi))\{\epsilon + \beta_\sigma, \epsilon' - \alpha - \beta_\sigma\},$$

then $\mathbb{H}$ is a pure $\mathbb{C}$-Hodge structure of weight $w = n - 1 - \alpha \in 2\mathbb{Z}$. We see that $\tilde{c}_{v,B}$ extends to a map of pure $\mathbb{C}$-Hodge structures:

$$\tilde{c}_{v,B} : \mathbb{H} \to {}^c\mathbb{H}$$

with $\tilde{c}_{v,B}^2 = 1$. Moreover we see that $\tilde{c}_{v,B}$ interchanges $\mathrm{Fil}^{w/2-1}\,\mathbb{H}$ and $\overline{\mathrm{Fil}}^{w/2-1}\mathbb{H}$, and that these two spaces have trivial intersection. We deduce that

$$\begin{aligned} |\operatorname{tr}\tilde{c}_{v,B}| &\le n - 2\dim_{\mathbb{C}} \mathrm{Fil}^{w/2-1}\,\mathbb{H} \\ &= \dim_{\mathbb{C}} \overline{\mathrm{Fil}}^{w/2}\mathbb{H} - \dim_{\mathbb{C}} \mathrm{Fil}^{w/2-1}\,\mathbb{H} \\ &= \dim_{\mathbb{C}} \mathrm{Fil}^{w/2}\,\mathbb{H} - \dim_{\mathbb{C}} \mathrm{Fil}^{w/2-1}\,\mathbb{H} \\ &= \dim_{\mathbb{C}} \mathrm{gr}^{w/2}\,\mathbb{H} = \mathrm{gr}^{w/2-\epsilon-\beta_\sigma}\,\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)). \end{aligned}$$

Cupping with $v_{\delta_{E/\mathbb{Q}}\circ\nu,B}$ shows that

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{gr}^{w/2-\epsilon-\beta_\sigma}\,\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)) \\ &\qquad = \dim_{\mathbb{C}} \mathrm{gr}^{w/2-\epsilon-\beta_\sigma}\,\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi \otimes (\delta_{E/\mathbb{Q}} \circ \nu), H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)). \end{aligned}$$

Thus it suffices to show that

$$\dim_{\mathbb{C}} \bigoplus_{\mathrm{BC}(\tilde{\pi})=(\psi^\infty, \Pi_F^\infty\otimes\phi^\infty)} \mathrm{gr}^{w/2-\epsilon-\beta_\sigma}\,\mathrm{Hom}_{G(\mathbb{A}^\infty)}(\pi, H^{n-1}(X_\sigma(\mathbb{C}), \mathbb{L}_\xi)) \le 2.$$

However the proof of Corollary 6.7 of [Shin 2011] shows this. (Note that the constant $C_G = \tau(G)\#\ker^1(\mathbb{Q}, G)$ of [Shin 2011] in our case equals 2.) So we have finally completed the proof of Proposition 1.2.

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