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The Chow ring of double EPW sextics

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A conjecture of Beauville and Voisin states that for an irreducible symplectic variety X the subring of $CH^*(X)$ generated by divisors goes injectively into the cohomology of X, via the cycle map. We prove this for a very general double Eisenbud–Popescu–Walter sextic.

1. Introduction

A difficult problem in algebraic geometry is to characterize the kernel (and the image) of the cycle map

$$c : CH^*(X) \to H^*(X, \mathbb{Z})$$

for a smooth projective variety X over $\mathbb{C}$. When X is an *irreducible symplectic variety* there is this general conjecture:

Conjecture 1 [Beauville 2007]. *Let X be an irreducible symplectic variety, and let $DCH(X) \subset CH^*_{\mathbb{Q}}(X)$ be the subalgebra generated by the divisors. Then the cycle map*

$$c : CH^*(X)_{\mathbb{Q}} \to H^*(X, \mathbb{Q})$$

is injective when restricted to $DCH(X)$.

We refer to the original article for the motivation of the conjecture and its link with the conjectures of Bloch and Beilinson; we just remark that the conjecture was known to hold when X is a K3 surface from [Beauville and Voisin 2004]. Conjecture 1 explicitly means the following: any polynomial relation

$$P(D_1, \ldots, D_k) = 0$$

in the fundamental classes of divisors which holds in $H^*(X)$ already holds inside $CH^*_{\mathbb{Q}}(X)$.

This has been extended by Claire Voisin:

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Conjecture 2 [Voisin 2008]. *Let X be an irreducible symplectic variety. Any polynomial relation*

$$P(D_1, \dots, D_k, c_i(X)) = 0$$

in the fundamental classes of divisors and in the Chern classes of X *that holds in $H^*(X)$ already holds in $CH^*_{\mathbb{Q}}(X)$.*

Theorem [Voisin 2008]. *Conjecture 2 holds when*

- *$X = S^{[n]}$, for some K3 surface S, and $n \leq 2b_2(S)_{\mathrm{tr}} + 4$, where $b_2(S)_{\mathrm{tr}}$ is the rank of the transcendental part of $H^2(S)$, that is, the orthogonal of the Néron–Severi lattice, or*
- *X is the Fano variety of lines on a cubic fourfold $Y \subset \mathbb{P}^5$.*

To our knowledge, no other cases of the conjecture have been verified. We prove it to be true for double EPW sextics, a class of irreducible symplectic varieties introduced in [O'Grady 2008a] and named after Eisenbud, Popescu and Walter. (We will review their construction shortly.) More precisely:

Theorem 1.1. *Let X be a double EPW sextic and $f : X \to Y$ its associated double covering. Let*

$$h = f^*\mathcal{O}_Y(1)$$

be the natural polarization. Then every polynomial relation between h and the Chern classes of X which holds in $H^*(X, \mathbb{Q})$ already holds in $CH^*(X)_{\mathbb{Q}}$.

In particular, if X is very general, Conjecture 2 holds for X.

Theorem 1.1 is the main result of my Ph.D. thesis [Ferretti 2009]. Some facts that are only cited in the present paper are described there in more detail.

EPW sextics. In this section we recall some known facts about EPW sextics and we fix the notation that we shall use. The results here are from [O'Grady 2006; 2008a]; see also [Ferretti 2009] for a detailed introduction.

We start with a six-dimensional vector space V over the field $\mathbb{C}$. The space $\bigwedge^6 V$ is one-dimensional, so we choose once and for all an isomorphism

$$\mathrm{vol} : \bigwedge^6 V \to \mathbb{C}.$$

This endows $\bigwedge^3 V$ with a symplectic form, given by

$$(\alpha, \beta) = \mathrm{vol}(\alpha \wedge \beta),$$

for $\alpha, \beta \in \bigwedge^3 V$, so $\bigwedge^3 V$ becomes a symplectic vector space of dimension 20.

For each nonzero $v \in V$ we can consider the Lagrangian subspace

$$F_v = \{v \wedge \alpha \mid \alpha \in \bigwedge^2 V\}.$$

This is clearly isotropic, and the isomorphism

$$\begin{aligned} \varphi_v : F_v &\xrightarrow{\cong} \bigwedge^2(V/\langle v\rangle) \\ v\wedge\alpha &\longmapsto [\alpha] \end{aligned} \tag{1-1}$$

shows that $\dim F_v = \binom{5}{2} = 10$.

Since the subspace F_v only depends on the class $[v]\in\mathbb{P}(V)$, the subspaces F_v fit together, giving rise to a *Lagrangian subbundle* F of the trivial symplectic bundle $\mathbb{P}(V)\times\bigwedge^3 V$. The maps in (1-1) then yield an isomorphism

$$F\cong\mathcal{S}\otimes\bigwedge^2\mathcal{Q},$$

where $\mathcal{Q}$ is the tautological quotient bundle on $\mathbb{P}(V)$ and $\mathcal{S}$ the tautological subbundle. From this a standard computation gives

$$c_1(F) = c_1(\bigwedge^2\mathcal{Q}) + \mathrm{rk}(F)c_1(\mathcal{S}) = -6H, \tag{1-2}$$

where $H = c_1(\mathcal{O}(1))$ is the hyperplane class on $\mathbb{P}(V)$.

We are now ready to define the EPW sextics, as follows. Fix a Lagrangian subspace $A\subset\bigwedge^3 V$. Note that the symplectic form gives a canonical identification

$$\bigwedge^3 V/A\cong A^\vee.$$

Let

$$\lambda_A : F\to\mathcal{O}_{\mathbb{P}(V)}\otimes A^\vee \tag{1-3}$$

be the inclusion $F\hookrightarrow\mathcal{O}_{\mathbb{P}(V)}\otimes\bigwedge^3 V$ followed by the projection modulo A. The map λ_A is a map of vector bundles of equal rank 10.

Definition 1.2. We set

$$Y_A = Z(\det\lambda_A),$$

the zero locus of the determinant of λ_A. This is a subscheme of $\mathbb{P}(V)$; when it is not the whole of $\mathbb{P}(V)$, Y_A is called an *EPW sextic*. (It is indeed a sextic by (1-2).)

The support of the scheme Y_A is by definition the locus

$$\{[v]\in\mathbb{P}(V)\mid\dim(F_v\cap A)\geq 1\}.$$

We then set

$$Y_A[k] = \{[v]\in\mathbb{P}(V)\mid\dim(F_v\cap A)\geq k\},$$

so that $Y_A = Y_A[1]$, at least set-theoretically. The loci $Y_A[k]$ also have a natural scheme structure, given by the vanishing of the determinants of the $(11-k)\times(11-k)$ minors of λ_A.

The natural parameter space for EPW sextics is the Lagrangian Grassmannian $\mathbb{LG}(\bigwedge^3 V)$, or more precisely the Zariski open set parametrizing those A for which

$Y_A \subsetneq \mathbb{P}(V)$. We recall that $\mathbb{LG}(\bigwedge^3 V) \subset \mathrm{Gr}(10, \bigwedge^3 V)$ is the subvariety of Lagrangian subspaces; it is a smooth variety of dimension 55.

Double EPW sextics. Assume that Y_A is not the whole of $\mathbb{P}(V)$. The map of vector bundles λ_A in (1-3) is an *injective homomorphism of sheaves*, whose cokernel is supported on Y_A. If we denote by

$$i_A : Y_A \to \mathbb{P}(V)$$

the inclusion, then we have an exact sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee \longrightarrow i_{A*}(\xi_A) \longrightarrow 0 \tag{1-4}$$

for some sheaf ξ_A on Y_A. For a generic Lagrangian subspace A the locus

$$Y_A[2] = \left\{ [v] \in \mathbb{P}(V) \mid \dim(F_v \cap A) \geq 2 \right\}$$

is properly contained in Y_A; it follows that ξ_A is generically free of rank 1. If

$$\zeta_A = \xi_A^\vee(3),$$

there is a natural *multiplication* map

$$m_A : \zeta_A \otimes \zeta_A \to \mathcal{O}_{Y_A}.$$

More precisely:

Lemma 1.3 [O'Grady 2008a]. *The map m_A is symmetric and associative, and gives an isomorphism between $\zeta_A \otimes \zeta_A$ and $\mathcal{O}_{Y_A}$.*

Thanks to the lemma we see that the sheaf

$$\mathcal{O}_{Y_A} \oplus \zeta_A$$

has the structure of an $\mathcal{O}_{Y_A}$-algebra, so we have an associated double covering.

Definition 1.4. We denote by X_A this double covering; the scheme X_A is called a *double EPW sextic*. We denote by

$$f_A : X_A \to Y_A$$

the covering map.

The scheme X_A is endowed with a polarization $h_A = f_A^* \mathcal{O}_{Y_A}(1)$.

Remark 1.5. The ramification locus of the map f_A is $Y_A[2]$. To see this we just need to observe that by construction the ramification locus is the locus where the sheaf ζ_A, or equivalently the sheaf ξ_A, is not locally free. Since $i_{A*}(\xi_A)$ is the cokernel of the map

$$\lambda_A : F \to \mathcal{O}_{\mathbb{P}(V)} \otimes A^\vee,$$

we see that the rank of ξ_A jumps exactly along $Y_A[2]$, hence our claim.

Definition 1.6 [O'Grady 2010]. We let

$$\Sigma = \left\{A \in \mathbb{LG}(\bigwedge^3 V) \mid \bigwedge^3(W) \subset A \text{ for some } W \subset V,\ \dim W = 3\right\}.$$

In other words, Σ is the set of Lagrangian subspaces of $\bigwedge^3 V$ containing a decomposable form. We also let

$$\Delta = \left\{A \in \mathbb{LG}(\bigwedge^3 V) \mid Y_A[3] \neq \varnothing\right\} \subset \mathbb{LG}(\bigwedge^3 V).$$

Finally we define

$$\mathbb{LG}(\bigwedge^3 V)^0 = \mathbb{LG}(\bigwedge^3 V) \setminus (\Sigma \cup \Delta).$$

We recall the following characterization of smooth double EPW sextics:

Theorem 1.7 [O'Grady 2010]. *The double covering X_A is smooth if and only if*

$$A \in \mathbb{LG}(\bigwedge^3 V)^0.$$

The relevance of these double coverings stems from the following result.

Theorem 1.8 [O'Grady 2010]. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$. Then X_A is an irreducible symplectic variety. The Beauville–Bogomolov form on $H^2(X, \mathbb{Z})$ is the same as that of $S^{[2]}$, where S is a $K3$ surface, and the Fujiki constant of X_A is 3.*

Let $Z_A = f_A^{-1}(Y_A[2])$; this is the branch locus for the 2:1 covering, hence it is isomorphic to $Y_A[2]$ itself. Since the covering involution is antisymplectic, the symplectic form restricts to 0 on Z_A, that is, Z_A is isotropic. Under mild assumptions Z_A is a surface, hence a Lagrangian surface inside X_A. More precisely:

Proposition 1.9 [O'Grady 2010]. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$. Then $Y_A[2]$ is a smooth connected surface of degree 40, with $\chi_{\text{top}}(Y_A[2]) = 192$.*

We will need the following relation in the Chow group. It appears in [O'Grady 2008b, Proposition 4.9], with a different proof from the one given below.

Proposition 1.10. *Let $A \in \mathbb{LG}(\bigwedge^3 V)^0$, $Z = Z_A$. The canonical class of Z satisfies*

$$2K_Z = \mathcal{O}_Z(6)$$

in $CH^*(Z)$.

Remark 1.11. The proposition determines K_Z only up to 2-torsion. Namely we can rewrite it as

$$K_Z = \mathcal{O}_Z(3) + \kappa,$$

where κ is a 2-torsion class. One can use the deformation argument in [Ferretti 2012] and the results of [Welters 1981] to show that the class κ is really nonzero.

Proof. For simplicity let $W = f(Z)$ denote the singular set of Y. We know that on W the map λ has constant rank 8, so we get the following exact sequence of *vector bundles* on W:

$$0 \longrightarrow \mathcal{K} \longrightarrow F \xrightarrow{\lambda|_W} \mathcal{O}_W \otimes (\bigwedge^3 V/A) \longrightarrow \zeta|_W \longrightarrow 0. \tag{1-5}$$

Here $\mathcal{K}$ is defined to be the kernel of $\lambda|_W$; it has rank 2. Identifying W with its preimage $Z \subset X$, we claim that the following isomorphisms hold:

$$\zeta|_W \cong \mathcal{N}_{Z/X}, \tag{1-6}$$

$$\mathcal{K} \cong \mathcal{N}^\vee_{Z/X}. \tag{1-7}$$

Assuming (1-6) and (1-7) for a moment, the exact sequence in (1-5) gives

$$c_1(\mathcal{N}^\vee_{Z/X}) - c_1(F) - c_1(\mathcal{N}_{Z/X}) = 0,$$

hence

$$2c_1(\mathcal{N}_{Z/X}) = -c_1(F) = \mathcal{O}_Z(6).$$

Since X has trivial canonical class, it follows that

$$2K_Z = 2c_1(\mathcal{N}_{Z/X}) = \mathcal{O}_Z(6),$$

as desired.

So we now turn to the proof of (1-6) and (1-7). Let $p \in Z$; then the covering involution φ fixes p, so φ^* acts on T_pX. This gives a decomposition

$$T_pX = (T_pX)_+ \oplus (T_pX)_-$$

in eigenspaces for φ^*, with eigenvalues ± 1. Since Z is the fixed locus of φ,

$$(T_pX)_+ = T_pZ.$$

On the other hand, since

$$X = \operatorname{Spec}(\mathcal{O}_Y \oplus \zeta),$$

we can identify

$$(T_pX)_- \cong \zeta_{f(p)}.$$

It follows that

$$(\mathcal{N}_{Z/X})_p \cong \zeta_{f(p)};$$

this fiber-wise identification is easily seen to globalize, hence yielding the isomorphism in (1-6).

For the other, we show that $\mathcal{K} \cong \zeta|^\vee_W$. Indeed observe that over W we have

$$\mathcal{K}_v = F_v \cap A \quad \text{and} \quad \zeta_v = \bigwedge^3 V/(F_v + A).$$

The symplectic form identifies $\mathcal{K}_v^\vee$ with the quotient $\bigwedge^3 V/(F_v \cap A)^\perp$, and since both A and F_v are Lagrangian we have

$$(F_v \cap A)^\perp = F_v^\perp + A^\perp = F_v + A,$$

thereby proving the isomorphism (1-7). □

Corollary 1.12. *For $A \in \mathbb{LG}(\bigwedge^3 V)^0$ the surface $Z_A \cong Y_A[2]$ is of general type.*

Plan of the paper. Before turning to the proof of Theorem 1.1, we explain the overall plan. Let $X = X_A$ be a smooth double EPW sextic, and denote its polarization $h = h_A$. The symplectic form gives an isomorphism

$$T_X \cong \Omega_X^1,$$

hence the odd Chern classes vanish. So we only need to consider $c_2(X)$ and $c_4(X)$. Moreover if A is very general in $\mathbb{LG}(\bigwedge^3 V)$, the group $\mathrm{Pic}(X_A)$ is cyclic, generated by h_A, so the second conclusion of Theorem 1.1 follows from the first.

The only relations in cohomology can be in degree 4, 6, or 8. Lemma 2.3 excludes the existence of relations of degree 4, hence we are left with relations in degree 6 or 8; these are listed in Propositions 2.1 and 2.2.

Since h^4, $c_2(X)\cdot h$, $c_2(X)^2$, and $c_4(X)$ are all proportional in cohomology, there must be some distinguished 0-cycle θ on X, such that all these classes are multiples of θ in $CH^4(X)$. We shall define θ as the class of any point on a suitable surface inside X_A; actually it will be easier to work with Y_A and pull back everything to X_A later.

Hence we look for a surface $S \subset X$ such that $CH^2(S)$ is trivial, so each point on S is rationally equivalent to each other. For instance, in the proof of the conjecture in the case where X is the Fano variety of a cubic fourfold, Voisin [2008] used a rational surface. In that case there is a family of Lagrangian surfaces on X, which are simply the Fano varieties of hyperplane sections of the cubic; if the section is singular enough, its Fano variety turns out to be rational.

In our case this construction is a delicate point: the analogue of S is an Enriques surface, but exhibiting it is complicated. The construction that we need is provided in [Ferretti 2012].

So the plan is as follows. In Section 2 we carry out the cohomology computations on X. In Section 3 we use the surface constructed in [Ferretti 2012] to define the class θ. In the rest of the paper we find enough relations in the Chow ring to finish the proof of the main theorem.

2. Cohomology computations

Let $X = X_A$ be a smooth double EPW sextic. In this section we compute the cohomological invariants of X, partly following [O'Grady 2008b]. We shall find

all relations in cohomology between h and the Chern classes of X. In later sections we shall show that these relations hold in the Chow ring.

Let σ be the symplectic form on X. Since the canonical of X is trivial

$$H^{4,0}(X) = H^0(X, \Omega_X^4)$$

is generated by σ^2. Moreover it is known that $H^3(X) = 0$, so we can compute the Euler characteristic

$$\chi(X, \mathbb{O}_X) = h^{0,0}(X) + h^{2,0}(X) + h^{4,0}(X) = 3.$$

The symplectic form on X gives an isomorphism

$$T_X \cong \Omega_X^1,$$

hence the odd Chern classes vanish. The Hirzebruch–Riemann–Roch theorem for X simplifies to

$$3 = \chi(X, \mathbb{O}_X) = \tfrac{1}{240}\left(c_2(X)^2 - \tfrac{1}{3}c_4(X)\right). \tag{2-1}$$

We introduce some more notation. Let us call

$$q \in \mathrm{Sym}^2(H^2(X, \mathbb{Q})^\vee)$$

the Beauville–Bogomolov form of X. Since it is nondegenerate, it allows us to give an identification

$$H^2(X, \mathbb{Q}) \cong H^2(X, \mathbb{Q})^\vee;$$

hence we obtain a dual quadratic form

$$q^\vee \in \mathrm{Sym}^2(H^2(X, \mathbb{Q})).$$

From [Verbitsky 1996, Theorem 1.5] (together with the computation of the Betti numbers in [Göttsche 1990]) we know that the cup product yields an isomorphism between $\mathrm{Sym}^2(H^2(X, \mathbb{Q}))$ and $H^4(X, \mathbb{Q})$, so we can regard $q^\vee$ as an element of $H^4(X, \mathbb{Q})$.

O'Grady [2008b] proves that we have the relation

$$q^\vee = \tfrac{5}{6}c_2(X), \tag{2-2}$$

and that for any $\alpha, \beta \in H^2(X, \mathbb{Q})$ we have

$$q^\vee \cdot \alpha \cdot \beta = 25q(\alpha, \beta). \tag{2-3}$$

We now work out the relations in the cohomology of X. Let

$$h = c_1(f^*\mathbb{O}_Y(1)) \in H^2(X).$$

Proposition 2.1. *In the cohomology ring $H^*(X, \mathbb{Q})$ we have*

$$h^4 = 12, \quad h^2 \cdot c_2(X) = 60, \quad c_2(X)^2 = 828, \quad c_4(X) = 324.$$

Proof. The first and the last relations are easily handled. Indeed,

$$h^4 = 2\deg(Y) = 12.$$

As for the last one we have

$$c_4(X) = \chi(X),$$

and since X is a deformation of $S^{[2]}$, where S is a K3, we have

$$\chi(X) = \chi(S^{[2]}) = 324.$$

By O'Grady's computations (2-3) and (2-2) we also have

$$c_2(X) \cdot h^2 = \tfrac{6}{5} q^\vee \cdot h^2 = \frac{25 \cdot 6}{5} q(h, h) = 60.$$

Finally we can use (2-1) to obtain $c_2(X)^2 = 828$. □

In degree 6 the only possible relation is a linear dependency between h^3 and $c_2(X) \cdot h$, and indeed we have:

Proposition 2.2. *There is a relation*

$$c_2(X) \cdot h = 5h^3$$

$H^6(X, \mathbb{Q})$.

Proof. From O'Grady's relation (2-3) we get

$$6q^\vee \cdot h \cdot \alpha = 6 \cdot 25 q(h, \alpha)$$

for all $\alpha \in H^2(X)$. On the other hand, by polarization of Fujiki's relation we obtain

$$25h^3 \cdot \alpha = 25 \cdot 3 \cdot q(h, h) q(h, \alpha) = 6 \cdot 25 q(h, \alpha).$$

So Poincaré duality implies that

$$25h^3 = 6q^\vee \cdot h$$

modulo torsion, and using (2-2) we get the thesis. □

We can instead exclude relations in degree 4:

Lemma 2.3. *The classes h^2 and $c_2(X)$ are linearly independent inside $H^2(X)$.*

This fact appears inside the proof of [O'Grady 2008b, Claim 3.1], but we add a short proof for completeness.

Proof. We can substitute $c_2(X)$ with its multiple $q^\vee$. Assume that we have a relation

$$h^2 + \lambda q^\vee = 0$$

for some $\lambda \in \mathbb{C}$. Then we get

$$h^2\alpha^2 = -25\lambda q(\alpha, \alpha)$$

for all $\alpha \in H^2(X)$. By polarization of the Fujiki formula we also obtain

$$h^2\alpha^2 = q(\alpha, \alpha)q(h, h) + 2q(h, \alpha)^2.$$

So if $q(\alpha, \alpha) = 0$ we obtain $q(h, \alpha) = 0$. This means that q is degenerate (the quadric defined by q would be contained in a hyperplane of $\mathbb{P}H^2(X)$), which is a contradiction. □

Finally, it will be useful to write out the explicit form of Hirzebruch–Riemann–Roch, using the above computations for the characteristic classes of X. We let

$$\mathcal{O}_X(1) = f^*\mathcal{O}_Y(1).$$

Then $\mathcal{O}_X(n)$ is ample on X, and since K_X is trivial, Kodaira vanishing yields

$$\chi(X, \mathcal{O}_X(n)) = h^0(X, \mathcal{O}_X(n)).$$

The formula of Hirzebruch–Riemann–Roch then reads

$$h^0(X, \mathcal{O}_X(n)) = \frac{h^4}{24}n^4 + \frac{c_2(X)\cdot h^2}{24}n^2 + \chi(\mathcal{O}_X) = \tfrac{1}{2}n^4 + \tfrac{5}{2}n^2 + 3. \qquad (2\text{-}4)$$

3. Definition of the class θ

Let $X = X_A$ be a double EPW sextic, $f : X \to Y$ the double covering. Our first task is to define a class

$$\theta \in CH^4(X)$$

of degree 1. Then we will show that the relations

$$h^4 = 12\theta, \quad h^2c_2(X) = 60\theta, \quad c_2(X)^2 = 828\theta, \quad c_4(X) = 324\theta$$

hold.

It will actually be easier to work on Y, so we should find the relationship between $CH(X)$ and $CH(Y)$.

Remark 3.1. The map $f : X \to Y$ induces a push-forward morphism

$$f_* : CH(X) \to CH(Y),$$

because f is proper (for the construction of Chow rings and morphisms between them see [Fulton 1984, Chapter 1]). On the other hand f^* is usually defined for flat

maps with fibers of constant dimension, or when the target is smooth, and neither is the case.

Following [Fulton 1984, Example 1.7.6] we can define f^* in our situation. Indeed, Fulton shows that if

$$Y = X/G$$

is the quotient of X by the action of a finite group G, we have a canonical isomorphism

$$CH(Y)_{\mathbb{Q}} \cong CH(X)^G_{\mathbb{Q}},$$

where as usual $CH(Y)_{\mathbb{Q}} = CH(Y) \otimes \mathbb{Q}$. So if f is the quotient map we can define f^* by the composition

$$CH(Y)_{\mathbb{Q}} \xrightarrow{\cong} CH(X)^G_{\mathbb{Q}} \hookrightarrow CH(X)_{\mathbb{Q}}.$$

Fulton also shows that the composition

$$CH(Y)_{\mathbb{Q}} \xrightarrow{f^*} CH(X)_{\mathbb{Q}} \xrightarrow{f_*} CH(Y)_{\mathbb{Q}}$$

is the multiplication map by $\sharp G$.

In our situation $G = \langle \varphi \rangle$, where φ is the covering involution, and the composition above is multiplication by 2.

Moreover we observe that

$$CH(Y)_{\mathbb{Q}} \cong CH(X)^{\mathbb{Z}/2\mathbb{Z}}_{\mathbb{Q}}$$

is a subring of $CH(X)_{\mathbb{Q}}$, so we can multiply cycle classes on Y even if it is singular.

We now recall the main result in [Ferretti 2012]. There we produce a surface $S \subset Y$ such that $CH^2(S) = \mathbb{Z}$; this will be the starting point for our investigation of $CH^*(Y)$. Let us fix a Lagrangian subspace $A \in \mathbb{LG}(\bigwedge^3 V)$. If B is another Lagrangian subspace such that

$$\dim(A \cap B) = 9, \tag{3-1}$$

it follows from the Grassmann formula that $Y_B[2] \subset Y_A$.

Theorem [Ferretti 2012]. *Let A be a Lagrangian subspace in $\mathbb{LG}(\bigwedge^3 V)^0$. Then there exists another Lagrangian subspace B such that $\dim(A \cap B) \geq 9$, and such that $S := Y_B[2]$ is either a (singular) Enriques surface, or a degeneration of such surfaces. In either case we have*

$$CH^2(S) \cong \mathbb{Z}.$$

Moreover we have the relation

$$[S] = \big[Y_A[2]\big] \tag{3-2}$$

in $CH^2(X_A)$.

So for such a choice of B we know that all points on $S = Y_B[2]$ are rationally equivalent. We define $\bar{\theta}$ as the class of any such point. We need to do some checks in order to show that this is actually well-defined. We also define

$$\theta = \tfrac{1}{2} f^*(\bar{\theta}) \in CH^4(X_A)_{\mathbb{Q}}.$$

Lemma 3.2. *Let* $B, B' \in \mathbb{LG}(\bigwedge^3 V)$ *such that* (3-1) *holds. Then*

$$Y_B[2] \cap Y_{B'}[2] \neq \varnothing. \tag{3-3}$$

Proof. It is enough to show that

$$Y_B[2] \cdot Y_{B'}[2] \neq 0$$

in $CH^*(Y_A)$. Thanks to (3-2) it will be enough to prove that

$$Y_A[2]^2 \neq 0.$$

By the definition of the ring structure on $CH^*(Y_A)$ we need to prove that

$$Z_A^2 \neq 0 \quad \text{in } CH^*(X_A).$$

But actually $Z_A^2 \neq 0$ already in cohomology. Indeed, using the fact that Z_A is Lagrangian, we have

$$Z_A^2 = c_2(\mathcal{N}_{Z_A/X_A}) = c_2(\Omega^1_{Z_A}) = c_2(Z_A) = \chi_{\text{top}}(Z_A) = 192$$

by Proposition 1.9. □

By the previous lemma we see that the class of $\bar{\theta} \in CH^4(Y_A)$ is actually independent of the chosen Lagrangian subspace B such that (3-1) holds.

4. Relations coming from vector bundle morphisms

In this section we omit A from the notation, for clarity. Fix a Lagrangian subspace $A \in \mathbb{LG}(\bigwedge^3 V)^0$ and denote $X = X_A$, $Y = Y_A$, and $Z = Z_A$. We shall exhibit a number of relations coming from exact sequences of sheaves on X and Y.

Lemma 4.1. *The following relation holds in* $CH(X)$:

$$3Z = 15h^2 - c_2(X).$$

Proof. We consider f as a map $X \to \mathbb{P}^5$, so it induces a morphism of vector bundles over X:

$$df : T_X \to f^* T_{\mathbb{P}^5}.$$

We notice that df is injective outside Z, so we can see Z as a degeneracy locus for this morphism. We then apply the Thom–Porteous formula in the form stated in [Fulton 1984, §14.4]. In their notation we have $e = 4$, $f = 5$, and $k = 3$.

This yields a cycle class

$$\mathbb{D}_3(df) \in CH^2(Z)$$

whose support is Z, and such that the image of $\mathbb{D}_3(df)$ in $CH^2(X)$ is

$$\Delta_2^{(1)}(c(f^*T_{\mathbb{P}^5} - T_X)) = c_2(f^*T_{\mathbb{P}^5} - T_X).$$

Here the total Chern class

$$c(f^*T_{\mathbb{P}^5} - T_X)$$

is defined formally in such a way that Whitney's formula holds, that is,

$$c(T_X) \cdot c(f^*T_{\mathbb{P}^5} - T_X) = c(f^*T_{\mathbb{P}^5}).$$

From the last equation and the fact that $c_1(T_X) = 0$ (since X is symplectic) we can obtain

$$c_2(f^*T_{\mathbb{P}^5} - T_X) = f^*c_2(T_{\mathbb{P}^5}) - c_2(T_X) = 15h^2 - c_2(X).$$

Since $\mathbb{D}_3(df)$ has support on Z, which is irreducible, we find that

$$kZ = 15h^2 - c_2(X) \tag{4-1}$$

for some $k \in \mathbb{Z}$. To find the right k, we observe that, again by [Fulton 1984, Theorem 14.4(c)], we have

$$\mathbb{D}_3(df) = [D_3(df)],$$

where $D_3(df)$ is the degeneracy locus of df. In other words $D_3(df)$ is just Z, with the scheme structure given by the vanishing of all 4×4 minors of df.

The map

$$f : X \to Y \subset \mathbb{P}^5$$

has, in suitable analytic coordinates around a point of Z, the local form

$$f(x, y, z, t) \underset{\text{loc}}{=} (x^2, xy, y^2, z, t).$$

The differential of f is then

$$df \underset{\text{loc}}{=} \begin{pmatrix} 2x & 0 & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 2y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

equating to 0 the determinants of its 3×3 minors yields

$$D_3(df) \underset{\text{loc}}{=} V(x^2, xy, y^2).$$

So we see that $D_3(df)$ has multiplicity 3 at each point of Z, hence $k = 3$.

Alternatively we could multiply (4-1) by h^2 to find

$$kZ \cdot h^2 = 15h^4 - c_2(X) \cdot h^2.$$

If we look at this relation in cohomology it becomes, thanks to Proposition 2.1,

$$40k = 15 \cdot 12 - 60,$$

so $k = 3$. □

We take a closer look at the differential of

$$f : X \to \mathbb{P}^5.$$

As a map of vector bundles, this is not injective exactly on Z. Hence it is always injective on stalks; in other words

$$df : T_X \to f^* T_{\mathbb{P}^5}$$

is an injective map *of sheaves*. Let $\mathcal{R}$ denote its cokernel; this is locally free of rank 1 outside Z. So we have the exact sequence

$$0 \longrightarrow T_X \longrightarrow f^* T_{\mathbb{P}^5} \longrightarrow \mathcal{R} \longrightarrow 0. \tag{4-2}$$

We now dualize it applying $\mathcal{H}om(\cdot, \mathcal{O}_X)$. We remark that

$$\mathcal{H}om(\mathcal{R}, \mathcal{O}_X)$$

is torsion-free, of rank one, and one can check in local coordinates that it is a line bundle. By (4-2) we get $c_1(\mathcal{R}) = 6h$, hence

$$\mathcal{H}om(\mathcal{R}, \mathcal{O}_X) \cong \mathcal{O}_X(-6).$$

Then we note that

$$\mathcal{E}xt^1(f^*(T_{\mathbb{P}^5}), \mathcal{O}_X) = 0,$$

because both sheaves are locally free. So if we let

$$\mathcal{Q} = \mathcal{E}xt^1(\mathcal{R}, \mathcal{O}_X),$$

the dual of (4-2) becomes

$$0 \longrightarrow \mathcal{O}_X(-6) \longrightarrow f^*(\Omega^1_{\mathbb{P}^5}) \xrightarrow{df^T} \Omega^1_X \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{4-3}$$

We remark that $\mathcal{Q}$ is set-theoretically supported on Z, because both $\mathcal{R}$ and $\mathcal{O}_X$ are locally free outside Z. Actually the schematic support of $\mathcal{Q}$ is $2Z$, that is, the subscheme of X defined by the ideal $\mathcal{I}_Z^2$. This follows from:

Lemma 4.2. *Let $\mathcal{Q}$ be as above; then* $\mathrm{Ann}(\mathcal{Q}) = \mathcal{I}_Z^2$.

Proof. We only need to prove this locally. As in the proof of Lemma 4.1 we can choose local coordinates on X such that

$$f(x, y, z, t) \underset{\text{loc}}{=} (x^2, xy, y^2, z, t);$$

then df^T has the matrix

$$df^T \underset{\text{loc}}{=} \begin{pmatrix} 2x & y & 0 & 0 & 0 \\ 0 & x & 2y & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

hence we have the presentation

$$\mathcal{Q} \underset{\text{loc}}{=} \frac{\langle dx, dy \rangle}{\langle x\,dx, x\,dy + y\,dx, y\,dy \rangle}.$$

A given $h(x, y) \in \mathbb{C}[x, y]$ then annihilates $\mathcal{Q}$ if and only if both $h\,dx$ and $h\,dy$ belong to the $k[x, y]$-module generated by $x\,dx$, $x\,dy + y\,dx$, and $y\,dy$.

Let us make this more explicit. Assume that

$$h(x, y)dx = a(x, y)x\,dx + b(x, y) \cdot (x\,dy + y\,dx) + c(x, y)y\,dy.$$

This yields

$$\begin{aligned} h(x, y) &= xa(x, y) + yb(x, y), \\ 0 &= xb(x, y) + yc(x, y). \end{aligned}$$

The second equation implies $b(x, y) = yb'(x, y)$, so the first becomes

$$h(x, y) = xa(x, y) + y^2 b'(x, y).$$

If h can be written this way, then we can choose c so that the second condition is satisfied. In short

$$h(x, y)\,dx \in \langle x\,dx, x\,dy + y\,dx, y\,dy \rangle_{k[x,y]}$$

if and only if $h \in (x, y^2)$.

We have the symmetric condition for $h(x, y)\,dy$, so we conclude that $h \in \mathrm{Ann}(\mathcal{Q})$ if and only if

$$h \in (x, y^2) \cap (x^2, y) = (x^2, xy, y^2).$$

The last equality between ideals can be proved, for instance, by the remark that both $(x, y^2) \cap (x^2, y)$ and (x^2, xy, y^2) consist of the polynomials h such that

$$h(0,0) = \frac{\partial h}{\partial x}(0,0) = \frac{\partial h}{\partial y}(0,0) = 0.$$

Finally, (x^2, xy, y^2) is exactly the square of the ideal (x, y) which locally defines Z. □

We now produce another exact sequence involving $\mathfrak{Q}$. Let

$$i : Z \hookrightarrow X$$

denote the inclusion. Recall that we have a canonical identification

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \cong i_* \mathcal{N}_{Z/X}^\vee : \tag{4-4}$$

locally the function g vanishing on Z corresponds to the normal covector dg. Consider the natural projection

$$\pi : \Omega_X^1|_Z \to \mathcal{N}_{Z/X}^\vee;$$

we see this as a map on X:

$$\pi : \Omega_X^1 \to \mathcal{I}_Z/\mathcal{I}_Z^2.$$

Lemma 4.3. *We have* $\pi \circ df^T = 0$.

Proof. We keep the notation of the proof of Lemma 4.2. We need only to verify the thesis on Z. The image of df^T is generated by

$$x\,dx,\ x\,dy + y\,dx,\ y\,dy,\ dz,\ dt.$$

The first three elements vanish on Z, while the latter two are in the kernel of π. □

The above lemma and the exact sequence in (4-3) provide us a surjective map

$$\alpha : \mathfrak{Q} \to i_*(\mathcal{N}_{Z/X}^\vee).$$

Lemma 4.4. *The kernel of* α *is* $i_*(\det T_Z)$.

Proof. We can see this explicitly in local coordinates. Keeping the notation of the above proofs, $\mathfrak{Q}$ is locally generated, on Z, by dx, dy, and $x\,dy = -y\,dx$. The conormal bundle $\mathcal{N}_{Z/X}^\vee$ is generated by dx and dy, and α is the obvious projection.

The kernel of α is then generated by $x\,dy$. Under the identification in (4-4) this corresponds to the generator $dx \wedge dy$ of $\bigwedge^2 \mathcal{N}_{Z/X}^\vee$.

So

$$\ker \alpha = i_*(\det \mathcal{N}_{Z/X}^\vee) \cong i_*(\det T_Z),$$

since Z is Lagrangian. □

Thanks to the lemma we get the exact sequence we are looking for:

$$0 \longrightarrow i_*(\det T_Z) \longrightarrow \mathcal{Q} \longrightarrow i_* T_Z \longrightarrow 0. \tag{4-5}$$

We can now find new relations in the Chow ring of X.

Proposition 4.5. *In $CH(X)_{\mathbb{Q}}$ we have*

$$c_2(X) \cdot h = 5h^3$$

and $c_4(X)$ is a linear combination of h^4, $c_2(X) \cdot h^2$, and $c_2(X)^2$.

Proof. This is just a matter of putting together the relations that come from the exact sequences (4-3) and (4-5).

We start from (4-3), which yields

$$(1-6h) \cdot (1 + c_2(X) + c_4(X)) = (1-h)^6 \cdot (1 + c_1(\mathcal{Q}) + c_2(\mathcal{Q}) + c_3(\mathcal{Q}) + c_4(\mathcal{Q})).$$

Comparing the terms in degree up to 2 we get:

$$c_1(\mathcal{Q}) = 0, \quad c_2(\mathcal{Q}) = c_2(X) - 15h^2 = -3Z, \tag{4-6}$$

where the last equality is Lemma 4.1. Then in degree 3 we have

$$c_3(\mathcal{Q}) = 6h(c_2(\mathcal{Q}) - c_2(X)) + 20h^3 = 6h \cdot (-15h^2) + 20h^3 = -70h^3, \tag{4-7}$$

using the second of (4-6). Finally in degree 4 we get, using (4-6) and (4-7),

$$c_4(X) = 15h^4 + 15h^2 \cdot c_2(\mathcal{Q}) - 6h \cdot c_3(\mathcal{Q}) + c_4(\mathcal{Q}) = 15h^4 - 45h^2 \cdot Z + 420h^4 + c_4(\mathcal{Q}),$$

hence

$$c_4(\mathcal{Q}) = c_4(X) - 435h^4 + 45h^2 \cdot Z. \tag{4-8}$$

Next we look at the relations coming from (4-5). To do this we shall use Grothendieck–Riemann–Roch, which for the closed embedding

$$i : Z \hookrightarrow X$$

takes the form

$$\mathrm{ch}(i_* \mathcal{F}) = i_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(\mathcal{N}_{Z/X})^{-1}),$$

for any $\mathcal{F} \in \mathrm{Coh}(Z)$. This is because in our situation we have

$$R^k i_*(\mathcal{F}) = 0$$

for all such $\mathcal{F}$, thanks to [Hartshorne 1977, Corollary III.11.2].

Using that Z is Lagrangian we have $\mathcal{N}_{Z/X} \cong \Omega^1_Z$, so we can compute

$$\mathrm{td}(\mathcal{N}_{Z/X}) = 1 - \tfrac{1}{2} c_1(Z) + \tfrac{1}{12}(c_1(Z)^2 + c_2(Z)),$$
$$\mathrm{td}(\mathcal{N}_{Z/X})^{-1} = 1 + \tfrac{1}{2} c_1(Z) + \tfrac{1}{6} c_1(Z)^2 - \tfrac{1}{12} c_2(Z).$$

Then we have

$$\begin{aligned}\mathrm{ch}(\det T_Z) &= 1 + c_1(Z) + \tfrac{1}{2}c_1(Z)^2,\\ \mathrm{ch}(T_Z) &= 2 + c_1(Z) + \tfrac{1}{2}\left(c_1(Z)^2 - c_2(Z)\right).\end{aligned}$$

So Grothendieck–Riemann–Roch for these sheaves becomes

$$\begin{aligned}\mathrm{ch}(i_*\det T_Z) &= i_*\big(1 + \tfrac{3}{2}c_1(Z) + \tfrac{7}{6}c_1(Z)^2 - \tfrac{1}{12}c_2(Z)\big),\\ \mathrm{ch}(i_*T_Z) &= i_*\big(2 + 2c_1(Z) + \tfrac{4}{3}c_1(Z)^2 - \tfrac{7}{6}c_2(Z)\big).\end{aligned}$$

Next we use the fact that in $CH(Z)_{\mathbb{Q}}$ we have

$$c_1(Z) = -K_Z = -3i^*(h),$$

thanks to Proposition 1.10. So we obtain

$$\begin{aligned}\mathrm{ch}(i_*\det T_Z) &= Z - \tfrac{9}{2}h\cdot Z + \tfrac{21}{2}h^2\cdot Z - \tfrac{1}{12}Z^2,\\ \mathrm{ch}(i_*T_Z) &= 2Z - 6h\cdot Z + 12h^2\cdot Z - \tfrac{7}{6}Z^2.\end{aligned}$$

We can use this to recover the Chern classes of $i_*(\det T_Z)$ and $i_*(T_Z)$. These are:

$$\begin{aligned}c_1(i_*\det T_Z) &= 0, & c_2(i_*\det T_Z) &= -Z,\\ c_3(i_*\det T_Z) &= -9h\cdot Z, & c_4(i_*\det T_Z) &= Z^2 - 63h^2\cdot Z,\end{aligned}$$

and

$$\begin{aligned}c_1(i_*T_Z) &= 0, & c_2(i_*T_Z) &= -2Z,\\ c_3(i_*T_Z) &= -12h\cdot Z, & c_4(i_*T_Z) &= 9Z^2 - 72h^2\cdot Z.\end{aligned}$$

Finally we use the exact sequence (4-5) to get the Chern classes of $\mathbb{Q}$. The first two are

$$c_1(\mathbb{Q}) = 0, \quad c_2(\mathbb{Q}) = -3Z,$$

in accordance with (4-6). Then we get

$$c_3(\mathbb{Q}) = -21h\cdot Z,$$

and comparing with (4-7) we obtain

$$-3h\cdot Z = -10h^3.$$

Using Lemma 4.1 this is equivalent to

$$c_2(X)\cdot h = 5h^3.$$

Finally we get

$$c_4(\mathbb{Q}) = 12Z^2 - 135h^2\cdot Z;$$

comparing with (4-8) this yields

$$12Z^2 - 135h^2 \cdot Z = c_4(X) - 435h^4 + 45h^2 \cdot Z,$$

and using again Lemma 4.1 to write Z as a rational combination of $c_2(X)$ and h^2, we get the second claim of the thesis. □

5. Conclusion of the proof

First we recall that we have defined the class

$$\theta = \tfrac{1}{2} f^*(\bar{\theta}).$$

Here $\bar{\theta}$ is the class of any point on a surface $S \subset Y_A$. By (3-2) we know that

$$\big[Y_A[2]\big] = [S] \quad \text{in } CH^2(Y_A).$$

We also let $\bar{h} = \mathcal{O}_Y(1)$, so that $h = f^*(\bar{h})$.

Lemma 5.1. *There exists a line $L_0 \subset Y$ which meets $Y_B[2]$.*

Proof. Let V be the union of lines contained in Y.

Step 1: $\dim V \geq 2$. Let $R \subset \mathrm{Gr}(2, V)$ be the locus of lines $\ell \subset Y_A$. We can obtain R as follows. Let

$$Y_A = V(g),$$

where g is a degree-6 polynomial, and let $\mathcal{S}$ be the tautological subbundle on $\mathrm{Gr}(2, V)$, so that $\mathrm{Sym}^6(\mathcal{S}^\vee)$ is the fiber bundle whose fiber at ℓ is the vector space of homogeneous polynomials of degree 6 on ℓ.

Then we can define a section

$$s \in H^0(\mathrm{Gr}(2, V), \mathrm{Sym}^6(\mathcal{S}^\vee))$$

by the condition

$$s(\ell) = g|_\ell.$$

By definition R is the zero locus of s. It follows that

$$\dim R \geq \dim \mathrm{Gr}(2, V) - \mathrm{rk}\, \mathrm{Sym}^6(\mathcal{S}^\vee) = 8 - 7 = 1,$$

provided R is not empty. But we can show that $R \neq \varnothing$ by computing the fundamental class

$$[R] = c_7(\mathrm{Sym}^6(\mathcal{S}^\vee)) = 432 \cdot 134\sigma_{4,3}.$$

Here the notation is that of Schubert calculus, see for instance [Griffiths and Harris 1978, §1.5].

Since $V = \bigcup_{\ell \in R} \ell$ is birational to a $\mathbb{P}^1$-bundle over R, it follows that $\dim V \geq 2$.

Step 2: There exists B' such that $A \cap B = A \cap B'$ and $Y_{B'}[2]$ meets V. Let

$$U = A \cap B.$$

In [Ferretti 2012] we have shown that there exists a divisor $D_U \subset Y_A$ which is covered by the surfaces $[Y_{B'}[2]]$ as B' ranges through the Lagrangian subspaces which satisfy

$$A \cap B' \supset U.$$

In particular D_U has dimension 3; since two varieties of dimensions 2 and 3 in $\mathbb{P}^5$ always meet, it follows that

$$D_U \cap V \neq \varnothing.$$

So there exists a Lagrangian subspace B' such that $B' \cap A = U$ and

$$Y_{B'}[2] \cap V \neq \varnothing.$$

Step 3: B meets V. We lift everything to X_A, which is smooth, so intersection theory applies. Let

$$\widetilde{V}_1 = f^{-1}(V) \quad \text{and} \quad \widetilde{V}_2 = f^{-1}(Y_{B'}[2]).$$

One easily sees that on X

$$\widetilde{V}_1 \cdot \widetilde{V}_2 \neq 0.$$

Since $f^{-1}(Y_B[2])$ and $\widetilde{V}_2$ have the same homology class, it follows that

$$\widetilde{V}_1 \cdot f^{-1}(Y_B[2]) \neq 0;$$

in particular $\widetilde{V}_1$ must meet $f^{-1}(Y_B[2])$, and so

$$V \cap Y_B[2] \neq \varnothing. \qquad \square$$

Using Lemma 5.1 we can start proving that

$$h^4 = 6\theta \quad \text{in } CH(X). \tag{5-1}$$

Indeed let L_0 be any line meeting S and let Λ be any plane containing L_0. Then $\bar{h}^3$ is represented by the intersection

$$\Lambda \cdot Y = L_0 + C,$$

where C is a quintic on Λ. Multiplying by $\bar{h}$ we obtain

$$\bar{h}^4 = L_0 \cdot \bar{h} + C \cdot \bar{h}.$$

We claim that this is represented by a 0-cycle supported on L_0. This is clear for the first addend; for the second we represent $\bar{h}$ by a hyperplane containing L_0 and transverse to Λ. It follows that $C \cdot \bar{h}$ is supported on $C \cap L_0$.

Since L_0 is rational, $CH^1(L_0) \cong \mathbb{Z}$, so $\bar{h}^4$ is rationally equivalent to a multiple of a point of L_0. Finally $L_0 \cap S \neq \varnothing$, so we get

$$\bar{h}^4 = k\bar{\theta} \quad \text{in } CH^4(Y)_{\mathbb{Q}}$$

for some $k \in \mathbb{Q}$.

Pulling back this relation to X and using $f^*(\bar{h}) = h$, $f^*(\bar{\theta}) = 2\theta$ we see that $h^4 = 2k\theta$ in $CH^4(X)_{\mathbb{Q}}$. Since in cohomology we have $h^4 = 12$ we must have $k = 6$, and so (5-1) is proved.

Next we show that

$$h^2 \cdot c_2(X) = 60\bar{\theta}. \tag{5-2}$$

We start from Lemma 4.1; pushing forward that relation we get

$$3\big[Y_A[2]\big] = 15 \cdot 4\bar{h}^2 - f_* c_2(X) \quad \text{in } CH^2(Y). \tag{5-3}$$

Multiplying (5-3) by $\bar{h}^2$ we get

$$\bar{h}^2 \cdot f_* c_2(X) = 60\bar{h}^4 - 3\bar{h}^2 \cdot \big[Y_A[2]\big].$$

We already proved that $\bar{h}^4$ is a multiple of $\bar{\theta}$, and the cycle class

$$\bar{h}^2 \cdot \big[Y_A[2]\big] = \bar{h}^2 \cdot [S]$$

is supported on S, hence it is a rational multiple of $\bar{\theta}$ too.

We conclude that the relation (5-2) holds up to a multiple, that is,

$$\bar{h}^2 \cdot f_* c_2(X) = k\bar{\theta}.$$

As before, we pull back this relation to X in order to make computations in cohomology. We get

$$h^2 \cdot 2c_2(X) = 2k\bar{\theta}.$$

Since in cohomology we have $h^2 \cdot c_2(X) = 60$, we must have $k = 60$, and (5-2) is proved.

In a similar way, we can rewrite (5-3) as

$$f_* c_2(X) = 15\bar{h}^2 - 3\big[Y_A[2]\big]$$

and take squares to write $(f_* c_2(X))^2$ as a combination of $\bar{h}^4$ and a 0-cycle supported on $Y_B[2]$. This shows that $(f_* c_2(X))^2$ is a rational multiple of $\bar{\theta}$.

As usual a cohomology computation yields the precise form

$$c_2(X)^2 = 828\theta.$$

Now we can use Proposition 4.5 to conclude that $c_4(X) = k\theta$, and finally we get $k = 324$ by comparison with the analogous computation in cohomology. This takes care of all relations in degree 8.

The only relation in degree 6 comes from Proposition 2.2, and is

$$c_2(X) \cdot h = 5h^3.$$

We already proved that the same holds in $CH^*(X)$ in Proposition 4.5, so this ends the proof of the main theorem.

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