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On the refined ramification filtrations in the equal characteristic case

Liang Xiao

Let k be a complete discrete valuation field of equal characteristic $p > 0$. Using the tools of p-adic differential modules, we define refined Artin and Swan conductors for a representation of the absolute Galois group G_k with finite local monodromy; this leads to a description of the subquotients of the ramification filtration on G_k. We prove that our definition of the refined Swan conductors coincides with that given by Saito, which uses étale cohomology. We also study its relation with the toroidal variation of Swan conductors.

Introduction

The ramification theory for a complete discrete valuation field k with possibly imperfect residue field κ_k was first studied by K. Kato [1989]; he used étale cohomology and Milnor K-theory to give a detailed description of the ramification of a character of the absolute Galois group G_k, or equivalently of its maximal abelian quotient G_k^{ab}. A. Abbes and T. Saito [2002; 2003] extended Kato's work by providing G_k with the *ramification filtration* $\mathrm{Fil}^a G_k$ and the *log ramification filtration* $\mathrm{Fil}^a_{\log} G_k$ satisfying certain properties. Saito [2009] later defined a natural injective homomorphism

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{Fil}^a_{\log} G_k / \mathrm{Fil}^{a+}_{\log} G_k, \mathbb{F}_p) \to \Omega^1_{\mathbb{O}_k}(\log) \otimes_{\mathbb{O}_k} \pi_k^{-a} \kappa_k^{\mathrm{alg}}$$

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for each $a \in \mathbb{Q}_{>0}$, where $\mathbb{O}_k$ is the ring of integers of k, π_k is a uniformizer, κ_k is the residue field, and $\Omega^1_k(\log)$ is the module of logarithmic differentials; he called it the *refined Swan conductor homomorphism.* This provides some further information about the subquotients for the log ramification filtration on G_k.

Along a different path, G. Christol, B. Dwork, S. Matsuda, Z. Mebhkout, and their collaborators used p-adic differential modules to give an interpretation of the Swan conductors of representations of G_k when the residue field κ_k is perfect. They associated a p-adic differential module over an annulus to any continuous representation of G_k, and proved that the Swan conductor of the representation is related to the radii of convergence of the local solutions for the differential module. K. Kedlaya [2007] generalized this approach to include the case in which the residue field is imperfect, by giving the definitions of Artin conductors and Swan conductors for a representation of G_k. The author [Xiao 2010] verified that this pair of definitions coincide with those naturally associated to the ramification filtration and log ramification filtration of Abbes and Saito [2002; 2003]. An important consequence of this comparison result is the *Hasse–Arf theorem* for the ramification filtration and the log one [Xiao 2010, Theorem 4.4.1], which states that the Artin conductors and Swan conductors are all integers.

In this paper, we give an alternative definition of the refined Swan conductor homomorphism as well as their nonlog counterparts, using p-adic differential modules, and we will compare our definition with that of Saito. Let us describe the basic idea of the definition. In this introduction, we assume for simplicity that κ_k has a finite p-basis $\{\bar{b}_1, \dots, \bar{b}_m\}$. Let K be the fraction field of the Cohen ring of κ_k with respect to $\bar{b}_1, \dots, \bar{b}_m$. Let $B_1, \dots, B_m$ denote the canonical lifts of $\bar{b}_1, \dots, \bar{b}_m$ to K, respectively. Let $A^1_K(\eta_0, 1)$ be the annulus over K with coordinate T and with radii in $(\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. By the aforementioned series of work, one can associate to an irreducible p-adic representation ρ of G_k with finite image a differential module $\mathscr{E}$ over $A^1_K(\eta_0, 1)$ for the differential operators $\partial_0 = \partial/\partial T$ and $\partial_1 = \partial/\partial B_1, \dots, \partial_m = \partial/\partial B_m$. Let $\boldsymbol{\pi} = -p^{1/(p-1)}$ denote a *Dwork pi* and put $K' = K(\boldsymbol{\pi})$. When ρ is of pure ramification break b, that is, when $\rho(\mathrm{Fil}^{b+} G_k)$ is trivial but $\rho(\mathrm{Fil}^{b} G_k)$ is not, the following naïve picture is helpful as a guide to intuition. Suppose that there exists a basis of $\mathscr{E} \otimes_K K'$, with respect to which $\partial_0, \partial_1, \dots, \partial_m$ act per the prescription:

$$\partial_0 = \boldsymbol{\pi} T^{-b-1} N_0, \quad \partial_1 = \boldsymbol{\pi} T^{-b} N_1, \dots, \partial_m = \boldsymbol{\pi} T^{-b} N_m, \tag{0.0.1}$$

where $N_0, \dots, N_m$ are matrices in $\mathbb{O}_{K'}[\![T]\!]$. For each $j \in \{0, \dots, m\}$, we use $\overline{N}_j$ to denote reduction of N_j modulo the ideal $(\boldsymbol{\pi}, T)$; these matrices commute and have coefficients in κ_k. Take a common (generalized) eigenbasis $e_1, \dots, e_d$ for all $\overline{N}_j$; set $\theta_{i,j}$ to be the (generalized) eigenvalue of $\overline{N}_j$ associated to e_i, viewed as an element in κ_k^{alg}. One may then define the multiset of *refined Swan conductors* of ρ

to be

$$\Big\{\pi_k^{-b}\Big(\theta_{i,0}\frac{d\pi_k}{\pi_k}+\theta_{i,1}d\bar{b}_1+\cdots+\theta_{i,m}d\bar{b}_m\Big) : i=1,\ldots,d\Big\}\subset\Omega^1_{\mathbb{O}_k}(\log)\otimes_{\mathbb{O}_k}\pi_k^{-b}\kappa_k^{\mathrm{alg}}.$$

(A multiset is a set where we allow elements to have multiplicity.) Of course, such a nice basis of $\mathscr{E}\otimes_K K'$ over the annulus $A^1_{K'}[\eta_0,1)$ with the described properties might not exist in general. In practice, we need the following two technical arguments to read off the multiset of refined Swan conductors.

(a) The above picture can be better described over a field. Namely, we have the description of the actions of $\partial_0,\ldots,\partial_m$ as in (0.0.1) over the completion of $K(T)$ with respect to the η-Gauss norm for any $\eta\in[\eta_0,1)$. By taking common eigenvalues as explained above, we can define a version of refined Swan conductors, called the *refined radii*, of the differential module at each radius η. We then show that the refined radii, as we vary the radius of the Gauss norm, also vary in a nice way when η is sufficiently close to 1: they form a unique multiset consisting of elements of $\Omega^1_{\mathbb{O}_k}(\log)\otimes_{\mathbb{O}_k}\pi_k^{-b}\kappa_k^{\mathrm{alg}}$, independent of the choice of η. We then just simply define this multiset to be the multiset of *refined Swan conductors* of the representation ρ; this does not require any good matrices representing the actions of ∂_j over the entire annulus.

(b) When the spectral norms of the differential operators are smaller than their operator norms over the base field, the description (0.0.1) requires some modification. Over the completion of $K(T)$ for the η-Gauss norm, we may find a basis such that the matrix for $\partial_j^{p^r}$ with an appropriate $r\in\mathbb{N}$ acts by some nice matrix as in (0.0.1). We then take the common eigenvalues of those matrices and define the refined radii to be the p^r-th roots of these eigenvalues. When trying to prove results in this case, we use a technique called *Frobenius antecedents* developed in [Kedlaya and Xiao 2010], which reduces the question at hand to the case when the spectral norms are bigger than the operator norms.

We can also define the notion of refined Artin conductors using a variant of the definition of the refined Swan conductors, in which the effect of log structure is removed, which amounts to replacing the factor T^{-b-1} by T^{-b} in (0.0.1).

Part of the content in this paper on refined Swan conductors has been already included in the author's thesis [Xiao 2009]. However, we feel the present paper provides a better context for our development of refined Swan conductors. We also fill in some gaps in the thesis.

To compare our definition of refined Swan conductors with Saito's, we proceed as in [Xiao 2010] by introducing the thickening spaces which tie the p-adic differential equations together with the rigid analytic spaces considered by Abbes and Saito. More precisely, we may first realize a finite Galois extension l of k as the corresponding extension of the function fields of a finite étale extension of smooth affine varieties $Y \to X$. We may further assume that both X and Y lift to smooth formal schemes $\boldsymbol{X}$ and $\boldsymbol{Y}$. The differential module associated to a p-adic representation of $\mathrm{Gal}(l/k)$ lives over the a subspace of the tube of X embbeded diagonally in $\boldsymbol{X} \times \boldsymbol{X}$, which is a rigid analytic subspace of the generic fiber of $\boldsymbol{X} \times \boldsymbol{X}$ and is called the *thickening space*. We carefully study the construction of the differential module and compare that with Saito's description of the special fiber of the formal scheme $\boldsymbol{Y}$. The core of the comparison result is to identify the data defining an Artin–Schreier cover of $\mathbb{A}^m_{\kappa_k}$ with the data coming from the associated Dwork isocrystals as a differential module.

We also remark that when k is an n-dimensional higher local field of characteristic $p > 0$, the refined conductors induce a ramification filtration on G_k indexed by $\mathbb{Q}^n$ with lexicographic order. This is expected to be compatible with certain filtration on the Milnor K-groups via Kato's class field theory.

Finally, we study the relation of the refined Swan conductors with the variation of intrinsic radii (certain form of Swan conductors) over a polyannulus. We prove that the valuations of the refined Swan conductors at a vertex of the polygon associated to the polyannulus encode some information about the slopes of the log-affine functions of the intrinsic radii at that vertex. For the precise statement, we refer to Proposition 4.3.13.

Plan of the paper. Section 1 is devoted to developing the theory of refined radii, the analog of refined conductors over a complete nonarchimedean field. In the first two subsections, we set up notation and recall some basic results on differential modules from [Kedlaya and Xiao 2010]. We define the refined radii in Section 1.3 and prove a decomposition result (Theorem 1.3.26) that separates pieces with different refined radii in a differential module. In Section 1.4 we consider the case where we allow multiple derivations to interact. In Section 1.5 we study how the refined radii vary on an annulus or a disc, when the radii are log-affine functions. We then define the refined conductors for solvable differential modules over an annulus in Section 1.6.

In Section 2 we apply the theory of refined conductors for solvable differential modules to define refined conductors for Galois representations. In the first two subsections we recall the construction of differential modules following [Kedlaya 2007], and deduce some basic properties. In Section 2.3 we define the refined conductor homomorphism. Section 2.4 briefly discusses an application to higher local fields.

In Section 3 we compare our definition with that of Saito, which is reviewed in Section 3.1. In Section 3.2 we realize the extension of fields as a finite étale cover of varieties and lift them to rigid analytic spaces over K. In Section 3.3 we do a crucial calculation on the differential module structure of Dwork isocrystals to determine their refined radii; this calculation forms the heart of our proof of the comparison theorem. We wrap up Section 3 with a proof of the comparison Theorem 3.4.1 in Section 3.4.

In Section 4 we focus on the interplay of refined Swan conductors with the toroidal variation of Swan conductors. A few technical lemmas are discussed in Section 4.2, and the main theorems are proved in Section 4.3.

1. Theory of differential modules

Our systematic study of differential modules proceeds in two stages: first over a complete nonarchimedean field, and then over an annulus over a complete nonarchimedean field. In the former case, the spectral norm, or equivalently *the radius of convergence*, of the differential operator is a very important invariant; when the differential module has pure radii, we will focus on certain secondary information of the differential module, called the *refined radii*. In the latter case, it was proved in [Kedlaya and Xiao 2010] that the radii of convergence of a differential module over an annulus give rise to piecewise log-affine functions as one varies the radii on the annulus; we will again focus on the secondary data: the refined radii. In the case when the aforementioned piecewise log-affine functions are in fact log-affine, we prove that the multisets of refined radii of the differential module at all radii are the *same*, if we naturally identify the spaces where these refined radii live.

1.1. ***Setup.*** This subsection is mainly to explain our convention on notations; however, the commutative algebra Lemma 1.1.10 will become a very useful tool later as explained in Remark 1.1.11.

Notation 1.1.1. By a *multiset* S, we mean a set where we allow elements to have multiplicity. For $s \in S$, the *multiplicity* of s in S is denoted by $\mathrm{multi}_s(S)$. When S consists of a single element (with multiplicity), we call it *pure*.

Notation 1.1.2. For any field K that will be considered in this paper, K^{alg} will denote a fixed algebraic closure. We let K^{sep} denote the separable closure of K inside K^{alg}. Set $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. For a finite Galois extension L/K (inside K^{sep}), we denote its Galois group by $G_{L/K} = \mathrm{Gal}(L/K)$.

For $e \in \mathbb{N}$, we use μ_e to denote the set of e-th roots of unity in K^{alg}.

Notation 1.1.3. By a *nonarchimedean* field, we mean a field K equipped with a nonarchimedean norm $|\cdot| = |\cdot|_K : K^\times \to \mathbb{R}_+^\times$. A subring of K (with the induced norm and topology) is called a *nonarchimedean ring*.

For a nonarchimedean field K, denote the ring of integers of K by

$$\mathbb{O}_K = \{x \in K : |x| \leq 1\}$$

and the maximal ideal of $\mathbb{O}_K$ by $\mathfrak{m}_K = \{x \in K : |x| < 1\}$; denote the residue field of K by $\kappa_K = \mathbb{O}_K/\mathfrak{m}_K$. We reserve the letter p for the characteristic of κ_K. If char $\kappa_K = p > 0$ and char $K = 0$, we normalize the norm on K so that $|p| = 1/p$. For an element $a \in \mathbb{O}_K$, we denote its image in κ_K under the reduction map by $\bar{a}$. In case K is discretely valued, let π_K denote a uniformizer of $\mathbb{O}_K$ and let $v_K(\cdot)$ be the corresponding valuation on K, normalized so that $v_K(\pi_K) = 1$.

For a nonarchimedean field K and $s \in \mathbb{R}$, we set

$$\mathfrak{m}_K^{(s)} = \{x \in K : |x| \le \mathrm{e}^{-s}\}, \quad \mathfrak{m}_K^{(s)+} = \{x \in K : |x| < \mathrm{e}^{-s}\}, \quad \kappa_K^{(s)} = \mathfrak{m}_K^{(s)}/\mathfrak{m}_K^{(s)+}.$$

If $s \in -\log|K^\times|$, there exists a noncanonical isomorphism $\kappa_K \simeq \kappa_K^{(s)}$. For $a \in K$ with $|a| \le \mathrm{e}^{-s}$, we sometimes denote its image in $\kappa_K^{(s)}$ by $\bar{a}^{(s)}$. In particular, $\kappa_K^{(0)} = \kappa_K$ and $\bar{a}^{(0)} = \bar{a}$ if $v(a) \ge 0$.

Notation 1.1.4. Let J be an index set. We use e_J to denote a tuple $(e_j)_{j\in J}$. For another tuple u_J, set $u_J^{e_J} = \prod_{j\in J} u_j^{e_j}$, if all but finitely many of the e_j are equal to 0. We also use $\sum_{e_J=0}^{n}$ to denote the sum over $e_j \in \{0, 1, \dots, n\}$ for each $j \in J$ provided $e_j \ne 0$ for only finitely many j; for notational simplicity, we may suppress the range of the summation when it is clear. If J is finite, put

$$|e_J| = \sum_{j\in J} |e_j| \quad \text{and} \quad (e_J)! = \prod_{j\in J} (e_j!).$$

Convention 1.1.5. Throughout this paper, all derivations on topological modules will be assumed to be continuous; in particular, $\Omega^1_{R/S}$ will denote the module of continuous differentials on the (topological) ring R relative to the (topological) base ring S; we may suppress S from the notation when $S = \mathbb{F}_p$, $\mathbb{Z}$ or $\mathbb{Z}_p$. Moreover, all derivations on nonarchimedean rings will be assumed to be bounded (that is, to have bounded operator norms). All connections considered will be assumed to be integrable.

Notation 1.1.6. For a matrix $A = (A_{ij})$ with coefficients in a nonarchimedean ring, we use $|A|$ to denote the supremum among the norms of the entries A_{ij} of A.

Hypothesis 1.1.7. For the rest of this subsection, we assume that K is a complete nonarchimedean field.

Notation 1.1.8. Let $I \subset [0, +\infty)$ be an interval and let $n \in \mathbb{N}$. Let

$$A_K^n(I) = \{(x_1, \dots, x_n) \in K^{\mathrm{alg}} : |x_i| \in I \text{ for } i = 1, \dots, n\}$$

denote the polyannulus of dimension n with radii in I. (We do not impose any rationality condition on the endpoints of I, so this space should be viewed as an analytic space in the sense of Berkovich [1990].) If I is written explicitly in terms of its endpoints (e.g., $[\alpha, \beta]$), we suppress the parentheses around I (e.g., $A_K^n[\alpha, \beta]$).

Notation 1.1.9. Let $0 < \alpha \leq \beta < +\infty$. We put

$$K\langle \alpha/t, t/\beta \rangle = \left\{ \sum_{n\in\mathbb{Z}} a_n t^n \;:\; |a_n|\eta^n \to 0 \text{ as } n \to \pm\infty,\ \text{for any } \eta \in [\alpha, \beta] \right\},$$

$$K\langle \alpha/t, t/\beta \}\!\} = \left\{ \sum_{n\in\mathbb{Z}} a_n t^n \;:\; |a_n|\eta^n \to 0 \text{ as } n \to \pm\infty,\ \text{for any } \eta \in [\alpha, \beta) \right\},$$

$$K\{\!\{\alpha/t, t/\beta]\!]_0 = \left\{ \sum_{n\in\mathbb{Z}} a_n t^n \;:\; |a_n|\eta^n \to 0 \text{ and } |a_n|\beta^n \text{ is bounded as } n \to \pm\infty,\ \text{for any } \eta \in (\alpha, \beta) \right\}.$$

$$K\langle t/\beta \rangle = \left\{ \sum_{n=0}^{\infty} a_n t^n \;:\; |a_n|\beta^n \to 0 \text{ as } n \to +\infty \right\},$$

$$K\{\!\{t/\beta\}\!\} = \left\{ \sum_{n=0}^{\infty} a_n t^n \;:\; |a_n|\eta^n \to 0 \text{ as } n \to +\infty,\ \text{for any } \eta \in [0, \beta) \right\},$$

$$K[\![t/\beta]\!]_0 = \left\{ \sum_{n=0}^{\infty} a_n t^n \;:\; |a_n|\beta^n \text{ is bounded as } n \to \infty \right\}.$$

For $I = \{1, \dots, n\}$ and a nonarchimedean ring R, we use $R\langle u_I \rangle$ to denote the Tate algebra, consisting of formal power series $\sum_{e_I \geq 0} a_{e_I} u_I^{e_I}$ with $a_{e_I} \in R$ and $|a_{e_I}| \to 0$ as $|e_I| \to +\infty$. For $(\eta_i)_{i\in I} \in (0, +\infty)^n$, the η_I*-Gauss norm* on the polynomial ring $R[t_I]$ is the norm $|\cdot|_{\eta_I}$ given by

$$\left| \sum_{e_I} a_{e_I} t_I^{e_I} \right|_{\eta_I} = \max_{e_I} \{ |a_{e_I}| \cdot \eta_I^{e_I} \};$$

this norm extends uniquely to multiplicative norms on $\mathrm{Frac}(R[t_I])$, and on $R\langle t_I \rangle$ in case $|\eta_i| \leq 1$ for any $i \in I$.

For $\eta \in [\alpha, \beta]$, the η-Gauss norm on $K[t]$ extends to multiplicative norms on $K\langle \alpha/t, t/\beta \rangle$ and $K[\![t/\beta]\!]_0$, on $K\langle \alpha/t, t/\beta \}\!\}$ in case $\eta \neq \beta$, and on $K\{\!\{\alpha/t, t/\beta]\!]_0$ in case $\eta \neq \alpha$.

We record here a lemma in commutative algebra which will be frequently used (implicitly) when gluing decompositions.

Lemma 1.1.10. *Let*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ T & \longrightarrow & U \end{array}$$

be a commuting diagram of inclusions of integral domains, such that the intersection $S \cap T$ within U is equal to R. Let M be a finite locally free R-module. Then the intersection of $M \otimes_R S$ and $M \otimes_R T$ within $M \otimes_R U$ is equal to M.

Proof. See [Kedlaya and Xiao 2010, Lemma 2.3.1]. □

Remark 1.1.11. We explain how this lemma is used in this paper. We often apply this lemma to the R-module $\mathrm{End}(M)$ over R for a differential module M. More precisely, we often encounter the situation when we can write both $M \otimes_R S$ and $M \otimes_R T$ as direct sums of two submodules such that both direct sum decompositions, when tensored with U, give the same direct sum decomposition of $M \otimes_R U$. We view the projections constituting the direct sum decompositions as elements in $\mathrm{End}(M) \otimes_R S$, $\mathrm{End}(M) \otimes_R T$, and $\mathrm{End}(M) \otimes_R U$, respectively. By Lemma 1.1.10, we see that the projections above are actually the images of one element of $\mathrm{End}(M)$ under the natural maps; this element defines a direct sum decomposition of M which when tensored with S or T yields the given direct sum decomposition of $M \otimes_R S$ or $M \otimes_R T$, respectively. In other words, we can "glue" the direct sum decompositions of $M \otimes_R S$ and of $M \otimes_R T$ along $M \otimes_R U$ to get a direct sum decomposition of M (over R).

1.2. ***Differential modules and radii of convergence.*** The starting point of the theory of nonarchimedean differential modules is to understand differential modules over a nonarchimedean field. One of the important tools is the *Newton polygon* associated to a *cyclic vector*, which gives much numerical information if the spectral norm of the differential operator is strictly bigger than the operator norm on the base field. To extend interesting results across the threshold imposed by the operator norm mentioned above, we restrict ourselves to the case when the differential operator is of *rational type*, that is, its metric properties resemble d/dX acting on the completion of $\mathbb{Q}_p(X)$ with respect to the 1-Gauss norm; in this case, we may entirely remove the restriction on spectral norms by considering the *Frobenius antecedents* of the differential modules.

Definition 1.2.1. Let K be a differential ring, that is, a ring equipped with a derivation ∂. Let $K\{T\}$ denote the (noncommutative) ring of twisted polynomials over K [Ore 1933]; its elements are finite formal sums $\sum_{i\geq 0} a_i T^i$ with $a_i \in K$, multiplied according to the rule $Ta = aT + \partial(a)$ for $a \in K$.

A ∂-*differential module* over K is a finite projective K-module V equipped with an action of ∂ (subject to the Leibniz rule); any ∂-differential module over K inherits a left action of $K\{T\}$ where T acts via ∂. The *rank* of V is the rank of V as a K-module. The module dual $V^\vee = \mathrm{Hom}_K(V, K)$ of V may be viewed as a ∂-differential module by setting $(\partial f)(\boldsymbol{v}) = \partial(f(\boldsymbol{v})) - f(\partial(\boldsymbol{v}))$. We say V is *free* if V is free as a module over K. We say V is *trivial* if it is isomorphic to $K^{\oplus d}$ for some $d \in \mathbb{N}$ as a ∂-differential module.

For a ∂-differential module V free of rank d over K, an element $\boldsymbol{v} \in V$ is called a *cyclic vector* if $\boldsymbol{v}, \partial\boldsymbol{v}, \dots, \partial^{d-1}\boldsymbol{v}$ form a basis of V as a K-module. A cyclic vector defines an isomorphism $V \simeq K\{T\}/K\{T\}P$ of ∂-differential modules, where $P \in K\{T\}$ is some monic twisted polynomial of degree d, and the ∂-action on

$K\{T\}/K\{T\}P$ is the left multiplication by T. If K is a differential field of characteristic 0, V always has a cyclic vector; see [Dwork et al. 1994, Theorem III.4.2; Kedlaya 2010, Theorem 5.4.2].

For a ∂-differential module V, we put $H^0_\partial(V) = \operatorname{Ker}\partial$.

Hypothesis 1.2.2. For the rest of this subsection, we assume that K is a complete nonarchimedean field of characteristic zero, equipped with a derivation ∂ with operator norm $|\partial|_K < \infty$, and that V is a nonzero ∂-differential module over K.

Definition 1.2.3. Let p denote the residual characteristic of K; we conventionally set

$$\omega = \begin{cases} 1 & \text{if } p = 0, \\ p^{-1/(p-1)} & \text{if } p > 0. \end{cases}$$

The *spectral norm of* ∂ *on* V is defined to be $|\partial|_{\mathrm{sp},V} = \lim_{n\to\infty} |\partial^n|_V^{1/n}$ for any fixed K-compatible norm $|\cdot|_V$ on V. Define the *generic* ∂*-radius* of V to be $R_\partial(V) = \omega|\partial|^{-1}_{\mathrm{sp},V}$; note that $R_\partial(V) > 0$. Let $V_1, \dots, V_d$ be the Jordan–Hölder constituents of V as a $K\{T\}$-module. We define the multiset $\mathfrak{R}_\partial(V)$ of *(extrinsic) subsidiary* ∂*-radii* of V to be the collection of $R_\partial(V_i)$ with multiplicity $\dim V_i$ for $i = 1, \dots, d$. Let $R_\partial(V; 1) \le \cdots \le R_\partial(V; \dim V)$ denote the elements of $\mathfrak{R}_\partial(V)$ in nondecreasing order. We say that V has *pure* ∂*-radii* if $\mathfrak{R}_\partial(V)$ is pure as a multiset; in other words, it consists of $\dim V$ copies of $R_\partial(V)$.

Definition 1.2.4. Let R be a complete K-algebra. For $\boldsymbol{v} \in V$ and $x \in R$, we define the ∂*-Taylor series* of $\boldsymbol{v}$ with respect to x to be

$$\mathbb{T}(\boldsymbol{v}; \partial; x) = \sum_{n=0}^{\infty} \frac{\partial^n(\boldsymbol{v})}{n!} x^n \in V \otimes_K R, \tag{1.2.5}$$

in case this series converges. When $V = K$, the ∂-Taylor series (1.2.5) with respect to a fixed $x \in R$ gives a homomorphism $K \to R$ of rings, if it converges for all $\boldsymbol{v} \in V = K$. For general V, the ∂-Taylor series (1.2.5) with respect to the same fixed $x \in R$ gives a homomorphism of K-modules $V \to V \otimes_K R$ respecting the aforementioned ring homomorphism, if both homomorphisms converge.

Lemma 1.2.6. *Let V, V_1, and V_2 be nonzero ∂-differential modules over K.*

(a) *If $0 \to V_1 \to V \to V_2 \to 0$ is exact, then we have $\mathfrak{R}_\partial(V) = \mathfrak{R}_\partial(V_1) \cup \mathfrak{R}_\partial(V_2)$.*

(b) *We have $\mathfrak{R}_\partial(V^\vee) = \mathfrak{R}_\partial(V)$.*

(c) *We have $R_\partial(V_1 \otimes V_2) \ge \min\{R_\partial(V_1), R_\partial(V_2)\}$. If V_1 is irreducible and $R_\partial(V_1) < R_\partial(V_2)$, then $V_1 \otimes V_2$ has pure ∂-radius $R_\partial(V_1)$.*

(d) *Let $f : K \to K[\![T/u]\!]_0$ be the homomorphism given by $f(x) = \mathbb{T}(x; \partial; T)$. Then $f^*V = V \otimes_{K,f} K[\![T/u]\!]_0$ is a $\partial_T = \partial/\partial T$-differential module over*

$K[\![T/u]\!]_0$. For $r \in (0, R_\partial(K))$, $R_\partial(V) \geq r$ if and only if f^*V restricts to a trivial ∂_T-differential module over $A^1_K[0, r)$.

Proof. The statements (a)–(c) are [Kedlaya and Xiao 2010, Lemma 1.2.9] and the statement (d) is [ibid., Proposition 1.2.14]. □

Definition 1.2.7. For $P(T) = \sum_i a_i T^i \in K[T]$ or $K\{T\}$ a nonzero (possibly twisted) polynomial, define the *Newton polygon* of P as the lower convex hull of the set $\{(-i, -\log|a_i|)\} \subset \mathbb{R}^2$.

Proposition 1.2.8 (Christol–Dwork). *Suppose that $V \simeq K\{T\}/K\{T\}P$, and let s be the lesser of $-\log|\partial|_K$ and the least slope of the Newton polygon of P. Then $\max\{|\partial|_K, |\partial|_{\mathrm{sp},V}\} = \mathrm{e}^{-s}$. More generally, the multiplicity of any $s' < -\log|\partial|_K$ as a slope of the Newton polygon of P coincides with the multiplicity of $\omega\mathrm{e}^{s'}$ in $\mathfrak{R}_\partial(V)$.*

Proof. This is [Kedlaya 2010, Theorem 6.5.3]. □

Definition 1.2.9. We say a derivation ∂ on K is of *rational type* if there exists $u \in K$ such that the following conditions hold (in this case, we call u a *rational parameter* for ∂):

(i) we have $\partial(u) = 1$ and $|\partial|_K = |u|^{-1}$, and

(ii) for each positive integer n, $|\partial^n/n!|_K \leq |\partial|_K^n$.

If ∂ is of rational type, the inequalities in (ii) are in fact equalities, which yields that $|\partial|_{\mathrm{sp},K} = \omega|\partial|_K$; see [Kedlaya and Xiao 2010, Definition 1.4.1].

Lemma 1.2.10. *Let ∂ be a derivation on K of rational type with u as a rational parameter and let L/K be a finite tamely ramified extension. Then the unique extension of ∂ to L is of rational type with u again as a rational parameter.*

Proof. This is [Kedlaya and Xiao 2010, Lemma 1.4.5]. □

Remark 1.2.11. We sometimes need to replace K by the completion of $K(x)$ with respect to the η-Gauss norm for some $\eta \in \mathbb{R}_{>0}$, where x is transcendental over K and we set $\partial x = 0$. The derivation ∂ is again of rational type when acting on the new field.

Definition 1.2.12. When ∂ is of rational type, it is more convenient to consider ∂-radii with a different normalization, as follows. For V a ∂-differential module, we define the *intrinsic ∂-radius* of V to be $IR_\partial(V) = |\partial|_{\mathrm{sp},K}/|\partial|_{\mathrm{sp},V} = |\partial|_K \cdot R_\partial(V)$. We define the multiset of *intrinsic subsidiary ∂-radii* to be $\mathfrak{IR}_\partial(V) = |\partial|_K \cdot \mathfrak{R}_\partial(V)$. We put $IR_\partial(V; i) = |\partial|_K \cdot R_\partial(V; i)$ for $i = 1, \dots, \dim V$. We say that V has *pure intrinsic ∂-radii* if $\mathfrak{IR}_\partial(V)$ is pure as a multiset.

Hypothesis 1.2.13. For the rest of this subsection, we assume that K is a complete nonarchimedean field of characteristic zero and residual characteristic p, equipped with a derivation ∂ of rational type. We *fix* $u \in K$ a rational parameter of ∂. We also assume $p > 0$ unless otherwise specified.

Construction 1.2.14. We construct the ∂-Frobenius as follows. If K contains a primitive p-th root of unity ζ_p, we may define an isometric action of the group $\mathbb{Z}/p\mathbb{Z}$ on K using ∂-Taylor series:

$$x^{(i)} = \mathbb{T}(x; \partial; (\zeta_p^i - 1)u) \qquad (i \in \mathbb{Z}/p\mathbb{Z},\ x \in K);$$

in particular, $u^{(i)} = \zeta_p^i u$. Let $K^{(\partial)}$ be the fixed subfield of K under this action; in particular, $u^p \in K^{(\partial)}$. Hence, we have a Galois extension $K/K^{(\partial)}$ generated by u with Galois group $\mathbb{Z}/p\mathbb{Z}$. If K does not contain all p-th roots of unity, we may still define $K^{(\partial)}$ by first constructing $(K(\mu_p))^{(\partial)}$ and then applying the Galois descent; in this case, the extension $K/K^{(\partial)}$ may not be Galois.

We call the inclusion $\varphi^{(\partial)*} : K^{(\partial)} \hookrightarrow K$ the *∂-Frobenius morphism* (*homomorphism*). We view $K^{(\partial)}$ as being equipped with the derivation $\partial' = \partial/(pu^{p-1})$; it is a derivation on $K^{(\partial)}$ because a simple calculation shows that $(\partial(x))^{(i)} = \zeta_p^i \partial(x^{(i)})$ for any $x \in K$, yielding that $\partial'(x)$ is invariant under the $\mathbb{Z}/p\mathbb{Z}$-action if $x \in K^{(\partial)}$. By [Kedlaya and Xiao 2010, Lemma 1.4.9], ∂' is of rational type on $K^{(\partial)}$.

We sometimes use $\varphi^{(\partial,n)} : K^{(\partial,n)} \hookrightarrow K$ to denote the *p^n-th ∂-Frobenius* (*homomorphism*) obtained by applying the above construction n times; if K contains a primitive p^n-th root of unity ζ_{p^n}, this is the same as the fixed field for the natural action of $\mathbb{Z}/p^n\mathbb{Z}$ on K given by $x^{(i)} = \mathbb{T}(x; \partial; (\zeta_{p^n}^i - 1)u)$ for $i \in \mathbb{Z}/p^n\mathbb{Z}$.

Remark 1.2.15. We point out that the definitions of ∂-Frobenius and $K^{(\partial)}$ depend on the choice of the rational parameter u.

Lemma 1.2.16. *The residue field $\kappa_{K^{(\partial)}}$ contains κ_K^p.*

Proof. We know that K is generated by u over $K^{(\partial)}$. If $|u| \notin |K^{(\partial)\times}|$, $K^{(\partial)}$ will have the same residue field as K does. If $|u| \in |K^{(\partial)\times}|$, let $x \in K^{(\partial)}$ be an element such that $|x| = |u|$. Then κ_K is generated over $\kappa_{K^{(\partial)}}$ by $\overline{u/x}$, whose p-th power lies in $\kappa_{K^{(\partial)}}$. The statement follows. □

Definition 1.2.17. Given a ∂'-differential module V' over $K^{(\partial)}$, its *∂-Frobenius pullback* is the ∂-differential module $\varphi^{(\partial)*}V' = V' \otimes_{K^{(\partial)}} K$ over K, where

$$\partial(\boldsymbol{v}' \otimes x) = pu^{p-1}\partial'(\boldsymbol{v}') \otimes x + \boldsymbol{v}' \otimes \partial(x) \qquad (\boldsymbol{v}' \in V', x \in K).$$

For a ∂-differential module V over K, we define the *∂-Frobenius descendant* of V to be the $K^{(\partial)}$-module $\varphi_*^{(\partial)}V$ obtained from V by restriction along $\varphi^{(\partial)*} : K^{(\partial)} \to K$ and viewed as a ∂'-differential module over $K^{(\partial)}$ with the action given by $\partial'(v) = \partial(v)/pu^{p-1}$ for any $v \in V$.

Let V be a ∂-differential module over K such that $IR_\partial(V) > p^{-1/(p-1)}$. A *$\partial$-Frobenius antecedent* of V (which always exists as is shown in Lemma 1.2.18(c)) is a ∂'-differential module V' over $K^{(\partial)}$ such that $V \cong \varphi^{(\partial)*}V'$ and $IR_{\partial'}(V') > p^{-p/(p-1)}$.

Lemma 1.2.18. *The ∂-Frobenius pullbacks and descendants have the following properties.*

(a) *For V' a ∂'-differential module over $K^{(\partial)}$, we have*

$$IR_\partial(\varphi^{(\partial)*}V') \geq \min\{IR_{\partial'}(V')^{1/p},\, p\, IR_{\partial'}(V')\}.$$

Moreover, if $IR_{\partial'}(V') \neq p^{-p/(p-1)}$, the above inequality is in fact an equality.

(b) *For V a ∂-differential module over K, there is a canonical isomorphism $\varphi^{(\partial)*}\varphi_*^{(\partial)}V \cong V^{\oplus p}$.*

(c) *For $i = 0, \dots, p-1$, let $W_i^{(\partial)}$ be the ∂'-differential module over $K^{(\partial)}$ with one generator $\boldsymbol{v}$ (which is a proxy of u^i), such that $\partial'(\boldsymbol{v}) = (i/p)u^{-p}\boldsymbol{v}$. Then $IR_{\partial'}(W_i^{(\partial)}) = p^{-p/(p-1)}$ for $i = 1, \dots, p-1$. For any ∂-differential module V over K, we have canonical isomorphisms $\iota_i : (\varphi_*^{(\partial)}V) \otimes W_i^{(\partial)} \cong \varphi_*^{(\partial)}V$ for $i = 0, \dots, p-1$. Moreover, a submodule U of $\varphi_*^{(\partial)}V$ is itself the ∂-Frobenius descendant of a submodule of V if and only if $\iota_i(U \otimes W_i^{(\partial)}) = U$ for $i = 0, \dots, p-1$.*

For V_1 and V_2 ∂-differential modules over K, we have

$$\varphi_*^{(\partial)}V_1 \otimes \varphi_*^{(\partial)}V_2 = \big(\varphi_*^{(\partial)}(V_1 \otimes V_2)\big)^{\oplus p}.$$

For V' a ∂'-differential module over $K^{(\partial)}$, we have

$$\varphi_*^{(\partial)}\varphi^{(\partial)*}V' \cong V' \oplus \bigoplus_{i=1}^{p-1} V' \otimes W_i^{(\partial)}.$$

(d) (*Christol–Dwork*) *Let V be a ∂-differential module over K such that*

$$IR_\partial(V) > p^{-1/(p-1)}.$$

Then there exists a unique ∂-Frobenius antecedent V' of V. Moreover, we have $IR_{\partial'}(V') = IR_\partial(V)^p$.

(e) *Let V be a ∂-differential module over K. Then*

$$\mathfrak{IR}_{\partial'}(\varphi_*^{(\partial)}V) = \bigcup_{r \in \mathfrak{IR}_\partial(V)} \begin{cases} \{r^p, \underbrace{p^{-p/(p-1)}, \dots, p^{-p/(p-1)}}_{p-1 \text{ times}}\} & \text{if } r > p^{-1/(p-1)}, \\ \{\underbrace{p^{-1}r, \dots, p^{-1}r}_{p \text{ times}}\} & \text{if } r \leq p^{-1/(p-1)}. \end{cases}$$

In particular, we have $IR_{\partial'}(\varphi_*^{(\partial)}V) = \min\{p^{-1}IR_\partial(V),\ p^{-p/(p-1)}\}$.

Proof. For (a), see [Kedlaya and Xiao 2010, Lemma 1.4.11 and Corollary 1.4.20]. (b) and (c) are straightforward. For (d), see [Kedlaya 2010, Theorem 10.4.2]. For (e), see [Kedlaya and Xiao 2010, Theorem 1.4.19]. □

Remark 1.2.19. As in [Kedlaya 2010, Theorem 10.4.4], one can form a version of Lemma 1.2.18(d) for differential modules over discs or annuli.

For the following theorem, we do not assume $p > 0$.

Theorem 1.2.20. *Let V be a ∂-differential module over K. Then there exists a unique decomposition of ∂-differential modules*:

$$V = \bigoplus_{r \in (0,1]} V_r,$$

where every subquotient of V_r has pure intrinsic ∂-radii r. Moreover, $V_r = 0$ *if $r \notin |K^\times|^{\mathbb{Q}}$.*

Proof. For the decomposition, see [Kedlaya and Xiao 2010, Theorem 1.4.21]. The rationality of those r such that $V_r \neq 0$ follows from Proposition 1.2.8 when $r < \omega$ and from taking ∂-Frobenius antecedents in the general case. □

Definition 1.2.21. We call $\oplus_{r \in (0,\omega)} V_r$ the *visible part* of V and $\oplus_{r \in [\omega,1]} V_r$ the *nonvisible part* of V. If V consists of only its visible part, we say V has *visible (intrinsic) ∂-radii*; similarly, if V consists of only its nonvisible part, we say V has *nonvisible (intrinsic) ∂-radii*.

Remark 1.2.22. Let V be a ∂-differential module over K with pure intrinsic ∂-radii $IR_\partial(V) > p^{-1/(p-1)}$. By Lemma 1.2.18(d), V has a ∂-Frobenius antecedent V'. By Lemma 1.2.18(c),

$$\varphi_*^{(\partial)} V = \varphi_*^{(\partial)} \varphi^{(\partial)*} V' \cong V' \oplus \left(\bigoplus_{i=1}^{p-1} V' \otimes W_i^{(\partial)} \right).$$

This decomposition coincides with the one obtained by applying Theorem 1.2.20 to $\varphi_*^{(\partial)} V$.

1.3. ***Refined radii.*** When a ∂-differential module V has *pure* ∂-radii, we will define the multiset of *refined ∂-radii*, certain secondary information for the differential module. Similar to the case of ∂-radii, we may canonically write V as a direct sum of ∂-differential submodules such that the multiset of refined ∂-radii for each direct summand consists of elements pairwise-conjugate under the action of $\mathrm{Gal}(K^{\mathrm{alg}}/K)$.

Hypothesis 1.3.1. In this subsection, let K be a complete nonarchimedean field of characteristic zero and residual characteristic p (possibly $p = 0$), equipped with a derivation ∂ of rational type. We fix $u \in K$ a rational parameter for ∂. Unless otherwise specified, we assume that V is a ∂-differential module of rank d over K with *pure* intrinsic ∂-radii $IR_\partial(V)$. Put $s = -\log(\omega R_\partial(V)^{-1}) = -\log |\partial|_{\mathrm{sp},V}$.

Notation 1.3.2. For $P(T) = T^d + a_1 T^{d-1} + \cdots + a_d \in K[T]$ a polynomial whose Newton polygon has pure slope s, the multiset of the *reduced roots* of P consists of the reductions of the roots of P in $\kappa^{(s)}_{K^{\mathrm{alg}}}$, counted with multiplicity. If P is the characteristic polynomial of a matrix $A \in \mathrm{Mat}(\mathfrak{m}^{(s)}_K)$, we call the reduced roots of P *the reduced eigenvalues* of A.

Notation 1.3.3. For $\boldsymbol{b} \in (0, 1]$ (a proxy of $IR_\partial(V)$), we define $\lambda = \lambda(\boldsymbol{b})$ and $r = r(\boldsymbol{b})$ as follows.

(i) When $\boldsymbol{b} < \omega$ (which happens if V has pure visible intrinsic ∂-radii), we let $\lambda(\boldsymbol{b}) = 0$ and $r(\boldsymbol{b}) = 1$.

(ii) When $\boldsymbol{b} \in [\omega, 1)$ and hence $p > 0$ (which happens if V has pure nonvisible ∂-radii), let $\lambda(\boldsymbol{b})$ denote the unique positive integer such that

$$\boldsymbol{b} \in \Big[p^{\frac{-1}{p^{\lambda(\boldsymbol{b})-1}(p-1)}},\, p^{\frac{-1}{p^{\lambda(\boldsymbol{b})}(p-1)}} \Big),$$

and put $r(\boldsymbol{b}) = p^{\lambda(\boldsymbol{b})}$.

(iii) When $\boldsymbol{b} = 1$, we let $\lambda(\boldsymbol{b}) = r(\boldsymbol{b}) = \infty$.

Definition 1.3.4. Let $\boldsymbol{b} \in (0, 1]$. A K-norm $|\cdot|_V$ on V is called ***b**-good* (or simply *good* if $\boldsymbol{b} = IR_\partial(V)$), if it admits an orthogonal (not necessarily orthonormal) basis, and

(i) when $\boldsymbol{b} < \omega$ (which happens when $\boldsymbol{b} = IR_\partial(V)$ for V visible), we have $|\partial|_V \leq \omega(\boldsymbol{b}|u|)^{-1}$,

(ii) when $\boldsymbol{b} \in [\omega, 1)$ and hence $p > 0$ (which happens when $\boldsymbol{b} = IR_\partial(V)$ for V nonvisible), we have

$$\left|\frac{\partial^i}{i!}\right|_V \leq |\partial|_K^i \text{ for } i = 1, \ldots, r-1, \qquad \left|\frac{\partial^r}{r!}\right|_V \leq p^{-1/(p-1)}(\boldsymbol{b}|u|)^{-r}, \tag{1.3.5}$$

(iii) when $\boldsymbol{b} = 1$, we have $|\partial^i/i!|_V \leq |\partial|_K^i$ for all $i \geq 0$.

One may summarize the conditions (i)–(iii) by writing

(iv) $\left|\partial^i/i!\right|_V \leq \max\left\{|\partial|_K^i, (\omega \boldsymbol{b}^{-1}|u|^{-1})^i/|i!|\right\}$ for $i = 1, \ldots, r$.

Indeed, the equivalence of (1) or (iii) with (iv) is straightforward and the equivalence of (ii) and (iv) (when necessarily $p > 0$) follows from the observation that the maximum above is equal to $|\partial|_K^i$ if $i < r$ and to $p^{-1/(p-1)}(\boldsymbol{b}|u|)^{-r}$ if $i = r$. From condition (iv), it is obvious that a $\boldsymbol{b}$-good norm is also $\boldsymbol{b}'$-good for any $\boldsymbol{b}' \leq \boldsymbol{b}$.

For the rest of this definition, we assume that $\boldsymbol{b} = IR_\partial(V) < 1$. By Lemma 1.3.9 below there exists a good norm for V.

Using this good norm, we define the multiset of *refined ∂-radii* of V, denoted by $\Theta_\partial(V)$, as follows. Enlarge the value group of K in the sense of Remark 1.2.11 so that V admits an orthonormal basis. Let N_r be the matrix of ∂^r with respect to the chosen basis. If $\alpha_1, \dots, \alpha_d$ are the reduced eigenvalues of N_r, viewed as elements in $\kappa^{(rs)}_{K^{\mathrm{alg}}}$, we put

$$\Theta_\partial(V, |\cdot|) = \{\alpha_1^{1/r}, \dots, \alpha_d^{1/r}\}$$

as the multiset consisting of elements in $\kappa^{(s)}_{K^{\mathrm{alg}}}$ (note that there is no ambiguity of taking r-th roots for elements in $\kappa^{(rs)}_{K^{\mathrm{alg}}}$ when $p > 0$). We will see in Lemmas 1.3.11 and 1.3.12 that the multiset of refined ∂-radii is independent of the choices of the good norm and the orthonormal basis of V. After these lemmas, we will abbreviate $\Theta_\partial(V, |\cdot|)$ to $\Theta_\partial(V)$. When $\Theta_\partial(V)$ is pure as a multiset, we say that V has *pure refined ∂-radii*.

We remark that $\Theta_\partial(V)$ does not depend on the choice of the rational parameter u. But it is sometimes convenient to use the multiset of *intrinsic refined ∂-radii* $\mathcal{I}\Theta_\partial(V) = u\Theta_\partial(V)$ for a fixed rational parameter $u \in K$.

Finally, in the case when $IR_\partial(V) = 1$, we conventionally define $\Theta_\partial(V)$ and $\mathcal{I}\Theta_\partial(V)$ to be the multisets consisting of 0 with multiplicity $\dim V$.

Remark 1.3.6. In the definition of refined ∂-radii, we first enlarged K to K', the completion of $K(x_1, \dots, x_n)$ for some $(\eta_1, \dots, \eta_n)$-Gauss norm. However, the multiset of refined ∂-radii $\Theta_\partial(V, |\cdot|)$ is still composed of elements in $\kappa^{(s)}_{K^{\mathrm{alg}}}$. Indeed, since the construction is canonical, for any $\theta \in \Theta_\partial(V, |\cdot|)$, we have $g\theta \in \Theta_\partial(V, |\cdot|)$ for any automorphism g of K' fixing K. But $\Theta_\partial(V, |\cdot|)$ is a finite multiset. So it can consist only of elements in $\kappa^{(s)}_{K^{\mathrm{alg}}}$. Alternatively, we can carefully keep track of the new variables we introduced in the computation of reduced eigenvalues; from this, we can also see that the multiset of refined ∂-radii is composed of elements in $\kappa^{(s)}_{K^{\mathrm{alg}}}$.

Remark 1.3.7. We also remark that when $p > 0$ and $\boldsymbol{b} = \omega^{1/p^\lambda}$, the condition (1.3.5) for $i = 1, \dots, p^{\lambda-1}$ is equivalent to (1.3.5) for $i = 1, \dots, p^\lambda$. But we need the matrix N_{p^λ} to define refined ∂-radii. For example, when $\boldsymbol{b} = IR_\partial(V) = \omega$, we will see in Lemma 1.3.9 below that the twisted polynomial from Proposition 1.2.8 gives us a good norm on V. However, one *cannot* compute the refined ∂-radii by taking the reduced roots of this twisted polynomial. Instead, one has to find the matrix for ∂^p.

Remark 1.3.8. For a good norm, one can show that the inequalities in (1.3.5) are in fact equalities, but we will not use this fact later; see [Kedlaya 2010, Lemma 6.2.4] for a proof of similar flavor.

Lemma 1.3.9. *Let V be as in Hypothesis 1.3.1. Assume that $\boldsymbol{b} \le IR_\partial(V)$, and that $\boldsymbol{b} < 1$ if $p > 0$. Then V admits a $\boldsymbol{b}$-good norm. In particular, any V with pure intrinsic radii $IR_\partial(V) < 1$ admits a good norm.*

Proof. We first assume that $\boldsymbol{b} \le \omega$. We take a cyclic vector $\boldsymbol{v} \in V$ with twisted polynomial P. By Proposition 1.2.8, the lesser of $-\log|\partial|_K$ and the least slope of the

Newton polygon of P equals $\min\{s, -\log|\partial|_K\} \geq -\log(\omega \boldsymbol{b}^{-1}|u|^{-1})$. Then we can define a $\boldsymbol{b}$-good norm on V by taking the orthogonal basis to be $\boldsymbol{v}, \partial\boldsymbol{v}, \dots, \partial^{d-1}\boldsymbol{v}$ with $|\partial^i \boldsymbol{v}| = \omega^i(\boldsymbol{b}|u|)^{-i}$ for $i = 0, \dots, d-1$. When $\boldsymbol{b} = \omega$, as pointed out in Remark 1.3.7, our bound $|\partial|_V \leq |u|^{-1}$ alone implies condition (1.3.5) for $r = 1, \dots, p$ when $p > 0$, and condition (iii) in Definition 1.3.4 when $p = 0$.

The remaining case is when $p > 0$ and $\boldsymbol{b} \in (p^{-1/(p-1)}, 1)$. We let $n = \lambda - 1$ if $\boldsymbol{b} = p^{-1/(p^{\lambda-1}(p-1))}$ and $n = \lambda$ otherwise. In other words, n is the unique nonpositive integer such that

$$\boldsymbol{b} \in \Big(p^{\frac{-1}{p^{n-1}(p-1)}}, p^{\frac{-1}{p^n(p-1)}}\Big].$$

Let $\varphi^{(\partial,n)} : K^{(\partial,n)} \to K$ be the p^n-th ∂-Frobenius and let $\tilde{\partial} = \partial/(p^n u^{p^n-1})$ be the corresponding derivation on $K^{(\partial,n)}$. Since $IR_\partial(V) \geq \boldsymbol{b} > p^{-1/(p^{n-1}(p-1))}$, by repeatedly applying Lemma 1.2.18(d), we obtain an n-fold ∂-Frobenius antecedent W over $K^{(\partial,n)}$; it has intrinsic $\tilde{\partial}$-radii

$$IR_{\tilde{\partial}}(W) = IR_\partial(V)^{p^n} \geq \boldsymbol{b}^{p^n} \in \Big(p^{\frac{-p}{p-1}}, p^{\frac{-1}{p-1}}\Big].$$

In particular, W has a $\boldsymbol{b}^{p^n}$-good norm by the argument in the previous paragraph. We have

$$\begin{aligned} &\big|u^{p^n}\tilde{\partial}\big|_W \leq p^{-1/(p-1)}\boldsymbol{b}^{-p^n} \in [1, p) \\ \Rightarrow\ &\big|u\partial\big|_W = p^{-n}\big|u^{p^n}\tilde{\partial}\big|_W \begin{cases} < p^{-n}\cdot p = p^{\lambda-1} & \text{when } n = \lambda, \\ \leq p^{-n}\cdot 1 = p^{\lambda-1} & \text{when } n = \lambda - 1. \end{cases} \end{aligned} \tag{1.3.10}$$

This norm on W gives rise to a K-norm $|\cdot|_V$ on V, which we will show is $\boldsymbol{b}$-good. By (1.3.10), we have $|u\partial - i|_V = |u\partial - i|_W \leq |i|$ for $i = 1, \dots, p^\lambda - 1$. Hence we have, for $i = 1, \dots, p^\lambda$,

$$\begin{aligned} \Big|\frac{u^i\partial^i}{i!}\Big|_V &= \Big|\frac{u^i\partial^i}{i!}\Big|_W = \Big|\frac{u\partial(u\partial-1)\cdots(u\partial-(i-1))}{i!}\Big|_W \leq \Big|\frac{u\partial}{i}\Big|_W \\ &= \Big|\frac{p^n}{i}u^{p^n}\tilde{\partial}\Big|_W \begin{cases} \leq 1 & \text{if } i = 1, \dots, p^\lambda - 1, \\ \leq p^{-1/(p-1)}\boldsymbol{b}^{-p^n} = p^{-1/(p-1)}\boldsymbol{b}^{-p^\lambda} & \text{if } i = p^\lambda \text{ and } n = \lambda, \\ \leq p^{-p/(p-1)}\boldsymbol{b}^{-p^n} = p^{-1} & \text{if } i = p^\lambda \text{ and } n = \lambda - 1. \end{cases} \end{aligned}$$

This verifies (1.3.5). □

Lemma 1.3.11. *Assume that $IR_\partial(V) < 1$. Let $|\cdot|$ be a good norm on V. Then the multiset of refined ∂-radii $\Theta_\partial(V, |\cdot|)$ is well-defined.*

Proof. By possibly enlarging K in the sense of Remark 1.2.11, we have two orthonormal bases $\underline{\boldsymbol{e}}$ and $\underline{\boldsymbol{e}}'$ for $|\cdot|_V$ such that $\underline{\boldsymbol{e}}' = \underline{\boldsymbol{e}}A$ for a transition matrix $A \in \mathrm{GL}_d(\mathbb{O}_K)$. For $i = 1, \dots, r$, let N_i denote the matrix of ∂^i with respect to $\underline{\boldsymbol{e}}$; by (1.3.5), we have $|N_i/i!| \le |\partial|_K^i$ for $i = 1, \dots, r-1$. Then

$$\frac{\partial^r(\underline{e}')}{r!} = \frac{\partial^r(\underline{e}A)}{r!} = \sum_{i=0}^{r} \frac{\partial^i(\underline{e})}{i!}\frac{\partial^{r-i}(A)}{(r-i)!} = \underline{e}'A^{-1}\biggl(\sum_{i=0}^{r} \frac{N_i}{i!}\frac{\partial^{r-i}(A)}{(r-i)!}\biggr).$$

If $A^{-1}MA$ denote the matrix of $\partial^r/r!$ with respect to $\underline{e}'$, we have

$$M = \frac{N_r}{r!} + \sum_{i=0}^{r-1} \frac{N_i}{i!}\frac{\partial^{r-i}(A)A^{-1}}{(r-i)!}.$$

Note that $|N_i/i!| \le |\partial|_K^i$ and

$$|\partial^{r-i}(A)A^{-1}/(r-i)!| \le |\partial|_K^{r-i}|A|\,|A^{-1}| \le |\partial|_K^{r-i}$$

imply that $|M - N_r/r!| \le |\partial|_K^r < \omega R_\partial(V)^{-r}$, which is smaller than any singular value of $N_r/r!$. By [Kedlaya 2010, Theorem 4.2.2], the reduced eigenvalues of $N_r/r!$ coincide with those of $A^{-1}MA$. Therefore, $\Theta_\partial(V, |\cdot|)$ does not depend on the choice of an orthogonal basis for $|\cdot|$. □

Lemma 1.3.12. *Assume that $IR_\partial(V) < 1$. Let $|\cdot|_1$ and $|\cdot|_2$ be two good norms on V. Then $\Theta_\partial(V, |\cdot|_1) = \Theta_\partial(V, |\cdot|_2)$.*

Proof. By possibly enlarging K as in Remark 1.2.11, we may choose orthonormal bases $\underline{e}$ and $\underline{f}$ of $|\cdot|_1$ and $|\cdot|_2$, respectively, so that $\underline{e}A = \underline{f}$ with $A = \mathrm{Diag}\{a_{11}, \dots, a_{dd}\}$.

Let N_i denote the matrix of ∂^i with respect to $\underline{e}$; by (1.3.5), we have $|N_i/i!| \le 1$ for $i = 1, \dots, r-1$. Then

$$\frac{\partial^r(\underline{f})}{r!} = \frac{\partial^r(\underline{e}A)}{r!} = \sum_{i=0}^{r} \frac{\partial^i(\underline{e})}{i!}\frac{\partial^{r-i}(A)}{(r-i)!} = \underline{f}A^{-1}\biggl(\sum_{i=0}^{r} \frac{N_i}{i!}\frac{\partial^{r-i}(A)}{(r-i)!}A^{-1}\biggr)A.$$

It suffices to show that $N_r/r!$ has the same reduced eigenvalues as

$$\sum_{i=0}^{r} \frac{N_i}{i!}\frac{\partial^{r-i}(A)}{(r-i)!}A^{-1}.$$

This is true by [Kedlaya 2010, Theorem 4.4.2] since

$$\biggl|\frac{N_i}{i!}\frac{\partial^{r-i}(A)}{(r-i)!}A^{-1}\biggr| = \biggl|\frac{N_i}{i!}\biggr| \cdot \biggl|\mathrm{Diag}\Bigl(\frac{\partial^{r-i}(a_{11})}{(r-i)!}a_{11}^{-1}, \dots, \frac{\partial^{r-i}(a_{dd})}{(r-i)!}a_{dd}^{-1}\Bigr)\biggr|$$
$$\le |\partial|_K^i \cdot |\partial|_K^{r-i} < \omega R_\partial(V)^{-1},$$

for $i = 0, \dots, r-1$. □

Corollary 1.3.13. *Assume that V has pure visible ∂-radii. For any cyclic vector $\boldsymbol{v} \in V$, the multiset of the reduced roots of the twisted polynomial associated to $\boldsymbol{v}$ is* *exactly the multiset of refined ∂-radii of V. In particular, this multiset is composed of nonzero elements of $\kappa^{(s)}_{K^{\mathrm{alg}}}$.*

More generally, we may drop the hypothesis that V has pure ∂-radii, and only assume that V has visible ∂-radii $R_\partial(V) = \omega \mathrm{e}^s$. Let h denote the multiplicity of $R_\partial(V)$ in the multiset $\mathfrak{R}_\partial(V)$. In this case, for any cyclic vector $\boldsymbol{v} \in V$, if we write the associated monic twisted polynomial as $X^d + a_1 X^{d-1} + \cdots + a_d$, then $|a_i| \leq \mathrm{e}^{-is}$ for $i \leq h$ and $|a_h| = \mathrm{e}^{-hs}$. Moreover, if $V_{\omega \mathrm{e}^s}$ is the direct summand of V with pure ∂-radii $\omega \mathrm{e}^s$ as given by Theorem 1.2.20, then $\Theta_\partial(V_{\omega \mathrm{e}^s})$ consists of the reduced roots of the polynomial $X^h + a_1 X^{h-1} + \cdots + a_h = 0$.

Proof. As already pointed out in Remark 1.3.7, we emphasize again that the case $IR_\partial(V) = \omega$ is not included in the statement. We first treat the case when V has pure visible ∂-radii. We can construct the good norm using the twisted polynomial as in Lemma 1.3.9. This twisted polynomial is then exactly the characteristic polynomial of the matrix of ∂ with respect to this basis. The claim follows.

For V not necessarily pure of ∂-radii, the bound for norms on a_i follows from Proposition 1.2.8. For the statement about refined ∂-radii, we need to dig into the proof of Theorem 1.2.20 a bit more. By [Kedlaya 2009, Corollary 3.2.4], we can write $P = QR$ where Q and R are monic twisted polynomials such that the Newton polygon of $Q = X^h + a'_1 X^{h-1} + \cdots + a'_h$ has pure slopes s and the Newton polygon of R has slope strictly bigger than s. Moreover, we have $V_{\omega \mathrm{e}^s} = K\{T\}/QK\{T\}$. The upshot is that the formal multiplication satisfies $|a_i - a'_i| < \mathrm{e}^{is}$ for any $i = 1, \dots, h$. Therefore, the reduced roots of $X^h + a_1 X^{h-1} + \cdots + a_h = 0$ are the same as the reduced roots of $X^h + a'_1 X^{h-1} + \cdots + a'_h = 0$, which are the same as the elements of $\Theta_\partial(V)$ by the discussion in the previous paragraph. □

Lemma 1.3.14. *Fix $\boldsymbol{b} \in (0, 1)$ and set $r = r(\boldsymbol{b})$, $\lambda = \lambda(\boldsymbol{b})$, and $s = -\log(\omega(\boldsymbol{b}|u|)^{-1})$. Let V' be a ∂-differential module over K of rank d, equipped with a basis $\underline{\boldsymbol{e}}$, with respect to which the action of ∂ satisfies the conditions in Definition 1.3.4 with the chosen $\boldsymbol{b}$. Assume that the reduced eigenvalues of the matrix $N_r \in \mathrm{Mat}(\mathfrak{m}_K^{(rs)})$ of ∂^r on V' are all nonzero in $\kappa^{(rs)}_{K^{\mathrm{alg}}}$. Then V' has pure intrinsic ∂-radii $\boldsymbol{b}$. As a consequence, $\Theta_\partial(V')$ is exactly the multiset of the reduced eigenvalues of N_r.*

Proof. Since $N_r \in \mathrm{Mat}(\mathfrak{m}_K^{(rs)})$, we have $IR_\partial(V') \geq \boldsymbol{b}$. Suppose that V' does not have pure intrinsic ∂-radii $\boldsymbol{b}$. Enlarging K as in Remark 1.2.11 if needed, we may apply Theorem 1.2.20 and Lemma 1.3.9 to V' and its Jordan–Hölder constituents to find a basis $\underline{\boldsymbol{f}}$ for which the conditions in Definition 1.3.4 hold and the matrix $\widetilde{N}_r \in \mathrm{Mat}(\mathfrak{m}_K^{(rs)})$ of ∂^r is degenerate modulo $\mathfrak{m}^{(rs)+}_{K^{\mathrm{alg}}}$ (when identifying $\kappa_K^{(rs)}$ with κ_K). Now, the same argument in the proof of Lemma 1.3.12 implies that N_r and $\widetilde{N}_r$ must have the same multiset of reduced eigenvalues. But zero is a reduced eigenvalue of $\widetilde{N}_r$ but not of N_r, which is a contradiction. Hence V' has pure intrinsic ∂-radii $\boldsymbol{b}$. The last statement is Definition 1.3.4. □

Lemma 1.3.15. *We have* $\Theta_\partial(V^\vee) = -\Theta_\partial(V) = \{-\theta \mid \theta \in \Theta_\partial(V)\}$.

Proof. This is straightforward. □

We will prove in Theorem 1.3.26 a direct sum decomposition of V parametrized by the multiset of refined ∂-radii. For this, we start with some basic properties of refined ∂-radii when V has visible ∂-radii.

Lemma 1.3.16. *Let V and W be two ∂-differential modules over K with pure and visible ∂-radii $R_\partial(V) = R_\partial(W)$. Then the following two statements are equivalent.*

(1) *The refined ∂-radii of V and W are distinct, that is, $\Theta_\partial(V) \cap \Theta_\partial(W) = \varnothing$.*

(2) *The tensor product $V \otimes W^\vee$ has pure ∂-radii $R_\partial(V)$.*

Moreover, if either statement holds, we have an equality of multisets:

$$\Theta_\partial(V \otimes W^\vee) = \{\theta_1 - \theta_2 : \theta_1 \in \Theta_\partial(V), \theta_2 \in \Theta_\partial(W)\}.$$

As corollaries, we have the following:

(a) *If $\Theta_\partial(V) \cap \Theta_\partial(W) = \varnothing$, then any homomorphism $f : W \to V$ of ∂-differential modules is zero.*

(b) *If $\Theta_\partial(W)$ has pure refined ∂-radii $\theta \in \kappa^{(s)}_{K^{\mathrm{alg}}}$, then $\theta \in \Theta_\partial(V)$ if and only if $V \otimes W^\vee$ does not have pure ∂-radii $R_\partial(V)$.*

(c) *If $\Theta_\partial(V)$ and $\Theta_\partial(W)$ both have the same pure ∂-radii $\theta \in \kappa^{(s)}_{K^{\mathrm{alg}}}$, then we have $R_\partial(V \otimes W^\vee) > R_\partial(V)$.*

Proof. By Lemma 1.3.15, we have $\Theta_\partial(W^\vee) = -\Theta_\partial(W)$. We may enlarge K as in Remark 1.2.11 so that we have good norms on both V and $W^\vee$ given by orthonormal bases. Equip $V \otimes W^\vee$ with the tensor product norm. Let $N_0, N_1 \in \mathrm{Mat}(\mathfrak{m}_K^{(s)})$ be the matrices of ∂ acting on V and $W^\vee$ with respect to the given bases, respectively. By Definition 1.3.4, $\Theta_\partial(V)$ and $-\Theta_\partial(W)$ are the multisets of reduced eigenvalues of N_0 and N_1, respectively. Then the multiset of reduced eigenvalues of the matrix $N = N_0 \otimes 1 + 1 \otimes N_1$ is exactly $\{\theta_1 - \theta_2 : \theta_1 \in \Theta_\partial(V), \theta_2 \in \Theta_\partial(W)\}$.

If (1) holds, then all reduced eigenvalues of N are nonzero and hence $|N^n| = \mathrm{e}^{-ns}$ for all $n \in \mathbb{N}$. Moreover, the reduction of N^n in $M_d\big(\kappa^{(ns)}_{K^{\mathrm{alg}}}\big)$ has full rank if we identify $\kappa^{(ns)}_{K^{\mathrm{alg}}}$ with $\kappa_{K^{\mathrm{alg}}}$. Therefore, $V \otimes W^\vee$ has pure ∂-radii $R_\partial(V)$ by Lemma 1.3.14, proving (2).

Conversely, if (2) holds, then the tensor product norm is a good norm on $V \otimes W^\vee$ already and the multiset of reduced eigenvalues of N is the multiset of refined ∂-radii of $V \otimes W^\vee$. By Corollary 1.3.13, $0 \notin \Theta_\partial(V \otimes W^\vee)$. This implies (1).

We now prove (a). Since $V \otimes W^\vee$ has pure ∂-radii $R_\partial(V) < \omega$, we have $H^0_\partial(V \otimes W^\vee) = 0$ and hence there is no nonzero homomorphism of ∂-differential modules from W to V.

Statement (b) is just (a special case of) the inverse statement of (1) $\Leftrightarrow$ (2).

For (c), we know that N_0 and N_1 have pure reduced eigenvalues θ and $-\theta$, respectively. Hence $N = N_0 \otimes 1 + 1 \otimes N_1$ reduces to a matrix in $\kappa^{(s)}_{K^{\mathrm{alg}}}$ with zero eigenvalues (if we identify $\kappa^{(s)}_{K^{\mathrm{alg}}}$ with $\kappa_{K^{\mathrm{alg}}}$). It is then nilpotent, that is,

$$N^n \in \mathrm{Mat}\big(\mathfrak{m}^{(ns)+}_{K^{\mathrm{alg}}}\big) \quad \text{for } n \geq \dim V \cdot \dim W.$$

This implies that $R_\partial(V \otimes W^\vee) > R_\partial(V)$. □

Lemma 1.3.17. *Let V and W be two ∂-differential modules over K. Assume that V has pure and visible ∂-radii and $R_\partial(V) < R_\partial(W)$. Then $V \otimes W^\vee$ has pure ∂-radii $R_\partial(V)$ and multiset of refined ∂-radii is composed of* $\dim W$ *copies of* $\Theta_\partial(V \otimes W^\vee)$.

Proof. By Theorem 1.2.20, we may assume that W has pure ∂-radii. By Lemma 1.3.9 we may find a $\boldsymbol{b}$-good norm on W with $\boldsymbol{b} = \min\{IR_\partial(W), \omega\} > IR_\partial(V)$.

We proceed as in Lemma 1.3.16. If N_0 and N_1 are the matrices of ∂ with respect to some orthonormal bases of V and $W^\vee$, respectively, then we have $N_1 \in \mathrm{Mat}\big(\mathfrak{m}^{(s)+}_K\big)$ and that N_0 has pure reduced eigenvalue $\Theta_\partial(V)$. Hence the multiset of reduced eigenvalues of $N_0 \otimes 1 + 1 \otimes N_1$ is simply composed of $\dim W$ copies of the set of reduced eigenvalues of N_0. The lemma follows. □

The refined ∂-radii of a nonvisible ∂-differential module is closely related to the ∂'-radii of its Frobenius antecedent; we can save much computation by using this fact. To establish this relation explicitly, it is more convenient to work with the refined *intrinsic* ∂-radii.

Proposition 1.3.18. *Assume $p > 0$. Let $\varphi^{(\partial)} : K^{(\partial)} \to K$ be the ∂-Frobenius with respect to the parameter u.*

(a) *Assume that $IR_\partial(V) \in (p^{-1/(p-1)}, 1)$, and then Lemma 1.2.18(d) implies that $V = \varphi^{(\partial)*}W$ for some ∂'-differential module W on $K^{(\partial)}$ such that*

$$IR_{\partial'}(W) = IR_\partial(V)^p.$$

We have

$$\Theta_\partial(V) = \big\{-(p\theta')^{1/p} : \theta' \in \Theta_{\partial'}(W)\big\}.$$

(b) *Assume that $IR_\partial(V) = p^{-1/(p-1)}$, and then $\varphi^{(\partial)}_*(V)$ has pure intrinsic ∂'-radii $p^{-p/(p-1)}$. The elements in $\mathcal{I}\Theta_{\partial'}(\varphi^{(\partial)}_*(V))$ can be grouped into p-tuples*

$$\Big(\frac{\theta}{p}, \frac{\theta+1}{p}, \dots, \frac{\theta+p-1}{p}\Big)$$

with $\theta \in \kappa_{K^{\mathrm{alg}}}$, and $\mathcal{I}\Theta_\partial(V)$ is the multiset composed of $(\theta^p - \theta)^{1/p} \in \kappa_{K^{\mathrm{alg}}}$ for each p-tuple above.

(c) *Assume* $IR_\partial(V) < p^{-1/(p-1)}$. *Then we have*

$$\mathcal{I}\Theta_{\partial'}(\varphi_*^{(\partial)}V) = \big\{\underbrace{p^{-1}\theta, \ldots, p^{-1}\theta}_{p \text{ times}} : \theta \in \mathcal{I}\Theta_\partial(V)\big\}.$$

Proof. (a) By Lemma 1.3.9 and by possibly enlarging K as in Remark 1.2.11, we can take an orthonormal basis $\underline{e}$ on W which defines a good norm. The norm induces a good norm on V by the explicit construction in Lemma 1.3.9. Let λ be as in Notation 1.3.3. We have

$$\begin{aligned} u^{p^\lambda}\partial^{p^\lambda} &= u\partial(u\partial-1)\cdots(u\partial-p^\lambda+1) = pu^p\partial'(pu^p\partial'-1)\cdots(pu^p\partial'-p^\lambda+1) \\ &= p^{p^{\lambda-1}}u^{p^\lambda}\partial'^{p^{\lambda-1}}\prod_{i=1,p\nmid i}^{p^\lambda-1}(pu^p\partial'-i); \end{aligned}$$

this operator also acts on W. Since $|u^p\partial'|_W \leq \max\{1, p^{-1/(p-1)}IR_{\partial'}(W)\} < p$, we have

$$\big|u^{p^\lambda}\partial^{p^\lambda} - p^{p^{\lambda-1}}\big((-1)\cdots(-p+1)\big)^{p^{\lambda-1}}u^{p^\lambda}\partial'^{p^\lambda}\big|_W < \big|u^{p^\lambda}\partial^{p^\lambda}\big|_W.$$

Therefore, the matrix of ∂^{p^λ} with respect to $\underline{e}$ is congruent to the matrix of

$$(-1)^{p^{\lambda-1}(p-1)}(p!)^{p^{\lambda-1}}\partial'^{p^{\lambda-1}} \quad \text{modulo } \mathfrak{m}_K^{(p^\lambda s)+}.$$

We then must have

$$\Theta_\partial(V, |\cdot|) = \big\{\big((-1)^{(p-1)}(p!)\theta\big)^{1/p} \,\big|\, \theta \in \Theta_{\partial'}(W)\big\} = \big\{-\big(p\theta\big)^{1/p} \,\big|\, \theta \in \Theta_{\partial'}(W)\big\}.$$

(b) When $IR_\partial(V) = p^{-1/(p-1)}$, Lemma 1.2.18(e) implies that $\varphi_*^{(\partial)}V$ has pure intrinsic ∂'-radii $p^{-p/(p-1)}$. By Lemma 1.2.18(e) and Lemma 1.3.16, the elements in $\mathcal{I}\Theta_{\partial'}(\varphi_*^{(\partial)}(V))$ can be grouped into p-tuples

$$\Big(\frac{\theta}{p}, \frac{\theta+1}{p}, \ldots, \frac{\theta+p-1}{p}\Big)$$

with $\theta \in \kappa_{K^{\mathrm{alg}}}$. (Note: explicit computation shows that $\mathcal{I}\Theta_{\partial'}(W_i^{(\partial)}) = \{\frac{i}{p}\}$.) By possibly enlarging K in the sense of Remark 1.2.11, we may assume that $\varphi_*^{(\partial)}V$ admits a good norm defined by an orthonormal basis $\underline{e}$. Let N be the matrix of $pu^p\partial'$ with respect to $\underline{e}$. Then $u^p\partial^p$ acts on $\varphi^{(\partial)*}\varphi_*^{(\partial)}(V) = V^{\oplus p}$ according to

$$u\partial(u\partial-1)\cdots(u\partial-p+1) = pu^p\partial'(pu^p\partial'-1)\cdots(pu^p\partial'-p+1).$$

Hence the matrix for this action with respect to $\underline{e}$ is congruent to the product $N(N-1)\cdots(N-p+1)$ modulo $p\mathbb{O}_{K^{(\partial)}}$ since $|pu^p\partial'|_{K^{(\partial)}} = p^{-1}$; then the multiset of its reduced eigenvalues is composed of $\theta^p - \theta$ with multiplicity p for each tuple

$$\Big(\frac{\theta}{p}, \frac{\theta+1}{p}, \ldots, \frac{\theta+p-1}{p}\Big)$$

in the multiset of reduced eigenvalues of N. The statement follows.

(c) By Lemma 1.2.18(e), $\varphi_*^{(\partial)}V$ has pure intrinsic ∂'-radii

$$p^{-1}IR_\partial(V) \leq p^{-p/(p-1)}.$$

Since $u^p\partial' = u\partial/p$, we can take a good norm of $\varphi_*^{(\partial)}V$ and deduce that

$$\mathcal{I}\Theta_{\partial'}(\varphi_*^{(\partial)}V) = \frac{1}{p}\mathcal{I}\Theta_\partial\big(\varphi^{(\partial)*}\varphi_*^{(\partial)}V\big),$$

which in turn equals $\frac{1}{p}\mathcal{I}\Theta_\partial(V^{\oplus p})$ by Lemma 1.2.18(b). The statement follows. □

Proposition 1.3.19. *Lemma 1.3.16 holds only assuming* $IR_\partial(V) = IR_\partial(W) < 1$ *instead of the visible hypothesis. Similarly, Lemma 1.3.17 holds with only assuming* $IR_\partial(V) < 1$ *instead of the visible hypothesis on* V.

Proof. It suffices to check the remaining cases: $p > 0$ and $IR_\partial(V) \geq p^{-1/(p-1)}$. If $IR_\partial(V) > p^{-1/(p-1)}$, the statements for V and W follow from the statements on their ∂-Frobenius antecedents by Proposition 1.3.18(a). If $IR_\partial(V) = p^{-1/(p-1)}$, the statements for V and W follow from the statements on their ∂-Frobenius descendants by Proposition 1.3.18(b) and Lemma 1.2.18(c). □

The following is an example of ∂-differential modules with pure refined ∂-radii. It will serve as a testing object later.

Our convention is to use Gothic letter $\mathfrak{s}$ instead of s when discussing intrinsic radii of convergence; we will never use both s and $\mathfrak{s}$ together.

Example 1.3.20. Fix $\mathfrak{s} \in -\log|K^\times|^{\mathbb{Q}}$ such that $\mathfrak{s} < 0$ if $p = 0$, and $\mathfrak{s} < \frac{1}{p}\log p$ if $p > 0$. Let $\theta \in \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})}$ be a nonzero element.

(1) If $p = 0$, then we have $\mathfrak{s} \in -\log|(K')^\times|$ and $\theta \in \kappa_{K'}^{(\mathfrak{s})}$ for some finite *tamely ramified* extension K' of K. Let x be a lift of θ to $\mathfrak{m}_{K'}^{(\mathfrak{s})}$. Put $d = 1$ and $n = 0$.

(2) If $p > 0$, there exists $n \in \mathbb{N}$ such that $\theta^{p^n} \in (\kappa_{K'}^{(p^{n-1}\mathfrak{s})})^p$ with $p^{n-1}\mathfrak{s} \in -\log|(K')^\times|$ for some finite *tamely ramified* extension K' of K. By Lemma 1.2.16, we may find a lift
$$x \in u^{-p^n}\mathfrak{m}_{K'(\partial)}^{(p^n\mathfrak{s})}$$
of $u^{-p^n}\theta^{p^n}$, where the extra u^{-p^n} reflects the different normalizations of refined intrinsic ∂-radii and refined ∂-radii. Put $d = p^n$.

Let $\mathcal{L}_{x,(n)}$ be the ∂-differential module over K' of rank d with basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d\}$, where the ∂-action is given by $\partial\boldsymbol{e}_i = \boldsymbol{e}_{i+1}$ for $i = 1, \dots, d-1$ and $\partial\boldsymbol{e}_d = x\boldsymbol{e}_1$.

Remark 1.3.21. When $p > 0$, we point out that $\mathfrak{s} < \frac{1}{p}\log p$ also includes some part of the nonvisible range. The restriction $\mathfrak{s} < \frac{1}{p}\log p$ in Example 1.3.20 is linked with the choice $x \in u^{-p^n}\mathfrak{m}_{K'(\partial)}^{(p^n\mathfrak{s})}$. In general, we may extend the range of $\mathfrak{s}$ to be

$$\Big(-\infty, \Big(\frac{1}{p-1} - \frac{1}{p^c(p-1)}\Big) \log p\Big)$$

for some $c \in \mathbb{N}$, but the price we pay is to take $x \in u^{-p^n} \mathfrak{m}_{K'^{(\partial,c)}}^{(p^n \mathfrak{s})}$ lifting $u^{-p^n}\theta^{p^n}$ for some $n \in \mathbb{N}$ and some finite tamely ramified extension K' of K. However, as c gets larger, we need to enlarge n to guarantee the existence of such a lift x. This is why we may not assume that $\mathfrak{s} < \frac{1}{p-1} \log p$.

Remark 1.3.22. In the nonvisible case, one can construct a ∂-differential module with pure refined ∂-radii by simply pulling back a ∂'-differential module over $K^{(\partial)}$ with appropriate refined ∂'-radii. However, such a naïve construction does not help our later study of the one-dimensional variation of refined ∂-radii. We will construct Example 1.5.7, a family version of Example 1.3.20, which looks similar in both visible and *nonvisible* ranges.

Lemma 1.3.23. *Keep the notation as in Example 1.3.20. Then $\mathscr{L}_{x,(n)}$ has pure intrinsic ∂-radii $IR_\partial(\mathscr{L}_{x,(n)}) = \omega \mathrm{e}^{\mathfrak{s}}$ and pure refined intrinsic ∂-radii θ.*

Proof. We may replace K by the completion of $K(z)$ with respect to the $|u|^{-1}\mathrm{e}^{-\mathfrak{s}}$-Gauss norm (and set $\partial z = 0$).

We first assume that either we have $p = 0$ or we have $p > 0$ and $\mathfrak{s} < 0$, that is, we consider the visible ∂-radii case. We note that $\boldsymbol{e}_1, z^{-1}\boldsymbol{e}_2, \dots, z^{-(d-1)}\boldsymbol{e}_d$ together define a good norm on $\mathscr{L}_{x,(n)}$; it is a straightforward computation to check that the statement in this case.

We now tackle the case when $p > 0$ and $\mathfrak{s} \in \big[0, \frac{1}{p} \log p\big)$. For $i = 1, \dots, p$, we have

$$\partial^i \boldsymbol{e}_l = \boldsymbol{e}_{i+l} \text{ when } i + l \leq p^n, \quad \text{and } \partial^i \boldsymbol{e}_{p^n - l} = \partial^{i-l}(x\boldsymbol{e}_1) \text{ when } i \geq l.$$

We will show that $\{\boldsymbol{e}_1, z^{-1}\boldsymbol{e}_2, \dots, z^{-(p^n-1)}\boldsymbol{e}_{p^n}\}$ defines a good norm on $\mathscr{L}_{x,(n)}$. Indeed, for $i = 1, \dots, p$, the matrix of ∂^i with respect to this basis is

$$N_i = \begin{pmatrix} 0 & 0 & \cdots & z^i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & z^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & z^i \\ z^{-p^n+i}x & 0 & \cdots & 0 & 0 & \cdots & 0 \\ z^{-p^n+i}\partial x & z^{-p^n+i}x & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z^{-p^n+i}\partial^{i-1}x & (i-1)z^{-p^n+i}\partial^{i-2}x & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{1.3.24}$$

Note that

$$|\partial|_{K^{(\partial)}} = p^{-1}|u|^{p-1}|\partial'|_{K^{(\partial)}} = p^{-1}|u|^{-1} \leq \omega|z| < |z|.$$

Hence, modulo $\mathfrak{m}_K^{(-\log|z|)+}$, the nonzero terms of N_i are the z^i and $z^{-p^n+i}x$ in (1.3.24); they form a 2-by-2 block matrix

$$\overline{N_i}^{(-\log|z|)} = \begin{pmatrix} 0 & z^i \cdot I_{(p^n-i)\times(p^n-i)} \\ z^{-p^n+i}x \cdot I_{i\times i} & 0 \end{pmatrix} \in \mathrm{Mat}_{p^n\times p^n}\big(\kappa_K^{(-\log|z|)}\big).$$

Note that $|z^{-p^n+i}x| = |z|^i$. By Lemma 1.3.14, we have $IR_\partial(\mathscr{L}_{x,(n)}) = \omega e^{\mathfrak{s}}$ and that this basis defines a good norm on V. Moreover, the multiset of reduced eigenvalues of N_p is composed of the element $x^{1/p^{n-1}}$ with multiplicity p^n. This implies that $\mathscr{I}\Theta_\partial(V) = \{\theta\ (p^n \text{ times})\}$ by the choice of x in Example 1.3.20. □

Lemma 1.3.25. *Let V be a ∂-differential module over K with pure visible ∂-radii $R_\partial(V) = \omega e^s$. Then we have the following.*

(a) *For any subquotient V_0 of V, all elements in $\Theta_\partial(V_0)$ appear in $\Theta_\partial(V)$.*

(b) *For any $\theta \in \kappa_{K^{\mathrm{alg}}}^{(s)}$, there is a unique maximal ∂-differential submodule of V which has pure refined ∂-radii θ.*

Proof. For $\theta \in \kappa_{K^{\mathrm{alg}}}^{(s)}$ such that $\theta \notin \Theta_\partial(V)$, let $\mathscr{L}_{x,(n)}$ be the ∂-differential module constructed in Example 1.3.20. By Lemmas 1.3.23 and 1.3.16, $V \otimes \mathscr{L}_{x,(n)}^\vee$ has pure ∂-radii $R_\partial(V)$, and so does $V_0 \otimes \mathscr{L}_{x,(n)}^\vee$. By the same lemmas again, we have $\theta \notin \Theta_\partial(V_0)$. This proves (a). We point out that this, however, does not prove the inclusion $\Theta_\partial(V_0) \subseteq \Theta_\partial(V)$ as a multiset, which will be a corollary of Theorem 1.3.26 below.

The second statement follows from the observation that if two submodules V_1 and V_2 of V both have pure refined ∂-radii θ, so does their sum $V_1 + V_2$ because it is a quotient of $V_1 \oplus V_2$. □

Similarly to the direct sum decomposition by intrinsic ∂-radii, we have a direct sum decomposition by refined intrinsic ∂-radii. The latter is in fact deduced from the former by twisting ∂-differential modules of the form $\mathscr{L}_{x,(n)}$.

Theorem 1.3.26. *Let K and V be as in Hypothesis 1.3.1. Then V admits a unique direct sum decomposition*

$$V = \bigoplus_{\{\theta\}\subset\kappa_{K^{\mathrm{alg}}}^{(s)}} V_{\{\theta\}}, \tag{1.3.27}$$

where the direct sum runs through all $\mathrm{Gal}(K^{\mathrm{alg}}/K)$*-orbits $\{\theta\}$ in $\kappa_{K^{\mathrm{alg}}}^{(s)}$, such that the refined ∂-radii of $V_{\{\theta\}}$ is a multiset consisting of the* $\mathrm{Gal}(K^{\mathrm{alg}}/K)$*-orbit $\{\theta\}$ with appropriate multiplicities.*

Moreover, if K' is a finite tamely ramified extension of K such that all the θ in the above decomposition belong to $\bigcup_n \big(\kappa_{K'}^{(p^n s)}\big)^{1/p^n}$, then we have a unique direct *sum decomposition*

$$V \otimes_K K' = \bigoplus_{\theta \in \kappa^{(s)}_{K'^{\mathrm{alg}}}} V_\theta$$

of ∂*-differential modules over* K' *such that each* V_θ *has pure refined* ∂*-radii* θ.

Proof. The statement is void if $IR_\partial(V) = 1$. We assume $IR_\partial(V) < 1$ from now on. We first replace K by the K' in the theorem; using the uniqueness of such a direct sum decomposition and Galois descent, we may recover the statement over K. Note that Lemma 1.2.10 implies that ∂ is still a derivation of rational type.

We first assume that either $p = 0$, or $p > 0$ and $IR_\partial(V) < p^{-1/(p-1)}$. For each $\theta \in \Theta_\partial(V)$, we construct $\mathscr{L}_{x,(n)}$ as in Example 1.3.20, which is a rank d ∂-differential module with pure ∂-radii $R_\partial(V)$ and pure refined radii θ. By Lemma 1.3.16(b), $V \otimes \mathscr{L}^\vee_{x,(n)}$ does not have pure radii $R_\partial(V)$. Theorem 1.2.20 then gives rise to a decomposition $V \otimes \mathscr{L}^\vee_{x,(n)} = W_0 \oplus W_1$, where $R_\partial(W_0) > R_\partial(V)$ and W_1 has pure ∂-radii $R_\partial(V)$.

Put $\widetilde{W}_0 = W_0 \otimes \mathscr{L}_{x,(n)}$ and $\widetilde{W}_1 = W_1 \otimes \mathscr{L}_{x,(n)}$. Consider the following homomorphisms of ∂-differential modules:

$$V \underset{j}{\overset{i}{\rightleftarrows}} V \otimes \mathscr{L}^\vee_{x,(n)} \otimes \mathscr{L}_{x,(n)} \xrightarrow{\ \sim\ } \widetilde{W}_0 \oplus \widetilde{W}_1,$$

where i is induced by the diagonal embedding $K \hookrightarrow \mathscr{L}^\vee_{x,(n)} \otimes \mathscr{L}_{x,(n)}$ and j is induced by the trace map $\mathscr{L}^\vee_{x,(n)} \otimes \mathscr{L}_{x,(n)} \twoheadrightarrow K$ normalized so that $ji = \mathrm{id}$. Let p_0 and p_1 be the projections from $V \otimes \mathscr{L}^\vee_{x,(n)} \otimes \mathscr{L}_{x,(n)}$ to the factors $\widetilde{W}_0$ and $\widetilde{W}_1$, respectively, viewed as submodules of the source. We then have $p_0^2 = p_0$, $p_1^2 = p_1$, and $p_0 + p_1 = 1$.

We claim that jp_0i and jp_1i are projectors on V. Indeed, Lemma 1.3.16(c) implies that $R_\partial(\mathscr{L}^\vee_{x,(n)} \otimes \mathscr{L}_{x,(n)}) > R_\partial(V)$. By Lemma 1.3.17, $V \otimes \mathscr{L}^\vee_{x,(n)} \otimes \mathscr{L}_{x,(n)}$ and hence $\widetilde{W}_0$ and $\widetilde{W}_1$ have pure ∂-radii $R_\partial(V)$. Lemma 1.3.17 also implies that $\Theta_\partial(\widetilde{W}_0)$ consists of solely θ, and by the "moreover" part of Lemma 1.3.16, we have

$$\Theta_\partial(\widetilde{W}_1) = \bigl\{\theta_1 + \theta \text{ (with multiplicity } d) \bigm| \theta_1 \in \Theta_\partial(W_1)\bigr\}.$$

In particular, we have $\theta \notin \Theta_\partial(\widetilde{W}_1)$. Hence any homomorphism of ∂-differential modules between $\widetilde{W}_0$ and $\widetilde{W}_1$ has to be zero by Lemma 1.3.16(a). In particular, $p_1ijp_0 = p_0ijp_1 = 0$. Thus, we have

$$\begin{aligned}
&(jp_0i)(jp_0i) = jp_0ij(1-p_1)i = jp_0i(ji) - j(p_0ijp_1)i = jp_0i,\\
&(jp_1i)(jp_1i) = jp_1ij(1-p_0)i = jp_1i(ji) - j(p_1ijp_0)i = jp_1i,\\
&jp_0i + jp_1i = j(p_0+p_1)i = ji = 1.
\end{aligned}$$

This proves $V = jp_0i(V) \oplus jp_1i(V)$. Moreover, by Lemma 1.3.25(i), $\Theta_\partial(jp_0i(V))$ consists of only θ since it is a quotient of $\widetilde{W}_0$, and $\Theta_\partial(jp_1i(V))$ does not contain θ since it is a quotient of $\widetilde{W}_1$. Applying this process to each of $\theta \in \Theta_\partial(V)$ gives the desired decomposition (1.3.27).

The uniqueness of the direct sum decomposition follows from Lemma 1.3.25(b).

Now if $p > 0$ and $IR_\partial(V) = p^{-1/(p-1)}$, the decomposition (1.3.27) comes from the decomposition of its ∂-Frobenius descendent, via the relation described in Proposition 1.3.18(b). If $p > 0$ and $IR_\partial(V) > p^{-1/(p-1)}$, the decomposition (1.3.27) comes from the decomposition of its ∂-Frobenius antecedent, via the relation described in Proposition 1.3.18(a). □

Now we prove some fundamental properties for tensor products of ∂-differential modules with pure ∂-radii and pure refined ∂-radii. One can combine this with Theorems 1.2.20 and 1.3.26 to obtain corresponding results for general ∂-differential modules.

Proposition 1.3.28. *Let V and W be two ∂-differential modules over K with pure ∂-radii $R_\partial(V) = R_\partial(W) < |u|^{-1}$ and pure refined ∂-radii θ_V and θ_W, respectively.*

(a) *Then $W^\vee$ has pure refined ∂-radii $-\theta_W$.*

(b) *If $\theta_V = \theta_W$, then we have $R_\partial(V \otimes W^\vee) > R_\partial(V)$.*

(c) *If $\theta_V \neq \theta_W$, then $V \otimes W^\vee$ has pure ∂-radii $R_\partial(V)$ and pure refined ∂-radii $\theta_V - \theta_W$.*

(d) *Moreover, if we do not assume that V and W have pure refined ∂-radii and let U denote the maximal submodule of $V \otimes W^\vee$ that has ∂-radii strictly larger than $R_\partial(V)$, then we have*

$$\dim U = \sum_{\theta \in \kappa^{(s)}_{K^{\mathrm{alg}}}} \mathrm{multi}_\theta(\Theta_\partial(V)) \cdot \mathrm{multi}_\theta(\Theta_\partial(W)).$$

Proof. (a) is straightforward, and (d) follows from (b) and (c) by the decomposition (1.3.27).

When $IR_\partial(V) = IR_\partial(W) < \omega$, (b) follows from Lemma 1.3.16(c), and (c) follows from the "moreover" part of the same lemma.

When $p > 0$ and $IR_\partial(V) = IR_\partial(W) > p^{-1/(p-1)}$, (b) and (c) for V and W follow from the same statement for the ∂-Frobenius antecedents of V and W, by the relation described in Proposition 1.3.18(a).

We now prove (b) and (c) in the case when $p > 0$ and $IR_\partial(V) = IR_\partial(W) = p^{-1/(p-1)}$. First, Lemma 1.2.18(3) implies that

$$\varphi_*^{(\partial)} V \otimes \big(\varphi_*^{(\partial)} W\big)^\vee = \big(\varphi_*^{(\partial)}(V \otimes W^\vee)\big)^{\oplus p}.$$

Note that Proposition 1.3.18(2) implies that the multiset of refined intrinsic ∂-radii of V is composed of all the solutions to

$$\left(\frac{x}{p}\right)^p - \frac{x}{p} = u\theta_V,$$

each with multiplicity $\dim V$, and that the multiset of refined intrinsic ∂-radii of W is composed of all the solutions to

$$\left(\frac{x}{p}\right)^p - \frac{x}{p} = u\theta_W,$$

each with multiplicity $\dim W$. If $\theta_V \neq \theta_W$, by (c) in the visible case together with Theorem 1.3.26, the multiset of refined intrinsic ∂'-radii of $\varphi_*^{(\partial)} V \otimes (\varphi_*^{(\partial)} W)^\vee$ consists of roots of

$$\left(\frac{x}{p}\right)^p - \frac{x}{p} = u(\theta_V - \theta_W),$$

each with multiplicity $p \dim V \dim W$. Statement (c) then follows from Proposition 1.3.18(b). If $\theta_V = \theta_W$, by (b) in the visible case together with Theorem 1.3.26, $\varphi_*^{(\partial)} V \otimes \varphi_*^{(\partial)} W$ has a submodule of dimension $(p-1)\dim V \dim W$ whose intrinsic ∂'-radius is strictly larger than $p^{-p/(p-1)}$. By Lemma 1.2.18(e), this can happen only if $IR_\partial(V \otimes W) > p^{-1/(p-1)}$, which is what we need to prove in (b). □

Remark 1.3.29. We remark that if we do not assume that ∂ is of rational type but assume that $R_\partial(V) < |\partial|_K^{-1}$ instead, all the results in the subsection still hold (note that we do not need Frobenius antecedent in the visible case).

1.4. ***Multiple derivations.*** Having studied the situation of one single derivation, we now let multiple commuting derivations interact. This essentially amounts to putting the information from each derivation together. To give the refined radii for multiple derivations a more canonical definition, we will represent the multiset of refined radii as a multiset of differential forms.

Notation 1.4.1. In this subsection, we put $J = \{1, \dots, m\}$.

Definition 1.4.2. Let K be a differential ring of order m, that is, a ring equipped with m commuting derivations $\partial_1, \dots, \partial_m$. A *$\partial_J$-differential module*, or simply a *differential module*, is a finite projective K-module V equipped with commuting actions of $\partial_1, \dots, \partial_m$. We will apply the results in previous subsections to each ∂_j separately.

Definition 1.4.3. Let K and V be as above, and let R be a complete K-algebra. For $\boldsymbol{v} \in V$ and $T_1, \dots, T_m \in R$, we define the *∂_J-Taylor series* to be

$$\mathbb{T}(\boldsymbol{v}; \partial_J; T_1, \dots, T_m) = \sum_{e_J=0}^{\infty} \frac{\partial_J^{e_J}(\boldsymbol{v})}{(e_J)!} T_J^{e_J} \in V \otimes_K R,$$

if it converges.

We will need the following tautological lemma in the proof of Theorem 1.4.20.

Lemma 1.4.4. *Let $\partial = \alpha_1\partial_1 + \cdots + \alpha_m\partial_m$ be another derivation, with $\alpha_1, \dots, \alpha_m$ in K. To simplify the notation, we formally write $\alpha_j = \partial(u_j)$ for any $j \in J$ (and one can check that the formula* (1.4.5) *can be written with no reference to u_j). Then, for any $x \in V$, we have*

$$\mathbb{T}\big(x; \partial_J; \mathbb{T}(u_1; \partial; \delta) - u_1, \dots, \mathbb{T}(u_m; \partial; \delta) - u_m\big) = \mathbb{T}(x; \partial; \delta), \tag{1.4.5}$$

as formal power series in $V \otimes_K K[\![\delta]\!]$.

Proof. Since (1.4.5) is a tautological statement, we may assume that K is $\mathbb{Z}$-torsion free. It suffices to show that (1.4.5) is true modulo δ^n for any ∂_J-differential module V and for any $x \in V$, by induction on n. This is clear for $n = 1$. Assume that we have proved this claim for n and we need to prove it for $n+1$. It suffices to prove the equality

$$\frac{\partial}{\partial\delta}\mathbb{T}\big(x; \partial_J; \mathbb{T}(u_1; \partial; \delta) - u_1, \dots, \mathbb{T}(u_m; \partial; \delta) - u_m\big) = \frac{\partial}{\partial\delta}\mathbb{T}(x; \partial; \delta) = \mathbb{T}(\partial(x); \partial; \delta)$$

modulo δ^n (note that the derivation reduces the exponents on δ by 1). We compute the left hand side as follows.

$$\begin{aligned}
&\frac{\partial}{\partial\delta}\mathbb{T}\big(x; \partial_J; \mathbb{T}(u_1; \partial; \delta) - u_1, \dots, \mathbb{T}(u_m; \partial; \delta) - u_m\big) \\
&\quad= \sum_{e_J=0}^{\infty} \frac{\partial_J^{e_J}(x)}{(e_J)!}\frac{\partial}{\partial\delta}\Big(\big(\mathbb{T}(u_1; \partial; \delta) - u_1\big)^{e_1} \cdots \big(\mathbb{T}(u_m; \partial; \delta) - u_m\big)^{e_m}\Big) \\
&\quad= \sum_{e_J=0}^{\infty} \frac{\partial_J^{e_J}(x)}{(e_J)!}\Big(\sum_{j\in J} e_j \cdot \big(\mathbb{T}(u_1; \partial; \delta) - u_1\big)^{e_1} \cdots \big(\mathbb{T}(u_j; \partial; \delta) - u_j\big)^{e_j-1} \\
&\qquad\qquad \cdots \big(\mathbb{T}(u_m; \partial; \delta) - u_m\big)^{e_m} \cdot \frac{\partial}{\partial\delta}\mathbb{T}(u_j; \partial; \delta)\Big) \\
&\quad= \sum_{j\in J}\sum_{e_J=0}^{\infty} \frac{\partial_J^{e_J}\big(\partial_j(x)\big)}{(e_J)!}\Big(\big(\mathbb{T}(u_1; \partial; \delta) - u_1\big)^{e_1} \\
&\qquad\qquad \cdots \big(\mathbb{T}(u_m; \partial; \delta) - u_m\big)^{e_m} \cdot \frac{\partial}{\partial\delta}\mathbb{T}(u_j; \partial; \delta)\Big)
\end{aligned}$$

By the induction hypothesis, modulo δ^n, this is congruent to

$$\begin{aligned}
\sum_{j\in J}\mathbb{T}(\partial_j(x); \partial; \delta) \cdot \frac{\partial}{\partial\delta}\mathbb{T}(u_j; \partial; \delta) &= \sum_{j\in J}\mathbb{T}(\partial_j(x); \partial; \delta) \cdot \mathbb{T}(\partial(u_j); \partial; \delta) \\
&= \mathbb{T}\Big(\sum_{j\in J}\partial_j(x)\partial(u_j); \partial; \delta\Big) = \mathbb{T}(\partial(x); \partial; \delta).
\end{aligned}$$

This finishes the induction and proves the lemma. □

Definition 1.4.6. Let K be a complete nonarchimedean differential field of order m and characteristic zero, and let V be a nonzero ∂_J-differential module over K. Define the *intrinsic radius* of V to be

$$IR(V) = \min_{j\in J}\{IR_{\partial_j}(V)\} = \min_{j\in J}\{|\partial_j|_{\mathrm{sp},K}/|\partial_j|_{\mathrm{sp},V}\}.$$

For $j \in J$, we say ∂_j is *dominant* for V if $IR_{\partial_j}(V) = IR(V)$. We define the multiset of *intrinsic subsidiary radii* $\mathfrak{IR}(V) = \{IR(V; 1), \dots, IR(V; \dim V)\}$ by collecting and ordering intrinsic radii from the Jordan–Hölder constituents, as in Definition 1.2.3. We again say that V has *pure intrinsic radii* if $\mathfrak{IR}(V)$ is pure as a multiset.

We similarly define the *extrinsic radius* $ER(V)$ to be the minimum of $R_{\partial_j}(V)$ and the multiset of *extrinsic subsidiary radii* $\mathfrak{ER}(V) = \{ER(V; 1), \dots, ER(V; \dim V)\}$ by collecting and ordering extrinsic radii from the Jordan–Hölder constituents.

Definition 1.4.7. Let K be a complete nonarchimedean differential field of order m and characteristic zero. We say that K is of *rational type* with respect to a set of parameters $\{u_j : j \in J\}$ if each ∂_j is of rational type with respect to u_j, and $\partial_i(u_j) = 0$ for $i \neq j$ in J.

Hypothesis 1.4.8. For the rest of this subsection, let K be a complete nonarchimedean field of characteristic zero, equipped with commuting derivations ∂_J of rational type with respect to parameters u_J. Let V be a ∂_J-differential module with pure ∂_j-radii for each $j \in J$. We assume moreover that $IR(V) < 1$.

Notation 1.4.9. For each j, put $s_j = -\log(\omega R_{\partial_j}(V)^{-1})$, $\lambda_j = \lambda(IR_{\partial_j}(V))$, and $r_j = r(IR_{\partial_j}(V))$. By Theorem 1.2.20, we have $s_j \in \mathbb{Q} \cdot \log|K^\times|$ for any j.

Definition 1.4.10. By Theorem 1.3.26, we may replace K by a finite tamely ramified extension such that V admits a direct sum decomposition $V = \bigoplus V_{\theta_J}$, where each direct summand V_{θ_J} has pure refined ∂_j-radii θ_j for any $j \in J$. Define the multiset of *refined radii* of V, denoted by $\Theta(V)$, to be the collection of $\vartheta = \sum_{j\in J} \theta_j \, du_j$ with multiplicity $\dim V_{\theta_J}$, where ϑ is viewed as an element of

$$\bigoplus_{j\in J} \kappa_{K^{\mathrm{alg}}}^{(s_j)} \, du_j.$$

The reason that we write the refined radii in the form of differentials will be justified later, in Theorem 1.4.20.

We will also consider cases where the derivations with larger radii of convergence are ignored.

(i) Let $\mathcal{I}\Theta(V)$ be the multiset consisting of elements $\sum \theta_j \, du_j$ with multiplicity $\dim V_{\theta_J}$, where the sum is taken over those j such that $IR_{\partial_j}(V_{\theta_J}) = IR(V_{\theta_J})$; this is called the multiset of *refined intrinsic radii*. Often, we view it as a multiset of elements in

$$\bigoplus_{j\in J} \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{du_j}{u_j} \quad \text{for } \mathfrak{s} = -\log(\omega IR(V)^{-1}).$$

We remark that this definition does not depend on the field extension of K we made earlier.

(ii) Let $\mathscr{E}\Theta(V)$ be the multiset consisting of elements $\sum \theta_j du_j$ with multiplicity $\dim V_{\theta_J}$, where the sum is only taken over those j such that $R_{\partial_j}(V_{\theta_J}) = R(V_{\theta_J})$. We call it the *refined extrinsic radii*.

Definition 1.4.11. Let $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_m) \in (0, 1]^m$. A norm $|\cdot|_V$ on V is $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_m)$-*good* (or simply *good* if $\boldsymbol{b}_j = IR_{\partial_j}(V)$ for all $j \in J$) if it is $\boldsymbol{b}_j$-good with respect to ∂_j for all $j \in J$.

Remark 1.4.12. In contrast to the single derivation case, we do not know if a good norm exists in general, unless we assume that K is discretely valued, in which case, Lemma 1.4.14 below gives an affirmative answer. Hypothesis 1.4.13 below may not be necessary for some of the results later in this subsection, as one might get around using some approximation process. Since we will work with complete discrete valuation field in most applications, we restrict ourselves here to this case.

Hypothesis 1.4.13. For the rest of this subsection, we assume that K is discretely valued.

Lemma 1.4.14. *Assume that $\boldsymbol{b}_j \in (0, IR_{\partial_j}(V)]$ for any $j \in J$, and that $\boldsymbol{b}_j < 1$ for all j if $p > 0$. Then the differential module V admits a $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_m)$-good norm.*

Proof. We first remark that if $IR_{\partial_j}(V) < 1$, Theorem 1.2.20 implies $IR_{\partial_j}(V) \in |K^\times|^{\mathbb{Q}}$. To prove the lemma, we may assume $\boldsymbol{b}_j = IR_{\partial_j}(V)$.

By the same argument as in Lemma 1.3.9 using Frobenius antecedent, it suffices to prove the lemma under the assumption that $\boldsymbol{b}_j \leq \omega$ for any $j \in J$. Note that the ∂_j-Frobenius antecedent is compatible with $\partial_{j'}$ for $j' \neq j$. Let K' be the completion of $K(x_J)$ with respect to the e^{-s_J}-Gauss norm, where we set $\partial_j(x_{j'}) = 0$ for all $j, j' \in J$ and $s_j = -\log(\omega(\boldsymbol{b}|u|)^{-1})$. In particular, K' is discretely valued since $\mathrm{e}^{-s_j} \in |K^\times|^{\mathbb{Q}}$ for any $j \in J$.

We first show that $V' := V \otimes K'$ has a $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$-good norm. For this, it suffices to show that given any norm $|\cdot|_{V'}$ with orthonormal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_d$, the submodule M' of V' generated by

$$\left\{ x_J^{a_J} \partial_J^{a_J} \boldsymbol{e}_i : a_j \in \mathbb{Z}_{\geq 0} \text{ for any } j \in J \text{ and } i \in \{1, \dots, d\} \right\}$$

over $\mathcal{O}_{K'}$ is a finite $\mathcal{O}_{K'}$-module; if so, M' gives rise to a norm on V', under which $|\partial_j| \leq |x_j| = \mathrm{e}^{-s_j}$ for all j verify the conditions of $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$-good norm in Definition 1.3.4. To prove that M' is a finite $\mathcal{O}_{K'}$-submodule, it suffices to prove that $|x_j^n \partial_j^n|_{V'}$ is bounded for each j as $n \to +\infty$ (we used here the fact that K' is discretely valued, otherwise boundness may not imply finiteness). It is then enough to verify this boundness condition for any K'-norm on V'. In particular, for each of ∂_j, we can choose a $\boldsymbol{b}_j$-good norm by Lemma 1.3.9, for which $|x_j^n \partial_j^n|_{V'} \leq 1$. Thus M' is finite over $\mathcal{O}_{K'}$ and hence we have a $(\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$-good norm on V'.

This norm restricts to a K-norm on V satisfying all the norm conditions in Definition 1.3.4. We use the following lattice lemma to show that it admits an orthogonal basis. □

Lemma 1.4.15. *Let F be a complete discrete valuation field and let V be a finite dimensional vector space, equipped with a norm compatible with F. Assume moreover that the valuation group* $\log|V - \{0\}|_V$ *of V is also discrete. Then V admits an orthogonal basis.*

Proof. The proof is almost the same as [Kedlaya 2010, Lemma 1.3.7]. For completeness and the convenience of the reader, we reproduce it here.

We use induction on the dimension $n = \dim V$. When $n = 1$, the statement is obvious; any nonzero vector forms an orthogonal basis. Now assuming the statement for $n-1$, we will prove it for an n-dimensional F-normed vector space $(V, |\cdot|_V)$ whose valuation group is discrete. Pick a nonzero vector $v_1 \in V$ and denote $W = V/Fv_1$, provided with the quotient norm $|\cdot|_W$; this is again F-compatible and has discrete valuation group. By the inductive hypothesis, W admits an orthogonal basis $\bar{v}_2, \dots, \bar{v}_n$. For $i = 2, \dots, n$, we pick $v_i \in V$ that lifts $\bar{v}_i \in W$ such that $|v_i|_V = |\bar{v}_i|_W$ (this is possible because V has discrete valuation group). We claim that $v_1, \dots, v_n$ form an orthogonal basis of V.

We need to prove that $|v|_V = \max_i\{|x_i||v_i|_V\}$ for any $v = x_1v_1 + \cdots + x_nv_n \in V$. It is clear that $|v|_V$ is less than or equal to the right hand side; we need to show $|v|_V \geq \max_i\{|x_i||v_i|_V\}$. We prove it the following two cases separately.

(i) If the maximum above is achieved by some $i \geq 2$, we have

$$|v|_V \geq |v \bmod Fv_1|_W = |x_2\bar{v}_2 + \cdots + x_n\bar{v}_n|_W = \max_{i=2}^n\{|x_i||\bar{v}_i|_W\} = \max_{i=1}^n\{|x_i||v_i|_V\}.$$

(ii) We have $|x_1||v_1| > |x_i||v_i|$ for all $i = 2, \dots, n$. In this case, we have $|v| = |x_1||v_1| = \max_i\{|x_i||v_i|_V\}$.

This shows that $v_1, \dots, v_n$ form an orthogonal basis of V and finishes the proof of the lemma. □

Remark 1.4.16. One may hope to find an analog of Example 1.3.20 for ∂_J-differential modules. This, however, amounts to carefully choosing the element x in Example 1.3.20 so that the actions of ∂_J *commutes*. For this, we might need to restrict the possible intrinsic refined radii to a subset of

$$\bigoplus_{j\in J} \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{du_j}{u_j}, \quad \text{where } \mathfrak{s} = -\log(\omega IR(V)^{-1}).$$

Unfortunately, we do not know how to identify this subset in general. Proposition 1.4.17 below partly answers this question.

It would be interesting to know, when $p > 0$, whether any element in

$$\bigoplus_{j\in J} \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{du_j}{u_j}$$

can appear in the multiset of refined intrinsic radii of some differential module. The referee also pointed out that the reduction of ∂_j may give rise to a $\mathscr{D}$-module in characteristic p. We do not know if this construction is independent of the choice of good norms. But we suspect that this is related to the reduction of some arithmetic $\mathscr{D}$-module when the differential module comes from one.

Proposition 1.4.17. *Assume that $IR(V) < \omega$ and that either $p = 0$ or $d = \mathrm{rank}\, V = 1$. Let $\mathfrak{s} = -\log(\omega IR(V)^{-1})$. Note that the action of $u_j\partial_j$ on K induces a derivation on $\kappa_{K^{\mathrm{unr}}}^{(\mathfrak{s})}$. If*

$$\vartheta = \sum_{j\in J} \theta_j \frac{du_j}{u_j} \in \mathscr{I}\Theta(V),$$

then for $i, j \in J$, we have $u_i\partial_i\theta_j = u_j\partial_j\theta_i$ in $\kappa_{K^{\mathrm{unr}}}^{(\mathfrak{s})}$.

Proof. By possibly replacing K by a finite tamely ramified extension, we reduce to the case when V is irreducible with a good norm given by an orthonormal basis, and when V has pure refined intrinsic radii $\sum_{j\in J} \theta_j(du_j/u_j)$. The $u_j\partial_j$-action with respect to this basis is given by a matrix $N_j \in \mathrm{Mat}_{d\times d}(\mathfrak{m}_K^{(\mathfrak{s})})$. Since ∂_i and ∂_j commute with each other for any $i, j \in J$, we have

$$N_iN_j + u_i\partial_i(N_j) = N_jN_i + u_j\partial_j(N_i). \tag{1.4.18}$$

Taking the trace of (1.4.18) gives $d\cdot u_i\partial_i\theta_j = d\cdot u_j\partial_j\theta_i$, which yields the proposition because d is invertible in κ_K. □

Before proceeding, we need some notation to use in Theorem 1.4.20 below.

Notation 1.4.19. If $p > 0$, we can write an integer $n \in \mathbb{N}$ as $n = a_0 + pa_1 + \cdots + p^k a_k$ with $a_1, \dots, a_k \in \{0, \dots, p-1\}$. Put $\sigma_p(n) = a_0 + \cdots + a_k$ if $p > 0$, and $\sigma_p(n) = 0$ if $p = 0$. It is straightforward to check that $\sigma_p(n_1) + \sigma_p(n_2) \geq \sigma_p(n_1 + n_2)$ for $n_1, n_2 \in \mathbb{N}$, and that $|n!| = \omega^{n-\sigma_p(n)}$ for $n \in \mathbb{N}$.

The following theorem explains how refined radii change when we consider a different set of derivations, and hence justifies the reason we wrote refined radii in the form of differentials in Definition 1.4.10.

Theorem 1.4.20. *Assume that V has pure refined ∂_j-radii $\theta_j \in \kappa^{(s_j)}_{K^{\mathrm{alg}}}$ for any $j \in J$. Let K' be a complete discrete valuation field containing K. Let ∂ be a derivation on K', extending the action of $\alpha_1\partial_1 + \cdots + \alpha_m\partial_m$ on K to K', where $\alpha_1, \ldots, \alpha_m \in K'$. In fact, we have $\alpha_j = \partial(u_j)$ for any $j \in J$. We assume that ∂ is a derivation of rational type on K'. Set $s = \min_{j\in J}\{s_j - \log|\alpha_j|\}$ and let J_0 be a subset of J consisting of j for which $s = s_j - \log|\alpha_j|$. Assume moreover that $IR_j(V) < 1$ if $j \in J_0$. Put $\theta = \sum_{j\in J_0}\alpha_j\theta_j \in \kappa^{(s)}_{K'^{\mathrm{alg}}}$.*

Then $R_\partial(V \otimes_K K') \leq \omega e^s$, and the equality is achieved if and only if $\theta \neq 0$ in $\kappa^{(s)}_{K'^{\mathrm{alg}}}$. Moreover, when equivalent statement is verified, $V \otimes_K K'$ has pure ∂-radii ωe^s and pure refined ∂-radii θ.

Proof. For $j \in J$, the equality $\alpha_j = \partial(u_j)$ follows from applying ∂ to u_j.

By Lemma 1.4.14 and by possibly enlarging K and K', we may assume that V admits a norm given by some orthonormal basis $\underline{e}$ such that, for any $j \in J$,

(i) if $IR_j(V) < 1$, the norm is good with respect to ∂_j, and

(ii) if $IR_j(V) = 1$, the norm is $\boldsymbol{b}_j$-good with respect to ∂_j for some $\boldsymbol{b}_j$ in

$$(|\alpha_j|e^{s-s_j}, 1) \cap |K^\times|^{\mathbb{Q}}.$$

In this case, instead of taking the usual definitions of r_j, λ_j, and s_j, we set $r_j = r(\boldsymbol{b}_j)$, $\lambda_j = \lambda(\boldsymbol{b}_j)$, and $s_j = s - \log(\boldsymbol{b}_j|\alpha_j|^{-1})$. Note that $s_j - \log|\alpha_j| > s$ still holds.

Similarly to Notation 1.3.3, we define integers r and λ as follows.

(x) When $|\partial|_{K'}\omega e^s < \omega$ we denote $\lambda = 0$ and $r = 1$.

(xx) When $|\partial|_{K'}\omega e^s \in [\omega, 1)$ and $p > 0$, let λ denote the unique nonnegative integer such that

$$|\partial|_{K'}\omega e^s \in \big[p^{-1/p^{\lambda-1}(p-1)}, p^{-1/p^{\lambda}(p-1)}\big),$$

and put $r = p^\lambda$. In this case, we have $(|\partial|_{K'}\omega e^s)^{p^k} \leq \omega$ for $k < \lambda$ and hence $(|\partial|_{K'}\omega e^s)^i \leq \omega^{\sigma_p(i)}$ for $i = 1, \ldots, r-1$.

For each $j \in J$, we have

$$\left|\frac{\partial_j^i}{i!}\right|_V \leq |\partial_j|_K^i, \text{ for } i = 1, \ldots, r_j - 1, \quad \text{and } |\partial_j^{r_j}|_V \leq |u_j|^{-r_j}e^{-r_js_j}.$$

For $i = 1, \ldots, r$, the action of ∂^i on an element x of $\underline{e}$ can be expressed in terms of the actions of ∂_J, according to the coefficients of δ^i on the left hand side of (1.4.5), applied to x. More precisely, for any $j \in J$ and any $i \in \mathbb{N}$, the coefficient of δ^i in $\mathbb{T}(u_j; \partial; \delta) - u_j$ has norm less than or equal to $|\partial(u_j)|\,|\partial|_{K'}^{i-1} = |\alpha_j|\,|\partial|_{K'}^{i-1}$. For any coefficient that arises in the ∂_J-Taylor series expansion, if we put $e_j = c_j + d_j r_j$ with $c_j \in \{0, \dots, r_j - 1\}$ and $d_j \in \mathbb{Z}_{\geq 0}$ for any $j \in J$, then we have

$$\left|\frac{\partial_J^{e_J}(x)}{(e_J)!}\right|_V \leq \prod_{j\in J}\left|\frac{\partial_j^{d_j r_j}}{(d_j r_j)!}\right|_V \cdot \prod_{j\in J}\left|\frac{\partial_j^{c_j}(x)}{(c_j)!}\right|_V$$
$$\leq |x|_V \cdot \prod_{j\in J}|\partial_j|_K^{c_j} \cdot \prod_{j\in J}\bigl(\mathrm{e}^{-d_j r_j s_j}\omega^{-d_j r_j + \sigma_p(d_j r_j)}\bigr),$$

Putting these two bounds together, we see that if a δ^i-term on the left hand side of (1.4.5) arises in a term that includes $\partial_J^{e_J}(x)/(e_J)!$ (which particularly implies that $i \geq e_1 + \cdots + e_m$), then its norm is smaller than or equal to

$$|x|\,|\partial|_{K'}^{i-e_1-\cdots-e_m}\prod_{j\in J}|\alpha_j|^{e_j}\prod_{j\in J}|\partial_j|_K^{c_j}\cdot\prod_{j\in J}\bigl(\mathrm{e}^{-d_j r_j s_j}\omega^{-d_j r_j+\sigma_p(d_j r_j)}\bigr)$$
$$= |x|\,|\partial|_{K'}^{i-e_1-\cdots-e_m}\prod_{j\in J}\bigl(|\partial_j|_K|\alpha_j|\bigr)^{c_j}\cdot\prod_{j\in J}\Bigl(\bigl(|\alpha_j|_K\mathrm{e}^{-s_j}\bigr)^{d_j r_j}\omega^{-d_j r_j+\sigma_p(d_j r_j)}\Bigr)$$
$$\leq |x|\,|\partial|_{K'}^{i-e_1-\cdots-e_m}\prod_{j\in J}|\partial|_{K'}^{c_j}\cdot\prod_{j\in J}\bigl(\mathrm{e}^{-d_j r_j s}\omega^{-d_j r_j+\sigma_p(d_j r_j)}\bigr)$$
$$\text{(note } |\partial|_{K'} \geq |\partial(u_j)|\,|u_j|^{-1} = |\alpha_j|\,|\partial_j|_K)$$
$$\leq |x|\,|\partial|_{K'}^{i}(|\partial|_{K'}\omega\mathrm{e}^{s})^{-\sum_j d_j r_j}\omega^{\sigma_p(\sum_j d_j r_j)}.$$

When $i = 1, \dots, r-1$, the coefficient of this δ^i-term has norm less than or equal to $|\partial|_{K'}^i|x|$ by condition (xx). When $i = r$, this δ^i-term has norm less than or equal to $|\partial|_{K'}^r\bigl((|\partial|_{K'}\omega\mathrm{e}^s)^{-r}\omega\bigr)|x| = \omega^{-r+1}\mathrm{e}^{-rs}|x|$; the equality can happen only when $\sum_j d_j r_j = r$ and $\sigma_p(\sum_j d_j r_j) = \sum_j \sigma_p(d_j r_j)$, which together yield $e_j = r$ for some $j \in J_0$ and $e_{j'} = 0$ for $j' \neq j$. When equality of norms is achieved, the corresponding δ^i-term is $\alpha_j^r\partial_j^r(x)/r!$. Therefore, modulo $\mathfrak{m}_{K'}^{(rs)+}$, the matrix of ∂^r with respect to $\underline{e}$ is congruent to $\sum_{j\in J_0}\alpha_j^r\partial_j^r$; this is a sum of matrices with single eigenvalues $\alpha_j^r\theta_j^r$ for $j \in J_0$ (note that, again, $IR_{\partial_j}(V) < 1$ for all $j \in J_0$). By Lemma 1.3.14, we have $R_\partial(V) \leq \omega\mathrm{e}^s$ and this is an equality if and only if $\sum_{j\in J_0}\alpha_j^r\theta_j^r \neq 0$ in $\kappa_{K'^{\mathrm{alg}}}^{(rs)}$, which is equivalent to $\sum_{j\in J_0}\alpha_j\theta_j \neq 0$ in $\kappa_{K'^{\mathrm{alg}}}^{(s)}$; note that r is always 1 or a power of p. Moreover, if the equivalent condition is satisfied, V has pure refined ∂-radii

$$\Bigl(\sum_{j\in J_0}\theta_j^r\alpha_j^r\Bigr)^{1/r} = \sum_{j\in J_0}\theta_j\alpha_j = \theta \in \kappa_{K'^{\mathrm{alg}}}^{(s)}. \qquad \square$$

Corollary 1.4.21. *Let V be a ∂-differential module over K and let*

$$f = \mathbb{T}(\,\cdot\,;\partial;T) : K \to K[\![T/u]\!]_0$$

and f^*V be as in Lemma 1.2.6(d). For $\eta \in [0, |u|)$, let F_η denote the completion of $K(T)$ with respect to the η-Gauss norm.

(a) *If $\eta \in (0, R_\partial(V)]$, $f^*V \otimes F_\eta$ has pure intrinsic ∂_T-radius 1; if $\eta \in (R_\partial(V), |u|)$, $f^*V \otimes F_\eta$ has (extrinsic) ∂_T-radius $R_\partial(V)$.*

(b) *When $\eta \in (R_\partial(V), |u|)$, we have $\Theta_{\partial_T}(f^*V \otimes F_\eta) = \Theta_\partial(V)$.*

Proof. For any $x \in V$, $f^*(\partial(x)) = \partial_T(f^*(x))$. The first statement follows from this immediately, and the second statement follows from Theorem 1.4.20. (When $IR_\partial(V) = 1$, (b) is void.) □

Remark 1.4.22. Similar to Remark 1.3.29, if we do not assume that $\partial_1, \dots, \partial_n$ are of rational type (but only commuting), the results from this subsection still hold if, for any ∂_j for which the refined ∂_j-radii are relevant, we have $R_{\partial_j}(V) \le |\partial_j|_K^{-1}$.

1.5. ***One-dimensional variation of refined radii.*** Having established the results for differential modules over a field, we now study the case of a differential module over a rigid analytic annulus or a rigid analytic disc. It is particularly interesting to study how the multisets of (subsidiary) radii of the differential module with respect to different Gauss norms vary as we change the radii which define the Gauss norm. Kedlaya and the author had proved various results on this in [Kedlaya 2010, Chapter 11; Kedlaya and Xiao 2010, Section 2], essentially stating that the (subsidiary) radii are piecewise log-affine functions in the radii of the annulus. In this subsection, we will characterize how the refined radii change as we change the radii for the Gauss norm, in the case when the functions given by the subsidiary radii are in fact log-affine.

Hypothesis 1.5.1. Throughout this subsection, we assume that K is a complete nonarchimedean field of characteristic zero and residual characteristic p. We also assume that K is equipped with derivations $\partial_1, \dots, \partial_m$ of rational type with respect to $u_1, \dots, u_m$.

Notation 1.5.2. Put $J = \{1, \dots, m\}$ and $J^+ = J \cup \{0\}$. For $\eta > 0$, let F_η denote the completion of $K(t)$ under the η-Gauss norm $|\cdot|_\eta$. Set $\partial_0 = d/dt$ on $K[t]$; it extends by continuity to F_η and ring of functions on discs or annuli. The derivations ∂_{J^+} are of rational type on F_η.

Notation 1.5.3. Fix $j \in J^+$ and an interval $I \subseteq [0, \infty)$. We say that I is an *open interval* in $[0, \infty)$ if it is of the form $[0, \beta)$ or (α, β), where $0 < \alpha < \beta$. Put $\dot{I} = I \backslash \{0\}$. For M a ∂_j-differential module of rank d over $A_K^1(I)$, $r \in -\log \dot{I}$, and $i \in \{1, \dots, d\}$, we put

$$f_i^{(j)}(M, r) = -\log R_{\partial_j}(M \otimes F_{e^{-r}}; i), \quad F_i^{(j)}(M, r) = f_1^{(j)}(M, r) + \cdots + f_i^{(j)}(M, r).$$

Theorem 1.5.4. *Fix $j \in J^+$ and an interval $I \subseteq [0, +\infty)$. Let M be a ∂_j-differential module of rank d over $A_K^1(I)$.*

(a) (linearity) *For $i = 1, \dots, d$, the functions $f_i^{(j)}(M, r)$ and $F_i^{(j)}(M, r)$ are continuous. They are piecewise affine on the locus where $f_i^{(j)}(M, r) > -\log|u_j|$ if $j \in J$; and they are piecewise affine on all of $-\log \dot{I}$ if $j = 0$.*

(b) (weak integrality)

(b1) *Suppose $p = 0$ or $j = 0$. If $i = d$ or $f_{i+1}^{(j)}(M, r_0) < f_i^{(j)}(M, r_0)$, the slopes of $F_i^{(j)}(M, r)$ in some neighborhood of $r = r_0$ belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i^{(j)}(M, r)$ and $F_i^{(j)}(M, r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{d}\mathbb{Z}$.*

(b2) *Suppose $p > 0$ and $j \in J$. If $f_i^{(j)}(M, r_0) > 1/(p^n(p-1)) \log p - \log|u_j|$ for some $n \in \mathbb{Z}_{\geq 0}$, then the slopes of each $f_i^{(j)}(M, r)$ and $F_i^{(j)}(M, r)$ in some neighborhood of r_0 belong to $\frac{1}{p^n d!}\mathbb{Z}$.*

(c) (monotonicity) *Suppose $0 \in I$ and suppose either $j \in J$, or $j = 0$ and $f_i^{(0)}(M, r_0) > r_0$. Then the slopes of $F_i^{(j)}(M, r_0)$ are nonpositive in a neighborhood of r_0.*

(d) (convexity) *For $i = 1, \dots, d$, the function $F_i^{(j)}(M, r)$ is convex.*

(e) (decomposition) *Assume that I is an open interval in $[0, +\infty)$. Suppose that for some $i \in \{1, \dots, d\}$, $F_i^{(j)}(M, r)$ is affine and $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ for $r \in -\log I$. Then we can write M uniquely as the direct sum of two ∂_j-differential submodules M_1 and M_2, such that, for any $\eta \in I$, the multiset of ∂_j-radii of $M_1 \otimes F_\eta$ exactly consists of the smallest i elements in the multiset of ∂_j-radii of $M \otimes F_\eta$.*

Proof. This is [Kedlaya and Xiao 2010, Theorems 2.2.5, 2.2.6, and 2.3.5]. □

Notation 1.5.5. Let $I \subseteq [0, +\infty)$ be an interval and let M be a ∂_{J^+}-differential module of rank d on $A_K^1(I)$. For $r \in -\log \dot{I}$ and $i \in \{1, \dots, d\}$, we put

$$f_i(M, r) = -\log IR(M \otimes F_{e^{-r}}; i) \quad \text{and} \quad F_i(M, r) = f_1(M, r) + \cdots + f_i(M, r).$$

Suppose that $I \subseteq [0, 1)$ and that $|u_j| = 1$ for any $j \in J$, we put

$$\hat{f}_i(M, r) = -\log ER(M \otimes F_{e^{-r}}; i) \quad \text{and} \quad \hat{F}_i(M, r) = \hat{f}_1(M, r) + \cdots + \hat{f}_i(M, r).$$

Theorem 1.5.6. *Fix an interval $I \subseteq [0, +\infty)$. Let M be a ∂_{J^+}-differential module of rank d over $A_K^1(I)$.*

(a) (linearity) *For $i = 1, \dots, d$, the functions $f_i(M, r)$ and $F_i(M, r)$ are continuous and piecewise affine.*

(b) (integrality) *If $i = d$ or $f_i(M, r_0) > f_{i+1}(M, r_0)$, then the slopes of $F_i(M, r)$ in some neighborhood of r_0 belong to $\mathbb{Z}$. Consequently, the slopes of each $f_i(M, r)$ and $F_i(M, r)$ belong to $\frac{1}{1}\mathbb{Z} \cup \cdots \cup \frac{1}{d}\mathbb{Z}$.*

(c) (monotonicity) *Suppose that $0 \in I$. Then the slopes of $F_i(M, r)$ are nonpositive, and each $F_i(M, r)$ is constant for r sufficiently large.*

(d) (convexity) *For* $i = 1, \dots, d$, *the function* $F_i(M, r)$ *is convex.*

(e) (decomposition) *Suppose that* I *is an open interval in* $(0, +\infty)$, *and suppose that, for some* $i \in \{1, \dots, d-1\}$, *the function* $F_i(M, r)$ *is affine and* $f_i(M, r) > f_{i+1}(M, r)$ *for* $r \in -\log \dot{I}$. *Then* M *can be uniquely written as the direct sum of two* ∂_{J^+}*-differential submodules* M_1 *and* M_2 *such that, for any* $\eta \in \dot{I}$, *the multiset of intrinsic radii of* $M_1 \otimes F_\eta$ *exactly consists of the smallest* i *elements in the multiset of intrinsic radii of* $M \otimes F_\eta$.

(f) (dichotomy) *Suppose that* I *is an open interval in* $[0, +\infty)$ *and that* M *is not the direct sum of two nonzero* ∂_{J^+}*-differential submodules. If* $f_1(M, r)$ *is affine for* $r \in -\log \dot{I}$, *then, for each* $j \in J^+$,

  (1) *either* $M \otimes F_\eta$ *has pure intrinsic* ∂_j*-radii and the intrinsic* ∂*-radius equals* $IR(M \otimes F_\eta)$ *for all* $\eta \in \dot{I}$, *or*

  (2) *we have* $IR_{\partial_j}(M \otimes F_\eta) > IR(M \otimes F_\eta)$ *for all* $\eta \in \dot{I}$.

Moreover, if $|u_j| = 1$ *for any* $j \in J$ *and if* $I \subseteq [0, 1)$, *then the same statements above except* (*c*) *hold for* $\hat{f}_i(M, r)$ *and* $\hat{F}_i(M, r)$ *in place of* $f_i(M, r)$ *and* $F_i(M, r)$, *respectively. In this case, the following statement holds.*

(c′) (monotonicity) *Suppose that* $0 \in I$. *For* $i = 1, \dots, d$, *for any point* r_0 *where* $\hat{f}_i(M, r_0) > r_0$, *the slopes of* $\hat{F}_i(M, r)$ *are nonpositive in some neighborhood of* r_0. *We also have* $\hat{f}_i(M, r) = r$ *for* r *sufficiently large.*

Proof. Statements (a)–(e) for $f_i(M, r)$ and $F_i(M, r)$ are proved in [Kedlaya and Xiao 2010, Theorems 2.4.4 and 2.5.1]. Statements (a), (b), (c′), (d), and (e) for $\hat{f}_i(M, r)$ and $\hat{F}_i(M, r)$ can be proved similarly as follows.

Let $\widetilde{K}$ denote the completion of $K(x_J)$ with respect to the $(1, \dots, 1)$-Gauss norm. For $I = [\alpha, \beta) \subseteq [0, 1)$, the Taylor series defines an injective continuous homomorphism $\tilde{f}^* : K\langle \alpha/t, t/\beta\}\} \to \widetilde{K}\langle \alpha/t, t/\beta\}\}$ such that $\tilde{f}^*(u_j) = u_j + x_j t$ (as in [Kedlaya and Xiao 2010, Notation 2.4.1]). For $\eta \in (\alpha, \beta)$, we use $\widetilde{F}_\eta$ to denote the completion of $\widetilde{K}(t)$ with respect to the η-Gauss norm. Then $\tilde{f}^*$ extends to an injective isometric homomorphism $\tilde{f}^* : F_\eta \hookrightarrow \widetilde{F}_\eta$.

We view $\tilde{f}^* M$ as a ∂_0-differential module on $A^1_{\widetilde{K}}[\alpha, \beta)$. Since

$$\partial_0|_{\tilde{f}^* M} = \partial_0|_M + \sum_{j \in J} x_j \partial_j|_M,$$

we have

$$R_{\partial_0}(M \otimes \widetilde{F}_\eta) = \min_{j \in J^+} \big\{ R_{\partial_j}(M \otimes F_\eta) \big\} = ER(M \otimes F_\eta), \text{ for any } \eta \in [\alpha, \beta).$$

In other words, $f_i^{(0)}(\tilde{f}^* M, r) = \hat{f}_i(M, r)$ for $r \in (-\log \beta, -\log \alpha)$. The theorem follows from Theorem 1.5.4; to obtain the decomposition in (e), we use Lemma 1.1.10 and Remark 1.1.11 to glue the decompositions over $A^1_{\widetilde{K}}[\alpha, \beta)$ and over F_η for some $\eta \in (\alpha, \beta)$.

We now prove (f) for the intrinsic radii; the proof for the extrinsic radii is similar. Fix $j \in J^+$. Assume that we are not in case (2). Then $IR_{\partial_j}(M \otimes F_\eta) = IR(M \otimes F_\eta)$ for some $\eta \in \dot{I}$. By Theorem 1.5.4(d), the that that $f_1^{(j)}(M, r)$ is convex forces $IR_{\partial_j}(M \otimes F_\eta) = IR(M \otimes F_\eta)$ for all $\eta \in \dot{I}$. Now, if $IR_{\partial_j}(M \otimes F_\eta; 2) > IR(M \otimes F_\eta)$ for all $\eta \in (\alpha, \beta)$, the decomposition (e) would imply that M is decomposable, which contradicts the assumption. Therefore, we have $IR_{\partial_j}(M \otimes F_\eta; 2) = IR(M \otimes F_\eta)$ for some $\eta \in \dot{I}$. By Theorem 1.5.4(d) again, we have the equality for all $\eta \in \dot{I}$. Continuing this argument for the third smallest and other subsidiary ∂_j-radii leads us to case (1). □

Next, we discuss how the multiset of refined ∂_j-radii of the ∂_j-differential module M changes when we base change the ∂_j-differential module M to the completions with respect to different Gauss norms, in the case when

$$f_1^{(j)}(M, r) = \cdots = f_{\operatorname{rank} M}^{(j)}(M, r)$$

is *affine*. Before proving general results, we first look at an example of ∂_j-differential module with pure refined ∂_j-radii when base changed to any completion with respect to the Gauss norm. It is a 1-dimensional family analog of Example 1.3.20.

Example 1.5.7. Let $j \in J^+$ and let $(\alpha, \beta) \subseteq (0, \infty)$ be an open interval. Fix $b \in \mathbb{Q}$ and $\theta \in \kappa_{K^{\mathrm{alg}}}^{(a)}$, where $a \in -\log |K^\times|^{\mathbb{Q}}$. Assume that

$$\mathrm{e}^a \alpha^b, \mathrm{e}^a \beta^b < \begin{cases} 1 & \text{if } p = 0, \\ p^{1/p} & \text{if } p > 0. \end{cases} \tag{1.5.8}$$

We will see that this *includes some nonvisible radii*. As noted in Remark 1.3.21, we cannot loosen the restriction in (1.5.8) from $p^{1/p}$ to $p^{1/(p-1)}$.

Let e be the prime-to-p part of the denominator of b. We have the following:

(i) If $p = 0$, then $a \in -\log |(K')^\times|$ and $\theta \in \kappa_{K'}^{(a)}$ for some finite *tamely ramified* extension K'/K. Let $x \in \mathfrak{m}_{K'}^{(a)}$ be a lift of θ. We set $n = 0$ and $d = 1$ in this case.

(ii) If $p > 0$ and $j = 0$, there exists $n \in \mathbb{N}$ such that

$$\theta^{p^n} \in \kappa_{K'}^{(p^n a)} \quad \text{with } p^n a \in -\log |(K')^\times| \text{ and } p^n eb \in p\mathbb{Z},$$

for some finite *tamely ramified* extension K'/K. Let $x \in \mathfrak{m}_{K'}^{(p^n a)}$ be a lift of θ^{p^n}. We set $d = p^n$.

(ii′) If $p > 0$ and $j \in J$, there exists $n \in \mathbb{N}$ such that $\theta^{p^n} \in (\kappa_{K'}^{(p^{n-1} a)})^p$ and $p^n eb \in \mathbb{Z}$ with $p^{n-1} a \in -\log |(K')^\times|$ for some finite *tamely ramified* extension K'/K. Let $x \in \mathfrak{m}_{K'^{(\partial_j)}}^{(p^n a)}$ be a lift of θ^{p^n}; this is possible by Lemma 1.2.16.

Let $A^1_{K'}(\alpha^{1/e},\beta^{1/e})$ be the open annulus with coordinate $t^{1/e}$. Let $\mathscr{L}^{(j)}_{x,b,(n)}$ denote the ∂_j-differential module over $A^1_{K'}(\alpha^{1/e},\beta^{1/e})$ of rank d with basis $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_d\}$, on which ∂_j acts per description

$$\partial_j\boldsymbol{e}_i=\boldsymbol{e}_{i+1}\text{ for }i=1,\dots,d-1\quad\text{and}\quad\partial_j\boldsymbol{e}_d=\begin{cases}xt^{-db}u_j^{-d}\boldsymbol{e}_1 & \text{if } j\in J,\\ xt^{-d(b+1)}\boldsymbol{e}_1 & \text{if } j=0.\end{cases}$$

We added u_j^{-d} and t^{-d} in the definition to balance the different normalizations on intrinsic ∂_j-radii.

Lemma 1.5.9. *Keep the notation as in Example 1.5.7. If we set $F'_{\mathrm{e}^{-r}}=F_{\mathrm{e}^{-r}}(t^{1/e})$, then for any $r\in(-\log\beta,-\log\alpha)$, $\mathscr{L}^{(j)}_{x,b,(n)}\otimes F'_{\mathrm{e}^{-r}}$ has pure intrinsic ∂_j-radii $\omega\mathrm{e}^{a-br}$ and pure refined ∂_j-radii θt^{-b}.*

Proof. Comparing this with Example 1.3.20 shows that for any r, $\mathscr{L}^{(j)}_{x,b,(n)}\otimes F'_{\mathrm{e}^{-r}}$ is isomorphic to $\mathscr{L}_{xt^{-db}u_j^{-d},(n)}$ if $j\in J$, and to $\mathscr{L}_{xt^{-d(b+1)},(n)}$ if $j=0$. Applying Lemma 1.3.23 to this ∂_j-differential module yields the result; note that the condition (1.5.8) corresponds to the condition on $\mathfrak{s}$ in Example 1.3.20. □

Theorem 1.5.10. *Fix $j\in J^+$. Let M be a ∂_j-differential module over an open annulus $A^1_K(\alpha,\beta)$ such that $M\otimes F_{\mathrm{e}^{-r}}$ has pure intrinsic ∂_j-radii $\omega\mathrm{e}^{a-br}<1$ for any $r\in(-\log\beta,-\log\alpha)$ (this implies that $f_1^{(j)}(M,r)=\dots=f^{(j)}_{\dim M}(M,r)$ is an affine function with slope b). Let e be the prime-to-p part of the denominator of b. Then there exists a unique direct sum decomposition*

$$M=\bigoplus_{\{\mu_e\theta\}\subseteq\kappa^{(a)}_{K^{\mathrm{alg}}}}M_{\{\mu_e\theta\}}$$

of ∂_j-differential modules over $A^1_K(\alpha,\beta)$ where the sum is over all $\mu_e\rtimes\mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits of $\kappa^{(a)}_{K^{\mathrm{alg}}}$, and the refined ∂_j-radii of $M_{\{\mu_e\theta\}}\otimes F_\eta$ for any $\eta\in(\alpha,\beta)$ is a multiset consisting of the $\mu_e\rtimes\mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits $\{t^{-b}\theta\}$ with appropriate multiplicities.

Moreover, if K' is a finite tamely ramified tension of K such that all the θ in the above decomposition belong to $\bigcup_n(\kappa^{(p^ns)}_{K'})^{1/p^n}$, then we have a unique direct sum decomposition

$$M\otimes_{K\{\{\alpha/t,t/\beta\}\}}K'\{\{\alpha^{1/e}/t^{1/e},t^{1/e}/\beta^{1/e}\}\}=\bigoplus_{\theta\in\kappa^{(a)}_{K^{\mathrm{alg}}}}M_\theta$$

of ∂_j-differential modules over $A^1_{K'}(\alpha^{1/e},\beta^{1/e})$ such that $M_\theta\otimes K'F'_\eta$ has pure refined ∂_j-radii $t^{-b}\theta$ for any $\eta\in(\alpha,\beta)$.

Proof. First of all, since defining a ∂_j-differential module only needs finite data, we may assume that $\mathbb{Q}\cdot\log|K^\times|\neq\mathbb{R}$.

The decomposition as stated in the theorem, if it exists, is determined by the decomposition of $M \otimes F_{e^{-r}}$ for each $r \in (-\log\beta, -\log\alpha)$; it is hence unique. We may always replace M by $M \otimes_{K\{\{\alpha/t, t/\beta\}\}} K'\{\{\alpha^{1/e}/t^{1/e}, t^{1/e}/\beta^{1/e}\}\}$ for e and any finite tamely ramified extension K' of K, and we may recover the result for M using Galois descent. In particular, we may assume that $e = 1$. Moreover, using Lemma 1.1.10 and Remark 1.1.11, it suffices to first obtain the decomposition in a neighborhood of each radius in (α, β) and then glue the decompositions on overlaps.

Let $r_0 \in (-\log\beta, -\log\alpha)$ be a point. We first assume that $IR_{\partial_j}(M \otimes F_{e^{-r_0}}) < 1$ when $p = 0$, and $IR_{\partial_j}(M \otimes F_{e^{-r_0}}) < p^{-1/p(p-1)}$ when $p > 0$ (note that this restriction still allows some nonvisible radii). By shrinking the interval (α, β) to a smaller neighborhood of r_0, we may assume that the condition above at r_0 holds for all points in $(-\log\beta, -\log\alpha)$. Pick a point $r_1 \in (-\log\beta, -\log\alpha)$ which *does not belong to* $\mathbb{Q} \cdot \log|K^\times|$.

Let $\theta t^{-b} \in \mathcal{I}\Theta_{\partial_j}(M \otimes F_{e^{-r_1}})$ be an element in the multiset of refined intrinsic ∂_j-radii, with multiplicity μ. Since $M \otimes F_{e^{-r_1}}$ has pure intrinsic ∂_j-radii ωe^{a-br_1}, we have

$$\theta t^{-b} \in \kappa^{(a-br_1)}_{F^{\mathrm{alg}}_{e^{-r_1}}} \cong t^{-b}\kappa^{(a)}_{K^{\mathrm{alg}}};$$

here the latter isomorphism follows from our choice $r_1 \notin \mathbb{Q} \cdot \log|K^\times|$. We may replace K by a finite tamely ramified extension so that

$$\theta \in \bigcup_n \big(\kappa_K^{(p^n a)}\big)^{1/p^n}.$$

The construction in Example 1.5.7 gives a ∂_j-differential module $\mathcal{L}^{(j)}_{x,b,(n)}$ over $A^1_K(\alpha, \beta)$ such that $\mathcal{L}^{(j)}_{x,b,(n)} \otimes F_{e^{-r}}$ has pure ∂_j-radii ωe^{a-br} and pure intrinsic ∂_j-radii θt^{-b} for any $r \in (-\log\beta, -\log\alpha)$.

If we set $N = M \otimes (\mathcal{L}^{(j)}_{x,b,(n)})^\vee$, then we have $IR_{\partial_j}(N \otimes F_{e^{-r}}) \le \omega e^{a-br}$ for any $r \in (-\log\beta, -\log\alpha)$. Moreover, Proposition 1.3.19 and Theorem 1.3.26 together imply that

$$f^{(j)}_1(M, r_1) = f^{(j)}_1(N, r_1) = f^{(j)}_{(\dim M-\mu)d}(N, r_1) > f^{(j)}_{(\dim M-\mu)d+1}(N, r_1).$$

By Theorem 1.5.6(d), the same inequality holds for all $r \in (-\log\beta, -\log\alpha)$ in place of r_1 because a convex function below a linear function is same as the linear function if and only if the two functions touch at some point. By Theorem 1.5.4(e), we have a unique decomposition of ∂_j-differential modules $N = N_0 \oplus N_1$ such that, for any $r \in (-\log\beta, -\log\alpha)$, $N_0 \otimes F_{e^{-r}}$ has pure intrinsic ∂_j-radii ωe^{a-br} and $IR_{\partial_j}(N_1 \otimes F_{e^{-r}}) > \omega e^{a-br}$. By the same argument as in Theorem 1.3.26, this implies that M admits a decomposition of ∂_j-differential modules $M = M_\theta \oplus M'$ over $A^1_K(\alpha, \beta)$ such that $M_\theta \otimes (\mathcal{L}^{(j)}_{x,b,(n)})^\vee = N_1$ and $M' \otimes (\mathcal{L}^{(j)}_{x,b,(n)})^\vee = N_0$. By

Proposition 1.3.28 and Lemma 1.5.9, for any $r \in (-\log\beta, -\log\alpha)$, $M_\theta \otimes F_{e^{-r}}$ has pure refined intrinsic ∂_j-radii θt^{-b}, and the multiset of refined intrinsic ∂_j-radii of $M' \otimes F_{e^{-r}}$ does not contain θt^{-b}. We obtain the decomposition in the theorem by applying this argument to every θ.

To finish the proof, it suffices to consider the case when $p > 0$ and

$$IR_{\partial_j}(M \otimes F_{e^{-r}}) \in \big[p^{-1/p(p-1)}, 1\big).$$

But in this case, the ∂_j-Frobenius antecedent of M exists over the annulus with radii in a neighborhood of r. The decomposition follows from the decomposition of the ∂_j-Frobenius antecedents of M (applied iteratively until the intrinsic ∂_j-radii fall in the range above). □

Remark 1.5.11. The artificial reduction to the case $\mathbb{Q} \cdot \log|K^\times| \neq \mathbb{R}$ is to deduce $\theta \in \kappa^{(a)}_{K^{\mathrm{alg}}}$. This fact can also be proved using Newton polygons if the $f_1^{(j)}(M, r)$ is not constantly $p^{-1/(p-1)}$, in which case one may alternatively use the Frobenius pushforward to reduce to the visible case.

Theorem 1.5.12. *Let I be an open interval of $[0, +\infty)$ and let M be a ∂_{J^+}-differential module over $A^1_K(I)$ such that $M \otimes F_{e^{-r}}$ has pure intrinsic radii $\omega e^{a-br} < 1$ for $r \in -\log(\dot{I})$. Let e denote the prime-to-p part of the denominator of b. Then there exists a unique direct sum decomposition $M = \bigoplus_{\{\mu_e\vartheta\}} M_{\{\mu_e\vartheta\}}$ of ∂_{J^+}-differential modules over $A^1_K(I)$, where the sum is taken over all $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits of*

$$\bigoplus_{j\in J} \kappa^{(a)}_{K^{\mathrm{alg}}} \frac{du_j}{u_j} \oplus \kappa^{(a)}_{K^{\mathrm{alg}}} \frac{dt}{t},$$

and the refined intrinsic radii of $M_{\{\mu_e\vartheta\}} \otimes F_\eta$ for any $\eta \in -\log\dot{I}$ is a multiset consisting of the $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits $\{t^{-b}\vartheta\}$ with appropriate multiplicities.

Moreover, there exists a finite tamely ramified tension K' of K such that we have a unique direct sum decomposition

$$M \otimes_{K[t]} K'[t^{1/e}] = \bigoplus_{\vartheta \in \bigoplus_{j\in J} \kappa^{(a)}_{K^{\mathrm{alg}}} \frac{du_j}{u_j} \oplus \kappa^{(a)}_{K^{\mathrm{alg}}} \frac{dt}{t}} M_\vartheta \tag{1.5.13}$$

of ∂_{J^+}-differential modules over $A^1_{K'}(I^{1/e})$ such that $M_\vartheta \otimes K'F'_\eta$ has pure refined intrinsic radii $t^{-b}\vartheta$ for any $\eta \in -\log\dot{I}$.

Proof. We first treat the case when $0 \notin I$. Without loss of generality, we assume that M is not a direct sum of two nonzero sub-∂_{J^+}-modules, which implies the dichotomy given by Theorem 1.5.6(f). We may apply Theorem 1.5.10 to the ∂_j for which case (f1) of Theorem 1.5.6 holds for M and note that the decompositions for different ∂_j given by Theorem 1.5.10 are compatible. This gives rise to the desired decomposition.

Now, we consider the case when $I = [0, \beta)$. Since we have already proved the theorem over (α, β) for any $\alpha > 0$, it suffices to find the decomposition for $I = [0, \alpha)$ for some $\alpha \in (0, 1)$. Note that when α is sufficiently small, $M \otimes A_K^1[0, \alpha)$ is trivial as a ∂_0-differential module and hence is the pullback of a ∂_J-differential module M_0 over K along the natural morphism $K \to K\{\{t/\alpha\}\}$. The decomposition (1.5.13) follows from the decomposition of M_0 given by Theorem 1.3.26. □

We have a similar result for refined extrinsic radii, but only over $A_K^1(I)$; this is because adjoining $t^{1/e}$ would change the extrinsic radii. This subtlety also comes up when considering differential modules over discs (as opposed to annuli) and trying to extend the decomposition into the center of the disc: this is only possible if the functions defined by the extrinsic radii are "constant".

Theorem 1.5.14. *Assume that $|u_j| = 1$ for all $j \in J$. Let M be a ∂_{J^+}-differential module over an open annulus $A_K^1(I)$ with $I \subseteq (0, 1)$. Assume that $M \otimes F_{e^{-r}}$ has pure extrinsic radii $\omega e^{a-br} < e^{-r}$ for $r \in -\log(\dot{I})$. Let e denote the prime-to-p part of the denominator of b. Then there exists a unique direct sum decomposition*

$$M = \bigoplus_{\{\mu_e \hat{\vartheta}\}} M_{\{\mu_e \hat{\vartheta}\}} \tag{1.5.15}$$

of ∂_{J^+}-differential modules over $A_K^1(I)$, where the direct sum is taken over all $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits

$$\{\mu_e \hat{\vartheta}\} \quad in \quad \bigoplus_{j \in J} \kappa_{K^{\mathrm{alg}}}^{(a)} \, du_j \ \oplus \ \kappa_{K^{\mathrm{alg}}}^{(a)} \, dt,$$

and the multiset of refined extrinsic radii of $M_{\{\mu_e \hat{\vartheta}\}} \otimes F_\eta$ exactly consists of the $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits $\{t^{-b} \mu_e \hat{\vartheta}\}$ with appropriate multiplicities, for any $\eta \in \dot{I}$.

Proof. The proof is the same as Theorem 1.5.12. □

Proposition 1.5.16. *Fix $j \in J^+$. Let M be a ∂_j-differential module over an open disc $A_K^1[0, \alpha)$ such that $M \otimes F_\eta$ for any η in a neighborhood of $\eta = \alpha$ has pure ∂_j-radii ωe^s, where ωe^s is independent of η, and is strictly less than $|u_j|$ if $j \in J$ and less than α if $j = 0$. Then there exists a unique direct sum decomposition $M = \bigoplus_{\{\theta\} \subset \kappa_K^{(s)}} M_{\{\theta\}}$ of ∂_j-differential modules over $A_K^1[0, \alpha)$, where the direct sum is taken over all $\mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits $\{\theta\}$ of $\kappa_K^{(s)}$, and the multiset of refined ∂_j-radii of $M_{\{\theta\}} \otimes F_\eta$ consists of the $\mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits $\{\theta\}$ with appropriate multiplicities, for any $\eta \in (0, \alpha)$ if $j \in J$ and for any $\eta \in (\omega e^s, \alpha)$ if $j = 0$.*

Proof. Theorem 1.5.4(c) implies that $M \otimes F_\eta$ has pure ∂_j-radii ωe^s, for any $\eta \in (0, \alpha]$ if $j \in J$ and for any $\eta \in (\omega e^s, \alpha]$ if $j = 0$. The proposition then follows from the same argument as in Theorem 1.5.10, but invoking [Kedlaya and Xiao 2010, Theorem 2.3.10] in place of Theorem 1.5.4(e) when making the decomposition by extrinsic radii. Note also that we will only make use of the ∂_j-differential module $\mathscr{L}^{(j)}_{x,0,(n)}$ in the proof which is defined over the entire disc $A^1_K[0,\alpha)$. □

Proposition 1.5.17. *Assume that* $|u_j| = 1$ *for any* $j \in J$. *Let* M *be a* ∂_{J^+}*-differential module over an open disc* $A^1_K[0,\alpha)$ *with* $\alpha < 1$. *Assume that* $M \otimes F_{e^{-r}}$ *has pure extrinsic radii* $\min\{\omega e^s, e^{-r}\}$ *for any* $r > -\log\alpha$, *where* $\omega e^s < \alpha$. *Then there exists a unique direct sum decomposition* $M = \bigoplus_{\{\hat\vartheta\}} M_{\{\hat\vartheta\}}$ *of* ∂_{J^+}*-differential modules over* $A^1_K[0,\alpha)$, *where the direct sum is taken over all* $\mathrm{Gal}(K^{\mathrm{alg}}/K)$*-orbits* $\{\hat\vartheta\}$ *in*

$$\bigoplus_{j\in J} \kappa^{(s)}_{K^{\mathrm{alg}}}\, du_j \;\oplus\; \kappa^{(s)}_{K^{\mathrm{alg}}}\, dt,$$

such that the multiset of refined extrinsic radii of $M_{\{\hat\vartheta\}} \otimes F_\eta$ *exactly consists of the* $\mathrm{Gal}(K^{\mathrm{alg}}/K)$*-orbits* $\{\hat\vartheta\}$ *with appropriate multiplicities*, *for any* $\eta > \omega e^s$.

Proof. Without loss of generality, we assume that M is not a direct sum of two nonzero ∂_{J^+}-differential modules. We first show a dichotomy, similar to Theorem 1.5.6(f): for each ∂_j, either $M \otimes F_\eta$ has pure ∂_j-radii ωe^s for all $\eta > \omega e^s$, or $R_{\partial_j}(M \otimes F_\eta) < \omega e^s$ for all $\eta > \omega e^s$. Assume that we are not in the latter case. Then $R_{\partial_j}(M \otimes F_\eta) = ER(M \otimes F_\eta)$ for some $\eta \in (\omega e^s, \alpha)$. By parts (c) and (d) of Theorem 1.5.4, the monotonicity and convexity of $f^{(j)}_1(M, r)$ forces $R_{\partial_j}(M\otimes F_\eta) = ER(M\otimes F_\eta)$ for all $\eta \in (0,\alpha)$. Now, if $R_{\partial_j}(M\otimes F_\eta; 2) > ER(M\otimes F_\eta)$ for all $\eta \in (\omega e^s, \alpha)$, we may use [Kedlaya and Xiao 2010, Theorem 2.3.10] to decompose M to split off the smallest ∂_j-radii, which contradicts the indecomposability assumption on M. Therefore, $R_{\partial_j}(M \otimes F_\eta; 2) = ER(M \otimes F_\eta)$ for some $\eta \in (\omega e^s, \alpha)$. Continuing this argument for the third and other subsidiary ∂_j-radii leads us to the former case of the claim. The proposition now follows from applying Proposition 1.5.16 to each ∂_j that satisfies the former condition of the claim. □

Remark 1.5.18. We do not expect a decomposition theorem analogous to Proposition 1.5.17 in the case when the functions for extrinsic radii are linear with negative slopes. The reason is that, when η is sufficiently close to 0, $ER(M \otimes F_\eta)$ is always the same as η, and hence no information about the ∂_j-radii of $M \otimes F_\eta$ is reflected in the extrinsic radii. In contrast, in the situation of Proposition 1.5.17 if the functions of extrinsic radii stay constant before the they become equal to $-\log\eta$, all dominant ∂_j must have constant ∂_j-radii by the monotonicity (Theorem 1.5.4(c)).

1.6. ***Refined differential conductors.*** Differential modules defined over an open annulus with outer radius 1 are historically considered very important, in particular those whose intrinsic radii approach 1, as we base change to the completion with respect to the Gauss norms with radii approaching to 1; this is known as the solvable case. In particular, the rate of the such change of intrinsic radii is related to the Swan conductors if the differential modules come from a Galois representation of $G_{\mathbb{F}_p((t))}$. In this subsection, we focus on this situation and define differential conductors, as well as refined differential conductors if the differential module has pure differential conductors.

We continue to assume Hypothesis 1.5.1. Moreover, we assume $p > 0$ in this subsection.

Definition 1.6.1. Let M be a ∂_{J^+}-differential module of rank d over $A_K^1(\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. We say that M is *solvable* if $IR(M \otimes F_\eta) \to 1$ as $\eta \to 1^-$.

Theorem 1.6.2. *Suppose M is a solvable ∂_{J^+}-differential module of rank d over $A_K^1(\eta_0, 1)$, for some $\eta_0 \in (0, 1)$. Then by making η_0 sufficiently close to 1, there exists a unique direct sum decomposition $M = M_1 \oplus \cdots \oplus M_\gamma$ over $A_K^1(\eta_0, 1)$ and nonnegative distinct rational numbers $b_1, \ldots, b_\gamma$ with $b_i \cdot \operatorname{rank}(M_i) \in \mathbb{Z}$, such that $M_i \otimes F_\eta$ has pure intrinsic radii η^{b_i} for any $i = 1, \ldots, \gamma$ and any $\eta \in (\eta_0, 1)$.*

Keep the same hypothesis and assume moreover that $|u_j| = 1$ for all $j \in J$. Then by making η_0 sufficiently close to 1, there exists a unique direct sum decomposition $M = \hat{M}_1 \oplus \cdots \oplus \hat{M}_{\hat{\gamma}}$ over $A_K^1(\eta_0, 1)$ and nonnegative distinct rational numbers $\hat{b}_1, \ldots, \hat{b}_{\hat{\gamma}}$ with $\hat{b}_i \cdot \operatorname{rank}(\hat{M}_i) \in \mathbb{Z}$, such that $\hat{M}_i \otimes F_\eta$ has pure extrinsic radii $\eta^{\hat{b}_i}$ for any $i = 1, \ldots, \hat{\gamma}$ and any $\eta \in (\eta_0, 1)$.

Proof. By parts (a), (b), and (d) of Theorem 1.5.6, for $l = 1, \ldots, d$, the functions $d!\,F_l(M, r)$ and $d!\,\hat{F}_l(M, r)$ on $(0, -\log \eta_0)$ are continuous, convex, and piecewise affine with integer slopes. The assumption $d!\,F_l(M, r) \to 0$ also implies that $d!\,\hat{F}_l(M, r) \to 0$ as $r \to 0^+$; because of this and the fact that $d!\,F_l(M, r) \geq 0$ and $d!\,\hat{F}_l(M, r) \geq 0$ for all r, the slopes of $F_l(M, r)$ and $\hat{F}_l(M, r)$ are forced to be nonnegative. Hence there is a least such slope; that is, $d!\,F_l(M, r)$ and $d!\,\hat{F}_l(M, r)$ are linear in a right neighborhood of $r = 0$.

We can thus choose $\eta_0 \in (0, 1)$ so that $d!\,F_l(M, r)$ and $d!\,\hat{F}_l(M, r)$ are linear on $(0, -\log \eta_0)$ for $l = 1, \ldots, d$. The desired decomposition is constructed in Theorem 1.5.6(e) and the integrality of $b_i \cdot \operatorname{rank}(M_i)$ and $\hat{b}_i \cdot \operatorname{rank}(\hat{M}_i)$ follows from the fact that $F_{\dim M_i}(M_i, r)$ and $\hat{F}_{\dim \hat{M}_i}(\hat{M}_i, r)$ have integral slopes, again by Theorem 1.5.6(b). □

Definition 1.6.3. Let M be a solvable ∂_{J^+}-differential module of rank d over $A_K^1(\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. Define the multiset of *differential log-breaks* of M to be the multiset consisting of b_i from Theorem 1.6.2 with multiplicity rank M_i; we use $b_{\log}(M; 1) \geq \cdots \geq b_{\log}(M; d)$ to denote the differential log-breaks in decreasing order. We define the *differential Swan conductor* of M to be the sum of the differential log-breaks, that is, $\operatorname{Swan}(M) = \sum_{i=1}^r b_i \cdot \operatorname{rank}(M_i)$; it is a nonnegative integer by Theorem 1.6.2. We say that M has *pure differential log-breaks* if all differential log-breaks are equal.

When M has pure differential log-breaks, we define the multiset of *refined Swan conductors* of M, denoted by $\mathcal{I}\Theta(M)$, to be the multiset consisting of ϑ in (1.5.13) with multiplicity rank M_ϑ.

Similarly, when $|u_j| = 1$ for all $j \in J$, we define the multiset of *differential nonlog-breaks* to be the multiset consisting of $\hat{b}_i$ from Theorem 1.6.2 with multiplicity rank $\hat{M}_i$; we use $b_{\mathrm{nlog}}(M; 1) \geq \cdots \geq b_{\mathrm{nlog}}(M; d)$ to denote the differential nonlog-breaks in decreasing order. We define the *differential Artin conductor* of M to be the sum of the differential nonlog-breaks; it is also a nonnegative integer by Theorem 1.6.2. We say that M has *pure differential nonlog-breaks* if all differential nonlog-breaks are equal.

When M has pure differential nonlog-breaks, we define the multiset of *refined Artin conductors* of M, denoted by $\mathscr{C}\Theta(M)$, to be the multiset of $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{sep}}/K)$-orbits $\{\mu_e\hat{\vartheta}\}$ in (1.5.15) with multiplicity equal to the multiplicities of $\{t^{-b}\mu_e\hat{\vartheta}\}$ in $M_{\{\mu_e\hat{\vartheta}\}} \otimes F_\eta$ for any $\eta \in (\eta_0, 1)$.

2. Refined differential conductors for Galois representations

One of the most important applications of p-adic differential modules is to provide an interpretation of the Swan conductors of representations of G_k, where k is a complete discrete valuation field of equal characteristic $p > 0$ with perfect residue field. This idea was later generalized by Kedlaya [2007] to the case when the residue field of k need not to be perfect, and by the author [Xiao 2010] to relate the differential modules to the Swan conductors in the sense of Abbes and Saito [2002]. In this section, we further develop the theory on the differential module side to incorporate the study of refined differential conductors, which will be related to Saito's definition [2009] of refined Swan conductors, as proved in the next section.

Throughout this section, we assume that $p > 0$ is a prime number.

2.1. ***Construction of differential modules.*** This subsection is dedicated to the construction of the differential modules associated to representations of G_k, where k is a complete discrete valuation field of equal characteristic $p > 0$.

Definition 2.1.1. For a field κ of characteristic $p > 0$, a *p-basis* of κ is a set $(b_j)_{j\in J} \subset \kappa$ such that the products $b_J^{e_J}$, where $e_j \in \{0, 1, \ldots, p-1\}$ for all $j \in J$ and $e_j = 0$ for all but finitely many j, form a basis of the vector space κ over κ^p.

Notation 2.1.2. Let k be a complete discrete valuation field of characteristic $p > 0$. Let π_k be a uniformizer of k, generating the maximal ideal $\mathfrak{m}_k$ in the ring of integers $\mathbb{O}_k$. Let $\kappa = \kappa_k$ denote the residue field. Let $\bar{\kappa} = \kappa^{\mathrm{alg}}$ denote an algebraic closure of κ. We choose and fix a noncanonical isomorphism $k \simeq \kappa((\pi_k))$. We fix a p-basis $\bar{b}_J$ of κ and let $b_J \subset k$ be the preimage of them via the isomorphism above. Then $\{b_J, \pi_k\}$ form a p-basis of k, which we refer to as a *lifted p-basis*. Let $k_0 = \bigcap_{n\in\mathbb{N}} \kappa^{p^n} = \bigcap_{n\in\mathbb{N}} k^{p^n}$. We know that $d\pi_k$ and db_J form a basis of $\Omega^1_{\mathbb{O}_k}$ over $\mathbb{O}_k$.

Let $\mathbb{O}_K$ denote the Cohen ring of κ with respect to $\bar{b}_J$ and let $B_J \subset \mathbb{O}_K$ be the canonical lifts of the p-basis. Put $K = \operatorname{Frac} \mathbb{O}_K$. We use $\mathbb{O}_{K_0}$ to denote the ring of Witt vectors of k_0, viewed as a subring of $\mathbb{O}_K$ and we put $K_0 = \mathbb{O}_{K_0}\left[\frac{1}{p}\right]$.

Notation 2.1.3. For an extension k'/k of a complete discrete valuation field, the *(naïve) ramification degree* of k'/k is simply the index of the valuation of k in that of k'.

We say that k'/k is *tamely ramified* if $p \nmid e$ and the residue field extension $\kappa_{k'}/\kappa_k$ is separable, that is, $\kappa_{k'}$ is algebraic and separable over $\kappa_k(x_\alpha; \alpha \in \Lambda)$ for some transcendental elements x_α and an index set Λ. If moreover, $e = 1$, we say k'/k is *unramified*.

Notation 2.1.4. By a *representation* of G_k, we mean a continuous homomorphism $\rho : G_k \to \mathrm{GL}(V_\rho)$, where V_ρ is a vector space over a (topological) field F of characteristic zero. We say that ρ is a *p-adic* if F is a finite extension of $\mathbb{Q}_p$.

Let F be a finite extension of $\mathbb{Q}_p$, let $\mathbb{O}$ denote its ring of integers, and let $\mathbb{F}_q$ denote the residue field of $\mathbb{O}$, where q is a power of p. Put $\mathbb{Z}_q = W(\mathbb{F}_q)$ and $\mathbb{Q}_q = \mathbb{Z}_q\left[\frac{1}{p}\right]$. By an *$\mathbb{O}$-representation* of G_k, we mean a continuous homomorphism $\rho : G_k \to \mathrm{GL}(\Lambda_\rho)$ with Λ_ρ a finite free $\mathbb{O}$-module.

For ρ a p-adic representation or an $\mathbb{O}$-representation, we say that ρ has *finite local monodromy* if the image of the inertia group I_k is finite.

We assume that $\mathbb{F}_q \subseteq k_0$. Put $K' = KF$. Since $F/\mathbb{Q}_q$ is totally ramified, we have $\mathbb{O}_{K'} \cong \mathbb{O}_K \otimes_{\mathbb{Z}_q} \mathbb{O}$.

Notation 2.1.5. We put $\mathcal{R}^\eta_{K'} = K'\langle \eta/T, T\}\}$ for $\eta \in (0, 1)$ and $\mathcal{R}_{K'} = \bigcup_{\eta \in (0,1)} \mathcal{R}^\eta_{K'}$; the latter ring is commonly called the *Robba ring* over K'. Let $\mathcal{R}^{\mathrm{int}}_{K'}$ be the subring of $\mathcal{R}_{K'}$ consisting of elements whose 1-Gauss norm is bounded by 1; it is a Henselian discrete valuation ring, with residue field k if we identify the reduction of T with π_k. For $\eta \in (0, 1)$, we use F'_η to denote the completion of $K'(T)$ with respect to the η-Gauss norm.

A *Frobenius lift* ϕ is an endomorphism of $\mathcal{R}^{\mathrm{int}}_{K'}$ which lifts the natural q-th power Frobenius on k. Any Frobenius lift extends by continuity to an action on $\mathcal{R}_{K'}$. A *standard Frobenius lift* is a Frobenius lift which sends T to T^p and B_j to B_j^p for any $j \in J$.

The differentials

$$\Omega^1_{\mathcal{R}^{\mathrm{int}}_{K'}}, \quad \Omega^1_{\mathcal{R}_{K'}} \quad \text{and} \quad \Omega^1_{\mathcal{R}^\eta_{K'}}$$

for any $\eta \in (0, 1)$ admit a basis given by dB_J and dT. We set $\partial_0 = \partial/\partial T$, $\partial_j = \partial/\partial B_j$ with $j \in J$ for the dual basis. Then a ∇-module over $\mathcal{R}_{K'}$ is just a ∂_{J^+}-differential module.

Definition 2.1.6. Let ϕ be a Frobenius lift. For $R = \mathcal{R}_{K'}$, $\mathcal{R}^{\eta}_{K'}$, or $\mathcal{R}^{\mathrm{int}}_{K'}$, a (ϕ, ∇)-*module* M over R is a ∂_{J^+}-differential module together with an isomorphism $\Phi : \phi^* M \to M$ of ∂_{J^+}-differential modules.

Theorem 2.1.7. *For any Frobenius lift ϕ, we have an equivalence of categories between the category of $\mathbb{O}$-representations with finite local monodromy and the category of (ϕ, ∇)-modules over $\mathcal{R}^{\mathrm{int}}_{K'}$. Moreover, all (ϕ, ∇)-modules can be realized over $\mathcal{R}^{\eta}_{K'}$ for some $\eta \in (0, 1)$. This (ϕ, ∇)-module is independent of the choice of the p-basis.*

Proof. The functor is constructed in [Kedlaya 2007, Section 3; Xiao 2010, Section 2.2]. □

Definition 2.1.8. For a p-adic representation ρ of G_k with finite local monodromy, we choose an $\mathbb{O}$-lattice Λ_ρ of V_ρ, stable under the action of G_k; this gives an $\mathbb{O}$-representation of G_k. Theorem 2.1.7 then produces a (ϕ, ∇)-module over $\mathcal{R}^{\mathrm{int}}_{K'}$, whose base change to $\mathcal{R}_{K'}$ is called the *differential module associated to* ρ, denoted by $\mathscr{E}_\rho$. This $\mathscr{E}_\rho$ does not depend on the choice of the lattice Λ_ρ.

For the rest of this subsection, we assume the following.

Hypothesis 2.1.9. The residue field κ has a finite p-basis $\bar{b}_J$, where $J = \{1, \dots, m\}$. We put $J^+ = J \cup \{0\}$.

Proposition 2.1.10. *Let ϕ be the standard Frobenius lift on $\mathcal{R}^{\mathrm{int}}_{K'}$. Then the Frobenius $\phi : F'_{\eta^q} \to F'_\eta$ is the same as the iterative Frobenius $\varphi^{(\partial_0,\lambda)} \circ \cdots \circ \varphi^{(\partial_m,\lambda)}$ in Construction 1.2.14, where $q = p^\lambda$.*

Proof. We may assume that K' contains ζ_q, a q-th root of unity. It suffices to show that the image $\phi(F'_{\eta^q})$ is stable under the action of $(\mathbb{Z}/q\mathbb{Z})^{m+1}$ in the sense of Construction 1.2.14, where each ∂_j-Frobenius corresponds to a factor $\mathbb{Z}/q\mathbb{Z}$, and that the degree of F'_η over $\phi(F'_{\eta^q})$ is q^{m+1}.

For $\underline{i} = (i_0, \dots, i_m) \in (\mathbb{Z}/q\mathbb{Z})^{m+1}$, we have $T^{(\underline{i})} = \zeta_q^{i_0} T$ and $(B_j)^{(\underline{i})} = \zeta_q^{i_j} B_j$ for any $j \in J$. Hence $(\,\cdot\,)^{(\underline{i})} \circ \phi$ for all $\underline{i}$ are continuous homomorphisms from $\mathbb{O}_K[\![T]\!]$ to itself, sending B_j to B_j^q and T to T^q. By the functoriality of Cohen rings (see [Xiao 2010, Proposition 2.1.8]), these homomorphisms are all the same. Hence the image of ϕ is stable under the $(\mathbb{Z}/q\mathbb{Z})^{m+1}$-action. It is evident that F'_η has rank q^{m+1} over $\phi(F'_{\eta^q})$; this forces the two homomorphisms to be the same. □

Proposition 2.1.11. *Let ϕ be the standard Frobenius lift on $\mathcal{R}^{\mathrm{int}}_{K'}$ and let $\mathscr{E}$ be a (ϕ, ∇)-module over $A^1_{K'}[\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. Then $\mathscr{E}$ is solvable.*

Proof. This is well-known to the experts; we include a proof for the convenience of the reader. By Lemma 1.2.18(a), we have

$$\begin{aligned} &f_i(\phi^* M, r) \\ &\qquad = \max\left\{p^{-\lambda} f_i(M, qr),\, p^{1-\lambda}(f_i(M, qr) - \log p), \dots, f_i(M, qr) - \lambda \log p\right\}, \end{aligned}$$

where $\lambda = \log_p q$. Since $\phi^* M \xrightarrow{\sim} M$, the function $g_i(M) = \limsup_{r\to 0^+} f_i(M, r)$ satisfies

$$g_i(M) = \max\left\{p^{-\lambda} g_i(M),\, p^{1-\lambda}(g_i(M) - \log p), \dots, g_i(M) - \lambda \log p\right\}.$$

This forces $g_i(M)$ to be zero. By the continuity of $f_i(M, r)$ and the convexity of $F_i(M, r)$ in Theorem 1.5.6, $\lim_{r\to 0^+} f_i(M, r) = 0$. In other words, $\mathscr{E}$ is solvable. □

Proposition 2.1.12. *Let ϕ be the standard Frobenius lift and let ϕ' be another Frobenius lift on $\mathcal{R}^{\mathrm{int}}_{K'}$. Assume that $\mathscr{E}$ is a (ϕ, ∇)-module over $A^1_{K'}[\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. Then the restriction of $\mathscr{E}$ to $A^1_{K'}[\eta, 1)$ for some $\eta \in [\eta_0, 1)$ is naturally equipped with a (ϕ', ∇)-module structure.*

Proof. Define the Frobenius structure for ϕ' by Taylor series as follows. For $\boldsymbol{v} \in \mathscr{E}$,

$$\phi'(\boldsymbol{v}) = \sum_{e_{J^+}=0}^{\infty} \frac{(\phi'(T) - \phi(T))^{e_0} \prod_{j\in J}(\phi'(B_j) - \phi(B_j))^{e_j}}{(e_{J^+})!} \phi\Big(\frac{\partial^{e_0}}{\partial T^{e_0}} \frac{\partial^{e_1}}{\partial B_1^{e_1}} \cdots \frac{\partial^{e_m}}{\partial B_m^{e_m}}(\boldsymbol{v})\Big).$$

Since $|\phi'(T) - \phi(T)|_1 < 1$ and $|\phi'(B_j) - \phi(B_j)|_1 < 1$ for all $j \in J$, we have the same inequality using η-Gauss norm when $\eta \in [\eta'_0, 1]$ for some η'_0 sufficiently close to 1. Hence the expression for ϕ' converges on $A^1_{K'}[\eta'_0, 1)$ and gives the restriction of $\mathscr{E}$ to $A^1_{K'}[\eta'_0, 1)$ a structure of (ϕ', ∇)-module. □

Remark 2.1.13. One may also approach the results of this subsection without referring to the standard Frobenius but instead using a generalized version of Lemma 1.2.18(a) for noncentered Frobenius. This point of view is taken in [Kedlaya 2010, Chapter 17].

2.2. *Differential conductors.* Combining the results from Section 1.6 and Proposition 2.1.11, we can define differential conductors for a representation of G_k with finite local monodromy. To make this definition more robust, we will introduce the break with respect to each element of the p-basis, and the break of the differential module is just the maximum among all breaks for each element of the p-basis, after appropriate normalization. This point of view is in particular useful when we try to understand how the conductors change when restricting a Galois representation to G_l for some (explicit) finite extension l of k.

Definition 2.2.1. We first assume that k satisfies Hypothesis 2.1.9. Let ρ be a representation of G_k with finite local monodromy. The *log-breaks* of ρ are defined to be the differential log-breaks of $\mathscr{E}_\rho$, as a solvable ∂_{J^+}-differential module. Put $b_{\log}(\rho; l) = b_{\log}(\mathscr{E}_\rho; l)$ for $l = 1, \dots, \dim \rho$. Similarly, the *nonlog-breaks* of ρ are defined to be the differential nonlog-breaks of $\mathscr{E}_{\rho/\rho^{I_k}}$ together with the element 0 with multiplicity $\dim \rho^{I_k}$, where ρ^{I_k} is the maximal subrepresentation of ρ on which I_k acts trivially. Put $b_{\mathrm{nlog}}(\rho; l) = b_{\mathrm{nlog}}(\mathscr{E}_{\rho/\rho^{I_k}}; l)$ for $l = 1, \dots, \dim(\rho/\rho^{I_k})$, and $b_{\mathrm{nlog}}(\rho; \dim(\rho/\rho^{I_k}) + 1) = \cdots = b_{\mathrm{nlog}}(\rho; \dim \rho) = 0$.

For simplicity, we also put $b_{\mathrm{nlog}}(\rho) = b_{\mathrm{nlog}}(\rho; 1)$ and $b_{\log}(\rho) = b_{\log}(\rho; 1)$; they are called the *highest nonlog-break* and the *highest log-break*, respectively.

We now consider a general k. For a p-adic representation ρ of G_k with finite local monodromy, let l be the extension of k corresponding to $\operatorname{Ker} \rho$ via Galois theory. We may choose a p-basis $\{c_J, \pi_l\}$ of l such that π_l is a uniformizer and $c_J \subset \mathbb{O}_l^\times$, and such that $c_{J \setminus J_0} \subset \mathbb{O}_k$ for some finite subset $J_0 \subset J$. If we use $k^\wedge$ to denote the completion of $k(c_{J\setminus J_0}^{1/p^n}; n \in \mathbb{N})$, then $k^\wedge$ verifies Hypothesis 2.1.9. We define the *nonlog-breaks* and *log-breaks* of ρ to be, respectively, those of $\rho|_{G_{k^\wedge}}$. Their sums are called the *Artin conductors* and, respectively, *Swan conductors* of ρ, denoted by $\mathrm{Art}(\rho)$ and $\mathrm{Swan}(\rho)$. These do not depend on the choice of the p-basis or of J_0, by [Kedlaya 2007, Proposition 2.6.6].

Definition 2.2.2. Put $\mathrm{Fil}^0 G_k = G_k$ and $\mathrm{Fil}^a G_k = I_k$ for $a \in (0, 1]$. For $a > 1$, let R_a be the set of finite image representations ρ with nonlog-break strictly less than a. Put $\mathrm{Fil}^a G_k = \bigcap_{\rho \in R_a} \big(I_k \cap \ker(\rho)\big)$ and set $\mathrm{Fil}^{a+} G_k$ to be the closure of $\bigcup_{b>a} \mathrm{Fil}^b G_k$. This defines a filtration on G_k such that for any representation ρ with finite image, $\rho(\mathrm{Fil}^a G_k)$ is trivial if and only if $\rho \in R_a$.

Similarly, put $\mathrm{Fil}^0_{\log} G_k = G_k$. For $a > 0$, let $R_{a,\log}$ be the set of finite image representations ρ with log-break less than a. Put $\mathrm{Fil}^a_{\log} G_k = \bigcap_{\rho \in R_{a,\log}} \big(I_k \cap \ker(\rho)\big)$ and set $\mathrm{Fil}^{a+}_{\log} G_k$ to be the closure of $\bigcup_{b>a} \mathrm{Fil}^b_{\log} G_k$. This defines a filtration on G_k such that for any representation ρ with finite image, $\rho(\mathrm{Fil}^a_{\log} G_k)$ is trivial if and only if $\rho \in R_{a,\log}$.

For a finite Galois extension l of k, the above filtrations induce filtrations on the Galois group $G_{l/k}$ by

$$G^a_{l/k,(\log)} = G_l \mathrm{Fil}^a_{(\log)} G_k / G_l \quad \text{and} \quad G^{a+}_{l/k,(\log)} = G_l \mathrm{Fil}^{a+}_{(\log)} G_k / G_l,$$

for $a \geq 0$. We define the *(log-)ramification breaks* of the extension l/k to be the numbers b for which $G^b_{l/k(,\log)} \neq G^{b+}_{l/k(,\log)}$. We order them as

$$b_{(\mathrm{n})\log}(l/k) = b_{(\mathrm{n})\log}(l/k; 1) \geq b_{(\mathrm{n})\log}(l/k; 2) \geq \cdots .$$

In particular, if ρ is a *faithful* representation of $G_{l/k}$, we have $b_{(\mathrm{n})\log}(\rho) = b_{(\mathrm{n})\log}(l/k)$.

Theorem 2.2.3. *The differential conductors satisfy the following properties*:

(a) *For any representation ρ of finite local monodromy*,

$$\mathrm{Art}(\rho) = \sum_{a \in \mathbb{Q}_{\geq 0}} a \cdot \dim\big(V_\rho^{\mathrm{Fil}^{a+} G_k} / V_\rho^{\mathrm{Fil}^a G_k}\big) \in \mathbb{Z}_{\geq 0},$$
$$\mathrm{Swan}(\rho) = \sum_{a \in \mathbb{Q}_{\geq 0}} a \cdot \dim\big(V_\rho^{\mathrm{Fil}^{a+}_{\log} G_k} / V_\rho^{\mathrm{Fil}^a_{\log} G_k}\big) \in \mathbb{Z}_{\geq 0}.$$

(b) *Let k'/k be a* (*not necessarily finite*) *extension of complete discretely valued* *fields. If k'/k is unramified, then* $\mathrm{Fil}^a G_{k'} = \mathrm{Fil}^a G_k$ *for* $a > 0$. *If k'/k is tamely ramified with naïve ramification index* $e < \infty$, *then* $\mathrm{Fil}^{ea}_{\log} G_{k'} = \mathrm{Fil}^a_{\log} G_k$ *for* $a > 0$.

(c) *For* $a > 0$, *we have* $\mathrm{Fil}^{a+1} G_k \subseteq \mathrm{Fil}^a_{\log} G_k \subseteq \mathrm{Fil}^a G_k$.

(d) *For graded pieces we have, for* $a > 1$,

$$\mathrm{Fil}^a G_k/\mathrm{Fil}^{a+} G_k = \begin{cases} 0 & \text{if } a \notin \mathbb{Q}, \\ \text{an abelian group killed by } p & \text{if } a \in \mathbb{Q}, \end{cases}$$

and for $a > 0$,

$$\mathrm{Fil}^a_{\log} G_k/\mathrm{Fil}^{a+}_{\log} G_k = \begin{cases} 0 & \text{if } a \notin \mathbb{Q}, \\ \text{an abelian group killed by } p & \text{if } a \in \mathbb{Q}. \end{cases}$$

(e) *These filtrations on* G_k *agree with the ones defined in [Abbes and Saito 2002, 2003].*

Proof. Using the comparison [Xiao 2010, Theorem 4.4.1] of the arithmetic and differential conductors, this follows from their basic properties as stated in [Xiao 2010, Theorem 2.4.1 and Proposition 4.1.7]. We refer to [Xiao 2010; Abbes and Saito 2002] for the definition of Abbes and Saito's filtrations. □

We now study the break for each element of the p-basis. We assume the validity of Hypothesis 2.1.9 for the rest of the subsection.

Proposition 2.2.4. *For each* $j \in J^+$, *there is a ramification break* $b_j(\rho)$ *associated to* b_j $(j \in J)$ *or* π_k $(j = 0)$, *such that* $R_{\partial_j}(\mathscr{E}_\rho \otimes F'_\eta) = \eta^{b_j(\rho)}$ *for any* $\eta \in (\eta_0, 1)$ *with some* $\eta_0 < 1$. *Moreover,*

$$b_{\mathrm{nlog}}(\rho) = \max_{j \in J^+}\{b_j(\rho)\}, \quad b_{\log}(\rho) = \max\{b_0(\rho) - 1;\, b_j(\rho) \text{ for } j \in J\}.$$

Proof. By applying the same argument of Proposition 2.1.11 to intrinsic ∂_j-radii, we know that

$$IR_{\partial_j}(\mathscr{E}_\rho \otimes F'_{\eta^q}) = IR_{\partial_j}(\mathscr{E}_\rho \otimes F'_\eta)^q$$

as $\eta \to 1^-$. Therefore, by the convexity given by Theorem 1.5.4(d), $f_1^{(j)}(\mathscr{E}_\rho, r)$ is affine as $r \to 0^+$. The proposition follows. □

Definition 2.2.5. We call $b_{J^+}(\rho)$ the *breaks by p-basis* of ρ with respect to the lifted p-basis b_J and the uniformizer π_k.

Remark 2.2.6. Rigorously speaking, the breaks by p-basis depend on the choice of the dual basis $\partial_0, \dots, \partial_m$ of the differential forms. So when we change the choices of the lifted p-basis and the uniformizer, the breaks by basis $b_j(\rho)$ may change accordingly.

Lemma 2.2.7. *Fix $j_0 \in J$. Let $b'_{J^+}(\rho)$ be the breaks by p-basis of ρ with respect to the lifted p-basis $\{b_{J\setminus\{j_0\}}, b_{j_0}+\pi_k\}$ and the uniformizer π_k. Then $b'_j(\rho)=b_j(\rho)$ for $j \in J$ and*

$$b'_0(\rho)\begin{cases}=\max\{b_0(\rho), b_{j_0}(\rho)\} & \text{if } b_0(\rho)\neq b_{j_0}(\rho),\\ \leq b_0(\rho) & \text{if } b_0(\rho)=b_{j_0}(\rho).\end{cases}$$

Proof. Let ∂'_{J^+} denote the derivations dual to the basis $dB_{J\setminus\{j_0\}}, dT, d(B_{j_0}+T)$ of $\Omega^1_{\mathcal{R}^{\text{int}}_{K'}}$. Then $\partial'_J=\partial_J$ and $\partial'_0=\partial_0-\partial_{j_0}$. The lemma follows immediately. □

Remark 2.2.8. This lemma is in fact much stronger than it appears. Applying the same argument to $b_{j_0}+\alpha\pi_k$ for all $\alpha\in k_0$ implies that, for *all but possibly one* $\alpha\in k_0$, $b'_0(\rho)\geq b_{j_0}(\rho)$. So, vaguely speaking, the equality $b_0(\rho)=b(\rho)$ holds "generically"; this motivates the following lemma.

Lemma 2.2.9. *Fix $j_0\in J$. Let $\tilde{k}$ be the completion of $k(x)$ with respect to the 1-Gauss norm, equipped with the lifted p-basis $\{b_{J\setminus\{j_0\}}, b_{j_0}+x\pi_k, x\}$. Let $\tilde{\rho}$ be the representation $G_{\tilde{k}}\to G_k \xrightarrow{\rho} GL(V_\rho)$. Let $\tilde{b}_{J^+\cup\{m+1\}}(\tilde{\rho})$ denote the breaks by p-basis with respect to the aforementioned lifted p-basis and the uniformizer π, where $\tilde{b}_{J\setminus\{j_0\}}(\tilde{\rho})$ corresponds to $b_{J\setminus\{j_0\}}$, $\tilde{b}_{j_0}(\tilde{\rho})$ corresponds to $b_{j_0}+x\pi_k$, $\tilde{b}_0(\tilde{\rho})$ corresponds to π_k, and $\tilde{b}_{m+1}(\tilde{\rho})$ corresponds to x. Then $\tilde{b}_j(\rho')=b_j(\rho)$ for $j\in J$, $\tilde{b}_{m+1}(\tilde{\rho})=b_{j_0}(\rho)-1$, and $\tilde{b}_0(\tilde{\rho})=\max\{b_0(\rho), b_{j_0}(\rho)\}$. In particular, $\tilde{b}_{\text{nlog}}(\tilde{\rho})=b_{\text{nlog}}(\rho)$.*

Proof. Let $\widetilde{K}'$ denote the completion of $K'(X)$ with respect to the 1-Gauss norm, where X is the canonical lift of x. Let $f: A^1_{\widetilde{K}'}[\eta_0,1)\to A^1_{K'}[\eta_0,1)$ be the natural morphism. Then $f^*\mathscr{E}_\rho$ is the differential module associated to ρ'. Let $\tilde{\partial}_{J^+\cup\{m+1\}}$ be the differential operators corresponding to the p-basis $(b_{J\setminus\{j_0\}}, b_{j_0}+x\pi_k, \pi_k)$. Then under the identification by f^*, we have

$$\tilde{\partial}_J=\partial_J,\quad \tilde{\partial}_{m+1}=T\partial_{j_0},\quad \tilde{\partial}_0=\partial_0-X\partial_{j_0}. \tag{2.2.10}$$

The lemma follows from this because X is transcendental over K'. □

Lemma 2.2.11. *Fix $j_0\in J$. Set $k'=k(b_{j_0}^{1/p})$, equipped with the lifted p-basis $\{b_{J\setminus\{j_0\}}, b_{j_0}^{1/p}\}$. Let $b'_{J^+}(\rho|_{G_{k'}})$ be the breaks by p-basis of $\rho|_{G_{k'}}$ with respect to the aforementioned p-basis and uniformizer π_k. Then $b'_j(\rho|_{G_{k'}})=b_j(\rho)$ for $j\in J^+\setminus\{j_0\}$ and $b'_{j_0}(\rho|_{G_{k'}})=\frac{1}{p}b_{j_0}(\rho)$.*

Proof. Replacing k by k' is equivalent to pulling back the differential module $\mathscr{E}_\rho$ along $\varphi^{(\partial_j)}$. The lemma follows from applying Lemma 1.2.18(a) to $\mathscr{E}\otimes F'_\eta$ when $\eta\to 1^-$. □

Lemma 2.2.12. *Fix $j_0\in J$. Let k' denote the completion of $k(b_{j_0}^{1/p^n}; n\in\mathbb{N})$ equipped with lifted p-basis $b_{J\setminus\{j_0\}}$. Let $b'_{J^+}(\rho|_{G_{k'}})$ be the breaks by p-basis of $\rho|_{G_{k'}}$ with* *respect to this p-basis and the uniformizer* π_k. *Then we have* $b'_j(\rho|_{G_{k'}}) = b_j(\rho)$ *for* $j \in J^+ \backslash \{j_0\}$.

Proof. Replacing k with k' is equivalent to simply forgetting the j_0-direction. □

Situation 2.2.13. Now, we study a particular case of base change, which will be useful in the comparison Theorem 3.4.1. This type of base change was first considered by Saito [2009].

Fix $e \in \mathbb{N}$ possibly divisible by p. Let k be as above, and let k' be the completion of $k(x)$ with respect to the 1-Gauss norm, with uniformizer $\pi_{k'} = \pi_k$. Put $\tilde{k} = k'[u]/(u^e - x^{-1}\pi_k)$. The residue field of $\tilde{k}$ is $\kappa(\bar{x})$; we consider the p-basis $(b_J, \bar{x})$ and the uniformizer $\pi_{\tilde{k}} = u$ of $\tilde{k}$. We choose the unique isomorphism $\kappa(\bar{x})((u)) \simeq \tilde{k}$ that is compatible with the chosen isomorphism $\kappa(\pi_k) \simeq k$ in Notation 2.1.2 and that sends $\bar{x}$ to x. This gives rise to the lifted p-basis (b_J, x, u) of $\tilde{k}$.

Proposition 2.2.14. *The natural homomorphism* $G_{\tilde{k}} \to G_k$ *induces a homomorphism* $\mathrm{Fil}^{ea}_{\log} G_{\tilde{k}} \to \mathrm{Fil}^{a}_{\log} G_k$ *for any* $a \in \mathbb{Q}_{\geq 0}$. *Moreover, the induced homomorphism*

$$\mathrm{Fil}^{ea}_{\log} G_{\tilde{k}} / \mathrm{Fil}^{ea+}_{\log} G_{\tilde{k}} \to \mathrm{Fil}^{a}_{\log} G_k / \mathrm{Fil}^{a+}_{\log} G_k$$

is surjective for any $a \in \mathbb{Q}_{>0}$.

Proof. It suffices to show that, for a p-adic representation of G_k with finite local monodromy and pure log-break $b_{\log}(\rho)$, the induced representation

$$\tilde{\rho} : G_{\tilde{k}} \to G_k \to GL(V_\rho)$$

also has the same log-break. Let $\widetilde{K}'$ be the completion of $K'(X)$ with respect to the 1-Gauss norm, where X is the canonical lift of $\bar{x}$. We then have a natural map $f : A^1_{\widetilde{K}'}[\eta^{1/e}, 1) \to A^1_{K'}[\eta, 1)$ for $\eta \to 1^-$, sending T to XU^e, where U is the coordinate of the former annulus.

Let $\tilde{b}_0(\tilde{\rho}), \dots, \tilde{b}_{m+1}(\tilde{\rho})$ be the breaks by p-basis with respect to b_J, x and the uniformizer $\pi_{\tilde{k}} = u$. Then $f^*\mathscr{E}_\rho$ is the differential module associated to $\tilde{\rho}$, with the actions of $\tilde{\partial}_0 = \partial/\partial U$, $\tilde{\partial}_J = \partial/\partial B_J$, and $\tilde{\partial}_{m+1} = \partial/\partial X$. We have

$$\tilde{\partial}_J = \partial_J, \quad \tilde{\partial}_0 = eXU^{e-1}\partial_0, \quad \text{and} \quad \tilde{\partial}_{m+1} = U^e \partial_0. \tag{2.2.15}$$

By Theorem 1.4.20, we have $\tilde{b}_J(\tilde{\rho}) = eb_J(\rho)$, $\tilde{b}_0(\tilde{\rho}) \leq eb_0(\rho) - (e-1)$, and $\tilde{b}_{m+1}(\tilde{\rho}) = eb_0(\rho) - e$ (when e is prime to p, the inequality becomes an equality). In particular, we have $\tilde{b}_{m+1}(\tilde{\rho}) \geq \tilde{b}_0(\tilde{\rho}) - 1$. Hence we conclude that

$$b_{\log}(\tilde{\rho}) = \max\{\tilde{b}_0(\tilde{\rho}) - 1, \tilde{b}_J(\tilde{\rho}), \tilde{b}_{m+1}(\tilde{\rho})\} = \max\{eb_J(\rho), eb_0(\rho) - e\} = eb_{\log}(\rho).$$

This proves the proposition. □

2.3. ***Refined differential conductors.*** In this subsection, we define the refined differential conductors, which provides additional information about the subquotient $\mathrm{Fil}^a_{(\log)} G_k / \mathrm{Fil}^{a+}_{(\log)} G_k$ of the ramification filtrations.

We keep the notation as in previous subsections but we drop Hypothesis 2.1.9.

Notation 2.3.1. Fix a Dwork pi $\boldsymbol{\pi} = (-p)^{1/(p-1)}$.

Notation 2.3.2. We put $\Omega^1_{\mathcal{O}_k}(\log) = \Omega^1_{\mathcal{O}_k} + \mathcal{O}_k \frac{d\pi_k}{\pi_k} \subset \Omega^1_k$. If we choose a p-basis $\bar{b}_J$ of κ as in Notation 2.1.2, we have

$$\Omega^1_{\mathcal{O}_k}(\log) = \mathcal{O}_k \frac{d\pi_k}{\pi_k} \oplus \Big(\bigoplus_{j \in J} \mathcal{O}_k \, db_j \Big).$$

Construction 2.3.3. Let ρ be a p-adic representation of G_k with finite local monodromy and with pure break $b = b_{\mathrm{nlog}}(\rho)$ or log-break $b = b_{\log}(\rho)$. We may replace k by the completion of an inseparable extension as in Definition 2.2.1 and then assume Hypothesis 2.1.9. Let $\mathcal{E}_\rho$ denote the (ϕ, ∇)-module associated to ρ. By Theorem 1.5.6(e), there exists $\eta_0 \in (0, 1)$ such that $\mathcal{E}_\rho \otimes F'_\eta$ has pure extrinsic or intrinsic, respectively, radii η^b for any $\eta \in [\eta_0, 1)$.

We define the multiset of *refined Artin conductors* of ρ to be

$$\mathrm{rar}(\rho) = \Big\{ \frac{1}{\boldsymbol{\pi}} \vartheta \pi_k^{-b} : \vartheta \in \mathcal{I}\Theta(\mathcal{E}_\rho) \Big\} \subset \Omega^1_{\mathcal{O}_k} \otimes_{\mathcal{O}_k} \pi_k^{-b} \bar{\kappa}.$$

Similarly, we define the multiset of *refined Swan conductors* of ρ to be

$$\mathrm{rsw}(\rho) = \Big\{ \frac{1}{\boldsymbol{\pi}} \vartheta \pi_k^{-b} : \vartheta \in \mathcal{I}\Theta(\mathcal{E}_\rho) \Big\} \subset \Omega^1_{\mathcal{O}_k}(\log) \otimes_{\mathcal{O}_k} \pi_k^{-b} \bar{\kappa}.$$

Remark 2.3.4. There is a unique primitive p-th root of unity ζ_p such that

$$\boldsymbol{\pi} \equiv (\zeta_p - 1) \bmod (\zeta_p - 1)^2.$$

The definition of refined conductors above is unchanged if we replace $\boldsymbol{\pi}$ by $\zeta_p - 1$.

Lemma 2.3.5. *In Construction 2.3.3, the definition of the refined Artin and Swan conductors does not depend on the choices of the lifted p-basis of k and the uniformizer π_k.*

Proof. We may assume Hypothesis 2.1.9 since only finitely many elements in the p-basis appear in the refined Artin and Swan conductors.

For another choice of lifted p-bases and uniformizers, we will consider another set of differential operators: $\partial'_j = \partial / \partial B'_j$ for $j \in J$ and $\partial'_0 = \partial / \partial T'$. We put

$$dB_j = \sum_{j' \in J} \alpha_{j,j'} dB'_{j'} + \alpha_{j,0} dT' \quad \text{for } j \in J, \quad \text{and} \quad dT = \sum_{j' \in J} \alpha_{0,j'} dB_{j'} + \alpha_{0,0} dT',$$

where $\alpha_{j,j'} \in \mathcal{O}_{K'}[\![T]\!]$ for $j, j' \in J^+$. Moreover, we have $\alpha_{0,j} \in T \cdot \mathcal{O}_{K'}[\![T]\!]$.

We may assume that $\mathscr{E}_\rho$ has pure differential nonlog-break, or pure differential log-break. So there exists $\eta_0 \in (0, 1)$ such that $\mathscr{E}_\rho \otimes F'_\eta$ has pure extrinsic (resp. intrinsic) radii η^b for all $\eta \in [\eta_0, 1)$.

Consider $\eta \in (\eta_0, 1) \cap p^{\mathbb{Q}}$ so that F'_η is discretely valued. Theorem 1.4.20 then implies that, for any $j \in J^+$ such that $R_{\partial'_j}(V \otimes F'_\eta) = ER(V \otimes F'_\eta)$, we have

$$\Theta_{\partial'_j}(\mathscr{E}_\rho \otimes F'_\eta) = \big\{\boldsymbol{\pi} T^{-b}(\alpha_{0,j}\theta_0 + \cdots + \alpha_{m,j}\theta_m) : \\ \boldsymbol{\pi} T^{-b}(\theta_0 dT + \theta_1 dB_1 + \cdots + \theta_m dB_m) \in \mathscr{E}\Theta(\mathscr{E}_\rho \otimes F'_\eta)\big\},$$

and for any $j \in J^+$ such that $IR_{\partial'_j}(V \otimes F'_\eta) = IR(V \otimes F'_\eta)$,

$$\Theta_{\partial'_j}(\mathscr{E}_\rho \otimes F'_\eta) = \Big\{\boldsymbol{\pi} T^{-b}\Big(\frac{\alpha_{0,j}}{T}\theta_0 + \cdots + \alpha_{m,j}\theta_m\Big) : \\ \boldsymbol{\pi} T^{-b}\Big(\theta_0 \frac{dT}{T} + \theta_1 dB_1 + \cdots + \theta_m dB_m\Big) \in \mathscr{I}\Theta(\mathscr{E}_\rho \otimes F'_\eta)\Big\}.$$

Note also that

$$\big(\alpha_{0,0}\theta_0 + \cdots + \alpha_{m,0}\theta_m\big)dT' + \sum_{j\in J}\big(\alpha_{0,j}\theta_0 + \cdots + \alpha_{m,j}\theta_m\big)dB'_j \\ = \theta_0 dT + \theta_1 dB_1 + \cdots + \theta_m dB_m.$$

Combining these two formulas, we conclude that $\mathscr{E}\Theta(V)$ (resp. $\mathscr{I}\Theta(V)$) for ∂_{J^+} is the same as that for ∂'_{J^+}. Hence the refined Artin and Swan conductors are well-defined. □

Lemma 2.3.6. *Let k'/k be a tamely ramified extension of ramification degree $e = e_{k'/k}$ and let ρ be a p-adic representation of G_k with finite local monodromy and with pure log-break $b = b_{\log}(\rho)$. Then $\rho|_{G_{k'}}$ has pure log-break eb. Moreover, if we identify $\Omega^1_{\mathbb{O}_k}(\log) \otimes_{\mathbb{O}_k} \pi_k^{-b}\bar{\kappa}$ with $\Omega^1_{\mathbb{O}_{k'}}(\log) \otimes_{\mathbb{O}_{k'}} \pi_{k'}^{-eb}\bar{\kappa}$, then* $\mathrm{rsw}(\rho)$ *is the same as* $\mathrm{rsw}(\rho|_{g_{k'}})$.

Proof. This follows immediately from the fact that $\mathscr{E}_{\rho|_{G_{k'}}}$ is just the base change of $\mathscr{E}_\rho$ along $A^1_{K'}[\eta^{1/e}, 1) \to A^1_{K'}[\eta, 1)$, where the coordinate for the first annulus is $t^{1/e}$. □

Theorem 2.3.7. *Let k be a complete discrete valuation field of equal characteristic $p > 0$.*

(a) *Let ρ be a p-adic representation of G_k with finite local monodromy and with pure log-break $b = b_{\log}(\rho) > 0$. Then there exists a unique direct sum decomposition of ρ as $\rho \cong \oplus_{\{\vartheta\}\subset \mathrm{rsw}(\rho)}\rho_{\{\vartheta\}}$, where the direct sum is taken over all $\mu_e \rtimes G_k$-orbits $\{\vartheta\}$ in* $\mathrm{rsw}(\rho)$, *and* $\mathrm{rsw}(\rho_{\{\vartheta\}})$ *consists of the Galois orbits $\{\vartheta\}$ with appropriate multiplicities. Moreover, there exists a finite tamely ramified extension k'/k of naïve ramification degree e such that we have a unique direct sum decomposition of representations of $G_{k'}$ over some finite* *extension F' of F: $\rho|_{G_{k'}} \otimes F' \cong \bigoplus_{\vartheta \in \mathrm{rsw}(\rho)} \rho_\vartheta$, such that ρ_ϑ has pure refined Swan conductors*

$$\vartheta \in \Omega^1_{\mathcal{O}_{k'}}(\log) \otimes_{\mathcal{O}_{k'}} \pi_{k'}^{-eb} \bar{\kappa} \cong \Omega^1_{\mathcal{O}_k}(\log) \otimes_{\mathcal{O}_k} \pi_k^{-b} \bar{\kappa}.$$

(b) *Choose the p-th root of unity ζ_p as in Remark 2.3.4. Then there exists an injective homomorphism for any $b \in \mathbb{Q}_{>0}$,*

$$\mathrm{rsw} = \mathrm{rsw}_k : \mathrm{Hom}\big(\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k, \mathbb{F}_p\big) \to \Omega^1_{\mathcal{O}_k}(\log) \otimes_{\mathcal{O}_k} \pi_k^{-b} \bar{\kappa}, \qquad (2.3.8)$$

such that, when viewing the left hand side as a subset of

$$\mathrm{Hom}\big(\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k, \mathbb{Q}_p(\zeta_p)^\times\big)$$

via the identification of $1 \in \mathbb{F}_p$ *with ζ_p, we have, for any p-adic representation ρ of G_k with finite local monodromy and with pure log-break b, the images of the summands of $\rho|_{\mathrm{Fil}^b_{\log} G_k}$ under the homomorphism* rsw *exactly form the multiset of refined Swan conductors of ρ. Moreover, the homomorphism* (2.3.8) *does not depend on the choices of the Dwork pi.*

Proof. For both (a) and (b), we may assume that Hypothesis 2.1.9 holds, since only finitely many elements in a p-basis matter.

(a) Using the identification given in Lemma 2.3.6, we may first replace k and $\mathrm{Frac}\,\mathcal{O}$ by a tamely ramified extension of k and a finite extension of $\mathrm{Frac}\,\mathcal{O}$, respectively, so that the decomposition of the ∇-module $\mathcal{E}_\rho$ given by (1.5.13) of $\mathcal{E}_\rho$ can be realized over $\mathcal{R}_{K'}$, and that $\mathbb{F}_q \subseteq k_0$. Since this decomposition is canonical, it is also a decomposition of (ϕ, ∇)-modules. By the slope filtration [Kedlaya 2007, Theorem 3.4.6], the Frobenius action on each direct summand of $\mathcal{E}_\rho$ is étale, yielding the decomposition of the representation via the equivalence of categories in Theorem 2.1.7.

(b) The following are immediate corollaries of Proposition 1.3.19.

(i) For any p-adic representations ρ and ρ' of G_k with finite local monodromy, same pure log-break b, and same pure refined Swan conductor ϑ, the log-break of $\rho \otimes \rho'^\vee$ is strictly smaller than b.

(ii) For any p-adic representations ρ and ρ' of G_k with finite local monodromy, same pure log-break b, but different pure refined Swan conductor $\vartheta \neq \vartheta'$, respectively, $\rho \otimes \rho'^\vee$ has pure log-break b and pure refined Swan conductor $\vartheta - \vartheta'$.

We also need the following easy fact about Galois representations.

(iii) For any homomorphism $\chi : \mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k \to \mathbb{F}_p \hookrightarrow \mathbb{Q}_p(\zeta_p)^\times$, there exist a finite tamely ramified extension k' of k with naïve ramification degree e and a representation ρ_χ of $G_{k'}$ with finite local monodromy, pure log-break eb, and pure refined Swan conductor, such that $\rho_\chi|_{\mathrm{Fil}^b_{\log} G_k/\mathrm{Fil}^{b+}_{\log} G_k}$ contains χ as a direct summand.

Proof of (iii): The chosen p-th root of unity ζ_p in Remark 2.3.4 promotes χ to the homomorphism

$$\chi : \mathrm{Fil}^b_{\log} G_k/\mathrm{Fil}^{b+}_{\log} G_k \to \mathbb{F}_p \to \mathbb{Q}_p(\zeta_p)^\times$$

by identifying 1 with ζ_p. Since $G_k/\mathrm{Fil}^{b+}_{\log} G_k$ is a profinite group, there exists a normal subgroup H of G_k of finite index containing $\mathrm{Fil}^{b+}_{\log} G_k$, such that χ factors through

$$I = \mathrm{Fil}^b_{\log} G_k/(H \cap \mathrm{Fil}^b_{\log} G_k).$$

Put $\rho' = \mathrm{Ind}_I^{G_k/H} \chi$; then $\rho'|_{\mathrm{Fil}^b G_k}$ contains χ as a direct summand. We may use (a) to write $\rho'|_{G_{k'}}$ for some finite tamely ramified extension k' of k as the direct sum of representations with pure refined Swan conductors. Then χ appears in at least one of the direct summand, which we take to be our chosen ρ_χ.

Having established (iii), we define rsw to be the morphism sending χ to the unique refined Swan conductor of ρ_χ, which is an element of

$$\Omega^1_{\mathbb{O}_{k'}}(\log) \otimes_{\mathbb{O}_{k'}} \pi_{k'}^{-eb} \bar{\kappa} \cong \Omega^1_{\mathbb{O}_k}(\log) \otimes_{\mathbb{O}_k} \pi_k^{-b} \bar{\kappa},$$

via the identification in Lemma 2.3.6. This map is well-defined by (iv) below and it is clearly a homomorphism. Its injectivity will follow from (v).

(iv) For any two representations ρ_χ and ρ'_χ satisfying (iii), they must have the same refined Swan conductor.

Suppose the contrary, that is, ρ_χ and ρ'_χ have distinct pure refined Swan conductors ϑ and ϑ'. This in particular implies that $\rho_\chi \otimes \rho'^\vee_\chi$ has pure Swan conductor b by (ii). However, the construction of ρ_χ and ρ'_χ implies that $\rho_\chi \otimes \rho'^\vee_\chi|_{G_{k'}}$ contains a direct summand trivial on $\mathrm{Fil}^{eb}_{\log} G_{k'}$; this is a contradiction.

(v) For two distinct homomorphisms $\chi, \chi' : \mathrm{Fil}^b_{\log} G_k/\mathrm{Fil}^{b+}_{\log} G_k \to \mathbb{F}_p$, the representations ρ_χ and $\rho_{\chi'}$ given by (iii) have distinct refined Swan conductors.

Suppose the contrary. Then (i) implies that $\rho_\chi \otimes \rho^\vee_{\chi'}$ would have log-break strictly less than eb. However, $\rho_\chi \otimes \rho^\vee_{\chi'}$, when restricted to

$$\mathrm{Fil}^b_{\log} G_k/\mathrm{Fil}^{b+}_{\log} G_k = \mathrm{Fil}^{eb}_{\log} G_{k'}/\mathrm{Fil}^{eb+}_{\log} G_{k'},$$

has a direct summand isomorphic to $\chi \otimes \chi'^\vee$, which is nontrivial. This is a contradiction.

We now prove the independence on the choice of the Dwork pi. If we choose another Dwork pi, we would need to use another primitive p-th root of unity ζ_p^i for some $i \in 1, \dots, p-1$. On one hand, the refined Swan conductor is multiplied by $(\zeta_p^i - 1)/(\zeta_p - 1) \equiv i \bmod (\zeta_p - 1)$. On the other hand, the p-adic representation

$$\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k \to \mathbb{Q}_p(\zeta_p)^\times$$

becomes χ^i. Hence we need to take $\rho_\chi^{\otimes i}$ as our p-adic representation of $G_{k'}$ to define the homomorphism rsw. This representation has refined Swan conductor $\mathrm{rsw}(\rho_\chi^{\otimes i}) = i \cdot \mathrm{rsw}(\rho_\chi)$, which is the same as the refined Swan conductor of ρ computed using the old Dwork pi. □

Remark 2.3.9. It is interesting to point out that the choice of a Dwork pi is related to the choice of the Artin–Scheier ℓ-adic sheaf in [Saito 2009]; they both amount to choosing a primitive p-th root of unity. The difference is that we consider it as an element in $\overline{\mathbb{Q}}_p$ whereas Saito viewed it as an element in $\overline{\mathbb{Q}}_l$.

Proposition 2.3.10. *Let k be a complete discrete valuation field of equal characteristic $p > 0$. Then for $b \in \mathbb{Q}_{>0}$, the conjugation action of*

$$\mathrm{Fil}^{0+}_{\log} G_k / \mathrm{Fil}^b_{\log} G_k \quad \textit{on} \quad \mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k$$

is trivial. In other words, $\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k$ *lies in the center of* $\mathrm{Fil}^{0+}_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k$.

Proof. This proposition is proved in [Abbes and Saito 2003, Theorem 1]. We give an alternative proof using differential modules.

It suffices to prove the following: for a p-adic representation ρ of G_k with finite local monodromy and with pure log-break b, if it is absolutely irreducible under any tamely ramified extension, then

$$\rho|_{\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k}$$

is a direct sum of a *single* character $\chi : \mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k \to \mathbb{O}^\times$. This is equivalent to showing that the action of $\mathrm{Fil}^b_{\log} G_k$ on $\rho \otimes \rho^\vee$ is trivial, and hence to showing that the log-break of $\rho \otimes \rho^\vee$ is strictly smaller than b.

As usual, we may assume Hypothesis 2.1.9. By Theorem 2.3.7(a), the irreducibility condition on ρ implies that ρ must have pure refined Swan conductor and hence the log-break $\rho \otimes \rho^\vee$ must be strictly less than b. We are done. □

Proposition 2.3.11. *Keep the notation as in Situation 2.2.13. Then the refined Swan conductor homomorphism* rsw_k *for k factors as*

$$\mathrm{Hom}(\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k, \mathbb{F}_p) \longrightarrow \mathrm{Hom}(\mathrm{Fil}^{e_{\tilde{k}/k} b}_{\log} G_{\tilde{k}} / \mathrm{Fil}^{e_{\tilde{k}/k} b+}_{\log} G_{\tilde{k}}, \mathbb{F}_p)$$
$$\xrightarrow{\mathrm{rsw}_{\tilde{k}}} \Omega^1_{\mathbb{O}_{\tilde{k}}}(\log) \otimes_{\mathbb{O}_{\tilde{k}}} \pi_{\tilde{k}}^{-eb} \kappa_{\tilde{k}^{\mathrm{alg}}}. \quad (2.3.12)$$

Proof. Keep the notation as in Proposition 2.2.14, let $\widetilde{F}'_\eta$ be the completion of $\widetilde{K}'(U)$ with respect to the $\eta^{1/e}$-Gauss norm in U. Fix $\eta_0 \in (0, 1)$ such that $IR(\mathscr{E}_\rho \otimes F'_\eta) = \eta^b$

for $\eta \in [\eta_0, 1)$. Then (2.2.15) implies that, for any $\eta \in [\eta_0, 1) \cap p^{\mathbb{Q}}$ and for any $j \in \{0, \dots, m+1\}$ such that $IR_{\partial_j}(f^*\mathscr{E}_\rho \otimes \widetilde{F}'_\eta) = IR(\mathscr{E}_\rho \otimes \widetilde{F}'_\eta)$, we have

$$\Theta_{\partial_j}(f^*\mathscr{E}_\rho \otimes \widetilde{F}'_\eta) = \begin{cases} \Theta_{\partial_j}(\mathscr{E} \otimes F'_\eta) & \text{if } j \in J, \\ eXU^{e-1}\Theta_{\partial_0}(\mathscr{E} \otimes F'_\eta) & \text{if } j = 0 \text{ and hence } p \nmid e, \\ U^e\Theta_{\partial_{m+1}}(\mathscr{E} \otimes F'_\eta) & \text{if } j = m+1, \end{cases}$$

using Theorem 1.4.20 to compute the refined radii. The proposition follows. □

One may want to prove analogs of Theorem 2.3.7 and Proposition 2.3.10 for refined Artin conductors. This however needs to take a bit more effort because there may not be a representation of G_k with pure refined Artin conductor. Instead, we reduce to the classical case, where the results for refined Artin conductors follows from those for refined Swan conductors.

Theorem 2.3.13. *Let k be a complete discrete valuation field of equal characteristic $p > 0$.*

(a) *Choose the p-th root of unity ζ_p as in Remark 2.3.4. Then there exists an injective homomorphism for any $b \in \mathbb{Q}_{>1}$,*

$$\mathrm{rar} = \mathrm{rar}_k : \mathrm{Hom}(\mathrm{Fil}^b G_k/\mathrm{Fil}^{b+} G_k, \mathbb{F}_p) \to \Omega^1_{\mathcal{O}_k} \otimes_{\mathcal{O}_k} \pi_k^{-b}\bar{k}, \tag{2.3.14}$$

such that, when viewing the left hand side as a subset of

$$\mathrm{Hom}(\mathrm{Fil}^b G_k/\mathrm{Fil}^{b+} G_k, \mathbb{Q}_p(\zeta_p)^\times)$$

via the identification of $1 \in \mathbb{F}_p$ with ζ_p, we have, for any p-adic representation ρ of G_k with finite local monodromy and with pure nonlog-break b, the images of the summands of $\rho|_{\mathrm{Fil}^b G_k}$ under rar *exactly form the multiset of refined Artin conductors of ρ. Moreover, this homomorphism does not depend on the choices of the Dwork pi.*

(b) *For any $b \in \mathbb{Q}_{>1}$, the conjugation action of*

$$\mathrm{Fil}^{1+} G_k/\mathrm{Fil}^b G_k \quad \text{on} \quad \mathrm{Fil}^b G_k/\mathrm{Fil}^{b+} G_k$$

is trivial. That is, $\mathrm{Fil}^b G_k/\mathrm{Fil}^{b+} G_k$ lies in the center of $\mathrm{Fil}^{1+} G_k/\mathrm{Fil}^{b+} G_k$.

Proof. For both (a) and (b), we may assume Hypothesis 2.1.9. Moreover, we assume that J is not empty because otherwise we are in the classical case, and both (a) and (b) follow from their log-version counterpart: Theorem 2.3.7 and Proposition 2.3.10, respectively.

We perform a base change similar to the one in Lemma 2.2.9. Let k' be the completion of $k(x_1, \dots, x_m)$ with respect to the $(1, \dots, 1)$-Gauss norm and let $\tilde{k}$ be the completion of

$$k'\big((b_j + x_j\pi_k)^{1/p^n}, x_j^{1/p^n}; n \in \mathbb{N}; j \in J\big),$$

equipped with the uniformizer $\pi_{\tilde{k}} = \pi_k$. It is in fact a complete discrete valuation field with *perfect* residue field. By Lemmas 2.2.9 and 2.2.11, the natural homomorphism $G_{\tilde{k}} \to G_k$ induces a *surjective* homomorphism

$$\mathrm{Fil}^a G_{\tilde{k}} / \mathrm{Fil}^{a+} G_{\tilde{k}} \longrightarrow \mathrm{Fil}^a G_k / \mathrm{Fil}^{a+} G_k.$$

Dualizing this gives an *injective* homomorphism

$$\mu : \mathrm{Hom}(\mathrm{Fil}^a G_k / \mathrm{Fil}^{a+} G_k, \mathbb{F}_p) \to \mathrm{Hom}(\mathrm{Fil}^a G_{\tilde{k}} / \mathrm{Fil}^{a+} G_{\tilde{k}}, \mathbb{F}_p).$$

For ρ a representation of G_k with finite local monodromy and with pure nonlog-break b we let $\tilde{\rho}$ denote the representation $G_{\tilde{k}} \to G_k \xrightarrow{\rho} GL(V_\rho)$. Let K'' denote the completion of $K'(X_J)$ with respect to the $(1, \dots, 1)$-Gauss norm, where X_j is a lift of x_j for $j \in J$. Let $\widetilde{K}$ denote the completion of

$$K''\big((B_j + X_j T)^{1/p^n}, X_j^{1/p^n}; n \in \mathbb{N}, j \in J\big).$$

Let $f : A^1_{\widetilde{K}}[\eta_0, 1) \to A^1_{K'}[\eta_0, 1)$ denote the natural morphism. Then $f^*\mathscr{E}_\rho$ is the differential module associated to $\tilde{\rho}$. Let $\tilde{\partial}$ denote the differential operator on $f^*\mathscr{E}_\rho$ dual to the basis dT. Similar to (2.2.10), we have

$$\tilde{\partial} = \partial_0 - X_1 \partial_1 - \cdots - X_m \partial_m.$$

If we let $\widetilde{F}_\eta$ denote the completion of $\widetilde{K}(T)$ with respect to the η-Gauss norm, we have

$$R_{\tilde{\partial}}(f^*\mathscr{E} \otimes \widetilde{F}_\eta) = \min_{j \in J^+} \big\{ R_{\partial_j}(\mathscr{E} \otimes F'_\eta) \big\}.$$

Hence $\tilde{\rho}$ has pure nonlog-break b and, by Theorem 1.4.20, its multiset of refined Artin conductors is

$$\mathrm{rar}(\tilde{\rho}) = \big\{ (\theta_0 - X_1\theta_1 - \cdots - X_m\theta_m) d\pi_k \,\big|\, \theta_0 d\pi_k + \theta_1 db_1 + \cdots + \theta_m db_m \in \mathrm{rar}(\rho) \big\}.$$

In other words, if we consider the $\bar{\kappa}$-linear injective homomorphism

$$\lambda : \Omega^1_{\mathbb{O}_k} \otimes_{\mathbb{O}_k} \pi_k^{-b} \overline{\kappa} \to \pi_k^{-b} \kappa_{\tilde{k}^{\mathrm{alg}}} d\pi_k$$

given by $\lambda(db_j) = -X_j d\pi_k$ and $\lambda(d\pi_k) = d\pi_k$, then $\mathrm{rar}(\tilde{\rho}) = \lambda(\mathrm{rar}(\rho))$. This together with the injectivity of μ reduce (a) and (b) for G_k to that of $G_{\tilde{k}}$, which is already known as we explained earlier. In particular, we have $\lambda \circ \mathrm{rsw}_k = \mathrm{rsw}_{\tilde{k}} \circ \mu$. □

2.4. *Multi-indexed ramification filtrations for higher local fields.* When k is an n-dimensional local field, the refined Artin and Swan conductors give more refined filtrations on the Galois group G_k, indexed by $\mathbb{Q}^n$ with lexicographic order. We restrict ourselves to the equal characteristic $p > 0$ case.

Definition 2.4.1. We say that a complete discrete valuation field k of characteristic $p > 0$ is an $(m+1)$-*dimensional local field* if there is a chain of fields $k = k_{m+1}, k_m, \dots, k_0$, where k_{i+1} is a complete discrete valuation field with residue field k_i for $i = 0, \dots, m$. Contrary to most literature, we do *not* assume that k_0 is a perfect field. Let $\{b_j\}_{j\in J}$ be a set of lifts of a p-basis of k_0 to $\mathbb{O}_k$.

An $(m+1)$-tuple of elements $t_0, \dots, t_m \in k$ is called a *system of local parameters* of k if $t_i \in \mathbb{O}_k$ is a lift of a uniformizer of k_{m+1-i} all the way up to k. Such a choice gives a (noncanonical) isomorphism $k \simeq k_0((t_m))\cdots((t_0))$. In this case, we have

$$\Omega^1_{\mathbb{O}_{k'}}(\log) = \bigoplus_{i=0}^m \mathbb{O}_{k'} \frac{dt_i}{t_i} \oplus \bigoplus_{j\in J} \mathbb{O}_{k'} \frac{db_j}{b_j} \quad \text{and} \quad \Omega^1_{\mathbb{O}_{k'}} = \bigoplus_{i=0}^m \mathbb{O}_{k'}\, dt_i \oplus \bigoplus_{j\in J} \mathbb{O}_{k'}\, db_j.$$

Equip $\mathbb{Q}^{m+1}$ with the lexicographic order: $\boldsymbol{i} = (i_1, \dots, i_{m+1}) < \boldsymbol{j} = (j_1, \dots, j_{m+1})$ if and only if, for some $l \le m+1$,

$$i_l < j_l, \quad i_{l+1} = j_{l+1}, \quad \dots, \quad i_{m+1} = j_{m+1}.$$

For $a \in \mathbb{Q}$, we use $\mathbb{Q}^{m+1}_{>a}$ to denote the subset of $\mathbb{Q}^{m+1}$ consisting of $\boldsymbol{i} = (i_1, \dots, i_{m+1})$ such that $i_{m+1} > a$.

Given a system of local parameters, we define a multi-indexed valuation as follows, denoted by $\boldsymbol{v} = (v_1, \dots, v_{m+1}) : k^\times \to \mathbb{Z}^{m+1} \subset \mathbb{Q}^{m+1}$, where $v_{m+1} = v_{k_{m+1}}$ and recursively we have, downwards from $i = m+1$ to $i = 1$, that $v_{i-1}(\alpha) = v_{k_{i-1}}(\alpha_{i-1})$ with α_{i-1} equal to the reduction of $\alpha_i t_{m+1-i}^{-v_i(\alpha_i)}$ in k_{i-1}. Note that the definition of $\boldsymbol{v}$ depends on the choice of local parameters $t_0, \dots, t_m$.

Definition 2.4.2. For $\lambda = \sum_{i=0}^m \alpha_i dt_i + \sum_{j\in J} \beta_j db_j \in \Omega^1_{\mathbb{O}_k} \otimes_{\mathbb{O}_k} k$, we set

$$\boldsymbol{v}_{\mathrm{nlog}}(\lambda) = \min\{\boldsymbol{v}(\alpha_0), \dots, \boldsymbol{v}(\alpha_m), \boldsymbol{v}(\beta_j);\ j \in J\}.$$

This gives a multi-indexed valuation on $\Omega^1_{\mathbb{O}_k} \otimes_{\mathbb{O}_k} t_0^{-i_{m+1}}\overline{\kappa}$ for $i_{m+1} \in \mathbb{Q}$.

For $\lambda = \sum_{i=0}^m \alpha_i \frac{dt_i}{t_i} + \sum_{j\in J} \beta_j \frac{db_j}{b_j} \in \Omega^1_{\mathbb{O}_k}(\log) \otimes_{\mathbb{O}_k} k$, we set

$$\boldsymbol{v}_{\log}(\lambda) = \min\{\boldsymbol{v}(\alpha_0), \dots, \boldsymbol{v}(\alpha_m), \boldsymbol{v}(\beta_j);\ j \in J\}.$$

This gives a multi-indexed valuation on $\Omega^1_{\mathbb{O}_k}(\log) \otimes_{\mathbb{O}_k} t_0^{-i_{m+1}}\overline{\kappa}$ for $i_{m+1} \in \mathbb{Q}$.

For $\boldsymbol{i} = (i_1, \dots, i_{m+1}) \in \mathbb{Q}^{m+1}_{>1}$, we define $\mathrm{Fil}^{\boldsymbol{i}} G_k$ to be the subgroup of $\mathrm{Fil}^{i_{m+1}} G_k$ given by the intersection of the kernels of characters

$$\chi : \mathrm{Fil}^{i_{m+1}} G_k \to \mathrm{Fil}^{i_{m+1}} G_k / \mathrm{Fil}^{i_{m+1}+} G_k \to \mathbb{F}_p$$

for which $\boldsymbol{v}_{\mathrm{nlog}}(\chi) > -\boldsymbol{i}$. We similarly define $\mathrm{Fil}^{\boldsymbol{i}} G_k$ for $\boldsymbol{i} = (i_1, \dots, i_{m+1}) \in \mathbb{Q}^{m+1}_{>0}$ by adding subscripts log to the definition above.

Remark 2.4.3. The abstract filtrations do not depend on the choices of local parameters, but the indexings do. Set $O_K = \{x \in K : \boldsymbol{v}(x) \geq (0, \ldots, 0)\}$. It might be more natural to index the above filtrations by "rational powers of fractional ideals of K" of the form $I^{1/n}$, where I is an O_K-submodule of K containing O_K, n is an integer, and $I^{1/n}$ is equivalent to $I'^{1/n'}$ if $I^{n'} = I'^n$ as O_K-submodules of K.

Remark 2.4.4. When k_0 is a finite field, this filtration is expected to be compatible with an easily defined filtration on the Milnor K-groups via class field theory for higher local fields. This may be verified by comparing the filtration on the Milnor K-groups with Kato's refined Swan conductors, which is equivalent to Saito's definition by [Abbes and Saito 2009, Theorem 9.1.1] and hence to our definition by Theorem 3.4.1 proved later. For more along this line, the reader may refer to the recipe in Kato's masterpiece [Kato 1989].

3. Comparison with Saito's definition

In this section, we compare our definition of the refined Swan conductor homomorphism with the one given by Saito in [Saito 2009]. Since the reader who is only interested in one side of the story may use this result (Theorem 3.4.1) as a black box, we present the proof assuming that the reader is familiar with the definition of arithmetic ramification filtrations; see for instance [Saito 2009, Section 1; Xiao 2010].

The proof of the comparison theorem is of a geometric nature. We explain the rough idea here. We first realize the given finite extension l of k as the corresponding extension of function fields of a finite étale extension of smooth affine varieties $Y \to X$. Our main object is some version of infinitesimal neighborhood of the generic fiber over k of the diagonal embedding of Y into $Y \times Y$, viewed as a rigid analytic space over k. The refined Swan conductor homomorphism defined by Saito makes use of the stable formal model of such an object, whereas our definition using differential modules is closely related to some object over the generic point of a smooth model over $\mathbb{O}_K$ lifting the aforementioned rigid space. The crucial calculation we performed in Section 3.3 relates these objects, in which case it boils down to some explicit computation on a higher dimensional analog of the Artin–Scheier cover, and on the associated ℓ-adic sheaves and overconvergent F-isocrystals.

We assume $p > 0$ is a prime number throughout this section.

3.1. ***Review of Saito's definition.*** In this subsection, we review the definition of the ramification filtrations and the refined Swan conductors defined by Abbes and Saito in [Abbes and Saito 2002; 2003; Saito 2009]. Instead of introducing the general construction, we will focus on a special case which is used in the comparison theorem. For more details and a complete treatment, one may consult [Saito 2009].

Construction 3.1.1. Let l be a finite Galois extension of k. We consider a closed immersion $\operatorname{Spec}\mathbb{O}_l \to P$ into a smooth (affine) scheme P over $\operatorname{Spec}\mathbb{O}_k$. Put $\mathcal{I} = \operatorname{Ker}(\mathbb{O}_P \to \mathbb{O}_l)$.

For $r = a/b \in \mathbb{Q}$ with $a, b > 0$, let $P^{[a/b]}_{\mathbb{O}_k} \to P$ be the blowup at the ideal $\mathcal{I}^b + \mathfrak{m}_k^a \mathbb{O}_P$ and let $P^{(a/b)}_{\mathbb{O}_k} \subset P^{[a/b]}_{\mathbb{O}_k}$ be the complement of the support of

$$\big(\mathcal{I}^a \mathbb{O}_{P^{[a/b]}_{\mathbb{O}_k}} + \mathfrak{m}_k^b \mathbb{O}_{P^{[a/b]}_{\mathbb{O}_k}}\big)/\mathfrak{m}_k^b \mathbb{O}_{P^{[a/b]}_{\mathbb{O}_k}}.$$

Let $P^{(r)}_{\mathbb{O}_k}$ be the normalization of $P^{(a/b)}_{\mathbb{O}_k}$; it does not depend on a and b but only on their ratio. Let $P^{(r)}_k$ and $P^{(r)}_\kappa$ denote the generic fiber and the special fiber of $P^{(r)}_{\mathbb{O}_k}$, respectively. Let $\widehat{P}^{(r)}_k$ denote the generic fiber of completing $P^{(r)}_{\mathbb{O}_k}$ along $P^{(r)}_\kappa$. The immersion $\operatorname{Spec}\mathbb{O}_l \to P$ is uniquely lifted to an immersion $\operatorname{Spec}\mathbb{O}_l \to P^{(r)}_{\mathbb{O}_k}$.

By the finiteness theorem of Grauert–Remmert cited in [Abbes and Saito 2003, Theorem 1.10], there exists a finite *separable* extension k'/k of naïve ramification degree $e = e_{k'/k}$ such that the normalization $P^{(er)}_{\mathbb{O}_{k'}}$ of $P^{(r)}_{\mathbb{O}_k} \times_{\mathbb{O}_k} \mathbb{O}_{k'}$ has reduced geometric fibers over $\operatorname{Spec}\mathbb{O}_{k'}$, which we call a *stable model* of $P^{(r)}_{\mathbb{O}_k}$. We put

$$P^{(r)}_{\bar\kappa} = P^{(er)}_{\mathbb{O}_{k'}} \times_{\mathbb{O}_{k'}} \bar\kappa;$$

this is called the *stable special fiber* of $P^{(r)}_{\mathbb{O}_k}$ and it does not depend on the choice of k'.

We defer the discussion of the properties of this construction until later when we have a concrete example at hand.

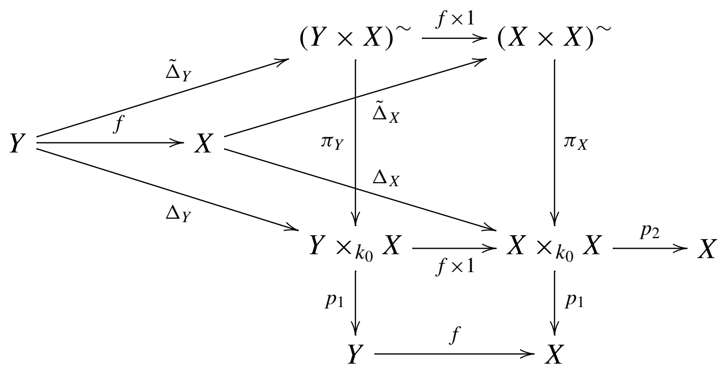
For the rest of this section, we make the following geometric assumption.

Hypothesis 3.1.2 (Geom). There exists an affine smooth variety X over k_0 and an irreducible divisor D, smooth over k_0 with generic point ξ, such that $\mathbb{O}_k \cong \mathbb{O}^\wedge_{X,\xi}$, where the latter is the completion of the local ring at ξ. In particular, Hypothesis 2.1.9 is fulfilled.

Remark 3.1.3. This hypothesis is essentially the same as the hypothesis of the same name in [Saito 2009, p. 786], except that our k is the completion of the Henselian local field considered in Saito's paper.

Construction 3.1.4. After replacing X (and hence D) by an étale neighborhood of ξ if necessary, there exists a finite flat morphism $f : Y \to X$ of smooth varieties over k_0 such that $V = Y \times_X U \to U = X\backslash D$ is finite étale with Galois group $G_{l/k}$ and that $Y \times_X \operatorname{Spec}\mathbb{O}^\wedge_{X,\xi} = \operatorname{Spec}\mathbb{O}_l$.

Let $(X \times X)'$ be the blowup of $X \times_{k_0} X$ along $D \times_{k_0} D$, and let $(X \times X)^\sim$ denote the complement of the proper transforms of $X \times_{k_0} D$ and $D \times_{k_0} X$ in $(X \times X)'$. The diagonal embedding $\Delta_X : X \to X \times_{k_0} X$ naturally lifts to an embedding $\tilde\Delta_X : X \to (X \times X)^\sim$. Now, pulling back the whole picture along $f : Y \to X$ gives the commutative diagram



(3.1.5)

where $(Y \times X)^{\sim}$ is the fiber product of the big square, and all parallelograms are Cartesian.

Put

$$P = (X \times X)^{\sim} \times_{p_2 \circ \pi_X, X} \operatorname{Spec} \mathcal{O}^{\wedge}_{X,\xi} \quad \text{and} \quad Q = (Y \times X)^{\sim} \times_{p_2 \circ \pi_X \circ (f \times 1), X} \operatorname{Spec} \mathcal{O}^{\wedge}_{X,\xi}.$$

Taking the Cartesian product of the top part of (3.1.5) with $\operatorname{Spec} \mathcal{O}^{\wedge}_{X,\xi} = \operatorname{Spec} \mathcal{O}_k$ over $X \times_{k_0} X$ along p_2 then gives the following commutative diagram.

$$\begin{array}{ccccc} \operatorname{Spec} \mathcal{O}_l & \xrightarrow{\tilde{\Delta}_Y} & Q & & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times 1} & & \\ \operatorname{Spec} \mathcal{O}_k & \xrightarrow{\tilde{\Delta}_X} & P & \xrightarrow{p_2} & \operatorname{Spec} \mathcal{O}_k \end{array} \tag{3.1.6}$$

Let $\mathcal{J}$ denote the ideal of the immersion $\tilde{\Delta}_X$. We will view P and Q as schemes over $\mathcal{O}_k$ via p_2.

We can now apply Construction 3.1.1 to the embeddings $\tilde{\Delta}_X$ and $\tilde{\Delta}_Y$ to define

$$P^{(er)}_{\mathcal{O}'_k}, P^{(er)}_{k'}, \widehat{P}^{(er)}_{k'}, P^{(er)}_{\bar{\kappa}} \quad \text{and} \quad Q^{(er)}_{\mathcal{O}'_k}, Q^{(er)}_{k'}, \widehat{Q}^{(er)}_{k'}, Q^{(er)}_{\bar{\kappa}},$$

respectively, where k'/k is a finite separable extension of naïve ramification degree e. We still use p_1 to denote the morphism $P^{(er)}_{\mathcal{O}_{k'}} \to P \xrightarrow{p_1} \operatorname{Spec} \mathcal{O}_k$. By functoriality of Construction 3.1.1, we have a morphism $f^{(r)} : Q^{(er)}_{\mathcal{O}_{k'}} \to P^{(er)}_{\mathcal{O}_{k'}}$.

Remark 3.1.7. The field extension k' serves as the role of a "coefficient field"; we only use it to provide reasonable integral structures of our spaces over $\mathcal{O}_{k'}$, and also to make er an integer. We can make k' as large as we need.

In contrast, the extension l/k pulled back from $p_1 : X \times_{k_0} X \to X$ encodes the arithmetic information.

We collect together some properties of these spaces.

Proposition 3.1.8. *Let k'/k be a finite separable extension of naïve ramification degree e.*

(a) *When er is an integer, the space $P^{(er)}_{\mathbb{O}_{k'}}$ is defined to be $\sum_{i\geq 0}\pi_{k'}^{-ier}\cdot\mathcal{I}^i\subset\mathbb{O}_P\otimes_{\mathbb{O}_k}k'$. It is smooth over $\mathbb{O}_{k'}$, and its closed fiber $P^{(er)}_{\kappa_{k'}}$ can be canonically identified with the $\kappa_{k'}$-vector space $\Omega^1_{\mathbb{O}_k}(\log)\otimes_{\mathbb{O}_k}\pi_{k'}^{-er}\kappa_{k'}$. The rigid space*

$$\widehat{P}^{(er)}_{k'}\quad \textit{is isomorphic to}\quad \mathrm{Sp}\left(k'\langle\pi_{k'}^{-er-e}\delta_0,\pi_{k'}^{-er}\delta_J\rangle\right),$$

where $\delta_0,\dots,\delta_m$ form a dual basis of $\Omega^1_{\mathbb{O}_k}$.

(b) *The generic fiber $Q^{(er)}_{k'}$ of $Q^{(er)}_{\mathbb{O}_{k'}}$ is isomorphic to $P^{(er)}_{k'}\otimes_{p_1,k}l$. In particular, $Q^{(er)}_{k'}$ is finite and étale over $P^{(er)}_{k'}$ with Galois group $G_{l/k}$, and the same is true for*

$$\widehat{Q}^{(er)}_{k'}\quad\textit{over}\quad\widehat{P}^{(er)}_{k'}.$$

(c) *Let $\mathrm{Spf}\,\mathbb{O}_{Q^\wedge}$ be the completion of Q along $\mathrm{Spec}\,\mathbb{O}_l$. If er is an integer, then*

$$\widehat{Q}^{(er)}_{k'}\quad\textit{is the affinoid variety}\quad X^j_{\log}(\mathbb{O}_{Q^\wedge}\to\mathbb{O}_l)_{k'}$$

defined in [Abbes and Saito 2003, Section 4.2] *for $j=r$.*

(d) *If the highest log-break $b_{\log}(l/k)$ is less than or equal to r, then $Q^{(r)}_{\bar\kappa}$ is an element of the category $(\mathrm{FE}/P^{(r)}_{\bar\kappa})^{\mathrm{alg}}$, defined below in Definition 3.1.9.*

(e) *The highest log-break $b_{\log}(l/k)$ is strictly less than r if and only if the number of connected components of $Q^{(r)}_{\bar\kappa}$ is $[l:k]$.*

Proof. For (a), see [Saito 2009, Lemma 1.10]. The claim (b) follows from the fact that $f:V\to U$ is finite and étale with Galois group $G_{l/k}$. For (c), see [ibid., Example 1.21]. The statements (d) and (e) follow from [ibid., Lemma 1.13 and Theorem 1.24]. □

Definition 3.1.9. For an $\bar\kappa$-vector space W of finite dimensional, let $(\mathrm{FE}/W)^{\mathrm{alg}}$ be the full subcategory of (FE/W) whose objects are finite étale morphisms $g:Z\to W$ such that Z admits a structure of algebraic group scheme and such that g is a morphism of algebraic groups.

Remark 3.1.10. By the argument just before [ibid., Lemma 1.23], the category $(\mathrm{FE}/W)^{\mathrm{alg}}$ is a Galois category associated to the Galois group $\pi_1^{\mathrm{alg}}(W)$, which is a quotient of the fundamental group $\pi_1(W)$. This group can be identified with the Pontrjagin dual of the extension group $\mathrm{Ext}^1(W,\mathbb{F}_p)$ in the category of smooth algebraic groups over $\bar\kappa$. The map $W^\vee=\mathrm{Hom}_{\bar\kappa}(W,\bar\kappa)\to\mathrm{Ext}^1(W,\mathbb{F}_p)$ sending a linear form $f:W\to\mathbb{A}^1_{\bar\kappa}$ to the pullback along f of the Artin–Scheier sequence

$$0\to\mathbb{F}_p\to\mathbb{A}^1_{\bar\kappa}\xrightarrow{t\to t^p-t}\mathbb{A}^1_{\bar\kappa}\to 0$$

is an isomorphism.

Proposition 3.1.11. *We have a surjective homomorphism*

$$\pi_1^{\mathrm{alg}}(P_{\bar\kappa}^{(b)}) \twoheadrightarrow \mathrm{Fil}_{\log}^{b} G_k/\mathrm{Fil}_{\log}^{b+} G_k;$$

it induces an injective homomorphism

$$\mathrm{rsw}' : \mathrm{Hom}(\mathrm{Fil}_{\log}^{b} G_k/\mathrm{Fil}_{\log}^{b+} G_k, \mathbb{F}_p) \longrightarrow \Omega^1_{\mathbb{O}_k}(\log) \otimes_{\mathbb{O}_k} \pi_k^{-b}\bar\kappa.$$

Proof. For the first half of the proposition, see [Saito 2009, Theorem 1.24]. The second half follows from Remark 3.1.10. □

In the following special case, we give a more detailed description of these spaces.

Situation 3.1.12. Let l/k be a finite totally ramified Galois extension, which is *not* tamely ramified. Assume that the highest log-break $b = b_{\log}(l/k)$ is a positive integer. Assume moreover that $\mathrm{Fil}_{\log}^{b-1} G_k/(\mathrm{Fil}_{\log}^{b-1} G_k \cap G_l) \simeq \mathbb{F}_p$; in particular, the second highest log-break $b_{\log}(l/k; 2)$ is strictly less than $b_{\log}(l/k) - 1$. By Proposition 3.1.11, $Q_{\bar\kappa}^{(b)}$ consists of $[l:k]/p$ copies of the *same* Artin–Scheier cover of $P_{\bar\kappa}^{(b)}$, at least if we forget about the algebraic group structure. Assume that this cover is given by

$$\bar z^p - \bar z + \big(\bar\alpha_0 \pi_k^{-b-1}\delta_0 + \bar\alpha_1 \pi_k^{-b}\delta_1 + \cdots + \bar\alpha_m \pi_k^{-b}\delta_m\big) = 0 \tag{3.1.13}$$

for some $\bar\alpha_{J^+} \in \bar\kappa$, where the coordinates of $P_{\bar\kappa}^{(b)}$ are given by $\pi_k^{-b-1}\delta_0$ and $\pi_k^{-b}\delta_J$. These elements $\bar\alpha_0, \dots, \bar\alpha_m$ are determined up to multiplication by $i \in \mathbb{F}_p^\times$, in accordance with the choice of $\bar z$ up to multiplication by the same $i \in \mathbb{F}_p^\times$.

Let k'/k be a finite separable extension of ramification degree $e > 1$, such that $Q_{\mathbb{O}_{k'}}^{(eb)}$ is a stable model. By possibly enlarging k', we may assume that $\bar\alpha_{J^+} \in \kappa_{k'}$ and that $Q_{\kappa_{k'}}^{(eb)}$ is the disjoint union of $[l:k]/p$ copies of the aforementioned Artin–Scheier cover of $P_{\kappa_{k'}}^{(eb)}$.

Lemma 3.1.14. *The space $Q_{\mathbb{O}_{k'}}^{(eb)}$ is the disjoint union of $[l:k]/p$ copies of the same space $R_{\mathbb{O}_{k'}}^{(eb)}$. Let $\widehat R_{\mathbb{O}_{k'}}^{(eb)}$ denote the completion of $R_{\mathbb{O}_{k'}}^{(eb)}$ along its special fiber and let $\widehat R_{k'}^{(eb)}$ denote the generic fiber, viewed as a rigid analytic space. Then $\widehat Q_{k'}^{(eb-1)}$ is the disjoint union of $[l:k]/p$ copies of a same space $\widehat R_{k'}^{(eb-1)}$, which is the normal closure of $\widehat P_{k'}^{(eb-1)}$ in $\widehat R_{k'}^{(eb)}$ and is finite and étale over $\widehat P_{k'}^{(eb-1)}$.*

Proof. There is a $G_{l/k}$-equivariant one-to-one correspondence between the connected components of $Q_{\kappa_{k'}}^{(eb)}$ and the connected components of $Q_{\mathbb{O}_{k'}}^{(eb)}$.

Since the second highest log-break $b_{\log}(l/k; 2)$ is strictly less than $b_{\log}(l/k) - 1$, by [Abbes and Saito 2002, Remark 3.13], the number of connected components of $\widehat Q_{k'}^{(eb-1)}$ is $[l:k]/p$. Note that each connected component of $\widehat Q_{k'}^{(eb-1)}$, which is automatically finite and étale over $\widehat{P}_{k'}^{(eb-1)}$, can be also characterized as the normal closure of $\widehat{P}_{k'}^{(eb-1)}$ in $\widehat{R}_{k'}^{(eb)}$; this normal closure is the space $\widehat{R}_{k'}^{(eb-1)}$ we sought. □

Proposition 3.1.15. *Let $\alpha_{J^+} \subset \mathbb{O}_{k'}$ lift $\bar{\alpha}_{J^+} \subset \kappa_{k'}$. We can choose a lift z of $\bar{z}$ to $\widehat{R}_{\mathbb{O}_{k'}}^{(eb)}$ such that its minimal polynomial over $\widehat{P}_{\mathbb{O}_{k'}}^{(eb)} = \operatorname{Spf} \mathbb{O}_{k'}\langle \pi_{k'}^{-eb-e}\delta_0, \pi_{k'}^{-eb}\delta_J\rangle$ is*

$$z^p - z + \left(\alpha_0 \pi_{k'}^{-eb-e}\delta_0 + \alpha_1 \pi_{k'}^{-eb}\delta_1 + \cdots + \alpha_m \pi_{k'}^{-eb}\delta_m\right) = 0. \tag{3.1.16}$$

Then the element z generates $\widehat{R}_{\mathbb{O}_{k'}}^{(eb)}$ over $\widehat{P}_{\mathbb{O}_{k'}}^{(eb)}$. Moreover, the element z extends to a section over $\widehat{R}_{k'}^{(eb-1)}$ and it generates $\widehat{R}_{k'}^{(eb-1)}$ over $\widehat{P}_{k'}^{(eb-1)}$.

Proof. We first pick any lift z' of $\bar{z}$ to $\widehat{R}_{\mathbb{O}_{k'}}^{(eb)}$; it must satisfy an equation of the form $z'^p + a_1 z'^{p-1} + \cdots + a_p = 0$, where $a_1, \ldots, a_p \in \mathbb{O}_{k'}\langle \pi_{k'}^{-eb-e}\delta_0, \pi_{k'}^{-eb}\delta_J\rangle$ and the reduction of this equation is exactly (3.1.13). For the given $\alpha_{J^+} \subset \mathbb{O}_{k'}$, we have

$$\epsilon = z'^p - z' + \left(\alpha_0 \pi_{k'}^{-eb-e}\delta_0 + \alpha_1 \pi_{k'}^{-eb}\delta_1 + \cdots + \alpha_m \pi_{k'}^{-eb}\delta_m\right) \in \pi_{k'}\mathbb{O}_{\widehat{R}_{\mathbb{O}_{k'}}^{(eb)}}.$$

Now, $z = z' + \epsilon + \epsilon^p + \epsilon^{p^2} + \cdots$ converges and satisfies (3.1.16).

Since z generates a subalgebra of $\mathbb{O}_{\widehat{R}_{\mathbb{O}_{k'}}^{(eb)}}$ which is finite and étale over $\mathbb{O}_{\widehat{P}_{\mathbb{O}_{k'}}^{(eb)}}$ of the same degree p, this subalgebra has to equal $\mathbb{O}_{\widehat{R}_{\mathbb{O}_{k'}}^{(eb)}}$.

For the similar statement for $eb-1$ in place of eb, we argue as follows. Since $\widehat{R}_{k'}^{(eb-1)}$ is the normal closure of $\widehat{P}_{k'}^{(eb-1)}$ in $\widehat{R}_{k'}^{(eb)}$ by Lemma 3.1.14, the element z extends to a section over $\widehat{R}_{k'}^{(eb-1)}$ with the *same* minimal polynomial (3.1.16). Again, since z generates a subalgebra of $\mathbb{O}_{\widehat{R}_{k'}^{(eb-1)}}$ which is finite and étale over $\mathbb{O}_{\widehat{P}_{k'}^{(eb-1)}}$ of same degree, it has to generate the whole ring. This finishes the proof. □

3.2. *Lifting rigid spaces.* The definition of the refined Swan conductor homomorphism using differential modules makes use of spaces and modules over the field K. Following the idea of [Xiao 2010], we formally lift the picture of the previous subsection from k to some annulus $A_K^1[\eta, 1)$. This construction is a local version of Berthelot's definition [1996] of rigid cohomology.

Construction 3.2.1. Replacing X by an open Zariski neighborhood of ξ if necessary, there exists a finite morphism $\boldsymbol{f} : \boldsymbol{Y} \to \boldsymbol{X}$ between two affine smooth formal schemes of topologically finite type over $\mathbb{O}_{K_0}$, such that $\boldsymbol{f}$ reduces to f modulo p and such that the induced map $\boldsymbol{Y} \setminus f^{-1}(D) \to \boldsymbol{X} \setminus D$ is finite étale with Galois group $G_{l/k}$. In particular, the special fibers of $\boldsymbol{X}$ and $\boldsymbol{Y}$ are X and Y, respectively.

Let $\blacktriangle_X : \boldsymbol{X} \to \boldsymbol{X} \times_{\operatorname{Spf} \mathbb{O}_{K_0}} \boldsymbol{X}$ be the diagonal embedding, and put $\blacktriangle_Y = (\mathrm{id}, \boldsymbol{f}) : \boldsymbol{Y} \to \boldsymbol{Y} \times_{\operatorname{Spf} \mathbb{O}_{K_0}} \boldsymbol{X}$. Let p_1 and p_2 denote the projections from $\boldsymbol{X} \times_{\operatorname{Spf} \mathbb{O}_{K_0}} \boldsymbol{X}$ to the first and the second factors, respectively.

Let $\boldsymbol{X}^\wedge$ denote the completion of $\boldsymbol{X} \times_{\operatorname{Spf} \mathbb{O}_{K_0}} \boldsymbol{X}$ along the diagonal embedding $\blacktriangle_X$; it can be identified with the completion of the cotangent bundle of $\boldsymbol{X}$ along its zero section. Set $\boldsymbol{Y}^\wedge = \boldsymbol{X}^\wedge \otimes_{p_1, \boldsymbol{X}} \boldsymbol{Y}$; it is the same as the completion of $\boldsymbol{Y} \times_{\operatorname{Spf} \mathbb{O}_{K_0}} \boldsymbol{X}$ along the embedding $\blacktriangle_Y$.

For $\eta \in (0, 1)$, we set $\mathcal{R}^{\mathrm{int}}_{K,\eta}$ to be the subring of $\mathcal{R}^{\eta}_{K}$ consisting of elements having 1-Gauss norm ≤ 1; it is complete with respect to the η'-Gauss norm for $\eta' \in [\eta, 1]$. On one hand, this ring does not give rise to a formal scheme; on the other hand, it is good to keep the geometric intuition. Hence we introduce the *geometric incarnation* $\operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{K,\eta}$, which is just a symbol. Any morphism between geometric incarnations should be thought of as ring homomorphisms; in particular, the fiber product is simply the (completed) tensor product. We also point out that we will only consider affine schemes and there is no question of gluing.

We may compare the following commutative diagram with (3.1.5).

$$\begin{array}{ccccccc}
\boldsymbol{Y} & \xrightarrow{f} & \boldsymbol{X} & & & & \\
\downarrow{\scriptstyle \blacktriangle_Y} & & \downarrow{\scriptstyle \blacktriangle_X} & & & & \\
\boldsymbol{Y}^\wedge & \xrightarrow{f\times 1} & \boldsymbol{X}^\wedge & \xrightarrow{p_2} & \boldsymbol{X} & \xleftarrow{i} & \operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{K,\eta} \\
\downarrow & & \downarrow{\scriptstyle p_1} & & & & \\
\boldsymbol{Y} & \xrightarrow{f} & \boldsymbol{X} & & & &
\end{array} \tag{3.2.2}$$

where $i : \operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{K,\eta} \to \boldsymbol{X}$ is the geometric incarnation of the natural homomorphism $\mathbb{O}^\wedge_{\boldsymbol{X},\xi} \to \mathcal{R}^{\mathrm{int}}_{K,\eta}$, for some $\eta \in (0, 1) \cap p^{\mathbb{Q}}$. We have

$$\operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{K,\eta} \times_{\boldsymbol{X}} \boldsymbol{Y} = \operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{L,\eta^{1/e_{l/k}}}$$

for η sufficiently close to 1^-. Put

$$\boldsymbol{P}_\eta = \boldsymbol{X}^\wedge \times_{p_2, \boldsymbol{X}, i} \operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{K,\eta} \text{ and } \boldsymbol{Q}_\eta = \boldsymbol{Y}^\wedge \times_{p_2 \circ (f \times 1), \boldsymbol{X}, i} \operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{K,\eta}.$$

Again, both $\boldsymbol{P}_\eta$ and $\boldsymbol{Q}_\eta$ should be thought of as geometric incarnations of $\mathbb{O}_{\boldsymbol{P}_\eta}$ and $\mathbb{O}_{\boldsymbol{Q}_\eta}$, the completed tensor products of corresponding rings of functions. We then have the following Cartesian diagram

$$\begin{array}{ccc}
\operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{L,\eta^{1/e_{l/k}}} & \xrightarrow{\blacktriangle_Y} & \boldsymbol{Q}_\eta \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times 1} \\
\operatorname{Sp} \mathcal{R}^{\mathrm{int}}_{K,\eta} & \xrightarrow{\blacktriangle_X} & \boldsymbol{P}_\eta
\end{array} \tag{3.2.3}$$

Lemma 3.2.4. *The morphism* $p_1 : \boldsymbol{P}_\eta \to \operatorname{Sp}(\mathcal{R}^{\mathrm{int}}_{K,\eta})$ *is given by the continuous homomorphism* $\psi : \mathcal{R}^{\mathrm{int}}_{K,\eta} \to \mathcal{R}^{\mathrm{int}}_{K,\eta}[\![\delta_0/T, \delta_1, \dots, \delta_m]\!]$ *such that* $\psi(T) = T + \delta_0$,

$\psi(B_j) = B_j + \delta_j$ for $j \in J$. More precisely, for $x \in \mathscr{R}^{\text{int}}_{K,\eta}$, we have

$$\psi(x) = \sum_{e_{J^+}=0}^{+\infty} \frac{\partial_{J^+}^{e_{J^+}}(x)}{(e_{J^+})!} \delta_{J^+}^{e_{J^+}}.$$

Proof. The first statement follows from the description of $\boldsymbol{X}^\wedge$ above and the second statement follows by the uniqueness of such a homomorphism. □

Construction 3.2.5. Let k'/k be a finite separable extension of naïve ramification degree e. Since $\mathscr{R}^{\text{int}}_K$ is Henselian, there exists $\mathscr{R}^{\text{int}}_{K'}$ corresponding to the extension k'/k, where K' is the fraction field of a Cohen ring of $\kappa_{k'}$. For η sufficiently close to 1^-, the extension $\mathscr{R}^{\text{int}}_{K'}$ of $\mathscr{R}^{\text{int}}_K$ descends to a finite étale algebra $\mathscr{R}^{\text{int}}_{K',\eta^{1/e}}$ over $\mathscr{R}^{\text{int}}_{K,\eta}$ for some η sufficiently close to 1. Fix such an η. Let T' denote the coordinate of $\mathscr{R}^{\text{int}}_{K',\eta^{1/e}}$.

Let $r \in \mathbb{N}$ (be a proxy of eb or $eb - 1$). Let

$$\boldsymbol{P}^{(r)}_{K',\eta} = \operatorname{Sp}\left(\mathscr{R}^{\text{int}}_{K',\eta^{1/e}}\langle T'^{-r-e}\delta_0, T'^{-r}\delta_J\rangle\right)$$

be the geometric incarnation of a closed-disc bundle (with changing radii) over $\operatorname{Sp}\mathscr{R}^{\text{int}}_{K',\eta^{1/e}}$; it may be viewed as a subspace of $\boldsymbol{P}_\eta$ (in the sense of geometric incarnation). Let $\boldsymbol{Q}^{(r)}_{K',\eta}$ be the preimage (in the sense of geometric incarnation) of $\boldsymbol{P}^{(r)}_{K',\eta}$ under the morphism $\boldsymbol{Q}_\eta \to \boldsymbol{P}_\eta$.

Proposition 3.2.6. *Let ρ be a p-adic representation of $G_{l/k}$. Let*

$$\mathscr{F}_\rho = \left((\boldsymbol{f} \times 1)_* \mathscr{O}_{\boldsymbol{Q}_\eta} \otimes V_\rho\right)^{G_{l/k}}$$

be the differential module over $\boldsymbol{P}_\eta$ and for $r \in \mathbb{N}$, let

$$\mathscr{F}^{(r)}_{\rho,K'} = \left((\boldsymbol{f} \times 1)_* \mathscr{O}_{\boldsymbol{Q}^{(r)}_{K',\eta}} \otimes V_\rho\right)^{G_{l/k}}$$

be the corresponding differential module over $\boldsymbol{P}^{(r)}_{K',\eta}$. Then $\mathscr{F}_\rho$ and $\mathscr{F}^{(r)}_{\rho,K'}$ are the pullbacks of $\mathscr{E}_\rho$ along $p_1 : \boldsymbol{P}_\eta \to \operatorname{Sp}\mathscr{R}^{\text{int}}_{K,\eta}$ and $p_1 : \boldsymbol{P}^{(r)}_{K',\eta} \to \operatorname{Sp}\mathscr{R}^{\text{int}}_{K,\eta}$, respectively.

Proof. This follows from the following $G_{l/k}$-equivariant Cartesian diagram of geometric incarnated morphisms.

$$\begin{array}{ccccc}
\boldsymbol{Q}^{(r)}_{K',\eta} & \longrightarrow & \boldsymbol{Q}_\eta & \xrightarrow{p_1} & \operatorname{Sp}\mathscr{R}^{\text{int}}_{L,\eta^{1/e_{l/k}}} \\
\downarrow{\scriptstyle \boldsymbol{f}\times 1} & & \downarrow{\scriptstyle \boldsymbol{f}\times 1} & & \downarrow{\scriptstyle \boldsymbol{f}} \\
\boldsymbol{P}^{(r)}_{K',\eta} & \longrightarrow & \boldsymbol{P}_\eta & \xrightarrow{p_1} & \operatorname{Sp}\mathscr{R}^{\text{int}}_{K,\eta}
\end{array}$$

□

Corollary 3.2.7. *For $a \in \mathbb{Q}_{<b}$ and $\eta \in (0, 1) \cap p^{\mathbb{Q}}$, let $F_{\eta,a}$ denote the completion of $K(T, \delta_{J^+})$ with respect to the $(\eta, \eta^{a+1}, \eta^a, \dots, \eta^a)$-Gauss norm and let*

$F'_{\eta,a} = F_{\eta,a} \otimes_{\mathscr{R}^{\mathrm{int}}_{K,\eta}} \mathscr{R}^{\mathrm{int}}_{K',\eta^{1/e}}$. *Assume that ρ has pure log-break b and pure refined Swan conductor*

$$\vartheta = \pi_k^{-b}\Big(\bar{\alpha}_0 \frac{d\pi_k}{\pi_k} + \bar{\alpha}_1 \frac{d\bar{b}_1}{\bar{b}_1} + \cdots + \bar{\alpha}_m \frac{d\bar{b}_m}{\bar{b}_m}\Big),$$

where $\bar{\alpha}_{J^+} \in \bar{\kappa}$. If $r < ea < eb$ and η is sufficiently close to 1^-, then $\mathscr{F}_\rho \otimes F'_{\eta,a} = \mathscr{F}^{(r)}_{\rho,K'} \otimes F'_{\eta,a}$ as a $\partial/\partial\delta_{J^+}$-differential module has pure intrinsic radii η^b and pure refined intrinsic radii

$$T^{-b}\Big(\bar{\alpha}_0 \frac{d\delta_0}{T} + \bar{\alpha}_1 \frac{d\delta_1}{B_1} + \cdots + \bar{\alpha}_m \frac{d\delta_m}{B_m}\Big).$$

Proof. By Lemma 3.2.4 and Proposition 3.2.6, $\mathscr{F}^{(r)}_{\rho,K'}$ is the pullback of $\mathscr{E}_\rho$ along the multidimensional analog of the generic point homomorphism as in Corollary 1.4.21. However, the calculation of the refined ∂_j-radii can be computed independently for each of the ∂_j. Hence the statement follows from Corollary 1.4.21. □

Before proceeding, we briefly recall the lifting construction in [Xiao 2010, Section 1], which lifts a rigid analytic space over $\kappa_{k'}$ to a rigid analytic space over $A^1_{K'}[\eta^{1/e}, 1)$ for $\eta \in p^{\mathbb{Q}} \cap (0, 1)$ sufficiently close to 1^-.

Construction 3.2.8. Let Z be a rigid analytic space over k' with ring of analytic functions $A_{k'} = k'\langle u_1, \ldots, u_s\rangle/I_{k'}$. Let $I_{K'} \subset \mathbb{O}_{K'}\langle u_1, \ldots, u_s\rangle((T'))$ be an ideal such that $\mathbb{O}_{K'}\langle u_1, \ldots, u_s\rangle((T'))/I_{K'}$ is *flat* over $\mathbb{O}_{K'}$ and $I_{K'} \otimes_{\mathbb{O}_{K'}} k' = I_{k'}$. We call $\boldsymbol{X}_\eta = \mathrm{Spf}\big(\mathscr{R}^{\mathrm{int}}_{K',\eta}\langle u_1, \ldots, u_s\rangle/I_{K'}\big)$ a *lifting space* of X.

Proposition 3.2.9. *Fix $r \in \mathbb{N}$.*

(a) *The space $\boldsymbol{Q}^{(r)}_{K',\eta}$ is a lifting space of $\widehat{Q}^{(r)}_{k'}$.*

(b) *Suppose $Q^{(r)}_{\mathbb{O}_{k'}}$ is a stable model and $r = eb$ or $eb - 1$. Then for η sufficiently close to 1^-, $\boldsymbol{Q}^{(r)}_{K',\eta}$ has $[l : k]/p$ connected components, each of which is isomorphic to a formal scheme $\boldsymbol{R}^{(r)}_{K',\eta}$ finite and étale over $\boldsymbol{P}^{(r)}_{K',\eta}$ of degree p.*

(c) *Fix a Dwork pi $\boldsymbol{\pi} = (-p)^{1/(p-1)}$ and fix $\boldsymbol{\alpha}_{J^+} \subset \mathscr{R}^{\mathrm{int}}_{K'(\boldsymbol{\pi}),\eta^{1/e}}$ lifts of α_{J^+}. By making η closer to 1^- if needed, we may choose a lift $\boldsymbol{z}$ of z to $\boldsymbol{R}^{(r)}_{K'(\boldsymbol{\pi}),\eta}$ whose minimal polynomial over $\boldsymbol{P}^{(r)}_{K',\eta}$ is of the form*

$$\frac{1}{p\boldsymbol{\pi}}\big((1+\boldsymbol{\pi}\boldsymbol{z})^p - 1 - p\boldsymbol{\pi}(\boldsymbol{\alpha}_0 T'^{-eb-e}\delta_0 + \boldsymbol{\alpha}_1 T'^{-eb} + \cdots + \boldsymbol{\alpha}_m T'^{-eb})\big) = 0. \quad (3.2.10)$$

Proof. The first statement follows from the construction. The second statement follows from [Xiao 2010, Proposition 1.2.11]; the fact that all the connected components are isomorphic to the same $\boldsymbol{R}^{(r)}_{K'(\boldsymbol{\pi}),\eta}$ is a corollary of (c), proved below.

For (c), pick a lift $\boldsymbol{z}_1$ of z to $\boldsymbol{R}^{(r)}_{K'(\boldsymbol{\pi}),\eta}$ whose minimal polynomial reduces to (3.1.16) modulo $\boldsymbol{\pi}$. (Note that K is absolutely unramified.) We define the following substitution process. Assume that we have defined z_i. We set

$$\boldsymbol{\lambda}_i = \frac{1}{p\boldsymbol{\pi}}\big((1+\boldsymbol{\pi} z_i)^p - 1 - p\boldsymbol{\pi}(\boldsymbol{\alpha}_0 T'^{-eb-e}\delta_0 + \boldsymbol{\alpha}_1 T'^{-eb} + \cdots + \boldsymbol{\alpha}_m T'^{-eb})\big).$$

and set $z_{i+1} = z_i - \boldsymbol{\lambda}_i$. Hence we have

$$\begin{aligned}\boldsymbol{\lambda}_{i+1} &= \frac{1}{p\boldsymbol{\pi}}\big((1+\boldsymbol{\pi} z_i - \boldsymbol{\pi}\boldsymbol{\lambda}_i)^p - (1+\boldsymbol{\pi} z_i)^p + p\boldsymbol{\pi}\boldsymbol{\lambda}_i\big) \\ &= \big(1-(1+\boldsymbol{\pi} z_i)^{p-1}\big)\boldsymbol{\lambda}_i + \sum_{n=2}^{p-1}\frac{1}{p\boldsymbol{\pi}}\binom{p}{n}(1+\boldsymbol{\pi} z_i)^{p-n}(-\boldsymbol{\pi}\boldsymbol{\lambda}_i)^n + (-1)^{p-1}\boldsymbol{\lambda}_i^p.\end{aligned}$$

Since $|\boldsymbol{\lambda}_1|_1 \leq p^{-1/(p-1)}$, by continuity, $|\boldsymbol{\lambda}_1|_\eta < 1$ for $\eta \in [\eta_0, 1]$ for some η_0 sufficiently close to 1^-. Thus,

$$|\boldsymbol{\lambda}_{i+1}|_\eta \leq \max\big\{p^{-1/(p-1)}|\boldsymbol{\lambda}_i|_\eta, |\boldsymbol{\lambda}_i|_\eta^p\big\} \quad \text{for } \eta \in [\eta_0, 1).$$

As a consequence, this substitution process converges with respect to all η-Gauss norms for $\eta \in [\eta_0, 1]$. The limit $z = \lim_{i\to+\infty} z_i$ satisfies (3.2.10). By the same argument as in Proposition 3.1.15, the limit z generates $\boldsymbol{R}^{(r)}_{K'(\boldsymbol{\pi}),\eta}$ over $\boldsymbol{P}^{(r)}_{K'(\boldsymbol{\pi}),\eta}$ when η is sufficiently close to 1^-. □

3.3. *Dwork isocrystals.* In this subsection, we single out a calculation of refined radii for the differential modules coming from a higher dimensional Artin–Scheier cover. This is the heart of the comparison Theorem 3.4.1. We will state it in a slightly general form because it has its own interest in the study of differential modules.

Hypothesis 3.3.1. In this subsection, let K be a complete discrete valuation field of characteristic zero, containing $\boldsymbol{\pi}$. Let κ denote its residue field, which has characteristic $p > 0$.

Situation 3.3.2. Let $\boldsymbol{P}$ denote the formal scheme $\operatorname{Spf}\mathcal{R}^{\text{int}}_{K,\eta}\langle\delta_0, \dots, \delta_m\rangle$, and let T be the coordinate of $R^{\text{int}}_{K,\eta}$. Let $\boldsymbol{R}$ be a finite extension of $\boldsymbol{P}$ generated by z satisfying the relation

$$(1+\boldsymbol{\pi} z)^p = 1 + p\boldsymbol{\pi} T^{-r}(\boldsymbol{\alpha}_0\delta_0 + \cdots + \boldsymbol{\alpha}_m\delta_m),$$

where $r \in \mathbb{N}$ and $\boldsymbol{\alpha}_j \in \mathcal{R}^{\text{int}}_{K,\eta}$ for $j = 1, \dots, m$. Let $\alpha_j \in \kappa$ be the reduction of $\boldsymbol{\alpha}_j$ for any j. We assume that not all α_j are zero. Let $f : \boldsymbol{R} \to \boldsymbol{P}$ be the natural morphism, which is finite and étale.

Construction 3.3.3. We reproduce a multidimensional version of the construction in [Kedlaya 2005, Lemma 5.4.7]. The pushforward $f_*\mathcal{O}_{\boldsymbol{Q}}$ decomposes as the direct sum of p differential modules of rank 1, with respect to $\partial_j = \partial/\partial\delta_j$ for $j = 0, \dots, m$.

Let $\mathscr{E}_i$ be the differential module given by $(1+\boldsymbol{\pi} z)^i$ for $i = 1, \dots, p-1$. (The trivial submodule of $f_*\mathcal{O}_{\boldsymbol{Q}}$ is not of interest to us.)

Notation 3.3.4. For $\eta \in (0, 1)$, let $\boldsymbol{F}_\eta$ be the completion of $K(T, \delta_0, \dots, \delta_m)$ with respect to the $(\eta, 1, \dots, 1)$-Gauss norm.

Proposition 3.3.5. *For η sufficiently close to 1^-, the intrinsic radius $IR(\mathscr{E}_i \otimes \boldsymbol{F}_\eta)$ is equal to η^r and the refined intrinsic radius of $\mathscr{E}_i$ for $i = 1, \dots, p-1$ is given by*

$$\mathscr{I}\Theta(\mathscr{E}_i \otimes \boldsymbol{F}_\eta) = \big\{ i\boldsymbol{\pi} T^{-r}(\alpha_0 d\delta_0 + \cdots + \alpha_m d\delta_m) \big\}.$$

Proof. Since

$$p\,\frac{d(1+\boldsymbol{\pi} z)^i}{(1+\boldsymbol{\pi} z)^i} = i\,\frac{d\big(1 + p\boldsymbol{\pi} T^{-r}(\boldsymbol{\alpha}_0\delta_0 + \cdots + \boldsymbol{\alpha}_m\delta_m)\big)}{1 + p\boldsymbol{\pi} T^{-r}(\boldsymbol{\alpha}_0\delta_0 + \cdots + \boldsymbol{\alpha}_m\delta_m)},$$

$\mathscr{E}_i$ is isomorphic to a differential module given by

$$\nabla \boldsymbol{v} = i\boldsymbol{\pi} T^{-r}\big(1 + p\boldsymbol{\pi} T^{-r}(\boldsymbol{\alpha}_0\delta_0 + \cdots + \boldsymbol{\alpha}_m\delta_m)\big)^{-1}\boldsymbol{v} \otimes \big(\boldsymbol{\alpha}_0 d\delta_0 + \cdots + \boldsymbol{\alpha}_m d\delta_m\big).$$

Fix $j = 0, \dots, m$. Using the proof of [Kedlaya 2005, Lemma 5.4.7], when η is sufficiently close to 1^- (e.g., $\eta > p^{-1/r}$), viewed as a ∂_j-differential module, this is the same as

$$\partial_j \boldsymbol{w}_j = i\boldsymbol{\pi}\boldsymbol{\alpha}_j T^{-r}\boldsymbol{w}_j,$$

where $\boldsymbol{w}_j$ is a section of $\mathscr{E}_i$, dependent on j. Hence $\partial_j^n(\boldsymbol{w}_j) = (i\boldsymbol{\pi}\boldsymbol{\alpha}_j T^{-r})^n \boldsymbol{w}_j$, and the proposition follows immediately. □

3.4. *Comparison.* In this subsection, we assemble the results from previous subsections to prove the following comparison theorem.

Theorem 3.4.1. *Assume Hypothesis (Geom). Then for $b \in \mathbb{Q}_{>0}$, the homomorphism* $\mathrm{rsw} : \mathrm{Hom}(\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k, \mathbb{F}_p) \to \Omega^1_k(\log) \otimes \pi_k^{-b}\bar{\kappa}$ *in Theorem 2.3.7 is the same as the homomorphism* rsw' *in Proposition 3.1.11.*

Proof. Let $\tilde{k}$ be as in Proposition 2.2.14. By [Saito 2009, Lemma 1.22], rsw' for k factors as

$$\mathrm{Hom}(\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k, \mathbb{F}_p) \to \mathrm{Hom}(\mathrm{Fil}^{e_{\tilde{k}/k}b}_{\log} G_{\tilde{k}} / \mathrm{Fil}^{(e_{\tilde{k}/k}b)+}_{\log} G_{\tilde{k}}, \mathbb{F}_p)$$
$$\xrightarrow{\mathrm{rsw}'_{\tilde{k}}} \Omega^1_{\mathbb{O}_{\tilde{k}}}(\log) \otimes_{\mathbb{O}_{\tilde{k}}} \pi_{\tilde{k}}^{-e_{\tilde{k}/k}b} \kappa_{\tilde{k}^{\mathrm{alg}}}.$$

The same factorization is also valid for rsw as in (2.3.12). Hence we may choose $e_{\tilde{k}/k}$ divisible by the denominator of b and reduce to the case when b is an integer. We also remark that, for the same reason, we may feel free to replace k by a finite tamely ramified extension.

Fix ζ_p a p-th root of unity. Let $\chi : \mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k \to \mathbb{F}_p$ be a nontrivial character and put

$$\mathrm{rsw}'(\chi) = \pi_k^{-b}\Big(\bar{\alpha}_0 \frac{d\pi_k}{\pi_k} + \bar{\alpha}_1 d\bar{b}_1 + \cdots + \bar{\alpha}_m d\bar{b}_m\Big),$$

where $\bar{\alpha}_0, \dots, \bar{\alpha}_m \in \bar{\kappa}$. By identifying $1 \in \mathbb{F}_p$ with $\zeta_p \in \mathbb{Q}_p(\zeta_p)$, we get a homomorphism

$$\mathrm{Fil}^b_{\log} G_k / \mathrm{Fil}^{b+}_{\log} G_k \xrightarrow{\chi} \mathbb{F}_p \to \mathbb{Q}_p(\zeta_p)^\times;$$

we still use χ to denote the composition. By the argument and the result of Theorem 2.3.7 and by possibly replacing k by a finite tamely ramified extension, we can find a p-adic representation ρ of G_k with finite image and pure log-break b such that $\rho|_{\mathrm{Fil}^b_{\log} G_k}$ is a direct sum of copies of χ. Moreover, we may assume that ρ is irreducible when restricted to any finite tamely ramified extension of k' of k. The representation ρ factors exactly through l/k, a finite Galois extension. It must be true that $\mathrm{Fil}^b_{\log} G_k / G_l \cap \mathrm{Fil}^b_{\log} G_k \simeq \mathbb{F}_p$. By possibly making another tamely ramified extension of k, we may assume that the second highest log-break of l/k is strictly less than $b-1$; thus $\mathrm{Fil}^{b-1}_{\log} G_k / G_l \cap \mathrm{Fil}^{b-1}_{\log} G_k \simeq \mathbb{F}_p$.

We shall now use the results and notation from previous subsections. By Proposition 3.2.9, $\boldsymbol{Q}^{(eb-1)}_{K',\eta}$ is a disjoint union of $[l:k]/p$ copies of $\boldsymbol{R}^{(eb-1)}_{K',\eta}$, which is finite and étale over $\boldsymbol{P}^{(eb-1)}_{K',\eta}$, generated by $\boldsymbol{z}$ with minimal polynomial (3.2.10). (Here, we made a choice of z and $\boldsymbol{z}$ in accordance with the algebraic group structure on $Q^b_{\bar{\kappa}}$; see the remarks after (3.1.13).) By Proposition 3.3.5, this implies that $\mathscr{F}^{eb-1}_{\rho,K'} \otimes F'_{\eta,b-1/2e}$ as $\eta \to 1^-$ has pure refined intrinsic radii

$$\boldsymbol{\pi} T^{-b}\Big(\bar{\alpha}_0 \frac{d\delta_0}{T} + \bar{\alpha}_1 d\delta_1 + \cdots + \bar{\alpha}_m d\delta_m\Big).$$

(Here we made a choice of Dwork pi $\boldsymbol{\pi}$ so that $\boldsymbol{\pi} \equiv \zeta_p - 1 \bmod (\zeta_p - 1)^2$ as in Remark 2.3.4.) By Corollary 3.2.7, the refined Swan conductor of $\mathscr{E}_\rho$ has to be $\pi_k^{-b}(\bar{\alpha}_0 \frac{d\pi_k}{\pi_k} + \bar{\alpha}_1 d\bar{b}_1 + \cdots + \bar{\alpha}_m d\bar{b}_m)$, the same as rsw'. □

Remark 3.4.2. By [Abbes and Saito 2009, Theorem 9.1.1], the two definitions of refined Swan conductors above are the same as Kato's definition in [Kato 1989], when the representation is one-dimensional. So all three definitions agree. This result is also implicitly contained in [Chiarellotto and Pulita 2009].

4. Refined Swan conductors and variation of intrinsic radii on polyannuli

When we have a differential module over a polyannulus or a polydisc, similar to the one-dimensional situation, we may study how the multiset of intrinsic radii of the differential module changes as we complete the module with respect to different Gauss norms. Kedlaya and the author [2010] proved that the partial sums of the log of intrinsic radii form continuous convex piecewise affine functions. The purpose of this section is to prove that the slopes at some point of such affine functions are related to the refined intrinsic radii of the differential module, completed with respect to the corresponding Gauss norm. Again, the proof proceeds in two steps, first over an annulus or a disc (Section 4.2) and then over a polyannulus or a polydisc

(Section 4.3). The first subsection focuses on some technical results which will be used in the following two subsections.

Hypothesis 4.0.1. We assume Hypothesis 1.5.1 and keep the notation of Section 1. We also assume that K is discretely valued throughout this section. We do not insist $p > 0$ in this section unless otherwise specified.

4.1. ***Partial decomposition for differential modules.*** In Section 1.5, we deliberately restricted ourselves to the situation over open annuli. In many applications, it is equally important to understand the theory of differential modules over a bounded analytic ring, for example $K\{\{\alpha/t, t\rrbracket_0$. This subsection is devoted to developing a parallel theory in this case, which is not addressed in [Kedlaya and Xiao 2010].

We fix some $\alpha \in (0, 1)$ for this subsection.

Notation 4.1.1. We define E to be the completion of $\operatorname{Frac}\big(K\{\{\alpha/t, t\rrbracket_0\big)$ with respect to the 1-Gauss norm; it is isomorphic to the p-adic completion of $\mathcal{O}_K((t))[\frac{1}{p}]$, and it contains F_1 as a subfield.

If $s \in -\log|K^\times|$, we can find an element $x \in K^\times$ with $|x| = \mathrm{e}^{-s}$. This x defines an isomorphism

$$\kappa_E^{(s)} \xrightarrow{\cdot x^{-1}} \kappa_E \cong \kappa_K((t)).$$

Hence we have a canonical valuation $\mathrm{v}_s(\,\cdot\,)$ on $\kappa_E^{(s)}$ given by the t-valuation; this does not depend on the choice of $x \in K^\times$. This valuation extends naturally to $\kappa_{E^{\mathrm{alg}}}^{(s)}$ for $s \in \mathbb{Q} \cdot \log|K^\times|$.

Notation 4.1.2. Let $j \in J^+$. For M a ∂_j-differential module over $K\{\{\alpha/t, t\rrbracket_0$ of rank d and $i \in \{1, \dots, d\}$, we put

$$f_i^{(j)}(M, 0) = -\log R_{\partial_j}(M \otimes E; i), \quad F_i^{(j)}(M, 0) = f_1^{(j)}(M, 0) + \cdots + f_i^{(j)}(M, 0).$$

We similarly define $f_i(M, 0)$ and $F_i(M, 0)$ if M is a ∂_{J^+}-differential module over $K\{\{\alpha/t, t\rrbracket_0$.

Proposition 4.1.3. *Fix $j \in J^+$. Let M be a ∂_j- (resp. ∂_{J^+}-) differential module of rank d over $K\{\{\alpha/t, t\rrbracket_0$.*

(a) *The functions $f_i^{(j)}(M, r)$ and $F_i^{(j)}(M, r)$ are continuous, and are affine if $f_i^{(j)}(M, 0) > -\log|u_j|$; the functions $f_i(M, r)$ and $F_i(M, r)$ are affine.*

(b) *Suppose for some $i \in \{1, \dots, d-1\}$, the function $F_i^{(j)}(M, r)$ (resp. $F_i(M, r)$ is affine), and $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ (resp. $f_i(M, r) > f_{i+1}(M, r)$) for $r \in [0, -\log\alpha)$. Then M admits a unique direct sum decomposition $M_0 \oplus M_1$ over $K\{\{\alpha/t, t\rrbracket_0$ such that*

(i) *for any* $\eta \in (0, -\log \alpha)$, *the multisets of subsidiary* ∂_j*-radii* (*resp. intrinsic radii*) *of* $M_0 \otimes F_\eta$ *exactly consist of the* i *smallest elements of the multisets of subsidiary* ∂_j*-radii* (*resp. intrinsic radii*) *of* $M \otimes F_\eta$, *and*

(ii) *the multisets of subsidiary* ∂_j*-radii* (*resp. intrinsic radii*) *of* $M_0 \otimes E$ *exactly consist of the* i *smallest elements of the multisets of subsidiary* ∂_j*-radii* (*resp. intrinsic radii*) *of* $M \otimes E$.

Proof. The statement (a) for ∂_j-radii follows from the exact same argument as [Kedlaya and Xiao 2010, Theorem 2.2.6(a)], which follows immediately from the corresponding properties of the associated twisted polynomial. We now explain how we deduce (a) for intrinsic radii. Firstly, by parts (a), (b) and (d) of Theorem 1.5.6, $d! \cdot F_i(M, r)$ is convex and piecewise affine of integer slopes for $r \in (0, -\log \alpha)$. We need only to check continuity at $r = 0$, which follows from exactly the same argument as in Step 1 of the proof of [ibid., Theorem 2.3.9].

The statement (b) is proved in [ibid., Theorems 2.3.9, 2.5.5, and Remarks 2.3.11, 2.5.7]. □

Note that the statement (b) of the above proposition *excludes* the case when $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ for $r \in (0, -\log \alpha)$ and $f_i^{(j)}(M, 0) = f_{i+1}^{(j)}(M, 0)$, and the similar case with the superscript (j) removed. The rest of this subsection is devoted to extending the conclusion of (b) to this case.

Notation 4.1.4. Set $\mathscr{R} = \bigcap_{\alpha \in (0,1)} K\{\{\alpha/t, t\}\}$ and $\mathscr{R}^{\mathrm{bd}} = \bigcap_{\alpha \in (0,1)} K\{\{\alpha/t, t]\!]_0$, where the latter can be identified with the subring of the former consisting of elements with finite 1-Gauss norm.

Hypothesis 4.1.5. We assume that $|u_j| = 1$ for $j \in J$.

This hypothesis is just to make our presentation simpler. We can always reduce to this case by replacing K by the completion of $K(x_1, \dots, x_m)$ with respect to the $(|u_1|, \dots, |u_m|)$-Gauss norm and by replacing u_j by u_j/x_j, where $\partial_j(x_{j'}) = 0$ for $j, j' \in J$. Note that K is still discretely valued.

Lemma 4.1.6. *The ring* $\mathscr{R}^{\mathrm{bd}}$ *is a field. A sequence* $(f_n)_{n \in \mathbb{N}} \subset K\{\{\alpha/t, t]\!]_0$ *is convergent if it is convergent for the* r*-Gauss norm for all* $r \in (\alpha, 1)$ *and is bounded for the* 1*-Gauss norm.*

Proof. The first statement is well-known; see [Kedlaya 2005, Lemma 3.5.2]. We remark that this would be false if K were not discretely valued. To see the second statement, we observe that $(f_n)_{n \in \mathbb{N}}$ converges in $K\{\{\alpha/t, t\}\}$. The limit has bounded coefficients and hence lies in $K\{\{\alpha/t, t]\!]_0$. □

Lemma 4.1.7. *Fix* $j \in J^+$. *Let* $\mathscr{R}^{\mathrm{bd}}\{T\}$ *be the ring of twisted polynomials as in Definition 1.2.1, where* T *stands for* ∂_j *if* $j \in J$ *and for* d/dt *if* $j = 0$. *Let* $P = T^d + a_i T^{d-1} + \cdots + a_d \in \mathscr{R}^{\mathrm{bd}}\{T\}$ *be a monic twisted polynomial whose Newton* *polygon has pure slope* $s < 1$. *Let* $\{b_1, \dots, b_r\}$ *be the set of* v_s*-valuations of the reduced roots of* P (*not counting multiplicity, with either increasing or decreasing order*), *when we view* P *as a twisted polynomial in* $E\{T\}$. *Then* P *admits a unique factorization* $P = Q_1 \cdots Q_r$ *as products of monic twisted polynomials such that all the reduced roots of* Q_i, *when viewed as twisted polynomials in* $E\{T\}$, *have* v_s*-valuations* b_i.

Proof. We assume that $b_1, \dots, b_r$ are in decreasing order. It then suffices to show that we can write $P = QR$ as a product of two monic polynomials such that the reduced roots of Q and R, when viewed as twisted polynomials in $E\{T\}$, have pure v_s-valuations b_1 and strictly less than b_1, respectively. We can also write it as $P = RQ$ satisfying the same condition, but with different Q and R. By Lemma 4.1.6, the claim follows from [Kedlaya 2009, Proposition 3.2.2] because the sequences $\{P_l\}$ and $\{Q_l\}$ there are bounded under the 1-Gauss norm. □

Lemma 4.1.8. *Fix* $j \in J$. *Let* M *be a* ∂_j*-differential module of rank* d *over* $K\{\{\alpha/t, t]\!]_0$ *such that* $M \otimes E$ *has pure intrinsic* ∂_j*-radii* $IR_{\partial_j}(M \otimes E) < \omega$. *By choosing a cyclic vector of* $M \otimes \mathcal{R}^{\mathrm{bd}}$, *we may identify* $M \otimes \mathcal{R}^{\mathrm{bd}}$ *with* $\mathcal{R}^{\mathrm{bd}}\{T\}/\mathcal{R}^{\mathrm{bd}}\{T\}P$, *where* P *is a twisted polynomial in* $\mathcal{R}^{\mathrm{bd}}\{T\}$. *Then for* η *sufficiently close to* 1^-, *the slopes of the Newton polygon of* P (*for the* η*-Gauss norm*) *are the log of the subsidiary* ∂_j*-radii of* $M \otimes F_\eta$ *minus* $\log \omega$.

Proof. The identification $M \otimes \mathcal{R}^{\mathrm{bd}} \simeq \mathcal{R}^{\mathrm{bd}}\{T\}/\mathcal{R}^{\mathrm{bd}}\{T\}P$ descends to

$$M \otimes K\{\{\beta/t, t]\!]_0 \simeq K\{\{\beta/t, t]\!]_0\{T\}/K\{\{\beta/t, t]\!]_0\{T\}P$$

for β sufficiently close to 1^-. Note that for η sufficiently close to 1^-, all ∂_j-radii of $M \otimes F_\eta$ are visible. The lemma follows from Proposition 1.2.8. □

The following theorem also holds without assume Hypothesis 4.1.5.

Theorem 4.1.9. *Fix* $j \in J^+$. *Let* M *be a* ∂_j- (*resp.* ∂_{J^+}-) *differential module of rank* d *over* $K\{\{\alpha/t, t]\!]_0$ *such that* $M \otimes E$ *has pure intrinsic* ∂_j*-radii* $IR_{\partial_j}(M \otimes E) < 1$ (*resp. intrinsic radii* $IR(M \otimes E) < 1$). *Suppose that for some* $i \in \{1, \dots, d-1\}$, *the function* $F_i^{(j)}(M, r)$ (*resp.* $F_i(M, r)$) *is affine and* $f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r)$ (*resp.* $f_i(M, r) > f_{i+1}(M, r)$) *for any* $r \in (0, -\log \alpha)$. *Then* M *admits a unique direct sum decomposition* $M_0 \oplus M_1$ *of* ∂_j- (*resp.* ∂_{J^+}-) *differential module over* $K\{\{\alpha/t, t]\!]_0$ *such that, for any* $\eta \in (0, -\log \alpha)$, *the multiset of* ∂_j*-radii* (*resp. intrinsic radii*) *of* $M_0 \otimes F_\eta$ *exactly consists of the smallest* i *elements of the multiset of* ∂_j*-radii* (*resp. intrinsic radii*) *of* $M \otimes F_\eta$.

Proof. We first deduce the ∂_j-differential module case. By Theorem 1.5.4(e), it suffices to obtain the decomposition over $K\{\{\beta/t, t]\!]_0$ for $\beta \in (\alpha, 1)$ sufficiently close to 1 and then we may apply Lemma 1.1.10 and Remark 1.1.11 to glue this decomposition with the decomposition given by Theorem 1.5.4(e).

To start, we assume that $IR_{\partial_j}(M \otimes E) < \omega$. By making β closer to 1, we may assume that $IR_{\partial_j}(M \otimes F_\eta) < \omega$ for all $\eta \in (\beta, 1)$ as well. It is also very easy to reduce to the case when Hypothesis 4.1.5 holds. Since $\mathscr{R}^{\mathrm{bd}}$ is a field, we can find a cyclic vector to identify $M \otimes \mathscr{R}^{\mathrm{bd}}$ with $\mathscr{R}^{\mathrm{bd}}\{T\}/\mathscr{R}^{\mathrm{bd}}\{T\}P$ for a monic twisted polynomial P as in Lemma 4.1.7. Applying Lemma 4.1.7 to $M \otimes \mathscr{R}^{\mathrm{bd}}$ with the b's in decreasing order, we can find a submodule M_0 of M such that the multiset of ∂_j-radii of $M_0 \otimes F_\eta$ exactly consists of the smallest i elements in the multiset of ∂_j-radii of $M \otimes F_\eta$ when η sufficiently close to 1^-. Applying Lemma 4.1.7 again with the b's increasing, we can find a quotient M_0' of M satisfying exactly the same condition on M_0 as above. Then the kernel of $M \to M_0'$ together with M_0 gives the direct sum decomposition required in the theorem.

We next assume that $p > 0$ and $IR_{\partial_j}(M \otimes E) = p^{-1/(p-1)}$. If $j \in J$, the ∂_j-Frobenius $\varphi^{(\partial_j)} : K^{(\partial_j)} \to K$ naturally extends to

$$\varphi^{(\partial_j)} : K^{(\partial_j)}\{\{\alpha/t, t]\!]_0 \to K\{\{\alpha/t, t]\!]_0;$$

if $j = 0$, we have $\varphi^{(\partial_0)} : K\{\{\alpha^p/t^p, t^p]\!]_0 \to K\{\{\alpha/t, t]\!]_0$. Then the desired decomposition follows from the decomposition of $\varphi_*^{(\partial_j)} M$. Note that $\varphi^{(\partial_j)*}\varphi_*^{(\partial_j)} M \cong M^{\oplus p}$.

If $p > 0$ and $IR_{\partial_j}(M \otimes E) > p^{-1/(p-1)}$, we may assume that

$$IR_{\partial_j}(M \otimes F_\eta) > p^{\frac{-1}{p-1}}$$

for all $\eta \in (\beta, 1)$, and the decomposition follows from that of the ∂_j-Frobenius antecedent of M.

Finally, we show that the ∂_{J^+}-differential module case follows from the ∂_j-differential module case. By Theorem 1.5.6(e), it suffices to find the decomposition over $K\{\{\beta/t, t]\!]_0$ for $\beta \in (\alpha, 1)$ sufficiently close to 1 and then, to glue the decompositions using Lemma 1.1.10 and Remark 1.1.11. By Proposition 4.1.3(a) and Theorem 1.5.4(a), there exists $\beta \in (\alpha, 1)$ such that, if $IR_{\partial_j}(M \otimes E; i) < 1$ for some j, then the function $f_i^{(j)}(M, r)$ for this j is affine over $[0, -\log\beta)$. By the decompositions given by Proposition 4.1.3(b) and this theorem for ∂_j, the restriction of M to $K\{\{\beta/t, t]\!]_0$ is the direct sum of ∂_{J^+}-differential modules M_l such that, for any $j \in J^+$ with $IR_{\partial_j}(M_l \otimes E) < 1$, the ∂_j-differential module $M_l \otimes F_\eta$ has pure ∂_j-radii for any $\eta \in (\beta, 1)$. Since we already know that $M \otimes E$ has pure intrinsic radii < 1, we may take β sufficiently close to 1 such that each direct summand above has pure intrinsic radii equal to the ∂_j-radii for some j, when tensored with F_η for any $\eta \in (\beta, 1)$. Hence regrouping the direct summands gives the direct sum decomposition we are looking for. □

Remark 4.1.10. The condition $IR_{\partial_j}(M \otimes E) < 1$ is crucial. As pointed out in [Kedlaya 2010, Remark 12.5.4], one may give counterexamples in the case $IR_{\partial_j}(M \otimes$

$E) = 1$ using the theory of crystals. However, in the presence of a Frobenius, one may still get the decomposition.

Proposition 4.1.11. *Let M be a ∂_{J^+}-differential module over $K\{\{\alpha/t, t]\!]_0$ (resp. $K[\![t]\!]_0$) of rank d. We put $\hat{f}_i(M, 0) = -\log ER(M \otimes E; i)$ and*

$$\hat{F}_i(M, 0) = \hat{f}_1(M, 0) + \cdots + \hat{f}_i(M, 0) \quad \text{for } i = 1, \ldots, d.$$

(a) *The functions $\hat{f}_i(M, r)$ and $\hat{F}_i(M, r)$ are affine at $r = 0$.*

(b) *Suppose for some $i \in \{1, \ldots, d-1\}$, the function $\hat{F}_i(M, r)$ is affine and $\hat{f}_i(M, r) > \hat{f}_{i+1}(M, r)$ for $r \in (0, -\log\alpha)$ (resp. whenever $\hat{f}_i(M, r) > r$), and suppose that $\hat{f}_i(M, 0) > 0$. Then M admits a unique direct sum decomposition $M_0 \oplus M_1$ over $K\{\{\alpha/t, t]\!]_0$ (resp. $K[\![t]\!]_0$) such that the multiset of extrinsic radii of $M \otimes F_\eta$ for any $\eta \in (0, -\log\alpha)$ (resp. for any $\eta > 0$ such that $\hat{f}_i(M, r) > r$) consists of the smallest i elements of the multiset of extrinsic radii of $M \otimes F_\eta$.*

Proof. (a) follows from exactly the same argument as in Proposition 4.1.3. We now prove (b). By the extrinsic version of Theorem 1.5.6(e), it suffices to find the decomposition over $K\{\{\beta/t, t]\!]_0$ for $\beta \in (\alpha, 1)$ sufficiently close to 1 and then we may apply Lemma 1.1.10 and Remark 1.1.11 to glue the decompositions. By Proposition 4.1.3(b) and Theorem 4.1.9 for ∂_j-differential modules, there exists $\beta \in (\alpha, 1)$ such that when we tensor M with $K\{\{\beta/t, t]\!]_0$, it is a direct sum of differential modules M_l such that either for any $j \in J^+$ with $R_{\partial_j}(M_l \otimes E) < 1$, $M_l \otimes F_\eta$ has pure ∂_j-radii for all $\eta \in (\beta, 1)$, or we have $ER(M_l \otimes E) = 1$. The proposition then follows from regrouping these direct summands. □

4.2. ***Refined radii and the log-slopes of the radii.*** For a differential module over an annulus or a disc, the slopes of the functions coming from the radii of convergence are determined by the multiset of refined radii for the differential module completed for the corresponding Gauss norm. We also give a refined radii decomposition result for differential modules over bounded analytic rings.

Theorem 4.2.1. *Fix $j \in J^+$ and let M be a ∂_j-differential module over $K\{\{\alpha/t, t]\!]_0$ of rank d. Assume that $f_i^{(j)}(M, r)$ for all i are the same and are affine of slope b in $r \in [0, -\log\alpha)$. Moreover, we assume that $R_{\partial_j}(M \otimes E) = \omega e^s$ is strictly less than $|u_j|^{-1}$ if $j \in J$ and is strictly less than 1 if $j = 0$. Then the v_s-valuation of any element in the multiset of refined ∂_j-radii of $M \otimes E$ is $-b$.*

Proof. We may assume that $|u_j| = 1$. We first consider the case when $M \otimes E$ has pure visible intrinsic ∂_j-radii $IR_{\partial_j}(M \otimes E) < \omega$. By making α closer to 1^-, we may assume that the function $f_i^{(j)}(M, r) > -\log\omega$ for each i is affine over $[0, -\log\alpha)$.

As in Theorem 4.1.9, we may identify $M \otimes \mathcal{R}_K^{\mathrm{bd}}$ with $\mathcal{R}^{\mathrm{bd}}\{T\}/\mathcal{R}^{\mathrm{bd}}\{T\}P$ for some twisted polynomial $P = T^d + a_1T^{d-1} + \cdots + a_d \in \mathcal{R}^{\mathrm{bd}}\{T\}$. Since $M \otimes E$ has pure ∂_j-radii ωe^s, the Newton polygon of P with respect to the 1-Gauss norm has pure slope s and the multiset $\Theta_{\partial_j}(M \otimes E)$ is just the multiset of reduced roots of this twisted polynomial. We put

$$\overline{P} = T^d + \bar{a}_1^{(s)} T^{d-1} + \cdots + \bar{a}_d^{(ds)},$$

where $\bar{a}_i^{(is)} \in \kappa_K^{(is)}((t))$.

When η is sufficiently close to 1^-, the Newton polygon of P with respect to the η-Gauss norm is determined by the Newton polygon of $\overline{P}$ in the following sense: it is the lower convex hull of the set $\{(-i, -\log|a_i|_1 - \mathsf{v}(\bar{a}_i^{(is)}) \log \eta)\}$. By Lemma 4.1.8, this implies that the collection of all slopes of functions $f_i^{(j)}(M, r)$ for all i at $r = 0$ is exactly the collection of the v_s-valuations of the roots of $\overline{P}$, which in turn equals the collection of the v_s-valuations of the elements of the multiset of refined ∂_j-radii of $M \otimes E$.

Now, it suffices to reduce to the case above using ∂_j-Frobenius. Assume $p > 0$ from now on. It is easier to work with intrinsic radii and refined intrinsic radii. So we put $g_i(M, r) = f_i^{(j)}(M, r) + \log|u_j|$ if $j \in J$ and $g_i(M, r) = f_i^{(j)}(M, r) - r$ if $j = 0$. We will use $g_i'(M, \cdot)$ to denote the derivative of the function $g_i(M, \cdot)$. Moreover, we set $s' = -\log(\omega IR_{\partial_j}(V)^{-1})$.

If $IR_{\partial_j}(M \otimes E) = \omega = p^{-1/(p-1)}$, we set $M_1 = \varphi_*^{(\partial_j)} M$. Then Lemma 1.2.18(d) implies that if $j \in J$,

$$\{g_i'(M_1, 0)\} = \begin{cases} \{p g_i'(M, 0) \ (d \text{ times}), 0 \ ((p-1)d \text{ times})\} & \text{if } g_i'(M, 0) < 0, \\ \{g_i'(M, 0) \ (pd \text{ times})\} & \text{if } g_i'(M, 0) \geq 0, \end{cases}$$

and if $j = 0$,

$$\{g_i'(M_1, 0)\} = \begin{cases} \{g_i'(M, 0), 0 \ (p-1 \text{ times})\} & \text{if } g_i'(M, 0) < 0, \\ \{\frac{1}{p} g_i'(M, 0) \ (p \text{ times})\} & \text{if } g_i'(M, 0) \geq 0. \end{cases}$$

By Proposition 1.3.18, the elements in the multiset $\mathcal{I}\Theta_{\partial_j'}(M_1 \otimes E^{(\partial_j)})$ can be grouped into p-tuples

$$\Big(\frac{\theta}{p}, \frac{\theta+1}{p}, \ldots, \frac{\theta+p-1}{p}\Big),$$

and the multiset $\mathcal{I}\Theta_\partial(M \otimes E)$ is composed of $(\theta^p - \theta)^{1/p}$ for each p-tuple above with the same multiplicity, where $\theta \in \kappa_{E^{\mathrm{alg}}}$. Elementary calculation shows the following relation between the v_0-valuations of $(\theta^p - \theta)^{1/p}$ and the $\mathsf{v}_{-\log p}$-valuation of θ:

- when $\mathsf{v}_0(\theta) < 0$, we have $\mathsf{v}_{-\log p}((\theta + l)/p) = \mathsf{v}_0(\theta)$ for $l = 0, \ldots, p-1$, and $\mathsf{v}_0((\theta^p - \theta)^{1/p}) = \mathsf{v}_0(\theta)$;
- when $\mathsf{v}_0(\theta) \geq 0$, we have $\mathsf{v}_{-\log p}((\theta + l)/p) = 0$ for $l = 1, \ldots, p-1$, and $\mathsf{v}_0((\theta^p - \theta)^{1/p}) = (1/p)\mathsf{v}_0(\theta)$.

Hence the statement for M_1 with $\mathsf{v}_{-\log p}$ implies that for M with v_0.

If $IR_{\partial_j}(M \otimes E) > \omega$, by Lemma 1.2.18(d) and Remark 1.2.19, M has a ∂_j-Frobenius antecedent M_0 if α is sufficiently close to 1^-. By Lemma 1.2.18(d) and Proposition 1.3.18, we have

$$\begin{aligned} & g_i(M_0, r) = pg_i(M, r) \text{ for any } i, \\ & \qquad \text{and } \mathcal{I}\Theta_{\partial'_j}(M_0 \otimes E^{(\partial_j)}) = \big\{(-\theta)^p/p : \theta \in \mathcal{I}\Theta_{\partial_j}(M \otimes E)\big\}, \text{ if } j \in J; \\ & g_i(M_0, pr) = pg_i(M, r) \text{ for any } i, \\ & \qquad \text{and } \mathcal{I}\Theta_{\partial'_j}(M_0 \otimes E^{(\partial_j)}) = \big\{(-\theta)^p/p : \theta \in \mathcal{I}\Theta_{\partial_j}(M \otimes E)\big\}, \text{ if } j = 0. \end{aligned}$$

Since $\mathsf{v}_{(ps'-\log p)}((-\theta)^p/p) = p\mathsf{v}_{s'}(\theta)$, the statement for M with $\mathsf{v}_{s'}(-\log p)$ follows from the statement for M_0 with $\mathsf{v}_{ps'-\log p}$ if $j \in J$ and with $\frac{1}{p}\mathsf{v}_{ps'-\log p}$ if $j = 0$ (note that t^p is the coordinate in the latter case). □

Corollary 4.2.2. *Fix $j \in J^+$ and let M be a ∂_j-differential module over $K\{\{\alpha/t, t]\!]_0$. Assume that $M \otimes E$ has pure ∂_j-radii $R_{\partial_j}(M \otimes E) = \omega e^s$, which is strictly less than $|u_j|^{-1}$ if $j \in J$ and is strictly less than 1 if $j = 0$. Then the following two multisets are the same*:

(i) *the multiset composed of the v_s-valuations of the elements in the multiset of refined ∂_j-radii of $M \otimes E$, that is, $\big\{\mathsf{v}_s(\theta) : \theta \in \Theta_{\partial_j}(M \otimes E)\big\}$;*

(ii) *the multiset composed of the negatives of the slopes of $f_i^{(j)}(M, r)$ at $r = 0$, for $i = 1, \dots, d$.*

Proof. This follows from combining Theorems 4.1.9 and 4.2.1. □

Notation 4.2.3. For any $\mathfrak{s} \in \mathbb{Q}\cdot\log|K^\times|$, the valuation $\mathsf{v}_\mathfrak{s}$ on $\kappa_E^{(\mathfrak{s})}$ induces a valuation on

$$\kappa_E^{(\mathfrak{s})}\frac{dt}{t} \oplus \bigoplus_{j\in J}\kappa_E^{(\mathfrak{s})}\frac{du_j}{u_j},$$

still denoted by $\mathsf{v}_\mathfrak{s}$, by setting

$$\mathsf{v}_\mathfrak{s}\Big(\theta_0\frac{dt}{t} + \theta_1\frac{du_1}{u_1} + \dots + \theta_m\frac{du_m}{u_m}\Big) = \min_{j\in J^+}\big\{\mathsf{v}_\mathfrak{s}(\theta_j)\big\}, \text{ for } \theta_0, \dots, \theta_m \in \kappa_E^{(\mathfrak{s})}.$$

Corollary 4.2.4. *Let M be a ∂_{J^+}-differential module over $K\{\{\alpha/t, t]\!]_0$. Assume that $M \otimes E$ has pure intrinsic radii $IR(M \otimes E) = \omega e^\mathfrak{s} < 1$. Then the following two multisets are the same*:

(i) *the valuations of the refined intrinsic radii of $M \otimes E$, $\big\{\mathsf{v}_\mathfrak{s}(\theta) : \theta \in \mathcal{I}\Theta(M \otimes E)\big\}$;*

(ii) *the negatives of the slopes of $f_i(M, r)$ at $r = 0$, for $i = 1, \dots, d$.*

Proof. This follows from combining Theorems 4.1.9 and 4.2.1. □

Similar to Theorem 1.3.26, we have the following decomposition by refined radii.

Theorem 4.2.5. *Fix $j \in J^+$ and let M be a ∂_j-differential module of rank d over $K\{\{\alpha/t, t]\!]_0$. Assume that $M \otimes F_\eta$, for $\eta \in (\alpha, 1)$, and $M \otimes E$ all have pure ∂_j-radii, and assume that the function $f_1^{(j)}(M, r)$ is affine with slope b for $r \in [0, -\log \alpha)$. Let e be the prime-to-p part of the denominator of b. Moreover, assume that $R_{\partial_j}(M \otimes E) = \omega \mathrm{e}^s$ is strictly less than $|u_j|^{-1}$ if $j \in J$ and is strictly less than 1 if $j = 0$. Then there exists a finite tamely ramified extension K' of K and a unique direct sum decomposition*

$$M \otimes K'\{\{\alpha^{1/e}/t^{1/e}, t^{1/e}]\!]_0 = \bigoplus_{\theta \in \kappa_{K^{\mathrm{alg}}}^{(s)}} M_\theta$$

of ∂_j-differential modules such that

(i) *$M_\theta \otimes F_\eta$ has pure refined ∂_j-radii θt^{-b} for all $\eta \in (\alpha, 1)$, and*

(ii) *every element in the multiset of refined ∂_j-radii of $M_\theta \otimes E$ is congruent to θt^{-b} modulo elements in $\kappa_{K^{\mathrm{alg}}}^{(s)}$ with v_s-valuation strictly bigger than $\mathrm{v}_s(\theta t^{-b}) = -b$.*

Moreover, this decomposition descents to a unique decomposition of M itself by Galois descent, satisfying analogous properties, but in the fashion stated in terms of $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits.

Proof. The proof is identical to that of Theorem 1.5.10, except that we use decomposition Theorem 4.1.9 in place of Theorem 1.5.4. □

Theorem 4.2.6. *Let M be a ∂_{J^+}-differential module of rank d over $K\{\{\alpha/t, t]\!]_0$. Assume that $M \otimes F_\eta$, for $\eta \in (\alpha, 1)$, and $M \otimes E$ all have pure intrinsic radii, and assume that the function $f_1(M, r)$ is affine with slope b for $r \in [0, -\log \alpha)$. Let e be the prime-to-p part of the denominator of b. Moreover, assume that $IR(M \otimes E) = \omega \mathrm{e}^{\mathfrak{s}} < 1$. Then there exists a finite tamely ramified extension K' of K and a unique direct sum decomposition*

$$M \otimes K'\{\{\alpha^{1/e}/t^{1/e}, t^{1/e}]\!]_0 = \bigoplus_{\vartheta \in \bigoplus_{j \in J} \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{du_j}{u_j} \oplus \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{dt}{t}} M_\vartheta$$

of ∂_{J^+}-differential modules such that

(i) *$M_\vartheta \otimes F_\eta$ has pure refined intrinsic radii ϑt^{-b} for all $\eta \in (\alpha, 1)$, and*

(ii) *every element in the multiset of refined intrinsic radii of $M_\vartheta \otimes E$ is congruent to ϑt^{-b} modulo those elements in*

$$\bigoplus_{j \in J} \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{du_j}{u_j} \oplus \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{dt}{t}$$

with $\mathrm{v}_{\mathfrak{s}}$-valuation strictly bigger than $\mathrm{v}_{\mathfrak{s}}(\vartheta t^{-b}) = -b$.

Moreover, this decomposition descents to a unique decomposition of M itself by Galois descent, satisfying analogous properties, but in the fashion stated in terms of $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbits.

Proof. The proof is identical to that of Theorem 1.5.12, except that we invoke Theorem 4.2.5 in place of Theorem 1.5.4. □

Corollary 4.2.7. *Let M be a ∂_{J^+}-differential module of rank d over $K\{\{\alpha/t, t]\!]_0$. Assume that $M \otimes E$ has pure intrinsic radii $IR(M \otimes E) = \omega e^{\mathfrak{s}} < 1$ and that the function $f_i(M, r)$ for each $i = 1, \ldots, d$ is affine over $[0, -\log\alpha)$. Let $M = \bigoplus_{b\in\mathbb{Q}} M_b$ be the unique direct sum decomposition of M over $A^1_K(\alpha, 1)$ such that $f_1(M_b, r) = \cdots = f_{\dim M_b}(M_b, r)$ has slope b. Then the following two multisets are the same:*

(i) *The multiset composed of all elements in*

$$\mathscr{I}\Theta(M_b \otimes F_\eta) \subset \bigoplus_{j\in J^+} t^{-b}\kappa^{(\mathfrak{s})}_{K^{\mathrm{alg}}}\frac{du_j}{u_j} \oplus t^{-b}\kappa^{(\mathfrak{s})}_{K^{\mathrm{alg}}}\frac{dt}{t}$$

for all b and for some fixed $\eta \in (\alpha, 0)$ (this is independent of the choice of η);

(ii) *The multiset composed of $\bar{\vartheta}$ for all $\vartheta \in \Theta_{\partial_j(V)}$, where $\bar{\vartheta}$ is the reduction of*

$$\vartheta \in \bigoplus_{j\in J^+} t^{-b}\kappa^{(\mathfrak{s})}_{K^{\mathrm{alg}}}\frac{du_j}{u_j} \oplus t^{-b}\kappa^{(\mathfrak{s})}_{K^{\mathrm{alg}}}\frac{dt}{t}$$

modulo those elements with v_s-valuation strictly bigger than $\mathsf{v}_s(\vartheta)$.

Proof. It follows from the decomposition Theorems 4.1.9 and 4.2.6. □

We have similar results for extrinsic radii.

Theorem 4.2.8. *Assume that $|u_j| = 1$ for all $j \in J$. For $s \in \mathbb{R}$, let $\hat{\mathsf{v}}_s$ be the valuation on $\kappa^{(s)}_E dt \oplus \bigoplus_{j\in J} \kappa^{(s)}_E du_j$ given by*

$$\hat{\mathsf{v}}_s\big(\theta_0 dt + \theta_1 du_1 + \cdots + \theta_m du_m\big) = \min_{j\in J^+}\big\{\mathsf{v}_s(\theta_j)\big\}.$$

Let M be a ∂_{J^+}-differential module of rank d over $K\{\{\alpha/t, t]\!]_0$. Assume that $M \otimes F_\eta$, for $\eta \in (\alpha, 1)$, and $M \otimes E$ all have pure extrinsic radii, and assume that the function $\hat{f}_1(M, r)$ is affine with slope b for $r \in [0, -\log\alpha)$. Let e be the prime-to-p part of the denominator of b. Moreover, assume that $ER(M \otimes E) = \omega e^s < 1$. Then there exists a unique direct sum decomposition $M = \bigoplus_{\{\mu_e\hat{\vartheta}\}} M_{\{\mu_e\hat{\vartheta}\}}$ of ∂_{J^+}-differential modules over $K\{\{\alpha/t, t]\!]_0$, where the direct sum runs through all $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{sep}}/K)$-orbits $\{\mu_e\hat{\vartheta}\}$ in $\bigoplus_{j\in J} \kappa^{(s)}_{K^{\mathrm{alg}}} du_j \oplus \kappa^{(s)}_{K^{\mathrm{alg}}} dt$ such that

(i) *for all $\eta \in (\alpha, 1)$, the multiset of refined extrinsic radii of $M_{\{\mu_e\vartheta\}} \otimes F_\eta$ is composed of the $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{alg}}/K)$-orbit $\{\mu_e\hat{\vartheta}t^{-b}\}$ with the appropriate multiplicity, and*

(ii) *the multiset consisting of the reductions of elements in the multiset of refined extrinsic radii of* $M_{\{\mu_e\hat{\vartheta}\}} \otimes E$ *modulo those elements with* $\hat{\mathrm{v}}_s$*-valuation is strictly bigger than* $-b$, *is composed of the* $\mu_e \rtimes \mathrm{Gal}(K^{\mathrm{alg}}/K)$*-orbit* $\{\mu_e\hat{\vartheta}t^{-b}\}$ *with the appropriate multiplicity.*

Proof. The proof is identical to that of Theorem 1.5.14, except that we use invoke Theorem 4.2.5 in place of Theorem 1.5.4. □

Corollary 4.2.9. *Assume that* $|u_j| = 1$ *for all* $j \in J$. *Let* M *be a* ∂_{J^+}*-differential module of rank* d *over* $K[\![t]\!]_0$. *Assume that* $ER(M \otimes E) = \omega e^s < 1$. *Let* M_e *denote the unique* ∂_{J^+}*-differential submodule of* $M \otimes E$ *that has pure extrinsic radii* $ER(M \otimes E)$; *put* $l = \dim M_e$. *Then*:

(a) *The* $\hat{\mathrm{v}}_s$*-valuations of elements in* $\Theta(M_e \otimes E)$ *are all nonnegative.*

(b) *There exists a unique direct sum decomposition*

$$M = \bigoplus_{\{\hat{\vartheta}\}} M_{\{\hat{\vartheta}\}} \oplus M_0$$

of ∂_{J^+}*-differential modules over* $K[\![t]\!]_0$, *where the first direct sum is taken over all* $\mathrm{Gal}(\kappa_K^{\mathrm{sep}}/\kappa_K)$*-orbits* $\{\hat{\vartheta}\} \subset \bigoplus_{j\in J} \kappa_{K^{\mathrm{alg}}}^{(s)}\, du_j \oplus \kappa_{K^{\mathrm{alg}}}^{(s)}\, dt$ *such that*

(i) *for all* $\eta < 1$, $M_{\{\hat{\vartheta}\}} \otimes F_\eta$ *has pure extrinsic radii* $\min\{\omega e^s, \eta\}$ *and, when* $\eta \in (\omega e^s, 1)$, *the multiset* $\Theta(M_{\{\vartheta\}} \otimes F_\eta)$ *is composed of* $\{\hat{\vartheta}\}$ *with multiplicity*,

(ii) *the multiset consisting of reductions of elements in the multiset of refined extrinsic radii of* $M_{\{\vartheta\}} \otimes E$ *modulo those elements with positive* $\hat{\mathrm{v}}_s$*-valuation is composed of* $\{\hat{\vartheta}\}$ *with appropriate multiplicity*, *and*

(iii) *for any* $r > 0$ *satisfying* $\hat{f}_1(M_0, r) < r$, *we have* $\hat{f}_1(M_0, r) < \omega e^s$.

Proof. (a) By Proposition 4.1.11(a) together with Theorem 1.5.6(c′), we know that the functions $f_1(M, r), \dots, f_l(M, r)$ are linear in a neighborhood of r with nonpositive slopes. Then applying the decomposition in Proposition 4.1.11(b) and Theorem 4.2.8 together with description (ii) in Theorem 4.2.8, we conclude that the $\hat{\mathrm{v}}_s$-valuations of elements in $\Theta(M_e)$ are all nonnegative.

(b) Let l' denote the number of elements in $\Theta(M_e)$ whose $\hat{\mathrm{v}}_s$-valuation is zero. By the proof of (a), we see that the derivatives $\hat{f}_1'(M, 0) = \cdots = \hat{f}_{l'}'(M, 0)$ are equal to 0, and that $f_{l'+1}'(M, 0) > 0$ or $\hat{f}_{l'+1}(M, 0) > \hat{f}_l(M, 0)$ in case $l = l'$. By items (c′) and (d) of Theorem 1.5.6, we know that

$$\hat{f}_1(M, 0) = \hat{f}_1(M, r) = \cdots = \hat{f}_{l'}(M, r) > \hat{f}_{l'+1}(M, r)$$

for any $r < \hat{f}_1(M, 0)$. We may then apply Proposition 4.1.11 to split off the desired M_0. Now, we may apply the standard technique (Lemma 1.1.10 and Remark 1.1.11) to glue the decomposition given by Theorem 4.2.8 and Proposition 1.5.17; this gives the further decompositions by $M_{\{\theta\}}$. □

4.3. ***Variation over polyannuli.*** In this subsection, we study differential modules over a polyannulus or a polydisc. In particular, we are interested in studying the functions coming from the radii of convergence when we complete the differential module with respect to various Gauss norms. We relate the slopes of such functions with the valuations of the refined intrinsic radii.

In this subsection, we assume Hypothesis 1.5.1 and we assume that K is discretely valued.

Definition 4.3.1. A subset $C \subseteq \mathbb{R}^n$ is called *nondegenerate* if it contains an open subset of $\mathbb{R}^n$. Its interior is denoted by C^{int}.

An *integral affine functional* on $\mathbb{R}^n$ is a map $\lambda : \mathbb{R}^n \to \mathbb{R}$ of the form

$$\lambda(x_1, \dots, x_n) = a_1x_1 + \cdots + a_nx_n + b$$

for some $a_1, \dots, a_n \in \mathbb{Z}$ and $b \in -\log|K^\times|^{\mathbb{Q}}$.

A subset $C \subseteq \mathbb{R}^n$ is *rational polyhedral* (or *RP* for short) if it is *bounded* and there exist integral affine functionals $\lambda_1, \dots, \lambda_r$ such that

$$C = \{x \in \mathbb{R}^n : \lambda_i(x) \geq 0 \text{ for } i = 1, \dots, r\}.$$

For $C \subseteq \mathbb{R}^n$ an RP subset of $\mathbb{R}^n$, a function $f : C \to \mathbb{R}^n$ is *integral polyhedral* if there exist finitely many integral affine functionals $\lambda'_1, \dots, \lambda'_d$ such that $f(x) = \max\{\lambda'_1(x), \dots, \lambda'_d(x)\}$ for any $x \in C$.

Remark 4.3.2. Our convention slightly differs from that of [Kedlaya and Xiao 2010], where RP subsets are not assumed to be bounded. However, some of the statements below still hold for unbounded RP, and they are often simple corollaries of the statements in the bounded case. We leave this as an exercise for the reader.

Notation 4.3.3. We put $I = \{1, \dots, n\}$. We use $\underline{a}$ to denote the n-tuple $(a, \dots, a)$.

Definition 4.3.4. For a subset $C \subseteq \mathbb{R}^n$, let e^{-C} denote the *closure* in $\mathbb{R}^n$ of the subset $\{\mathrm{e}^{-r_I} : r_I \in C\}$. A subset S of $[0, +\infty)^n$ is called *log-RP* if $S = \mathrm{e}^{-C}$ for some RP subset C of $\mathbb{R}^n$; it is called *nondegenerate* if C is so.

For S a log-RP subset of $[0, +\infty)^n$, define $A_K(S^{\text{int}})$ to be the subspace of the (Berkovich) analytic n-space with coordinates $t_1, \dots, t_n$ satisfying the condition $(|t_1|, \dots, |t_n|) \in \mathrm{e}^{-C^{\text{int}}}$. We use $K\{\{S\}\}$ to denote its ring of functions, and use $K[\![S]\!]_0$ to denote the subring of $K\{\{S\}\}$ consisting of functions that are bounded on $|t_I| \in \mathrm{e}^{-C^{\text{int}}}$.

One cannot literally equate S^{int} with $\mathrm{e}^{-C^{\text{int}}}$; the problem is that we cannot take the log for a zero coordinate in S-space. But, in practice, one can view the two spaces the same, just being careful when stating a result.

Notation 4.3.5. Let S be a nondegenerate log-RP subset of $[0, +\infty)^n$ and let R denote either $K\{\{S\}\}$ or $K[\![S]\!]_0$. Let M be a $\partial_{I\cup J}$-differential module over R of rank

d, with respect to the derivations $\partial_1, \dots, \partial_m$ and $\partial_{m+1} = \partial/\partial t_1, \dots, \partial_{m+n} = \partial/\partial t_n$. For an element η_I in $(\eta_1, \dots, \eta_n) \in S$ (S^{int} if $R = K\{\{S\}\}$), let F_{η_I} be the completion of $\mathrm{Frac}(R)$ with respect to the η_I-Gauss norm. We remark that for η_I on the boundary of S, F_{η_I} "looks different" (more like E than F_η in the 1-dimensional case).

For an element r_I in $-\log S$ (or $-\log S^{\mathrm{int}}$ if $R = K\{\{S\}\}$), put

$$f_l(M, r_I) = -\log IR(M \otimes F_{\mathrm{e}^{-r_I}}; l) \quad \text{and} \quad F_l(M, r_I) = f_1(M, r_I) + \cdots + f_l(M, r_I)$$

for $l = 1, \dots, d$.

Theorem 4.3.6. *Keep the notation as above. We have the following*:

(a) (*Polyhedrality*) *The functions* $d!\,F_l(M, r_I)$, *for* $l = 1, \dots, d-1$, *and* $F_d(M, r_I)$ *are integral polyhedral functions.*

(b) (*Decomposition*) *Suppose that for some* $l \in \{1, \dots, d\}$, *the function* $F_l(M, r_I)$ *is affine, and suppose that* $f_l(M, r_I) > f_{l+1}(M, r_I)$ *for any* $r_I \in -\log S$. *Then* M *admits a unique direct sum decomposition* $M \cong M_0 \oplus M_1$ *of differential modules such that for any* $\eta_I \in -\log S^{\mathrm{int}}$, *the multiset of intrinsic radii of* M_0 *exactly consists of the smallest* l *elements in the multiset of intrinsic radii of* $M \otimes F_{\eta_I}$.

(c) (*Refined radii*) *Assume that* $R = K\{\{S\}\}$ *and that*

$$f_1(M, r_I) = \cdots = f_d(M, r_I) = -\log \omega - \mathfrak{s} + b_1 r_1 + \cdots + b_n r_n$$

are affine functions on $-\log S^{\mathrm{int}}$. *Let* e_i *denote the prime-to-*p *part of the denominator of* b_i *for all* $i \in I$. *Then there exists a finite tamely ramified extension* K' *of* K *and a multiset*

$$\mathcal{J}\Theta(M) \subset \bigoplus_{i \in I} \kappa_{K'}^{(\mathfrak{s})} \frac{dt_i}{t_i} \oplus \bigoplus_{j \in J} \kappa_{K'}^{(\mathfrak{s})} \frac{du_j}{u_j}$$

such that we have a unique direct sum decomposition of differential modules

$$M \otimes_R R[t_1^{1/e_1}, \dots, t_n^{1/e_n}] = \bigoplus_{\vartheta \in \mathcal{J}\Theta(M)} M_\vartheta,$$

such that each $M_\vartheta \otimes F_{\eta_I}[t_1^{1/e_1}, \dots, t_n^{1/e_n}]$ *has pure refined intrinsic radii* $t_I^{-b_I}\vartheta$.

Proof. For (a) and (b), see [Kedlaya and Xiao 2010, Theorems 3.3.9 and 3.4.4, and Remark 3.4.7]. (c) follows from the same argument but using Theorem 1.5.12 as the decomposition tool. □

To extend (c) of the theorem above to the boundary is a little tricky. We will prove it in a special case and leave the general case as an exercise for the reader.

Situation 4.3.7. Consider the subset $C = \{(x_I) \subset \mathbb{R}^n : x_I \geq 0,\ x_1 + \cdots + x_n \leq 1\}$. Put $S = \mathrm{e}^{-C}$, and $R = K[\![S]\!]_0$. Let M be a differential module over $K[\![S]\!]_0$. Assume moreover that $f_1(M,\underline{0}) = \cdots = f_d(M,\underline{0}) = -\log\omega - \mathfrak{s}$ with $\mathfrak{s} < 0$. We define the following two multisets.

(1) Choose $x \in \mathfrak{m}_K^{(\mathfrak{s})} \backslash \mathfrak{m}_K^{(\mathfrak{s})+}$ to identify $\kappa_{F_{\underline{1}}}^{(\mathfrak{s})} \xrightarrow{\cdot x^{-1}} \kappa_{F_{\underline{1}}}$ and embed the latter into the higher local field $\kappa_K((t_1))\cdots((t_n))$, which is equipped with a multi-indexed valuation with respect to the parameters $(t_n,\dots,t_1)$. This gives rise to a valuation $\boldsymbol{v}_{\mathfrak{s}} : \kappa_{F_{\underline{1}}}^{(\mathfrak{s})} \to \mathbb{Z}^n \subset \mathbb{Q}^n$, where the latter is equipped with the lexicographical order; this does not depend on the choice of x. Define the following valuation on

$$\bigoplus_{i\in I} \kappa_{F_{\underline{1}}^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{dt_i}{t_i} \oplus \bigoplus_{j\in J} \kappa_{F_{\underline{1}}^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{du_j}{u_j},$$

still denoted by $\boldsymbol{v}_{\mathfrak{s}}$, by taking the minimum of $\boldsymbol{v}_{\mathfrak{s}}$ over the coefficients. We consider the multiset $A = \{(\boldsymbol{v}(\vartheta), \bar{\vartheta}) \,|\, \vartheta \in \mathcal{I}\Theta(M \otimes F_{\underline{1}})\}$, where $\bar{\vartheta}$ is the reduction of $t_I^{-\boldsymbol{v}_{\mathfrak{s}}(\vartheta)}\vartheta$ to

$$\bigoplus_{i\in I} \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{dt_i}{t_i} \oplus \bigoplus_{j\in J} \kappa_{K^{\mathrm{alg}}}^{(\mathfrak{s})} \frac{du_j}{u_j}.$$

(2) By Theorem 4.3.6(a), there exists an RP subset C' of C which is adjacent to the cells $t_1 = \cdots = t_i = 0$ for $i = 1,\dots,n-1$, such that the function $f_l(M, r_I)$ for each l is affine in r_I over C'. Then, over $\mathrm{e}^{-C'^{\mathrm{int}}}$, we have a unique direct sum decomposition of differential modules $M = \bigoplus_{b_I\in\mathbb{Q}^n} M_{b_I}$ such that

$$f_1(M_{b_I}, r_I) = \cdots = f_{\dim M_{b_I}}(M_{b_I}, r_I) = -\log\omega - \mathfrak{s} + b_1 r_1 + \cdots + b_n r_n.$$

We put

$$B = \{(-b_1,\dots,-b_n,\vartheta) : b_I \in \mathbb{Q}^n,\ t_1^{-b_1}\cdots t_n^{-b_n}\vartheta \in \mathcal{I}\Theta(M \otimes F_{\eta_I})\},$$

for some $\eta_I \in C'^{\mathrm{int}}$ and this set does not depend on the choice of η_I by Theorem 4.3.6(c).

Choose integers $e_1,\dots,e_n \in \mathbb{N}$ coprime to p such that $e_i b_i \in \mathbb{Z}$ for any i and for any $(-b_1,\dots,-b_n,\vartheta) \in B$. Put $R' = K[\![C']\!]_0[t_1^{1/e_1},\dots,t_n^{1/e_n}]$.

Theorem 4.3.8. *The two multisets A and B are the same (for any C' that satisfies the condition in (2)). Moreover, there exists a finite tamely ramified extension K'/K and a unique direct sum decomposition*

$$M \otimes R' \otimes K' = \bigoplus_{(b_I,\vartheta)\in B} M_{(b_I,\vartheta)}$$

such that, if we put $F'_{\mathrm{e}^{-r_I}} = F_{\mathrm{e}^{-r_I}}[t_1^{1/e_1},\dots,t_n^{1/e_n}] \otimes K'$,

(i) *for all* $r_I \in C'^{\mathrm{int}}$, $M_{(b_I,\vartheta)} \otimes F'_{\mathrm{e}^{-r_I}}$ *has pure intrinsic radii* $\omega \mathrm{e}^{-b_1r_1-\cdots-b_nr_n+\mathfrak{s}}$ *and pure refined intrinsic radii* $t_I^{-b_I}\vartheta$, *and*

(ii) *any element in* $\mathcal{I}\Theta(M \otimes F'_{\underline{1}})$ *is congruent to* $t_I^{-b_I}\vartheta$ *modulo elements with* $\boldsymbol{v}_s$*-valuation strictly bigger than* $(-b_1, \dots, -b_n)$.

Proof. We first construct the decomposition that satisfies condition (i). For this, we may replace K by a finite tamely ramified extension such that all ϑ appearing in B lie in

$$\bigoplus_{i\in I} \kappa_K^{(s)} \frac{dt_i}{t_i} \oplus \bigoplus_{j\in J} \kappa_K^{(s)} \frac{du_j}{u_j}$$

for an appropriate s. In this case, we construct the decomposition of $M \otimes R'$ using the same argument as in [Kedlaya and Xiao 2010, Theorem 3.4.4] by invoking Theorems 4.1.9 and 4.2.6 at appropriate places.

Now we check condition (ii) for this direct sum decomposition; this is equivalent to identifying the multisets A with B for each $M_{b_I,\vartheta}$. Note that we already know that $M_{b_I,\vartheta} \otimes F_{e^{-r_I}}$ has pure intrinsic radii $\omega e^{-b_1r_1-\cdots-b_nr_n+s}$. For simplicity, we put $M = M_{b_I,\vartheta}$. We do induction on the dimension n. When $n = 0$ there is nothing to prove. We assume that the theorem is proved for $n-1$. Let D denote the face $t_1 = 0$ of C. Put $\widetilde{C} = C \cap D$, $\widetilde{C}' = C' \cap D$, $\widetilde{S} = \mathrm{e}^{-\widetilde{C}}$, and $\widetilde{R} = \widetilde{K}[\![\widetilde{S}]\!]_0$ with coordinates $t_2, \dots, t_n$, where $\widetilde{K}$ is the completion of $\mathrm{Frac}\big(K[\![t_1]\!]_0\big)$ with respect to the 1-Gauss norm.

By applying the induction hypothesis to $\widetilde{M} = M \otimes_R \widetilde{R}$, the multiset A is equal to

$$A' = \Big\{ \big(\mathsf{v}_s(\vartheta'), -b_2, \dots, -b_n, \overline{t_1^{-\mathsf{v}_s(\vartheta')}\vartheta'}\big) \;\big|\; (-b_2, \dots, -b_n) \in \mathbb{Q}^{n-1}, \\ t_2^{-b_2} \cdots t_n^{-b_n}\vartheta' \in \mathcal{I}\Theta(M \otimes F_{\eta_I}) \Big\},$$

for any $(r_2, \dots, r_n) \in \widetilde{C}'$, where v_s is the valuation on

$$\bigoplus_{i\in I} \kappa_{\widetilde{K}^{\mathrm{alg}}}^{(s)} \frac{dt_i}{t_i} \oplus \bigoplus_{j\in J} \kappa_{\widetilde{K}^{\mathrm{alg}}}^{(s)} \frac{du_j}{u_j}$$

as in Notation 4.2.3, and $\overline{t_1^{-\mathsf{v}_s(\vartheta')}\vartheta'}$ is the reduction of $t_1^{-\mathsf{v}_s(\vartheta')}\vartheta'$ in

$$\bigoplus_{i\in I} \kappa_{K^{\mathrm{alg}}}^{(s)} \frac{dt_i}{t_i} \oplus \bigoplus_{j\in J} \kappa_{K^{\mathrm{alg}}}^{(s)} \frac{du_j}{u_j}.$$

It suffices to identify the multiset A' with B. When $r_I \in \mathbb{Q}^n \cap C'$, this follows from applying Corollary 4.2.7 to the line which passes through the point r_I and is parallel to the t_1-axis. In particular, this says that for any ϑ' above,

$$\overline{t_1^{-\mathsf{v}_s(\vartheta')}\vartheta'} \text{ is the same as } \vartheta.$$

When r_I is not rational, the same statement follows from the "continuity" result in Theorem 4.3.6(c). □

Remark 4.3.9. One can also describe the intrinsic radii of $M_{b_I,\vartheta}$ at the point $(r_I) \in C'$ with $r_1 = \cdots = r_l = 0$ for some $l \in \{1, \dots, d-1\}$. We leave this as an exercise for interested readers.

Next we consider the situation for solvable differential modules.

Definition 4.3.10. Let $C = \{(x_I) \subset \mathbb{R}^n : x_I \geq 0, x_1 + \cdots + x_n = 1\}$. For $[\alpha, \beta] \in (0, 1)$, we put $S_{[\alpha,\beta]} = \{\rho^C : \rho \in [\alpha, \beta]\}$ and $R_{[\alpha,\beta]} = K[\![S_{[\alpha,\beta]}]\!]_0$. For $\alpha \in (0, 1)$, we put $R_\alpha = \bigcap_{\beta \in (\alpha,1)} R_{[\alpha,\beta]}$.

Fix $\alpha \in (0, 1)$. Let M be a differential module over R_α. Assume that M is *solvable*, that is, for each $x_I \in C$, we have $f_1(M, \rho^{x_I}) \to 0$ as $\rho \to 1^-$.

By Theorem 1.6.2, for $x_I \in C$, there exists $b_1(M, x_I), \dots, b_d(M, x_I)$ such that $f_l(M, -x_I \log \rho) = \rho^{b_l(M,x_I)}$ when $\rho \to 1^-$, for $l = 1, \dots, d$. Put

$$B_l(M, x_I) = b_1(M, x_I) + \cdots + b_l(M, x_I)$$

for $l = 1, \dots, d$.

Proposition 4.3.11. *Keep the notation as above. Then the functions $d!B_l(M, x_I)$ and $B_d(M, x_I)$ are integral polyhedral functions.*

Proof. See [Kedlaya 2011, Theorem 3.3.3]. The proposition also follows from Theorem 4.3.6(a). □

Construction 4.3.12. Keep the notation as above.

Let $\underline{x} = (0, \dots, 1) \in C$ be the point. Let $\mathfrak{F}$ be the completion of the fraction field of $\mathbb{O}_K((t_1)) \cdots ((t_{n-1}))$; it is a higher dimensional local field. We have a natural embedding $R_\alpha \hookrightarrow \mathfrak{F}\{\{\eta/t_n, t_n\}\} = \widetilde{\mathfrak{F}}_\eta$, if $\eta \in (\alpha, 1)$. This means to restrict the picture to the line $(0, \dots, 0, \rho)$ for $\rho \in (\eta, 1)$. We assume that $M \otimes \widetilde{\mathfrak{F}}_\eta$ has pure-log break b.

Recall that, as in Situation 4.3.7, we have a valuation

$$\boldsymbol{v} : \bigoplus_{i \in I} \kappa_{\mathfrak{F}^{\mathrm{alg}}} \frac{dt_i}{t_i} \oplus \bigoplus_{j \in J} \kappa_{\mathfrak{F}^{\mathrm{alg}}} \frac{du_j}{u_j} \to \mathbb{Q}^n.$$

Proposition 4.3.13. *Keep the notation as above. The following two multisets of $(n-1)$-tuples are the same.*

(i) *The multiset composed of valuations $\boldsymbol{v}$ of the elements of $(1/\boldsymbol{\pi})\mathcal{I}\Theta(M \otimes \widetilde{\mathfrak{F}}_\eta)$, where $\boldsymbol{\pi}$ is a Dwork pi.*

(ii) *The multiset of slopes of $b_l(M, x_I)$, for $l = 1, \dots, d$, on a RP subset of C which is adjacent to the cells $\{t_1 = \cdots = t_i = 0, t_{i+1} + \cdots + t_n = 1\}$ for all $i = 1, \dots, n$.*

Proof. It follows from Theorem 4.3.8. □

Remark 4.3.14. One may interpret the above proposition geometrically, as in [Kedlaya 2011]. We will come back to this discussion in a future work.

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lxiao@math.uchicago.edu *Department of Mathematics, University of Chicago, 5734 S. University Ave, Chicago, IL 60637, United States*

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On common values of $\phi(n)$ and $\sigma(n)$, II

Kevin Ford and Paul Pollack

For each positive-integer valued arithmetic function f, let $\mathscr{V}_f \subset \mathbb{N}$ denote the image of f, and put $\mathscr{V}_f(x) := \mathscr{V}_f \cap [1, x]$ and $\mathscr{V}_f(x) := \#\mathscr{V}_f(x)$. Recently Ford, Luca, and Pomerance showed that $\mathscr{V}_\phi \cap \mathscr{V}_\sigma$ is infinite, where ϕ denotes Euler's totient function and σ is the usual sum-of-divisors function. Work of Ford shows that $V_\phi(x) \asymp V_\sigma(x)$ as $x \to \infty$. Here we prove a result complementary to that of Ford et al. by showing that most ϕ-values are not σ-values, and vice versa. More precisely, we prove that, as $x \to \infty$,

$$\#\{n \leqslant x : n \in \mathscr{V}_\phi \cap \mathscr{V}_\sigma\} \leqslant \frac{V_\phi(x) + V_\sigma(x)}{(\log\log x)^{1/2+o(1)}}.$$

1. Introduction

1A. ***Summary of results.*** For each positive-integer valued arithmetic function f, let $\mathscr{V}_f$ denote the image of f, and put $\mathscr{V}_f(x) := \mathscr{V}_f \cap [1, x]$ and $V_f(x) := \#\mathscr{V}_f(x)$. In this paper we are primarily concerned with the cases when $f = \phi$, the Euler totient function, and when $f = \sigma$, the usual sum-of-divisors function. When $f = \phi$, the study of the counting function V_f goes back to Pillai [1929], and was subsequently taken up in [Erdős 1935; 1945; Erdős and Hall 1973; 1976; Pomerance 1986; Maier and Pomerance 1988; Ford 1998a] (with an announcement in [Ford 1998b]). From the sequence of results obtained in these papers, we mention Erdős's asymptotic formula [1935] for $\log(V_f(x)/x)$, namely

$$V_f(x) = \frac{x}{(\log x)^{1+o(1)}} \quad (x \to \infty),$$

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and the much more intricate determination of the precise order of magnitude by Ford,

$$V_f(x) \asymp \frac{x}{\log x} \exp\bigl(C(\log_3 x - \log_4 x)^2 + D \log_3 x - (D + \tfrac{1}{2} - 2C) \log_4 x\bigr). \quad (1\text{-}1)$$

Here $\log_k$ denotes the k-th iterate of the natural logarithm, and the constants C and D are defined as follows: Let

$$F(z) := \sum_{n=1}^{\infty} a_n z^n, \quad \text{where } a_n = (n+1)\log(n+1) - n \log n - 1. \quad (1\text{-}2)$$

Since each $a_n > 0$ and $a_n \sim \log n$ as $n \to \infty$, it follows that $F(z)$ converges to a continuous, strictly increasing function on (0, 1), and $F(z) \to \infty$ as $z \uparrow 1$. Thus, there is a unique real number ϱ for which

$$F(\varrho) = 1 \quad (\varrho = 0.542598586098471021959\ldots). \quad (1\text{-}3)$$

In addition, F' is strictly increasing, and $F'(\varrho) = 5.697758\ldots$. Then

$$C = \frac{1}{2\,|\log \varrho|} = 0.817814\ldots$$
$$D = 2C(1 + \log F'(\varrho) - \log(2C)) - \tfrac{3}{2} = 2.176968\ldots.$$

In [Ford 1998a], it is also shown that (1-1) holds for a wide class of ϕ-like functions, including $f = \sigma$. Consequently, $V_\phi(x) \asymp V_\sigma(x)$.

Erdős [1959, p. 172] asked if it could be proved that infinitely many natural numbers appear in both $\mathscr{V}_\phi$ and $\mathscr{V}_\sigma$ (see also [Erdős and Graham 1980]). This question was recently answered by Ford, Luca, and Pomerance [Ford et al. 2010]. Writing $V_{\phi,\sigma}(x)$ for the number of common elements of $\mathscr{V}_\phi$ and $\mathscr{V}_\sigma$ up to x, they proved that

$$V_{\phi,\sigma}(x) \geqslant \exp((\log\log x)^c)$$

for some positive constant $c > 0$ and all large x (in [Garaev 2011] this is shown for *all* constants $c > 0$). This lower bound is probably very far from the truth. For example, if p and $p+2$ form a twin prime pair, then $\phi(p+2) = p+1 = \sigma(p)$; a quantitative form of the twin prime conjecture then implies that $V_{\phi,\sigma}(x) \gg x/(\log x)^2$. In Part I of this article [Ford and Pollack 2011], we showed that a stronger conjecture of the same type allows for an improvement. Roughly, our result is as follows:

Theorem A. *Assume a strong uniform version of Dickson's prime k-tuples conjecture. Then as $x \to \infty$,*

$$V_{\phi,\sigma}(x) = \frac{x}{(\log x)^{1+o(1)}}.$$

Theorem A suggests that $V_{\phi,\sigma}(x)$ is much larger than we might naively expect. This naturally leads one to inquire about what can be proved in the opposite direction; for instance, could it be that a positive proportion of ϕ-values are also σ-values?

N	$V_\phi(N)$	$V_\sigma(N)$	$V_{\phi,\sigma}(N)$	$\frac{V_{\phi,\sigma}(N)}{V_\phi(N)}$	$\frac{V_{\phi,\sigma}(N)}{V_\sigma(N)}$
10000	2374	2503	1368	0.5762426	0.5465441
100000	20254	21399	11116	0.5488299	0.5194635
1000000	180184	189511	95145	0.5280436	0.5020553
10000000	1634372	1717659	841541	0.5149017	0.4899348
100000000	15037909	15784779	7570480	0.5034264	0.4796063
1000000000	139847903	146622886	69091721	0.4940490	0.4712206

Table 1. Data on ϕ-values, σ-values, and common values up to $N = 10^k$, from $k = 5$ to $k = 9$.

The numerical data up to 10^9, exhibited in Table 1, suggests that the proportion of common values is decreasing, but the observed rate of decrease is rather slow.

Our principal result is the following estimate, which implies in particular that almost all ϕ-values are not σ-values, and vice versa.

Theorem 1.1. *As $x \to \infty$,*

$$V_{\phi,\sigma}(x) \leqslant \frac{V_\phi(x) + V_\sigma(x)}{(\log\log x)^{1/2+o(1)}}.$$

The proof of Theorem 1.1 relies on the detailed structure theory of totients as developed in [Ford 1998a]. It would be interesting to know the true rate of decay of $V_{\phi,\sigma}(x)/V_\phi(x)$.

1B. *Sketch.* Since the proof of Theorem 1.1 is rather intricate and involves a number of technical estimates, we present a brief outline of the argument in this section.

We start by discarding a sparse set of undesirable ϕ and σ-values. More precisely, we identify (in Lemma 3.2) convenient sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$ with the property that almost all ϕ-values less than or equal to x have all their preimages in $\mathscr{A}_\phi$ and almost all σ-values less than or equal to x have all their preimages in $\mathscr{A}_\sigma$. This reduces us to studying how many ϕ and σ-values arise as solutions to the equation

$$\phi(a) = \sigma(a'), \quad \text{where} \quad a \in \mathscr{A}_\phi,\, a' \in \mathscr{A}_\sigma.$$

Note that to show that $V_{\phi,\sigma}(x)/V_\phi(x) \to 0$, we need only count the number of common ϕ-σ-values of this kind, and not the (conceivably much larger) number of pairs $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$ corresponding to these values.

What makes the sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$ convenient for us? The properties imposed in the definitions of these sets are of two types, *anatomical* and *structural*. By anatomical considerations, we mean general considerations of multiplicative structure as commonly appear in elementary number theory (for example, consideration of the number and size of prime factors). By structural considerations, we mean those depending for their motivation on the fine structure theory of totients developed by Ford [1998a].

Central to our more anatomical considerations is the notion of a *normal prime*. Hardy and Ramanujan [1917] showed that almost all natural numbers $\leqslant x$ have $\sim \log\log x$ prime factors, and Erdős [1935] showed that the same holds for almost all shifted primes $p-1 \leqslant x$. Moreover, sieve methods imply that if we list the prime factors of $p-1$ on a double-logarithmic scale, then these are typically close to uniformly distributed in $[0, \log\log p]$. Of course, all of this remains true with $p+1$ in place of $p-1$. We assume that the numbers belonging to $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$ have all their prime factors among this set of normal primes.

If we assume that numbers n all of whose prime factors are normal generate "most" f-values (for $f \in \{\phi, \sigma\}$), we are led to a series of linear inequalities among the (double-logarithmically renormalized) prime factors of n. These inequalities are at the heart of the structure theory of totients as developed in [Ford 1998a]. As one illustration of the power of this approach, mapping the L largest prime factors of n (excluding the largest) to a point in $\mathbb{R}^L$, the problem of estimating $V_f(x)$ reduces to the problem of finding the volume of a certain region of $\mathbb{R}^L$, called the fundamental simplex. In broad strokes, this is how one establishes Ford's bound (1-1). We incorporate these linear inequalities into our definitions of $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$. One particular linear combination of renormalized prime factors appearing in the definition of the fundamental simplex is of particular interest to us (see condition (8) in the definition of $\mathscr{A}_f$ in Section 3 below); that we can assume this quantity is less than 1 is responsible for the success of our argument.

Suppose now that we have a solution to $\phi(a) = \sigma(a')$, where $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$. We write $a = p_0 p_1 p_2 \cdots$ and $a' = q_0 q_1 q_2 \cdots$, where the sequences of p_i and q_j are nonincreasing. We cut the first of these lists in two places; at the k-th prime p_k and at the L-th prime p_L. The precise choice of k and L is somewhat technical; one should think of the primes p_i larger than p_k as the "large" prime divisors of a, those smaller than p_k but larger than p_L as "small", and those smaller than p_L as "tiny". The equation $\phi(a) = \sigma(a')$ can be rewritten in the form

$$(p_0-1)(p_1-1)(p_2-1)\cdots(p_{k-1}-1) f d \\ = (q_0+1)(q_1+1)(q_2+1)\cdots(q_{k-1}+1)e, \quad (1\text{-}4)$$

where

$$f := \phi(p_k \cdots p_{L-1}), \quad d := \phi(p_L p_{L+1} \cdots), \quad \text{and} \quad e := \sigma(q_k q_{k+1} \cdots). \quad (1\text{-}5)$$

To see that (1-4) correctly expresses the relation $\phi(a) = \sigma(a')$, we recall that the primes $p_1, \ldots, p_k$ are all large, so that by the "anatomical" constraints imposed in the definition of $\mathscr{A}_\phi$, each appears to the first power in the prime factorization of a. An analogous statement holds for the primes $q_1, \dots, q_k$; this follows from the general principle, established below, that $p_i \approx q_i$ provided that either side is not too small. There is one respect in which (1-5) may not be quite right: Since p_L is tiny, we cannot assume a priori that $p_L \neq p_{L-1}$, and so it may be necessary to amend the definition of d somewhat; we ignore this (ultimately minor) difficulty for now.

To complete the argument, we fix d and estimate from above the number of solutions (consisting of $p_0, \dots, p_{k-1}, q_0, \dots, q_{k-1}, e, f$) to the relevant equations of the form (1-4); then we sum over d. The machinery facilitating these estimates is encoded in Lemma 4.1, which is proved by a delicate, iterative sieve argument of a kind first introduced in [Maier and Pomerance 1988] and developed in [Ford 1998a, §5]. The hypotheses of that lemma include several assumptions about the p_i and q_j, and about e, f, and d. All of these rather technical hypotheses are, in our situation, consequences of our definitions of $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$; we say more about some of them in a remark following Lemma 4.1.

Notation. Let $P^+(n)$ denote the largest prime factor of n, understood so that $P^+(1) = 1$, and let $\Omega(n, U, T)$ denote the total number of prime factors p of n such that $U < p \leqslant T$, counted according to multiplicity. Constants implied by the Landau O and the Vinogradov $\ll$ and $\gg$ symbols are absolute unless otherwise specified. Symbols in boldface type indicate vector quantities.

2. Preliminaries

2A. ***Anatomical tools.*** We begin with two tools from the standard chest. The first is a form of the upper bound sieve and the second concerns the distribution of smooth numbers.

Lemma 2.1 (see, e.g., [Halberstam and Richert 1974, Theorem 4.2]). *Suppose $A_1, \dots, A_h$ are positive integers and $B_1, \dots, B_h$ are integers such that*

$$E = \prod_{i=1}^{h} A_i \prod_{1 \leqslant i < j \leqslant h} (A_i B_j - A_j B_i) \neq 0.$$

Then

$$\#\{n \leqslant x : A_i n + B_i \text{ prime } (1 \leqslant i \leqslant h)\} \ll \frac{x}{(\log x)^h} \prod_{p \mid E} \frac{1 - \frac{\nu(p)}{p}}{(1 - \frac{1}{p})^h} \ll \frac{x (\log_2(|E| + 2))^h}{(\log x)^h},$$

where $\nu(p)$ is the number of solutions of the congruence $\prod (A_i n + B_i) \equiv 0 \pmod{p}$, and the implied constant may depend on h.

Let $\Psi(x, y)$ denote the number of $n \leqslant x$ for which $P^+(n) \leqslant y$. The following estimate is due to Canfield, Erdős, and Pomerance:

Lemma 2.2 [Canfield et al. 1983]. *Fix $\epsilon > 0$. If $2 \leqslant y \leqslant x$ and $u = \dfrac{\log x}{\log y}$, then*

$$\Psi(x, y) = x/u^{u+o(u)}$$

for $u \leqslant y^{1-\epsilon}$, as $u \to \infty$.

The next lemma supplies an estimate for how often $\Omega(n)$ is unusually large; this may be deduced from the theorems in Chapter 0 of [Hall and Tenenbaum 1988].

Lemma 2.3. *The number of integers $n \leqslant x$ for which $\Omega(n) \geqslant \alpha \log_2 x$ is*

$$\ll_\alpha \begin{cases} x(\log x)^{-Q(\alpha)} & \text{if } 1 < \alpha < 2, \\ x(\log x)^{1-\alpha\log 2}\log_2 x & \text{if } \alpha \geqslant 2, \end{cases}$$

where $Q(\lambda) = \int_1^\lambda \log t\, dt = \lambda\log(\lambda) - \lambda + 1$.

In the remainder of this section, we give a precise meaning to the term "normal prime" alluded to in the introduction and draw out some simple consequences. For $S \geqslant 2$, a prime p is said to be *S-normal* if the following two conditions hold for each $f \in \{\phi, \sigma\}$:

$$\Omega(f(p), 1, S) \leqslant 2\log_2 S,$$

and, for every pair of real numbers (U, T) with $S \leqslant U < T \leqslant f(p)$, we have

$$\left|\Omega(f(p), U, T) - (\log_2 T - \log_2 U)\right| < \sqrt{\log_2 S \log_2 T}. \tag{2-1}$$

This definition is slightly weaker than the corresponding definition on [Ford 1998a, p. 13], and so the results from that paper remain valid in our context. As a straightforward consequence of the definition, if p is S-normal, $f \in \{\phi, \sigma\}$, and $f(p) \geqslant S$, then

$$\Omega(f(p)) \leqslant 3\log_2 f(p). \tag{2-2}$$

The following lemma is a simple consequence of [Ford 1998a, Lemma 2.10] and (1-1):

Lemma 2.4. *For each $f \in \{\phi, \sigma\}$, the number of f-values less than or equal to x which have a preimage divisible by a prime that is not S-normal is*

$$\ll V_f(x)(\log_2 x)^5(\log S)^{-1/6}.$$

We also record the observation that if p is S-normal, then $P^+(f(p))$ cannot be too much smaller than p, on a double-logarithmic scale.

Lemma 2.5. *If $5 \leqslant p \leqslant x$ is an S-normal prime and $f(p) \geqslant S$, then*

$$\frac{\log_2 P^+(f(p))}{\log_2 x} \geqslant \frac{\log_2 p}{\log_2 x} - \frac{\log_3 x + \log 4}{\log_2 x}.$$

Proof. We have

$$P^{+}(f(p)) \geqslant f(p)^{\frac{1}{\Omega(f(p))}} \geqslant f(p)^{\frac{1}{3\log_2 f(p)}} \geqslant p^{\frac{1}{4\log_2 x}}.$$

The result follows upon taking the double logarithm of both sides. □

2B. *Structural tools.* In this section, we describe more fully some components of the structure theory of totients alluded to in the introduction. Given a natural number n, write $n = p_0(n)p_1(n)p_2(n)\cdots$, where $p_0(n) \geqslant p_1(n) \geqslant p_2(n) \geqslant \cdots$ are the primes dividing n (with multiplicity). For a fixed x, we put

$$x_i(n; x) = \begin{cases} (\log_2 p_i(n))/(\log_2 x) & \text{if } i < \Omega(n) \text{ and } p_i(n) > 2, \\ 0 & \text{if } i \geqslant \Omega(n) \text{ or } p_i(n) = 2. \end{cases}$$

Suppose $L \geqslant 2$ is fixed and that $\xi_i \geqslant 0$ for $0 \leqslant i \leqslant L-1$. Recall the definition of the a_i from (1-2) and let $\mathscr{S}_L(\boldsymbol{\xi})$ be the set of $(x_1, \ldots, x_L) \in \mathbb{R}^L$ with $0 \leqslant x_L \leqslant x_{L-1} \leqslant \cdots \leqslant x_1 \leqslant 1$ and

$$\begin{array}{ll} (I_0) & a_1x_1 + a_2x_2 + \cdots + a_Lx_L \leqslant \xi_0, \\ (I_1) & a_1x_2 + a_2x_3 + \cdots + a_{L-1}x_L \leqslant \xi_1x_1, \\ \vdots & \qquad\vdots \\ (I_{L-2}) & a_1x_{L-1} + a_2x_L \leqslant \xi_{L-2}x_{L-2}. \end{array}$$

Define $T_L(\boldsymbol{\xi})$ as the volume (L-dimensional Lebesgue measure) of $\mathscr{S}_L(\boldsymbol{\xi})$. For convenience, let $\mathbf{1} = (1, 1, \ldots, 1)$, $\mathscr{S}_L = \mathscr{S}_L(\mathbf{1})$ (the "fundamental simplex"), and let T_L be the volume of $\mathscr{S}_L$. Let

$$L_0(x) := \lfloor 2C(\log_3 x - \log_4 x) \rfloor,$$

where C is defined as in the introduction. The next lemma allows us to locate the preimages of almost all f-values within suitable sets of the form $\mathscr{S}_L(\boldsymbol{\xi})$.

Lemma 2.6 [Ford 1998a, Theorem 15]. *Write $L_0 = L_0(x)$. Suppose $0 \leqslant \Psi < L_0$, $L = L_0 - \Psi$, and let*

$$\xi_i = \xi_i(x) = 1 + \frac{1}{10(L_0 - i)^3} \qquad (0 \leqslant i \leqslant L-2).$$

The number of f-values $v \leqslant x$ with a preimage n for which

$$(x_1(n; x), \ldots, x_L(n; x)) \notin \mathscr{S}_L(\boldsymbol{\xi}) \quad \textit{is} \quad \ll V_f(x)\exp(-\Psi^2/4C).$$

For future use, we collect here some further structural lemmas from [Ford 1998a]. The next result, which follows immediately from our (1-1) and Lemma 4.2 of that article, concerns the size of sums of the shape appearing in the definition of inequality (I_0) above.

Lemma 2.7. *Suppose that $L \geqslant 2$, $0 < \omega < \frac{1}{10}$, and x is sufficiently large. The number of f-values $v \leqslant x$ with a preimage satisfying*

$$a_1 x_1(n; x) + \cdots + a_L x_L(n; x) \geqslant 1 + \omega$$

is $\ll V_f(x)(\log_2 x)^5 (\log x)^{-\omega^2/(150L^3 \log L)}$.

We will make heavy use of the following (purely geometric) statement about the simplices $\mathscr{S}_L(\boldsymbol{\xi})$, which appears as [Ford 1998a, Lemma 3.10]. Recall from (1-3) that $\varrho = 0.542598\ldots$ denotes the unique real number with $\sum_{n=1}^{\infty} a_n \varrho^n = 1$.

Lemma 2.8. *If $\mathbf{x} \in \mathscr{S}_L(\boldsymbol{\xi})$ and $\xi_0^L \xi_1^{L-1} \cdots \xi_{L-2}^2 \leqslant 1.1$, then $x_j \leqslant 3\varrho^{j-i} x_i$ when $i < j$, and $x_j < 3\varrho^j$ for $1 \leqslant j \leqslant L$.*

Define $\mathscr{R}_L(\boldsymbol{\xi}; x)$ as the set of integers n with $\Omega(n) \leqslant L$ and

$$(x_0(n; x), x_1(n; x), \ldots, x_{L-1}(n; x)) \in \mathscr{S}_L(\boldsymbol{\xi}).$$

For $f \in \{\phi, \sigma\}$, put

$$R_L^{(f)}(\boldsymbol{\xi}; x) = \sum_{n \in \mathscr{R}_L(\boldsymbol{\xi}, x)} \frac{1}{f(n)}.$$

The next lemma, extracted from [Ford 1998a, Lemma 3.12], relates the magnitude of $R_L^{(f)}(\boldsymbol{\xi}; x)$ to the volume of the fundamental simplex T_L, whenever $\boldsymbol{\xi}$ is suitably close to $\mathbf{1}$. In that article, it plays a crucial role in the proof of the upper-bound aspect of (1-1).

Lemma 2.9. *If $1/(1000k^3) \leqslant \omega_{L_0-k} \leqslant 1/(10k^3)$ for $1 \leqslant k \leqslant L_0$, $\xi_i = 1 + \omega_i$ for each i, and $L \leqslant L_0$, then*

$$R_L^{(f)}(\boldsymbol{\xi}; x) \ll (\log_2 x)^L T_L$$

for both $f = \phi$ and $f = \sigma$.

While only the case $f = \phi$ of Lemma 2.9 appears in the statement of [ibid., Lemma 3.12], the $f = \sigma$ case follows trivially, since $\sigma(n) \geqslant \phi(n)$. In order to apply Lemma 2.9, we need estimates for the volume T_L; this is handled by the next lemma, extracted from [ibid., Corollary 3.4].

Lemma 2.10. *Assume $1 \leqslant \xi_i \leqslant 1.1$ for $0 \leqslant i \leqslant L-2$ and that $\xi_0^L \xi_1^{L-1} \cdots \xi_{L-2}^2 \asymp 1$. If $L = L_0 - \Psi > 0$, then*

$$(\log_2 x)^L T_L(\boldsymbol{\xi}) \ll Y(x) \exp(-\Psi^2/4C).$$

Here

$$Y(x) := \exp\bigl(C(\log_3 x - \log_4 x)^2 + D \log_3 x - (D + \tfrac{1}{2} - 2C) \log_4 x\bigr). \tag{2-3}$$

We conclude this section with the following technical lemma, which will be needed when we select the sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$ in Section 3.

Lemma 2.11. *For $f \in \{\phi, \sigma\}$ and $y \geqslant 20$,*

$$\sum_{\substack{v \in \mathscr{V}_f \\ P^+(v) \leqslant y}} \frac{1}{v} \ll \frac{\log_2 y}{\log_3 y} Y(y), \tag{2-4}$$

where Y is as defined in (2-3). *Moreover, for any $b > 0$,*

$$Y(\exp((\log x)^b)) \ll_b Y(x) \Big(\frac{\log_3 x}{\log_2 x}\Big)^{-2C \log b}. \tag{2-5}$$

Proof. We split the left-hand sum in (2-4) according to whether or not $v \leqslant y^{\log_2 y}$. The contribution of the large v is $O(1)$ and so is negligible: Indeed, for $t > y^{\log_2 y}$, we have $\log t/\log y > \log_2 y$. Thus, by Lemma 2.2, we have $\Psi(t, y) \ll t/(\log t)^2$ (say), and the $O(1)$ bound follows by partial summation. We estimate the sum over small v by ignoring the smoothness condition. Put $X = y^{\log_2 y}$. Since $V_f(t) \asymp (t/\log t) Y(t)$, partial summation gives that

$$\sum_{\substack{v \in \mathscr{V}_f \\ v \leqslant X}} \frac{1}{v} \ll 1 + \int_3^X \frac{Y(t)}{t \log t}\, dt = (1 + o(1)) Y(X) \frac{\log_2 X}{\log_3 X},$$

as $y \to \infty$. (The last equality follows, for instance, from L'Hôpital's rule.) Since $\log_2 X/\log_3 X \sim \log_2 y/\log_3 y$ and $Y(X) \sim Y(y)$, we have (2-4). Estimate (2-5) follows from the definition of Y and a direct computation; here it is helpful to note that if we redefine $X := \exp((\log x)^b)$, then $\log_3 X = \log_3 x + \log b$ and $\log_4 X = \log_4 x + O_b(1/\log_3 x)$. □

3. Definition of the sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$

We continue fleshing out the introductory sketch, giving precise definitions to the preimage sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$. Put

$$L := \big\lfloor L_0(x) - 2\sqrt{\log_3 x}\big\rfloor, \quad \xi_i := 1 + \frac{1}{10(L_0 - i)^3} \quad (1 \leqslant i \leqslant L). \tag{3-1}$$

The next lemma is a final technical preliminary.

Lemma 3.1. *Let $f \in \{\phi, \sigma\}$. The number of f-values $v \leqslant x$ with a preimage n for which*

(i) *$(x_1(n; x), \ldots, x_L(n; x)) \in \mathscr{S}_L(\boldsymbol{\xi})$ and*

(ii) *n has fewer than $L + 1$ odd prime divisors* (*counted with multiplicity*)

is $\ll V_f(x)/\log_2 x$.

Proof. We treat the case when $f = \phi$; the case when $f = \sigma$ requires only small modifications. We can assume that $x/\log x \leqslant n \leqslant 2x \log_2 x$, where the last inequality follows from known results on the minimal order of the Euler function. By Lemma 2.3, we can also assume that $\Omega(n) \leqslant 10 \log_2 x$. Put $p_i := p_i(n)$, as defined in Section 2B. Since $(x_1(n; x), \dots, x_L(n; x)) \in \mathscr{S}_L(\boldsymbol{\xi})$ by hypothesis, Lemma 2.8 gives that $x_2 < 3\varrho^2 < 0.9$, and so $p_2 \leqslant \exp((\log x)^{0.9})$. Thus,

$$n/(p_0 p_1) = p_2 p_3 \cdots \leqslant \exp(10(\log_2 x)(\log x)^{0.9}) = x^{o(1)},$$

and so $p_0 \geqslant x^{2/5}$ (say) for large x. In particular, we can assume that $p_0^2 \nmid n$.

Suppose now that n has exactly $L_0 - k + 1$ odd prime factors, where we fix $k > L_0 - L$. Then

$$v = (p_0 - 1)\phi(p_1 p_2 \cdots p_{L_0-k})2^s$$

for some integer $s \geqslant 0$. Using the prime number theorem to estimate the number of choices for p_0 given $p_1 \cdots p_{L_0-k}$ and 2^s, we obtain that the number of v of this form is

$$\ll \frac{x}{\log x} \sum_{p_1 \cdots p_{L_0-k}} \frac{1}{\phi(p_1 \cdots p_{L_0-k})} \sum_{s \geqslant 0} \frac{1}{2^s}.$$

(We use here that $x/(\phi(p_1 \cdots p_{L_0-k})2^s) \gg p_0 \geqslant x^{2/5}$.) The sum over s is $\ll 1$. To handle the remaining sum, we observe that $p_1 \cdots p_{L_0-k}$ belongs to the set $\mathscr{R}_{L_0-k}(\boldsymbol{\xi}_k, x)$, where $\boldsymbol{\xi}_k := (\xi_0, \dots, \xi_{L_0-k-2})$. Thus, the remaining sum is bounded by

$$R^{(\phi)}_{L_0-k}(\boldsymbol{\xi}_k; x) = \sum_{m \in \mathscr{R}_{L_0-k}(\boldsymbol{\xi}_k, x)} \frac{1}{\phi(m)}.$$

So by Lemmas 2.9 and 2.10, both of whose hypotheses are straightforward to verify,

$$R^{(\phi)}_{L_0-k}(\boldsymbol{\xi}_k; x) \ll (\log_2 x)^{L_0-k} T_{L_0-k} \leqslant (\log_2 x)^{L_0-k} T_{L_0-k}(\boldsymbol{\xi}_k) \ll Y(x) \exp(-k^2/4C).$$

Collecting our estimates, we obtain a bound of

$$\ll \frac{x}{\log x} Y(x) \exp(-k^2/4C) \ll V_\phi(x) \exp(-k^2/4C).$$

Now since $L_0 - L > 2\sqrt{\log_3 x}$, summing over $k > L_0 - L$ gives a final bound which is

$$\ll V_\phi(x) \exp(-(\log_3 x)/C) \ll V_\phi(x)/\log_2 x,$$

as desired. □

For the rest of this paper, we fix $\epsilon > 0$ and assume that $x \geqslant x_0(\epsilon)$. Put

$$S := \exp((\log_2 x)^{36}), \quad \delta := \sqrt{\frac{\log_2 S}{\log_2 x}}, \quad \omega := (\log_2 x)^{-1/2+\epsilon/2}. \tag{3-2}$$

For $f \in \{\phi, \sigma\}$, let $\mathscr{A}_f$ be the set of $n = p_0(n)p_1(n)\cdots$ satisfying $f(n) \leqslant x$ and

(0) $n \geqslant x/\log x$,

(1) every squarefull divisor m of n or $f(n)$ satisfies $m \leqslant \log^2 x$,

(2) all of the primes $p_j(n)$ are S-normal,

(3) $\Omega(f(n)) \leqslant 10\log_2 x$ and $\Omega(n) \leqslant 10\log_2 x$,

(4) if $d \parallel n$ and $d \geqslant \exp((\log_2 x)^{1/2})$, then $\Omega(f(d)) \leqslant 10\log_2 f(d)$,

(5) $(x_1(n; x), \dots, x_L(n; x)) \in \mathscr{S}_L(\boldsymbol{\xi})$,

(6) n has at least $L + 1$ odd prime divisors,

(7) $P^+(f(p_0)) \geqslant x^{1/(\log_2 x)}$, $p_1(n) < x^{1/(100\log_2 x)}$,

(8) $a_1x_1(n; x) + \cdots + a_Lx_L(n; x) \leqslant 1 - \omega$.

The following lemma asserts that a generic f-value has all of its preimage in $\mathscr{A}_f$.

Lemma 3.2. *For each $f \in \{\phi, \sigma\}$, the number of f-values $\leqslant x$ with a preimage $n \notin \mathscr{A}_f$ is*

$$\ll V_f(x)(\log_2 x)^{-1/2+\epsilon}.$$

Remarks. (i) The $\mathscr{A}_f$ not only satisfy Lemma 3.2 but do so economically. In fact, from condition (5) and the work of [Ford 1998a, §4], we have that $\#\mathscr{A}_f \ll V_f(x)$. Thus, on average, an element of $\mathscr{V}_f(x)$ has only a bounded number of preimages from $\mathscr{A}_f$. So when we turn in Sections 4 and 5 to counting ϕ-values arising from solutions to $\phi(a) = \sigma(a')$, with $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$, we expect not to be (excessively) overcounting.

(ii) Of the nine conditions defining $\mathscr{A}_f$, conditions (0)–(4) are, in the nomenclature of the introduction, purely *anatomical*, while conditions (5)–(8) depend to some degree on the fine structure theory of [Ford 1998a]. Condition (8) is particularly critical. It is (8) which ensures that the sieve bounds developed in Section 4 result in a nontrivial estimate for $V_{\phi,\sigma}(x)$. Our inability to replace the exponent $\frac{1}{2}$ on $\log_2 x$ in Lemma 3.2 (or in Theorem 1.1) by a larger number is also rooted in (8).

Proof. It is clear that the number of values of $f(n)$ corresponding to n failing (0) or (1) is $\ll x\log_2 x/\log x$, which (recalling (1-1)) is permissible for us. By Lemma 2.4 and our choice of S, the number of values of $f(n)$ coming from n failing (2) is $\ll V_f(x)/\log_2 x$. The same holds for values coming from n failing (3), by Lemma 2.3.

Suppose now that n fails condition (4). Then n has a unitary divisor $d \geqslant \exp((\log_2 x)^{1/2})$ with $\Omega(f(d)) \geqslant 10\log_2 f(d)$. Put $w := f(d)$. Then $w \mid f(n)$, and

$f(n) \ll x \log_2 x$. So if $w \geqslant x^{1/2}$, then the number of possibilities for $f(n)$ is

$$\ll x \log_2 x \sum_{\substack{w \geqslant x^{1/2} \\ \Omega(w) \geqslant 10 \log_2 w}} \frac{1}{w} \ll \frac{x \log_2 x}{\log x},$$

using Lemma 2.3 to estimate sum over w. If $w \leqslant x^{1/2}$, we observe that $f(n)/w = f(n/d) \in \mathscr{V}_f$; hence, with $Y(x)$ defined as in (2-3), the number of corresponding values of $f(n)$ is

$$\ll \sum_{\substack{\exp((\log_2 x)^{1/3}) \leqslant w \leqslant x^{1/2} \\ \Omega(w) \geqslant 10 \log_2 w}} V_f(x/w) \ll \frac{x}{\log x} Y(x) \sum_{\substack{w \geqslant \exp((\log_2 x)^{1/3}) \\ \Omega(w) \geqslant 10 \log_2 w}} \frac{1}{w} \ll \frac{V_f(x)}{\log_2 x}.$$

By Lemma 2.6, the number of f-values with a preimage failing (5) is

$$\ll \frac{V_f(x)}{\log_2 x}.$$

According to Lemma 3.1, the number of f-values with a preimage satisfying (5) but not (6) is also $\ll V_f(x)/\log_2 x$.

Suppose now that n satisfies (0)–(6). In what follows, we write $x_i = x_i(n; x)$. From (5), we have $\xi_0 \geqslant a_1 x_1 + a_2 x_2 \geqslant (a_1 + a_2) x_2$, and so $x_2 \leqslant 0.8$. So from (3),

$$\frac{n}{p_0(n) p_1(n)} = p_2(n) p_3(n) \cdots \leqslant \exp(10(\log_2 x)(\log x)^{0.8}) < x^{1/100}. \tag{3-3}$$

In particular, $p_0 > x^{1/3} + 1$ and $f(p_0) > x^{1/3}$, so that $v := f(p_1 p_2 \cdots) \leqslant x^{2/3}$. The prime p_0 satisfies $f(p_0) \leqslant x/v$. For z with $x^{1/3} < z \leqslant x$, the number of primes p_0 with $f(p_0) \leqslant z$ and $P^+(f(p_0)) \leqslant x^{1/\log_2 x}$ is (crudely) bounded by $\Psi(z, x^{1/\log_2 x}) \ll z/(\log x)^2$, by Lemma 2.2. So the number of values of $f(n)$ coming from n with $P^+(f(p_0)) \leqslant x^{1/\log_2 x}$ is

$$\ll \sum_{\substack{v \leqslant x^{2/3} \\ v \in \mathscr{V}_f}} \sum_{\substack{p:\ f(p) \leqslant x/v \\ P^+(f(p)) \leqslant x^{1/\log_2 x}}} 1 \ll \frac{x}{(\log x)^2} \sum_{\substack{v \leqslant x^{2/3} \\ v \in \mathscr{V}_f}} \frac{1}{v} \ll \frac{x}{(\log x)^{2-\epsilon}}.$$

To handle the second condition in (7), observe that since $f(p_0) \leqslant x/v$, the prime number theorem (and the bound $v \leqslant x^{2/3}$) shows that given v, the number of possibilities for p_0 is $\ll x/(v \log x)$. Suppose that $p_1(n) > x^{1/(100 \log_2 x)}$. Then $x_1 = x_1(n; x) \geqslant 0.999$, and we conclude from $\sum_{i \geqslant 1} a_i x_i \leqslant \xi_0$ that either $x_2 \leqslant \varrho^{3/2}$ or $x_3 \leqslant \varrho^{5/2}$. Writing v_2 for $f(p_2 p_3 \cdots)$ and v_3 for $f(p_3 p_4 \cdots)$, we see that the number of such f-values is

$$\ll \frac{x}{\log x}\sum_{p_1}\frac{1}{p_1}\Bigg(\sum_{\substack{P^+(v_2)\\ \leqslant\exp((\log x)^{\varrho^{3/2}})}}\frac{1}{v_2}+\sum_{p_2}\frac{1}{p_2}\sum_{\substack{P^+(v_3)\\ \leqslant\exp((\log x)^{\varrho^{5/2}})}}\frac{1}{v_3}\Bigg)$$

$$\ll \frac{x}{\log x}\log_3 x\Big(Y(x)\Big(\frac{\log_3 x}{\log_2 x}\Big)^{1/2}+(\log_2 x)Y(x)\Big(\frac{\log_3 x}{\log_2 x}\Big)^{3/2}\Big)\ll \frac{V_f(x)}{(\log_2 x)^{1/2-\epsilon}},$$

using Lemma 2.11 to estimate the sums over v_2 and v_3.

Finally, we consider n for which (0)–(7) hold but where condition (8) fails. By Lemma 2.7, we can assume that $a_1x_1+\cdots+a_Lx_L<1+\omega$, since the number of exceptional f-values is

$$\ll V(x)\exp(-(\log_2 x)^{\epsilon/2})\ll \frac{V(x)}{\log_2 x}.$$

Thus,

$$1-\omega<a_1x_1+\cdots+a_Lx_L<1+\omega, \tag{3-4}$$

while by condition (I_1) in the definition of $\mathscr{S}_L(\boldsymbol{\xi})$, $a_1x_2+\cdots+a_{L-1}x_L\leqslant\xi_1x_1$. We claim that if J is fixed large enough depending on ϵ, then there is some $2\leqslant j\leqslant J$ with $x_j\leqslant\varrho^{j-\epsilon/3}$. If not, then for large enough J,

$$\xi_1x_1\geqslant\sum_{j=1}^{J-1}a_jx_{j+1}\geqslant\varrho^{1-\epsilon/3}(a_1\varrho+a_2\varrho^2+\cdots+a_J\varrho^J)>\varrho^{1-\epsilon/4}.$$

Thus, $x_1\geqslant\varrho^{1-\epsilon/4}\xi_1^{-1}\geqslant\varrho^{1-\epsilon/5}$, and so $\xi_0\geqslant\varrho^{-\epsilon/5}(a_1\varrho+a_2\varrho^2+\cdots+a_J\varrho^J)\geqslant\varrho^{-\epsilon/6}$, which is false. This proves the claim. We assume below that $j\in[2,J]$ is chosen as the smallest index with $x_j\leqslant\varrho^{j-\epsilon/3}$; by condition (1), this implies that all of $p_1,\dots,p_{j-1}$ appear to the first power in the prime factorization of n.

Now given $x_2,\dots,x_L$, we have from (3-4) that $x_1\in[\alpha,\alpha+2\omega]$ for a certain α. Thus,

$$\sum_{p_1}\frac{1}{p_1}\ll\omega\log_2 x=(\log_2 x)^{1/2+\epsilon/2}.$$

So the number of f-values that arise from n satisfying (0)–(7) but failing (8) is

$$\ll\frac{x}{\log x}\sum_{j=2}^{J}\sum_{p_1,\dots,p_{j-1}}\frac{1}{p_1\cdots p_{j-1}}\sum_{\substack{P^+(v)\leqslant\exp((\log x)^{\varrho^{j-\epsilon/3}})\\ v\in\mathscr{V}_f}}\frac{1}{v}$$

$$\ll\frac{x}{\log x}\sum_{j=2}^{J}(\log_2 x)^{j-3/2+\epsilon/2}Y(x)\Big(\frac{\log_3 x}{\log_2 x}\Big)^{-1+j-\epsilon/3}\ll V_f(x)(\log_2 x)^{-1/2+\epsilon}.$$

This completes the proof of Lemma 3.2. □

As a corollary of Lemma 3.2, we have that $V_{\phi,\sigma}(x)$ is bounded, up to an additive error of $\ll (V_\phi(x)+V_\sigma(x))/(\log_2 x)^{1/2-\epsilon}$, by the number of values $\phi(a)$ that appear in solutions to the equation

$$\phi(a)=\sigma(a'), \quad \text{where} \quad (a,a')\in\mathscr{A}_\phi\times\mathscr{A}_\sigma.$$

In Sections 4 and 5, we develop the machinery required to estimate the number of such values. Ultimately, we find that it is smaller than $(V_\phi(x)+V_\sigma(x))/(\log_2 x)^A$ for any fixed A, which immediately gives Theorem 1.1.

4. The fundamental sieve estimate

Lemma 4.1. *Let y be large, $k\geqslant 1$, $l\geqslant 0$, $30\leqslant S\leqslant v_k\leqslant v_{k-1}\leqslant\cdots\leqslant v_0=y$, and $u_j\leqslant v_j$ for $0\leqslant j\leqslant k-1$. Put $\delta=\sqrt{\log_2 S/\log_2 y}$, $\nu_j=\log_2 v_j/\log_2 y$, $\mu_j=\log_2 u_j/\log_2 y$. Suppose that d is a natural number for which $P^+(d)\leqslant v_k$. Moreover, suppose that both of the following hold:*

(a) *For $2\leqslant j\leqslant k-1$, either $(\mu_j,\nu_j)=(\mu_{j-1},\nu_{j-1})$ or $\nu_j\leqslant\mu_{j-1}-2\delta$. Also, $\nu_k\leqslant\mu_{k-1}-2\delta$.*

(b) *For $1\leqslant j\leqslant k-2$, we have $\nu_j>\nu_{j+2}$.*

The number of solutions of

$$(p_0-1)\cdots(p_{k-1}-1)fd=(q_0+1)\cdots(q_{k-1}+1)e\leqslant y \tag{4-1}$$

in $p_0,\dots,p_{k-1},q_0,\dots,q_{k-1},e,f$ satisfying

(i) *p_i and q_i are S-normal primes,*

(ii) *$u_i\leqslant P^+(p_i-1)$, $P^+(q_i+1)\leqslant v_i$ for $0\leqslant i\leqslant k-1$,*

(iii) *neither $\phi(\prod_{i=0}^{k-1}p_i)$ nor $\sigma(\prod_{i=0}^{k-1}q_i)$ is divisible by r^2 for a prime $r\geqslant v_k$,*

(iv) *$P^+(ef)\leqslant v_k$; $\Omega(f)\leqslant 4l\log_2 v_k$,*

(v) *p_0-1 has a divisor $\geqslant y^{1/2}$ which is composed of primes $>v_1$*

is

$$\ll\frac{y}{d}(c\log_2 y)^{6k}(k+1)^{\Omega(d)}(\log v_k)^{8(k+l)\log(k+1)+1}(\log y)^{-2+\sum_{i=1}^{k-1}a_i\nu_i+E},$$

where $E=\delta\sum_{i=2}^{k}(i\log i+i)+2\sum_{i=1}^{k-1}(\nu_i-\mu_i)$. Here c is an absolute positive constant.

Remarks. Since the lemma statement is very complicated, it may be helpful to elaborate on how it will be applied in Section 5 below. Given $(a,a')\in\mathscr{A}_\phi\times\mathscr{A}_\sigma$ satisfying $\phi(a)=\sigma(a')$, rewrite the corresponding equation in the form (1-4), with d, e, and f as in (1-5). (Here L is as in (3-1), and k, given more precisely in the next section, satisfies $k\approx L/2$.) We are concerned with counting the number of values $\phi(a)$ which arise from such solutions. We partition the solutions according to the value of d, which describes the contribution of the "tiny" primes to $\phi(a)$, and by the rough location of the primes p_i and q_i, which we encode in the selection of intervals $[u_i, v_i]$ (cf. Lemma 2.5). Finally, we apply Lemma 4.1 and sum over both d and the possible selections of intervals; this gives an estimate for the number of $\phi(a)$ which is smaller than $(V_\phi(x) + V_\sigma(x))/(\log_2 x)^A$, for any fixed A.

In our application, conditions (i)–(v) of Lemma 4.1 are either immediate from the definitions, or are readily deduced from the defining properties of $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$. Conditions (a) and (b) are rooted in the observation that while neighboring primes in the prime factorization of a (or a') may be close together (requiring us to allow $[u_{i+1}, v_{i+1}] = [u_i, v_i]$), the primes $p_i(a)$ and $p_{i+2}(a)$ are forced to be far apart on a double-logarithmic scale. Indeed, since $(x_1(a; x), \dots, x_L(a; x)) \in \mathscr{S}_L(\boldsymbol{\xi})$, Lemma 2.8 shows that $x_{i+2} < 3\varrho^2 x_i < 0.9 x_i$.

Proof. We consider separately the prime factors of each shifted prime lying in each interval $(v_{i+1}, v_i]$. For $0 \leqslant j \leqslant k-1$ and $0 \leqslant i \leqslant k$, let

$$s_{i,j}(n) = \prod_{\substack{p^a \| (p_j - 1) \\ p \leqslant v_i}} p^a, \qquad s'_{i,j}(n) = \prod_{\substack{p^a \| (q_j + 1) \\ p \leqslant v_i}} p^a, \qquad s_i = df \prod_{j=0}^{k-1} s_{i,j} = e \prod_{j=0}^{k-1} s'_{i,j}.$$

Also, for $0 \leqslant j \leqslant k-1$ and $1 \leqslant i \leqslant k$, let

$$t_{i,j} = \frac{s_{i-1,j}}{s_{i,j}}, \qquad t'_{i,j} = \frac{s'_{i-1,j}}{s'_{i,j}}, \qquad t_i = \prod_{j=0}^{k-1} t_{i,j} = \prod_{j=0}^{k-1} t'_{i,j}.$$

For each solution $\mathscr{A} = (p_0, \dots, p_{k-1}, f, q_0, \dots, q_{k-1}, e)$ of (4-1), let

$$\sigma_i(\mathscr{A}) = \{s_i; s_{i,0}, \dots, s_{i,k-1}, f; s'_{i,0}, \dots, s'_{i,k-1}, e\},$$
$$\tau_i(\mathscr{A}) = \{t_i; t_{i,0}, \dots, t_{i,k-1}, 1; t'_{i,0}, \dots, t'_{i,k-1}, 1\}.$$

Defining multiplication of $(2k + l + 2)$-tuples component-wise, we have

$$\sigma_{i-1}(\mathscr{A}) = \sigma_i(\mathscr{A}) \tau_i(\mathscr{A}). \tag{4-2}$$

Let $\mathfrak{S}_i$ denote the set of $\sigma_i(\mathscr{A})$ arising from solutions $\mathscr{A}$ of (4-1) and $\mathfrak{T}_i$ the corresponding set of $\tau_i(\mathscr{A})$. By (4-2), the number of solutions of (4-1) satisfying the required conditions is

$$|\mathfrak{S}_0| = \sum_{\sigma_1 \in \mathfrak{S}_1} \sum_{\substack{\tau_1 \in \mathfrak{T}_1 \\ \sigma_1 \tau_1 \in \mathfrak{S}_0}} 1. \tag{4-3}$$

First, fix $\sigma_1 \in \mathfrak{S}_1$. By assumption (v) in the lemma, $t_{1,0} \geqslant y^{1/2}$. Also,

$$t_1 = t_{1,0} = t'_{1,0} \leqslant y/s_1,$$

t_1 is composed of primes $> v_1$, and $s_{1,0}t_1+1$ and $s'_{1,0}t_1-1$ are prime. Write $t_1=t'_1Q$, where $Q=P^+(t_1)$. Since p_0 is an S-normal prime, (2-2) gives that

$$Q \geqslant t_1^{1/\Omega(t_1)} \geqslant t_1^{1/\Omega(p_0-1)} \geqslant y^{1/(2\Omega(p_0-1))} \geqslant y^{1/(6\log_2 y)},$$

Given t'_1, Lemma 2.1 implies that the number of Q is $O(y(\log_2 y)^6/(s_1t'_1\log^3 y))$. Moreover,

$$\sum \frac{1}{t'_1} \leqslant \prod_{v_1<p\leqslant y}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) \ll \frac{\log y}{\log v_1}=(\log y)^{1-\nu_1}.$$

Consequently, for each $\sigma_1\in\mathfrak{S}_1$,

$$\sum_{\substack{\tau_1\in\mathfrak{T}_1\\ \sigma_1\tau_1\in\mathfrak{S}_0}} 1 \ll \frac{y(\log_2 y)^6}{s_1(\log y)^{2+\nu_1}}. \tag{4-4}$$

Next, suppose $2\leqslant i\leqslant k$. We now apply an iterative procedure: If $v_i<v_{i-1}$, we use the identity

$$\sum_{\sigma_{i-1}\in\mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} = \sum_{\sigma_i\in\mathfrak{S}_i}\frac{1}{s_i}\sum_{\substack{\tau_i\in\mathfrak{T}_i\\ \sigma_i\tau_i\in\mathfrak{S}_{i-1}}}\frac{1}{t_i}. \tag{4-5}$$

If $v_i=v_{i-1}$, then (4-5) remains true but contains no information, and in this case we use the alternative identity

$$\sum_{\sigma_{i-1}\in\mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} = \sum_{\sigma_{i+1}\in\mathfrak{S}_{i+1}}\frac{1}{s_{i+1}}\sum_{\substack{\tau_{i+1}\in\mathfrak{T}_{i+1}\\ \sigma_{i+1}\tau_{i+1}\in\mathfrak{S}_{i-1}}}\frac{1}{t_{i+1}}. \tag{4-6}$$

We consider first the simpler case when $v_i<v_{i-1}$. Suppose $\sigma_i\in\mathfrak{S}_i$, $\tau_i\in\mathfrak{T}_i$ and $\sigma_i\tau_i\in\mathfrak{S}_{i-1}$. By assumption (ii), $t_i=t_{i,0}\cdots t_{i,i-1}=t'_{i,0}\cdots t'_{i,i-1}$. In addition, $s_{i,i-1}t_{i,i-1}+1=p_{i-1}$ and $s'_{i,i-1}t'_{i,i-1}-1=q_{i-1}$ are prime. Let $Q:=P^+(t_{i,i-1})$, $Q':=P^+(t'_{i,i-1})$, $b:=t_{i,i-1}/Q$ and $b':=t'_{i,i-1}/Q'$.

We consider separately $\mathfrak{T}_{i,1}$, the set of τ_i with $Q=Q'$ and $\mathfrak{T}_{i,2}$, the set of τ_i with $Q\neq Q'$. First,

$$\Sigma_1 := \sum_{\substack{\tau_i\in\mathfrak{T}_{i,1}\\ \sigma_i\tau_i\in\mathfrak{S}_{i-1}}}\frac{1}{t_i} \leqslant \sum_t \frac{h(t)}{t}\max_{b,b'}\sum_Q\frac{1}{Q},$$

where $h(t)$ denotes the number of solutions of $t_{i,0}\cdots t_{i,i-2}b=t=t'_{i,0}\cdots t'_{i,i-2}b'$, and in the sum on Q, $s_{i,i-1}bQ+1$ and $s'_{i,i-1}b'Q-1$ are prime. By Lemma 2.1, the number of $Q\leqslant z$ is $\ll z(\log z)^{-3}(\log_2 y)^3$ uniformly in b, b'. By partial summation,

$$\sum_{Q\geqslant u_{i-1}}\frac{1}{Q} \ll (\log_2 y)^3(\log y)^{-2\mu_{i-1}}.$$

Also, $h(t)$ is at most the number of dual factorizations of t into i factors each, that is, $h(t) \leqslant i^{2\Omega(t)}$. By (2-1), $\Omega(t) \leqslant i(\nu_{i-1} - \nu_i + \delta)\log_2 y =: I$. Also, by assumption (iii), t is squarefree. Thus,

$$\sum_t \frac{h(t)}{t} \leqslant \sum_{j \leqslant I} \frac{i^{2j} H^j}{j!},$$

where

$$\sum_{\nu_i < p \leqslant \nu_{i-1}} \frac{1}{p} \leqslant (\nu_{i-1} - \nu_i)\log_2 y + 1 =: H.$$

By assumption (a), $\nu_{i-1} - \nu_i \geqslant 2\delta$, hence $I \leqslant \frac{3}{2} i H \leqslant \frac{3}{4} i^2 H$. Hence,

$$\sum_t \frac{h(t)}{t} \leqslant \left(\frac{i^2 H}{I}\right)^I \sum_{j \leqslant I} \frac{I^j}{j!} < i^I \exp(I) = (\log y)^{(i + i\log i)(\nu_{i-1} - \nu_i + \delta)}. \tag{4-7}$$

This gives

$$\Sigma_1 \ll (\log_2 y)^3 (\log y)^{-2\mu_{i-1} + (i + i\log i)(\nu_{i-1} - \nu_i + \delta)}.$$

For the sum over $\mathfrak{T}_{i,2}$, set $t_i = tQQ'$. Note that $tQ' = t_{i,0} \cdots t_{i,i-2} b$ and $tQ = t'_{i,0} \cdots t'_{i,i-2} b'$, so $Q \mid t'_{i,0} \cdots t'_{i,i-2} b'$ and $Q' \mid t_{i,0} \cdots t_{i,i-2} b$. If we fix the factors divisible by Q and by Q', then the number of possible ways to form t is $\leqslant i^{2\Omega(t)}$ as before. Then

$$\Sigma_2 := \sum_{\substack{\tau_i \in \mathfrak{T}_{i,2} \\ \sigma_i \tau_i \in \mathfrak{S}_{i-1}}} \frac{1}{t_i} \leqslant \sum_t \frac{i^{2\Omega(t)+2}}{t} \max_{b,b'} \sum_{Q,Q'} \frac{1}{QQ'},$$

where $s_{i,i-1} bQ + 1$ and $s'_{i,i-1} b'Q' - 1$ are prime. By Lemma 2.1, the number of $Q \leqslant z$ (respectively $Q' \leqslant z$) is $\ll z(\log z)^{-2}(\log_2 y)^2$. Hence,

$$\sum_{Q,Q'} \frac{1}{QQ'} \ll (\log_2 y)^4 (\log y)^{-2\mu_{i-1}}.$$

Combined with (4-7), this gives $\Sigma_2 \ll i^2 (\log_2 y)^4 (\log y)^{-2\mu_{i-1} + (i + i\log i)(\nu_{i-1} - \nu_i + \delta)}$. From (a) and (b), $i^2 \leqslant k^2 \leqslant (\log_2 y)^2$. Adding Σ_1 and Σ_2 shows that for each σ_i,

$$\sum_{\substack{\tau_i \in \mathfrak{T}_i \\ \sigma_i \tau_i \in \mathfrak{S}_{i-1}}} \frac{1}{t_i} \ll (\log_2 y)^6 (\log y)^{-2\mu_{i-1} + (i\log i + i)(\nu_{i-1} - \nu_i + \delta)}. \tag{4-8}$$

We consider now the case when $\nu_i = \nu_{i-1}$. Set $Q_1 := P^+(t_{i+1,i-1})$, $Q_2 := P^+(t_{i+1,i})$, $Q_3 := P^+(t'_{i+1,i-1})$, and $Q_4 := P^+(t'_{i+1,i})$. From (iii), we have that $Q_1 \neq Q_2$ and $Q_3 \neq Q_4$. Moreover, letting b_i denote the cofactor of Q_i in each case, we have that

$$\begin{aligned} s_{i+1,i-1}b_1Q_1+1&=p_{i-1}, & s'_{i+1,i-1}b_3Q_3-1&=q_{i-1},\\ s_{i+1,i}b_2Q_2+1&=p_i, & s'_{i+1,i}b_4Q_4-1&=q_i. \end{aligned} \tag{4-9}$$

Since there are now several ways in which the various Q_i may coincide, the combinatorics is more complicated than in the case when $v_i < v_{i-1}$. We index the cases by fixing the incidence matrix (δ_{ij}) with $\delta_{ij}=1$ if $Q_i=Q_j$ and $\delta_{ij}=0$ otherwise.

Write $D=\gcd(Q_1Q_2, Q_3Q_4)$, and let $Q:=Q_1Q_2/D$ and $Q':=Q_3Q_4/D$, so that D, Q, and Q' are formally determined by (δ_{ij}). Then $QQ'\mid t_{i+1}$, and writing $t_{i+1}/D=tQQ'$, we have

$$tQ=t_{i+1,0}t_{i+1,1}\cdots t_{i+1,i-2}b_3b_4, \tag{4-10}$$

$$tQ'=t'_{i+1,0}t'_{i+1,1}\cdots t'_{i+1,i-2}b_1b_2. \tag{4-11}$$

We now choose which terms on the right-hand sides of (4-10) and (4-11) contain the prime factors of Q and Q', respectively; since $\Omega(Q)\leqslant 2$ and $\Omega(Q')\leqslant 2$, this can be done in at most $(i+1)^4$ ways. Having made this choice, the number of ways to form t is bounded by $(i+1)^{2\Omega(t)}$, and so

$$\sum_{\substack{\tau_{i+1}\in\mathfrak{T}_{i+1}\\ \sigma_{i+1}\tau_{i+1}\in\mathfrak{S}_{i-1}}}\frac{1}{t_{i+1}}\leqslant\sum_t\frac{(i+1)^{2\Omega(t)+4}}{t}\max_{b_1,b_2,b_3,b_4}\sum\frac{1}{DQQ'}. \tag{4-12}$$

It is easy to check that $DQQ'=\prod_{j\in\mathscr{J}}Q_j$, where $\mathscr{J}$ indexes the distinct Q_j. For each $j\in\mathscr{J}$, let n_j be the number of linear forms appearing in (4-9) involving Q_j. Since each of these n_j linear forms in Q_j is prime, as is Q_j itself, Lemma 2.1 implies that the number of possibilities for $Q_j\leqslant z$ is $\ll z(\log z)^{-n_j-1}(\log_2 y)^{n_j+1}$, and so

$$\sum_{Q_j\geqslant u_{i-1}}\frac{1}{Q_j}\ll(\log_2 y)^{n_j+1}(\log u_{i-1})^{-n_j}\ll(\log_2 y)^{n_j+1}(\log y)^{-n_j\mu_{i-1}},$$

uniformly in the choice of the b's. Since $\sum_{j\in\mathscr{J}}n_j=4$ and $\sum_{j\in\mathscr{J}}1\leqslant 4$,

$$\sum\frac{1}{DQQ'}\leqslant\prod_{j\in\mathscr{J}}\biggl(\sum_{Q_j\geqslant u_{i-1}}\frac{1}{Q_j}\biggr)\ll(\log_2 y)^8(\log y)^{-4\mu_{i-1}}. \tag{4-13}$$

The calculation (4-7), with i replaced by $i+1$, shows that

$$\sum_t\frac{(i+1)^{2\Omega(t)}}{t}\leqslant(\log y)^{((i+1)+(i+1)\log(i+1))(\nu_i-\nu_{i+1}+\delta)}. \tag{4-14}$$

Combining (4-12), (4-13), and (4-14) shows that

$$\sum_{\substack{\tau_{i+1}\in\mathfrak{T}_{i+1}\\ \sigma_{i+1}\tau_{i+1}\in\mathfrak{S}_{i-1}}} \frac{1}{t_{i+1}} \leqslant (i+1)^4(\log_2 y)^8(\log y)^{-4\mu_{i-1}+((i+1)+(i+1)\log(i+1))(\nu_i-\nu_{i+1}+\delta)}$$
$$\leqslant (\log_2 y)^{12}(\log y)^{-2\mu_{i-1}+(i\log i+i)(\nu_i-\nu_{i-1})-2\mu_i+((i+1)\log(i+1)+(i+1))(\nu_{i+1}-\nu_i+\delta)}, \tag{4-15}$$

where in the last line we use that $v_{i-1}=v_i$ and $(i+1)^4 \leqslant k^4 \leqslant (\log_2 y)^4$.

Using (4-3), (4-5), and (4-6) together with the inequalities (4-4), (4-8), and (4-15), we find that the number of solutions of (4-1) is

$$\ll y(c\log_2 y)^{6k}(\log y)^{-2-\nu_1+\sum_{i=2}^{k}(\nu_{i-1}-\nu_i+\delta)(i\log i+i)-2\mu_{i-1}} \sum_{\sigma_k\in\mathfrak{S}_k}\frac{1}{s_k},$$

where c is some positive constant. Note that the exponent of $\log y$ is

$$\leqslant -2+\sum_{i=1}^{k-1} a_i\nu_i + E.$$

It remains to treat the sum on σ_k. Given $s_k' = s_k/d$, the number of possible σ_k is at most the number of factorizations of s_k' into $k+1$ factors times the number of factorizations of ds_k' into $k+1$ factors, which is at most $(k+1)^{\Omega(ds_k')}(k+1)^{\Omega(s_k')}$. By assumptions (i) and (iv), $\Omega(s_k') \leqslant 4(k+l)\log_2 v_k$. Thus,

$$\sum_{\sigma_k\in\mathfrak{S}_k}\frac{1}{s_k} \leqslant \frac{(k+1)^{\Omega(d)}(k+1)^{8(k+l)\log_2 v_k}}{d}\sum_{P^+(s_k')\leqslant v_k}\frac{1}{s_k'}$$
$$\ll \frac{(k+1)^{\Omega(d)}(\log v_k)^{8(k+l)\log(k+1)+1}}{d}. \qquad \square$$

5. Counting common values: Application of Lemma 4.1

In this section we prove the following proposition, which combined with Lemma 3.2 immediately yields Theorem 1.1. Throughout the rest of this paper, we adopt the definitions of L, the ξ_i, S, δ, and ω from (3-1) and (3-2).

Proposition 5.1. *Fix $A>0$. For large x, the number of distinct values of $\phi(a)$ that arise from solutions to the equation*

$$\phi(a)=\sigma(a'), \quad \textit{with} \quad (a,a')\in\mathscr{A}_\phi\times\mathscr{A}_\sigma,$$

is smaller than $(V_\phi(x)+V_\sigma(x))/(\log_2 x)^A$.

Let us once again recall the strategy outlined in the introduction and in the remarks following Lemma 4.1. Let $(a,a')\in\mathscr{A}_\phi\times\mathscr{A}_\sigma$ be a solution to $\phi(a)=\sigma(a')$. Let $p_i := p_i(a)$ and $q_i := p_i(a')$, in the notation of Section 2B. We choose a cutoff

k so that all of $p_0, \dots, p_{k-1}$ and $q_0, \dots, q_{k-1}$ are "large". Then by condition (1) in the definition of the sets $\mathscr{A}_f$, neither $p_i^2 \mid a$ nor $q_i^2 \mid a'$, for $0 \leqslant i \leqslant k-1$. Fixing a notion of "small" and "tiny", we rewrite the equation $\phi(a) = \sigma(a')$ in the form

$$(p_0 - 1) \cdots (p_{k-1} - 1) f d = (q_0 + 1) \cdots (q_{k-1} + 1) e, \tag{5-1}$$

where f is the contribution to $\phi(a)$ from the "small" primes, d is the contribution from the "tiny" primes, and e is the contribution of both the "small" and "tiny" primes to $\sigma(a')$.

We then fix d and numbers u_i and v_i, chosen so that $u_i \leqslant P^+(p_i - 1)$, $P^+(q_i + 1) \leqslant v_i$ for each $0 \leqslant i \leqslant k-1$. With these fixed, Lemma 4.1 provides us with an upper bound on the number of corresponding solutions to (5-1). Such a solution determines the common value $\phi(a) = \sigma(a') \in \mathscr{V}_\phi \cap \mathscr{V}_\sigma$. We complete the proof of Proposition 5.1 by summing the upper bound estimates over all choices of d and all selections of the u_i and v_i.

We carry out this plan in four stages, each of which is treated in more detail below:

- Finalize the notions of "small" and "tiny", and so also the choices of d, e, and f.
- Describe how to choose the u_i and v_i so that the intervals $[u_i, v_i]$ capture $P^+(p_i - 1)$ and $P^+(q_i + 1)$ for all $0 \leqslant i \leqslant k-1$.
- Check that the hypotheses of Lemma 4.1 are satisfied.
- Take the estimate of Lemma 4.1 and sum over d and the choices of u_i and v_i.

5A. *"Small" and "tiny".* Suppose we are given a solution $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$ to $\phi(a) = \sigma(a')$. Set $x_j = x_j(a; x)$ and $y_j = x_j(a'; x)$, in the notation of Section 2B, so that (from the definition of $\mathscr{A}_f$) the sequences $\mathbf{x} = (x_1, \dots, x_L)$ and $\mathbf{y} = (y_1, \dots, y_L)$ belong to $\mathscr{S}_L(\boldsymbol{\xi})$.

Lemma 5.2. *With $\{z_j\}_{j=1}^{L}$ denoting either of the sequences $\{x_j\}$ or $\{y_j\}$, we have*

(i) $z_j < 3\varrho^j$ *for* $1 \leqslant j \leqslant L$,

(ii) $z_{L-j} \geqslant \frac{3}{100}\varrho^{-j} / \log_2 x$ *for* $0 \leqslant j < L$.

(iii) $z_{j+2} \leqslant 0.9 z_j$ *for* $1 \leqslant j \leqslant L-2$.

Proof. Claim (i) is repeated verbatim from Lemma 2.8. By the same lemma, $z_j \leqslant 3\varrho^{j-i} z_i$ for $1 \leqslant i < j \leqslant L$. This immediately implies (iii), since $\varrho^2 < 0.9$. Moreover, fixing $j = L$, condition (6) in the definition of $\mathscr{A}_f$ gives that

$$z_i \geqslant \tfrac{1}{3}\varrho^{i-L} z_L \geqslant \frac{\log_2 3}{3} \varrho^{i-L} / \log_2 x > \tfrac{3}{100}\varrho^{i-L} / \log_2 x,$$

which is (ii) up to a change of variables. $\square$

Lemma 5.3. *The minimal index* $k_0 \leqslant L$ *for which*

$$\log_2 P^+(p_{k_0} - 1) < (\log_2 x)^{1/2+\epsilon/10}$$

satisfies $k_0 \sim (1/2 - \epsilon/10)L$ *as* $x \to \infty$.

Proof. Lemma 5.2(i) shows that the least K with $\log_2 p_K < (\log_2 x)^{1/2+\epsilon/10}$ satisfies $K \leqslant (1/2 - \epsilon/10 + o(1))L$, as $x \to \infty$. Since $\log_2 P^+(p_K - 1) \leqslant \log_2 p_K$, this gives the asserted upper bound on k_0. The lower bound follows in a similar fashion from Lemma 5.2(ii) and Lemma 2.5. □

Recall the definition of δ from (3-2), and put

$$\eta := 10L\delta, \quad \text{so that} \quad \eta \asymp (\log_3 x)^{3/2}(\log_2 x)^{-1/2}.$$

We choose our "large"/"small" cutoff point k by taking $k = k_0$ if $x_{k_0-1} - x_{k_0} \geqslant 20\eta$, and taking $k = k_0 - 1$ otherwise. For future use, we note that with this choice of k,

$$x_{k-1} - x_k \geqslant 20\eta. \tag{5-2}$$

This inequality is immediate if $k = k_0$; in the opposite case, by Lemma 5.2(iii),

$$\begin{aligned} x_{k-1} - x_k = x_{k_0-2} - x_{k_0-1} &\geqslant x_{k_0-2} - x_{k_0} - 20\eta \\ &\geqslant 0.1x_{k_0-2} - 20\eta \geqslant 0.1(\log_2 x)^{-1/2+\epsilon/10} - 20\eta > 20\eta. \end{aligned}$$

Note that with this choice of k, we have $\log_2 p_i > (\log_2 x)^{1/2+\epsilon/10}$ for $0 \leqslant i \leqslant k-1$, and so condition (1) in the definition of $\mathscr{A}_\phi$ guarantees that each p_i divides a to the first power only, for $0 \leqslant i \leqslant k-1$. Moreover, from Lemmas 5.2(ii) and 5.3, we have $\log_2 q_i > (\log_2 x)^{1/2+\epsilon/11}$ for $0 \leqslant i \leqslant k-1$. So each q_i divides a' only to the first power, for $0 \leqslant i \leqslant k-1$. Now take

$$f := \phi(p_k p_{k+1} \cdots p_{L-1}), \quad d := \begin{cases} \phi(p_L p_{L+1} \cdots) & \text{if } p_{L-1} \neq p_L, \\ \frac{p_L}{\phi(p_L)} \phi(p_L p_{L+1} \cdots) & \text{if } p_{L-1} = p_L, \end{cases} \tag{5-3}$$

and

$$e := \sigma(q_k q_{k+1} \cdots),$$

and observe that (5-1) holds.

5B. ***Selection of the u_j and v_j.*** Rather than choose the u_j and v_j directly, it is more convenient to work with the μ_j and ν_j; then u_j and v_j are defined by $u_j := \exp((\log x)^{\mu_j})$ and $v_j := \exp((\log x)^{\nu_j})$. Put

$$\zeta_0 := 1 - \frac{\log_3 x + \log 100}{\log_2 x} \quad \text{and} \quad \zeta_j := \zeta_0 - j\eta \quad (j \geqslant 1), \tag{5-4}$$

and note that with $\nu_0 := 1$ and $\mu_0 := \zeta_0$, we have

$$u_0 = x^{1/(100 \log_2 x)} < x^{1/\log_2 x} \leqslant P^+(p_0 - 1),\ P^+(q_0 + 1) \leqslant x = v_0,$$

by condition (7) in the definitions of $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$. To choose the remaining μ_j and ν_j, it is helpful to know that p_j and q_j are close together (renormalized on a double logarithmic scale) for $1 \leqslant j \leqslant k$. This is the substance of the following lemma.

Lemma 5.4. *If $p_j \geqslant S$ and $q_j \geqslant S$, then $|x_j - y_j| \leqslant (2j+1)\delta < \eta$. These hypotheses hold if $L - j \geqslant 2C \log_4 x + 12$, and so in particular for $1 \leqslant j \leqslant k$.*

Proof. Suppose for the sake of contradiction that $y_j \geqslant x_j + (2j+1)\delta$; since the p_i and q_i are all S-normal, this would imply that

$$(j+1)(y_j - x_j - \delta) \leqslant \frac{\Omega(\sigma(a'), p_j, q_j)}{\log_2 x} = \frac{\Omega(\phi(a), p_j, q_j)}{\log_2 x} \leqslant j(y_j - x_j + \delta),$$

which is false. We obtain a similar contradiction if we suppose $x_j \geqslant y_j + (2j+1)\delta$. The second half of the lemma follows from Lemma 5.2 and a short calculation, together with the estimate $k \sim (1/2 - \epsilon/10)L$ of Lemma 5.3. □

We choose the intervals $[\mu_j, \nu_j]$ for $1 \leqslant j \leqslant k-1$ successively, starting with $j = 1$. (We select ν_k last, by a different method.) Say that the pair $\{x_j, x_{j+1}\}$ is *well-separated* if $x_j - x_{j+1} \geqslant 10\eta$, and *poorly separated* otherwise.

In the well-separated case, among all ζ_i (with $i \geqslant 0$), choose ζ minimal and ζ' maximal with

$$\zeta' \log_2 x \leqslant \log_2 \min\{P^+(p_j - 1), P^+(q_j + 1)\} \\ \leqslant \log_2 \max\{P^+(p_j - 1), P^+(q_j + 1)\} \leqslant \zeta \log_2 x,$$

and put

$$\mu_j := \zeta, \qquad \nu_j := \zeta'.$$

In the poorly separated case, $j < k-1$, by (5-2). We select $[\mu_j, \nu_j] = [\mu_{j+1}, \nu_{j+1}]$ by a similar recipe: Among all ζ_i (with $i \geqslant 0$), choose ζ minimal and ζ' maximal with

$$\zeta' \log_2 x \leqslant \log_2 \min\{P^+(p_j - 1), P^+(q_j + 1), P^+(p_{j+1} - 1), P^+(q_{j+1} + 1)\} \\ \leqslant \log_2 \max\{P^+(p_j - 1), P^+(q_j + 1), P^+(p_{j+1} - 1), P^+(q_{j+1} + 1)\} \leqslant \zeta \log_2 x,$$

and put

$$\nu_j = \nu_{j+1} = \zeta, \quad \text{and} \quad \mu_j = \mu_{j+1} = \zeta'.$$

To see that these choices are well-defined, note that by (7) in the definition of $\mathscr{A}_f$, we have $x_j, y_j \leqslant \zeta_0$, which implies that a suitable choice of ζ above exists in both cases. Also, for $1 \leqslant i \leqslant k$, we have $x_i, y_i \geqslant (\log_2 x)^{-1/2+\epsilon/11}$ (by Lemma 5.3 and 5.2(ii)). So by Lemma 2.5,

$$\log_2 \min\{P^+(p_i - 1), P^+(q_i + 1)\} / \log_2 x \geqslant (\log_2 x)^{-1/2+\epsilon/12},$$

say. Since neighboring ζ_i are spaced at a distance $\eta \asymp (\log_2 x)^{-1/2} (\log_3 x)^{3/2}$, a suitable choice of ζ' also exists in both cases.

For our application of Lemma 4.1, it is expedient to keep track at each step of the length of the intervals $[\mu_j, \nu_j]$, as well as the distance between the left-endpoint of the last interval chosen and the right-endpoint of the succeeding interval (if any). In the well-separated case, Lemmas 5.4 and 2.5 show that

$$\nu_j \leqslant \max\{x_j, y_j\} + \eta \leqslant x_j + 2\eta,$$

while

$$\mu_j \geqslant \min\{x_j, y_j\} - \frac{\log_3 x + \log 4}{\log_2 x} - \eta \geqslant x_j - 3\eta, \tag{5-5}$$

so that $\nu_j - \mu_j \leqslant 5\eta$. Also, if a succeeding interval exists (so that $j+1 \leqslant k-1$), then $\nu_{j+1} \leqslant \max\{x_{j+1}, y_{j+1}\} + \eta \leqslant x_{j+1} + 2\eta$, and the separation between μ_j and ν_{j+1} satisfies the lower bound

$$\mu_j - \nu_{j+1} \geqslant x_j - x_{j+1} - 5\eta \geqslant 5\eta. \tag{5-6}$$

In the poorly separated case, we have

$$\nu_j \leqslant \max\{x_j, y_j, x_{j+1}, y_{j+1}\} + \eta = \max\{x_j, y_j\} + \eta \leqslant x_j + 2\eta,$$

as before, but the lower bound on μ_j takes a slightly different form;

$$\begin{aligned}\mu_j \geqslant \min\{x_j, y_j, x_{j+1}, y_{j+1}\} &- \frac{\log_3 x + \log 4}{\log_2 x} - \eta \\ \geqslant (x_{j+1} - \eta) &- \frac{\log_3 x + \log 4}{\log_2 x} - \eta \geqslant x_{j+1} - 3\eta \geqslant x_j - 13\eta,\end{aligned} \tag{5-7}$$

so that $\nu_j - \mu_j \leqslant 15\eta$. In this case, since $\nu_j = \nu_{j+1}$ and $\mu_j = \mu_{j+1}$, the succeeding interval (if it exists) is $[\mu_{j+2}, \nu_{j+2}]$. By Lemma 5.2(iii),

$$x_j - x_{j+2} \geqslant 0.1 x_j \geqslant 0.1 (\log_2 x)^{-1/2+\epsilon/10} > 20\eta,$$

say. Thus, $\nu_{j+2} \leqslant \max\{x_{j+2}, y_{j+2}\} + \eta \leqslant x_{j+2} + 2\eta \leqslant x_j - 18\eta$, and so

$$\mu_{j+1} - \nu_{j+2} = \mu_j - \nu_{j+2} \geqslant (x_j - 13\eta) - (x_j - 18\eta) \geqslant 5\eta. \tag{5-8}$$

At this point we have selected intervals $[\mu_j, \nu_j]$, for all $0 \leqslant j \leqslant k-1$. We choose $\nu_k = \zeta$, where ζ is the minimal ζ_i satisfying $\zeta \geqslant x_k + \eta$. Note that

$$\log_2 S / \log_2 x = 36 \log_3 x / \log_2 x < (\log_2 x)^{-1/2+\epsilon/11} \leqslant x_k < \zeta = \nu_k \leqslant x_k + 2\eta.$$

Thus, $\upsilon_k > S$. From (5-5) and (5-7), $\mu_{k-1} \geqslant x_{k-1} - 3\eta$, so that also

$$\mu_{k-1} - \nu_k \geqslant x_{k-1} - x_k - 5\eta \geqslant 15\eta, \tag{5-9}$$

where the last estimate uses (5-2).

5C. ***Verification of hypotheses.*** We now check that Lemma 4.1 may be applied with $y = x$. By construction, $S \leqslant v_k \leqslant v_{k-1} \leqslant \cdots \leqslant v_0 = x$, and $u_i \leqslant v_i$ for all $0 \leqslant i \leqslant k-1$. Moreover, if $[\mu_j, \nu_j] \neq [\mu_{j-1}, \nu_{j-1}]$ (where $2 \leqslant j \leqslant k-1$), then from (5-6) and (5-8), $\mu_{j-1} - \nu_j \geqslant 5\eta = 50L\delta > 2\delta$, and from (5-9), $\mu_{k-1} - \nu_k \geqslant 15\eta > 2\delta$. Thus, condition (a) of Lemma 4.1 is satisfied. It follows from our method of selecting the μ_j and ν_j that if $\nu_j = \nu_{j+1}$, then (again by (5-8)) $\nu_{j+2} \leqslant \mu_{j+1} - 5\eta < \nu_{j+1} = \nu_j$, which shows that condition (b) is also satisfied. Moreover, since $\nu_k > x_k$, we have $P^+(d) \leqslant p_L \leqslant p_k < v_k$. So we may focus our attention on hypotheses (i)–(v) of Lemma 4.1. We claim that these hypotheses are satisfied with our choices of d, e, and f from Section 5A and with

$$l := L - k. \tag{5-10}$$

Property (i) is contained in (2) from the definition of $\mathscr{A}_f$. By construction,

$$u_i \leqslant P^+(p_i - 1),\ P^+(q_i + 1) \leqslant v_i$$

for all $0 \leqslant i \leqslant k-1$, which is (ii). Since $v_k \geqslant S > \log y$, property (iii) holds by (1) in the definition of $\mathscr{A}_f$. The verification of (iv) is somewhat more intricate. Recalling that $\nu_k > x_k$, it is clear from (5-3) that

$$P^+(f) < p_k \leqslant v_k.$$

To prove the same estimate for $P^+(e)$, we can assume $e \neq 1$. Let $r = P^+(e)$, and observe that $r \mid \sigma(R)$, for some prime power R with $R \parallel q_k q_{k+1} \cdots$. If R is a proper prime power, then from (1) in the definition of $\mathscr{A}_f$, we have

$$r \leqslant \sigma(R) \leqslant 2R \leqslant 2(\log x)^2 < v_k.$$

So we can assume that R is prime, and so $R \leqslant q_k$ and $r \leqslant P^+(R+1) \leqslant \max\{3, R\} \leqslant q_k$. But by Lemma 5.4,

$$\log_2 q_k / \log_2 x = y_k \leqslant x_k + (2k+1)\delta < x_k + \eta \leqslant \nu_k.$$

Thus, $P^+(e) = r \leqslant v_k$. Hence, $P^+(ef) \leqslant v_k$. Turning to the second half of (iv), write $p_k \cdots p_{L-1} = AB$, where A is squarefree, B is squarefull and $\gcd(A, B) = 1$. Recalling (2-2), we see that

$$\Omega(\phi(A)) \leqslant 3\Omega(A) \log_2 v_k \leqslant 3l \log_2 v_k,$$

with l as in (5-10). Let B' be the largest divisor of a supported on the primes dividing B, so that B' is squarefull and $B \mid B'$. By (1) in the definition of $\mathscr{A}_f$, we have $B' \leqslant (\log x)^2$. If $B' \leqslant \exp((\log_2 x)^{1/2})$, then (estimating crudely)

$$\Omega(\phi(B)) \leqslant \Omega(\phi(B')) \leqslant 2 \log \phi(B') \leqslant 2 \log B' \leqslant 2(\log_2 x)^{1/2}.$$

On the other hand, if $B' > \exp((\log_2 x)^{1/2})$, then by (4) in the definition of $\mathscr{A}_f$,

$$\Omega(\phi(B)) \leqslant \Omega(\phi(B')) \leqslant 10\log_2 \phi(B') \leqslant 10\log_2 B' \ll \log_3 x.$$

Since $\log_2 v_k = \nu_k \log_2 x > \eta \log_2 x > (\log_2 x)^{1/2}$, we have we have $\Omega(\phi(B)) \leqslant 2\log_2 v_k$ in either case. Hence,

$$\Omega(f) = \Omega(\phi(A)) + \Omega(\phi(B)) \leqslant (3l+2)\log_2 v_k \leqslant 4l \log_2 v_k,$$

which completes the proof of (iv). Finally, we prove (v): Suppose that $b \geqslant x^{1/3}$ is a divisor of $p_0 - 1$. Recalling again (2-2),

$$P^+(b) \geqslant b^{1/\Omega(p_0-1)} \geqslant b^{\frac{1}{3\log_2 x}} \geqslant x^{\frac{1}{9\log_2 x}} > x^{\frac{1}{100\log_2 x}} \geqslant v_1.$$

Thus, setting b to be the largest divisor of $p_0 - 1$ supported on the primes $\leqslant v_1$, we have $b < x^{1/3}$. From (3-3) and conditions (0) and (7) in the definition of $\mathscr{A}_\phi$,

$$p_0 = \frac{a}{p_1 p_2 p_3 \cdots} > \frac{x/\log x}{x^{1/100} p_1} > x^{0.95},$$

say. Thus, $(p_0 - 1)/b$ is a divisor of $p_0 - 1$ composed of primes greater than v_1 and of size at least $(p_0 - 1)x^{-1/3} > x^{9/10}x^{-1/3} > x^{1/2}$.

5D. *Denouement.* We are now in a position to establish Proposition 5.1 and so also Theorem 1.1. Suppose that k and the μ_i and ν_i are fixed, as is d; this also fixes $l = L - k$. By Lemma 4.1, whose hypotheses were verified above, the number of values $\phi(a)$ coming from corresponding solutions to $\phi(a) = \sigma(a')$, with $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$, is

$$\begin{aligned} &\ll \frac{x}{d}(c\log_2 x)^{6k}(k+1)^{\Omega(d)}(\log v_k)^{8(k+l)\log(k+1)+1}(\log x)^{-2+\sum_{i=1}^{k-1} a_i\nu_i + E} \\ &\qquad \leqslant \frac{x}{d}\exp(O((\log_3 x)^2))L^{\Omega(d)}(\log v_k)^{L^2}(\log x)^{-2+\sum_{i=1}^{k-1} a_i x_i + E'}, \quad (5\text{-}11) \end{aligned}$$

where

$$E' := E + \sum_{i=1}^{k-1} a_i(\nu_i - x_i) = \delta\sum_{i=2}^{k}(i\log i + i) + 2\sum_{i=1}^{k-1}(\nu_i - \mu_i) + \sum_{i=1}^{k-1} a_i(\nu_i - x_i).$$

By our choice of ν_i and μ_i in Section 5B, we have $\nu_i - \mu_i \ll \eta$ and $\nu_i - x_i \ll \eta$. Hence,

$$E' \ll \delta L^2 \log L + \eta\left(L + \sum_{i=1}^{k-1} a_i\right) \ll \delta L^2\log L + \eta L^2 \log L \ll \delta L^3 \log L.$$

In combination with (8) from the definition of $\mathscr{A}_\phi$, this shows that the exponent of $\log x$ on the right-hand side of (5-11) is at most $-1-\omega+E' \leqslant -1-\omega/2$, and so

$$(\log x)^{-2+\sum_{i=1}^{k-1} a_i x_i + E'} \leqslant (\log x)^{-1} \exp\bigl(-\tfrac{1}{2}(\log_2 x)^{1/2+\epsilon/2}\bigr).$$

Moreover, by Lemma 5.3 and Lemma 5.2(i),

$$\nu_k \leqslant x_k + 2\eta \leqslant (\log_2 x)^{-1/2+\epsilon/9} + 2\eta \leqslant (\log_2 x)^{-1/2+\epsilon/5}, \tag{5-12}$$

and hence

$$(\log v_k)^{L^2} = \exp(L^2(\log_2 x)\nu_k) \leqslant \exp((\log_2 x)^{1/2+\epsilon/4}).$$

Inserting all of this back into (5-11), we obtain an upper bound which is

$$\ll \frac{x}{\log x} \exp\bigl(-\tfrac{1}{3}(\log_2 x)^{1/2+\epsilon/2}\bigr)\frac{L^{\Omega(d)}}{d}. \tag{5-13}$$

Now we sum over the parameters previously held fixed. We have $k < L$; also, for $i > 0$, each μ_i and ν_i has the form ζ_j of (5-4). Thus, the number of possibilities for k and the μ_i and ν_i is

$$\leqslant L(1+\lfloor \eta^{-1}\rfloor)^{2L} \leqslant \exp(O((\log_3 x)^2)). \tag{5-14}$$

Next, we prove that

$$\Omega(d) \ll (\log_2 x)^{1/2} \tag{5-15}$$

uniformly for the d under consideration, so that

$$L^{\Omega(d)} \leqslant \exp(O((\log_2 x)^{1/2} \log_4 x)). \tag{5-16}$$

Put $m := p_L p_{L+1} \cdots$. Suppose first that $p_L \neq p_{L-1}$, so that m is a unitary divisor of a and $d = \phi(m)$. If $m \leqslant \exp((\log_2 x)^{1/2})$, then (5-15) follows from the crude bound $\Omega(d) \ll \log d$. On the other hand, if $m > \exp((\log_2 x)^{1/2})$, then from (4) in the definition of $\mathscr{A}_\phi$, we have $\Omega(d) = \Omega(\phi(m)) \ll \log_2 m$. But by (3) in the definition of $\mathscr{A}_\phi$ and Lemma 5.2(i),

$$\begin{aligned}\log_2 m \leqslant \log_2 p_L^{10\log_2 x} &\ll \log_3 x + \log_2 p_L \ll \log_3 x + \varrho^L \log_2 x \\ &\ll \log_3 x + \varrho^{-2\sqrt{\log_3 x}} \varrho^{L_0} \log_2 x \ll \varrho^{-2\sqrt{\log_3 x}} \log_3 x \ll \exp(O(\sqrt{\log_3 x})),\end{aligned}$$

which again gives (5-15). Suppose now that $p_L = p_{L-1}$. In this case, let m' be the largest divisor of a supported on the primes dividing m. Then $d \mid \phi(m')$, and so $\Omega(d) \leqslant \Omega(\phi(m'))$. Write $m' = p_L^j m''$, where $j \geqslant 2$ and $p_L \nmid m''$; both p_L^j and m'' are unitary divisors of a. We have $\Omega(\phi(m'')) \ll (\log_2 x)^{1/2}$, by mimicking the argument used for m in the case when $p_L \neq p_{L-1}$. Also, $\Omega(\phi(p_L^j)) \ll (\log_2 x)^{1/2}$

except possibly if $p_L^j > \exp((\log_2 x)^{1/2})$, in which case, invoking (1) and (4) in the definition of $\mathscr{A}_\phi$,

$$\Omega(\phi(p_L^j)) \leqslant 10 \log_2 \phi(p_L^j) \leqslant 10 \log_2 p_L^j \leqslant 10 \log_2 (\log^2 x) \ll \log_3 x.$$

So

$$\Omega(d) \leqslant \Omega(\phi(m')) = \Omega(\phi(p_L^j)) + \Omega(\phi(m'')) \ll (\log_2 x)^{1/2},$$

confirming (5-15).

Referring back to (5-13), we see that it remains only to estimate the sum of $1/d$. Since $P^+(d) \leqslant v_k$, (5-12) shows that every prime dividing d belongs to the set $\mathscr{P} := \{p : \log_2 p \leqslant (\log_2 x)^{1/2+\epsilon/5}\}$. Thus,

$$\sum \frac{1}{d} \leqslant \prod_{p \in \mathscr{P}} \Bigl(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\Bigr) \ll \exp((\log_2 x)^{1/2+\epsilon/5}). \tag{5-17}$$

Combining the estimates (5-13), (5-14), (5-16), and (5-17), we find that

$$\#\{\phi(a) : a \in \mathscr{A}_\phi, a' \in \mathscr{A}_\sigma, \phi(a) = \sigma(a')\} \ll \frac{x}{\log x} \exp\bigl(-\tfrac{1}{4}(\log_2 x)^{1/2+\epsilon/2}\bigr),$$

which completes the proof of Proposition 5.1 and of Theorem 1.1.

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ford@math.uiuc.edu *Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, United States*

pollack@uga.edu *Department of Mathematics, University of Georgia, Boyd Graduate Studies Research Center, Athens, GA 30602, United States*

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Galois representations associated with unitary groups over $\mathbb{Q}$

Christopher Skinner

We show that a cuspidal automorphic representation $\pi = \bigotimes_{\ell \leq \infty} \pi_\ell$ of a unitary similitude group $\mathrm{GU}(a, b)_{/\mathbb{Q}}$ with archimedean component π_∞ in a regular discrete series has an associated $(a+b)$-dimensional p-adic Galois representation with Frobenius eigenvalues given by the local base change parameters for all primes ℓ such that π_ℓ and $\mathrm{GU}(a, b)$ are unramified.

1. Introduction

In this paper we explain how results of Morel [2010] on the cohomology of the noncompact Shimura varieties associated to unitary similitude groups over $\mathbb{Q}$ can be combined with results of Shin [2011] on the cohomology of certain compact Shimura varieties and with certain analytic results — most notably the stability of the gamma factors arising from the doubling method for unitary groups [Lapid and Rallis 2005; Brenner 2008] — to prove that a cuspidal automorphic representation π of $\mathrm{GU}(a, b)_{/\mathbb{Q}}$ with archimedean component in a discrete series and regular (in a sense made precise below) has an associated $(a+b)$-dimensional p-adic Galois representation with Frobenius eigenvalues given by the local base change parameters for all primes ℓ such that π and $\mathrm{GU}(a, b)$ are unramified. Our motivation for this is the use in [Skinner and Urban 2010] of these p-adic Galois representations in the case $(a, b) = (2, 2)$ to prove the Iwasawa–Greenberg main conjecture for a large class of modular forms. The main results include Theorems A and B below, whose proofs are intertwined.

Let K be an imaginary quadratic field of discriminant d_K. Let $n = a + b$ be a partition of a positive integer n as the sum of two nonnegative integers a and b. Then

$$J_{a,b} := \begin{pmatrix} 1_a & \\ & -1_b \end{pmatrix}$$

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defines an Hermitian pairing on the space $V := K^n$. Let $G := \mathrm{GU}(a, b)_{/\mathbb{Q}}$ denote the unitary similitude group over $\mathbb{Q}$ of the Hermitian pair $(V, J_{a,b})$. The L-packets of discrete series representations of $G(\mathbb{R})$ are naturally indexed by the irreducible algebraic representations of $G_{/K}$ (see Section 4.1). By a *regular* discrete series representation of $G(\mathbb{R})$ we will mean one belonging to an L-packet indexed by a representation with regular highest weight.

Let $H := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m \times \mathrm{GL}_n)$. For any $\mathbb{Q}$-algebra R, let $(x, g) \mapsto (\bar{x}, \bar{g})$ be the involution of $H(R) = (R \otimes K)^\times \times \mathrm{GL}_n(R \otimes K)$ induced by the nontrivial automorphism of K, and let θ be the involution defined by $\theta((x, g)) = (\bar{x}, \bar{x}^t \bar{g}^{-1})$. Note that an irreducible admissible representation σ of $H(\mathbb{A}_\mathbb{Q})$ is given by a pair (ψ, τ) consisting of an admissible character ψ of $\mathbb{A}_K^\times$ and an irreducible admissible representation τ of $\mathrm{GL}_n(\mathbb{A}_K)$ and that $\sigma = (\psi, \tau)$ is θ-stable (that is, $\sigma^\theta \cong \sigma$) if and only if $\tau^\vee \cong \tau^c$ and $\psi = \psi^c \chi_\tau^c$, where χ_τ is the central character of τ and the superscripts '$\vee$' and 'c' denote, respectively, the contragredient and composition with the involution induced by the nontrivial automorphism of K. Let $\mathrm{BC} : {}^L G \to {}^L H$ be the base change morphism (see Section 2.3).

Theorem A (weak base change). *Let π be an irreducible cuspidal representation of $G(\mathbb{A}_\mathbb{Q})$ and let χ_π be its central character* (*a character of $\mathbb{A}_K^\times$*). *Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Suppose $ab \neq 0$ and π_∞ is a regular discrete series belonging to an L-packet indexed by a representation ξ. There exists an automorphic representation $\sigma = (\psi, \tau)$ of $H(\mathbb{A}_\mathbb{Q})$ such that*:

(a) $\sigma^\theta \cong \sigma$, $\psi = \chi_\pi^c$ *and* $\chi_\tau = \chi_\pi^c / \chi_\pi$.

(b) *For a prime $\ell \notin \Sigma(\pi)$, σ_ℓ is unramified, and if $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G$ is the Langlands parameter of π_ℓ then*

$$\psi_{\sigma_\ell} := \mathrm{BC} \circ \psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L H$$

is the Langlands parameter of σ_ℓ. In particular, for any idèle class character χ of $\mathbb{A}_K^\times$ there is equality of twisted standard L-functions

$$L_{\Sigma(\pi)}(s, \pi \times \chi) = L_{\Sigma(\pi)}(s, \tau \times \chi).$$

(c) σ_∞ *has the same infinitesimal character as* $\xi \otimes \xi^\theta$.

There is a natural identification of $G_{/K}$ with $\mathbb{G}_m \times \mathrm{GL}_n$ (see Section 2.2) and hence of $G(\mathbb{R} \otimes K)$ with $H(\mathbb{R})$, which then identifies ξ, and hence ξ^θ, as a representation of $H(\mathbb{R})$. The (partial) standard L-function of π is as defined as in [Li 1992, §3].

Let $\overline{K}$ be an algebraic closure of K and let $G_K := \mathrm{Gal}(\overline{K}/K)$. For each finite place v of K let $\overline{K}_v$ be an algebraic closure of K_v and fix an embedding $\overline{K} \hookrightarrow \overline{K}_v$. The latter identifies $G_{K_v} := \mathrm{Gal}(\overline{K}_v / K_v)$ with a decomposition group for v in G_K and hence the Weil group $W_{K_v} \subset G_{K_v}$ with a subgroup of G_K.

Let p be a prime and $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$. Let $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ be an isomorphism. Our conventions for Galois representations are geometric.

Theorem B (Galois representations). *Let π be an irreducible cuspidal representation of $G(\mathbb{A}_{\mathbb{Q}})$ and let χ_π be its central character. Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Suppose $ab \neq 0$ and π_∞ is a regular discrete series belonging to an L-packet indexed by the representation ξ. Let $\sigma = (\psi, \tau)$ be as in Theorem A. There exists a continuous, semisimple representation $\rho_\pi = \rho_{\pi,\iota} : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that:*

(a) $\rho_\pi^c \simeq \rho_\pi^\vee \otimes \rho_{\chi_\pi^{1+c}} \epsilon^{1-n}$.

(b) *ρ_π is unramified at all finite places not above primes in $\Sigma_p(\pi) := \Sigma(\pi) \cup \{p\}$, and for such a place w*

$$(\rho_\pi|_{W_{K_w}})^{ss} = \iota\mathrm{Rec}_w\big(\tau_w \otimes \psi_w |\cdot|_w^{(1-n)/2}\big).$$

In particular,

$$L_{\Sigma_p(\pi)}(s, \rho_\pi) = L_{\Sigma_p(\pi)}\Big(s + \frac{1-n}{2}, \tau \times \psi\Big).$$

(c) *For $v|p$, $\rho_\pi|_{G_{K_v}}$ is potentially semistable of Hodge–Tate-type ξ.*

(d) *If $p \notin \Sigma(\pi)$ then*

(d) *If $p \notin \Sigma(\pi)$ then for any $v|p$, $\rho_\pi|_{G_{K_v}}$ is crystalline. Moreover, for any j in $\mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$ the eigenvalues of the action of the $[K_v : \mathbb{Q}_p]$-th power of the crystalline Frobenius on*

$$D_{\mathrm{cris}}(\rho_\pi|_{G_{K_v}}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, j} \overline{\mathbb{Q}}_p$$

are the eigenvalues of the action of Frobenius on $\iota\mathrm{Rec}_v\big(\tau_v \otimes \psi_v |\cdot|_v^{(1-n)/2}\big)$.

For any irreducible admissible representation α of $\mathrm{GL}_n(K_w)$, $\mathrm{Rec}_w(\alpha)$ is the Weil–Deligne representation over $\mathbb{C}$ associated by the local Langlands correspondence, and $\iota\mathrm{Rec}_w(\alpha)$ is the representation over $\overline{\mathbb{Q}}_p$ obtained by change of scalars via ι. For $\rho_\pi|_{G_{K_v}}$ to be of Hodge–Tate type ξ means that the Hodge–Tate weights can be read off from ξ in a prescribed way (see Section 4.4).

As the proof of Theorem A shows, there is a partition $n = m_1 + \cdots + m_r$ such that the representation τ in Theorem A is of the form $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ with τ_i a cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_K)$ such that $\tau_i^c \cong \tau_i^\vee$ and $\sigma_i := \tau_i \otimes |\cdot|^{(m_i - n)/2}$ is regular algebraic in the sense of [Clozel 1990]. Then the representation ρ_π of Theorem B is just $\rho_\psi \otimes \big(\bigoplus_{i=1}^r \rho_{\sigma_i,\iota}\big)$, where $\rho_{\sigma_i,\iota}$ is the m_i-dimensional p-adic Galois representation associated to σ_i ($\rho_{\sigma_i,\iota}$ is obtained from [Shin 2011]).

The theory of pseudorepresentations in combination with congruences between automorphic forms allows the weakening of some of the hypotheses of Theorem B —

cases where $ab = 0$ or where ξ is not regular can be allowed. But we do not include this here.

If $\mathbb{Q}$ is replaced by a totally real field of degree greater than one, then the analogs of Theorems A and B are known, the weak base change having been proved by Labesse [2011]. Furthermore, versions of these theorems have been proved by Morel [2010], who proves Theorem A but with $\Sigma(\pi)$ replaced by an indeterminate set of primes, and by Harris and Labesse [2004], who require additional conditions at some finite primes. The work of Morel is the starting point of our proofs.

Our proofs of Theorems A and B proceed essentially as follows. By results of Morel, an automorphic representation $\sigma = (\psi, \tau)$ of $H(\mathbb{A}_\mathbb{Q})$ as in Theorem A exists but with $\Sigma(\pi)$ replaced by an indeterminate set $S \supseteq \Sigma(\pi)$. Furthermore, τ is a subquotient of an induced representation $\mathrm{Ind}_P^{\mathrm{GL}_n}\left(\bigotimes_{i=1}^r \tau_i\right)$ with $P \subset \mathrm{GL}_n$ the standard parabolic associated with a partition $n = m_1 + \cdots + m_r$ and each τ_i a discrete representation of $\mathrm{GL}_{m_i}(\mathbb{A}_\mathbb{Q})$ such that $\tau_i^c \cong \tau_i^\vee$. By considering absolute values of Satake parameters, it follows from the work of Mœglin and Waldspurger characterizing the discrete series representations of $\mathrm{GL}_{m_i}(\mathbb{A}_\mathbb{Q})$ that each τ_i is cuspidal, and a consideration of infinitesimal characters yields that $\sigma_i := \tau_i \otimes |\cdot|^{(n_i-n)/2}$ is algebraic with the same infinitesimal character as a regular irreducible representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_i}$. The regularity of ξ is used in both these arguments. Then $\rho_{\pi,\iota} := \rho_\psi \otimes \left(\bigoplus_{i=1}^r \rho_{\sigma_i,\iota}\right)$, with $\rho_{\sigma_i,\iota}$ being the representation deduced from the work of Shin, satisfies conclusions (a), (b), and (c) of Theorem B with $\Sigma(\pi)$ replaced by S. It then remains to show that (b) of Theorem A also holds for $\ell \in S$ but $\ell \notin \Sigma(\pi)$, for then (b) and (d) of Theorem B follow from the corresponding results for the $\rho_{\sigma_i,\iota}$. To obtain (b) of Theorem A for such an ℓ we first observe that the representation $\bigwedge^a \rho_{\pi,\iota}$ is unramified at the places $w|\ell$. This is because Morel has essentially shown that this representation appears in the intersection cohomology of a Shimura variety associated to π that has good reduction at $w|\ell$ (some argument is required to reduce to the nonendoscopic case); this is another point at which the regularity of ξ is used. Then the local-global compatibility satisfied by the $\rho_{\sigma_i,\iota}$ implies that there is a finite order character χ_ℓ of $K_\ell^\times$ such that each $\tau_{i,w} \otimes \chi_w$ is unramified, and hence a principal series representation of $\mathrm{GL}_{m_i}(K_w)$ with Satake parameters all having the same absolute values (again using regularity of ξ). This information is then combined with that coming from the γ-factors of the standard L-functions. Lapid and Rallis have defined local γ-factors $\gamma(s, \pi_v \times \chi_v)$ for the standard L-function of π such that

$$L_S(s, \pi \times \chi) = \prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \chi_v) \times L_S(1-s, \pi^\vee \times \chi^{-1}),$$

and Brenner has proved stability for these γ-factors at nonarchimedean places. Comparing this with the functional equation for $L_S(s, \tau \times \chi)$ and choosing a global character χ with ℓ-component χ_ℓ and with sufficiently ramified q-components χ_q for $\ell \neq q \in S$ yields an equality between γ-factors for π and

$$\tau : \gamma(s, \pi_\ell \times \chi_\ell) = \gamma(s, \tau_\ell \times \chi_\ell).$$

Comparing the definitions of these gamma factors and exploiting some freedom in the choice of χ_ℓ and χ then yields conclusion (b) of Theorem A.

After some preliminary remarks fixing notation for unitary and related groups in Section 2, in Section 3 we give the analytic arguments involving L-functions and γ-factors. In Section 4 we then recall the results of Morel and Shin and explain how Theorems A and B follow.

2. Preliminaries

We adopt the following notation and conventions.

2.1. ***Galois groups and representations.*** Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field of discriminant d_K. For $F = \mathbb{Q}$ or K, let $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Let W_F be a Weil group of F; this comes with a homomorphism to G_F. For each place v of F fix an algebraic closure $\overline{F}_v$ of F_v and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{F}_v$. The latter identifies $G_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$ with a decomposition group in G_F. Let W_{F_v} be the Weil group of F_v; for a finite place v, W_{F_v} is a subgroup of G_{F_v} and so is identified with a subgroup of G_F. Fix a homomorphism $W_{F_v} \to W_F$ compatible with the fixed inclusion $G_{F_v} \subset G_F$. We denote the action on K of the nontrivial automorphism in $\mathrm{Gal}(K/\mathbb{Q})$ by $x \mapsto \bar{x}$. For simplicity, we also fix an embedding $K \hookrightarrow \mathbb{C}$ (equivalently, an isomorphism $\overline{K}_\infty \cong \mathbb{C}$).

Let p be fixed prime and $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ a fixed isomorphism. Our conventions for p-adic Galois representations are geometric: L-functions of representations of G_F or G_{F_v} are defined by taking characteristic polynomials of *geometric* Frobenius elements.

For an algebraic Hecke character of $\mathbb{A}_F^\times$ (so $\chi_\infty(x) = \mathrm{sgn}(x)^r x^t$ if $F = \mathbb{Q}$ and $\chi_\infty(x) = x^r \bar{x}^t$ if $F = K$, for some $r, t \in \mathbb{Z}$) let

$$\rho_\chi = \rho_{\chi,\iota} : G_F \to \overline{\mathbb{Q}}_p^\times$$

be the p-adic Galois character such that $L_{\{p\}}(s, \rho_\chi) = L_{\{p\}}(s, \chi)$. Then $\epsilon : G_F \to \mathbb{Z}_p^\times$ is the p-adic character associated to the norm $|\cdot|_F$ character of $\mathbb{A}_F^\times$; this is the p-adic cyclotomic character: for a geometric Frobenius frob_v, $v \nmid p\infty$,

$$\epsilon(\mathrm{frob}_v) = \mathrm{Norm}(v)^{-1}.$$

2.2. ***The groups: G, G_0, H, and H_0.*** Let $n_1, \dots, n_k$ be positive integers and $n := n_1 + \cdots + n_k$. For each $i = 1, \dots, k$ let $n_i = a_i + b_i$ be a partition of n_i as a sum of two nonnegative integers. Let

$$J_i = J_{a_i,b_i} := \begin{pmatrix} 1_{a_i} & \\ & -1_{b_i} \end{pmatrix}.$$

Then J_i defines a Hermitian pairing on K^{n_i}. Let

$$G = G(U(a_1, b_1) \times \cdots \times U(a_k, b_k))_{/\mathbb{Q}}$$

and let $\mu : G \to \mathbb{G}_m$ be its similitude character. That is, for any $\mathbb{Q}$-algebra R,

$$G(R) = \left\{ g = (g_1, \ldots, g_k) \in \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R \otimes K) : \exists \lambda \in R^\times \text{ such that } g_i J_i^t \bar{g}_i = \lambda J_i \right\}$$

and $\mu(g) = \lambda$. Here $g \mapsto \bar{g}$ is the involution of $\mathrm{GL}_m(R \otimes K)$ defined by the action of the nontrivial automorphism of K. Let $G_0 := U(a_1, b_1) \times \cdots \times U(a_k, b_k)$ be the kernel of μ.

For any K-algebra R there is a natural isomorphism $R \otimes K \xrightarrow{\sim} R \times R$, $r \otimes x \mapsto (rx, r\bar{x})$. Using this, we identify $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^k \mathrm{GL}_{n_i}$:

$$g = (g_i', g_i'') \in G(R) \subset \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R \otimes K) = \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R) \times \mathrm{GL}_{n_i}(R)$$

is identified with $(\mu(g), (g_i')) \in R^\times \times \prod_{i=1}^k \mathrm{GL}_{n_i}(R)$. Then $G_{0/K}$ is identified with the subgroup $\prod_{i=1}^k \mathrm{GL}_{n_i}$.

Let $H := \mathrm{Res}_{K/\mathbb{Q}} G_{/K}$. Then $H_{/K}$ is identified with $G_{/K} \times G_{/K}$. The identification of $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^k \mathrm{GL}_{n_i}$ identifies H with

$$\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_{n_i}.$$

Let θ be the involution of H defined by

$$\theta(x, (g_i)) = (\bar{x}, (\bar{x}^t \bar{g}_i^{-1})).$$

Let $H_0 := \mathrm{Res}_{K/\mathbb{Q}} G_0$. Note that θ also defines an involution $(g_i) \mapsto ({}^t\bar{g}_i^{-1})$ of H_0. An irreducible admissible representation of $H(\mathbb{A}_\mathbb{Q})$ is given by a tuple $(\psi, (\tau_i))$ with ψ an admissible character of $\mathbb{A}_K^\times$ and each τ_i an irreducible admissible representation of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$.

2.3. Dual groups and *L*-groups. The identification of $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^k \mathrm{GL}_{n_i}$ also identifies the dual group $\widehat{G}$ with $\mathbb{C}^\times \times \prod_{i=1}^k \mathrm{GL}_{n_i}(\mathbb{C})$, with $G_\mathbb{Q}$ acting through the quotient $\mathrm{Gal}(K/\mathbb{Q})$ and the nontrivial automorphism $c \in \mathrm{Gal}(K/\mathbb{Q})$ acting by

$$c(x, (g_i)) = \left(x \prod_{i=1}^{k} \det g_i, (\Phi_{n_i}^{-1t} g_i^{-1} \Phi_{n_i}) \right),$$

where $\Phi_m := (\Phi_{m,ij}) = ((-1)^{i+1}\delta_{i,m-j+1})$. Put ${}^LG := \widehat{G} \rtimes W_{\mathbb{Q}}$. Similarly, $\widehat{G}_0 = \prod_{i=1}^k \mathrm{GL}_{n_i}(\mathbb{C})$ with the same action of $G_{\mathbb{Q}}$; let ${}^LG_0 := \widehat{G}_0 \rtimes W_{\mathbb{Q}}$. The L-homomorphism corresponding to taking an irreducible admissible $G_0(\mathbb{A}_{\mathbb{Q}})$-constituent of an irreducible admissible $G(\mathbb{A}_{\mathbb{Q}})$ representation is the projection

$$ {}^LG \to {}^LG_0, (x, (g_i)) \rtimes w \mapsto (g_i) \rtimes w. $$

Since $H_{/K} = G_{/K} \times G_{/K}$, $\widehat{H} = \widehat{G} \times \widehat{G}$ with the action of $G_{\mathbb{Q}}$ again factoring through $\mathrm{Gal}(K/\mathbb{Q})$ and with $c(x, y) = (c(y), c(x))$. Similarly, $\widehat{H}_0 = \widehat{G}_0 \times \widehat{G}_0$ with the same action of $G_{\mathbb{Q}}$. Put ${}^LH := \widehat{H} \rtimes W_{\mathbb{Q}}$ and ${}^LH_0 := \widehat{H}_0 \rtimes W_{\mathbb{Q}}$. The diagonal embedding $\widehat{G} \hookrightarrow \widehat{H} = \widehat{G} \times \widehat{G}$ is $G_{\mathbb{Q}}$-equivariant; its extension to L-groups

$$ \mathrm{BC} : {}^LG \to {}^LH $$

is the base change map. Let $\mathrm{BC} : {}^LG_0 \to {}^LH_0$ be the similarly defined map.

3. L-functions and γ-factors

In this section we prove the key analytic ingredient of our proof of Theorems A and B. We assume in the argument that $G_0 = U(a, b)$ (that is, k=1).

Let π be a cuspidal automorphic representation of $G_0(\mathbb{A}_{\mathbb{Q}})$. Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. By the principle of functoriality for the L-group homomorphism $\mathrm{BC} : {}^LG_0 \to {}^LH_0$ it is expected — at the very least — that there should be a *weak base change* of π to $H_0(\mathbb{A}_{\mathbb{Q}})$. That is, there should exist an automorphic representation τ of $H_0(\mathbb{A}_{\mathbb{Q}})$ (equivalently, of $\mathrm{GL}_n(\mathbb{A}_K)$) such that for $\ell \notin \Sigma(\pi)$, the Langlands parameter $\psi_{\tau_\ell} : W_{\mathbb{Q}_\ell} \to {}^LH_0$ of τ_ℓ is just $\mathrm{BC} \circ \psi_{\pi_\ell}$, with $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^LG_0$ the Langlands parameter of π_ℓ. We say that τ is a *very weak base change* of π if there is *some* set $S \supset \Sigma(\pi)$ such that this relation between Langlands parameters holds for all $\ell \notin S$.

Proposition 1. *Let π be a cuspidal automorphic representation of $G_0(\mathbb{A}_{\mathbb{Q}})$. Assume that there exists a very weak base change τ of π to $H_0(\mathbb{A}_{\mathbb{Q}})$. If τ is a tempered principal series at every finite place $\ell \notin \Sigma(\pi)$, then τ is a weak base change of π.*

We deduce the conclusion of this proposition by comparing L-functions and γ-factors. Let $R := \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$. Then $\widehat{R} = \mathbb{C}^\times \times \mathbb{C}^\times$ with $G_{\mathbb{Q}}$ acting through $\mathrm{Gal}(K/\mathbb{Q})$ and the nontrivial automorphism c of K acting as $c(x_1, x_2) = (x_2, x_1)$. Let ${}^LR := \widehat{R} \rtimes W_{\mathbb{Q}}$. Let ω be a Hecke character of $\mathbb{A}_K$. Then ω is an irreducible admissible representation of $R(\mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_K^\times$; we let $\psi_{\omega_\ell} : W_{\mathbb{Q}_\ell} \to {}^LR$ be the Langlands parameter associated with $\omega_\ell := \bigotimes_{v|\ell} \omega_v$ (coming from class field theory). The L-groups of $G_0 \times R$ and $H_0 \times R$ are ${}^L(G_0 \times R) = {}^LG_0 \times_{W_{\mathbb{Q}}} {}^LR = (\widehat{G}_0 \times \widehat{R}) \rtimes W_{\mathbb{Q}}$ and ${}^L(H_0 \times R) = {}^LH_0 \times_{W_{\mathbb{Q}}} {}^LR = (\widehat{H}_0 \times \widehat{R}) \rtimes W_{\mathbb{Q}}$, with $W_{\mathbb{Q}}$ acting on each factor.

Let π and τ be as in the proposition. The unramified local L-factors $L(s,\pi_\ell\times\omega_\ell)$ of the standard L-function of $\pi\times\omega$ are the L-factors associated with the representation $r_{st}: {}^L(G_0\times R)\to \mathrm{GL}_{2n}(\mathbb{C})$,

$$r_{st}((g,(x_1,x_2))\rtimes 1)=\begin{pmatrix} x_1g & \\ & x_2\Phi_n^{-1t}g^{-1}\Phi_n\end{pmatrix}\qquad r_{st}(1\rtimes c)=\begin{pmatrix} & 1_n\\ 1_n & \end{pmatrix}.$$

If $\ell\nmid d_K$ and π_ℓ and ω_ℓ are unramified, then

$$L(s,\pi_\ell\times\omega_\ell)=\det(1-\ell^{-s}r_{st}(\psi_{\pi_\ell}(\mathrm{frob}_\ell),\psi_{\omega_\ell}(\mathrm{frob}_\ell)))^{-1}.$$

Similarly, the local unramified L-factors $L(s,\tau_\ell\times\omega_\ell):=\prod_{v|\ell}L(s,\tau_v\times\omega_v)$ are the L-factors associated with the homomorphism $r'_{st}: {}^L(H_0\times R)\to\mathrm{GL}_{2n}(\mathbb{C})$,

$$r'_{st}(((g_1,g_2),(x_1,x_2))\rtimes 1)=\begin{pmatrix} x_1g_1 & \\ & x_2\Phi_n^{-1t}g_2^{-1}\Phi_n\end{pmatrix}\qquad r'_{st}(1\rtimes c)=\begin{pmatrix} & 1_n\\ 1_n & \end{pmatrix}.$$

In particular, $r_{st}=r'_{st}\circ(\mathrm{BC}\times id)$, so $L(s,\pi_\ell\times\omega_\ell)=L(s,\tau_\ell\times\omega_\ell)$ if $\ell\nmid d_K$ and π_ℓ, τ_ℓ, and ω_ℓ are unramified and $\psi_{\tau_\ell}:=\mathrm{BC}\circ\psi_{\pi_\ell}$ (so for all $\ell\notin S$).

Lemma 2. *Suppose $\ell\nmid d_K$ and π_ℓ are τ_ℓ are unramified. If*

$$L(s,\pi_\ell\times\omega_\ell)=L(s,\tau_\ell\times\omega_\ell)$$

for all unramified ω_ℓ, then $\psi_{\tau_\ell}=\mathrm{BC}\circ\psi_{\pi_\ell}$.

Proof. Let

$$\psi_{\pi_\ell}(\mathrm{frob}_\ell)=t\rtimes\mathrm{frob}_\ell,\quad t=\mathrm{diag}(t_1,\dots,t_n),\quad\text{and}\quad \psi_{\tau_\ell}(\mathrm{frob}_\ell)=(h,h)\rtimes\mathrm{frob}_\ell,$$

$h=\mathrm{diag}(h_1,\dots,h_n)$ (ψ_{τ_ℓ} must be of this form as $\tau_\ell^c\cong\tau_\ell^\vee$). Suppose first that ℓ does not split in K. As $\mathrm{frob}_\ell=c$ in $\mathrm{Gal}(K/\mathbb{Q})$, the condition that $L(s,\pi_\ell\times\omega_\ell)=L(s,\tau_\ell\times\omega_\ell)$ is just that $t_i/t_{n-i}=h_i/h_{n-i}$ (after possibly reordering the h_i). That is, $t=zh$ for some $z\in\mathbb{C}^\times$, and so $(z,1)\psi_{\tau_\ell}(\mathrm{frob}_\ell)(z^{-1},1)=\mathrm{BC}\circ\psi_{\pi_\ell}(\mathrm{frob}_\ell)$. Hence, ψ_{τ_ℓ} is equivalent to $\mathrm{BC}\circ\psi_{\pi_\ell}$.

Suppose that ℓ splits in K. Let $\psi_{\omega_\ell}(\mathrm{frob}_\ell)=(\alpha,\beta)\rtimes\mathrm{frob}_\ell$. As $\mathrm{frob}_\ell=1$ in $\mathrm{Gal}(K/\mathbb{Q})$, the equality $L(s,\pi_\ell\times\omega_\ell)=L(s,\tau_\ell\times\omega_\ell)$ means that

$$\mathrm{diag}(\alpha t,\beta\Phi_n^{-1}t^{-1}\Phi_n)\in\mathrm{GL}_{2n}(\mathbb{C})\quad\text{and}\quad\mathrm{diag}(\alpha h,\beta\Phi_n^{-1}h^{-1}\Phi_n)\in\mathrm{GL}_{2n}(\mathbb{C})$$

are equivalent. As α and β can be arbitrary, it follows that t and h are equivalent, so $\mathrm{BC}\circ\psi_{\pi_\ell}$ is equivalent to ψ_{τ_ℓ}. □

Let $S\supset\Sigma(\pi)$ be any finite set of primes such that $\psi_{\tau_\ell}=\mathrm{BC}\circ\psi_{\pi_\ell}$ for all $\ell\notin S$. The (partial) standard L-functions $L_S(s,\pi\times\omega)$ and $L_S(s,\tau\times\omega)$, given by the Euler products

$$L_S(s,\pi\times\omega)=\prod_{\ell\notin S}L(s,\pi_s\times\omega_\ell)\quad\text{and}\quad L_S(s,\tau\times\omega)=\prod_{\ell\notin S}L(s,\tau_\ell\times\omega_\ell)$$

for $\operatorname{Re}(s) \gg 0$, satisfy

$$L_S(s, \pi \times \omega) = L_S(s, \tau \times \omega).$$

The doubling method of Piatetski-Shapiro and Rallis provides an integral representation of $L_S(s, \pi \times \omega)$ as well as local γ-factors at all places; see [Gelbart et al. 1987, Part A] and especially [Lapid and Rallis 2005]. In particular, for each place v of $\mathbb{Q}$, Lapid and Rallis have defined local γ-factors $\gamma(s, \pi_v \times \omega_v) := \gamma_v(s, \pi_v \times \omega_v, \psi_v)$, ψ_v being the standard additive character of K_v and proved that the local γ-factors $\gamma(s, \pi_v \times \omega_v)$ are compatible with parabolic induction and are as expected in the unramified cases. The functional equation for $L_S(s, \pi \times \omega)$ is then

$$L_S(s, \pi \times \omega) = \prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \omega_v) \times L_S(1-s, \pi^\vee \times \omega^{-1}).$$

Comparing this with the usual functional equation for the standard GL_n L-function $L_S(s, \tau \times \omega)$ we find that

$$\prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \omega_v) = \prod_{v \in S \cup \{\infty\}} \prod_{w \mid v} \gamma(s, \tau_w \times \omega_w), \tag{3.1}$$

where w is a place of K and $\gamma(s, \tau_w \times \omega_w)$ is the γ-factor defined by Godement and Jacquet (again using the standard additive characters). For a place v of $\mathbb{Q}$, set

$$\gamma(s, \tau_v \times \omega_v) := \prod_{w \mid v} \gamma(s, \tau_w \times \omega_w).$$

We exploit *stability* of γ-factors. This says that if π_1 and π_2 are two irreducible admissible representations of $G_0(\mathbb{Q}_\ell)$, then for χ a sufficiently ramified character of $K_\ell^\times$, $\gamma(s, \pi_1 \times \chi) = \gamma(s, \pi_2 \times \chi)$. This has been proved by Brenner [2008]. Stability is also known for the Godement–Jacquet γ-factors for GL_n. Taking $\pi_1 = \pi_\ell$ and π_2 to be an unramified tempered principal series, we see that if ω_ℓ is sufficiently ramified then

$$\gamma(s, \pi_\ell \times \omega_\ell) = \gamma(s, \pi_2 \times \omega_\ell) = \gamma(s, \tau_2 \times \omega_\ell) = \gamma(s, \tau_\ell \times \omega_\ell), \tag{3.2}$$

where τ_2 is the representation of $H_0(\mathbb{Q}_\ell) = \mathrm{GL}_n(K_\ell)$ having Langlands parameter equal to the composition with BC of the parameter of π_2; τ_2 is also an unramified tempered principal series. The first and last equalities in (3.2) come from stability, and the middle comes from [Lapid and Rallis 2005, Theorem 4]: part 1 of this theorem, together with the hypothesis that π_2 is a principal series, reduces the equality to the minimal cases — the anisotropic cases, which are part 7 of the theorem, and the isotropic cases, which are part 8 — plus the analog of part 2 for the Godement–Jacquet γ-factors (compatibility with parabolic induction).

It is easy to see that given any finite set of primes S' it is possible to find a set $S'' \supset S \cup S'$ and a finite order Hecke character ω of $\mathbb{A}_K^\times$ such that ω_ℓ is arbitrary for all $\ell \in S'$, and ω_ℓ is sufficiently ramified at all primes $\ell \in S'' - S'$ and unramified at all primes not in S''. Taking $S' = \varnothing$, we deduce from (3.1) and (3.2) that $\gamma(s, \pi_\infty \times \omega_\infty) = \gamma(s, \tau_\infty \times \omega_\infty)$. Taking $S' = \{\ell\}$, any prime ℓ, we then deduce from (3.1) and (3.2) that

$$\gamma(s, \pi_\ell \times \omega_\ell) = \gamma(s, \tau_\ell \times \omega_\ell) \tag{3.3}$$

always.

Suppose now that $\ell \notin \Sigma(\pi)$. By hypothesis, τ_v is a tempered principal series for $v|\ell$. Suppose first that ℓ is inert in K. Then $\tau_\ell \cong \pi(\mu_1, \dots, \mu_n)$ with $|\mu_i(x)| = 1$ for all $x \in K_\ell^\times$. Fix j between 1 and n and choose ω_ℓ so that $\mu_j\omega_\ell$ is unramified. Let $I \subset \{1, \dots, n\}$ be the set of indices such that $\mu_i\omega_\ell$ is unramified. Then

$$\gamma(s, \tau_\ell \times \omega_\ell) = \prod_{i \in I} \frac{1 - \mu_i\omega_\ell(\ell)\ell^{-2s}}{1 - \mu_i^{-1}\omega_\ell^{-1}(\ell)\ell^{2s-2}} \times \prod_{i \notin I} \gamma(s, \mu_i\omega_\ell).$$

As $\mu_i\omega_\ell$ is ramified for $i \notin I$, $\gamma(s, \mu_i\omega_\ell)$ is holomorphic with no zeros. Furthermore, the temperedness of τ_ℓ ensures that there is no cancellation between the numerators and denominators of the factors coming from the $i \in I$. Therefore, $\gamma(s, \tau_\ell \times \omega_\ell)$ has $|I| \geq 1$ poles. However, if ω_ℓ is ramified, then, since π_ℓ is unramified, it follows from combining parts 1, 7, and 8 of [Lapid and Rallis 2005, Theorem 4] that $\gamma(s, \pi_\ell \times \omega_\ell)$ is holomorphic. So it must be that ω_ℓ — and hence μ_j — is unramified. But j was arbitrary, so each μ_i is unramified: τ_ℓ is an unramified principal series. Therefore, by (3.3),

$$\frac{L(1-s, \pi_\ell^\vee)}{L(s, \pi_\ell)} = \gamma(s, \pi_\ell) = \gamma(s, \tau_\ell) = \frac{L(1-s, \tau_\ell^\vee)}{L(s, \tau_\ell)}$$

(for the first equality, see part 3 of [Lapid and Rallis 2005, Thm. 4]). As τ_ℓ is tempered, the zeros of the right-hand side are those of $L(s, \tau_\ell)^{-1}$, while those of the left-hand side are *a priori* a subset of those of $L(s, \pi_\ell)^{-1}$. This means that $L(s, \tau_\ell)/L(s, \pi_\ell)$ is holomorphic. But each of $L(s, \tau_\ell)^{-1}$ and $L(s, \pi_\ell)^{-1}$ is a polynomial of degree n in ℓ^{-2s} with constant term 1, and so they must be equal. That is, $L(s, \pi_\ell) = L(s, \tau_\ell)$. Since an unramified ω_ℓ equals $|\cdot|_\ell^t$ for some $t \in \mathbb{C}$, it follows that $L(s, \pi_\ell \otimes \omega_\ell) = L(s+t, \pi_\ell) = L(s+t, \tau_\ell) = L(s, \tau_\ell \otimes \omega_\ell)$, which implies — by Lemma 2 — that $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$.

Suppose that $\ell = v\bar{v}$ splits in K. Viewing $\mathbb{Q}_\ell$ as a K-algebra via the embedding that induces v, $G_0(\mathbb{Q}_\ell)$ is identified with $\mathrm{GL}_n(K_v) = \mathrm{GL}_n(\mathbb{Q}_\ell)$ and π_ℓ with a representation π_v of $\mathrm{GL}_n(\mathbb{Q}_\ell)$. Let $\pi_{\bar{v}} = \pi_v^\vee$. Then

$$\begin{aligned}\gamma(s, \pi_v \times \omega_v)\gamma(s, \pi_{\bar{v}} \times \omega_{\bar{v}}) &= \gamma(s, \pi_\ell \times \omega_\ell)\\ &= \gamma(s, \tau_\ell \times \omega_\ell) = \gamma(s, \tau_v \times \omega_v)\gamma(s, \tau_{\bar{v}} \times \omega_{\bar{v}}).\end{aligned}$$

The first equality follows from part 8 of [Lapid and Rallis 2005, Theorem 4]. By choosing ω_ℓ so that $\omega_{\bar{v}}$ is sufficiently ramified but ω_v is unramified, $\gamma(s, \pi_{\bar{v}} \times \omega_{\bar{v}})$ and $\gamma(s, \tau_{\bar{v}} \times \omega_{\bar{v}})$ can be assumed to be holomorphic with no zeros. Arguing as in the nonsplit case then yields that τ_v is unramified and $L(s, \tau_v) = L(s, \pi_v)$ (recall that τ_v and $\tau_{\bar{v}}$ are assumed to be principal series and tempered). Reversing the role of ω_v and $\omega_{\bar{v}}$ then yields that $\tau_{\bar{v}}$ is unramified and $L(s, \tau_{\bar{v}}) = L(s, \pi_{\bar{v}})$. As $L(s, \pi_\ell) = L(s, \pi_v)L(s, \pi_{\bar{v}})$, it follows that $L(s, \pi_\ell \otimes \omega_\ell) = L(s, \tau_\ell \otimes \omega_\ell)$ for all unramified ω_ℓ, which — by Lemma 2 again — implies that $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$. This completes the proof of Proposition 1.

4. σ and ρ_π

In this section, k is arbitrary.

4.1. ***Algebraic representations and discrete series for $G(\mathbb{R})$.*** Let $T \subset G$ be the subgroup of diagonal elements. Then $T_{/K}$ is identified with the diagonal subgroup

$$\mathbb{G}_m^{1+n} = \mathbb{G}_m^{1+n_1+\cdots+n_k} \subset \mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{GL}_{n_i},$$

and the character group $X(T)$ is identified with $\mathbb{Z}^{1+n}$: to $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k) \in \mathbb{Z}^{1+n}$, $\underline{c}_i \in \mathbb{Z}^{n_i}$, corresponds the character

$$(t_0, (\mathrm{diag}(t_{i,1}, \ldots, t_{i,n_i})) \mapsto t_0^{c_0} \prod_{i=1}^{n} \prod_{j=1}^{n_i} t_{i,j}^{c_{i,j}}.$$

We take the dominant characters to be those that are dominant with respect to the upper-triangular Borel B; this is equivalent to $c_{i,1} \geq c_{i,2} \geq \cdots \geq c_{i,n_i}$. Regular dominant characters are those where the inequalities are strict. The (regular) irreducible algebraic representations of $G_{/K}$ are indexed by the (regular) dominant characters in $X(T)$: to the representation ξ corresponds its highest weight with respect to the pair (T, B).

The L-packets of discrete series representations of $G(\mathbb{R})$ are indexed by equivalence classes of elliptic Langlands parameters $\psi : W_\mathbb{R} \to {}^L G$. The restriction to $W_\mathbb{C} = \mathbb{C}^\times$ of such a ψ is equivalent to a representation of the form

$$z \mapsto ((z/\bar{z})^{p_0}, (\mathrm{diag}((z/\bar{z})^{p_{i,1}}, \ldots, (z/\bar{z})^{p_{i,r_i}}))) \rtimes z$$

with $p_0 \in \mathbb{Z}$ and $p_{i,j} \in (n_i - 1)/2 + \mathbb{Z}$; the ordering can be chosen so that $p_{i,1} > \cdots > p_{i,r_i}$. Let $c_{i,j} := p_{i,j} - (n_i - 2i + 1)/2$. Then $c_{i,1} \geq \cdots \geq c_{i,r_i}$, and $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k)$, $c_0 := p_0$ and $\underline{c}_i := (c_{i,1}, \ldots, c_{i,r_i})$, is a dominant character of $X(T)$ and so corresponds to an irreducible algebraic representation ξ of $G_{/K}$ of highest weight $\underline{c}$. This gives a parametrization of the discrete series L-packets by the irreducible algebraic representations of $G_{/K}$; we denote the L-packet indexed by ξ by $\Pi_d(\xi)$. By a regular discrete series we will mean one belonging to an L-packet $\Pi_d(\xi)$ with ξ having regular highest weight.

4.2. $\boldsymbol{\sigma}$**.** Suppose $a_i b_i \neq 0$ for all i. Let π be a cuspidal automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ with $\pi_\infty \in \Pi_d(\xi)$ for some regular algebraic representation ξ of $G_{/K}$. Let χ_π be the character of the scalar torus $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \subset G$ determined by π (an algebraic Hecke character of $\mathbb{A}_K^\times$). Let $\Sigma(\pi)$ be the finite set comprising the primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Let $\underline{c} \in X(T)$ be the (regular) highest weight of ξ. Put $i(\underline{c}) := (c_0', -\underline{c}_1', \ldots, -\underline{c}_k')$, where if $\underline{c}_i = (c_{i,1}, \ldots, c_{i,n_i})$ then $\underline{c}_i' := (c_{i,n_i}, \ldots, c_{i,1})$ and $c_0' := c_0 + \sum_{i=1}^k \sum_{j=1}^{n_i} c_{i,j}$. Then $i(\underline{c})$ is also a regular dominant character in $X(T)$.

The weight of an irreducible algebraic representation of $G_{/K}$ is the integer m such that the action of the central torus $\mathbb{G}_m \subset G$ is given by $x \mapsto x^m$; the weight of the representation ξ with highest weight $\underline{c} \in X(T)$ is $c_0 + c_0'$.

It follows from the proofs of Corollary 8.5.3 and Lemma 8.5.6 in [Morel 2010] — see especially the top paragraph on page 156 there — that there exist partitions $n_i = m_{i,1} + \cdots + m_{i,r_i}$ with each $m_{i,j} > 0$, irreducible automorphic representations $\tau_{i,j}$ of $\mathrm{GL}_{m_{i,j}}(\mathbb{A}_K)$, and a finite set of primes $S \supset \Sigma(\pi)$ satisfying the following conditions:

- $\tau_{i,j}$ is discrete.
- $\tau_{i,j}^c = \tau_{i,j}^\vee$.
- For $\ell \notin S$ and $v|\ell$, each $\tau_{i,j,v}$ is unramified.
- Let $\ell \notin S$, $v|\ell$, and let $\tau_{i,v}$ be the unramified irreducible subquotient of $\mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}}\big(\bigotimes_j \tau_{i,j,v}\big)$ and σ_ℓ the irreducible representation of $H(\mathbb{Q}_\ell)$ defined by the tuple $\big(\bigotimes_{v|\ell} \chi_\pi^c, \big(\bigotimes_{v|\ell} \tau_{i,v}\big)\big)$. If ψ_{π_ℓ} is the Langlands parameter of π_ℓ, then $\mathrm{BC} \circ \psi_\ell$ is the Langlands parameter of σ_ℓ.
- The infinitesimal character of $\tau_i := \mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}}\big(\bigotimes_j \tau_{i,j}\big)$ is the same as that of the absolutely irreducible algebraic character of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$ of highest weight $(\underline{c}_i, -\underline{c}_i')$; $\chi_\pi^c(z) = z^{c_0}\bar{z}^{c_0'}$.

Here, $P_i \subset \mathrm{GL}_{n_i}$ is the standard parabolic associated with the partition $n_i = m_{i,1} + \cdots + m_{i,r_i}$.

Recall that the infinitesimal character of an admissible representation of $\mathrm{GL}_m(\mathbb{C})$ is an element of $\mathfrak{a}_{m,\mathbb{C}}^\vee$ modulo the action of the Weyl group $W(\mathfrak{gl}_{m,\mathbb{C}}, \mathfrak{a}_{m,\mathbb{C}})$, where $\mathfrak{gl}_m := \mathrm{Lie}(\mathrm{GL}_m(\mathbb{C}))$ and $\mathfrak{a}_m := \mathrm{Lie}(A_m(\mathbb{C}))$ with $A_m := \mathbb{G}_m^m \subset \mathrm{GL}_m$ the diagonal torus. Identifying $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C} \times \mathbb{C}$ via $z \otimes w \mapsto (zw, \bar{z}w)$ and $\mathbb{C} = \mathrm{Lie}(\mathbb{C}^\times)$ (in the usual way, so the exponential map is $z \mapsto e^z$) identifies $\mathfrak{a}_{m,\mathbb{C}}$ with $\mathbb{C}^m \times \mathbb{C}^m$, and hence $\mathfrak{a}_{m,\mathbb{C}}^\vee := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}_{m,\mathbb{C}}, \mathbb{C}) = \mathbb{C}^m \times \mathbb{C}^m$ (using the dual basis); $W(\mathfrak{gl}_{m,\mathbb{C}}), \mathfrak{a}_{m,\mathbb{C}}^\vee)$ is then identified with $\mathfrak{S}_m \times \mathfrak{S}_m$. An absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ corresponds to its highest weight with respect to

$$(\mathrm{Res}_{K/\mathbb{Q}}A_m, \mathrm{Res}_{K/\mathbb{Q}}B_m),$$

$B_m \subset \mathrm{GL}_m$ being the upper-triangular Borel; this is an element of

$$X(\mathrm{Res}_{K/\mathbb{Q}}A_m) = X(A_m) \times X(A_m)$$

(the identification being via $\mathrm{Res}_{K/\mathbb{Q}}A_{m/K} = A_m \times A_m$) given by a pair of dominant characters of $X(A_m) = \mathbb{Z}^m$ (the last identification is the usual one: $\underline{c} = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ corresponds to the character $\mathrm{diag}(t_1, \ldots, t_m) \mapsto t_1^{c_1} \cdots t_m^{c_m}$; dominant characters satisfy $c_1 \geq \cdots \geq c_m$, and regular dominant characters are those where the inequalities are strict). The infinitesimal character of the irreducible representation of highest weight $(\underline{c}_1, \underline{c}_2)$ is $(\underline{c}_1, \underline{c}_2) + \rho_{\mathrm{GL}_m} \in \mathfrak{a}^\vee_{m,\mathbb{C}}$, where $\rho_{\mathrm{GL}_m} := ((m-1)/2, (m-3)/2, \ldots, (3-m)/2, (1-m)/2)$ is half the sum of the usual positive roots in $\mathfrak{gl}_m$.

As ξ is regular, if the weight of ξ is zero (that is, $c_0 + c_0' = 0$) then by [Morel 2010, Theorem 7.3.1], the Satake parameters of π_ℓ, $\ell \notin S$, all have absolute value 1. The same is then true of the Satake parameters of $\tau_{i,j,v}$ for any $v|\ell$ as $\psi_{\sigma_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$. For ξ having general weight $m \in \mathbb{Z}$, let π' and ξ' be the twists of π and ξ, respectively, by the character $\mu(\cdot)^{-m}$; then ξ' is regular of weight 0 and $\pi'_\infty \in \Pi_d(\xi')$. The representations of the $\mathrm{GL}_{n_i}(\mathbb{A}_K)$ associated to π' as above are the same as those associated to π: this can be seen by the relation between Langlands parameters at $\ell \notin S$. The case of general weight then follows immediately from that of weight zero. Therefore, we also have that

- for $\ell \notin S$, $v|\ell$, the Satake parameters of $\tau_{i,j,v}$ all have absolute value 1 - $\tau_{i,j,v}$ is tempered; furthermore, $\tau_{i,v} = \mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}}\left(\bigotimes_j \tau_{i,j,v}\right)$ and is a tempered principal series.

Lemma 3. *Each $\tau_{i,j}$ is cuspidal, and $\sigma_{i,j} := \tau_{i,j} \otimes |\cdot|^{(m_{i,j}-n_i)/2}$ is algebraic and has the same infinitesimal character as a regular absolutely irreducible algebraic representation $\xi_{i,j}$ of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$.*

Here $\sigma_{i,j}$ being algebraic automorphic representation of $\mathrm{GL}_{m_{i,j}}(\mathbb{A}_K)$ is as in [Clozel 1990, 1.2.3]: the infinitesimal character $\underline{b}_{i,j} \in \mathfrak{a}^\vee_{m_{i,j},\mathbb{C}} = \mathbb{C}^{m_{i,j}} \times \mathbb{C}^{m_{i,j}}$ of $\sigma_{i,\infty}$ satisfies $\underline{b}_{i,j} + (1-m_{i,j})/2 \in \mathbb{Z}^{m_{i,j}} \times \mathbb{Z}^{m_{i,j}}$.

Proof. As $\tau_{i,j}$ is discrete, by the main results of [Mœglin and Waldspurger 1989] there is a factorization $m_{i,j} = s_{i,j}r_{i,j}$ and an irreducible cuspidal automorphic representation $\alpha_{i,j}$ of $\mathrm{GL}_{s_{i,j}}(\mathbb{A}_K)$ such that $\tau_{i,j}$ is the unique irreducible quotient of

$$\mathrm{Ind}_{P_{i,j}}^{\mathrm{GL}_{m_{i,j}}} \beta_{i,j} \quad \beta_{i,j} = (\alpha_{i,j} \otimes |\cdot|^{(1-r_{i,j})/2}) \otimes \cdots \otimes (\alpha_{i,j} \otimes |\cdot|^{(r_{i,j}-1)/2}),$$

where $P_{i,j} \subset \mathrm{GL}_{m_{i,j}}$ is the standard parabolic associated with the partition $m_{i,j} = s_{i,j} + \cdots + s_{i,j}$ ($r_{i,j}$ summands). Since for all but finitely many v the Satake parameters of $\tau_{i,j,v}$ all have the same absolute value, it must then be that $r_{i,j} = 1$, and so $\tau_{i,j} = \alpha_{i,j}$ is cuspidal.[1]

Let $\underline{a}_{i,j} \in \mathfrak{a}^\vee_{m_{i,j},\mathbb{C}}$ be the infinitesimal character of $\tau_{i,j,\infty}$. Then the infinitesimal character of $\tau_{i,\infty}$ is $\underline{a}_i := (\underline{a}_{i,1}, \ldots, \underline{a}_{i,r_i}) \in \mathfrak{a}^\vee_{n_i,\mathbb{C}}$. In particular, there exist $L', L'' \subset \{1, \ldots, n_i\}$ of cardinality $m = m_{i,j}$ such that $\underline{a} = \underline{a}_{i,j} = (\underline{a}', \underline{a}'') \in \mathbb{C}^m \times \mathbb{C}^m$ with $\underline{a}'$ and $\underline{a}''$ equal to $(c_{i,\ell} + (n_i - 2\ell + 1)/2)_{\ell \in L'}$ and $(-c_{i,\ell} + (2\ell - n_i + 1)/2)_{\ell \in L''}$, respectively. Suppose $L' = \{\ell'_1, \ldots, \ell'_m\}$ with $\ell'_1 < \ell'_2 < \cdots < \ell'_m$ and $L'' = \{\ell''_1, \ldots, \ell''_m\}$ with $\ell''_1 > \ell''_2 > \cdots > \ell''_m$. Then the infinitesimal character $\underline{b} = \underline{b}_{i,j}$ of $\sigma_{i,j}$ is given by $\underline{b} = \underline{a} + (m - n_i)/2 = (\underline{d}', \underline{d}'') + \rho_{\mathrm{GL}_m}$, where

$$\underline{d}' = (c_{i,\ell'_k} + k - \ell'_k)_{1 \le k \le m} \quad \text{and} \quad \underline{d}'' = (-c_{i,\ell''_k} + \ell''_k - n_i + k)_{1 \le k \le m}.$$

As $\rho_{\mathrm{GL}_m} + (1-m)/2 \in \mathbb{Z}^m$, it follows that $\underline{b} + (1-m)/2 \in \mathbb{Z}^m \times \mathbb{Z}^m$, so $\sigma_{i,j}$ is algebraic. Also,

$$\begin{gathered} c_{i,\ell'_k} + k - \ell'_k - c_{i,\ell'_{k+1}} - k - 1 + \ell'_{k+1} = c_{i,\ell'_k} - c_{i,\ell'_{k+1}} - 1 + \ell'_{k+1} - \ell'_k \ge 1 \\ -c_{i,\ell''_k} + \ell''_k - n_i + k + c_{i,\ell''_{k+1}} - \ell''_{k+1} + n_i - k - 1 = c_{i,\ell''_{k+1}} - c_{i,\ell''_k} + \ell''_k - \ell''_{k+1} - 1 \ge 1, \end{gathered}$$

so $\underline{d}'$ and $\underline{d}''$ are both regular and dominant. Therefore,

$$\underline{d} := (\underline{d}', \underline{d}'') \in X(A_m) \times X(A_m)$$

corresponds to a regular absolutely irreducible algebraic representation $\xi_{i,j}$ of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ with infinitesimal character $\underline{d} + \rho_{\mathrm{GL}_m} = \underline{b}$. □

Corollary 4. *The cuspidal representations $\tau_{i,j}$ are tempered at all finite places. Furthermore, each τ_i is irreducible and tempered at all finite places.*

Proof. Choose an algebraic Hecke character χ of $\mathbb{A}_K^\times$ such that $\chi\chi^c = |\cdot|^{n_i - m_{i,j}}$. Then $\sigma_{i,j} \otimes \chi$ is a conjugate self-dual algebraic cuspidal representation with infinitesimal character that of a regular absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$. Therefore, $\sigma_{i,j} \otimes \chi$ is tempered at all finite places by [Shin 2011, Corollary 1.3]. The claims about $\tau_{i,j}$ and τ_i follow easily from this. □

Put

$$\psi := \chi_\pi^c \quad \text{and} \quad \sigma := (\psi, (\tau_i)). \tag{4.4}$$

Then σ is identified with an irreducible automorphic representation of $H(\mathbb{A}_\mathbb{Q})$. This is a very weak base change of π in the sense that the Langlands parameter ψ_{σ_ℓ} of σ_ℓ is $\mathrm{BC} \circ \psi_{\pi_\ell}$ for all $\ell \notin S$, ψ_{π_ℓ} being the Langlands parameter of π_ℓ.

[1] This can also be seen by considering infinitesimal characters.

Remark 5. Suppose $k = 1$. Let π_0 be an irreducible automorphic constituent of the restriction of π to $G_0(\mathbb{A}_\mathbb{Q})$. Then $\tau = \tau_1$ is a very weak base change of π_0 to $H_0(\mathbb{A}_\mathbb{Q})$ that is tempered at all finite places. By Proposition 1, to complete the proof of Theorem A it suffices to show that τ_v is a principal series for all $v|\ell$, $\ell \notin \Sigma(\pi)$. This is done in the following by analyzing certain Galois representations associated with τ.

4.3. $\boldsymbol{\rho_\pi}$**.** Let $\rho : G_K \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ be a continuous representation. Let ξ be an absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ with highest weight $(\underline{c}_1, \underline{c}_2) \in X(A_m) \times X(A_m) = \mathbb{Z}^m \times \mathbb{Z}^m$. Let $v|p$ be a place of K. Recall that $\rho_v := \rho|_{G_{\mathbb{Q}_v}}$ being Hodge–Tate means that the graded $(\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v)$-module $D_{\mathrm{HT},v}(\rho_v) := (\rho_v \otimes B_{\mathrm{HT},v})^{G_{K_v}}$, $B_{\mathrm{HT},v} := \bigoplus_{t\in\mathbb{Z}} \widehat{\overline{K}}_v(t)$, is a free $(\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v)$-module of rank m. By ρ_v being of Hodge–Tate type ξ we mean that for any $j \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$, the graded $\overline{\mathbb{Q}}_p$-module $D_{\mathrm{HT}}(\rho_v) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, j} \overline{\mathbb{Q}}_p$ is nonzero in degrees $i - 1 - c_{1,i}$, $i = 1, .., m$, if the restriction of j to K is the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and otherwise is nonzero in degrees $i - 1 - c_{2,i}$, $i = 1, \dots, m$.

Let $\sigma_{i,j}$ be as in Lemma 3. From [Shin 2011] we conclude that there exist representations $\rho_{i,j} = \rho_{\sigma_{i,j},\iota} : G_K \to \mathrm{GL}_{m_{i,j}}(\overline{\mathbb{Q}}_p)$ such that

- $\rho_{i,j}$ is continuous and semisimple,
- for $v \nmid p$, $\mathrm{WD}(\rho_{i,j}|_{G_{K_v}})^{\mathrm{Fr-ss}} = \iota\mathrm{Rec}_v\big(\sigma_{i,j,v} \otimes |\cdot|_v^{(1-m_{i,j})/2}\big)$,
- $\rho_{i,j}^c \cong \rho_{i,j}^\vee \otimes \epsilon^{1-n_i}$,
- for each $v|p$, $\rho_{i,j}|_{G_{K_v}}$ is potentially semistable of Hodge–Tate type $\xi_{i,j}$,
- for $v|p$, if $\sigma_{i,j,v}$ is unramified then $\rho_{i,j}|_{G_{K_v}}$ is crystalline and the eigenvalues of the $[K_v : \mathbb{Q}_p]$-th power of the crystalline Frobenius on
$$D_{\mathrm{cris}}(\rho_{i,j}|_{G_{K_v}}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v^0, \lambda} \overline{\mathbb{Q}}_p, \quad \text{any } \lambda \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v^0, \overline{\mathbb{Q}}_p),$$
are the Frobenius eigenvalues of $\iota\mathrm{Rec}_v\big(\sigma_{i,j,v} \otimes |\cdot|_v^{(1-m_{i,j})/2)}\big)$, where $K_v^0 \subset K_v$ is the maximal absolutely unramified extension.

Here $\mathrm{WD}(\rho_{i,j}|_{G_{K_v}})^{\mathrm{Fr-ss}}$ is the Frobenius semisimple Weil–Deligne representation associated to the $\rho_{i,j}|_{G_{K_v}}$.

The existence of $\rho_{i,j}$ follows from [Shin 2011, Theorem 1.2]: As in the proof of Corollary 4, choose an algebraic Hecke character χ of $\mathbb{A}_K$ such that $\sigma_{i,j} \otimes \chi$ is conjugate self-dual; such a character can be chosen to be unramified at any given finite set of finite places. Then [ibid., Theorem 1.2] applies to $\sigma_{i,j} \otimes \chi$ and we set $\rho_{i,j} := R_{p,\iota}(\sigma_{i,j}^\vee \otimes \chi^{-1}) \otimes \rho_{\chi,\iota}^\vee$ in Shin's notation (the contragredients are here because of the normalization of the local Langlands correspondence in [Shin 2011]). By varying the set of primes at which χ is unramified we obtain the compatibility with the local Langlands correspondence at all $v \nmid p$. A comparison between the eigenvalues of the $[K_v : \mathbb{Q}_p]$th-power of the crystalline Frobenius eigenvalues and the Frobenius eigenvalues of the Weil–Deligne representation is not stated explicitly in [ibid.] but can be obtained by appealing to the comparison theorem in [Katz and Messing 1974]: the arguments in [Shin 2011, §7] and especially [Taylor and Yoshida 2007, §2] explain that there is a solvable CM-extension L/K in which all places of K above p split and such that $\mathrm{BC}_{L/K}(\sigma_{i,j}^\vee \otimes \chi^{-1})$ is cuspidal and an algebraic Hecke character ψ of $\mathbb{A}_L^\times$ unramified at all primes above p such that some multiple of the p-adic G_L-representation $R_{p,\iota}(\mathrm{BC}_{L/K}(\sigma_{i,j}^\vee \otimes \chi^{-1})) \otimes \rho_{\psi,\iota}$ is cut out by correspondences acting on the cohomology with constant coefficients of a self-product of the universal abelian variety over a compact Shimura variety (with good reduction at v if $\sigma_{i,j,v} \otimes \chi_v$ is unramified). Here $\mathrm{BC}_{L/K}(\cdot)$ denotes the base change lift to $\mathrm{GL}_n(\mathbb{A}_L)$.

Put

$$\rho_i := \textstyle\bigoplus_{j=1}^{r_i} \rho_{i,j}, \quad i = 1, \dots, k,$$

and

$$\rho_\pi := \rho_\psi \otimes \big(\textstyle\bigoplus_{i=1}^{k} \rho_i\big). \tag{4.5}$$

Remark 6. Suppose $k = 1$. Then ρ_π satisfies the conclusions of Theorem B, but with S replacing $\Sigma(\pi)$ and with the additional condition that $p \notin S$ for part (d); the definition of ρ_π being of "Hodge–Tate type ξ" is given after Theorem 10 below.

Proposition 7. *For $v|\ell$, $\ell \notin \Sigma(\pi)$, the representations $\tau_{i,j,v}$ and $\tau_{i,v}$ are tempered principal series.*

Our proof of this proposition will come from an understanding of the ramification at $v|\ell$, $\ell \notin \Sigma_p(\pi)$, of the representation

$$r_\pi := \rho_\psi \otimes \bigotimes_{i=1}^{k} \textstyle\bigwedge^{a_i} \rho_i .$$

First, we explain what it means for π to be an endoscopic lift. This means that each n_i has a partition $n_i = n_i^+ + n_i^-$ as a sum of nonnegative integers with some $n_j^+ n_j^- \neq 0$ and such that $\sum_{i=1}^k n_i^-$ is even, and that there is a cuspidal automorphic representation γ of $G'(\mathbb{A}_\mathbb{Q})$, with

$$G' := G(U(a_1^+, b_1^+) \times U(a_1^-, b_1^-) \times \cdots \times U(a_k^+, b_k^+) \times U(a_k^-, b_k^-))$$

and

$$(a_i^\pm, b_i^\pm) = \Big(\Big\lfloor \frac{n_i^\pm + 1}{2} \Big\rfloor, \Big\lceil \frac{n_i^\pm - 1}{2} \Big\rceil\Big),$$

such that γ is unramified at each prime $\ell \notin \Sigma(\pi)$, and for each $\ell \notin \Sigma(\pi)$ the Langlands parameter ψ_{π_ℓ} of π_ℓ is the composition of the Langlands parameter ψ_{γ_ℓ}

of γ_ℓ with the endoscopic L-group homomorphism

$$\mathrm{End} : {}^LG' \to {}^LG,$$

is defined as follows (see also [Morel 2010, Proposition 2.3.2]. Let $\epsilon_{K/\mathbb{Q}}:W_{\mathbb{Q}}\to\{\pm 1\}$ be the nontrivial quadratic character factoring through $\mathrm{Gal}(K/\mathbb{Q})$; by class field theory this determines a quadratic character $\omega_{K/\mathbb{Q}} : \mathbb{A}^\times/\mathbb{Q}^\times \to \{\pm 1\}$. Fix a finite-order Hecke character ω_K of $\mathbb{A}_K^\times$ such that $\omega_K|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$ and let $\mu : W_K \to \mathbb{C}^\times$ be the character corresponding via class field theory. Let $c \in W_{\mathbb{Q}}$ be a lift of the nontrivial automorphism of K. Define $\varphi : W_{\mathbb{Q}} \to {}^LG$ by

$$\varphi(c) = \left(1, \left(\begin{pmatrix} \Phi_{n_i^+} & \\ & (-1)^{n_i^+}\Phi_{n_i^-} \end{pmatrix} \Phi_{n_i}^{-1}\right)\right) \rtimes c,$$

$$\varphi(w) = \left(1, \left(\begin{pmatrix} \mu^{n_i^-}(w) I_{n_i^+} & \\ & \mu^{-n_i^+}(w) I_{n_i^-} \end{pmatrix}\right)\right) \rtimes w, \quad w \in W_K.$$

The endoscopic map is then

$$\mathrm{End}((\lambda, (g_i^+, g_i^-)) \rtimes w) = \left(\lambda, \begin{pmatrix} g_i^+ & \\ & g_i^- \end{pmatrix}\right)\varphi(w).$$

Here $((\lambda, (g_i^+, g_i^-)) \in \widehat{G}' = \mathbb{C}^\times \times \prod_{i=1}^k \mathrm{GL}_{n_i^+}(\mathbb{C}) \times \mathrm{GL}_{n_i^-}(\mathbb{C})$.

Lemma 8. *Either π is an endoscopic lift of some γ with γ_∞ a regular discrete series or the representation r_π is unramified at all $v|\ell$, $\ell \notin \Sigma_p(\pi)$.*

Proof. By [Morel 2010, Theorem 7.2.2] (see also the proof of [ibid., Theorem 7.3.1]), either π is an endoscopic lift of some γ with γ_∞ a regular discrete series indexed by a representation with the same weight as ξ (see [ibid., Lem. 7.3.4]) or (some multiple of) $r_\pi^\vee$ occurs[2] in the middle degree intersection cohomology of a Shimura variety associated with G, ξ, and π. By [Lan 2008], this Shimura variety is known to have good reduction at all $v|\ell$, $\ell \notin \Sigma_p(\pi)$, so the representation r_π is unramified at such v. □

Proof of Proposition 7. Let $v|\ell$, $\ell \notin \Sigma(\pi)$. Suppose π is the endoscopic lift of some γ with γ_∞ a regular discrete series. Let $\sigma' = (\psi', (\tau_i^+, \tau_i^-))$ be the very weak base change of γ as in Section 4.2 (so $\tau_i^\pm$ is an irreducible automorphic representation of $\mathrm{GL}_{n_i^\pm}(\mathbb{A}_K)$). From the definition of π being an endoscopic lift of γ, it follows that

$$\tau_i = (\tau_i^+ \otimes \omega_K^{-n_i^-}) \boxplus (\tau_i^- \otimes \omega_K^{n_i^+})$$

[2]Theorem 7.2.2 of [Morel 2010] only applies to the case $k = 1$ as stated, but it is asserted at the start of [ibid., 7.2] that the results and proofs "would work the same way" for general k. Indeed, the result for the case $k > 1$ is stated and used in the proof of [ibid., Theorem 7.3.1].

(as $\tau_i^{\pm}$ is tempered by Corollary 4). We may therefore reduce to the case where π is not endoscopic, and hence, by Lemma 8, to the case where r_π is unramified at v.

Suppose that r_π is unramified at v. Consider the isogeny

$$G_1 := \mathrm{GL}_1 \times \prod_{i=1}^{k} \prod_{j=1}^{r_i} \mathrm{GL}_{m_{i,j}} \to G_2 := \mathrm{GL}_1 \otimes \bigotimes_{i=1}^{k} \mathrm{GL}_{\binom{n_i}{a_i}},$$

$$(\lambda, (g_{i,j})) \mapsto \lambda \otimes \bigotimes_{i=1}^{k} \textstyle\bigwedge^{a_i}(\mathrm{diag}(g_{i,1}, \ldots, g_{i,r_i})).$$

The kernel of this isogeny is central. As r_π is the composition of

$$\rho := \rho_\psi \oplus \bigoplus_{i=1}^{k} \rho_i : G_K \to G_1(\overline{\mathbb{Q}}_p)$$

with this isogeny, it then follows that since r_π is unramified at v, the image of inertia at v under ρ is contained in the center of $G_1(\overline{\mathbb{Q}}_p)$, and so the image of inertia at v under each $\rho_{i,j}$ is central. So some finite-order twist of each $\rho_{i,j}$ is unramified at v, which — by compatibility of $\rho_{i,j}$ with the local Langlands correspondence — implies that a finite-order twist of each $\sigma_{i,j,v}$, and hence of each $\tau_{i,j,v}$, is unramified. By Corollary 4, $\tau_{i,j,v}$ is also tempered. It follows that each $\tau_{i,j,v}$ is a tempered principal series, so each $\tau_{i,v}$ must also be a tempered principal series. □

4.4. ***The main results.*** We can now state our main results, of which Theorems A and B are special cases, and complete their proofs.

Theorem 9. *Let π be an irreducible cuspidal representation of $G(\mathbb{A}_\mathbb{Q})$ and let χ_π be the character of the scalar torus $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \subset G$ determined by π (a character of $\mathbb{A}_K^\times$). Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Suppose $a_i b_i \neq 0$, $i = 1, \ldots, k$, and π_∞ is a regular discrete series belonging to an L-packet $\Pi_d(\xi)$. There exists an automorphic representation $\sigma = (\psi, (\tau_i))$ of $H(\mathbb{A}_\mathbb{Q})$ such that*:

(a) $\sigma^\theta \cong \sigma$; $\psi = \chi_\pi^c$.

(b) *For a prime $\ell \notin \Sigma(\pi)$, σ_ℓ is unramified, and if $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G$ is the Langlands parameter of π_ℓ then*

$$\psi_{\sigma_\ell} := \mathrm{BC} \circ \psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L H$$

is the Langlands parameter of σ_ℓ.

(c) *σ_∞ has the same infinitesimal character as $\xi \otimes \xi^\theta$.*

Proof. Let $\sigma = (\psi, (\tau_i))$ be as in (4.4). Then part (a) holds. Furthermore, there exists a finite set of primes $S \supset \Sigma(\pi)$ such that part (b) holds with S replacing $\Sigma(\pi)$.

Let $\pi_0 \subset \pi$ be an irreducible automorphic representation of $G_0(\mathbb{A}_\mathbb{Q})$. Then π_0 is given by a tuple $(\pi_{0,i})_{1 \le i \le k}$ with $\pi_{0,i}$ an automorphic representation of $U(a_i, b_i)$.

For $\ell \notin \Sigma(\pi)$, each $\pi_{0,i,\ell}$ is unramified and the Langlands parameter $\psi_{\pi_{0,i,\ell}}$ of $\pi_{0,i,\ell}$ is given by composing $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^LG$ with the projection

$$^LG \to {}^LG_0 \to {}^LU(a_i, b_i) = \mathrm{GL}_{n_i}(\mathbb{C}) \rtimes W_{\mathbb{Q}}.$$

From part (b) holding for $\ell \notin S$ it then follows that for such ℓ the Langlands parameter of $\tau_{i,\ell}$ is $\mathrm{BC} \circ \psi_{\pi_{0,i,\ell}}$; that is, τ_i is a very weak base change of $\pi_{0,i}$. But by Proposition 7, $\tau_{i,v}$ is a tempered principal series for each $v|\ell$, $\ell \notin \Sigma(\pi)$, so it follows from Proposition 1 that τ_i is a weak base change of $\pi_{0,i}$. That (b) holds is then immediate from the relation between the Langlands parameters of π_ℓ and of the $\pi_{0,i,\ell}$.

To see that part (c) holds, we first recall that the infinitesimal character of an admissible representation of $H(\mathbb{R})$ is an element of $\mathfrak{s}_{\mathbb{C}}^\vee$ up to action of the Weyl group $W(\mathfrak{h}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}})$, where $\mathfrak{h} := \mathrm{Lie}(H(\mathbb{R}))$ and $\mathfrak{s} := \mathrm{Lie}(S(\mathbb{R}))$ with $S := \mathrm{Res}_{K/\mathbb{Q}} T_{/K} \subset G$ the group of diagonal matrices. Then $S_{/K} = T_{/K} \times T_{/K}$ and $X(S) = X(T) \times X(T)$. The irreducible algebraic representations of $H_{/K}$ correspond to pairs of dominant characters of $X(T)$ — the highest weight of the representation with respect to S and the upper-triangular Borel. In particular, the representation $\xi \otimes \xi^\theta$ corresponds to $(\underline{c}, i(\underline{c}))$ and has infinitesimal character $(\underline{c}, i(\underline{c})) + \rho_H$, where $\rho_H := (0, (\rho_{\mathrm{GL}_{n_i}}))$. On the other hand, $S(\mathbb{R}) = \mathbb{C}^\times \times \prod_{i=1}^k A_{n_i}$ so

$$\mathfrak{s}_{\mathbb{C}}^\vee = \mathbb{C}^2 \oplus \mathfrak{a}_{n_1,\mathbb{C}}^\vee \oplus \cdots \oplus \mathfrak{a}_{n_j,\mathbb{C}}^\vee = \mathbb{C}^{1+n} \times \mathbb{C}^{1+n},$$

and the infinitesimal character of σ_∞ is $(c_0, c_0') \bigoplus_{i=1}^k(\text{infinitesimal character of } \tau_i)$. Since the infinitesimal character of τ_i is $(\underline{c}_i, -\underline{c}_i') + \rho_{\mathrm{GL}_{n_i}}$, the infinitesimal character of σ_∞ is $((c_0, (\underline{c}_i + \rho_{\mathrm{GL}_{n_i}})), (c_0', (-\underline{c}_i' + \rho_{\mathrm{GL}_{n_i}}))) = (\underline{c}, i(\underline{c})) + \rho_H$. ☐

Theorem 10. *Let π be an irreducible cuspidal representation of $G(\mathbb{A}_{\mathbb{Q}})$ and let χ_π be the character of the scalar torus $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \subset G$ determined by π (a character of $\mathbb{A}_K^\times/K^\times$). Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell|d_K$. Suppose $a_ib_i \neq 0$, $i = 1, \ldots, k$, and π_∞ is a regular discrete series belonging to an L-packet $\Pi_d(\xi)$. Let $\sigma = (\psi, (\tau_i))$ be as in Theorem 9. There exists a continuous, semisimple representation*

$$\rho_\pi = \rho_{\pi,\iota} : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

such that:

(1) *ρ_π is unramified at all finite places not above primes in $\Sigma_p(\pi) := \Sigma(\pi) \cup \{p\}$, and for such a place w*

$$(\rho_\pi|_{W_{K_w}})^{ss} = \textstyle\bigoplus_{i=1}^k \iota\mathrm{Rec}_w\big(\tau_{i,w} \otimes \psi_w|\cdot|_w^{(1-n_i)/2}\big).$$

(b) *For $v|p$, $\rho_\pi|_{G_{K_v}}$ is potentially semistable of Hodge–Tate-type ξ.*

(c) *If $p \notin \Sigma(\pi)$ then for any $v|p$, $\rho_\pi|_{G_{K_v}}$ is crystalline; for any*

$$j \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$$

the eigenvalues of the action of the $[K_v : \mathbb{Q}_p]$-th power of the crystalline Frobenius on

$$D_{\mathrm{cris}}(\rho_\pi|_{G_{K_v}}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, j} \overline{\mathbb{Q}}_p$$

are the eigenvalues of the action of Frobenius on

$$\bigoplus_{i=1}^{k} \iota\mathrm{Rec}_v\big(\tau_{i,v} \otimes \psi_v|\cdot|_v^{(1-n_i)/2}\big).$$

Let $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k) \in X(T)$ be the highest weight of ξ. By $\rho_\pi|_{G_{K_v}}$ being of Hodge–Tate type ξ, we mean that ρ_π is of Hodge–Tate type $(c_0 + \underline{c}, \underline{c}'_0 + \underline{c}')$.

Proof. If we take ρ_π to be as in (4.5), then (a) is immediate from Theorem 9(b) and the definition of ρ_π as being the twist by ρ_ψ of the sum of the $\rho_{i,j}$. From the proof of Lemma 3, the character $\xi_{i,j}$ has highest weights

$$(c_{i,\ell'_t} + t - \ell'_t, -c_{i,\ell''_t} + \ell''_t - n_i + t)_{1 \le t \le m_{i,j}},$$

and so for $v|p$,

$$D_{\mathrm{HT},v}(\rho_{i,j}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, \zeta} \overline{\mathbb{Q}}_p$$

is nonzero in degrees $\ell'_t - 1 - c_{i,\ell'_t}$ if $\zeta \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$ induces the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}_p \cong \mathbb{C}$, and otherwise is nonzero in degrees $n_i - \ell''_t - 1 + c_{i,\ell''_t}$. That $\rho_\pi|_{G_{K_v}}$ is of Hodge–Tate type ξ then follows from this and the fact that $\psi_\infty(z) = z^{c_0}\bar{z}^{c'_0}$ and so ρ_ψ is of Hodge–Tate type (c_0, c'_0). That $\rho_\pi|_{G_{K_v}}$, $v|p$, is potentially semistable and even crystalline with the prescribed Frobenius eigenvalues if $v|p$ follows from the corresponding facts for ρ_ψ and the $\rho_{i,j}$. □

Theorems A and B are just the special cases where $k = 1$.

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cmcls@math.princeton.edu *Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, United States*

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Abelian varieties and Weil representations

Sug Woo Shin

The main goal of this article is to construct and study a family of Weil representations over an arbitrary locally noetherian scheme without restriction on characteristic. The key point is to recast the classical theory in the scheme-theoretic setting. As in work of Mumford, Moret-Bailly and others, a Heisenberg group (scheme) and its representation can be naturally constructed from a pair of an abelian scheme and a nondegenerate line bundle, replacing the role of a symplectic vector space. Once enough is understood about the Heisenberg group and its representations (e.g., the analogue of the Stone–von Neumann theorem), it is not difficult to produce the Weil representation of a metaplectic group (functor) from them. As an interesting consequence (when the base scheme is Spec $\overline{\mathbb{F}}_p$), we obtain the new notion of mod p Weil representations of p-adic metaplectic groups on $\overline{\mathbb{F}}_p$-vector spaces. The mod p Weil representations admit an alternative construction starting from a p-divisible group with a symplectic pairing.

We have been motivated by a few possible applications, including a conjectural mod p theta correspondence for p-adic reductive groups and a geometric approach to the (classical) theta correspondence.

1. Introduction

For a quick overview of contents and results, see Section 1H.

1A. ***Motivation from theta correspondence.*** The Heisenberg groups, their representations and the Weil representations (also called oscillator or metaplectic representations) play interesting roles in a wide range of mathematics. In the context of number theory and representation theory, they give rise to the theta correspondence, which enables us to relate automorphic forms or representations of one connected reductive group (or its covering group) to those of another group. It not only helps to establish instances of the Langlands functoriality but also reveals

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deep information about arithmetic invariants and as such has led to numerous profound applications. The theta correspondence has been developed very well in both local and global contexts (namely, for p-adic/real groups and adelic groups, respectively), though there are still many open questions, for representations on vector spaces over $\mathbb{C}$ (or an algebraically closed field of characteristic 0).

On the other hand, there has been growing interest in the representations of p-adic reductive groups on vector spaces over $\overline{\mathbb{F}}_l$ ($l \neq p$) and $\overline{\mathbb{F}}_p$ (as well as representations with p-adic analytic structure) in connection with Galois theory as part of the extended Langlands philosophy under the motto "mod l, mod p and p-adic local Langlands program". From the global perspective, one seeks the theta correspondence for mod p or p-adic automorphic forms.[1] Thus, it is a very natural question to ask whether there is a reasonable theory of local and global theta correspondence for representations on $\overline{\mathbb{F}}_l$ and $\overline{\mathbb{F}}_p$ vector spaces and more ambitiously for representations of p-adic analytic nature.

In the classical theory, the following basic ingredients are needed to formulate the local theta correspondence for p-adic groups. The global setup is similar. (Unfortunately the exceptional theta correspondence is not going to be considered in our work.) We need

(i) a p-adic Heisenberg group arising from a symplectic vector space $(V, \langle\cdot,\cdot\rangle)$ over $\mathbb{Q}_p$,

(ii) the Stone–von Neumann theorem and Schur's lemma for representations of the Heisenberg group,

(iii) the Weil representation of the p-adic metaplectic group $\mathrm{Mp}(V, \langle\cdot,\cdot\rangle)$, and

(iv) reductive dual pairs in $\mathrm{Sp}(V, \langle\cdot,\cdot\rangle)$.

It is natural to try to extend (i)–(iv) to a more general setting. The current paper will do this for (i)–(iii), leaving (iv) (and a conjectural mod p theta correspondence) to a sequel [Shin $\geq$ 2012].

1B. ***Mod p Weil representations, prelude.*** Let us briefly point out some difficulty when trying to construct the Weil representation of a p-adic metaplectic group on an $\overline{\mathbb{F}}_p$-vector space, which was not done before but is a special case of our results. There would be two naïve approaches. If one tries to define an $\overline{\mathbb{F}}_p$-version of a classical p-adic Heisenberg group (e.g., [Mœglin et al. 1987, Chapter 2]) by replacing the role of $\mathbb{C}$ by $\overline{\mathbb{F}}_p$, it is impossible to obtain a reasonable group, ruling out (ii) above. For instance, every continuous additive character $\mathbb{Q}_p \to \overline{\mathbb{F}}_p^\times$ is trivial.

[1] The p-adic version of the Shimura–Shintani correspondence was studied in [Stevens 1994; Ramsey 2009; Park 2010]. Their work interpolates the classical correspondence p-adically and does not require much use of representation theory. The author does not know yet whether or how their work could be interpreted in the framework of representation theory.

Another approach would be to take an explicit (Schrödinger or lattice) model for the Weil representation and switch the coefficient from $\mathbb{C}$ to $\overline{\mathbb{F}}_p$. Then the problem is that the group actions are no longer well defined. In the Schrödinger model, some group action is given by Fourier transform, which cannot be defined for $\overline{\mathbb{F}}_p$-valued functions on a p-adic group. (See Remark 7.9 for a related discussion.) In the lattice model, the formula involves p in the denominator, which no longer makes sense. It is not immediately clear how to fix these problems unless new ideas are introduced.

1C. *Geometric construction via Mumford's theory.* We remedy the situation by giving a uniform geometric construction of (i)–(iii) regardless of the characteristic of the coefficient field, starting from an abelian scheme $A \to S$ and a nondegenerate line bundle L instead of a symplectic vector space. In the local case, the effect is roughly to replace $(V, \langle \cdot , \cdot \rangle)$ by the rational p-adic Tate module V_pA of A with L-Weil pairing. (Here V_pA is regarded as an ind-group scheme as explained in Section 3A.) The construction makes sense even in characteristic p; it just behaves differently. (For an analogy, think about $A[p^\infty]$ in characteristic p and away from p.) Actually (i) and (ii) are basically treated in Mumford's theory of abelian varieties and theta functions. (As the results are often not in the desired generality in the literature, we fill the gaps along the way. See the next paragraph.) Once (i) and (ii) are done, (iii) is obtained without much difficulty. The theory is so flexible as to allow the construction of the objects (i)–(iii) over an arbitrary locally noetherian base scheme S.

Sections 2–4 of our paper follow the approach of [Mumford 2007, §§3–5] and [Moret-Bailly 1985, §5] closely while adapting several facts in the classical theory of theta correspondence (e.g., [Mœglin et al. 1987]) to the geometric setting. In [Mumford 2007, §§3–5], the Heisenberg groups and their representations are studied mostly over an (algebraically closed) field, and the scheme-theoretic approach in the relative setting is only sketched on a few pages. Moret-Bailly consistently works in the relative setting, but the theory is treated only at finite level. Our contribution is to carry out the construction and justify necessary facts (e.g., Theorems 1.1 and 1.2) at infinite level (in a p-adic or a finite adelic limit). As a byproduct we obtain the (dual) lattice model over a general locally noetherian base scheme and deduce the restriction property of Heisenberg and Weil representations (Section 4E and Lemma 5.10) from the Künneth formula. (It turns out that lattice models always exist, but Schrödinger models are often missing.) We can also make sense of matrix coefficients and dual representations in this generality. It is hoped that the geometric interpretation will shed light on some facts well known by other methods.

Our work is definitely not the first attempt toward a geometric construction of Weil representations. This was considered in an unpublished manuscript of Harris [1987]. (It appears that the manuscript was planned to include an application to some cases of the symplectic-orthogonal theta correspondence, but that part was not written to our knowledge.) His approach to Heisenberg groups and representations closely follows that of Mumford [1966, 1967a, 1967b] and works only in characteristic 0 (even though ideas are often generalizable). Hence, his setting is simpler than ours, and many scheme-theoretic issues do not arise there. His innovation is to construct a Weil representation in the way that it is closely tied with the geometry of Siegel modular varieties. On the other hand, our construction is so general that it applies to almost any families of abelian varieties, but when specialized to the universal abelian scheme over a Siegel modular variety, the two constructions of Weil representations by us and by Harris are orthogonal in some sense.

From a different perspective and motivation, [Gurevich and Hadani 2007] constructs classical Weil representations for finite metaplectic groups as perverse sheaves (Deligne's idea), and the function field analogue is dealt with in [Lysenko 2006; Lafforgue and Lysenko 2009], for instance. Their constructions are quite different from ours and do not seem to carry over to the number field case. In the converse direction, our construction does not work in the function field case either. The basic reason is that the p-adic symplectic (or metaplectic) group in our setting appears as the automorphism group of a rational p-adic Tate module, which is a vector space over $\mathbb{Q}_p$ rather than something like $\mathbb{F}_p((t))$.

In our setup, symplectic groups and metaplectic groups are defined as group functors varying over the base. By introducing a level structure, we can trivialize the rational Tate module (ind-scheme), which has the effect that those group functors may be identified with constant families of groups. When the base is $\operatorname{Spec}\mathbb{C}$, we precisely recover the classical notion of (i)–(iv) from our construction.

It is worth emphasizing that we have completely avoided the use of harmonic analysis. This is only natural for our method to work in all characteristics uniformly. In this regard, even when specialized to the classical case (over $\operatorname{Spec}\mathbb{C}$), our construction of the Weil representation is different from the classical treatment (e.g., [Mœglin et al. 1987]).

As the reader can see, one of our crucial observations was to realize that Mumford's theory had the key to the main question raised in Section 1A. This may appear to be a simple idea, but when we consulted a few experts on theta correspondence, we learned that the idea was largely unnoticed though a similar idea must have been conceived by some experts (e.g., [Harris 1987]).

1D. ***Mod p Weil representations.*** To study finite adelic objects, one may concentrate on one place at a time. So let us restrict ourselves to p-adic Heisenberg groups and p-adic metaplectic groups. By the Stone–von Neumann theorem (more precisely, its analogue in our setting), a family of Heisenberg representations, as well as that of Weil representations, tends to be a constant family (modulo the line bundle pulled back from the base). However, things do change when moving between points of different residue characteristic. Unsurprisingly, new phenomena essentially occur in characteristic p. (This is related to the fact that $A[p^\infty]$ is étale away from characteristic p.) It is worth noting that the Heisenberg and metaplectic groups vary significantly in characteristic p as the isogeny type of $A[p^\infty]$ varies over fibers. On the other hand, over a base ring like $\overline{\mathbb{Z}}_p$, a classical Weil representation (over the generic fiber) specializes to a mod p Weil representation (Section 7D).

In view of these new phenomena, we feel that it is fundamental to understand mod p Weil representations, namely when the base is $\operatorname{Spec} \overline{\mathbb{F}}_p$. In order to make their local nature more transparent, we present an alternative construction of mod p Weil representations using p-divisible groups instead of abelian schemes (Section 6D). Then lattice and Schrödinger models are studied in Sections 7B and 7C. There remains the question of whether the Schrödinger model exists in the nonordinary case (see the paragraph below Proposition 7.10). Another interesting question about the p-adic metaplectic group itself is whether it arises from a double covering of the p-adic symplectic group (see the questions in Section 5D).

1E. *Weil representations of real metaplectic groups.* Real Heisenberg groups and real metaplectic groups do not appear in this paper. This is not defective but quite natural if we want a uniform theory that works in positive characteristics as real groups are not expected to have nice representations on $\overline{\mathbb{F}}_p$-vector spaces. In the special case where the base is $\operatorname{Spec} \mathbb{C}$, it is possible to extend the Heisenberg representations to real places (thereby one can define the real Weil representation) as explained in [Mumford 2007, §5, Application I] (also see Proposition 3.2 of the book).

1F. *Summary of main results.* Let A be an abelian scheme over a locally noetherian scheme S. Let $f : L \to A$ be a symmetric nondegenerate line bundle of index i over A $(0 \le i \le \dim_S A)$. Following Mumford, we construct the adelic Heisenberg group $\widehat{\mathcal{G}}(L) = \widehat{\mathcal{G}}(A, L)$ fitting in a short exact sequence $1 \to \mathbb{G}_m \to \widehat{\mathcal{G}}(L) \to VA \to 1$. A weight-1 representation of $\widehat{\mathcal{G}}(L)$ is defined to be a quasicoherent $\mathcal{O}_S$-module equipped with $\widehat{\mathcal{G}}(L)$-action such that $\lambda \in \mathbb{G}_m$ acts by λ. An (adelic) Heisenberg representation of $\widehat{\mathcal{G}}(L)$ is an irreducible admissible and smooth $\widehat{\mathcal{G}}(L)$-representation of weight 1 that does not vanish anywhere on S. (Admissibility and smoothness are defined in Definitions 4.7 and 4.8.)

Theorem 1.1 (Stone–von Neumann theorem and Schur's lemma, Theorem 4.15). *For any Heisenberg representation $\mathcal{H}$ of $\widehat{\mathcal{G}}(L)$, there is an equivalence of categories*

$$\begin{pmatrix} \textit{weight-}1 \textit{ smooth} \\ \widehat{\mathcal{G}}(L)\textit{-representations} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \textit{quasicoherent} \\ \mathcal{O}_S\textit{-modules} \end{pmatrix}$$

given by $\mathcal{M} \mapsto \underline{\mathrm{Hom}}_{\widehat{\mathcal{G}}(L)}(\mathcal{H}, \mathcal{M})$ *and* $\mathcal{N} \mapsto \mathcal{H} \otimes \mathcal{N}$, *which are quasi-inverses of each other.*

Theorem 1.2 (Construction of Heisenberg representations, Corollary 4.14). *The* $\mathcal{O}_S$*-module* $\widehat{\mathcal{V}}(L) := \varinjlim_n R^i f_*(n^* L)$ *is a Heisenberg representation of* $\widehat{\mathcal{G}}(L)$.

Theorem 1.3 (Construction of Weil representations, Section 5A). *For any Heisenberg representation* $\mathcal{H}$ *of* $\widehat{\mathcal{G}}(L)$, *we can construct a "metaplectic" group functor* $\underline{\mathrm{Mp}}(VA, \hat{e}^L)$ *sitting in a sequence of group functors on* (Sch/S)

$$1 \to \mathbb{G}_m \to \underline{\mathrm{Mp}}(VA, \hat{e}^L) \to \underline{\mathrm{Sp}}(VA, \hat{e}^L) \to 1, \tag{1-1}$$

which is an exact sequence of groups upon evaluation at any locally noetherian S*-scheme.*

Theorem 1.4 (Comparison with classical theory, Section 6A). *In case* $S = \mathrm{Spec}\,\mathbb{C}$, *a choice of level structure for* VA *equipped with* L*-Weil pairing allows one to identify* $\widehat{\mathcal{G}}(L)$, $\widehat{\mathcal{V}}(L)$, $\underline{\mathrm{Sp}}(VA, \hat{e}^L)$ *and* $\underline{\mathrm{Mp}}(VA, \hat{e}^L)$ *with the following objects in the classical finite adelic setting*: *the Heisenberg group*, *Heisenberg representation*, *symplectic group and metaplectic group*, *respectively.* (*Here the metaplectic group is a central extension of the symplectic group by* $\mathbb{C}^\times$ *as can be seen from* (1-1).)

The preceding theorems are also valid in the p-adic setting instead of the finite adelic setting. (In particular, take limits over powers of p rather than all positive integers, and use $V_p A$ in place of VA.) Moreover, the analogous construction works for $(\Sigma, \langle \cdot\,, \cdot \rangle)$ in place of (A, L), where Σ is a p-divisible group over S with a symplectic pairing $\langle \cdot\,, \cdot \rangle$, granted that a Heisenberg representation exists for the Heisenberg group associated with $(\Sigma, \langle \cdot\,, \cdot \rangle)$. This is most interesting when S is an $\mathbb{F}_p$-scheme. A Heisenberg representation for $(\Sigma, \langle \cdot\,, \cdot \rangle)$ can be exhibited when $S = \mathrm{Spec}\, k$ for an algebraically closed field k of characteristic p (but the author does not know in what generality it exists) and leads to a construction of a mod p Weil representation of a p-adic group functor over $\mathrm{Spec}\, k$.

1G. ***Scope of applications and further developments.*** As we construct a family of Weil representations from a family of abelian varieties and line bundles, it would be natural to apply our results to the universal family of abelian varieties over moduli spaces such as Shimura varieties. This should be related to metaplectic automorphic forms and a worthy object already in characteristic 0. We hope that our results will be of some use when studying theta correspondence via Shimura varieties by methods in algebraic geometry, for instance in the context of Kudla's program [2002].

When there is a Weil representation, it is very natural to consider a reductive dual pair and the resulting theta correspondence (Section 1A). In the sequel [Shin

$\geq$ 2012], we do this for the newly constructed mod p Weil representation of a p-adic metaplectic group.

In order to access many cases of mod p Weil representations and theta correspondence, a necessary step would be to explicate the models in Section 7 further, especially in the case of supersingular abelian varieties (or p-divisible groups).

1H. ***Contents and organization of the paper.*** This article is naturally divided into two parts. Under each part we have listed some of the main contents. The sequel [Shin $\geq$ 2012] may be regarded as Part III.

Part I. Heisenberg groups and Heisenberg representations

- Construction of the p-adic or adelic Heisenberg group and Heisenberg representation from an abelian scheme A and a nondegenerate line L bundle over a locally noetherian scheme S. (Sections 2–4)
- A description of the Heisenberg group as $\mathbb{G}_m \times VA$ with a twisted group law, where VA is the "rational Tate module", when L is symmetric. (Section 3E)
- A study of the category of representations of the Heisenberg group, subsuming the Stone–von Neumann theorem and Schur's lemma. (Proposition 2.12 and Theorem 4.15)

Part II. Weil representations, level structures and explicit models

- Construction of the p-adic or adelic metaplectic group and the Weil representation over S. (Sections 5A and 5D)
- Comparison with classical theory via level structure. (Sections 6A–6B)
- Weil representations over $\overline{\mathbb{F}}_p$ of p-adic metaplectic group; Igusa level structure; an approach via a p-divisible group replacing the role of an abelian variety. (Sections 6C–6D)
- Study of lattice and Schrödinger models; examples. (Section 7)

1I. ***Notation and convention.*** If S is a scheme, denote by (Sch/S), (Flat/S) and $(\mathrm{LocNoeth}/S)$ the categories of S-schemes, flat S-schemes and locally noetherian S-schemes, respectively. All fppf sheaves on S in sets or groups are considered on a small fppf site. Their category is a full subcategory of the category functors from (Flat/S) to the category of sets or groups. An $\mathbb{O}_S$-module always means a *quasicoherent* $\mathbb{O}_S$-module in this article and is often viewed as an fppf sheaf on S as well. The category of $\mathbb{O}_S$-modules is denoted QCoh_S.

An object of (Flat/S) may be viewed as an fppf sheaf in sets on S, and this induces a fully faithful functor. The underlined notation such as $\underline{\mathrm{Hom}}$, $\underline{\mathrm{End}}$ and $\underline{\mathrm{Aut}}$ denotes a sheaf or a functor (rather than just a set, a group, a ring, etc.) in the appropriate category determined by the context. Often $\underline{\mathrm{Mp}}$ and $\underline{\mathrm{Sp}}$ are defined as group functors on (Sch/S).

In this article we will usually work in (Sch/S) for a base scheme S. In particular, any morphism of schemes is always assumed to be an S-morphism, and a fiber product is taken over S unless specified otherwise. The tensor product of two $\mathcal{O}_S$-modules is denoted by $\otimes$ (rather than $\otimes_{\mathcal{O}_S}$) if there is no danger of confusion.

2. Finite Heisenberg groups and their representations

We use the following notation:

- S is a scheme,
- $f : A \to S$ is an abelian scheme over S of relative dimension $g \geq 1$,
- $f^\vee : A^\vee \to S$ is the dual abelian scheme (cf. [Faltings and Chai 1990, I.1]),
- L is a line bundle over A,
- $T_x : A \times_S T \to A \times_S T$ is the translation by x, where T is an S-scheme and $x \in A(T)$,
- $\lambda_L : A \to A^\vee$ is the morphism sending x to $T_x^* L \otimes L^{-1}$.

When we think of L, we will often go between two equivalent viewpoints: either as an invertible sheaf $\mathscr{L}$ of $\mathcal{O}_S$-modules on A or as a line bundle equipped with a projection $\pi : L \to A$ (cf. [Mumford et al. 1994, I.3]). Given L, the corresponding $\mathscr{L}$ is described as $\mathscr{L}(U) = \{ s : U \to L \mid \pi \circ s = \mathrm{id}_U \}$ for each open subscheme U of A. By setting $L = \underline{\mathrm{Spec}}(\oplus_{n \leq 0} \mathscr{L}^{\otimes n})$ (relative spectrum over A), we recover L from $\mathscr{L}$. In order to avoid cumbersome switch of notation, we just write L for either L or the corresponding $\mathscr{L}$.

2A. *Nondegenerate line bundles.*

Definition 2.1. A line bundle L over A is (relatively) *nondegenerate* if $\lambda_L : A \to A^\vee$ is a finite morphism.

Lemma 2.2. *If L is nondegenerate, then*

(i) λ_L *is an isogeny* (*a surjective quasifinite homomorphism of group schemes*) *and*

(ii) $\ker \lambda_L$ *is a finite flat group scheme over S.*

Proof. We know that λ_L is compatible with the group scheme structures. Surjectivity and quasifiniteness follow from the case of $S = \mathrm{Spec}\, k$ for a field k when the result is well known (cf. [Bosch et al. 1990, Lemma 1, page 178]). Part (ii) is a consequence of the fact that any isogeny of abelian schemes is finite flat. □

Lemma 2.3. *The following are equivalent*:

(i) *L is nondegenerate in the above sense.*

(ii) *For every point $s \in S$, the fiber L_s over A_s is nondegenerate* (*i.e.*, λ_{L_s} *is finite*).

(iii) *For every geometric point $\overline{s} \in S$, the fiber $L_{\overline{s}}$ over $A_{\overline{s}}$ is nondegenerate* (*i.e.*, $\lambda_{L_{\overline{s}}}$ *is finite*).

Proof. It is obvious that (i) implies (ii). By the flat base change theorem (applied to the base extension from s to $\overline{s}$), (ii) and (iii) are equivalent. It remains to deduce (i) from (ii). Observe that (ii) implies that λ_L is quasifinite. An easy application of the valuative criterion shows that λ_L is proper. Hence, λ_L is finite. □

Lemma 2.4. *Suppose that A is defined over $S = \operatorname{Spec} k$, where k is a field. For a nondegenerate line bundle L, there exists a unique integer $0 \leq \operatorname{ind}(L) \leq g$ (the* ***index*** *of L) such that $H^{\operatorname{ind}(L)}(A, L) \neq 0$.*

Proof. See [Mumford 1974, §16] when k is algebraically closed. The general case is reduced to the algebraically closed case by the flat base change theorem. □

In general the following result is well known. We present a proof as we were incompetent in finding a handy reference.

Lemma 2.5. *Suppose that S is locally noetherian. Let L be a nondegenerate line bundle over A. The index function $s \mapsto \operatorname{ind}(L_s)$ from S to $\mathbb{Z}$ is locally constant* (*with Zariski topology on S*).

Proof. As the question is local, we may assume that S is noetherian and connected. We know that $s \mapsto \dim H^i(A_s, L_s)$ is upper semicontinuous and that $s \mapsto \chi(L_s)$ is constant. Let m be the maximum value of i such that the function $s \mapsto \dim H^i(A_s, L_s)$ is nonzero. (We know $m \leq g$.) The constancy of $\chi(L_s)$ and Lemma 2.4 imply that $\operatorname{ind}(L_s) \in \{m, m-2, m-4, \dots\}$ for all $s \in S$. Since the specialization map $\phi^{m+1}(s) : R^{m+1} f_* L \otimes k(s) \to H^{m+1}(A_s, L_s) = 0$ is trivially surjective, [Hartshorne 1977, Theorem III.12.11(a)] says that it is an isomorphism for every $s \in S$; hence, $R^{m+1} f_* L = 0$. Then Part (b) of the cited theorem implies that $\phi^m(s)$ is surjective for all $s \in S$. On the other hand, $\phi^{m-1}(s)$ is also trivially surjective for $s \in S$, so the same theorem shows that $R^m f_* L$ is locally free on S and that $\phi^m(s)$ is an isomorphism. Therefore, $\operatorname{ind}(L_s) = m$ for all $s \in S$, and we are done.[2] □

Definition 2.6. A line bundle L over A is nondegenerate of *index* $i \in \mathbb{Z}$ if $\operatorname{ind}(L_s) = i$ for all $s \in S$.

Remark 2.7. A nondegenerate line bundle of index 0 is none other than a relatively ample line bundle.

[2]We refer to [Hartshorne 1977] only for convenience as it has the exact form of the theorem we need. As it is written, it applies to a (locally) projective abelian scheme A over S. This is no problem as projectivity can be relaxed to properness by [Grothendieck 1963, III.7.7].

2B. ***Heisenberg groups.*** Define an S-subgroup scheme $K(L) := \ker \lambda_L$ of A. Concretely, the group $K(L)(T)$ for each S-scheme T consists of $x \in A(T)$ such that $T_x^*(L \times T) \simeq (L \times T) \otimes p_2^* M$ for some line bundle M on S, where $p_2 : A \times T \to T$ is the projection map. If L is nondegenerate, then $K(L)$ is a finite flat group scheme by Lemma 2.2.

Let us define a group-valued contravariant functor $\underline{\mathrm{Aut}}(L/A)$ on (Sch/S). The group $\underline{\mathrm{Aut}}(L/A)(T)$ consists of pairs (ψ, x), where $x \in A(T)$ and $\psi : L \times T \to L \times T$ is an isomorphism such that the following diagram commutes:

$$\begin{array}{ccc} L \times T & \xrightarrow{\psi} & L \times T \\ {\scriptstyle (\pi,1)}\downarrow & & \downarrow{\scriptstyle (\pi,1)} \\ A \times T & \xrightarrow{T_x} & A \times T \end{array}$$

The group law is provided by $(\psi_1, x_1)(\psi_2, x_2) = (\psi_1\psi_2, x_1 + x_2)$. The functor $\underline{\mathrm{Aut}}(L/A)$ is representable by a group scheme denoted $\mathcal{G}(L)$ and called a theta group or a *Heisenberg group* (*scheme*). There is a natural sequence of S-group schemes

$$1 \to \mathbb{G}_m \to \mathcal{G}(L) \to K(L) \to 1, \tag{2-1}$$

where the maps are respectively $\alpha \mapsto (\alpha, 0)$ and $(\psi, x) \mapsto x$ on T-valued points. We identified $\mathbb{G}_m$ with the automorphisms of L over A. The argument in the proof of [Mumford 1974, §23, Theorem 1] shows that (2-1) is exact as Zariski sheaves. The commutator map $\mathcal{G}(L) \times \mathcal{G}(L) \to \mathcal{G}(L)$ given by $(\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ has image in $\mathbb{G}_m$ and induces a bilinear pairing

$$e^L : K(L) \times K(L) \to \mathbb{G}_m.$$

Lemma 2.8. *If L is nondegenerate, then e^L is symplectic, namely alternating and nondegenerate.* (*The latter means that an isomorphism $K(L) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}_S}(K(L), \mathbb{G}_m)$ is induced by e^L.*)

Proof. See [Moret-Bailly 1985, IV.2.4(ii)]. □

2C. ***The Stone–von Neumann theorem and Schur's lemma.*** From here on we will always assume that L is *nondegenerate*, unless it is said otherwise.

Definition 2.9 [Moret-Bailly 1985, V.1.1]. Let G be a group scheme over S and $\mathcal{F}$ an $\mathcal{O}_S$-module (always assumed to be quasicoherent). We say that $\mathcal{F}$ is a G-*representation* (on an $\mathcal{O}_S$-module) when $\mathcal{F}$ is equipped with a morphism of fppf sheaves in groups $G \to \underline{\mathrm{Aut}}_{\mathcal{O}_S}(\mathcal{F})$. A morphism between two G-representations $\mathcal{F}_1$ and $\mathcal{F}_2$ is a morphism of $\mathcal{O}_S$-modules $\mathcal{F}_1 \to \mathcal{F}_2$ compatible with G-actions. The same definition makes sense when G is replaced with an fppf sheaf in groups.

Remark 2.10. When $\underline{G}$ is a group functor on (Sch/S), a representation of $\underline{G}$ will mean an $\mathcal{O}_S$-module $\mathcal{F}$ equipped with a morphism of group functors $\underline{G} \to \underline{\mathrm{Aut}}_{\mathcal{O}_S}(\mathcal{F})$, where $\underline{\mathrm{Aut}}_{\mathcal{O}_S}(\mathcal{F})$ is regarded as the group functor $T \mapsto \mathrm{Aut}_{\mathcal{O}_T}(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$ on (Sch/S).

Definition 2.11 [Moret-Bailly 1985, V.2.1, V.2.3]. A $\mathcal{G}(L)$-representation $\mathcal{F}$ is of *weight* $w \in \mathbb{Z}$ if $\mathbb{G}_m$ acts on $\mathcal{F}$ via the character $\lambda \mapsto \lambda^w$. A $\mathcal{G}(L)$-representation $\mathcal{F}$ is *irreducible* if every $\mathcal{G}(L)$-subrepresentation $\mathcal{F}'$ of $\mathcal{F}$ has the form $\mathcal{F}' = \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{I}$ for some ideal sheaf $\mathcal{I}$ of $\mathcal{O}_S$ (equipped with trivial $\mathcal{G}(L)$-action).

Most of the time our focus will be on representations of weight 1 or -1. The following result due to Moret-Bailly (but see Remark 2.14 below) is crucial in understanding weight-1 representations of $\mathcal{G}(L)$.

Proposition 2.12. *Let $\mathcal{F}$ be a $\mathcal{G}(L)$-representation of weight* 1. *Suppose that $\mathcal{F}$ is a locally free $\mathcal{O}_S$-module of rank* $\deg L$.

(i) *$\mathcal{F}$ is an irreducible $\mathcal{G}(L)$-representation.*

(ii) *There is an equivalence between the category of $\mathcal{O}_S$-modules and the category of $\mathcal{G}(L)$-representations of weight* 1 *on $\mathcal{O}_S$-modules given by $\mathcal{N} \mapsto \mathcal{F} \otimes \mathcal{N}$ and $\mathcal{M} \mapsto \underline{\mathrm{Hom}}_{\mathcal{G}(L)}(\mathcal{F}, \mathcal{M})$, which are canonically quasi-inverses of each other.* (*The composition of the two functors in any order is canonically isomorphic to the identity functor.*)

(iii) *If $\mathcal{F}'$ is another weight-*1 *$\mathcal{G}(L)$-representation that is locally free of rank* $\deg L$, *then there exists a unique* (*up to isomorphism*) *line bundle $\mathcal{M}$ on S such that*

$$\mathcal{F}' = \mathcal{F} \otimes \mathcal{M}.$$

(iv) $\underline{\mathrm{End}}_{\mathcal{G}(L)}(\mathcal{F}) \simeq \mathcal{O}_S$ *canonically.*

Proof. The first two assertions are contained in [Moret-Bailly 1985, V.2.4.2, V.2.4.3]. As for (iii), clearly (ii) implies that there is an $\mathcal{O}_S$-module $\mathcal{M}$ such that $\mathcal{F}' = \mathcal{F} \otimes \mathcal{M}$. As $\mathcal{F}'$ and $\mathcal{F}$ are locally free of the same rank, it follows that $\mathcal{M}$ is locally free of rank 1. Part (iv) is a consequence of (ii) since $\underline{\mathrm{End}}_{\mathcal{G}(L)}(\mathcal{F}) \simeq \underline{\mathrm{End}}_{\mathcal{O}_S}(\mathcal{O}_S) \simeq \mathcal{O}_S$. □

For each $0 \leq j \leq g$, note that $R^j f_* L$ is naturally a $\mathcal{G}(L)$-representation of weight 1 (which could be the zero sheaf) in the above sense. We will need the following fundamental result on $\mathcal{G}(L)$-representations:

Proposition 2.13. *Assume that S is locally noetherian and that L has index i.*

(i) *$R^j f_* L = 0$ unless $j = i$.*

(ii) *$R^i f_* L$ is locally free*, *and* $(\mathrm{rank}_{\mathcal{O}_S} R^i f_* L)^2 = \mathrm{rank}_{\mathcal{O}_S} K(L) = (\deg L)^2$. *In particular*, *it satisfies the condition of Proposition 2.12.*

(iii) *$(R^i f_* L)_s \simeq H^i(A_s, L_s)$ for each $s \in S$.*

Proof. When $i = 0$, (i) and (ii) were deduced in [Mumford et al. 1994, Chapter 0, §5] from [Grothendieck 1963, III.7.7.5, III.7.7.10, III.7.8.4.]. The same results of [Grothendieck 1963] imply (i) and (ii) for arbitrary i. Part (iii) amounts to the assertion that $\phi^i(s)$ is an isomorphism as shown in the proof of Lemma 2.5. □

Remark 2.14. When $S = \operatorname{Spec} k$ for an algebraically closed field k, the results of this subsection in this case were proved in the appendix of [Sekiguchi 1977]. (The proof is attributed to Mumford; cf. [Mumford 1966, §1].) The sheaf $R^i f_* L$ provides us with a k-vector space $H^i(A, L)$ with an action of $\mathscr{G}(L)$. Proposition 2.13 says that $H^i(A, L)$ is the unique (up to isomorphism) irreducible $\mathscr{G}(L)$-representation of weight 1. When $S = \operatorname{Spec} \mathbb{C}$, Proposition 2.12 implies the classical Stone von–Neumann theorem for finite Heisenberg groups on $\mathbb{C}$-vector spaces.

2D. ***Matrix coefficient map.*** Let $\mathscr{F}$ be as in Proposition 2.12. Then

$$\mathscr{F}^\vee = \underline{\mathrm{Hom}}_{\mathbb{O}_S}(\mathscr{F}, \mathbb{O}_S)$$

is a $\mathscr{G}(L)$-representation of weight -1 via

$$(\gamma \cdot v^\vee)(v) = v^\vee(\gamma^{-1} \cdot v), \quad \gamma \in \mathscr{G}(L),\ v \in \mathscr{F},\ v^\vee \in \mathscr{F}^\vee. \tag{2-2}$$

Equip $\underline{\mathrm{Hom}}_{\mathbb{G}_m}(\mathscr{G}(L), \mathbb{O}_S)$ with a structure of $\mathscr{G}(L) \times \mathscr{G}(L)$-representation via

$$((\gamma_1, \gamma_2)\cdot\phi)(\gamma) = \phi(\gamma_2^{-1}\gamma\gamma_1), \quad \gamma_1, \gamma_2, \gamma \in \mathscr{G}(L),\ \phi \in \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\mathscr{G}(L), \mathbb{O}_S). \tag{2-3}$$

Lemma 2.15. *The map sending $v \otimes v^\vee$ to $\gamma \mapsto v^\vee(\gamma v)$ yields an isomorphism of $\mathscr{G}(L) \times \mathscr{G}(L)$-representations*

$$\mathscr{F} \otimes \mathscr{F}^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\mathscr{G}(L), \mathbb{O}_S).$$

Proof. See [Moret-Bailly 1985, Theorem V.2.4.2(i)]. □

3. Adelic and p-adic Heisenberg groups

In this section we consider not only a single abelian scheme A but also coverings of A simultaneously in order to obtain a theory of p-adic and adelic Heisenberg groups. Although it would be natural to deal with a tower of abelian schemes in the sense of Mumford [1967b, §7], which involves all isogenies to A, we have chosen to work with only multiplication-by-n maps ($n \in \mathbb{Z}_{>0}$) in favor of simplicity and concreteness. (If one wishes to make the analogue of Mumford's polarized tower of abelian schemes in our context, one may relax the ampleness condition and allow line bundles to be nondegenerate.)

Keep the notation from the previous section. In particular, L is a nondegenerate line bundle over A. We do not assume that L is symmetric until Section 3E. No condition (such as being locally noetherian) is imposed on S in Section 3.

3A. ***Construction of T and V.*** This subsection is about a general construction. Let X be a commutative group scheme over S. For each $n \in \mathbb{Z}_{>0}$, let $n : X \to X$ denote the multiplication-by-n map by a slight abuse of notation. Assume that

$$\text{for every } n \geq 1, \text{ the map } n \text{ is finite and flat.} \tag{$*$}$$

Set $X[n] := \ker n$, which is a finite flat group scheme, and

$$TX := \varprojlim_n X[n]$$

with respect to $m : X[mn] \to X[n]$ for each $m, n \geq 1$. Since the latter maps are finite (thus affine) S-morphisms, [Grothendieck 1964, IV.8.2.3] implies that the limit TX exists in the category of S-group schemes. The underlying structure ring is $TX = \underline{\mathrm{Spec}}(\varinjlim \mathcal{O}_{X[n]})$, and as a group functor

$$TX(T) = \{ (x_r)_{r \geq 1} \mid x_r \in X[r](T),\ r x_{rs} = x_s,\ \text{if } r, s \geq 1 \}.$$

Define an ind-group scheme

$$VX := \varinjlim TX$$

with respect to $m : TX \to TX$ (from the n-th copy of TX to the mn-th copy for all $m, n \geq 1$). As a group functor, for each S-scheme T,

$$VX(T) = \{ (x_r)_{r \geq 1} \mid x_r \in X(T),\ N x_1 = 0 \text{ for some } N \geq 1,\ r x_{rs} = x_s,\ \forall r, s \geq 1 \}. \tag{3-1}$$

By a variant of Yoneda's lemma, VX is determined as an ind group scheme by the above description as a group functor. By allowing m and n to run over powers of a prime p, we can similarly define T_p and V_p. Note that there is a canonical isomorphism $TX \simeq \prod_p T_p X$, functorial in X.

There is a canonical action of $\widehat{\mathbb{Z}}$ on TX coming from the compatible canonical actions of $\mathbb{Z}/r\mathbb{Z}$ on $X[r]$ for $r \geq 1$. The $\widehat{\mathbb{Z}}$-action on TX patches to an action of $\mathbb{A}^\infty = \varinjlim_n \frac{1}{n}\widehat{\mathbb{Z}}$ on VX. Similarly $\mathbb{Z}_p$ and $\mathbb{Q}_p$ act on $T_p X$ and $V_p X$, respectively.

The construction of TX, VX, $T_p X$ and $V_p X$ is functorial in X and carries over to commutative flat ind-group schemes X satisfying $(*)$ above. For instance, $T_p X$ and $V_p X$ make sense for p-divisible groups X over S.

Example 3.1. There exist natural isomorphisms $T\mathbb{G}_m \simeq T\mu_\infty$, $T(\mathbb{Q}/\mathbb{Z}) \simeq \widehat{\mathbb{Z}}$, $T_p(\mathbb{Q}_p/\mathbb{Z}_p) \simeq \mathbb{Z}_p$ and $T_p\mathbb{G}_m \simeq T_p\mu_{p^\infty}$.

Example 3.2. We have $T_p\mu_{p^\infty} = \underline{\mathrm{Spec}}(\varinjlim \mathcal{O}_S[T]/(T^{p^n} - 1))$ with transition maps $f(T) \mapsto f(T^p)$.

Example 3.3. If X has bounded torsion (there exists $n \geq 1$ such that $X[n] = X[mn]$ for all $m \geq 1$), then TX, VX, $T_p X$ and $V_p X$ are all trivial group schemes.

An important case is when X is an abelian scheme. An isogeny of abelian scheme $\alpha : A' \to A$ is said to be *bounded* if $\ker\alpha \subset A[n]$ for some $n \geq 1$. (This condition is automatic if S has finitely many connected components because the fiberwise rank of $\ker\alpha$ is locally constant on S but not in general; suppose that $S = \coprod_{j\geq 1} S_j$ and that for each j, $S_j \neq \varnothing$ and α is the multiplication by j on $A \times_S S_j$. Then α is not bounded.) A map of ind-group schemes $\beta : VA' \to VA$ is said to be *bounded* if $mTA \subset \beta(nTA') \subset TA$ for some $m, n \geq 1$. The same notion is defined for a map $V_p\Sigma' \to V_p\Sigma$, where Σ' and Σ are p-divisible groups over S.

Lemma 3.4. *Let $\alpha : A' \to A$ be a bounded isogeny of abelian schemes. The induced map $V(\alpha) : VA' \to VA$ sending $(x_r)_{r\geq 1}$ to $(\alpha(x_r))_{r\geq 1}$ is a bounded isomorphism.*

Proof. We remark that the boundedness of α is needed to ensure that $V(\alpha)$ is an invertible map. Also note that if $\ker\alpha \subset A'[m]$, then $mTA \subset V(\alpha)(TA') \subset TA$. □

Remark 3.5. In the geometric theory of theta functions à la Mumford, one reason why VA naturally shows up is that an (ample) line bundle over A can be trivialized over VA. In this regard, VA is the analogue of the universal covering spaces for complex abelian varieties. Unsurprisingly, we will see VA appearing in the construction of adelic Heisenberg groups and Weil representations.

Lemma 3.6. *The scheme TA is flat over S and defines an fppf sheaf in groups on* (Flat/S). *The ind-scheme VA also defines an fppf sheaf in groups on* (Flat/S). *The same is true for T_pA and V_pA.*

Proof. Let us show that TA is flat over S. We may assume that S is affine. Let $S = \mathrm{Spec}\, C$ and $A[n] = \mathrm{Spec}\, B_n$ for $n \geq 1$. As $m : A[mn] \to A[n]$ is surjective, we see that $m^* : B_n \to B_{mn}$ is injective. Since B_n is a flat C-algebra, $\varinjlim B_n$ is also one. Hence, the assertion about TA follows. The fppf sheaf axiom for VA is easily deduced from that for TA. The case of T_pA and V_pA is proved in the same way. □

Consider a category of p-divisible groups over S in which morphisms are bounded isogenies, and then obtain a new category by inverting bounded isogenies. When Σ is a p-divisible group over S, let $\underline{\mathrm{Aut}}^{0,\mathrm{b}}_S(\Sigma)$ denote the automorphism group functor on (Sch/S) arising from the latter category. Let $\underline{\mathrm{Aut}}^{\mathrm{b}}_S(V_p\Sigma)$ be the group functor on (Sch/S) assigning bounded automorphisms. Define a map

$$\xi : \underline{\mathrm{Aut}}^{0,\mathrm{b}}_S(\Sigma) \to \underline{\mathrm{Aut}}^{\mathrm{b}}_S(V_p\Sigma)$$

by $\alpha \mapsto ((x_{p^r})_{r\geq 0} \mapsto (\alpha(x_{p^r}))_{r\geq 0})$, where $x_1 \in \Sigma$ and $x_{p^r} = px_{p^{r+1}}$.

Lemma 3.7. *The above map ξ is an isomorphism.*

Proof. It suffices to present the inverse map ξ^{-1} of ξ. Let $\alpha \in \underline{\mathrm{Aut}}^{\mathrm{b}}_S(V_p\Sigma)$ so that $p^mT_p\Sigma \subset p^n\alpha(T_p\Sigma) \subset T_p\Sigma$ for some $m \geq n \geq 0$. For each $r \geq 0$, $p^n\alpha$ maps $(1/p^r)T_p\Sigma$ to itself, thus inducing a map $\Sigma[p^r] \to \Sigma[p^r]$ by taking quotients by $T_p\Sigma$. By patching these maps, we obtain a map $\alpha' : \Sigma \to \Sigma$ such that $\ker\alpha'$ is killed by p^m; hence, α' is bounded. Then we define $\xi^{-1}(\alpha) = p^{-n}\alpha'$. It is routine to verify that ξ^{-1} is indeed the inverse map of ξ. □

3B. ***Construction of adelic and p-adic Heisenberg groups.*** Let $\widetilde{A}$ be the S-group scheme equipped with $u : \widetilde{A} \to A$, which is the inverse limit of the coverings $n : A \to A$ for all integers $n \geq 1$ (cf. [Mumford 2007, 4.27]). The limit $\widetilde{A}$ exists as a group scheme because the maps n are affine, again due to [Grothendieck 1964, IV.8.2.3]. We have that $TA = \ker u$ in the notation of Section 3A. Set $\frac{1}{n}TA := u^{-1}(A[n]) = \ker(nu)$. (We interpret $u^{-1}(A[n])$ as $\widetilde{A} \times_{u,A} A[n]$.)

Lemma 3.8. *For each $n \in \mathbb{Z}_{>0}$, $K(n^*L) \simeq A \times_{n^2,A} K(L)$ canonically. In other words,*

$$K(n^*L)(T) = \{\, x \in A(T) \mid n^2x \in K(L)(T) \,\}.$$

Proof. Set $L_T := L \times T$. For $x \in A(T)$,

$$T_x^* n^* L_T \otimes (n^*L_T)^{-1} \simeq n^*(T_{nx}L_T \otimes L_T^{-1}) \simeq T_{n^2x}^* L_T \otimes L_T^{-1},$$

where the second isomorphism results from the theorem of the square. Therefore, $x \in K(n^*L)(T)$ if and only if $n^2x \in K(L)(T)$. □

Set $T(A, L) := u^{-1}(K(L))$ and $\frac{1}{n}T(A, L) := (nu)^{-1}(K(L))$. There are canonical identifications (as schemes over A)

$$\frac{1}{n}TA \simeq \varprojlim_{m\geq 1} A[mn] \quad \text{and} \quad T(A, L) \simeq \varprojlim_{m\geq 1} K(m^*L).$$

We have a natural projection $u : \frac{1}{n}TA \to A[n]$. For $m, n \in \mathbb{Z}_{>0}$, $m : A \to A$ induces $\frac{1}{mn}TA \xrightarrow{\sim} \frac{1}{n}TA$. Its inverse map is denoted by $\frac{1}{m} : \frac{1}{n}TA \xrightarrow{\sim} \frac{1}{mn}TA$. Similarly, there are natural maps

$$\frac{1}{n^2}T(A, L) = T(A, n^*L) \twoheadrightarrow K(n^*L) \quad \text{and} \quad \frac{1}{m} : \frac{1}{n}T(A, L) \xrightarrow{\sim} \frac{1}{mn}T(A, L).$$

A concrete description of $T(A, L)$ is that for each S-scheme T,

$$T(A, L)(T) = \{\, (x_r)_{r\geq 1} \mid x_r \in A(T),\ x_1 \in K(L)(T),\ x_r = sx_{rs},\ \forall r, s \geq 1 \,\}.$$

The ind-group scheme VA (Section 3A) is canonically identified with the ind-scheme arising from $\{\frac{1}{n}TA\}_{n\geq 1}$ with the inclusions $\frac{1}{n}TA \hookrightarrow \frac{1}{mn}TA$ for $m, n \geq 1$ as can be seen by the commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & TA & \xrightarrow{\ m\ } & TA & \longrightarrow & \cdots \\
 & & n \uparrow \wr & & mn \uparrow \wr & & \\
\cdots & \longrightarrow & \frac{1}{n}TA & \hookrightarrow & \frac{1}{mn}TA & \longrightarrow & \cdots
\end{array}$$

There are natural inclusions $\frac{1}{n}TA \hookrightarrow VA$ and $\frac{1}{n}T(A, L) \hookrightarrow VA$ for each $n \geq 1$.

Set $\widetilde{\mathscr{G}}(n^*L) := \mathscr{G}(n^*L) \times_{K(n^*L)} \frac{1}{n^2}T(A, L)$. We will denote by j_n the canonical projection $\widetilde{\mathscr{G}}(n^*L) \to \frac{1}{n^2}T(A, L)$. The next lemma will endow us with an inclusion later. (See (3-4).)

Lemma 3.9. *Let T be an S-scheme, $x' \in A(T)$ and $(\psi, x) \in \mathscr{G}(n^*L)(T)$. Suppose that $x = mx'$. Then there exists a unique $\psi' : (mn)^*L \xrightarrow{\sim} (mn)^*L$ such that $(\psi', x') \in \mathscr{G}((mn)^*L)(T)$ and*

$$\begin{array}{ccc} (mn)^*L & \xrightarrow[\sim]{\psi'} & (mn)^*L \\ \downarrow & & \downarrow \\ n^*L & \xrightarrow[\sim]{\psi} & n^*L \end{array} \tag{3-2}$$

commutes, where the vertical maps are the projection maps (as $(mn)^*L$ is the fiber product of n^*L with A over $m : A \to A$).

Proof. Without loss of generality, we may assume that $n = 1$. The uniqueness is easy. If $(\psi', x'), (\psi'', x') \in \mathscr{G}(m^*L)(T)$ have the property as above, then the difference $(\psi''(\psi')^{-1}, 0)$ is in the image of some $t \in \mathbb{G}_m(T)$ under $\mathbb{G}_m \to \mathscr{G}(L)$. This means that (3-2) remains commutative after multiplying t to the top arrow. This implies that $t = 1$ and $\psi' = \psi''$.

Let us verify the existence of ψ'. The fact that $(\psi, x) \in \mathscr{G}(L)(T)$ induces an isomorphism $\xi : L \xrightarrow{\sim} T_x^*L$ making the top triangle in the left diagram commute. In the following two diagrams, the rectangles are cartesian squares. (We are abusing the notation to use A and L to denote $A \times_S T$ and $L \times_S T$.)

$$\begin{array}{ccc} L & & \\ & \searrow^{\xi}_{\sim} & \searrow^{\psi}_{\sim} \\ & T_x^*L & \xrightarrow{\sim} L \\ & \downarrow & \downarrow \\ & A & \xrightarrow[T_x]{\sim} A \end{array} \qquad \begin{array}{ccc} m^*L & & \\ & \searrow^{m^*\xi}_{\sim} & \searrow^{\psi'}_{\sim} \\ & T_{x'}^*m^*L & \xrightarrow{\sim} m^*L \\ & \downarrow & \downarrow \\ & A & \xrightarrow[T_x]{\sim} A \end{array}$$

Then $m^*L \overset{m^*\xi}{\simeq} m^*T_x^*L \simeq T_{x'}^*m^*L$. (The latter holds because $m \circ T_{x'} = T_x \circ m$.) Let $\psi' \in \underline{\mathrm{Aut}}(m^*L)(T)$ be the latter map composed with $T_{x'}^*m^*L \simeq m^*L$ in the above diagram. Then $(\psi', x') \in \mathscr{G}(m^*L)(T)$, so (3-2) commutes up to an automorphism of L fixing $L \to A$. Such an automorphism is a multiplication by $s \in \mathbb{O}_T^\times$. The commutativity of (3-2) is achieved by multiplying t to ψ'. □

Remark 3.10. (This remark is to be recalled in the proof of Lemma 4.5.) By associating ψ' to $x' \in A[m](T)$ with $x = 0$ and $\psi = \mathrm{id}$ in Lemma 3.9, we can define an action of $A[m]$ on $(mn)^*L$. This action is the same as the $A[m]$-action induced on $(mn)^*L$ via Proposition 4.1 (by taking $G = A[m]$ and ξ to be $n : A \to A$). This can be seen from the fact that the quotient of $(mn)^*L$ with respect to the former $A[m]$-action is n^*L, as shown in the proof of Lemma 3.15.

Corollary 3.11. *For each S-scheme T, the set $\widetilde{\mathscr{G}}(n^*L)(T)$ may be described as the set of $(\psi_r, x_r)_{r\geq 1}$ such that*

(i) $(\psi_r, x_r) \in \mathscr{G}((rn)^*L)(T)$ *for all* $r \geq 1$,

(ii) $x_r = s x_{rs}$ *for all* $r, s \geq 1$ *and*

(iii) *the following diagram commutes for all* $r, s \geq 1$:

$$\begin{array}{ccc} (rsn)^*L & \xrightarrow{\psi_{rs}} & (rsn)^*L \\ \downarrow & & \downarrow \\ rn^*L & \xrightarrow{\psi_r} & rn^*L \end{array}$$

Proof. The set of $(\psi_r, x_r)_{r\geq 1}$ in the corollary will be temporarily called $\widetilde{\mathscr{G}}_0(n^*L)(T)$. As $\widetilde{\mathscr{G}}(n^*L) = \mathscr{G}(n^*L) \times_{K(n^*L)} \frac{1}{n^2}T(A, L)$, we see that $\widetilde{\mathscr{G}}(n^*L)(T)$ consists of ψ_1 and $(x_r)_{r\geq 1}$ such that $(\psi_1, x_1) \in \mathscr{G}(n^*L)(T)$ and $x_r = s x_{rs}$ for all $r, s \geq 1$.

There is an obvious map

$$\widetilde{\mathscr{G}}_0(n^*L)(T) \to \widetilde{\mathscr{G}}(n^*L)(T) \tag{3-3}$$

forgetting ψ_r for $r \geq 2$. By Lemma 3.9, for a choice of ψ_1 and $(x_r)_{r\geq 1}$, there exists a sequence $(\psi_r)_{r\geq 2}$ satisfying (iii) of the corollary, and such a sequence is unique. Therefore, (3-3) is a bijection. □

Henceforth, $\widetilde{\mathscr{G}}(n^*L)$ is viewed as the S-group scheme whose associated group functor is described as in Corollary 3.11. It is flat over S and defines an fppf sheaf on S by essentially the same argument as in the proof of Lemma 3.6. Thanks to Lemma 3.9, we have a map of group schemes

$$i_{n,mn} : \widetilde{\mathscr{G}}(n^*L) \to \widetilde{\mathscr{G}}((mn)^*L) \tag{3-4}$$

sending $((\psi_r, x_r)_{r\geq 1}) \mapsto ((\psi'_r, x'_r)_{r\geq 1})$ on T-valued points, where $x'_r = x_{rm}$ and $\psi'_r = \psi_{rm}$.

Lemma 3.12. *The following diagram commutes, and its rows are fppf exact*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widetilde{\mathscr{G}}(n^*L) & \xrightarrow{j_n} & \frac{1}{n^2}T(A, L) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i_{n,mn}} & & \downarrow{\scriptstyle 1/m} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widetilde{\mathscr{G}}((mn)^*L) & \xrightarrow{j_{mn}} & \frac{1}{(mn)^2}T(A, L) & \longrightarrow & 1 \end{array}$$

Proof. Everything is obvious except perhaps the surjectivity. The map j_n is fppf surjective as it is a base change of the fppf surjective map $\mathscr{G}(n^*L) \to K(n^*L)$ (cf. Section 2B). Similarly, j_{mn} is fppf surjective. □

Let us define an ind-group scheme via $i_{n,mn}$:

$$\widehat{\mathscr{G}}(L) := \varinjlim_n \widetilde{\mathscr{G}}(n^*L).$$

Under the limit, the maps $nj_n : \widetilde{\mathscr{G}}(n^*L) \to \frac{1}{n}T(A, L)$ induce a map $\hat{j} : \widehat{\mathscr{G}}(L) \to VA$. The fact that $\widetilde{\mathscr{G}}(n^*L)$ are fppf sheaves shows that $\widehat{\mathscr{G}}(L)$ is also one. We have a commutative diagram where rows are fppf exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widetilde{\mathscr{G}}(n^*L) & \xrightarrow{nj_n} & \frac{1}{n}T(A, L) & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{natural}} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(L) & \xrightarrow{\hat{j}} & VA & \longrightarrow & 1 \end{array} \tag{3-5}$$

Lemma 3.13. *Let A' be an abelian scheme over S and $\alpha : A' \to A$ a bounded isogeny. Then α induces an isomorphism $\widehat{\mathscr{G}}(\alpha) : \widehat{\mathscr{G}}(\alpha^*L) \to \widehat{\mathscr{G}}(L)$ fitting in the commutative diagram below*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(\alpha^*L) & \longrightarrow & VA' & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \wr\big\downarrow{\scriptstyle \widehat{\mathscr{G}}(\alpha)} & & \wr\big\downarrow{\scriptstyle V(\alpha)} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(L) & \longrightarrow & VA & \longrightarrow & 1 \end{array}$$

Proof. The map $\widehat{\mathscr{G}}(\alpha)$ comes from the maps $\widetilde{\mathscr{G}}(\alpha^*n^*L) \to \widetilde{\mathscr{G}}(n^*L)$ for $n \geq 1$, which are constructed as $(\psi, x_r)_{r\geq 1} \mapsto (\phi, \alpha(x_r))_{r\geq 1}$. Here $\phi : L \xrightarrow{\sim} L$ is obtained from $\widetilde{\mathscr{G}}(\alpha^*n^*L)$ by taking the quotient of the diagram below by the action of $\ker \alpha$:

$$\begin{array}{ccc} \alpha^*n^*L & \xrightarrow[\sim]{\psi} & \alpha^*n^*L \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow[\sim]{T_{x_1}} & A \end{array}$$

It is straightforward to verify that $\widehat{\mathscr{G}}(\alpha)$ is compatible with the maps id and $V(\alpha)$ and thus an isomorphism. □

Now let L' be a line bundle over A' such that $L' \simeq \alpha^*L$. This induces an isomorphism $\widehat{\mathscr{G}}(L') \simeq \widehat{\mathscr{G}}(\alpha^*L)$. It is easy to check that the latter isomorphism is independent of the choice of the isomorphism $L' \simeq \alpha^*L$. By composing with $\widehat{\mathscr{G}}(\alpha)$, we obtain an isomorphism $\widehat{\mathscr{G}}(L') \simeq \widehat{\mathscr{G}}(L)$.

Lemma 3.14. *Let A, A', α, L and L' be as above. The isomorphism $\widehat{\mathscr{G}}(L') \simeq \widehat{\mathscr{G}}(L)$ fits into the following commutative diagram*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(L') & \longrightarrow & VA' & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(L) & \longrightarrow & VA & \longrightarrow & 1 \end{array}$$

Proof. In view of Lemma 3.13, it is enough to note the obvious commutativity:

$$\begin{array}{ccccc} \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(L') & \longrightarrow & VA' \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(\alpha^* L) & \longrightarrow & VA' \end{array}$$

□

3C. ***The map $\hat{\sigma}$.*** Define $\sigma_1 : TA \to \mathscr{G}(L)$ by $(x_r)_{r \ge 1} \mapsto (\mathrm{id}, x_1)$ and

$$\widetilde{\sigma}_1 : TA \to \widetilde{\mathscr{G}}(L) = \mathscr{G}(L) \times_{K(L)} T(A, L)$$

by σ_1 and the natural inclusion $TA \hookrightarrow T(A, L)$. For $n > 1$, set $\widetilde{\sigma}_n := i_{1,n} \circ \widetilde{\sigma}_1$. Further composing with the projection $\widetilde{\mathscr{G}}(n^* L) \to \mathscr{G}(n^* L)$, we obtain $\sigma_n : TA \to \mathscr{G}(n^* L)$. By construction, $\widetilde{\sigma}_n$s are compatible with the inclusions $i_{n,mn}$ for $m, n \ge 1$ and thus yield a map $\hat{\sigma} : TA \to \widehat{\mathscr{G}}(L)$. Note that $\widetilde{\sigma}_n$, σ_n ($n \ge 1$) and $\hat{\sigma}$ are morphisms of (ind-)group schemes and that $\mathbb{G}_m \cap \hat{\sigma}(nTA) = \{1\}$ in $\widehat{\mathscr{G}}(L)$ for every $n \ge 1$.

Lemma 3.15 [Mumford 2007, Proposition 4.13].

(i) $N_{\widehat{\mathscr{G}}(L)}(\hat{\sigma}(nTA)) = Z_{\widehat{\mathscr{G}}(L)}(\hat{\sigma}(nTA))$.

(ii) $Z_{\widehat{\mathscr{G}}(L)}(\hat{\sigma}(nTA)) \simeq \widetilde{\mathscr{G}}(n^* L)$ *canonically.*

(iii) *There is an isomorphism $N_{\widehat{\mathscr{G}}}(\hat{\sigma}(nTA))/\hat{\sigma}(nTA) \xrightarrow{\sim} \mathscr{G}(n^* L)$ induced by the canonical maps*

$$N_{\widehat{\mathscr{G}}(L)}(\hat{\sigma}(nTA)) \simeq \widetilde{\mathscr{G}}(n^* L) \twoheadrightarrow \mathscr{G}(n^* L).$$

Proof. As usual, we implicitly work on T-points for some S-scheme T. Let $X = (\phi_r, x_r)_{r \ge 1} \in N_{\widehat{\mathscr{G}}(L)}(\hat{\sigma}(nTA))$ and $Y = (\psi_r, y_r)_{r \ge 1} \in \hat{\sigma}(nTA)$. Part (i) follows from the fact that

$$XYX^{-1}Y^{-1} \in \mathbb{G}_m \cap \hat{\sigma}(nTA) = \{1\}.$$

Let us prove (ii). For some $m \ge 1$, $(\phi'_r, x'_r)_{r \ge 1} \in \widetilde{\mathscr{G}}((mn)^* L)$ may represent an element of $\widehat{\mathscr{G}}(L)$. It suffices to show that if $(\phi'_r, x'_r)_{r \ge 1}$ centralizes $\hat{\sigma}(nTA)$, then $(\phi'_r, x'_r)_{r \ge 1} = i_{n,mn}((\phi_r, x_r)_{r \ge 1})$ for some $(\phi_r, x_r)_{r \ge 1} \in \widetilde{\mathscr{G}}(n^* L)$. Consider the commutative diagram

$$\begin{array}{ccc} (rmn)^*L & \xrightarrow{\phi'_r} & (rmn)^*L \\ \downarrow & & \downarrow \\ A & \xrightarrow{T_{x'_r}} & A \end{array} \tag{3-6}$$

The image of $\hat{\sigma}(nTA)$ in $\widetilde{\mathcal{G}}((mn)^*L)$ is none other than $\widetilde{\sigma}_{mn}(nTA)$, which consists of $(\psi_r, y_r)_{r\geq 1}$ such that $y_r \in A[rm]$. Recall that $A[m]$ acts on $(rmn)^*L$ as explained at the beginning of Remark 3.10. Let us verify that the whole diagram (3-6) is $A[m]$-equivariant. (In fact it is even $A[rm]$-equivariant.) Since $(\phi'_r, x'_r)_{r\geq 1}$ commutes with elements of $\widetilde{\sigma}_{mn}(nTA)$, the top arrow in the diagram is $A[m]$-equivariant. The vertical maps are $A[m]$-equivariant by [Mumford 2007, Lemma 4.11]. The same fact is obvious for the bottom map. By taking quotients of (3-6) by $A[m]$, we obtain ϕ_r such that the following commutes:

$$\begin{array}{ccc} (rn)^*L & \xrightarrow{\phi_r} & (rn)^*L \\ \downarrow & & \downarrow \\ A & \xrightarrow{T_{mx'_r}} & A \end{array}$$

By Lemma 3.9, $(\phi'_r, x'_r)_{r\geq 1} = i_{n,mn}((\phi_r, mx'_r)_{r\geq 1})$. The proof of (ii) is complete.

For the proof of (iii), it is enough to note that the image of $\hat{\sigma}(nTA)$ in $\widetilde{\mathcal{G}}(n^*L)$ consists of $(\phi_r, x_r)_{r\geq 1}$ such that (ϕ_1, x_1) is the identity element. □

Lemma 3.16. *Let (A, L) be as before, $\alpha : A' \to A$ be a bounded isogeny and $L' = \alpha^*L$. Let $\hat{\sigma}' : TA' \to \widehat{\mathcal{G}}(L')$ denote the analogue of $\hat{\sigma}$ constructed from (A', L'). Then the following commutes*:

$$\begin{array}{ccc} TA' & \xrightarrow{\hat{\sigma}'} & \widehat{\mathcal{G}}(L') \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle \widehat{\mathcal{G}}(\alpha)}\ \wr \\ TA & \xrightarrow{\hat{\sigma}} & \widehat{\mathcal{G}}(L) \end{array}$$

Proof. Let $(x_r)_{r\geq 1} \in TA'$. Both $\widehat{\mathcal{G}}(\alpha) \circ \hat{\sigma}'$ and $\hat{\sigma} \circ \alpha$ map $(x_r)_{r\geq 1}$ to $(\mathrm{id}, \alpha((x_r)_{r\geq 1}))$ in $\widetilde{\mathcal{G}}(L)$. □

3D. ***The pairing $\hat{e}^L$.*** In analogy with e^L in Section 2B, we obtain a bilinear commutator pairing from the bottom row of (3-5),

$$\hat{e}^L : VA \times VA \to \mathbb{G}_m,$$

which is a morphism of ind-group schemes over S. On the other hand, the $\mathbb{Z}/n\mathbb{Z}$-linear Weil pairings

$$e_n^{L,\mathrm{Weil}} : A[n] \times A[n] \to \mu_n \hookrightarrow \mathbb{G}_m$$

for $n \geq 1$ are glued to an $\mathbb{A}^\infty$-linear pairing (cf. Section 3A)

$$\hat{e}^{L,\mathrm{Weil}} : VA \times VA \to V\mathbb{G}_m.$$

Concretely on the functors of points, the map is

$$((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) \mapsto \big(e^{L,\mathrm{Weil}}_{N^2 n}(x_{Nn}, y_{Nn})\big)_{n\geq 1},$$

where $N \geq 1$ is such that $x_1, y_1 \in A[N^2]$. The definition is independent of N. The right side is an element of $V\mathbb{G}_m$ since $e^{L,\mathrm{Weil}}_{N^2 mn}(x_{Nmn}, y_{Nmn})^m = e^{L,\mathrm{Weil}}_{N^2 n}(x_{Nn}, y_{Nn})$ for any $m, n \geq 1$. Let $\flat : V\mathbb{G}_m \to \mathbb{G}_m$ be the map $\flat((x_r)_{r\geq 1}) = x_1$ in the notation of (3-1).

Lemma 3.17. *The pairing $\hat{e}^L$ is nondegenerate, and $\hat{e}^L = \flat \circ \hat{e}^{L,\mathrm{Weil}}$.*

Proof. The nondegeneracy of $\hat{e}^L$ is deduced from the nondegeneracy of e^{n^*L} for all $n \geq 1$ (Lemma 2.8). Indeed, if $\hat{e}^L$ were degenerate, there would be an S-scheme T and a nonzero section $x \in \frac{1}{n}T(A, L)(T)$ such that $\hat{e}^L(x, y) = 1$ for any section y of VA in a T-scheme. Choose a large enough $m \geq 1$ such that $x \notin \hat{\sigma}(mnTA)$. Then x has nontrivial image $\bar{x}$ in $\frac{1}{mn}T(A, L)/\hat{\sigma}(mnTA) \simeq K((mn)^*L)$, but by the assumption, $\bar{x}$ pairs trivially with any section of $K((mn)^*L)$ via $e^{(mn)^*L}$. This contradicts the nondegeneracy of $e^{(mn)^*L}$.

Let us now prove the second assertion. Let $(\phi, x), (\psi, y) \in \widehat{\mathcal{G}}(L)$, and write $x = (x_n)_{n\geq 1}, y = (y_n)_{n\geq 1} \in VA$. For any $N \geq 1$ chosen as above,

$$\flat\big(\hat{e}^{L,\mathrm{Weil}}((x_n)_{n\geq 1}, (y_n)_{n\geq 1})\big) = e^{L,\mathrm{Weil}}_{N^2 n}(x_{Nn}, y_{Nn}) = e^L(x_N, y_N)^{N^2} = e^L(x_1, y_1).$$

(The second equality is standard. See Property (5) of [Mumford 1974, §23] for instance.) Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{G}((N^2)^*L) & \xrightarrow{\;j\;} & K((N^2)^*L) & \longrightarrow & 1 \\
 & & \Big\uparrow{\scriptstyle \mathrm{id}} & & \Big\uparrow & & \Big\uparrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widetilde{\mathcal{G}}((N^2)^*L) & \xrightarrow{\;j_{N^2}\;} & T(A, (N^2)^*L) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle N^2} & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathcal{G}}(L) & \xrightarrow{\;\hat{j}\;} & VA & \longrightarrow & 1
\end{array}$$

Let $x' = (x'_n)_{n\geq 1}$ and $y' = (y'_n)_{n\geq 1}$ be such that $x'_n = x_{N^2 n}$ and $y'_n = y_{N^2 n}$. Note that $x'_1, y'_1 \in A[N^4] \subset K((N^2)^*L)$; thus, $x', y' \in T(A, (N^2)^*L)$. The commutativity of the diagram allows us to equalize the commutator pairing for each row. The second assertion follows from

$$\hat{e}^L(x, y) = \phi\psi\phi^{-1}\psi^{-1} = e^{(N^2)^* L}(x'_1, y'_1) = e^L(Nx'_1, Ny'_1) = e^L(x_1, y_1). \qquad \square$$

3E. ***Symmetric line bundles and the map*** $\hat{\boldsymbol{\tau}}$. Our construction of $\hat{\tau}$ is based on [Mumford 2007, §4] as well as Step V in Appendix I of that book. From here on, assume that L is symmetric, i.e., $(-1)^* L \simeq L$. There is an isomorphism (e.g., appeal to Lemma 3.14 with $A = A'$, $\alpha = -1$ and $L' = L$)

$$\widehat{\mathscr{G}}((-1)^* L) \xrightarrow{\sim} \widehat{\mathscr{G}}(L) \tag{3-7}$$

uniquely characterized as follows. If $(\phi_r, x_r)_{r\geq 1}$ is mapped to $(\psi_r, -x_r)_{r\geq 1}$, then the diagram below commutes, where the vertical maps are induced by the pullback along $(-1) : A \to A$:

$$\begin{array}{ccc} (-1)^* L & \xrightarrow{\psi_r} & (-1)^* L \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi_r} & L \end{array}$$

A choice of an isomorphism $I : L \simeq (-1)^* L$ induces $\widehat{\mathscr{G}}(L) \xrightarrow{\sim} \widehat{\mathscr{G}}((-1)^* L)$. By composing with (3-7), we obtain an isomorphism

$$i^L : \widehat{\mathscr{G}}(L) \xrightarrow{\sim} \widehat{\mathscr{G}}(L)$$

and can show that it is independent of the choice of I (cf. [Mumford 2007, Proposition 4.16]). The situation may be understood through a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(L) & \xrightarrow{\hat{j}} & VA & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i^L} & & \downarrow{\scriptstyle -1} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathscr{G}}(L) & \xrightarrow{\hat{j}} & VA & \longrightarrow & 1 \end{array} \tag{3-8}$$

For each $n \geq 1$, clearly the map $x \mapsto x i^L(x)^{-1}$ from $\widetilde{\mathscr{G}}(n^* L)$ to $\widetilde{\mathscr{G}}(n^* L)$ factors as the composite of $j_n : \widetilde{\mathscr{G}}(n^* L) \to \frac{1}{n^2} T(A, L)$ and $h_n : \frac{1}{n^2} T(A, L) \to \widetilde{\mathscr{G}}(n^* L)$. We construct $\widetilde{\tau}_{2n}$ as the composite

$$\frac{1}{2n^2} T(A, L) \xrightarrow{\frac{1}{2}} \frac{1}{(2n)^2} T(A, L) \xrightarrow{h_{2n}} \widetilde{\mathscr{G}}((2n)^* L).$$

When n is odd, $\widetilde{\tau}_n : \frac{1}{n^2} T(A, L) \to \widetilde{\mathscr{G}}(n^* L)$ is defined as

$$\frac{1}{n^2} T(A, L) \overset{\frac{1}{2}}{\simeq} \frac{1}{n^2} T(A, L) \xrightarrow{h_n} \widetilde{\mathscr{G}}(n^* L).$$

It is readily checked that $\widetilde{\tau}_n$ are compatible with $i_{n,mn}$ for $m, n \geq 1$ so that they glue together to a map $\hat{\tau} : VA \to \widehat{\mathscr{G}}(L)$. (Note that $\widetilde{\tau}_n$ and $\widetilde{\tau}_{mn}$ provide sections in the diagram of Lemma 3.12.) By construction, $\hat{\tau}$ is a section of $\hat{j}$, namely

$$\hat{j} \circ \hat{\tau} = \mathrm{id}. \tag{3-9}$$

The map $\hat{\tau}$ enables us to identify $\widehat{\mathscr{G}}(L)$ with $\mathbb{G}_m \times VA$ equipped with a certain group law that resembles the classical Heisenberg group law. To be precise, define a group law on $\mathbb{G}_m \times VA$ by

$$(\lambda, x) \cdot (\mu, y) = (\lambda\mu \cdot \hat{e}^L(\tfrac{1}{2}x, y), x + y). \tag{3-10}$$

Lemma 3.18 [Mumford 2007, Proposition 4.18.B]. *The map*

$$\mathbb{G}_m \times VA \to \widehat{\mathscr{G}}(L), \quad (\lambda, x) \mapsto \lambda \cdot \hat{\tau}(x)$$

is an isomorphism of ind-group schemes over S.

Proof. The above map is readily seen to be an isomorphism of ind-schemes over S from the row exactness of (3-8) together with (3-9). It remains to check the homomorphism property. Set $\tilde{x} = \hat{\tau}(x/2)$ and $\tilde{y} = \hat{\tau}(y/2)$. Let $\lambda, \mu \in \mathbb{G}_m$. Then

$$\begin{aligned}
\lambda\hat{\tau}(x)\mu\hat{\tau}(y) &= \lambda\mu\hat{\tau}(x)\,\hat{\tau}(y) \\
&= \lambda\mu\tilde{x}i^L(\tilde{x})^{-1}\tilde{y}i^L(\tilde{y})^{-1} \\
&= \lambda\mu\tilde{x}i^L(\tilde{x})^{-1}\tilde{y}(\tilde{x}i^L(\tilde{x})^{-1}\tilde{y})^{-1}\tilde{y}\tilde{x}i^L(\tilde{x})^{-1}i^L(\tilde{y})^{-1} \\
&= \lambda\mu\hat{e}^L\big(\hat{j}(\tilde{x}i^L(\tilde{x})^{-1}), \hat{j}(\tilde{y})\big)\,\tilde{y}\tilde{x}i^L((\tilde{y}\tilde{x})^{-1}) \\
&= \lambda\mu\hat{e}^L(x, y/2)\,\hat{\tau}(\tilde{x} + \tilde{y}) = \lambda\mu\hat{e}^L(x/2, y)\,\hat{\tau}(\tilde{x} + \tilde{y}). \qquad \square
\end{aligned}$$

It is natural to ask about the difference between $\hat{\sigma}$ and $\hat{\tau}|_{TA}$, which are maps from TA to $\widehat{\mathscr{G}}(L)$. Consider the map

$$e^L_* : \tfrac{1}{2}TA \to \mathbb{G}_m, \quad e^L_*(x) = \hat{\sigma}^L(2x)\,\hat{\tau}(2x)^{-1}. \tag{3-11}$$

Lemma 3.19. *The map e^L_* is a quadratic form factoring as*

$$\tfrac{1}{2}TA \twoheadrightarrow A[2] \to \mu_2 \hookrightarrow \mathbb{G}_m,$$

where $\frac{1}{2}TA \twoheadrightarrow A[2]$ and $\mu_2 \hookrightarrow \mathbb{G}_m$ are canonical surjection and injection. In particular, $\hat{\sigma}$ and $\hat{\tau}$ coincide on $2 \cdot TA$. For all $x, y \in \frac{1}{2}TA$ (viewed as T-valued points for each S-scheme T),

$$e^L_*(x + y)\,e^L_*(x)^{-1}e^L_*(y)^{-1} = e^L(x, y)^2.$$

Proof. The proof of [Mumford 2007, Proposition 4.18.C] is easily adapted to the scheme-theoretic setting as in the proof of the last lemma. $\square$

Lemma 3.20. *Consider (A', L') and (A, L) with a bounded isogeny $\alpha : A' \to A$ such that $L' = \alpha^* L$. Suppose that L' and L are symmetric. Then the following diagram commutes, where the row isomorphisms are as in Lemma 3.18*:

$$\begin{CD} \mathbb{G}_m \times VA' @>\sim>> \widehat{\mathcal{G}}(L') \\ @V{\sim}V{(\mathrm{id}, V(\alpha))}V @V{\sim}V{\widehat{\mathcal{G}}(\alpha)}V \\ \mathbb{G}_m \times VA @>\sim>> \widehat{\mathcal{G}}(L) \end{CD}$$

Proof. The proof is reduced to checking $i^L \circ \widehat{\mathcal{G}}(\alpha) = \widehat{\mathcal{G}}(\alpha) \circ i^{L'}$, which amounts to the commutativity of the outer rectangle below:

$$\begin{CD} \widehat{\mathcal{G}}(L') @>I_1>> \widehat{\mathcal{G}}((-1)^* L') @>\widehat{\mathcal{G}}(-1)>> \widehat{\mathcal{G}}(L') \\ @VV{\widehat{\mathcal{G}}(\alpha)}V @VV{\widehat{\mathcal{G}}(\alpha)}V @VV{\widehat{\mathcal{G}}(\alpha)}V \\ \widehat{\mathcal{G}}(L) @>I_2>> \widehat{\mathcal{G}}((-1)^* L) @>\widehat{\mathcal{G}}(-1)>> \widehat{\mathcal{G}}(L) \end{CD}$$

The maps I_1 and I_2 are induced by any choice of isomorphisms $L' \simeq (-1)^* L'$ and $L \simeq (-1)^* L$ (since such isomorphisms allow us to make the identifications $\underline{\mathrm{Aut}}(L'/A) \simeq \underline{\mathrm{Aut}}((-1)^* L'/A)$ and $\underline{\mathrm{Aut}}(L'/A) \simeq \underline{\mathrm{Aut}}((-1)^* L'/A)$), and they are easily seen to be independent of the choice. The right half commutes because $\widehat{\mathcal{G}}(\alpha)\widehat{\mathcal{G}}(-1) = \widehat{\mathcal{G}}(-1)\widehat{\mathcal{G}}(\alpha) = \widehat{\mathcal{G}}(-\alpha)$. In order to verify that the left half commutes, one reduces to the situation where $\widehat{\mathcal{G}}$, L' and L are replaced with $\widetilde{\mathcal{G}}$, $n^* L'$ and $n^* L$, respectively. Then by using the description of Corollary 3.11, one checks that

$$\begin{CD} (\psi_r, x_r)_{r \geq 1} @>I_1>> (\psi_r, x_r)_{r \geq 1} \\ @VV{\widehat{\mathcal{G}}(\alpha)}V @VV{\widehat{\mathcal{G}}(\alpha)}V \\ (\overline{\psi}_r, \alpha(x_r))_{r \geq 1} @>I_2>> (\overline{\psi}_r, \alpha(x_r))_{r \geq 1} \end{CD}$$

where $\overline{\psi}_r$ is the induced automorphism of $L \to A$ obtained from taking quotient by $\ker \alpha$ of ψ_r. (The latter is an automorphism of $L' \to A'$.) □

3F. *p-adic Heisenberg groups.* It is easy to adapt the construction of this section to obtain p-adic analogues. There are obvious definitions of $\frac{1}{p^n} T_p(A, L)$ and $\widetilde{\mathcal{G}}_p((p^n)^* L)$. In order that $T_p(A, L)$ be contained in $V_p A$, we need to assume that $\deg L$ is a power of p (including $\deg L = 1$) or equivalently that $K(L)$ is a p-group. Then we have $\frac{1}{p^n} T_p(A, L) \hookrightarrow V_p A$ for all $n \geq 1$. Define

$$\widehat{\mathcal{G}}_p(L) := \varinjlim_n \widetilde{\mathcal{G}}_p((p^n)^* L).$$

There is a commutative diagram similar to (3-5):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widetilde{\mathcal{G}}_p((p^n)^*L) & \xrightarrow{p^n j_{p^n}} & \frac{1}{p^n}T(A,L) & \longrightarrow & 1 \\ & & \Big\downarrow \text{id} & & \Big\downarrow & & \Big\downarrow \text{natural} & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \widehat{\mathcal{G}}_p(L) & \xrightarrow{\hat{j}_p} & V_pA & \longrightarrow & 1 \end{array}$$

The commutator pairing yields $\hat{e}_p^L : V_pA \times V_pA \to \mathbb{G}_m$. A group morphism $\hat{\sigma}_p : T_pA \to \widehat{\mathcal{G}}_p(L)$ is constructed as before. Now suppose that L is symmetric. Then there is a map $\hat{\tau}_p : V_pA \to \widehat{\mathcal{G}}_p(L)$ (which is not compatible with group structure) such that $\hat{j}_p \circ \hat{\tau}_p = \text{id}$. If $p \neq 2$, then $\hat{\sigma}_p^L = \hat{\tau}_p|_{T_pA}$ and $e_{p,*}^L \equiv 1$. If $p = 2$, the map $e_{2,*}^L : \frac{1}{2}T_2A \to \mathbb{G}_m$ sending x to $\hat{\sigma}_2^L(2x)\, \hat{\tau}_2(2x)^{-1}$ factors through $\frac{1}{2}T_2A \twoheadrightarrow A[2] \to \mu_2 \hookrightarrow \mathbb{G}_m$ and satisfies the same formula as in Lemma 3.19. Using $\hat{\tau}_p$ we get an isomorphism $\mathbb{G}_m \times V_pA \xrightarrow{\sim} \widehat{\mathcal{G}}_p(L)$ (for any p including $p = 2$) if the group law on the left-hand side is as in (3-10).

4. Adelic and p-adic Heisenberg representations

As before, A is an abelian scheme over S, and L is a nondegenerate line bundle over A. Throughout Section 4, S is locally noetherian, but L is not assumed to be symmetric except briefly at the end of Section 4D.

4A. ***Some preliminaries on group actions.*** The following general notation will be used in Section 4A:

- G is a finite flat group scheme over S (not necessarily étale), and
- $\alpha : X \to S$ is an S-scheme of finite type equipped with a strictly free G-action (i.e., $G \times_S X \to X \times_S X$ via $(g, x) \mapsto (gx, x)$ is a closed immersion) such that every orbit is contained in an affine open set.

Then a general theorem of Grothendieck (cf. [Tate 1997, Theorem 3.4]) ensures that the quotient $Y := X/G$, along with $\beta : Y \to S$ and $\xi : X \to Y$, exists in the category of S-schemes. (This is a universal geometric quotient and an fppf quotient. See [van der Geer and Moonen $\geq$ 2012, Theorem 4.16, Theorem 4.35] for details.)

Proposition 4.1. *Let $\mathcal{F}'$ and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and a coherent $\mathcal{O}_Y$-module, respectively. The canonical maps $\mathcal{F} \to (\xi_*\xi^*\mathcal{F})^G$ and $\xi^*(\xi_*(\mathcal{F}')^G) \to \mathcal{F}'$ are isomorphisms.* (*The maps are given by the fact that ξ^* is the left adjoint of ξ_*.*) *The map $\mathcal{F} \mapsto \xi^*\mathcal{F}$ induces an equivalence of the category of coherent $\mathcal{O}_Y$-modules* (*resp. locally free $\mathcal{O}_Y$-modules of finite rank*) *with the category of coherent $\mathcal{O}_X$-modules with G-action* (*resp. locally free $\mathcal{O}_Y$-modules of finite rank with G-action*).

Proof. The statement and proof of [Mumford 1974, §12 Theorem 1] can be adapted to the relative setting over S. □

Lemma 4.2. *The category of G-representations on $\mathcal{O}_S$-modules has enough injectives.*

Proof. Let $\mathcal{F}$ be a G-representation on an $\mathcal{O}_S$-module. Since the category of $\mathcal{O}_S$-modules has enough injectives, there exists an injective $\mathcal{O}_S$-module $\mathcal{I}$ with $i : \mathcal{F} \hookrightarrow \mathcal{I}$. The $\mathcal{O}_S$-module $\widetilde{\mathcal{I}} := \underline{\mathrm{Hom}}_{\mathcal{O}_S}(G, \mathcal{I})$ is an injective object in the category of G-representations on $\mathcal{O}_S$-modules since, by the injectivity of $\mathcal{I}$, the functor

$$\mathcal{M} \mapsto \underline{\mathrm{Hom}}_G(\mathcal{M}, \underline{\mathrm{Hom}}_{\mathcal{O}_S}(G, \mathcal{I})) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{M}, \mathcal{I})$$

is exact. (The latter isomorphism is given by $\phi \mapsto (m \mapsto \phi(m)(e))$, where e is the identity of G.) The isomorphism for $\mathcal{M} = \mathcal{F}$ yields $\underline{\mathrm{Hom}}_G(\mathcal{F}, \widetilde{\mathcal{I}}) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{I})$, and the map $\tilde{i} : \mathcal{F} \to \widetilde{\mathcal{I}}$ corresponding to i is an injection. □

There is a functor $\mathcal{V} \mapsto \mathcal{V}^G$ from the category of G-representations on $\mathcal{O}_S$-modules to the category of $\mathcal{O}_S$-modules. (By [Moret-Bailly 1985, V.1.2], $\mathcal{V}^G$ is an $\mathcal{O}_S$-module.) For $i \geq 0$, let $\mathcal{H}^i(G, \mathcal{V})$ denote the ith right derived functor of the left exact functor $\mathcal{V} \mapsto \mathcal{V}^G$.

Lemma 4.3. *There is a spectral sequence* $E_2^{i,j} = \mathcal{H}^i(G, R^j\alpha_*(\xi^*\mathcal{F})) \Rightarrow R^{i+j}\beta_*\mathcal{F}$.

Proof. Let $\mathrm{Rep}(G)$ denote the category of G-representations on $\mathcal{O}_S$-modules. Consider the left exact functors $\mathrm{QCoh}_Y \to \mathrm{Rep}(G)$ and $\mathrm{Rep}(G) \to \mathrm{QCoh}_S$ given by $\mathcal{F} \mapsto \alpha_*(\xi^*\mathcal{F})$ and $\mathcal{V} \mapsto \mathcal{V}^G$, respectively. Note that

$$\alpha_*(\xi^*\mathcal{F})^G = \beta_*\mathcal{F}. \tag{4-1}$$

The desired spectral sequence is none other than the Grothendieck spectral sequence. We only need to show that the functor $\mathcal{F} \mapsto \alpha_*(\xi^*\mathcal{F})$ carries injective objects to acyclic objects.

Set $G_Y := G \times_S Y$. Note that $\alpha_*(\xi^*\mathcal{F}) = \beta_*\xi_*\xi^*\mathcal{F}$, $\xi_*\xi^*\mathcal{F} \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_S}(G, \mathcal{F})$ and

$$\beta_*\underline{\mathrm{Hom}}_{\mathcal{O}_S}(G, \mathcal{F}) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_S}(G, \beta_*\mathcal{F}).$$

For any $\mathcal{O}_S$-module $\mathcal{F}'$, $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(G, \mathcal{F}')$ is acyclic for taking G-invariants, so the proof is complete. (The argument is the same as the one showing the acyclicity of induced modules in group cohomology. Indeed, if $\mathcal{F}' \to \mathcal{I}_\bullet$ is an injective resolution in $\mathcal{O}_S$-modules, then $\underline{\mathrm{Hom}}_{\mathcal{O}_S}(G, \mathcal{F}') \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(G, \mathcal{I}_\bullet)$ is an injective resolution in $\mathrm{Rep}(G)$. When G-invariants are taken, the latter resolution becomes $\mathcal{F}' \to \mathcal{I}_\bullet$, which is exact.) □

Corollary 4.4. *Suppose that* $R^j\alpha_*(\xi^*\mathcal{F}) = R^j\beta_*\mathcal{F} = 0$ *for an integer* $q \geq 0$ *unless* $j = q$. *Then the canonical morphism* $R^q\beta_*\mathcal{F} \to (R^q\alpha_*(\xi^*\mathcal{F}))^G$ (*cf. Equation* (4-1)) *is an isomorphism.*

Proof. The spectral sequence of Lemma 4.3 degenerates at E_2 by the assumption and induces the desired isomorphism. □

4B. ***Construction of adelic Heisenberg representations.*** Temporarily, we make an assumption that L is nondegenerate of index i for some $i \geq 0$. (This will be removed at the end of this subsection.) Set

$$\mathcal{V}^{(n)}(L) := R^i f_*(n^*L).$$

Note that $\widetilde{\mathcal{G}}(n^*L)$ acts on $\mathcal{V}^{(n)}(L)$ through its projection onto $\mathcal{G}(n^*L)$, whose action was discussed in Section 2C. For $m, n \in \mathbb{Z}_{>0}$, there is a natural map functorial in L

$$f_*n^*L \to f_*m_*m^*(n^*L) \simeq f_*(mn)^*L$$

induced by the adjunction map $n^*L \to m_*m^*(n^*L)$. It works similarly with higher direct image of f (since m_* is exact). Let $\nu_{n,mn} : \mathcal{V}^{(n)}(L) \to \mathcal{V}^{(mn)}(L)$ be the functorial map $R^i f_*(n^*L) \to R^i f_*((mn)^*L)$. Clearly $\nu_{mn,mnk}\nu_{n,mn} = \nu_{n,mnk}$ for any $m, n, k \in \mathbb{Z}_{>0}$ as both sides are the functorial map with respect to $(mnk)^*L \to n^*L$ covering $mk : A \to A$. Also, $\nu_{n,mn}$ is compatible with $i_{n,mn} : \widetilde{\mathcal{G}}(n^*L) \hookrightarrow \widetilde{\mathcal{G}}((mn)^*L)$; namely, for all $\gamma \in \widetilde{\mathcal{G}}(n^*L)$ and $v \in \mathcal{V}^{(n)}(L)$, we have $i_{n,mn}(\gamma) \cdot \nu_{n,mn}(v) = \nu_{n,mn}(\gamma \cdot v)$. Indeed, this results from the commutativity of

$$\begin{array}{ccc} (mn)^*L & \xrightarrow{\;i_{n,mn}(\gamma)\;} & (mn)^*L \\ \downarrow & & \downarrow \\ n^*L & \xrightarrow{\;\gamma\;} & n^*L \end{array}$$

where γ and $i_{n,mn}(\gamma)$ act through their respective images in $\mathcal{G}(n^*L)$ and $\mathcal{G}((mn)^*L)$.

Lemma 4.5. *As $\mathcal{O}_S$-modules, for all $m, n \geq 1$,*

$$\mathcal{V}^{(n)}(L) \simeq (\mathcal{V}^{(mn)}(L))^{A[m]} = (\mathcal{V}^{(mn)}(L))^{\widetilde{\sigma}_{mn}(nTA)},$$

where the first isomorphism is induced by $\nu_{n,mn}$.

Proof. Take $\alpha = \beta = f$, $\xi = m$, $\mathcal{F} = n^*L$ and $q = i$ in Corollary 4.4 to obtain the first isomorphism

$$\mathcal{V}^{(n)}(L) \simeq (\mathcal{V}^{(mn)}(L))^{A[m]}. \tag{4-2}$$

We claim that

$$(\mathcal{V}^{(mn)}(L))^{A[m]} = (\mathcal{V}^{(mn)}(L))^{\widetilde{\sigma}_{mn}(nTA)}. \tag{4-3}$$

On the right-hand side, $\widetilde{\mathcal{G}}((mn)^*L)$ acts through $\mathcal{G}((mn)^*L)$. The action of the subgroup scheme $\widetilde{\sigma}_{mn}(nTA)$ of $\widetilde{\mathcal{G}}((mn)^*L)$ factors through

$$\widetilde{\sigma}_{mn}(nTA)/\widetilde{\sigma}_{mn}(mnTA) \simeq A[m].$$

In view of Corollary 3.11, the latter $A[m]$-action (on the right-hand side of (4-3)) is described as follows: $x' \in A[m]$ acts on $(mn)^*L$ via ψ', which is obtained from Lemma 3.9 by taking $x = 0$ and $\psi = \mathrm{id}$, and this induces the action of x' on $\mathcal{V}^{(mn)}(L)$.

The claim (4-3) follows from the fact that this $A[m]$-action is the same as the one used in Corollary 4.4 and thus used in (4-2). (See Remark 3.10.) □

Using the fact that $\nu_{n,mn}$ are compatible with $i_{n,mn}$ as explained above, we obtain

$$\widehat{\mathcal{V}}(L) := \varinjlim_{n} \mathcal{V}^{(n)}(L)$$

as a $\widehat{\mathcal{G}}(L)$-representation, where $\nu_{n,mn}$ are transition maps. The $\mathcal{O}_S$-module $\widehat{\mathcal{V}}(L)$ carries a weight-1 action by $\widehat{\mathcal{G}}(L)$. Its properties will be investigated in Section 4C.

Remark 4.6. A more concise definition of $\widehat{\mathcal{V}}(L)$ would be $R^i f_*(u^*L)$. (We defined u in Section 3B.) It is useful to view $\widehat{\mathcal{G}}(L)$ as a compatible system of $\widetilde{\mathcal{G}}(n^*L)$-actions on $\mathcal{V}^{(n)}(L)$ as this allows us to derive properties of $\widehat{\mathcal{V}}(L)$ from those of $\mathcal{V}^{(n)}(L)$.

Now we drop the assumption that the index of L is constant over S. Let g be the relative dimension of A over S. Since the index function $s \mapsto L_s$ is locally constant, we can decompose $S = \coprod_{i=0}^{g} S_i$ into open and closed subschemes such that the index function is constantly i on each S_i. The previous paragraphs construct $\mathcal{V}^{(n)}(L)$, $\widehat{\mathcal{V}}(L)$ and so on over each S_i; thereby, we obtain them over S.

4C. *Basic properties.* Just like at the end of the last subsection, we no longer assume that L has fixed index over S.

Definition 4.7. A $\widehat{\mathcal{G}}(L)$-representation $\mathcal{F}$ on an $\mathcal{O}_S$-module is *admissible* if $\mathcal{F}^{\hat{\sigma}(n\cdot TA)}$ is a coherent $\mathcal{O}_S$-module for every $n \geq 1$. (We always have that $\mathcal{F}^{\hat{\sigma}(nTA)}$ is an $\mathcal{O}_S$-submodule of $\mathcal{F}$.)

Definition 4.8. A $\widehat{\mathcal{G}}(L)$-representation $\mathcal{F}$ is *smooth* if

$$\mathcal{F} = \bigcup_{n \geq 1} \mathcal{F}^{\hat{\sigma}(n\cdot TA)}.$$

A useful observation is that $\mathcal{F}^{\hat{\sigma}(n\cdot TA)}$ is a module over $N_{\widehat{\mathcal{G}}}(\hat{\sigma}(nTA))/\hat{\sigma}(nTA) \simeq \mathcal{G}(n^*L)$, cf. Lemma 3.15. This will be exploited several times.

Remark 4.9. If L is a symmetric line bundle so that $\hat{\tau}$ is available, an equivalent criterion for smoothness is that $\mathcal{F} = \bigcup_{n\geq 1} \mathcal{F}^{\hat{\tau}(n\cdot TA)}$. The obvious reason is that $\hat{\sigma} = \hat{\tau}$ on $2TA$.

Remark 4.10. It is not inconceivable that any weight-1 $\widehat{\mathcal{G}}(L)$-representation is smooth, but we have not checked this.

Lemma 4.11. *Suppose that L is nondegenerate of index i. Then*

(i) $\widehat{\mathcal{V}}(L)^{\hat{\sigma}(nTA)} = \mathcal{V}^{(n)}(L)$, *and*

(ii) *the $\widehat{\mathcal{G}}(L)$-representation $\widehat{\mathcal{V}}(L)$ is admissible and smooth.*

Proof. Clearly (i) implies (ii). Part (i) is obtained from Lemma 4.5 by taking limit over m. Note that this works even if $\mathrm{ind}(L)$ is not constant on S, cf. the end of Section 4B. □

We give a tentative definition of a (finite) adelic Heisenberg representation. Perhaps a more satisfactory definition is (ii) of Proposition 4.19 below.

Definition 4.12. An adelic *Heisenberg representation* $\mathcal{H}$ of $\widehat{\mathcal{G}}(L)$ is a smooth $\widehat{\mathcal{G}}(L)$-representation of weight 1 such that $\mathcal{H}^{\hat{\sigma}(nTA)}$ is a locally free $\mathcal{O}_S$-module of rank $n^{2g} \cdot \deg L$ for all $n \geq 1$.

Lemma 4.13. *An adelic Heisenberg representation $\mathcal{H}$ of $\widehat{\mathcal{G}}(L)$ is a locally free $\mathcal{O}_S$-module*[3] *and irreducible.* (*The notion of irreducibility is as in Definition 2.11.*)

Proof. Let $\mathcal{H}' \subset \mathcal{H}$ be a smooth $\widehat{\mathcal{G}}(L)$-subrepresentation, so $(\mathcal{H}')^{\hat{\sigma}(nTA)} \subset \mathcal{H}^{\hat{\sigma}(nTA)}$ is a $\widetilde{\mathcal{G}}(n^*L)$- and $\mathcal{G}(n^*L)$-subrepresentation. By Proposition 2.12(i), there is an ideal sheaf $\mathcal{I}_n$ of $\mathcal{O}_S$ such that

$$(\mathcal{H}')^{\hat{\sigma}(nTA)} = \mathcal{I}_n \cdot \mathcal{H}^{\hat{\sigma}(nTA)}. \tag{4-4}$$

By taking $\hat{\sigma}(TA)$-invariants, where $\hat{\sigma}(TA)$ acts through its image in $\widetilde{\mathcal{G}}(n^*L)$, we get

$$(\mathcal{H}')^{\hat{\sigma}(TA)} = \mathcal{I}_n \cdot \mathcal{H}^{\hat{\sigma}(TA)}.$$

Since $\mathcal{H}^{\hat{\sigma}(TA)}$ is locally free, the comparison with (4-4) for $n = 1$ shows that $\mathcal{I}_n = \mathcal{I}_1$ for all $n \geq 1$. Hence, $\mathcal{H}' = \mathcal{H} \otimes \mathcal{I}_1$. The local freeness of $\mathcal{H}$ follows from the fact that

$$\mathcal{H}_s = \varinjlim_n (\mathcal{H}^{\hat{\sigma}(n!TA)})_s$$

is free over $\mathcal{O}_{S,s}$, as it is an increasing union of finite free modules. (Since each transition map has a section, a basis can be written down easily.) □

Corollary 4.14. *The $\widehat{\mathcal{G}}(L)$-representation $\widehat{\mathcal{V}}(L)$ is an adelic Heisenberg representation of $\widehat{\mathcal{G}}(L)$ in the sense of Definition 4.12.*

Proof. This follows from Proposition 2.13(ii) and Lemmas 4.11 and 4.13. □

Theorem 4.15. *Let $\mathcal{H}$ be a Heisenberg representation of $\widehat{\mathcal{G}}(L)$. Then there is an equivalence of categories*

$$\mathrm{Rep}^1_{\mathrm{sm}}(\widehat{\mathcal{G}}(L)) \xrightarrow{\sim} \mathrm{QCoh}_S$$

given by $\mathcal{M} \mapsto \underline{\mathrm{Hom}}_{\widehat{\mathcal{G}}(L)}(\mathcal{H}, \mathcal{M})$ and $\mathcal{N} \mapsto \mathcal{H} \otimes \mathcal{N}$, which are quasi-inverses of each other.

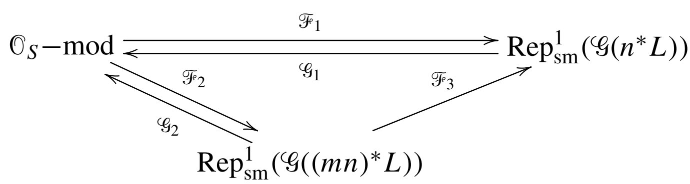
[3]Note that we are dealing with an $\mathcal{O}_S$-module, which is typically of infinite rank. We consider an $\mathcal{O}_S$-module $\mathcal{H}$ locally free if the Zariski localization $\mathcal{H}_s$ is a free $\mathcal{O}_{S,s}$-module for all $s \in S$. This does not automatically imply that $\mathcal{H}|_U$ is a free $\mathcal{O}_U$-module in some open neighborhood U of s for a given $s \in S$.

Proof. To simplify notation, let us write $\widehat{\mathscr{V}}$ for $\widehat{\mathscr{V}}(L)$. Suppose that the proposition is known for $\mathscr{H}=\widehat{\mathscr{V}}$. Then it implies that any Heisenberg representation $\mathscr{H}'$ is isomorphic to $\widehat{\mathscr{V}}\otimes\mathscr{F}$ for an $\mathbb{O}_S$-module $\mathscr{F}$. Since the $\hat{\sigma}(TA)$-invariants in $\mathscr{H}'$ and $\widehat{\mathscr{V}}$ are locally free of the same rank, we see that $\mathscr{F}$ is an invertible $\mathbb{O}_S$-module. By using this, the proposition for $\mathscr{H}'$ is easily deduced from the case for $\widehat{\mathscr{V}}$.

Consider the case $\mathscr{H}=\widehat{\mathscr{V}}$. We will show that the natural map

$$\widehat{\mathscr{V}}\otimes\underline{\mathrm{Hom}}_{\widehat{\mathscr{G}}(L)}(\widehat{\mathscr{V}},\mathscr{M})\to\mathscr{M} \tag{4-5}$$

sending $v\otimes f$ to $f(v)$, which is clearly functorial in $\mathscr{M}$, is an isomorphism in $\mathrm{Rep}^1_{\mathrm{sm}}(\widehat{\mathscr{G}}(L))$. Once this is shown, the same argument as on page 113 of [Moret-Bailly 1985] proves that $\mathscr{N}\to\underline{\mathrm{Hom}}_{\widehat{\mathscr{G}}(L)}(\widehat{\mathscr{V}},\widehat{\mathscr{V}}\otimes\mathscr{N})$ is a functorial isomorphism in QCoh_S, and we will be done. As a preparation, let us consider the functors



where $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{G}_1$ and $\mathscr{G}_2$ are the functors in Proposition 2.12(ii), which give equivalences of categories, and $\mathscr{F}_3$ is given by the rule $\mathscr{F}_3(\mathscr{M}_0)=\mathscr{M}_0^{A[m]}$. Then $\mathscr{F}_3\circ\mathscr{F}_2\simeq\mathscr{F}_1$ canonically. Indeed, in view of Lemma 4.11(i),

$$\mathscr{F}_1(\mathscr{M}_0)=\mathscr{M}_0\otimes\mathscr{V}^{(n)}\simeq(\mathscr{M}_0\otimes\mathscr{V}^{(mn)})^{A[m]}=\mathscr{F}_3(\mathscr{F}_2(\mathscr{M}_0)).$$

Now let $\mathscr{M}\in\mathrm{Rep}^1_{\mathrm{sm}}(\widehat{\mathscr{G}}(L))$, and set $\mathscr{M}^{(n)}:=\mathscr{M}^{\hat{\sigma}(nTA)}$ for $n\geq 1$. It is implied by $\mathscr{F}_3\circ\mathscr{F}_2\simeq\mathscr{F}_1$ that canonically

$$\mathscr{G}_1\mathscr{F}_3(\mathscr{F}_2\mathscr{G}_2)\simeq(\mathscr{G}_1\mathscr{F}_1)\mathscr{G}_2.$$

Thanks to Proposition 2.12(ii), we get a canonical isomorphism $\mathscr{G}_1\mathscr{F}_3\simeq\mathscr{G}_2$. Applying to $\mathscr{M}^{(mn)}$ and unraveling the functors, we have a canonical isomorphism

$$\underline{\mathrm{Hom}}_{\widetilde{\mathscr{G}}((mn)^*L)}(\mathscr{V}^{(mn)},\mathscr{M}^{(mn)})\xrightarrow{\sim}\underline{\mathrm{Hom}}_{\widetilde{\mathscr{G}}(n^*L)}(\mathscr{V}^{(n)},\mathscr{M}^{(n)}) \tag{4-6}$$

induced by the restriction to $\mathscr{V}^{(n)}$. (The right-hand side of (4-6) is a rewriting of $\underline{\mathrm{Hom}}_{\mathscr{G}(n^*L)}(\mathscr{V}^{(n)},\mathscr{M}^{(n)})$, where we use the fact that $\hat{\sigma}(nTA)$ is trivial on $\mathscr{V}^{(n)}$ and $\mathscr{M}^{(n)}$. What happens to the left-hand side is similar.) We obtain the following commutative diagram in which the vertical maps come from natural inclusions

$\mathcal{V}^{(n)} \hookrightarrow \mathcal{V}^{(mn)}$, $\mathcal{M}^{(n)} \hookrightarrow \mathcal{M}^{(mn)}$ and (4-6):

$$\begin{array}{ccc} \mathcal{V}^{(n)} \otimes \underline{\mathrm{Hom}}_{\widetilde{\mathcal{G}}(n^*L)}(\mathcal{V}^{(n)}, \mathcal{M}^{(n)}) & \xrightarrow{\sim} & \mathcal{M}^{(n)} \\ \downarrow & & \downarrow \\ \mathcal{V}^{(mn)} \otimes \underline{\mathrm{Hom}}_{\widetilde{\mathcal{G}}((mn)^*L)}(\mathcal{V}^{(mn)}, \mathcal{M}^{(mn)}) & \xrightarrow{\sim} & \mathcal{M}^{(mn)} \end{array}$$

By taking limit over n, we deduce that (4-5) is an isomorphism. □

Corollary 4.16. *The canonical map $\mathbb{O}_S \to \underline{\mathrm{End}}_{\widehat{\mathcal{G}}(L)}(\mathcal{H})$ (via the $\mathbb{O}_S$-module structure on $\mathcal{H}$) is an isomorphism.*

Proof. By Theorem 4.15, $\underline{\mathrm{End}}_{\widehat{\mathcal{G}}(L)}(\mathcal{H}) \simeq \underline{\mathrm{End}}_{\mathbb{O}_S}(\mathbb{O}_S) \simeq \mathbb{O}_S$. □

Corollary 4.17. *Let $\mathcal{H}$ be as in Lemma 4.13. If $\mathcal{H}'$ is another $\widehat{\mathcal{G}}(L)$-representation with the same property, then there exists an invertible $\mathbb{O}_S$-module $\mathcal{N}$ such that*

$$\mathcal{H}' \simeq \mathcal{H} \otimes \mathcal{N}.$$

Proof. This is proved as in the first paragraph of the proof of Theorem 4.15. □

Corollary 4.18. *Suppose that $S = \mathrm{Spec}\, R$ for a local ring R. Then any two Heisenberg representations are isomorphic.*

Proof. Immediate from Corollary 4.17. □

This subsection ends with an alternative characterization of Heisenberg representations. It will be used in Section 5A.

Proposition 4.19. *The following are equivalent*:

(i) *$\mathcal{H}$ is a Heisenberg representation of $\widehat{\mathcal{G}}(L)$. (See Definition 4.12.)*

(ii) *$\mathcal{H}$ is a weight-1 admissible smooth irreducible representation of $\widehat{\mathcal{G}}(L)$ on a locally free $\mathbb{O}_S$-module such that $\mathcal{H}$ does not vanish anywhere on S.*

Remark 4.20. In (ii) above, it is enough to require $\mathcal{H} \neq 0$ when S is connected. On the other hand, one could show that the admissibility in (ii) is superfluous by extending Lemma 4.22 to the case when $\mathcal{F}$ may be of infinite rank. That proof is easily reduced to the finite rank situation.

Proof. Lemma 4.13 says that (i) implies (ii). In order to show the other implication, let $\mathcal{H}$ be as in (ii) and $\mathcal{H}'$ be a Heisenberg representation of $\widehat{\mathcal{G}}(L)$ (in the sense of Definition 4.12). Theorem 4.15 tells us that $\mathcal{H} \simeq \mathcal{H}' \otimes \mathcal{N}$ for some $\mathbb{O}_S$-module $\mathcal{N}$. By taking invariants under $\hat{\sigma}(nTA)$ for a large enough n (so that the invariants are nontrivial), we see that $\mathcal{N}$ has to be a coherent $\mathbb{O}_S$-module. It suffices to show that $\mathcal{N}$ is locally free of rank 1.

Choose an arbitrary $s \in S$. The stalks at s are related by $\mathcal{H}_s \simeq \mathcal{H}'_s \otimes_{\mathbb{O}_{S,s}} \mathcal{N}_s$. We see that $\mathcal{N}_s$ is a projective $\mathbb{O}_{S,s}$-module as $\mathcal{H}'_s$ and $\mathcal{H}_s$ are free over $\mathbb{O}_{S,s}$. Since $\mathcal{N}_s$ is finitely generated over the noetherian ring $\mathcal{O}_{S,s}$, it is free of finite rank. Now let U be an open affine noetherian neighborhood of s in S. The proof will be complete if $\mathcal{N}|_U$ is shown to be an invertible $\mathcal{O}_U$-module.

Suppose this is not the case. Lemma 4.23 tells us that $\mathcal{N}|_U$ has an $\mathcal{O}_U$-submodule $\mathcal{M}$ that is not given as $\mathcal{N}|_U \otimes_{\mathcal{O}_U} \mathcal{I}$ for any ideal sheaf $\mathcal{I} \subset \mathcal{O}_U$. Applying Lemma 4.22, we obtain an $\mathcal{O}_S$-submodule $\mathcal{N}'$ of $\mathcal{N}$ such that $\mathcal{N}'|_U = \mathcal{M}$. Then it is impossible that $\mathcal{N}' = \mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{I}$ for an ideal sheaf $\mathcal{I} \subset \mathcal{O}_S$. (If it were possible, by restricting to U, we would get $\mathcal{M} = \mathcal{N}_U \otimes_{\mathcal{O}_U} \mathcal{I}|_U$, but this is a contradiction.) This means, via Theorem 4.15 (applicable to $\mathcal{H}'$), that $\mathcal{H}$ allows a $\widehat{\mathcal{G}}(L)$-subrepresentation $\mathcal{H}' \otimes \mathcal{N}'$ not given by an ideal sheaf, contradicting the assumption that $\mathcal{H}$ is irreducible. □

Corollary 4.21. *Consider (A', L') and (A, L) with a bounded isogeny $\alpha : A' \to A$ such that $L' = \alpha^* L$. Let $\widehat{\mathcal{G}}(\alpha) : \widehat{\mathcal{G}}(L') \xrightarrow{\sim} \widehat{\mathcal{G}}(L)$ be defined as in Lemma 3.13. If $\rho : \widehat{\mathcal{G}}(L) \to \underline{\mathrm{Aut}}_{\mathcal{O}_S}(\mathcal{H})$ is a Heisenberg representation, then $\rho \circ \widehat{\mathcal{G}}(\alpha)$ is a Heisenberg representation of $\widehat{\mathcal{G}}(L')$.*

Proof. This is clear from criterion (ii) of Proposition 4.19 and Lemma 3.16. (Thanks to the latter, the fact that ρ is admissible and smooth shows that $\rho \circ \widehat{\mathcal{G}}(\alpha)$ is also.) □

The following two lemmas were used in the proof of Proposition 4.19:

Lemma 4.22. *Let $\mathcal{F}$ be an $\mathcal{O}_S$-module and U an open affine subscheme of S. Let $\mathcal{M}$ be an $\mathcal{O}_U$-module defined by an $\mathcal{O}_S(U)$-submodule of $\mathcal{F}(U)$. Define a Zariski presheaf $\mathcal{F}'$ on S by*

$$\mathcal{F}'(V) = \left\{ a \in \mathcal{F}(V) \;\middle|\; a|_{U\cap V} \in \mathcal{M}(U \cap V) \right\}.$$

Then $\mathcal{F}'$ is a Zariski sheaf and an $\mathcal{O}_S$-submodule of $\mathcal{F}$. (Recall that every $\mathcal{O}_S$-module (likewise, every $\mathcal{O}_U$-module) is required to be quasicoherent in our convention.)

Proof. It is a routine check that $\mathcal{F}'$ is a Zariski sheaf. By construction, $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$. The verification that $\mathcal{F}'$ is an $\mathcal{O}_S$-module reduces to the affine case, in which case it is elementary. □

Lemma 4.23. *Let U be a noetherian scheme. Let $\mathcal{F}$ be a locally free $\mathcal{O}_U$-module of finite rank. Suppose that $\mathcal{F}$ has rank at least 1 at every point of U and rank greater than 1 at some point $u \in U$. (Note that U may not be connected.) Then there exists an $\mathcal{O}_U$-submodule $\mathcal{M} \subset \mathcal{F}$ that is not of the form $\mathcal{M} = \mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{I}$ for any ideal sheaf $\mathcal{I} \subset \mathcal{O}_U$.*

Proof. We can find an affine subscheme $V = \operatorname{Spec} B$ of U containing u on which $\mathcal{F}|_V$ is free of rank at least 2. Let M be any rank-1 free B-submodule of $\mathcal{F}(V)$, and denote by $\mathcal{M}'$ the corresponding $\mathcal{O}_V$-module. Extend $\mathcal{M}'$ to an $\mathcal{O}_U$-module $\mathcal{M}$ by the previous lemma. We claim that $\mathcal{M}$ satisfies the condition of the lemma. Indeed, if we had $\mathcal{M} = \mathcal{F} \otimes_{\mathcal{O}_U} \mathcal{I}$ for some ideal $\mathcal{I} \subset \mathcal{O}_U$, then we would reach a contradiction by taking stalk at u and computing the $k(u)$-dimension after tensoring $k(u) := \mathbb{O}_{U,u}/m_{U,u}$. (Here $m_{U,u}$ denotes the unique maximal ideal of $\mathbb{O}_{U,u}$.) □

4D. ***Dual Heisenberg representations and matrix coefficients.*** As we have seen in Section 2D, there are isomorphisms of $\mathscr{G}(n^*L) \times \mathscr{G}(n^*L)$-representations

$$\mathscr{V}(n^*L) \otimes \mathscr{V}(n^*L)^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\mathscr{G}(n^*L), \mathbb{O}_S) \tag{4-7}$$

for varying n. On the right-hand side, $\mathbb{G}_m$ acts on $\mathscr{G}(n^*L)$ and $\mathbb{O}_S$ by multiplication. We will promote (4-7) to an adelic isomorphism.

Definition 4.24. Define an $\mathbb{O}_S$-module

$$\begin{aligned}\underline{\mathrm{Hom}}^{\mathrm{sm}}_{\mathbb{G}_m}(\widehat{\mathscr{G}}(L), \mathbb{O}_S) &:= \bigcup_{n\geq 1} \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathscr{G}}(L)/\hat{\sigma}(nTA), \mathbb{O}_S)\\ &= \bigcup_{n\geq 1} \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathscr{G}}(L), \mathbb{O}_S)^{\hat{\sigma}(nTA)\times\{1\}}.\end{aligned}$$

A section ϕ of $\underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathscr{G}}(L), \mathbb{O}_S)$ is said to be *smooth* if it is a section of the above $\mathbb{O}_S$-module. (The definition is equivalent if $\hat{\tau}$ is used in place of $\hat{\sigma}$; cf. Remark 4.9.)

Lemma 4.25. *The map*

$$\widehat{\mathscr{V}}(L) \otimes \widehat{\mathscr{V}}(L)^\vee \to \underline{\mathrm{Hom}}^{\mathrm{sm}}_{\mathbb{G}_m}(\widehat{\mathscr{G}}(L), \mathbb{O}_S) \tag{4-8}$$

$$v \otimes v^\vee \mapsto (\gamma \mapsto v^\vee(\gamma v)) \tag{4-9}$$

is an isomorphism of $\widehat{\mathscr{G}}(L) \times \widehat{\mathscr{G}}(L)$*-representations. Here,* $\widehat{\mathscr{V}}(L)^\vee$ *is equipped with an action of* $\widehat{\mathscr{G}}(L)$ *by the same formula as* (2-2)*. On the right-hand side, the action is described by* $((\gamma_1, \gamma_2)\psi)(\gamma) = \psi(\gamma_2^{-1}\gamma\gamma_1)$ *for* $\psi \in \underline{\mathrm{Hom}}^{\mathrm{sm}}_{\mathbb{G}_m}(\widehat{\mathscr{G}}(L), \mathbb{O}_S)$ *and* $\gamma_1, \gamma_2 \in \widehat{\mathscr{G}}(L)$.

Proof. Recall from Lemma 3.15 that $\widetilde{\mathscr{G}}(n^*L)/\hat{\sigma}(nTA) \simeq \mathscr{G}(n^*L)$ naturally. Thus, (4-7) may be rewritten as

$$\begin{aligned}\mathscr{V}(n^*L) \otimes \mathscr{V}(n^*L)^\vee &\xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widetilde{\mathscr{G}}(n^*L), \mathbb{O}_S)^{\hat{\sigma}(nTA)\times\hat{\sigma}(nTA)}\\ &= \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widetilde{\mathscr{G}}(n^*L), \mathbb{O}_S)^{\hat{\sigma}(nTA)\times\{1\}},\end{aligned}$$

where the last equality holds thanks to Lemma 3.15(ii). By taking further invariant, we obtain

$$\mathscr{V}(L) \otimes \mathscr{V}(n^*L)^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widetilde{\mathscr{G}}(n^*L), \mathbb{O}_S)^{\hat{\sigma}(TA)\times\{1\}}$$

as maps of $\widetilde{\mathscr{G}}(L) \times \widetilde{\mathscr{G}}(n^*L)$-representations. (Note that $\mathscr{V}(L)$ is acted upon by $N_{\widetilde{\mathscr{G}}(n^*L)}(\hat{\sigma}(TA))/\hat{\sigma}(TA) = \widetilde{\mathscr{G}}(L)/\hat{\sigma}(TA)$.)

We patch these isomorphisms via inverse limit, which are compatible as n varies (as they are given by the same formula as (4-9)), to obtain an isomorphism of

$\widetilde{\mathcal{G}}(L)\times\widehat{\mathcal{G}}(L)$-representations

$$\mathcal{V}(L)\otimes\widehat{\mathcal{V}}(L)^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathcal{G}}(L),\mathbb{O}_S)^{\hat\sigma(TA)\times\{1\}}. \tag{4-10}$$

Likewise, there is an isomorphism of $\widetilde{\mathcal{G}}(n^*L)\times\widehat{\mathcal{G}}(L)$-representations

$$\mathcal{V}(n^*L)\otimes\widehat{\mathcal{V}}(L)^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathcal{G}}(L),\mathbb{O}_S)^{\hat\sigma(nTA)\times\{1\}}$$

given by the same formula as (4-9). By patching again, we arrive at the map (4-8) and see that it is an isomorphism. □

Corollary 4.26. *For any Heisenberg representation* $\mathcal{H}$, (4-9) *induces an isomorphism of* $\widehat{\mathcal{G}}(L)\times\widehat{\mathcal{G}}(L)$*-representations*

$$\mathcal{H}\otimes\mathcal{H}^\vee \simeq \underline{\mathrm{Hom}}^{\mathrm{sm}}_{\mathbb{G}_m}(\widehat{\mathcal{G}}(L),\mathbb{O}_S).$$

Proof. Corollary 4.17 tells us that $\mathcal{H}\otimes\mathcal{H}^\vee\simeq\widehat{\mathcal{V}}(L)\otimes\widehat{\mathcal{V}}(L)^\vee$ canonically. Composing this with (4-8), we derive the desired isomorphism. □

Definition 4.27. Set $\underline{\mathrm{Hom}}^{\mathrm{sm}}_{\mathbb{O}_S}(VA,\mathbb{O}_S) := \bigcup_{n\ge1}\underline{\mathrm{Hom}}_{\mathbb{O}_S}(VA/nTA,\mathbb{O}_S)$. A section of $\underline{\mathrm{Hom}}^{\mathrm{sm}}_{\mathbb{O}_S}(VA,\mathbb{O}_S)$ is said to be *smooth*.

From here until the end of this subsection, assume in addition that L is *symmetric*. There is a further isomorphism of $\mathbb{O}_S$-modules

$$\underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathcal{G}}(L),\mathbb{O}_S)\simeq\underline{\mathrm{Hom}}_{\mathbb{O}_S}(VA,\mathbb{O}_S) \tag{4-11}$$

by restricting from $\widehat{\mathcal{G}}(L)\simeq\mathbb{G}_m\times VA$ (Lemma 3.18) to $\{1\}\times VA$. Then (4-11) induces an isomorphism from $\underline{\mathrm{Hom}}^{\mathrm{sm}}_{\mathbb{G}_m}(\widehat{\mathcal{G}}(L),\mathbb{O}_S)$ onto $\underline{\mathrm{Hom}}^{\mathrm{sm}}(VA,\mathbb{O}_S)$, and the $\widehat{\mathcal{G}}(L)\times\widehat{\mathcal{G}}(L)$-action may be transported to them. This action will be used in the following corollary:

Corollary 4.28. *Suppose* $\deg L=1$*. There is a Heisenberg representation* $\mathcal{H}$ *so that*

$$\begin{aligned}\mathcal{H}^\vee &= \underline{\mathrm{Hom}}_{\mathbb{O}_S}(VA,\mathbb{O}_S)^{\hat\sigma(TA)\times\{1\}}\\ &=\left\{\phi\in\underline{\mathrm{Hom}}_{\mathbb{O}_S}(VA,\mathbb{O}_S)\;\middle|\;\phi(x)=e^L_*(\tfrac12 y)\,\hat e^L(\tfrac12 x,y)\cdot\phi(x+y),\ \forall x\in VA,\ y\in TA\right\}\\ &=\left\{\phi\in\underline{\mathrm{Hom}}^{\mathrm{sm}}_{\mathbb{O}_S}(VA,\mathbb{O}_S)\;\middle|\;\phi(x)=e^L_*(\tfrac12 y)\,\hat e^L(\tfrac12 x,y)\cdot\phi(x+y),\ \forall x\in VA,\ y\in TA\right\}.\end{aligned}$$

The action of $(\lambda,z)\in\mathbb{G}_m\times VA\simeq\widehat{\mathcal{G}}(L)$ *(cf. Lemma 3.18,* (3-10)*) is described by*

$$((\lambda,z)\phi)(x)=\lambda^{-1}\hat e^L(x,z/2)\cdot\phi(x-z).$$

Proof. Set $\mathcal{H}:=\widehat{\mathcal{V}}(L)\otimes\mathcal{V}(L)^\vee$. By the assumption $\mathcal{V}(L)$ is an invertible $\mathbb{O}_S$-module. The isomorphism (4-10) provides

$$\mathcal{H}^\vee\xrightarrow{\sim}\underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathcal{G}}(L),\mathbb{O}_S)^{\hat\sigma(TA)\times\{1\}}\simeq\underline{\mathrm{Hom}}_{\mathbb{O}_S}(VA,\mathbb{O}_S)^{\hat\sigma(TA)\times\{1\}}.$$

Let $y \in TA$. If $\phi \in \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathcal{G}}(L), \mathcal{O}_S)$, then $\hat{\sigma}(y) = \hat{\tau}(y)\, e_*^L(y/2)$ acts on ϕ as follows, where the elements of $\widehat{\mathcal{G}}(L)$ are written using Lemma 3.18:

$$\begin{aligned}(\hat{\sigma}(y)\cdot\phi)((1,x)) &= \phi(e_*^L(\tfrac{1}{2}y)(1,x)(1,y))\\ &= e_*^L(\tfrac{1}{2}y)\cdot\phi((\hat{e}^L(\tfrac{1}{2}x,y),x+y))\\ &= e_*^L(\tfrac{1}{2}y)\hat{e}^L(\tfrac{1}{2}x,y)\cdot\phi((1,x+y)).\end{aligned}$$

Thus, the condition that $\hat{\sigma}(y)\cdot\phi = \phi$ for all $y \in TA$ produces the transformation formula for ϕ. Such a ϕ is automatically smooth. Indeed, for any $x \in VA$, choose $n \geq 1$ such that $x \in \frac{1}{n}TA$. The transformation formula tells us that $\phi(x+y) = \phi(x)$ for all $y \in 2nTA$ since $e_*^L|_{2TA} \equiv 1$ and $\hat{e}^L|_{TA\times TA} \equiv 1$.

To compute the group action, let $\psi \in \underline{\mathrm{Hom}}_{\mathbb{G}_m}(\widehat{\mathcal{G}}(L), \mathcal{O}_S)$ be the map corresponding to ϕ via (4-11). Then (using Lemma 3.18 in the third equality)

$$\begin{aligned}((\lambda,z)\phi)(x) &= (\lambda\hat{\tau}(z)\psi)(\hat{\tau}(x)) = \lambda^{-1}\psi(\hat{\tau}(z)^{-1}\hat{\tau}(x))\\ &= \lambda^{-1}\psi(\hat{e}^L(z/2,x)^{-1}\hat{\tau}(x-z)) = \lambda^{-1}\hat{e}^L(z/2,x)^{-1}\psi(\hat{\tau}(x-z))\\ &= \lambda^{-1}\hat{e}^L(x,z/2)\cdot\phi(x-z). \qquad \square\end{aligned}$$

Remark 4.29. Corollary 4.28 may be thought of as presenting the (dual) lattice model for $\mathcal{H}^\vee$, whose dual gives rise to the lattice model for $\mathcal{H}$.

Remark 4.30. Although $\mathcal{H}$ is a smooth $\widehat{\mathcal{G}}(L)$-representation, there is no reason to expect $\mathcal{H}^\vee$ to be smooth in general. We caution the reader that the smoothness of ϕ in Corollary 4.28 does not imply that $\mathcal{H}^\vee$ is smooth as a $\widehat{\mathcal{G}}(L)$-representation.

Remark 4.31. Let us assume that L has index 0, namely that L is relatively ample. By choosing a particular section $l_0 \in H^0(S, \widehat{\mathcal{V}}(L)^\vee)$, one can associate theta functions for each element of $H^0(S, \widehat{\mathcal{V}}(L))$ as explained in [Mumford 2007, §5, Application 2]. More precisely, take l_0 to be "the evaluation at 0" map $\widehat{\mathcal{V}}(L) \to \mathcal{O}_S$. Then (4-8) (by taking $v^\vee = l_0$) and (4-11) induce a map $H^0(S, \widehat{\mathcal{V}}(L)) \to \mathrm{Hom}_{\mathcal{O}_S}(VA, \mathcal{O}_S)$, which is a geometric construction of theta functions.

4E. *An application of the Künneth formula, Part I.* For $r = 1, 2$, let $f_r : A_r \to S$ be an abelian scheme with a nondegenerate line bundle L_r of index i_r. Define $A := A_1 \times_S A_2$ with projections $p_r : A \to A_r$ and the structure map $f : A \to S$. Take $L := p_1^*L_1 \otimes p_2^*L_2$.

Lemma 4.32. *We have canonical isomorphisms*

$$R^j f_* L = \begin{cases} R^{i_1} f_{1,*}L_1 \otimes R^{i_2} f_{2,*}L_2 & \text{if } j = i_1 + i_2,\\ 0 & \text{if } j \neq i_1 + i_2.\end{cases}$$

In particular, L is nondegenerate of index $i_1 + i_2$.

Proof. This is a consequence of [Grothendieck 1963, Théorème 6.7.8]. (Take the two complexes of $\mathcal{O}_S$-modules there to be L_1 and L_2, where each of them is viewed as a complex concentrated in degree 0.) □

By checking that the isomorphisms in Lemma 4.32 for n^*L are compatible with transition maps for varying n (namely $\nu_{n,mn}$ in Section 3B and its analogues for (A_1, L_1) and (A_2, L_2)), we obtain a canonical isomorphism

$$\widehat{\mathcal{V}}(L_1) \otimes \widehat{\mathcal{V}}(L_2) \simeq \widehat{\mathcal{V}}(L). \tag{4-12}$$

Moreover, we have a natural embedding

$$\mathcal{G}(n^*L_1) \times \mathcal{G}(n^*L_2) \hookrightarrow \mathcal{G}(n^*L)$$

for each $n \geq 1$, sending $((\phi_1, x_1), (\phi_2, x_2))$ to $(p_1^*\phi_1 \otimes p_2^*\phi_2, (x_1, x_2))$. This map lifts to a map $\widetilde{\mathcal{G}}(n^*L_1) \times \widetilde{\mathcal{G}}(n^*L_2) \hookrightarrow \widetilde{\mathcal{G}}(n^*L)$ and patches to

$$\widehat{\mathcal{G}}(n^*L_1) \times \widehat{\mathcal{G}}(n^*L_2) \hookrightarrow \widehat{\mathcal{G}}(n^*L). \tag{4-13}$$

It is a routine check that (4-12) is equivariant with respect to (4-13). Namely, the restriction of the $\widehat{\mathcal{G}}(n^*L)$-representation $\widehat{\mathcal{V}}(L)$ to $\widehat{\mathcal{G}}(n^*L_1) \times \widehat{\mathcal{G}}(n^*L_2)$ via (4-13) is identified via (4-12) with the $\widehat{\mathcal{G}}(n^*L_1) \times \widehat{\mathcal{G}}(n^*L_2)$-representation on $\widehat{\mathcal{V}}(L_1) \otimes \widehat{\mathcal{V}}(L_2)$. In Section 5C, we will see an analogous result for Weil representations.

4F. ***Representations of p-adic Heisenberg groups.*** We return to the p-adic setup of Section 3F; in particular, $\deg L$ is assumed to be a power of a prime p. Define a $\widehat{\mathcal{G}}_p(L)$-representation

$$\widehat{\mathcal{V}}_p(L) := \varinjlim_n \mathcal{V}^{(p^n)}(L).$$

The admissibility and smoothness are defined for $\widehat{\mathcal{G}}_p(L)$-representations as in Definitions 4.7 and 4.8 by letting n run over powers of p. A Heisenberg representation of $\widehat{\mathcal{G}}_p(L)$ is defined exactly as in Proposition 4.19(ii) and induces an equivalence of categories as in Theorem 4.15. The representation $\widehat{\mathcal{V}}_p(L)$ is a Heisenberg representation of $\widehat{\mathcal{G}}_p(L)$, and any two Heisenberg representations differ by a tensoring with a line bundle over S. We also have the analogues of results in Section 4D and Section 4E.

5. Weil representations

As in the previous section, let A be an abelian scheme over a locally noetherian scheme S. Now L is a nondegenerate *symmetric* line bundle over A.

5A. ***Adelic Weil representations.*** Let $\rho : \widehat{\mathcal{G}}(L) \to \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathcal{H})$ be any adelic Heisenberg representation (Definition 4.12, cf. Proposition 4.19). Define a group functor $\underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)$ on (Sch/S) by

$$\underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)(T) = \{\, g \in \underline{\mathrm{Aut}}^{\mathrm{b}}_T(VA \times_S T) \mid \hat{e}^L \circ (g, g) = \hat{e}^L \,\}. \tag{5-1}$$

(The superscript b stands for "bounded".) Note that $g \in \underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)(T)$ acts on $\mathbb{G}_m(T) \times VA(T)$ by $g \cdot (\lambda, x) = (\lambda, gx)$ and that this action preserves the group law of (3-10). The automorphism $\widehat{\mathcal{G}}(L) \simeq \mathbb{G}_m \times VA$ of Lemma 3.18 allows us to transport the $\underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)$-action to the side of $\widehat{\mathcal{G}}(L)$.

Let T be a locally noetherian S-scheme. Write $L_T := L \times_S T$, and define $\rho_T : \widehat{\mathcal{G}}(L) \times T \to \underline{\mathrm{Aut}}_{\mathbb{O}_T}(\mathcal{H} \otimes \mathbb{O}_T)$ to be the representation induced from ρ by base extension. It can be seen from the construction of $\widehat{\mathcal{G}}(L)$ that $\widehat{\mathcal{G}}(L) \times T \simeq \widehat{\mathcal{G}}(L_T)$ canonically. Moreover, $\mathcal{H} \otimes \mathbb{O}_T$ is a Heisenberg representation of $\widehat{\mathcal{G}}(L_T)$. For each $g \in \underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)(T)$, define $\rho_T^g := \rho_T \circ g$, a weight-1 representation of $\widehat{\mathcal{G}}(L_T)$.

Lemma 5.1.

(i) *ρ_T^g is a Heisenberg representation of $\widehat{\mathcal{G}}(L_T)$.*

(ii) *$\rho_T^g \simeq \rho_T$ as $\widehat{\mathcal{G}}(L_T)$-representations.*

Proof. Without loss of generality, we may assume $T = S$. Since g is a bounded automorphism, there exist $m, m' \geq 1$ such that for every $n \geq 1$, $g(mnTA) \subset nTA$ and $g(nTA) \supset m'nTA$. Thus,

$$\mathcal{H}^{\rho^g(\hat{\tau}(mnTA))} \supset \mathcal{H}^{\rho(\hat{\tau}(nTA))} \quad \text{and} \quad \mathcal{H}^{\rho^g(\hat{\tau}(nTA))} \subset \mathcal{H}^{\rho(\hat{\tau}(m'nTA))}.$$

Therefore, ρ^g is smooth and admissible. Further, ρ^g is irreducible since any $\widehat{\mathcal{G}}(L)$-subrepresentation of ρ^g is also a $\widehat{\mathcal{G}}(L)$-subrepresentation of ρ, which is irreducible. Part (i) follows from Proposition 4.19.

Corollary 4.17 shows that $\rho^g \simeq \rho \otimes_{\mathbb{O}_S} \mathcal{N}$ as $\widehat{\mathcal{G}}(L)$-representations for some invertible sheaf $\mathcal{N}$ on S (equipped with trivial $\widehat{\mathcal{G}}(L)$-action). The isomorphism provides $f : \mathcal{H} \simeq \mathcal{H} \otimes \mathcal{N}$ as $\mathbb{O}_S$-modules. But f obviously induces an isomorphism $\rho \simeq \rho \otimes \mathcal{N}$ of $\widehat{\mathcal{G}}(L)$-representations. Therefore, $\rho^g \simeq \rho$. □

Now define a group functor $\underline{\mathrm{Mp}}^{\mathrm{b}}(VA, \hat{e}^L)$ on (Sch/S) such that for locally noetherian T,

$$\underline{\mathrm{Mp}}^{\mathrm{b}}(VA, \hat{e}^L)(T) = \left\{ (g, M) \in \underline{\mathrm{Sp}}^{\mathrm{b}}(VA \times T, \hat{e}^L) \times \underline{\mathrm{Aut}}_{\mathbb{O}_T}(\mathcal{H} \otimes \mathbb{O}_T) \;\middle|\; M \circ \rho_T \circ M^{-1} = \rho_T^g \right\} \tag{5-2}$$

with group law $(g_1, M_1)(g_2, M_2) = (g_1 g_2, M_1 M_2)$. (The definition is understood as a functor of points.) Similarly define $\underline{\mathrm{Mp}}^{\mathrm{b}}(TA, \hat{e}^L)$ with TA in place of VA. There is a sequence of group functors

$$1 \to \mathbb{G}_m \to \underline{\mathrm{Mp}}^{\mathrm{b}}(VA, \hat{e}^L) \to \underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L) \to 1. \tag{5-3}$$

The first map $\mathbb{G}_m \to \underline{\mathrm{Mp}}^{\mathrm{b}}(VA, \hat{e}^L)$ is given by $\alpha \mapsto (1, \alpha)$ using the canonical isomorphism $\mathbb{G}_m \simeq \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathcal{H})$, and the next map sends (g, M) to g.

We define variants $\underline{\mathrm{Sp}}(VA, \hat{e}^L)$ and $\underline{\mathrm{Mp}}(VA, \hat{e}^L)$, which are also group functors on (Sch/S) and $(\mathrm{LocNoeth}/S)$, respectively. For an S-scheme T, write $T = \coprod_{i \in I} T_i$ as a disjoint union of connected components. Set

$$\underline{\mathrm{Sp}}(VA, \hat{e}^L)(T) := \prod_{i \in I} \underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)(T_i)$$

and similarly for $\underline{\mathrm{Mp}}(VA, \hat{e}^L)(T)$. By the paragraph above Lemma 3.4, The boundedness condition is vacuous in $\underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)(T_i)$. As the analogue of (5-3), we have

$$1 \to \mathbb{G}_m \to \underline{\mathrm{Mp}}(VA, \hat{e}^L) \to \underline{\mathrm{Sp}}(VA, \hat{e}^L) \to 1. \tag{5-4}$$

Remark 5.2. In general, we do not address the issue of representability of $\underline{\mathrm{Sp}}$, $\underline{\mathrm{Mp}}$, $\underline{\mathrm{Sp}}^{\mathrm{b}}$ and $\underline{\mathrm{Mp}}^{\mathrm{b}}$ by ind-group schemes. When there is a level structure (Section 6), we will see that $\underline{\mathrm{Sp}}$ is often representable.

Lemma 5.3. *For any locally noetherian S-scheme T, the sequence of groups obtained from* (5-3) *by taking T-points is exact. The same is true for* (5-4).

Proof. It is enough to deal with (5-3), which implies the other case easily. The lemma is obvious except for the surjectivity, which we check now. Let $g \in \underline{\mathrm{Sp}}(VA, \hat{e}^L)(T)$. It suffices to show that $\underline{\mathrm{Hom}}_{\widehat{\mathcal{G}}(L_T)}(\rho_T, \rho_T^g)$ has a T-point. Since $\rho_T \simeq \rho_T^g$ by the preceding lemma, we have a (noncanonical) isomorphism $\mathrm{Hom}_{\widehat{\mathcal{G}}(L_T)}(\rho_T, \rho_T^g) \simeq \mathrm{Aut}_{\widehat{\mathcal{G}}(L_T)}(\rho_T)$. The latter is isomorphic to $\mathbb{G}_m(T)$ by Theorem 4.15, which is certainly nonempty. □

Remark 5.4. In the classical analogue of (5-3) (or (5-4)), the exactness in the middle results from the irreducibility of the Heisenberg representation and Schur's lemma. The surjectivity results from the Stone–von Neumann theorem.

Definition 5.5. The tautological representations $\underline{\mathrm{Mp}}^{\mathrm{b}}(VA, \hat{e}^L) \to \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathcal{H})$ and $\underline{\mathrm{Mp}}(VA, \hat{e}^L) \to \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathcal{H})$, respectively, given as a morphism of group functors on $(\mathrm{LocNoeth}/S)$ by $(g, M) \mapsto M$ is called the *Weil representation* or the oscillator representation (cf. Remark 2.10).

For the rest of Section 5, we mostly focus on $\underline{\mathrm{Mp}}^{\mathrm{b}}$ and $\underline{\mathrm{Sp}}^{\mathrm{b}}$. The results carry over to $\underline{\mathrm{Mp}}$ and $\underline{\mathrm{Sp}}$ easily (Section 5E). The Weil representation is independent of the choice of the Heisenberg representation $\mathcal{H}$ in a suitable sense, as we will soon see.

Lemma 5.6. *If* $\underline{\mathrm{Mp}}^{\mathrm{b}}_{\mathscr{H}}(VA, \hat{e}^L)$ *and* $\underline{\mathrm{Mp}}^{\mathrm{b}}_{\mathscr{H}'}(VA, \hat{e}^L)$ *denote the group functors arising from Heisenberg representations* $\mathscr{H}$ *and* $\mathscr{H}'$, *respectively, then there is a canonical isomorphism of metaplectic group functors sitting in a commutative diagram below*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \underline{\mathrm{Mp}}^{\mathrm{b}}_{\mathscr{H}}(VA, \hat{e}^L) & \longrightarrow & \underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L) & \longrightarrow & 1 \\ & & \| & & \sim \downarrow \mathrm{can} & & \| & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \underline{\mathrm{Mp}}^{\mathrm{b}}_{\mathscr{H}'}(VA, \hat{e}^L) & \longrightarrow & \underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L) & \longrightarrow & 1 \end{array}$$

Proof. By Corollary 4.17, $\mathscr{H}' = \mathscr{H} \otimes \mathscr{N}$ for an invertible $\mathbb{O}_S$-module $\mathscr{N}$. Thus, there is a canonical isomorphism $\alpha : \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathscr{H}) \simeq \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathscr{H}')$. Then $(g, M) \mapsto (g, \alpha(M))$ clearly induces the desired isomorphism. □

Corollary 5.7. *With the notation in the previous lemma, we have the following commutative diagram*:

$$\begin{array}{ccc} \underline{\mathrm{Mp}}^{\mathrm{b}}_{\mathscr{H}}(VA, \hat{e}^L) & \xrightarrow{\mathrm{Weil}} & \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathscr{H}) \\ \sim \downarrow \mathrm{can} & & \sim \downarrow \mathrm{can} \\ \underline{\mathrm{Mp}}^{\mathrm{b}}_{\mathscr{H}'}(VA, \hat{e}^L) & \xrightarrow{\mathrm{Weil}} & \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathscr{H}') \end{array}$$

Proof. This result follows immediately from the proof of Lemma 5.6. □

We would like to find a splitting of (5-3) over an "open compact subgroup" of $\underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)$. Let $m, n \geq 1$. Let $\underline{\mathrm{Sp}}^{\mathrm{b}}(\frac{1}{m}T(A, L); nTA, \hat{e}^L)$ denote the subgroup functor of $\underline{\mathrm{Sp}}^{\mathrm{b}}(VA, \hat{e}^L)$ consisting of g that stabilizes $\frac{1}{m}T(A, L)$ and nTA and induces the identity map on $\frac{1}{m}T(A, L)/nTA$. Note that $\underline{\mathrm{Sp}}^{\mathrm{b}}(\frac{1}{m}T(A, L); nTA, \hat{e}^L) = \underline{\mathrm{Sp}}^{\mathrm{b}}(T(A, L); mnTA, \hat{e}^L)$. (We will favor the expression on the left-hand side when it seems conceptually helpful.) Now suppose that $(g, M) \in \underline{\mathrm{Mp}}^{\mathrm{b}}(VA, \hat{e}^L)$ with $g \in \underline{\mathrm{Sp}}^{\mathrm{b}}(\frac{1}{2}T(A, L); 2TA, \hat{e}^L)$. The latter condition implies the g-action on $\widehat{\mathscr{G}}(L)$

- preserves $\hat{\tau}(2TA)$, which is equal to $\hat{\sigma}(2TA)$, and
- leaves $\widetilde{\mathscr{G}}(2^*L)$ stable and induces the identity map on

$$\mathscr{G}(2^*L) \simeq \widetilde{\mathscr{G}}(2^*L)/\hat{\tau}(2TA).$$

By restriction, M induces an isomorphism of representations

$$M_0 : \big(\rho|_{\widetilde{\mathscr{G}}(2^*L)}, \mathscr{H}^{\hat{\sigma}(2TA)}\big) \simeq \big(\rho^g|_{\widetilde{\mathscr{G}}(2^*L)}, \mathscr{H}^{\hat{\sigma}(2TA)}\big).$$

The representations factor through the quotient $\mathscr{G}(2^*L)$ of $\widetilde{\mathscr{G}}(2^*L)$. Since g acts as the identity on $\mathscr{G}(2^*L)$, we deduce that $\rho^g = \rho$ (not just an isomorphism) as $\mathscr{G}(2^*L)$-representations on $\mathscr{H}^{\hat{\sigma}(2TA)}$. Hence, by Proposition 2.12(iv),

$$M_0 \in \underline{\mathrm{Aut}}_{\mathscr{G}(L)}(\mathscr{H}^{\hat{\sigma}(2TA)}) \simeq \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathbb{O}_S) \simeq \mathbb{G}_m. \tag{5-5}$$

The former of the two canonical isomorphisms above is given by Proposition 2.12(ii). In light of (5-5), there is a unique choice of M (when g is fixed) that restricts to M_0. This leads to our next result.

Lemma 5.8. *There is a canonical splitting of* (5-3) *over* $\underline{\mathrm{Sp}}^{\mathrm{b}}(TA, \hat{e}^L)$. *Namely, there is a map of group functors*

$$\mathrm{spl} : \underline{\mathrm{Sp}}^{\mathrm{b}}(\tfrac{1}{2}T(A, L); 2TA, \hat{e}^L) \to \underline{\mathrm{Mp}}^{\mathrm{b}}(TA, \hat{e}^L)$$

such that if $\mathrm{spl}(g) = (g, M_g)$, *then* M_g *corresponds to the identity of* $\mathbb{G}_m$ *via* (5-5).

Proof. Let $\alpha : T \to S$ be an S-scheme. For each $g \in \underline{\mathrm{Sp}}^{\mathrm{b}}(\frac{1}{2}T(A, L); 2TA, \hat{e}^L)$, let us define M_g. As was seen in the proof of Lemma 5.3, there exists M'_g such that $(g, M'_g) \in \underline{\mathrm{Mp}}^{\mathrm{b}}(TA, \hat{e}^L)(T)$. Such an M'_g defines an automorphism $a \in \mathbb{G}_m(T)$ by (5-5). Set $M_g := a^{-1} \cdot M'_g$. Then $(g, M_g) \in \underline{\mathrm{Mp}}^{\mathrm{b}}(VA, \hat{e}^L)(T)$, and M_g corresponds to $1 \in \mathbb{G}_m(T)$ via (5-5). Moreover, it is straightforward to verify $(g_1g_2, M_{g_1g_2}) = (g_1g_2, M_{g_1}M_{g_2})$ as the images of $M_{g_1g_2}$ and $M_{g_1}M_{g_2}$ in $\mathbb{G}_m$ via (5-5) are both 1. □

Corollary 5.9. *Suppose that* $\deg L = 1$. *Then there is a canonical splitting of* (5-3) *over* $\underline{\mathrm{Sp}}^{\mathrm{b}}(TA, 4TA, \hat{e}^L)$.

Proof. Immediate, since $T(A, L) = TA$ and

$$\underline{\mathrm{Sp}}^{\mathrm{b}}(TA, 4TA, \hat{e}^L) = \underline{\mathrm{Sp}}^{\mathrm{b}}(\tfrac{1}{2}TA, 2TA, \hat{e}^L). \qquad \square$$

5B. ***Dual Weil representations.*** The dual Heisenberg representation $\mathcal{H}^\vee$ also plays the role of the dual Weil representation. Namely, $\underline{\mathrm{Mp}}^{\mathrm{b}}(A, \hat{e}^L)$ acts on $\mathcal{H}^\vee$ by the rule

$$((g, M) \cdot v^\vee)(v) = v^\vee(M^{-1}v), \quad v \in \mathcal{H},\ v^\vee \in \mathcal{H}^\vee.$$

5C. ***An application of the Künneth formula, Part II.*** We continue Section 4E with the same notation as in that subsection. Note that there is an obvious embedding

$$\underline{\mathrm{Sp}}^{\mathrm{b}}(A_1, \hat{e}^{L_1}) \times \underline{\mathrm{Sp}}^{\mathrm{b}}(A_2, \hat{e}^{L_2}) \hookrightarrow \underline{\mathrm{Sp}}^{\mathrm{b}}(A, \hat{e}^L).$$

The following is the analogue of a classical result [Mœglin et al. 1987, II.1(6)]:

Lemma 5.10. *The isomorphism*

$$i : \widehat{\mathcal{V}}(L_1) \otimes \widehat{\mathcal{V}}(L_2) \simeq \widehat{\mathcal{V}}(L)$$

of (4-12) *is an isomorphism of* $\underline{\mathrm{Mp}}^{\mathrm{b}}(VA_1, \hat{e}^{L_1}) \times \underline{\mathrm{Mp}}^{\mathrm{b}}(VA_2, \hat{e}^{L_2})$*-representations* (*the notion of representations as in Remark 2.10*) *if the action on the right-hand side is pulled back via*

$$\underline{\mathrm{Mp}}^{\mathrm{b}}(VA_1, \hat{e}^{L_1}) \times \underline{\mathrm{Mp}}^{\mathrm{b}}(VA_2, \hat{e}^{L_2}) \to \underline{\mathrm{Mp}}^{\mathrm{b}}(VA, \hat{e}^L)$$
$$((g_1, M_1), (g_2, M_2)) \mapsto (g_1 \otimes g_2, i(M_1 \otimes M_2)i^{-1}).$$

Proof. This is a tautology in view of the way the metaplectic group action is defined. □

5D. *Local Weil representations.* Let $\rho_p : \widehat{\mathcal{G}}_p(L) \to \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathcal{H}_p)$ be a p-adic Heisenberg representation. As in Section 5A, we define $\underline{\mathrm{Mp}}^{\mathrm{b}}(V_p A, \hat{e}^L_p)$ and $\underline{\mathrm{Sp}}^{\mathrm{b}}(V_p A, \hat{e}^L_p)$ and fit them into a sequence (cf. (5-3)) that is exact in the sense of Lemma 5.3:

$$1 \to \mathbb{G}_m \to \underline{\mathrm{Mp}}^{\mathrm{b}}(V_p A, \hat{e}^L_p) \to \underline{\mathrm{Sp}}^{\mathrm{b}}(V_p A, \hat{e}^L_p) \to 1. \tag{5-6}$$

The local Weil representation at p is the tautological representation

$$\underline{\mathrm{Mp}}^{\mathrm{b}}(V_p A, \hat{e}^L_p) \to \underline{\mathrm{Aut}}_{\mathbb{O}_S}(\mathcal{H}_p).$$

There is a splitting of (5-6) over $\underline{\mathrm{Sp}}^{\mathrm{b}}(T_p(A, L), T_p A, \hat{e}^L_p)$ if $p \neq 2$ and over $\underline{\mathrm{Sp}}^{\mathrm{b}}(\frac{1}{2}T_p(A, L); 2T_p A, \hat{e}^L)$ if $p = 2$ (cf. Lemma 5.8). Natural questions on the structure of $\underline{\mathrm{Mp}}^{\mathrm{b}}(V_p A, \hat{e}^L_p)$ are:

(i) When is (5-6) split?

(ii) If (5-6) is not split, does it come from a double cover? Namely, can we show that $\underline{\mathrm{Mp}}^{\mathrm{b}}(V_p A, \hat{e}^L_p)$ has a subgroup functor $\widetilde{\underline{\mathrm{Sp}}}^{\mathrm{b}}(V_p A, \hat{e}^L_p)$ that is an extension of $\underline{\mathrm{Sp}}^{\mathrm{b}}(V_p A, \hat{e}^L_p)$ by μ_2?

For the classical p-adic metaplectic group, it is known that the answers to (i) and (ii) are “never” and “yes”, respectively, at least when $p \neq 2$. The questions seem subtle if S is an $\mathbb{F}_p$-scheme and already when $S = \mathrm{Spec}\, \overline{\mathbb{F}}_p$. We will see a positive answer to (i) when A is an ordinary abelian variety (Corollary 7.7). We do not have a clue to (ii). See Example 6.9 for the case of supersingular abelian varieties.

5E. *From* $\underline{\mathbf{Mp}}^{\mathbf{b}}$ *to* $\underline{\mathbf{Mp}}$. Most results of Section 5 have been stated about $\underline{\mathrm{Mp}}^{\mathrm{b}}$ and $\underline{\mathrm{Sp}}^{\mathrm{b}}$. Everything we have proved or asked about $\underline{\mathrm{Mp}}^{\mathrm{b}}$ and $\underline{\mathrm{Sp}}^{\mathrm{b}}$ applies to $\underline{\mathrm{Mp}}$ and $\underline{\mathrm{Sp}}$. The proof is easily reduced to the case of connected base schemes, in which case $\underline{\mathrm{Mp}}^{\mathrm{b}}$ and $\underline{\mathrm{Mp}}$ coincide as well as $\underline{\mathrm{Sp}}^{\mathrm{b}}$ and $\underline{\mathrm{Sp}}$.

6. Level structures

In our context, a level structure is a trivialization of VA, $V_p A$ and so on. This allows us to compare our theory with the representation theory of the usual symplectic and metaplectic groups over number fields and p-adic fields (which are defined independently of abelian schemes and line bundles). This resembles the level structure arising naturally in the moduli-theoretic setting. It is interesting to note new characteristic p phenomena, which are not observed in the classical theory of Weil representations, when studying the Weil representation of a p-adic metaplectic group in characteristic p (Section 6C). Throughout Section 6, we assume that S is locally noetherian.

6A. ***Level structure on VA.*** Let S be a $\mathbb{Q}$-scheme and $(V, \langle\cdot,\cdot\rangle)$ be an even-dimensional $\mathbb{Q}$-vector space with a symplectic pairing. Let $\psi : \mathbb{A}^\infty \to \mathbb{G}_m$ be a nontrivial morphism of (ind-)group schemes over S. (This is the analogue of the additive character in the classical setting.) By composing, we obtain

$$\langle\cdot,\cdot\rangle_\psi : V \otimes \mathbb{A}^\infty \times V \otimes \mathbb{A}^\infty \to \mathbb{G}_m.$$

Suppose that there is an isomorphism of ind-group schemes over S

$$\eta : V \otimes \mathbb{A}^\infty \simeq VA$$

that carries $\langle\cdot,\cdot\rangle_\psi$ to $\hat{e}^L$. This forces ψ to factor through $\mu_\infty \hookrightarrow \mathbb{G}_m$ since $\hat{e}^L$ factors through $\mu_\infty \hookrightarrow \mathbb{G}_m$ (Lemma 3.17).

Lemma 3.18 together with η allows us to identify $\mathbb{G}_m \times (V \otimes \mathbb{A}^\infty) \simeq \widehat{\mathscr{G}}(L)$, where the left-hand side, to be denoted $\widehat{\mathscr{G}}(V, \langle\cdot,\cdot\rangle_\psi)$, has group law

$$(\lambda, x) \cdot (\mu, y) = (\lambda\mu \cdot \langle x/2, y\rangle_\psi, x + y). \tag{6-1}$$

Again via η, the exact sequences in (3-5) and (5-4) become

$$1 \to \mathbb{G}_m \to \widehat{\mathscr{G}}(V, \langle\cdot,\cdot\rangle_\psi) \to V \otimes \mathbb{A}^\infty \to 0,$$
$$1 \to \mathbb{G}_m \to \underline{\mathrm{Mp}}(V \otimes \mathbb{A}^\infty, \langle\cdot,\cdot\rangle_\psi) \to \underline{\mathrm{Sp}}(V \otimes \mathbb{A}^\infty, \langle\cdot,\cdot\rangle_\psi) \to 1.$$

(The analogue for $\underline{\mathrm{Mp}}^{\mathrm{b}}$ and $\underline{\mathrm{Sp}}^{\mathrm{b}}$ is also obtained from (5-3).) The group functor $\underline{\mathrm{Sp}}(V \otimes \mathbb{A}^\infty, \langle\cdot,\cdot\rangle_\psi)$ is represented by the constant group scheme associated with the usual symplectic group $\mathrm{Sp}(V \otimes \mathbb{A}^\infty, \langle\cdot,\cdot\rangle_\psi)$ while $\underline{\mathrm{Mp}}$ is defined by the same recipe as in (5-2) (using $\underline{\mathrm{Sp}}$ in place of $\underline{\mathrm{Sp}}^{\mathrm{b}}$).

Remark 6.1. Note that η does not exist unless S is in characteristic 0 because V_pA is not a constant ind-group scheme at any point $s \in S$ of residue characteristic p. See Section 6C for a different kind of level structure.

Remark 6.2. In the simple case when $S = \operatorname{Spec} k$ and k is an algebraically closed field of characteristic 0, a choice of $\chi : \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \mu_\infty$ over k gives rise to ψ in the following manner:

$$\psi : \mathbb{A}^\infty \to \mathbb{A}^\infty/\widehat{\mathbb{Z}} \overset{\text{can}}{\simeq} \mathbb{Q}/\mathbb{Z} \overset{\chi}{\simeq} \mu_\infty \hookrightarrow \mathbb{G}_m.$$

6B. ***Local level structure, Part I.*** We consider two kinds of level structures on V_pA. The first one is the local analogue of Section 6A. Let $(V_p, \langle\cdot,\cdot\rangle)$ be a symplectic $\mathbb{Q}_p$-vector space and $\psi : \mathbb{Q}_p \to \mathbb{G}_m$ a nontrivial morphism of (ind-)group schemes over S. Thereby, obtain $\langle\cdot,\cdot\rangle_\psi : V_p \times V_p \to \mathbb{G}_m$, where V_p is also viewed as a constant ind-group scheme over S. A level structure is a $\mathbb{Q}_p$-linear isomorphism

$$\eta : V_p \simeq V_pA$$

($\mathbb{Q}_p$ acts on V_pA as explained in Section 3A) carrying $\langle\cdot,\cdot\rangle_\psi$ to $\hat{e}^L_p$. As in Section 6A, this forces ψ to factor through $\mu_{p^\infty}\hookrightarrow\mathbb{G}_m$. The map η and the p-adic analogue of Lemma 3.18 enable us to identify $\widehat{\mathcal{G}}_p(V_p,\langle\cdot,\cdot\rangle_\psi):=\mathbb{G}_m\times V_p$ with $\widehat{\mathcal{G}}_p(L)$, where the former is equipped with the same group law as in (6-1). We obtain exact sequences

$$1\to\mathbb{G}_m\to\widehat{\mathcal{G}}_p(V_p,\langle\cdot,\cdot\rangle_\psi)\to V_p\to 0, \tag{6-2}$$
$$1\to\mathbb{G}_m\to\underline{\mathrm{Mp}}(V_p,\langle\cdot,\cdot\rangle_\psi)\to\underline{\mathrm{Sp}}(V_p,\langle\cdot,\cdot\rangle_\psi)\to 1.$$

6C. *Local level structure, Part II.* When S is in characteristic p, a different level structure is desirable (cf. Remark 6.1). Let k be a field extension of $\mathbb{F}_p$. Suppose that S is a k-scheme. Let $(\Sigma,\langle\cdot,\cdot\rangle_0)$ be a p-divisible group Σ over k with an alternating pairing $\langle\cdot,\cdot\rangle_0:\Sigma\times\Sigma\to\mu_{p^\infty}$. This can be promoted to

$$\langle\cdot,\cdot\rangle_1:V_p\Sigma\times V_p\Sigma\to V_p\mu_{p^\infty}$$

by the functoriality of V_p. Let $\flat_p:V_p\mu_{p^\infty}\to\mu_{p^\infty}$ be the p-adic analogue of $\flat$ in Section 3D. Set $\langle\cdot,\cdot\rangle:=\flat_p\circ\langle\cdot,\cdot\rangle_1$. Then a level structure is a $\mathbb{Q}_p$-linear isomorphism

$$\zeta:V_p\Sigma\times_k S\xrightarrow{\sim}V_pA$$

matching $\langle\cdot,\cdot\rangle$ and $\hat{e}^L_p$. Set $\widehat{\mathcal{G}}_p(\Sigma,\langle\cdot,\cdot\rangle_0):=\mathbb{G}_m\times V_p\Sigma$ with the group law

$$(\lambda,x)\cdot(\mu,y)=(\lambda\mu\cdot\langle\tfrac{1}{2}x,y\rangle,x+y). \tag{6-3}$$

In light of the p-adic analogue of Lemma 3.18, ζ induces an isomorphism

$$\widehat{\mathcal{G}}_p(\Sigma,\langle\cdot,\cdot\rangle_0)\simeq\widehat{\mathcal{G}}_p(L).$$

The p-adic analogues of exact sequences in (3-5) and (5-4) are identified via ζ with the following, where $\underline{\mathrm{Mp}}$ and $\underline{\mathrm{Sp}}$ are defined as in Section 5A:

$$1\to\mathbb{G}_m\to\widehat{\mathcal{G}}_p(\Sigma,\langle\cdot,\cdot\rangle_0)\to V_p\Sigma\to 0, \tag{6-4}$$
$$1\to\mathbb{G}_m\to\underline{\mathrm{Mp}}(V_p\Sigma,\langle\cdot,\cdot\rangle)\to\underline{\mathrm{Sp}}(V_p\Sigma,\langle\cdot,\cdot\rangle)\to 1.$$

A priori $\underline{\mathrm{Mp}}(V_p\Sigma,\langle\cdot,\cdot\rangle)$ depends not only on $(\Sigma,\langle\cdot,\cdot\rangle_0)$ but also on (A,L) because the definition involves the Heisenberg representation, which is constructed from (A,L). But Corollary 4.17 shows that two Heisenberg representations of $\widehat{\mathcal{G}}_p(\Sigma,\langle\cdot,\cdot\rangle_0)$ (constructed from two choices of (A,L)) differ by a tensoring with an invertible $\mathbb{O}_S$-module, so $\underline{\mathrm{Mp}}(V_p\Sigma,\langle\cdot,\cdot\rangle)$ and its Weil representation depend (up to isomorphism) only on $(\Sigma,\langle\cdot,\cdot\rangle_0)$ thanks to Corollary 5.7.

Remark 6.3. One can consider a variant when S is not entirely in characteristic p. For instance, if $\Sigma=(\mathbb{Q}_p/\mathbb{Z}_p\times\mu_{p^\infty})^g$ for some $g\geq 1$, which can be defined (together with $\langle\cdot,\cdot\rangle_0$) over $\operatorname{Spec}\mathbb{Z}$, one can take S to be any locally noetherian scheme, and the construction above goes through. On the other hand, if $(\Sigma, \langle\cdot,\cdot\rangle_0)$ is as above except that the base ring is not k but the integer ring $\mathbb{O}$ in an algebraic extension field of $\mathbb{Q}_p$, the discussion can be adapted to any $\mathbb{O}$-scheme S.

Remark 6.4. The level structure ζ is the analogue of the Igusa level structure used in the literature (e.g., [Katz and Mazur 1985; Harris and Taylor 2001; Hida 2004]).

Remark 6.5. It is an interesting phenomenon that the Heisenberg group and the metaplectic group at p heavily depend on the isogeny type of Σ (or $A[p^\infty]$) when S is in characteristic p. This is evident in (6-4), for instance. Each isogeny type gives rise to a different mod p Weil representation.

6D. ***Weil representations associated with p-divisible groups, without abelian varieties.*** Assume $p \neq 2$. Let $(\Sigma, \langle\cdot,\cdot\rangle_0)$ and $\langle\cdot,\cdot\rangle$ be as in Section 6C with $k = \overline{\mathbb{F}}_p$. For simplicity, assume that $\langle\cdot,\cdot\rangle_0$ is a perfect pairing. (In general it is enough to require $\langle\cdot,\cdot\rangle$ to be a perfect pairing.) We know that there exists an (A, L) such that there is a symplectic isomorphism $\zeta : V_p\Sigma \simeq V_pA$ thanks to Oort's result ([Rapoport 2005, Theorem 7.4], cf. [Oort 2001]) that any Newton polygon stratum in the mod p fiber of the Siegel modular variety with hyperspecial level at p is nonempty. Then Section 6C attaches the Heisenberg group/representation and Weil representation to $(\Sigma, \langle\cdot,\cdot\rangle_0)$. The goal of this subsection is to sketch an alternative approach without using (A, L) at all.

Recall that $\widehat{\mathcal{G}}_p := \widehat{\mathcal{G}}_p(\Sigma, \langle\cdot,\cdot\rangle_0)$ is already defined in Section 6C independently of (A, L). The key point will be to prove the existence of the Heisenberg representation of $\widehat{\mathcal{G}}_p$ without resorting to (A, L). In particular, we use the fact that any nondegenerate theta group possesses a weight-1 irreducible representation over an algebraically closed field [Moret-Bailly 1985, Chapter 5, Theorem 2.5.5].

Take $\hat{\sigma}_p : T_p\Sigma \to \widehat{\mathcal{G}}_p(\Sigma, \langle\cdot,\cdot\rangle_0)$ to be the natural embedding $x \mapsto (1, x)$. (The assumption $p \neq 2$ is used to ensure that the latter embedding preserves group structure.) It is easy to verify the analogue of Lemma 3.15 for $\widehat{\mathcal{G}}_p$, $\hat{\sigma}_p$, etc. (replacing $\widetilde{\mathcal{G}}((p^n)^*L)$ there with $\widetilde{\mathcal{G}}(p^n) := \mathbb{G}_m \times \frac{1}{p^n}T_p\Sigma$ in $\widehat{\mathcal{G}}_p$), in which $\hat{\sigma}_p(T_p\Sigma)$ embeds via the map $1 \times p^n$. Set $\mathcal{G}(p^n) := \widetilde{\mathcal{G}}(p^n)/\hat{\sigma}_p(T_p\Sigma)$, which is isomorphic to $\mathbb{G}_m \times \Sigma[p^{2n}]$ (which inherits the twisted group law). By the theorem of [Moret-Bailly 1985] cited above, each $\mathcal{G}(p^n)$ possesses a Heisenberg representation (irreducible representation over k of dimension p^n) for $n \geq 1$. The Heisenberg representation $\mathcal{H}$ of $\widehat{\mathcal{G}}_p$ is obtained by patching via the analogue of Lemma 4.5, and then one can check the analogues of Theorem 4.15, Proposition 4.19 and Corollaries 4.26 and 4.28. (Of course *TA*, *VA* and $\hat{e}^L$ should be replaced by $T_p\Sigma$, $V_p\Sigma$, and $\langle\cdot,\cdot\rangle$, and e^L_* should be ignored.) The construction of Section 5A carries over to $\underline{\mathrm{Mp}}(V_p\Sigma, \langle\cdot,\cdot\rangle_0)$ and its Weil representation on $\mathcal{H}$.

Remark 6.6. What we have denoted $\hat{\sigma}_p$ should be thought of as the analogue of $\hat{\tau}_p$ in the previous sections (although there is no distinction when $p \neq 2$). Perhaps one can still work with $p = 2$ if we select $e_* : \Sigma[2] \times \Sigma[2] \to \mu_2$, satisfying the properties of Lemma 3.19, to play the role of e_*^L. Then the above definition of $\hat{\sigma}_2$ should be multiplied by e_* (cf. (3-11)).

Remark 6.7. If $\mathrm{char}(k) \neq p$ and $k = \bar{k}$, then one can identify $V_p\Sigma$ with a symplectic $\mathbb{Q}_p$-vector space (as a constant group scheme), and the above construction still goes through without (A, L). When $k = \mathbb{C}$, this essentially recovers the classical construction.

Remark 6.8. We have worked with Σ over $\overline{\mathbb{F}}_p$ rather than over a more general scheme S. The only essential reason is that the existence of Heisenberg representations (i.e., the analogue of [Moret-Bailly 1985, Chapter 5, Theorem 2.5.5]) no longer holds in general. A sufficient condition for the existence of a Heisenberg representation is that Σ over S comes from some (A, L).

Example 6.9. Let $\Sigma_{1/2}$ denote a supersingular p-divisible group over $\overline{\mathbb{F}}_p$ of height 2 and dimension 1 equipped with a perfect pairing $(\cdot\,,\cdot) : \Sigma_{1/2} \times \Sigma_{1/2} \to \mu_{p^\infty}$. Let $D_{1/2}$ be a central quaternion algebra over $\mathbb{Q}_p$ of invariant 1/2. It is well known that $\mathrm{End}_{\overline{\mathbb{F}}_p}(\Sigma_{1/2})$ is isomorphic to the maximal order of $D_{1/2}$, so $\mathrm{End}_{\overline{\mathbb{F}}_p}(V\Sigma_{1/2}) \simeq D_{1/2}$. (In general, one can use Dieudonné theory to classify p-divisible groups Σ over $\overline{\mathbb{F}}_p$ up to isogeny and identify $\mathrm{End}_{\overline{\mathbb{F}}_p}(V\Sigma)$ as a semisimple $\mathbb{Q}_p$-algebra. See any standard reference such as [Demazure 1972].)

Set $\Sigma := (\Sigma_{1/2})^g$, and define $\langle\cdot\,,\cdot\rangle : \Sigma \times \Sigma \to \mu_{p^\infty}$ by

$$\langle (x_i)_{i=1}^g, (y_i)_{i=1}^g \rangle = \prod_{i=1}^{g} (x_i, y_i).$$

Then

$$\underline{\mathrm{Sp}}(V_p\Sigma, \langle\cdot\,,\cdot\rangle_0) \simeq \mathrm{Sp}_g(D_{1/2})$$

as constant group schemes over $\overline{\mathbb{F}}_p$. Observe that this group is an inner form of $\mathrm{Sp}_{2g}(\mathbb{Q}_p)$. The questions (i) and (ii) of Section 5D would be especially interesting to answer in this case. We would guess "no" to (i) and "yes" to (ii) in this case but without much evidence. The only heuristic reason is that this Σ is the unique p-divisible group over $\overline{\mathbb{F}}_p$ (up to isogeny) that is self-dual and isoclinic, so it makes harder for (5-6) (or the analogous sequence for Σ) to split. (For any other choice of a self-dual Σ, the group of $\overline{\mathbb{F}}_p$-points of $\underline{\mathrm{Sp}}(V_p\Sigma, \langle\cdot\,,\cdot\rangle_0)$ is isomorphic to the group of $\mathbb{Q}_p$-points of an inner form of a proper Levi subgroup of $\mathrm{Sp}_{2g}(\mathbb{Q}_p)$. This is a well known fact in the theory of isocrystals applied to Sp_{2g}, where the former group is often denoted $J_b(\mathbb{Q}_p)$. See [Kottwitz 1997] for instance.)

6E. *Global level structure.* It is clear how to put together local level structures to get a global one. Let $\mathbb{A}^{\infty,p}$ be the prime-to-p part of $\mathbb{A}^\infty$, namely $\widehat{\mathbb{Z}}^p := \prod_{l\neq p} \mathbb{Z}_l$ and $\mathbb{A}^{\infty,p} := \varinjlim_n \frac{1}{n}\widehat{\mathbb{Z}}^p$, where n runs over positive integers prime to p. When S is a $\mathbb{Q}$-scheme, this is done in the obvious manner by globalizing Section 6B. Let us say a few words when S is an $\mathbb{F}_p$-scheme. Consider the analogue $\psi^p : \mathbb{A}^{\infty,p} \to \mathbb{G}_m$ of ψ so that we have $\langle\cdot,\cdot\rangle_{\psi^p} : V \otimes \mathbb{A}^{\infty,p} \times V \otimes \mathbb{A}^{\infty,p} \to \mathbb{G}_m$ (cf. Section 6B). Let $(\Sigma, \langle\cdot,\cdot\rangle_0)$ be as in Section 6C, and set $\langle\cdot,\cdot\rangle_p := \flat_p \circ \langle\cdot,\cdot\rangle_1$ using notation there. A level structure in this setting is an $\mathbb{A}^\infty$-linear isomorphism

$$(\eta^p, \zeta) : V \otimes \mathbb{A}^{\infty,p} \times (V_p\Sigma \times_k S) \xrightarrow{\sim} VA$$

carrying $(\langle\cdot,\cdot\rangle_{\psi^p}, \langle\cdot,\cdot\rangle_p)$ to $\hat{e}^L$. We have exact sequences that look like (6-2) away from p and (6-4) at p.

7. Explicit models

In the study of Weil representations and the theta correspondence, it is important to find a good model on which the group action can be described explicitly. For p-adic or finite adelic metaplectic groups, the most popular models in the classical context are Schrödinger and lattice models. In Section 7, we focus on the p-adic setting and describe the models for Heisenberg and Weil representations in some simple cases. In those cases S is local, so the Heisenberg representation is unique up to isomorphism (Corollary 4.18). The mixed characteristic phenomenon of Section 7D is intriguing and begs further investigation.

Throughout Section 7, L is assumed to be symmetric and nondegenerate of degree 1. (The assumption on degree may not be essential but is very convenient. Degree 1 can be achieved over an algebraically closed field for any (A, L) without disturbing symmetry and nondegeneracy if we are allowed to modify (A, L) by an isogeny. See [Mumford 1974, §23, Theorem 4, cf. Corollary 1].) Let $C^\infty(\cdot, k)$ and $C_c^\infty(\cdot, k)$ denote the k-vector spaces of locally constant and, respectively, locally constant and compactly supported k-valued functions and $D^\infty(\cdot, k)$ denote the k-vector space dual of $C^\infty(\cdot, k)$. Throughout this section, a k-valued function is understood without further comments as a sheaf-theoretic homomorphism with target $\mathbb{O}_{\operatorname{Spec} k}$, but note that in the setting of Section 7A, this is no different from a function in the naïve sense.

7A. ***Over a field of characteristic not equal to p.*** Suppose that $S = \operatorname{Spec} k$, where k is algebraically closed of characteristic unequal to p. Therefore, V_pA is isomorphic to the constant ind-group scheme $\mathbb{Q}_p^{2g}$ over S. In this subsection, we may and will view V_pA as a $\mathbb{Q}_p$-vector space with symplectic pairing $\hat{e}_p^L : V_pA \times V_pA \to k^\times$. Similarly, T_pA is regarded as a free $\mathbb{Z}_p$-module sitting inside V_pA.

Corollaries 4.28 and 4.18, adapted to the local setting, tell us that the lattice model for the dual Heisenberg representation may be described as

$$\mathcal{H}^\vee_{\text{lattice}} = \big\{\phi \in C^\infty(V_pA, k) \;\big|\; \phi(x) = e_*^L(\tfrac{1}{2}y)\hat{e}^L(\tfrac{1}{2}x, y)\cdot\phi(x+y), \forall x \in V_pA, y \in T_pA\big\} \quad (7\text{-}1)$$

with $(\lambda, z) \in \mathbb{G}_m \times VA \simeq \widehat{\mathcal{G}}(L)$ acting as $((\lambda, z)\phi)(x) = \lambda^{-1}\hat{e}^L(x/2, z) \cdot \phi(x - z)$. Note that $e^L_* \equiv 1$ unless $p = 2$ (see Section 3F). The lattice model $\mathcal{H}_{\text{lattice}}$, the dual of $\mathcal{H}^\vee_{\text{lattice}}$, admits a concrete description

$$\mathcal{H}_{\text{lattice}} = \Big\{ \phi \in C^\infty_c(V_pA, k) \;\Big|\; \phi(x) = e^L_*(\tfrac{1}{2}y)\hat{e}^L(\tfrac{1}{2}x, y)^{-1} \cdot \phi(x+y), \forall x \in V_pA,\, y \in T_pA \Big\} \tag{7-2}$$

with the dual action; namely, (λ, z) acts as $((\lambda, z)\phi)(x) = \lambda\hat{e}^L(z/2, x) \cdot \phi(x + z)$. Indeed the pairing

$$\mathcal{H}_{\text{lattice}} \times \mathcal{H}^\vee_{\text{lattice}} \to k, \quad (f, g) \mapsto \sum_{x \in V_pA/T_pA} f(x)\, g(x)$$

is easily verified to be k-linear, perfect and $\widehat{\mathcal{G}}(L)$-equivariant. Refer to the literature such as [Mœglin et al. 1987, Chapter 2.II.8] (when $p \neq 2$) for a precise description of the Weil representation on $\mathcal{H}_{\text{lattice}}$. That reference treats the case $k = \mathbb{C}$, but the same formula applies if $p \neq \text{char}(k)$.

On the other hand, let $T_pA = \Lambda_1 \oplus \Lambda_2$ be a decomposition into free $\mathbb{Z}_p$-submodules that are totally isotropic for $\hat{e}^L_p$ and in perfect duality with respect to $\hat{e}^L_p$. Setting $V_i = \Lambda_i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ for $i = 1, 2$, we have $V_pA = V_1 \oplus V_2$ and may identify V_2 with $V_1^\vee$. When $p = 2$, we assume that

$$\forall x = (x_1, x_2) \in \Lambda_1 \oplus \Lambda_2, \quad e^L_*(x/2)\, \hat{e}^L_p(x_1/2, x_2) = 1. \tag{7-3}$$

The above condition amounts to assuming that L is even symmetric in the terminology of [Mumford 2007, Proposition 4.20]. This can always be achieved by pulling back L via the translation T_x for a suitable $x \in A[2](k)$. See [Mumford 2007, Corollary 4.24]. The Schrödinger model is (e.g., [Mœglin et al. 1987, Chapter 2.I.4.1], [Mumford 2007, Proposition 5.2.A])

$$\mathcal{H}_{\text{Sch}} = C^\infty_c(V_2, k), \quad ((\lambda, z_1, z_2) \cdot \phi)(x_2) = \lambda\hat{e}^L(x_2, z_1)\, \hat{e}^L(z_2/2, z_1) \cdot \phi(x_2 + z_2), \tag{7-4}$$

where we write $z = (z_1, z_2) \in V_1 \oplus V_2$. Corollary 4.18 ensures that $\mathcal{H}_{\text{lattice}} \simeq \mathcal{H}_{\text{Sch}}$ as $\widehat{\mathcal{G}}(L)$-representations on k-vector spaces. Refer to [Mumford 2007, Proposition 5.2], for example, to see an explicit isomorphism. Let us recall an explicit formula for the Weil representation on $\mathcal{H}_{\text{Sch}}$ to be compared to the mod p case later (Section 7C).

Proposition 7.1. *Consider $M_g \in \underline{\text{Aut}}_k(\mathcal{H}_{\text{Sch}})$ for $g \in \underline{\text{Sp}}(V_pA, \hat{e}^L_p)$ in the following three cases.* (*Here M_g and g are implicitly T-valued points for a locally noetherian k-scheme T. The matrices below are written with respect to $V_pA \simeq V_2^\vee \oplus V_2$. In* (iii), *we choose a k-valued Haar measure on V_2, which exists since $p \neq$* char(k).)

(i) $g = \begin{pmatrix} {}^tB^{-1} & 0 \\ 0 & B \end{pmatrix}$, $(M_g\phi)(x) = |\det B|_p^{-1/2}\phi(B^{-1}x)$ *for any* $B \in GL_k(V_2)$.

(ii) $g = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$, $(M_g\phi)(x) = \hat{e}^L_p(Cx, x)\,\phi(x)$, *where* $C \in \mathrm{Hom}_k(V_2, V_2^\vee)$ *is symmetric* (*i.e.*, $C = C^\vee$).

(iii) $g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $(M_g\phi)(x) = \int_{V_2} \hat{e}^L_p(x, y)\,\phi(y)\,dy.$

Then we have $(g, M_g) \in \underline{\mathrm{Mp}}(V_pA, \hat{e}^L_p)$ *in all three cases.*

Proof. This is proved by the same computation as in the proof of [Mumford 2007, Lemma 8.2] (cf. [Mœglin et al. 1987, Chapter 2.II.6]). □

Remark 7.2. Classically the factor $|\det B|_p^{-1/2}$ in (i) is inserted to make M_g a unitary operator. Of course $(g, M_g) \in \underline{\mathrm{Mp}}(V_pA, \hat{e}^L_p)$ still holds if $|\det B|_p^{-1/2}$ is erased.

Example 7.3. The classical Heisenberg and Weil representations (for p-adic groups) are obtained when $k = \mathbb{C}$ and $A = \mathbb{C}^g/\Lambda$ with $\Lambda = \mathbb{Z}^g + i\mathbb{Z}^g$, and L arises from a Riemann form $\Lambda \times \Lambda \to \mathbb{Z}$ defining a principal polarization.

Remark 7.4. In the definition of $\mathcal{H}_{\mathrm{Sch}}$, one cannot use $C^\infty(V_2, k)$ because the latter is not smooth with respect to the $\widehat{\mathcal{G}}(L)$-action (defined by the same formula). As for $\mathcal{H}_{\mathrm{lattice}}$, C_c^∞ cannot be replaced by C^∞ either for the same reason: the $\widehat{\mathcal{G}}(L)$-action on the C^∞-space is not smooth. Likewise, $\mathcal{H}^\vee_{\mathrm{lattice}}$ is nonsmooth.

7B. *Lattice model over* $\overline{\mathbb{F}}_p$. Suppose that $S = \mathrm{Spec}\, k$ with $k = \overline{\mathbb{F}}_p$. The dual lattice model $\mathcal{H}^\vee_{\mathrm{lattice}}$, which is again unique up to isomorphism, has the same description as Corollary 4.28 (cf. (7-1)). As before, $\mathcal{H}_{\mathrm{lattice}}$ is defined to be the dual of $\mathcal{H}^\vee_{\mathrm{lattice}}$ (and equipped with the dual action). Unlike (7-2), we do not have the notion of compact support on V_pA, so view $\mathcal{H}_{\mathrm{lattice}}$ just as a space of distributions. An interesting problem would be to find an explicit formula for the Weil representation on $\mathcal{H}_{\mathrm{lattice}}$.

7C. *Schrödinger model over* $\overline{\mathbb{F}}_p$. Let $k = \overline{\mathbb{F}}_p$ as before. Unlike lattice models, Schrödinger models do not always exist. The first obstruction is that V_pA or $A[p^\infty]$ is not always completely polarizable. For instance, if A is a supersingular elliptic curve, then $A[p^\infty]$ does not admit a product decomposition. According to Dieudonné theory, we can achieve

$$\zeta : \Sigma_1 \times \Sigma_2 \simeq A[p^\infty] \tag{7-5}$$

for mutually dual p-divisible groups Σ_1 and Σ_2 over k, by modifying A with an isogeny if necessary, if there are an exactly even number of simple p-divisible groups of slope $\frac{1}{2}$ in $A[p^\infty]$. Let us suppose that this is the case so that (7-5) exists. Also suppose that (7-5) is a complete polarization, i.e., $\hat{e}^L_p|_{\Sigma_1\times\Sigma_1} \equiv 1$, $\hat{e}^L_p|_{\Sigma_2\times\Sigma_2} \equiv 1$ and $\hat{e}^L_p$ defines a perfect pairing between Σ_1 and Σ_2. Then we also have $V_pA \simeq V_p(A[p^\infty]) \simeq V_p\Sigma_1 \times V_p\Sigma_2$. Now that there is a complete polarization, one can ask whether there is a Schrödinger model for $\mathscr{H}$. The answer is positive in the simplest case.

Proposition 7.5. *Suppose that A is ordinary; in other words, there exists an isomorphism $A[p^\infty] \simeq \Sigma_1 \times \Sigma_2$ with $\Sigma_1 = (\mu_{p^\infty})^g$ and $\Sigma_2 = (\mathbb{Q}_p/\mathbb{Z}_p)^g$. If $p = 2$, assume that* (7-3) *holds with $T_p\Sigma_1 \times T_p\Sigma_2$ in place of $\Lambda_1 \oplus \Lambda_2$. Then the k-vector space $\mathscr{H}_{\mathrm{Sch}} := C_c^\infty(V_p\Sigma_2, k)$ (where $V_p\Sigma_2$ is viewed as a $\mathbb{Q}_p$-vector space) on which $(\lambda, z_1, z_2) \in \mathbb{G}_m \times V_p\Sigma_1 \times V_p\Sigma_2 \simeq \widehat{\mathscr{G}}(L)$ acts by*

$$((\lambda, z_1, z_2)\cdot\phi)(x_2) = \lambda \hat{e}_p^L(x_2, z_1)\, \hat{e}_p^L(z_2/2, z_1)\cdot\phi(x_2 + z_2)$$

is a Heisenberg representation of $\widehat{\mathscr{G}}(L)$.

Remark 7.6. The above formula is the same as (7-4) except that it should be interpreted scheme-theoretically. On the other hand, the lemma does not generalize to the nonordinary case as $C_c^\infty(V_p\Sigma_2, k)$ has no natural meaning if Σ_2 is not étale.

Proof. Without loss of generality, we may assume $\hat{e}_p^L$ is the standard symplectic pairing (of the form (7-7)). Then it is easily verified that

$$\mathscr{H}_{\mathrm{Sch}}^{p^nT_p\Sigma} = C^\infty(\tfrac{1}{p^n}T_p\Sigma_2/p^nT_p\Sigma_2, k).$$

Hence, $\mathscr{H}_{\mathrm{Sch}}$ is smooth and admissible. By the Stone–von Neumann theorem (Theorem 4.15), $\mathscr{H}_{\mathrm{Sch}}$ is isomorphic to a Heisenberg representation tensored with a k-vector space. But the fact that $\dim_k \mathscr{H}_{\mathrm{Sch}}^{p^nT_p\Sigma} = p^{2n}$ shows that the latter vector space has dimension 1. Hence, $\mathscr{H}_{\mathrm{Sch}}$ is itself a Heisenberg representation. □

We introduce an ind k-group scheme

$$P := \left\{ \begin{pmatrix} (B^\vee)^{-1} & C \\ 0 & B \end{pmatrix} \;\middle|\; B \in \underline{\mathrm{Aut}}(V_p\Sigma_2),\ C \in \underline{\mathrm{Hom}}(V_p\Sigma_1, V_p\Sigma_2),\ C^\vee = C \right\}.$$

(The dual $\vee$ between $V_p\Sigma_1$ and $V_p\Sigma_2$ is taken with respect to $\hat{e}_p^L$.) Once a basis is chosen, we can identify $\underline{\mathrm{Aut}}(V_p\Sigma_2) \simeq GL_g(\mathbb{Q}_p)$ and $\underline{\mathrm{Hom}}(V_p\Sigma_1, V_p\Sigma_2) \simeq M_g(V_p\mu_{p^\infty})$ in view of (7-6) below. (We apologize for two different usages of M_g.)

Corollary 7.7. *In the setting of Proposition 7.5, we have*

(i) *a canonical isomorphism $\underline{\mathrm{Sp}}(V_pA, \hat{e}_p^L) \simeq P$ as group functors and*

(ii) *$\mathbb{G}_m \times P \simeq \underline{\mathrm{Mp}}(V_pA, \hat{e}_p^L)$ as group functors via $(\lambda, g) \mapsto \lambda M_g$, where M_g is defined on $\mathscr{H}_{\mathrm{Sch}}$ as*

- $g = \begin{pmatrix} (B^\vee)^{-1} & 0 \\ 0 & B \end{pmatrix}$, $(M_g\phi)(x) = \phi(B^{-1}x)$.
- $g = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$, $(M_g\phi)(x) = \hat{e}_p^L(Cx, x)\,\phi(x)$.

Remark 7.8. The action in (ii) above is the same as (i) and (ii) of Proposition 7.1. Since $\underline{\mathrm{Sp}}(V_pA, \hat{e}^L_p)$ is smaller when $\mathrm{char}(k) = p$, the action (iii) simply does not show up here. Also note that the above M_g-action does not involve $|\det B|_p^{-1/2}$, which does not make sense in k.

Proof. Part (i) is derived from the canonical isomorphisms

$$\begin{gathered}\underline{\mathrm{Hom}}_k(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{Z}/p^n\mathbb{Z}) \simeq \mathbb{Z}/p^n\mathbb{Z}, \quad \underline{\mathrm{Hom}}_k(\mathbb{Z}/p^n\mathbb{Z}, \mu_{p^n}) \simeq \mu_{p^n}, \\ \underline{\mathrm{Hom}}_k(\mu_{p^n}, \mathbb{Z}/p^n\mathbb{Z}) = 0, \quad \underline{\mathrm{Hom}}_k(\mu_{p^n}, \mu_{p^n}) \simeq \mathbb{Z}/p^n\mathbb{Z}.\end{gathered} \tag{7-6}$$

For (ii), the given action of (λ, g) obviously defines a splitting of (5-3). Since $\mathbb{G}_m \times \underline{\mathrm{Sp}}(V_pA, \hat{e}^L_p)$ is representable by a k-group scheme, the same is true for $\underline{\mathrm{Mp}}(V_pA, \hat{e}^L_p)$. □

Remark 7.9. If one naïvely attempts to find a mod p Weil representation, then one could guess that $C^\infty_c(\mathbb{Q}^g_p, \overline{\mathbb{F}}_p)$ is the right model just by imitating the classical Schrödinger model without using the Heisenberg representation (which may be difficult to come up with unless the Heisenberg group is defined scheme-theoretically). But then one gets into trouble in defining a projective representation of $\mathrm{Sp}_{2g}(\mathbb{Q}_p)$. Indeed, the group action in (iii) of Proposition 7.1, which amounts to the Fourier transform, does not make sense over $\overline{\mathbb{F}}_p$. (For instance, there is no $\overline{\mathbb{F}}_p$-valued Haar measure on V_2.) The virtue of our scheme-theoretic approach is that it renders a precise meaning to $C^\infty_c(\mathbb{Q}^g_p, \overline{\mathbb{F}}_p)$, which is but a special case of a mod p Weil representation corresponding to the ordinary p-divisible group. In addition, our approach explains why the Fourier transform action should disappear from the picture.

Denote by $D^\infty(V_p\Sigma_1, k)$ the dual k-vector space of $C^\infty(V_p\Sigma_1, k)$. The following proposition allows us to transport the Heisenberg representation structure from $C^\infty(V_p\Sigma_1, k)$ to $D^\infty(V_p\Sigma_1, k)$:

Proposition 7.10. *There is a canonical isomorphism of k-vector spaces*

$$C^\infty_c(V_p\Sigma_2, k) \simeq D^\infty(V_p\Sigma_1, k).$$

Proof. For a finite group scheme G and its dual $G^\vee$ over k, recall the standard fact that their rings of functions are canonically k-dual, namely $\mathcal{O}_G \simeq (\mathcal{O}_{G^\vee})^\vee$. When applied to $G = \frac{1}{p^n}\mathbb{Z}_p/\mathbb{Z}_p$, this provides a canonical isomorphisms $C(\frac{1}{p^n}\mathbb{Z}_p/\mathbb{Z}_p, k) \simeq D(\mu_{p^n}, k)$ for all $n \geq 1$, where D denotes the distribution. By taking inverse limit, $C_c(\mathbb{Q}_p/\mathbb{Z}_p, k) \simeq D^\infty(T_p\mu_{p^\infty}, k)$. Now by taking the direct limit along the maps on C_c and D^∞ induced by

$$\mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{p} \mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{p} \cdots, \quad T_p\mu_{p^\infty} \xrightarrow{p} T_p\mu_{p^\infty} \xrightarrow{p} \cdots,$$

we obtain $C^\infty_c(\mathbb{Q}_p, k) \simeq D^\infty(V_p\mu_{p^\infty}, k)$. The same argument with multiple copies of $\mathbb{Q}_p$ and $V_p\mu_{p^\infty}$ proves the proposition. □

So far we have considered only ordinary p-divisible groups Σ. For a general Σ with a complete polarization $\Sigma = \Sigma_1 \times \Sigma_2$ with respect to $\hat{e}_p^L$ (where $C_c^\infty(V_p\Sigma_2, k)$ does not make sense), it remains to be answered whether $D^\infty(V_p\Sigma_2, k)$ is a Heisenberg representation.

Remark 7.11. When $p > 2$, the material of this subsection can be rewritten in terms of only $(\Sigma, \langle\cdot,\cdot\rangle_0)$ by using Section 6D, getting rid of (A, L) from the picture. (See Remark 6.6 for $p = 2$.) We retained (A, L) to make the analogy with Section 7A more transparent and also not to make an exception $p \neq 2$.

7D. ***Over a ring of mixed characteristic* (0, *p*).** In this final example, consider the case when:

- K is a field extension of $\mathbb{Q}_p$ complete with respect to a p-adic valuation $v_p : K^\times \to \mathbb{R}$. Assume that $x^{p^n} - 1$ splits completely in K for all $n \geq 1$.
- $\mathcal{O}_K := \{ a \in K^\times \mid v_p(a) \geq 0 \} \cup \{0\}$.
- $S = \operatorname{Spec} \mathcal{O}_K$.
- $\Sigma = \Sigma_1 \times \Sigma_2$ with $\Sigma_1 = (\mu_{p^\infty})^g$ and $\Sigma_2 = (\mathbb{Q}_p/\mathbb{Z}_p)^g$ over S.
- $\langle\cdot,\cdot\rangle_0 : \Sigma \times \Sigma \to \mu_{p^\infty}$ is a symplectic pairing sending
$$\big(((x_i)_{i=1}^g, (y_i)_{i=1}^g), ((x_i')_{i=1}^g, (y_i')_{i=1}^g))\big) \mapsto \prod_{i=1}^g (x_i, y_i') \prod_{i=1}^g (x_i', y_i)^{-1}, \tag{7-7}$$
where $(\cdot,\cdot) : \mu_{p^\infty} \times \mathbb{Q}_p/\mathbb{Z}_p \to \mu_{p^\infty}$ is the canonical pairing and $\langle\cdot,\cdot\rangle$ is as in Section 6C.
- If $p = 2$, assume that (7-3) holds with $T_p\Sigma_1 \times T_p\Sigma_2$ in place of $\Lambda_1 \oplus \Lambda_2$.

As in the previous subsection, define an ind-group scheme over $\mathcal{O}_K$ by

$$P := \left\{ \begin{pmatrix} {}^tB^{-1} & C \\ 0 & B \end{pmatrix} \,\middle|\, B \in GL_g(\mathbb{Q}_p),\ C \in \underline{\operatorname{Hom}}(\mathbb{Q}_p^g, (V_p\mu_{p^\infty})^g),\ C^\vee = C \right\}.$$

Since (7-6) still holds with $\mathcal{O}_K$ in place of k, the exact analogue of Corollary 7.7 holds over $\mathcal{O}_K$. The P-representation on the free $\mathcal{O}_K$-module $\mathcal{H}_{\mathrm{Sch},\mathcal{O}_K} := C_c^\infty(V_p\Sigma_2, \mathcal{O}_K)$ is the Weil representation. It is instructive to note how this specializes to $\operatorname{Spec} K$ and $\operatorname{Spec} k$, where k now denotes the residue field of K. By passing to $\operatorname{Spec} k$, we recover the Weil representation of Corollary 7.7, which is again a P-representation. Over the generic fiber, Σ_1 becomes isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^g$ noncanonically. Therefore, $\underline{\mathrm{Sp}}(V_p\Sigma, \langle\cdot,\cdot\rangle_0)(K)$ is isomorphic to $\mathrm{Sp}_{2g}(\mathbb{Q}_p)$. The Weil representation $C_c^\infty(V_p\Sigma_2, K)$ over the generic fiber is the classical one described in Section 7A and contains $\mathcal{H}_{\mathrm{Sch},\mathcal{O}_K}$ as an "integral model". This example illustrates that the integral model may admit a smaller action than the generic fiber. It would be worthwhile to describe a similar phenomenon for Weil representations in the case of nonordinary p-divisible groups over $\mathbb{O}_K$.

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swshin@math.mit.edu *Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139, United States*

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Small-dimensional projective representations of symmetric and alternating groups

Alexander S. Kleshchev and Pham Huu Tiep

We classify the irreducible projective representations of symmetric and alternating groups of minimal possible and second minimal possible dimensions, and get a lower bound for the third minimal dimension. On the way we obtain some new results on branching which might be of independent interest.

1. Introduction

We denote by $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ the Schur double covers of the symmetric and alternating groups S_n and A_n (see Section 2C for the specific choice we make). The goal of this paper is to describe irreducible projective representations of symmetric and alternating groups of minimal possible and second minimal possible dimensions, or, equivalently the faithful irreducible representations of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ of two minimal possible dimensions. We also get a lower bound for the third minimal dimension.

Our ground field is an algebraically closed field $\mathbb{F}$ of characteristic $p \neq 2$. If $p = 0$, then the irreducible representations of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ over $\mathbb{F}$ are roughly labeled by the strict partitions of n, i.e., the partitions of n with distinct parts. To be more precise to each strict partition of n, one associates one or two representations of $\hat{\mathsf{S}}_n$ (of the same dimension if there are two) and similarly for $\hat{\mathsf{A}}_n$.

Now, when $p = 0$, the representations corresponding to the partition (n) are called *basic*, while the representations corresponding to the partition $(n-1, 1)$ are called *second basic*. To define the basic and the second basic representations of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ in characteristic $p > 0$, one needs to reduce the first and second basic representations in characteristic zero modulo p and take appropriate composition factors. This has been worked out in detail by Wales [1979]. Again, there are one or two basic representations for $\hat{\mathsf{S}}_n$ and one or two basic representations for $\hat{\mathsf{A}}_n$ (of the same dimension if there are two), and similarly for the second basic.

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The dimensions of the basic and the second basic representations have also been computed in [Wales 1979]. To state the result, set

$$\kappa_n := \begin{cases} 1 & \text{if } p \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\kappa_n = 0$ if $p = 0$. Then the dimensions of the basic representations for $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ are:

$$a(\hat{\mathsf{S}}_n) := 2^{\lfloor \frac{n-1-\kappa_n}{2} \rfloor}, \quad a(\hat{\mathsf{A}}_n) := 2^{\lfloor \frac{n-2-\kappa_n}{2} \rfloor}.$$

The dimensions of the second basic representations for $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ are:

$$b(\hat{\mathsf{S}}_n) := 2^{\lfloor \frac{n-2-\kappa_{n-1}}{2} \rfloor}(n-2-\kappa_n-2\kappa_{n-1}),$$
$$b(\hat{\mathsf{A}}_n) := 2^{\lfloor \frac{n-3-\kappa_{n-1}}{2} \rfloor}(n-2-\kappa_n-2\kappa_{n-1}).$$

Main Theorem. *Let $n \geq 12$, $G = \hat{\mathsf{S}}_n$ or $\hat{\mathsf{A}}_n$, and V be a faithful irreducible representation of G over $\mathbb{F}$. If* $\dim V < 2b(G)$, *then V is either a basic representation* (*of dimension $a(G)$*) *or a second basic representation* (*of dimension $b(G)$*).

The assumption $n \geq 12$ in the Main Theorem is necessary — for smaller n there are counterexamples. On the other hand, this assumption is not very important, since dimensions of all irreducible representations of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ are known for $n \leq 11$ anyway; see [Jansen et al. 1995].

We prove the Main Theorem by induction, for which we need to establish some new results on branching (see Sections 3–5). These results might be of independent interest. We establish other useful results on the way. For example, we find the labels for second basic representations in the modular case (see Section 3). Such labels were known so far only for basic representations.

The scheme of our inductive proof of the Main Theorem is as follows. First of all, it turns out that the treatment is much more streamlined if, instead of G-modules for $G \in \{\hat{\mathsf{S}}_n, \hat{\mathsf{A}}_n\}$, one works with *supermodules* over certain *twisted groups algebras* $\mathcal{T}_n$ and $\mathcal{U}_n$. This framework is prepared in Section 2. Consider now a faithful irreducible G-module W which is neither a basic nor a second basic module. Then there is an irreducible $\mathcal{T}_n$-supermodule V such that W is a composition factor of the G-module V. We aim to show that the restriction of V to a natural subalgebra $\mathcal{T}_m$ with $m \in \{n-1, n-2, n-3\}$, contains enough "large" composition factors, i.e., composition factors which again are neither a basic nor a second basic supermodule of $\mathcal{T}_m$. In this case we can invoke the induction hypothesis to show that $\dim V$ is at least a certain bound, which guarantees that $\dim W \geq 2b(G)$ (cf. Section 6). Otherwise, our branching results (Sections 4, 5) imply that V is labeled by a so-called *Jantzen–Seitz partition*, in which case we have to restrict V further down to a natural subalgebra $\mathcal{T}_m$ with $m \in \{n-6, n-7, n-8\}$, and again show that this restriction contains enough large composition factors.

The Main Theorem substantially strengthens Theorem A of [Kleshchev and Tiep 2004], which in turn strengthened [Wagner 1977], and fits naturally into the program of describing small dimension representations of quasisimple groups. For representations of symmetric and alternating groups results along these lines were obtained in [James 1983] and [Brundan and Kleshchev 2001b, Section 1]. For Chevalley groups, similar results can be found in [Landazuri and Seitz 1974; Seitz and Zalesskii 1993; Guralnick and Tiep 1999; Brundan and Kleshchev 2000; Hiss and Malle 2001; Guralnick et al. 2002; Guralnick and Tiep 2004] and many others.

Throughout the paper we assume that $n \geq 5$, unless otherwise stated. For small n symmetric and alternating groups are too small to be interesting.

2. Preliminaries

We keep the notation introduced in the Introduction.

2A. ***Combinatorics.*** We review combinatorics of partitions needed for projective representation theory of symmetric groups, referring the reader to [Kleshchev 2005, Part II] for more details. Let

$$\ell := \begin{cases} \infty & \text{if } p = 0, \\ (p-1)/2 & \text{if } p > 0; \end{cases} \qquad \text{and} \qquad I := \begin{cases} \mathbb{Z}_{\geq 0} & \text{if } p = 0, \\ \{0, 1, \dots, \ell\} & \text{if } p > 0. \end{cases}$$

For any $n \geq 0$, a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n is *p-strict* if $\lambda_r = \lambda_{r+1}$ for some r implies $p \mid \lambda_r$. A p-strict partition λ is *restricted* if in addition

$$\begin{cases} \lambda_r - \lambda_{r+1} < p & \text{if } p \mid \lambda_r, \\ \lambda_r - \lambda_{r+1} \leq p & \text{if } p \nmid \lambda_r, \end{cases}$$

for each $r \geq 1$. If $p = 0$, we interpret p-strict and restricted p-strict partitions as *strict partitions*, i.e., partitions all of whose nonzero parts are distinct. Let $\mathcal{RP}_p(n)$ denote the set of all restricted p-strict partitions of n. The *p'-height* $h_{p'}(\lambda)$ of $\lambda \in \mathcal{P}_p(n)$ is:

$$h_{p'}(\lambda) := \bigl|\{r \mid 1 \leq r \leq n \text{ and } p \nmid \lambda_r\}\bigr| \qquad (\lambda \in \mathcal{RP}_p(n)).$$

Let λ be a p-strict partition. We identify λ with its *Young diagram* consisting of certain nodes (or boxes). A node (r, s) is the node in row r and column s. We use the repeating pattern $0, 1, \dots, \ell-1, \ell, \ell-1, \dots, 1, 0$ of elements of I to assign (p-) *contents* to the nodes. For example, if $p = 5$ then $\lambda = (16, 11, 10, 10, 9, 5, 1) \in \mathcal{RP}_5$, and the contents of the nodes of λ are:

0	1	2	1	0	0	1	2	1	0	0	1	2	1	0	0
0	1	2	1	0	0	1	2	1	0	0					
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1							
0	1	2	1	0											
0															

The content of the node A is denoted by $\mathrm{cont}_p\, A$. Since the content of the node $A = (r, s)$ depends only on the column number s, we can also speak of $\mathrm{cont}_p\, s$ for any $s \in \mathbb{Z}_{>0}$.

Let λ be a p-strict partition and $i \in I$. A node $A = (r, s) \in \lambda$ is *i-removable* (for λ) if one of the following holds:

(R1) $\mathrm{cont}_p\, A = i$ and $\lambda_A := \lambda - \{A\}$ is again a p-strict partition.

(R2) The node $B = (r, s+1)$ immediately to the right of A belongs to λ, $\mathrm{cont}_p\, A = \mathrm{cont}_p\, B = i = 0$, and both $\lambda_B = \lambda - \{B\}$ and $\lambda_{A,B} := \lambda - \{A, B\}$ are p-strict partitions.

A node $B = (r, s) \notin \lambda$ is *i-addable* (for λ) if one of the following holds:

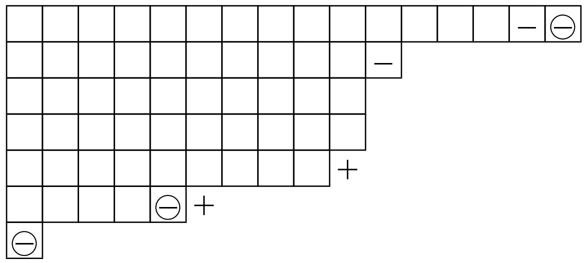
(A1) $\mathrm{cont}_p\, B = i$ and $\lambda^B := \lambda \cup \{B\}$ is again an p-strict partition.

(A2) The node $A = (r, s-1)$ immediately to the left of B does not belong to λ, $\mathrm{cont}_p\, A = \mathrm{cont}_p\, B = i = 0$, and both $\lambda^A = \lambda \cup \{A\}$ and $\lambda^{A,B} := \lambda \cup \{A, B\}$ are p-strict partitions.

Now label all i-addable nodes of λ by $+$ and all i-removable nodes of λ by $-$. The *i-signature* of λ is the sequence of pluses and minuses obtained by going along the rim of the Young diagram from bottom left to top right and reading off all the signs. The *reduced i-signature* of λ is obtained from the i-signature by successively erasing all neighboring pairs of the form $+-$. Nodes corresponding to $-$'s in the reduced i-signature are called *i-normal*. The rightmost i-normal node is called *i-good*. Define

$$\varepsilon_i(\lambda) = \#\{i\text{-normal nodes in } \lambda\} = \#\{-\text{'s in the reduced } i\text{-signature of } \lambda\}.$$

Continuing with the example above, the 0-addable and 0-removable nodes are labeled in the diagram at the top of the next page. The 0-signature of λ is $-, -, +, +, -, -, -$, and the reduced 0-signature is $-, -, -$. The nodes corresponding to the $-$'s in the reduced 0-signature have been circled in the diagram. The rightmost of them is 0-good.



Set

$$\tilde{e}_i\lambda = \begin{cases} \lambda_A & \text{if } A \text{ is the } i\text{-good node,} \\ 0 & \text{if } \lambda \text{ has no } i\text{-good nodes.} \end{cases}$$

The definitions imply that $\tilde{e}_i\lambda = 0$ or $\tilde{e}_i\lambda \in \mathcal{RP}_p(n-1)$ if $\lambda \in \mathcal{RP}_p(n)$.

2B. ***Crystal graph properties.*** We make $\mathcal{RP}_p := \bigsqcup_{n\geq 0} \mathcal{RP}_p(n)$ into an I-colored directed graph as follows: $\lambda \xrightarrow{i} \mu$ if and only if $\lambda = \tilde{e}_i\mu$. Kang [2003, Theorem 7.1] proves that this graph is isomorphic to $B(\Lambda_0)$, the crystal graph of the basic representation $V(\Lambda_0)$ of the twisted Kac–Moody algebra of type $A_{p-1}^{(2)}$ (interpreted as B_∞ if $p = 0$). The Cartan matrix $(a_{ij})_{i,j\in I}$ of this algebra is

$$\begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \quad \text{if } \ell \geq 2,$$

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \quad \text{if } \ell = 1,$$

$$\begin{pmatrix} 2 & -2 & 0 & & \\ -1 & 2 & -1 & 0 & \\ 0 & -1 & 2 & -1 & \\ & 0 & -1 & 2 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{if } \ell = \infty.$$

In view of Kang's result, we can use some nice properties of crystal graphs:

Lemma 2.1 [Stembridge 2003, Theorem 2.4]. *Let $i, j \in I$ and $i \neq j$. Then*

(i) *If $\varepsilon_i(\lambda) > 0$, then $0 \leq \varepsilon_j(\tilde{e}_i\lambda) - \varepsilon_j(\lambda) \leq -a_{ji}$.*

(ii) *If $\varepsilon_i(\lambda) > 0$ and $\varepsilon_j(\tilde{e}_i\lambda) = \varepsilon_j(\lambda) > 0$, then $\tilde{e}_i\tilde{e}_j\lambda = \tilde{e}_j\tilde{e}_i\lambda$.*

2C. ***Double covers and twisted group algebras.*** There are two double covers of the symmetric group but the corresponding group algebras over $\mathbb{F}$ are isomorphic, so it suffices to work with one of them. Let $\hat{\mathsf{S}}_n$ be the Schur double cover of the symmetric group S_n in which transpositions lift to involutions. It is known that $\hat{\mathsf{S}}_n$ is generated by elements $z, s_1, \dots, s_{n-1}$ subject only to the relations

$$zs_r = s_r z, \quad z^2 = 1, \quad s_r^2 = 1,$$
$$s_r s_{r+1} s_r = s_{r+1} s_r s_{r+1},$$
$$s_r s_t = z s_t s_r \ (|r-t| > 1)$$

for all admissible r, t. Then z has order 2 and generates the center of $\hat{\mathsf{S}}_n$. We have the natural map $\pi : \hat{\mathsf{S}}_n \to \mathsf{S}_\mathsf{n}$,

$$1 \to \langle z \rangle \to \hat{\mathsf{S}}_n \xrightarrow{\pi} \mathsf{S}_\mathsf{n} \to 1,$$

which maps s_r onto the simple transposition $(r, r+1) \in \mathsf{S}_\mathsf{n}$. The Schur double cover $\hat{\mathsf{A}}_n$ is $\pi^{-1}(\mathsf{A}_\mathsf{n})$. We introduce the *twisted group algebras*:

$$\mathcal{T}_n := \mathbb{F}\hat{\mathsf{S}}_n/(z+1), \quad \mathcal{U}_n := \mathbb{F}\hat{\mathsf{A}}_n/(z+1).$$

Spin representations of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ are representations on which z acts nontrivially. The irreducible spin representations are equivalent to the irreducible projective representations of S_n and A_n (at least when $n \neq 6, 7$). Moreover, z must act as -1 on the irreducible spin representations, so the irreducible spin representations of $\hat{\mathsf{S}}_n$ and $\hat{\mathsf{A}}_n$ are the same as the irreducible representations of the twisted group algebras $\mathcal{T}_n$ and $\mathcal{U}_n$, respectively. From now on we just work with $\mathcal{T}_n$ and $\mathcal{U}_n$.

We refer the reader to [Kleshchev 2005, Section 13.1] for basic facts on these twisted group algebras. In particular, $\mathcal{T}_n$ is generated by the elements $t_1, \dots, t_{n-1}$, where $t_r = s_r + (z+1)$, subject only to the relations

$$t_r^2 = 1, \quad t_r t_{r+1} t_r = t_{r+1} t_r t_{r+1}, \quad t_r t_s = -t_s t_r \ (|r-s| > 1).$$

Moreover, $\mathcal{T}_n$ has a natural basis $\{t_g \mid g \in \mathsf{S}_\mathsf{n}\}$ such that $\mathcal{U}_n = \operatorname{span}(t_g \mid g \in \mathsf{A}_\mathsf{n})$. This allows us to introduce a $\mathbb{Z}_2$-grading on $\mathcal{T}_n$ with $(\mathcal{T}_n)_{\bar{0}} = \mathcal{U}_n$ and $(\mathcal{T}_n)_{\bar{1}} = \operatorname{span}(t_g \mid g \in \mathsf{S}_\mathsf{n} \setminus \mathsf{A}_\mathsf{n})$. Thus $\mathcal{T}_n$ becomes a *superalgebra*, and we can consider its irreducible *supermodules*.

2D. ***Supermodules over $\mathcal{T}_n$ and $\mathcal{U}_n$.*** Here we review some known results on representation theory of $\mathcal{T}_n$ and $\mathcal{U}_n$ described in detail in [Kleshchev 2005, Chapter 22] following [Brundan and Kleshchev 2001a; 2002]. It is important that the different approaches of these last two papers are reconciled in [Kleshchev and Shchigolev 2012], where some additional branching results, which will be crucial for us here, are also established.

First of all, we consider the irreducible *supermodules* over $\mathcal{T}_n$. These are labeled by the partitions $\lambda \in \mathcal{RP}_p(n)$. It will be convenient to set

$$\sigma(m) := \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd;} \end{cases} \tag{2-1}$$

and

$$a(\lambda) := \sigma(n - h_{p'}(\lambda)). \tag{2-2}$$

The irreducible $\mathcal{T}_n$-supermodule corresponding to $\lambda \in \mathcal{RP}_p(n)$ will be denoted by D^λ, so that

$$\{D^\lambda \mid \lambda \in \mathcal{RP}_p(n)\}$$

is a complete and irredundant set of irreducible $\mathcal{T}_n$-supermodules up to isomorphism. Moreover, D^λ is of type $\mathtt{M}$ if $a(\lambda) = 0$ and D^λ is of type $\mathtt{Q}$ if $a(\lambda) = 1$. Recall the useful fact that $a(\lambda)$ has the same parity as the number of nodes in λ of nonzero content; see [Kleshchev 2005, (22.15)].

Let V be a $\mathcal{T}_n$-supermodule, $m_1, \dots, m_r \in \mathbb{Z}_{>0}$, and $\mu^1, \dots \mu^r \in \mathcal{RP}_p(n)$. We use the notation $m_1 D^{\mu^1} + \cdots + m_r D^{\mu^r} \in V$ to indicate that the multiplicity of each D^{μ^k} as a composition factor of V is at least m_k.

2E. *Modules over $\mathcal{T}_n$ and $\mathcal{U}_n$.* Now, we pass from supermodules over $\mathcal{T}_n$ to usual modules over $\mathcal{T}_n$ and $\mathcal{U}_n$. This is explained in detail in [Kleshchev 2005, Section 22.3]. Assume first that $a(\lambda) = 0$. Then D^λ is irreducible as a usual $\mathcal{T}_n$-module. We denote this $\mathcal{T}_n$-module again by D^λ. Moreover, D^λ splits into two nonisomorphic irreducible modules on restriction to $\mathcal{U}_n$: $\mathrm{res}^{\mathcal{T}_n}_{\mathcal{U}_n} D^\lambda = E^\lambda_+ \oplus E^\lambda_-$. On the other hand, let $a(\lambda) = 1$. Then, considered as a usual module, D^λ splits as two nonisomorphic $\mathcal{T}_n$-modules: $D^\lambda = D^\lambda_+ \oplus D^\lambda_-$. Moreover, $E^\lambda := \mathrm{res}^{\mathcal{T}_n}_{\mathcal{U}_n} D^\lambda_+ \cong \mathrm{res}^{\mathcal{T}_n}_{\mathcal{U}_n} D^\lambda_-$ is an irreducible $\mathcal{U}_n$-module. Now,

$$\{D^\lambda \mid \lambda \in \mathcal{RP}_p(n),\ a(\lambda) = 0\} \cup \{D^\lambda_+, D^\lambda_- \mid \lambda \in \mathcal{RP}_p(n),\ a(\lambda) = 1\}$$

is a complete irredundant set of irreducible $\mathcal{T}_n$-modules up to isomorphism, and

$$\{E^\lambda \mid \lambda \in \mathcal{RP}_p(n),\ a(\lambda) = 1\} \cup \{E^\lambda_+, E^\lambda_- \mid \lambda \in \mathcal{RP}_p(n),\ a(\lambda) = 0\}$$

is a complete irredundant set of irreducible $\mathcal{U}_n$-modules up to isomorphism.

We note that it is usually much more convenient to work with $\mathcal{T}_n$-supermodules, and then "desuperize" at the last moment using the theory described above to obtain results on usual $\mathcal{T}_n$-modules and $\mathcal{U}_n$-modules; see Remark 22.3.17 in [Kleshchev 2005]. For future use, we also point out that if V is an irreducible $\mathcal{T}_n$-supermodule and W is an irreducible constituent of V as a usual $\mathcal{T}_n$-module (or $\hat{\mathsf{S}}_n$-module), then

$$\frac{\dim V}{\dim W} = 2^{a(V)}.$$

2F. ***Weight spaces and superblocks.*** Let V be a $\mathcal{T}_n$-supermodule. We recall the notion of the formal character of V following [Brundan and Kleshchev 2003] and [Kleshchev 2005, Section 22.3]. Let $M_1, \dots, M_n$ be the Jucys–Murphy elements of $\mathcal{T}_n$; see [Kleshchev 2005, (13.6)]. The main properties of the Jucys–Murphy elements are as follows:

Theorem 2.2.

(i) [Kleshchev 2005, Lemma 13.1.1] *M_k^2 and M_l^2 commute for all $1 \le k, l \le n$.*

(ii) [Kleshchev 2005, Lemma 22.3.7] *If V is a finite-dimensional $\mathcal{T}_n$-supermodule, then for all $1 \le k \le n$, the eigenvalues of M_k^2 on V are of the form $i(i+1)/2$ for some $i \in I$.*

(iii) [Brundan and Kleshchev 2003, Theorem 3.2] *The even center of $\mathcal{T}_n$ is the set of all symmetric polynomials in the $M_1^2, \dots, M_n^2$.*

For an n-tuple $\boldsymbol{i} = (i_1, \dots, i_n) \in I^n$, the $\boldsymbol{i}$*-weight space* of a finite-dimensional $\mathcal{T}_n$-supermodule V is:

$$V_{\boldsymbol{i}} := \{v \in V \mid (M_k^2 - i_k(i_k+1)/2)^N v = 0 \text{ for } N \gg 0 \text{ and } k = 1, \dots, n\}.$$

By Theorem 2.2, we have $V = \bigoplus_{\boldsymbol{i} \in I^n} V_{\boldsymbol{i}}$. If $V_{\boldsymbol{i}} \neq 0$, we say that $\boldsymbol{i}$ is a *weight* of V.

We denote by $\varepsilon_i(V)$ the maximal nonnegative integer m such that D^λ has a nonzero $\boldsymbol{i}$-weight space with the last m entries of $\boldsymbol{i}$ equal to i.

The superblock theory of $\mathcal{T}_n$ is similar to the usual block theory but uses even central idempotents. Denote

$$\Gamma_n := \{\gamma : I \to \mathbb{Z}_{\ge 0} \mid \sum_{i \in I} \gamma(i) = n\}.$$

Also denote by ν_i the function from I to $\mathbb{Z}_{\ge 0}$ which maps i to 1 and j to 0 for all $j \neq i$. For $\gamma \in \Gamma_n$, we let

$$I^\gamma := \{\boldsymbol{i} = (i_1, \dots, i_n) \in I^n \mid \nu_{i_1} + \dots + \nu_{i_n} = \gamma\}.$$

If V is a finite-dimensional $\mathcal{T}_n$-supermodule, then by Theorem 2.2(iii),

$$V[\gamma] := \bigoplus_{\boldsymbol{i} \in I^\gamma} V_{\boldsymbol{i}}$$

is a $\mathcal{T}_n$-superblock component of V, referred to as the γ*-superblock component* of V, and the decomposition of V into the $\mathcal{T}_n$-superblock components (some of which might be zero) is:

$$V = \bigoplus_{\gamma \in \Gamma_n} V[\gamma].$$

The γ*-superblock* consists of all $\mathcal{T}_n$-supermodules V with $V[\gamma] = V$.

Let $\lambda \in \mathcal{RP}_p(n)$. For any $i \in I$ denote by $\gamma_i(\lambda)$ the number of nodes of λ of content i. Then we have a function

$$\gamma(\lambda) := \sum_{i \in I} \gamma_i(\lambda)\nu_i \in \Gamma_n.$$

Theorem 2.3 [Kleshchev 2005, Theorem 22.3.1 (iii)]. *Let $\lambda \in \mathcal{RP}_p(n)$ and $\gamma \in \Gamma_n$. Then D^λ is in the γ-superblock of $\mathcal{T}_n$ if and only if $\gamma(\lambda) = \gamma$.*

2G. *Branching rules.* Given a function $\gamma : I \to \mathbb{Z}_{\geq 0}$ and $i \in I$ we can consider the function $\gamma - \nu_i : I \to \mathbb{Z}_{\geq 0}$ if $\gamma(i) > 0$. Now, let $\lambda \in \mathcal{RP}_p(n)$. Denote

$$\text{res}_i\, D^\lambda := \left(\text{res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}}\, D^\lambda\right)[\gamma(\lambda) - \nu_i] \qquad (i \in I)$$

interpreted as zero if $\gamma_i(\lambda) = 0$. In other words,

$$\text{res}_i\, D^\lambda := \bigoplus_{\boldsymbol{i} \in I^n,\ i_n = i} D^\lambda_{\boldsymbol{i}} \qquad (i \in I). \tag{2-3}$$

We have

$$\text{res}^{\mathcal{T}_n}_{\mathcal{T}_{n-1}}\, D^\lambda = \bigoplus_{i \in I} \text{res}_i\, D^\lambda.$$

Moreover, either $\text{res}_i\, D^\lambda$ is zero, or $\text{res}_i\, D^\lambda$ is self-dual indecomposable, or $\text{res}_i\, D^\lambda$ is a direct sum of two self-dual indecomposable supermodules isomorphic to each other and denoted by $e_i D^\lambda$. If $\text{res}_i\, D^\lambda$ is zero or indecomposable we denote $e_i D^\lambda := \text{res}_i\, D^\lambda$. From now on, for any $\mathcal{T}_n$-supermodule V we will always denote

$$\text{res}_{n-j}\, V := \text{res}^n_{n-j}\, V := \text{res}^{\mathcal{T}_n}_{\mathcal{T}_{n-j}}\, V.$$

Theorem 2.4 [Kleshchev 2005, (22.14), Theorem 22.3.4; Kleshchev and Shchigolev 2012, Theorem A]. *Let $\lambda \in \mathcal{RP}_p(n)$. There exist $\mathcal{T}_{n-1}$-supermodules $e_i D^\lambda$ for each $i \in I$, unique up to isomorphism, satisfying the following conditions*:

(i) $\text{res}_{n-1}\, D^\lambda$ *is isomorphic to*

$$\begin{cases} e_0 D^\lambda \oplus 2e_1 D^\lambda \oplus \cdots \oplus 2e_\ell D^\lambda & \textit{if } a(\lambda) = 1, \\ e_0 D^\lambda \oplus e_1 D^\lambda \oplus \cdots \oplus e_\ell D^\lambda & \textit{if } a(\lambda) = 0. \end{cases}$$

(ii) *For each $i \in I$, $e_i D^\lambda \neq 0$ if and only if λ has an i-good node A, in which case $e_i D^\lambda$ is a self-dual indecomposable supermodule with irreducible socle and head isomorphic to D^{λ_A}.*

(iii) *If λ has an i-good node A, then the multiplicity of D^{λ_A} in $e_i D^\lambda$ is $\varepsilon_i(\lambda)$. Furthermore, $a(D^{\lambda_A})$ equals $a(D^\lambda)$ if and only if $i = 0$.*

(iv) *If $\mu \in \mathcal{RP}_p(n-1)$ is obtained from λ by removing an i-normal node then D^μ is a composition factor of $e_i D^\lambda$.*

(v) $e_i D^\lambda$ *is irreducible if and only if* $\varepsilon_i(\lambda) = 1$;

(vi) $\mathrm{res}_{n-1} D^\lambda$ *is completely reducible if and only if* $\varepsilon_i(\lambda) = 0$ *or* 1 *for all* $i \in I$.

(vii) $\varepsilon_i(D^\lambda) = \varepsilon_i(\lambda)$.

(viii) [Brundan and Kleshchev 2006, Theorem 1.2 (ii)] *Let A be the lowest removable node of λ such that $\lambda_A \in \mathcal{RP}_p(n-1)$. Assume that A has content i and that there are m i-removable nodes strictly below A in λ. Then the multiplicity of D^{λ_A} in $e_i D^\lambda$ is $m+1$.*

Finally, one rather special result:

Lemma 2.5 [Phillips 2004, Proposition 3.17]. *Let $p > 3$ and D, E be irreducible $\mathcal{T}_n$-supermodules such that* $\mathrm{res}_{n-1} D$ *and* $\mathrm{res}_{n-1} E$ *are both homogeneous with the same unique composition factor. Then $D \cong E$.*

2H. *Reduction modulo p.* To distinguish between the irreducible modules in characteristic 0 and p in this section we will write D_0^λ versus D_p^λ. We also distinguish between $I_0 = \mathbb{Z}_{\geq 0}$ and $I_p = \{0, 1, \dots, \ell\}$. To every $i \in I_0$ we associate $\bar{i} \in I_p$ via $\bar{i} := \mathrm{cont}_p\, i$. If $\boldsymbol{i} = (i_1, \dots, i_n) \in I_0^n$ then $\bar{\boldsymbol{i}} := (\bar{i}_1, \dots, \bar{i}_n) \in I_p^n$.

Denote reduction modulo p of a finite-dimensional $\mathcal{T}_n$-supermodule V in characteristic zero by $\overline{V}$. In particular we have $\overline{D_0^\lambda}$ for any strict partition λ of n.

In fact, let $(\mathbb{K}, R, \mathbb{F})$ be the splitting p-modular system which is used to perform reduction modulo p. In particular, $\mathbb{F} = R/(\pi)$ where (π) is the maximal ideal of R. So we have $\bar{V} = V_R \otimes_R \mathbb{F}$ for some $\mathcal{T}_n$-invariant superhomogeneous lattice V_R in V.

Recall that $\mathrm{char}\, \mathbb{F} \neq 2$ so we may assume that all $i(i+1)/2$ with $i \in I$ belong to the ring of integers R. As usual we consider elements of I_p as elements of $\mathbb{F}$. Then it is easy to see that

$$i(i+1)/2 + (\pi) = \bar{i}(\bar{i}+1)/2 \qquad (i \in I_0). \tag{2-4}$$

Let again V be an irreducible $\mathcal{T}_n$-supermodule in characteristic zero. When performing its reduction modulo p we can choose a $\mathcal{T}_n$-invariant R-lattice V_R of V that respects the weight space decomposition: $V_R = \bigoplus_{\boldsymbol{i} \in I_0^n} V_{\boldsymbol{i},R}$, where $V_{\boldsymbol{i},R} = V_R \cap V_{\boldsymbol{i}}$. Then $\overline{V_{\boldsymbol{i}}} := V_{\boldsymbol{i},R} \otimes_R \mathbb{F} \subseteq \overline{V}_{\bar{\boldsymbol{i}}}$. It follows that for an arbitrary $\boldsymbol{j} \in I_p^n$ we have

$$\overline{V}_{\boldsymbol{j}} = \bigoplus_{\substack{\boldsymbol{i} \in I_0^n \\ \bar{\boldsymbol{i}} = \boldsymbol{j}}} \overline{V_{\boldsymbol{i}}}. \tag{2-5}$$

This implies the following result (see the proof of [Kleshchev and Shchigolev 2012, Lemma 8.1.10]):

Proposition 2.6. *Let λ be a strict partition of n and D_0^λ be the corresponding irreducible $\mathcal{T}_n$-supermodule in characteristic zero. Then all composition factors of the reduction $\overline{D_0^\lambda}$ modulo p belong to the superblock γ, where $\gamma = \sum_{A\in\lambda} \nu_{\mathrm{cont}_p A}$, where the sum is over all nodes A of λ.*

We now use reduction modulo p to deduce some very special results on branching.

Lemma 2.7. *We have*:

(i) *if $p > 5$ and $n = p+1$, then* $\mathrm{res}_{n-1} D_p^{(p-1,2)}$ *has a composition factor D^μ with $\varepsilon_2(\mu) = 1$;*

(ii) *if $p > 3$ and $n = p+4$, then* $\mathrm{res}_{n-1} D_p^{(p+2,2)}$ *has a composition factor D^μ with $\varepsilon_0(\mu) = 2$.*

Proof. We will use the characterization of $\varepsilon_i(\lambda)$ given in Theorem 2.4(vii).

(i) Let $\gamma = 3\nu_1 + \nu_\ell + 2\sum_{i\neq 1,\ell} \nu_i$. Note that $D_0^{(p-1,2)}$ is the only ordinary irreducible in the γ-superblock, and $D_p^{(p-1,2)}$ is the only p-modular irreducible in the γ-superblock. It follows that

$$\overline{D_0^{(p-1,2)}} = m D_p^{(p-1,2)}$$

for some multiplicity m. So the restriction $\mathrm{res}_{n-1} D_p^{(p-1,2)}$ has the same composition factors as the reduction modulo p of the restriction

$$\mathrm{res}_{n-1} D_0^{(p-1,2)} = D_0^{(p-1,1)} \oplus D_0^{(p-2,2)}.$$

Now, note using (2-5) that $\varepsilon_2(\overline{D_0^{(p-2,2)}}) = 1$.

(ii) Let $\gamma = 4(\nu_0+\nu_1)+\nu_\ell+2\sum_{i\neq 0,1,\ell} \nu_i$. Note that $D_0^{(p+2,2)}$ is the only ordinary irreducible in the γ-superblock, and $D_p^{(p+2,2)}$ is the only p-modular irreducible in the γ-superblock. It follows that

$$\overline{D_0^{(p+2,2)}} = m D_p^{(p+2,2)}$$

for some multiplicity m. So the restriction $\mathrm{res}_{n-1} D_p^{(p+2,2)}$ has the same composition factors as the reduction modulo p of the restriction

$$\mathrm{res}_{n-1} D_0^{(p+2,2)} = D_0^{(p+2,1)} \oplus D_0^{(p+1,2)}.$$

Now, note using (2-5) that $\varepsilon_0(\overline{D_0^{(p+1,2)}}) = 2$. □

3. Basic and second basic modules

3A. ***Definition, properties, and dimensions.*** If the characteristic of the ground field is zero, then the *basic* supermodule A_n and the *second basic* supermodule

B_n over $\mathcal{T}_n$ are defined as

$$A_n := D^{(n)} \quad \text{and} \quad B_n := D^{(n-1,1)}.$$

If the ground field has characteristic $p > 0$, it follows from the results of [Wales 1979] that reduction modulo p of the characteristic zero basic supermodule has only one composition factor (which could appear with some multiplicity). We define the *basic* supermodule A_n in characteristic p to be this composition factor.

Moreover, again by [Wales 1979], reduction modulo p of the characteristic zero second basic supermodule will always have only one composition factor (with some multiplicity) which is not isomorphic to the basic supermodule — this new composition factor will be referred to as the *second basic* supermodule in characteristic p and denoted by B_n.

Thus we have defined the basic supermodule A_n and the second basic supermodule B_n for an arbitrary characteristic.

When $p > 0$, write n in the form

$$n = ap + b \qquad (a, b \in \mathbb{Z},\ 0 < b \leq p). \tag{3-1}$$

Define the functions $\gamma^{A_n}, \gamma^{B_n} \in \Gamma_n$ by

$$\gamma^{A_n} := a(2\nu_0 + \cdots + 2\nu_{\ell-1} + \nu_\ell) + \sum_{s=1}^{b} \nu_{\mathrm{cont}_p s},$$

$$\gamma^{B_n} := a(2\nu_0 + \cdots + 2\nu_{\ell-1} + \nu_\ell) + \sum_{s=1}^{b-1} \nu_{\mathrm{cont}_p s} + \nu_0.$$

Lemma 3.1. *A_n is in the γ^{A_n}-superblock and B_n is in the γ^{B_n}-superblock.*

Proof. This follows from the definitions of A_n and B_n above in terms of reductions modulo p and Proposition 2.6. □

Theorem 3.2 [Wales 1979].

(i) $\dim A_n = 2^{\lfloor \frac{n-\kappa_n}{2} \rfloor} = \begin{cases} 2^{\lfloor \frac{n}{2} \rfloor} & \text{if } p \nmid n, \\ 2^{\lfloor \frac{n-1}{2} \rfloor} & \text{if } p \mid n. \end{cases}$

(ii) *A_n is of type* $\mathtt{M}$ *if and only if n is odd and $p \nmid n$, or n is even and $p \mid n$.*

(iii) *The only possible composition factor of* $\mathrm{res}_{n-1} A_n$ *is* A_{n-1}.

Theorem 3.3 [Wales 1979].

(i) $\dim B_n = 2^{\lfloor \frac{n-1-\kappa_{n-1}}{2} \rfloor}(n - 2 - \kappa_n - 2\kappa_{n-1})$; *equivalently,*

$$\dim B_n = \begin{cases} 2^{\lfloor \frac{n-1}{2} \rfloor}(n-2) & \text{if } p \nmid n(n-1), \\ 2^{\lfloor \frac{n-1}{2} \rfloor}(n-3) & \text{if } p \mid n, \\ 2^{\lfloor \frac{n-2}{2} \rfloor}(n-4) & \text{if } p \mid (n-1). \end{cases}$$

(ii) *B_n is of type* M *if and only if n is odd and $p \mid (n-1)$, or n is even and $p \nmid (n-1)$.*

(iii) *The only possible composition factors of* $\operatorname{res}_{n-1} B_n$ *are A_{n-1} and B_{n-1}.*

Finally, we state two results concerning the weights of basic modules.

Lemma 3.4 [Phillips 2004, Corollary 3.12]. *The only weight appearing in A_n is*

$$(\operatorname{cont}_p 0, \operatorname{cont}_p 1, \dots, \operatorname{cont}_p(n-1)).$$

Lemma 3.5 [Phillips 2004, Lemma 3.13]. *Let $p > 3$ and D be an irreducible $\mathcal{T}_n$-supermodule. Suppose that there exist $i, j, k \in I$ (not necessarily distinct) such that every weight $\boldsymbol{i}$ appearing in D ends on ijk. Then D is basic.*

3B. *Labels.* It is important to identify the partitions which label the irreducible modules A_n and B_n in characteristic p. Recall the presentation (3-1). Define the partitions $\alpha_n \in \mathcal{RP}_p(n)$ as follows:

$$\alpha_n := \begin{cases} (p^a, b) & \text{if } b \neq p, \\ (p^a, p-1, 1) & \text{if } b = p, \end{cases}$$

and the partitions $\beta_n \in \mathcal{RP}_p(n)$ by

$$\beta_n := \begin{cases} (n-1, 1) & \text{if } n < p, \\ (p-2, 2) & \text{if } n = p, \\ (p-2, 2, 1) & \text{if } n = p+1, \\ (p+1, p^{a-1}, b-1) & \text{if } n > p+1 \text{ and } b \neq 1, \\ (p+1, p^{a-2}, p-1, 1) & \text{if } n > p+1 \text{ and } b = 1. \end{cases}$$

For technical reasons we will also need the partition $\gamma_n \in \mathcal{RP}_p(n)$ only defined for $n \not\equiv 0, 3 \pmod p$:

$$\gamma_n := \begin{cases} (n-2, 2) & \text{if } n < p \text{ or } n = p+1, \\ (p-1, 2, 1) & \text{if } n = p+2, \\ (p+2, p^{a-2}, p-1) & \text{if } n > p+2 \text{ and } b = 1, \\ (p+2, p^{a-2}, p-1, 1) & \text{if } n > p+2 \text{ and } b = 2, \\ (p+2, p^{a-1}, b-2) & \text{if } n > p+2 \text{ and } b \neq 1, 2, 3, p. \end{cases}$$

For $p = 3$ we define

$$\delta_n := (5, 3^{a-1}, 1) \qquad (\text{if } a \geq 2 \text{ and } b = 3).$$

Finally, for $p > 3$ we define (for $n \not\equiv 1, 4 \pmod p$)

$$\delta_n := \begin{cases} (n-3,3) \text{ or } (n-3,2,1) & \text{if } n \le p, \\ (p-1,3) & \text{if } n = p+2, \\ (p-1,3,1) \text{ or } (p,2,1) & \text{if } n = p+3, \\ (p+2,2,1) & \text{if } n = p+5 > 10, \\ (p+3,b-3) \text{ or } (p+2,b-3,1) & \text{if } a = 1 \text{ and } 5 < b < p, \\ (p+2,p-3,1) \text{ or } (p+2,p-2) & \text{if } n = 2p, \\ (p+3,p^{a-2},p-1) & \text{if } a \ge 2 \text{ and } b = 2, \\ (p+2,p^{a-1},1) \text{ or } (p+3,p^{a-2},p-1,1) & \text{if } a \ge 2 \text{ and } b = 3, \\ (p+2,p+1,p^{a-2},2) & \text{if } a \ge 2 \text{ and } b = 5 < p, \\ \begin{matrix}(p+3,p^{a-1},b-3) \text{ or} \\ (p+2,p+1,p^{a-2},b-3)\end{matrix} & \text{if } a \ge 2 \text{ and } 5 < b < p, \\ \begin{matrix}(p+2,p^{a-1},p-2) \text{ or} \\ (p+2,p+1,p^{a-2},p-3)\end{matrix} & \text{if } a \ge 2 \text{ and } b = p. \end{cases}$$

(In the cases where δ_n is not unique, this notation is used to refer to either of the two possibilities).

The cases where the formulas above do not produce a partition in $\mathcal{RP}_p(n)$ should be ignored. For example, if $p = 3$, there is no γ_5, because the second line of the definition of γ_n gives $(2,2,1) \notin \mathcal{RP}_3(5)$.

Theorem 3.6. *Let $\lambda \in \mathcal{RP}_p(n)$.*

(i) $A_n \cong D^{\alpha_n}$.

(ii) $B_n \cong D^{\beta_n}$.

(iii) *If $D^{\alpha_{n-1}}$ appears in the socle of* $\operatorname{res}_{n-1} D^\lambda$ *then $\lambda = \alpha_n$ or β_n.*

(iv) *If $D^{\beta_{n-1}}$ appears in the socle of* $\operatorname{res}_{n-1} D^\lambda$ *then $\lambda = \beta_n$ or γ_n. In particular, λ must be β_n if $n \equiv 0, 3 \pmod p$.*

(v) *If $D^{\gamma_{n-1}}$ appears in the socle of* $\operatorname{res}_{n-1} D^\lambda$ *then $\lambda = \gamma_n$ or δ_n. Conversely, $D^{\gamma_{n-1}}$ appears in the socle of* $\operatorname{res}_{n-1} D^{\delta_n}$.

Proof. (i) is proved in [Kleshchev 2005, Lemma 22.3.3].

(iii), (iv), and (v) come from Theorem 2.4 by analyzing how good nodes can be added to α_{n-1}, β_{n-1}, and γ_{n-1}, respectively.

(ii) If $n < p$ then the irreducible $\mathcal{T}_n$-supermodules in characteristic p are irreducible reductions modulo p of the irreducible modules in characteristic zero corresponding to the same partition. So the result is clear in this case. We now apply induction on n to prove the result for $n \ge p$. Let $B_n = D^\beta$. By Theorem 3.3(iii) and the inductive assumption, β can be obtained from α_{n-1} or β_{n-1} by adding a good node.

By (iii), the only partition other than α_n, which can be obtained out of α_{n-1} by adding a good node is β_n. Moreover, β_n can indeed be obtained out of α_{n-1} in such a way provided $n \not\equiv 0, 1 \pmod p$. This proves that $\beta = \beta_n$ unless $n \equiv 0, 1 \pmod p$.

By (iv), the only partition other than β_n, which can be obtained out of β_{n-1} by adding a good node is γ_n. Let $n \equiv 0 \pmod p$. Then there is no γ_n, and it follows that $\beta = \beta_n$ in this case also.

Finally, to complete the proof of the theorem, we just have to prove that $\beta = \beta_n$ when $n \equiv 1 \pmod p$. But we have only two options $\beta = \beta_n$ and $\beta = \gamma_n$, and the second one is impossible by Lemma 3.1. □

3C. ***Some branching properties.***

Lemma 3.7. *Let D be an irreducible $\mathcal{T}_n$-supermodule.*

(i) *If all composition factors of* $\operatorname{res}_{n-1} D$ *are isomorphic to* A_{n-1}, *then* $D \cong A_n$.

(ii) *If all composition factors of* $\operatorname{res}_{n-1} D$ *are isomorphic to* A_{n-1} *or* B_{n-1}, *then* $D \cong A_n$ *or* $D \cong B_n$, *with the following exceptions, when the result is indeed false*:

 (a) $p > 5, n = 5$, *and* $D = D^{(3,2)}$;
 (b) $p = 5, n = 6$, *and* $D = D^{(4,2)}$;
 (c) $p = 3, n = 7$, *and* $D = D^{(5,2)}$.

(iii) *Suppose that all composition factors of* $\operatorname{res}_m D$ *are isomorphic to* A_m *or* B_m *for some* $8 \le m \le n$. *Then* $D \cong A_n$ *or* $D \cong B_n$.

Proof. (i) is proved in [Kleshchev and Tiep 2004, Lemma 2.4]. For (ii), if A_{n-1} appears in the socle of $\operatorname{res}_{n-1} D$ then by Theorem 3.6(iii), D is isomorphic to A_n or B_n. Thus we may assume that the socle of D^λ is isomorphic to a direct sum of copies of $B_{n-1} = D^{\beta_{n-1}}$. By Theorem 3.6(iv) we just need to rule out the case $D = D^{\gamma_n}$.

When $n < p$ we have $\gamma_n = (n-2, 2)$, and $D^{(n-3,2)}$ is a composition factor of $\operatorname{res}_{n-1} D^{\gamma_n}$, unless $n = 5$, when we are in (a), and this is indeed an exception.

If $n > p$, let κ_{n-1} be the partition obtained from γ_n by removing the bottom removable node. It is easy to see using the explicit definitions of the partitions involved, that κ_{n-1} is a restricted p-strict partition of $n-1$ different from α_{n-1} and β_{n-1}, unless $n = p+1$ or $n = p+4$. Since the bottom removable node is always normal, in the nonexceptional cases we can apply Theorem 2.4(iv) to get a composition factor $D^{\kappa_{n-1}}$ in $\operatorname{res}_{n-1} D^{\gamma_n}$.

Now we deal with the exceptional cases $n = p+1$ and $n = p+4$. If $p = 3$, then the case $n = p+1$ does not arise since we are always assuming $n \ge 5$. If $n = p+4 = 7$, we are in the case (c), which is indeed an exception, as for $p = 3$ the only irreducible supermodules over $\mathcal{T}_6$ are basic and second basic.

Similarly, we get the exception (b) for $p = 5$, $n = p + 1$. All the other cases do not yield exceptions in view of Lemma 2.7.

To prove (iii), we proceed by induction on $k = n - m$, where the case $k = 0$ is obvious, and the case $k = 1$ follows from (ii). For the induction step, if U is any composition factor of $\mathrm{res}_{n-1} D$, then any composition factor of $\mathrm{res}_m U$ is isomorphic to A_m or B_m. By the induction hypothesis, U is isomorphic to A_{n-1} or B_{n-1}. Hence $D \cong A_n$ or $D \cong B_n$ by (ii). ☐

In the following two results, which are obtained applying Theorem 2.4, δ_n means any of the two possibilities for δ_n if δ_n is not uniquely defined.

Lemma 3.8. *Let $n \geq 6$, and denote $R := \mathrm{res}_{n-1} D^{\gamma_n}$. We have*:

(i) *If $n < p$, then $R \cong 2^{\sigma(n)}(D^{\gamma_{n-1}} \oplus D^{\beta_{n-1}})$.*

(ii) *If $n = p + 1$, then $D^{\alpha_{n-1}} + 2D^{\beta_{n-1}} \in R$.*

(iii) *If $a \geq 2$ and $b = 1$, then $2^{\sigma(n)}(2D^{\beta_{n-1}} + D^{\delta_{n-1}}) \in R$, except for the case $n = 7$, $p = 3$, when we have $4D^{\beta_{n-1}} \in R$.*

(iv) *If $b = 2$, then $2^{\sigma(n+1)} D^{\beta_{n-1}} + D^{\gamma_{n-1}} \in R$.*

(v) *If $a = 1$ and $b = 4$, then $4D^{\beta_{n-1}} \in R$.*

(vi) *If $a \geq 2$ and $b = 4$, then $2^{\sigma(n)}(2D^{\beta_{n-1}} + D^{\delta_{n-1}}) \in R$.*

(vii) *If $a \geq 1$ and $4 < b < p$, then $2^{\sigma(a+b)}(D^{\beta_{n-1}} + D^{\gamma_{n-1}}) \in R$.*

Notation. Let $\lambda \in \mathcal{RP}_p(n)$ and $j \in \mathbb{Z}_{>0}$. We denote by $d_j(\lambda)$ the number of composition factors (counting multiplicities) not isomorphic to A_{n-j}, B_{n-j} in $\mathrm{res}^n_{n-j} D^\lambda$.

Lemma 3.9. *We have $d_1(\delta_n) \geq 2$ and $d_2(\delta_n) \geq 3$, except possibly in one of the following cases*:

(i) *$n = 6$, $p > 5$, and $\delta_n = (3, 2, 1)$, in which case $\mathrm{res}_{n-1} D^{\delta_n} = D^{\gamma_{n-1}}$ and $\mathrm{res}_{n-2} D^{\delta_n} = 2D^{\beta_{n-2}}$.*

(ii) *$n = 7$, $p > 3$, and $\delta_n = (4, 3)$, in which case*

$$\begin{aligned} &\mathrm{res}_{n-1} D^{\delta_n} = 2D^{\gamma_{n-1}}, \\ &\mathrm{res}_{n-2} D^{\delta_n} = 2D^{\beta_{n-2}} + 2D^{\gamma_{n-2}} \quad \textit{if } p > 5, \\ &\mathrm{res}_{n-2} D^{\delta_n} \ni 4D^{\beta_{n-2}} + 2D^{\alpha_{n-2}} \quad \textit{if } p = 5; \end{aligned}$$

(iii) *$n = 7$, $p > 5$, and $\delta_n = (4, 2, 1)$, in which case*

$$\begin{aligned} &\mathrm{res}_{n-1} D^{\delta_n} = D^{\gamma_{n-1}} + D^{\delta_{n-1}}, \\ &\mathrm{res}_{n-2} D^{\delta_n} = D^{\beta_{n-2}} + 2D^{\gamma_{n-2}}. \end{aligned}$$

(iv) $p > 3$, $n = p + 3$, $\delta_n = (p, 2, 1)$, *in which case*

$$\mathrm{res}_{n-1} D^{\delta_n} \ni 2D^{\gamma_{n-1}} + D^{\alpha_{n-1}},$$
$$\mathrm{res}_{n-2} D^{\delta_n} \ni D^{\alpha_{n-2}} + 2D^{\beta_{n-2}} + 2D^{\gamma_{n-2}}.$$

(v) $p > 3$, $n = mp + 3$ *with* $m \geq 2$, $\delta_n = (p + 2, p^{m-1}, 1)$, *in which case*

$$\mathrm{res}_{n-1} D^{\delta_n} \ni 2D^{\gamma_{n-1}}, \quad \mathrm{res}_{n-2} D^{\delta_n} \ni 2 \cdot 2^{\sigma(m-1)} D^{\beta_{n-2}} + 2D^{\gamma_{n-2}}.$$

(vi) $p > 5$, $n = p + 6$, $\delta_n = (p + 3, 3)$, *in which case*

$$\mathrm{res}_{n-1} D^{\delta_n} \ni 2D^{\gamma_{n-1}}, \quad \mathrm{res}_{n-2} D^{\delta_n} \ni 2D^{\beta_{n-2}} + 2D^{\gamma_{n-2}}.$$

(vii) $p = 3$ *and* $\delta_n = (5, 3^{a-1}, 1)$, *in which case*

$$\mathrm{res}_{n-1} D^{\delta_n} \ni 2D^{\gamma_{n-1}}, \quad \mathrm{res}_{n-2} D^{\delta_n} \ni 2 \cdot 2^{\sigma(a-1)} D^{\beta_{n-2}} + 2D^{\gamma_{n-2}}.$$

(viii) $p > 3$, $n = pm$ *for an integer* $m \geq 2$, *and* $\delta_n = (p + 2, p^{m-2}, p - 2)$, *in which case* $\mathrm{res}_{n-1} D^{\delta_n} = 2^{\sigma(m)} D^{\gamma_{n-1}}$ *and*

$$\mathrm{res}_{n-2} D^{\delta_n} \ni \begin{cases} 2D^{\gamma_{n-2}} + 2D^{\beta_{n-2}} & \text{if } p > 5, \\ 2D^{\delta_{n-2}} + 4D^{\beta_{n-2}} & \text{if } p = 5 \text{ and } n > 10, \\ 4D^{\beta_{n-2}} & \text{if } p = 5, \text{ and } n = 10. \end{cases}$$

4. Results involving Jantzen–Seitz partitions

4A. ***JS-partitions.*** Let $\lambda \in \mathscr{RP}_p(n)$. We call λ a *JS-partition*, written $\lambda \in \mathrm{JS}$, if there is $i \in I$ such that $\varepsilon_i(\lambda) = 1$ and $\varepsilon_j(\lambda) = 0$ for all $j \in I \setminus \{i\}$. In this case we also write $\lambda \in \mathrm{JS}(i)$ or $D^\lambda \in \mathrm{JS}(i)$. The notion goes back to [Jantzen and Seitz 1992; Kleshchev 1994].

Note that if $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h > 0)$ is a JS-partition then the bottom removable node $A := (h, \lambda_h)$ is the only normal node of λ, and in this case we have $\lambda \in \mathrm{JS}(i)$, where $i = \mathrm{cont}\, A$.

Lemma 4.1. *Let* δ_n *be one of the explicit partitions defined in Section 3B. Then* $\delta_n \in \mathrm{JS}(i)$ *for some i if and only if* $p > 3$ *and one of the following happens*:

(i) $n = 6$, $p > 5$, *and* $\delta_n = (3, 2, 1)$; *in this case* $\delta_n \in \mathrm{JS}(0)$ *and* $a(\lambda) = 1$;

(ii) $n = 7$, $p > 3$, *and* $\delta_n = (4, 3)$; *in this case* $a(\lambda) = 1$ *and* $\delta_n \in \mathrm{JS}(2)$;

(iii) $n = mp$ *for* $m \geq 2$ *and* $\delta_n = (p + 2, p^{m-2}, p - 2)$; *in this case* $\delta_n \in \mathrm{JS}(2)$, $a(\lambda) = \sigma(m)$, *and*

$$\mathrm{res}_{n-2} D^{\delta_n} \ni \begin{cases} 2D^{\gamma_{n-2}} + 2D^{\beta_{n-2}} & \text{if } p > 5, \\ 2D^{\delta_{n-2}} + 4D^{\beta_{n-2}} & \text{if } p = 5 \text{ and } n > 10, \\ 4D^{\beta_{n-2}} & \text{if } p = 5, \text{ and } n = 10. \end{cases}$$

Proof. This is proved by inspection of the formulas for δ_n and applying the definition of the Jantzen–Seitz partitions. □

Now, we record some combinatorial results of A. Phillips.

Lemma 4.2 [Phillips 2004, Lemma 3.8]. *For* $\lambda \in \mathcal{RP}_p(n)$ *the following are equivalent*:

(i) $\lambda \in \mathrm{JS}(0)$;

(ii) $\lambda \in \mathrm{JS}(0)$ *and* $\tilde{e}_0\lambda \in \mathrm{JS}(1)$;

(iii) $\lambda \in \mathrm{JS}(i)$ *and* $\tilde{e}_i\lambda \in \mathrm{JS}(j)$ *for some* $i, j \in I$ *and exactly one of* i *and* j *is equal to* 0.

Lemma 4.3 [Phillips 2004, Lemma 3.14]. *Let* $\lambda \in \mathcal{RP}_p(n)$. *Then*:

(i) $\lambda = \alpha_n$ *and* $n \equiv 1 \pmod p$ *if and only if* $\varepsilon_i(\lambda) = 0$ *for all* $i \neq 0$ *and* $\tilde{e}_0(\lambda) \in \mathrm{JS}(0)$;

(ii) $\lambda = \alpha_n$ *and* $n \not\equiv 0, 1, 2 \pmod p$ *if and only if* $\lambda \in \mathrm{JS}(i)$ *and* $\tilde{e}_i\lambda \in \mathrm{JS}(j)$ *for some* $i, j \in I \setminus \{0\}$.

Lemma 4.4 [Phillips 2004, Lemma 3.7]. *Let* $\lambda = (l_1^{a_1}, \dots, l_m^{a_m}) \in \mathcal{RP}_p(n)$ *with* $l_1 > l_2 > \cdots > l_m > 0$. *Then* $\lambda \in \mathrm{JS}(0)$ *if and only if* $l_m = 1$ *and* $\mathrm{cont}_p\, l_s = \mathrm{cont}_p(l_{s+1} + 1)$ *for all* $s = 1, 2, \dots, m-1$.

4B. *Jantzen–Seitz partitions and branching.*

Lemma 4.5. *Let* $\lambda \in \mathrm{JS}(i)$ *and assume that* D^λ *is not basic. Then one of the following happens*:

(i) $i = 0$ *and* $\tilde{e}_0\lambda \in \mathrm{JS}(1)$;

(ii) $i = \ell$, $\varepsilon_{\ell-1}(\tilde{e}_\ell\lambda) \geq 2$ *and* $\varepsilon_j(\tilde{e}_\ell\lambda) = 0$ *for all* $j \neq \ell - 1$.

(iii) $i = 1$, $\varepsilon_0(\tilde{e}_1\lambda) \geq 2$ *and* $\varepsilon_j(\tilde{e}_1\lambda) = 0$ *for all* $j \neq 0$.

(iv) $p > 3$, $i \neq 0, \ell$, $\varepsilon_{i-1}(\tilde{e}_i\lambda) \geq 1$, $\varepsilon_{i+1}(\tilde{e}_i\lambda) = 1$ *and* $\varepsilon_j(\tilde{e}_i\lambda) = 0$ *for all* $j \neq i-1, i+1$. *Moreover, if in addition, we have* $i \neq 1$, *then* $\varepsilon_{i-1}(\tilde{e}_i\lambda) = 1$.

Proof. Assume first that $\tilde{e}_i\lambda \in \mathrm{JS}(j)$ for some j. Then by Lemma 4.3, exactly one of i, j is 0. Hence by Lemma 4.2, we are in (i).

Now, let $\tilde{e}_i\lambda \notin \mathrm{JS}$. Then, by Lemma 2.1, $\varepsilon_j(\tilde{e}_i\lambda) > 0$ implies that $j = i \pm 1$; moreover $\varepsilon_{i+1}(\tilde{e}_i\lambda) \leq 1$, and $\varepsilon_{i-1}(\tilde{e}_i\lambda) \leq 1$ if $i \neq 1, \ell$. If $i = \ell$, it now follows that we are in (ii). If $i = 1$ we are in (iii) or in (iv). If $i \neq 0, 1, \ell$, we are in (iv). □

Lemma 4.6. *Let* $\lambda \in \mathcal{RP}_p(n)$ *satisfy Lemma 4.5* (iv). *Then one of the following occurs*:

(i) $d_2(\lambda) \geq 4$.

(ii) $a(\lambda) = 0$, $i = 1$, *and* $d_2(\lambda) \geq 3$.

(iii) $D^\lambda \cong B_n$.

(iv) $p > 5$, $n = mp$ *for* $m \geq 2$, $\lambda = \delta_n = (p+2, p^{m-2}, p-2) \in \mathrm{JS}(2)$, *and* $\mathrm{res}_{n-2}\, D^{\delta_n} \ni 2D^{\gamma_{n-2}} + 2D^{\beta_{n-2}}$.

(v) $n = 5$, $p > 5$, *and* $\lambda = (3, 2)$.

(vi) $n = 7$, $p > 3$, *and* $\lambda = (4, 3)$.

Proof. We may assume that D^λ is not basic. We may also assume that D^λ is not second basic — otherwise we are in (iii). By Theorem 2.4 we have

$$\mathrm{res}_{n-1}\, D^\lambda = 2^{a(\lambda)} D^{\tilde{e}_i \lambda}.$$

Assume that $i \neq 1$. Then $i - 1 \neq 0$ and $a(\tilde{e}_i\lambda) + a(\lambda) = 1$, so we have

$$\mathrm{res}_{n-2}\, D^\lambda = 2(D^{\tilde{e}_{i-1}\tilde{e}_i\lambda} + D^{\tilde{e}_{i+1}\tilde{e}_i\lambda}).$$

If none of $D^{\tilde{e}_{i\pm1}\tilde{e}_i\lambda}$ is basic or second basic, we are in (i).

Suppose that $D^{\tilde{e}_{i\pm1}\tilde{e}_i\lambda} \cong A_{n-2}$. By Theorem 3.6, we may assume that $\lambda = \gamma_n$. But inspection shows that γ_n is never JS, unless $n = 5$ and $p > 5$, in which case, however, $\lambda \in \mathrm{JS}(1)$. Suppose now that $D^{\tilde{e}_{i\pm1}\tilde{e}_i\lambda} \cong B_{n-2}$. Then we may assume that $\lambda = \delta_n$. It follows from Lemma 4.1 that we are in the cases (iv) or (vi).

Now, let $i = 1$. Theorem 2.4 then gives

$$\mathrm{res}_{n-2}\, D^\lambda \ni 2^{a(\lambda)} e_0 D^{\tilde{e}_1\lambda} + 2D^{\tilde{e}_2\tilde{e}_1\lambda}.$$

If one of $D^{\tilde{e}_{1\pm1}\tilde{e}_1\lambda}$ is basic or second basic then $\lambda = \gamma_n$ or $\lambda = \delta_n$. If $\lambda = \gamma_n$ then we are in (v). The case $\lambda = \delta_n$ is impossible by Lemma 4.1. So we may assume that neither of $D^{\tilde{e}_{1\pm1}\tilde{e}_1\lambda}$ is basic or second basic.

If $\varepsilon_0(\tilde{e}_1\lambda) \geq 2$, then $D^{\tilde{e}_0\tilde{e}_1\lambda}$ appears in $e_0 D^{\tilde{e}_1\lambda}$ with multiplicity at least 2, and we are in (i). Finally, let $\varepsilon_0(\tilde{e}_1\lambda) = \varepsilon_2(\tilde{e}_1\lambda) = 1$. Then

$$\mathrm{res}_{n-2}\, D^\lambda = 2^{a(\lambda)} D^{\tilde{e}_0\tilde{e}_1\lambda} + 2D^{\tilde{e}_2\tilde{e}_1\lambda}.$$

If $a(\lambda) = 1$, we still get 4 composition factors, but if $a(\lambda) = 0$, we do get only 3 composition factors, which is case (ii). □

Lemma 4.7. *Let* $p > 3$ *and let* $\lambda \in \mathcal{RP}_p(n)$ *satisfy Lemma 4.5* (ii) *or* (iii). *Then one of the following occurs*:

(i) $d_2(\lambda) \geq 4$.

(ii) $D^\lambda \cong A_n$.

(iii) $p = 5$, $n = mp$ *for* $m \geq 2$, $\lambda = \delta_n = (p+2, p^{m-2}, p-2)$, *and*

$$\mathrm{res}_{n-2}\, D^{\delta_n} \ni \begin{cases} 2D^{\delta_{n-2}} + 4D^{\beta_{n-2}} & \text{if } n > 10, \\ 4D^{\beta_{n-2}} & \text{if } n = 10. \end{cases}$$

Proof. It follows from the assumption that all weights of D^λ are of the form $(*, i-1, i)$ and that D^λ has a weight of the form $(*, i-1, i-1, i)$. If all weights of D^λ are of the form $(*, i-1, i-1, i)$, then D^λ is basic by Lemma 3.5. If a weight of the form $(*, i, i-1, i)$ appears in D^λ, then so does $(*, i, i, i-1)$ or $(*, i-1, i, i)$ thanks to [Kleshchev 2005, Lemma 20.4.1], which leads to a contradiction. If $(*, j, i-1, i)$ appears with $j \neq i, i-2$, then $(*, i-1, j, i)$ also appears, again leading to a contradiction. So $i = \ell$ and weights of the form $(*, \ell-1, \ell-1, \ell)$ and $(*, \ell-2, \ell-1, \ell)$ appear in D^λ. In this case $a(\lambda)+a(\tilde{e}_\ell\lambda) = 1$, and so Theorem 2.4 yields a contribution of $4D^{\tilde{e}_{\ell-1}\tilde{e}_\ell\lambda}$ into $\mathrm{res}_{n-2} D^\lambda$. So, we are in (i) unless $\tilde{e}_{\ell-1}\tilde{e}_\ell\lambda = \alpha_{n-2}$ or β_{n-2}. If $\tilde{e}_{\ell-1}\tilde{e}_\ell\lambda = \alpha_{n-2}$, then $\lambda = \beta_n$ or γ_n, which never satisfy the assumptions of the lemma. If $\tilde{e}_{\ell-1}\tilde{e}_\ell\lambda = \beta_{n-2}$, then we may assume that $\lambda = \delta_n$, which by Lemma 4.1 leads to the case (iii). □

Note that if $p = 3$ then the cases (ii) and (iii) of Lemma 4.5 are the same.

Lemma 4.8. *Let $p = 3$ and $\lambda \in \mathcal{RP}_p(n)$ satisfy Lemma 4.5* (ii). *Then one of the following occurs*:

(i) $d_2(\lambda) \geq 4$;

(ii) λ *is of the form* $(*, 5, 4, 2)$, $a(\lambda) = 0$, *in which case* $\mathrm{res}_{n-2} D^\lambda$ *has composition factor* $D^{(*,5,3,1)} \not\cong A_{n-2}, B_{n-2}$ *with multiplicity* 3. *In particular*, $d_2(\lambda) \geq 3$.

(iii) $D^\lambda \cong A_n$ *or* B_n.

Proof. If λ is neither basic nor second basic, then the assumptions imply that λ has one of the following forms: $(*, 5, 4, 3^a, 2)$, $(*, 6, 4, 3^b, 2)$, or $(*, 5, 4, 2)$ with $a > 0$ and $b \geq 0$. In the first two cases, Theorem 2.4 gives at least 4 needed composition factors. So we may assume that we are in (ii). The rest now follows from Theorem 2.4. □

4C. *Class* JS(0). This is the most difficult case since modules $D^\lambda \in \mathrm{JS}(0)$ tend to branch with very small amount of composition factors.

Lemma 4.9. *Let $\lambda \in \mathcal{RP}_p(n)$ and assume that there exist distinct $i, j \in I \setminus \{0\}$ such that $\varepsilon_i(\lambda) = \varepsilon_j(\lambda) = 1$ and $\varepsilon_k(\lambda) = 0$ for all $k \neq i, j$. Then $\tilde{e}_i\tilde{e}_j\lambda \notin \mathrm{JS}(0)$.*

Proof. Assume first that $j \neq 1$. Then by Lemma 2.1, we have $\varepsilon_0(\tilde{e}_j\lambda) = 0$. Now, if $i \neq 1$ then similarly $\varepsilon_0(\tilde{e}_i\tilde{e}_j\lambda) = 0$, and $\tilde{e}_i\tilde{e}_j\lambda \notin \mathrm{JS}(0)$. If $i = 1$, we note by Lemma 4.2 that $\sum_k \varepsilon_k(\tilde{e}_j\lambda) > 1$. So there must exist $k \neq 0, 1$ such that $\varepsilon_k(\tilde{e}_j\lambda) \geq 1$. Now by Lemma 2.1, we have $\varepsilon_k(\tilde{e}_i\tilde{e}_j\lambda) \geq 1$, which shows that $\tilde{e}_i\tilde{e}_j\lambda \notin \mathrm{JS}(0)$.

Now assume that $j = 1$. Taking into account Lemma 2.1, we must have $\varepsilon_i(\tilde{e}_1\lambda) = \varepsilon_0(\tilde{e}_1\lambda) = 1$. By Lemma 4.4, $\tilde{e}_1\lambda$ is obtained from $\tilde{e}_i\tilde{e}_1\lambda$ by adding a box of content i to the first row. Now λ must be obtained from $\tilde{e}_1\lambda$ by adding a box of residue 1 to the last row, but then again by Lemma 4.4, we must have $\varepsilon_1(\lambda) \geq 2$. □

Our main result on branching of JS(0)-modules is as follows:

Proposition 4.10. *Let* $\lambda \in \mathcal{RP}_p(n)$ *belong to* JS(0) *and* $\lambda \neq \alpha_n, \beta_n$. *Assume in addition that*

(i) $n > 12$ *if* $p = 3$.

(ii) $n > 16$ *if* $p = 5$,

(iii) $n > 10$ *if* $p \geq 7$.

Then $d_6(\lambda) \geq 24$, *with three possible exceptions*:

(i) $p > 7$, $\lambda = (p-3, 3, 2, 1)$, *in which case we have*

$$4A_{p-3} + 20B_{p-3} + 16D^{(p-5,2)} + 4D^{(p-6,2,1)} \in \operatorname{res}^{p+3}_{p-3} D^{\lambda}.$$

(ii) $p \geq 7$, $\lambda = (p+2, p+1, p^a, p-1, 1)$ *with* $a \geq 0$, *in which case we have*

$$4D^{(p+2,p+1,p^a,p-6)} + 16D^{(p+2,p^{a+1},p-5)} + 4A_{n-6} + 20B_{n-6} \in \operatorname{res}_{n-6} D^{\lambda}.$$

(iii) $p = 5$, $n = 18$, *and* $\lambda = (7, 6, 4, 1)$, *in which case*

$$20D^{(7,4,1)} + 16B_{12} + 8A_{12} \in \operatorname{res}_{12} D^{\lambda}.$$

Proof. We will repeatedly use the notation $\lambda = (*, l_r^{a_r}, l_{r+1}^{a_{r+1}}, \dots, l_m^{a_m})$ if we only want to specify the last $m - r + 1$ lengths of the parts of λ.

First we consider the case $p = 3$. In this case, using Lemma 4.4 we see that λ is of the form $(*, 2, 1)$. Since $n > 12$ we could not have $* = \varnothing$, and by Lemma 4.4 again, we must have $\lambda = (*, 3^a, 2, 1)$ with $a > 1$ or $\lambda = (*, 4, 2, 1)$. We could not have $* = \varnothing$ since $\lambda \neq \alpha_n, \beta_n$, so by Lemma 4.4, we can get more information about λ, namely $\lambda = (*, 4, 3^a, 2, 1)$ or $\lambda = (*, 5, 4, 2, 1)$. Since $\lambda \neq \beta_n$ and $n > 12$, we conclude that $* \neq \varnothing$ in both cases.

Now, we get some information on the restriction $\operatorname{res}_{n-6} D^{\lambda}$ using Theorem 2.4. If $\lambda = (*, 4, 3^a, 2, 1)$, then $2^{a(\lambda)} D^{(*,4,3^a,1)} \in \operatorname{res}_{n-2} D^{\lambda}$. Now, the last node in the last row of length 3 in $(*, 4, 3^a, 1)$ satisfies the assumptions of Theorem 2.4(viii), so we conclude that $2D^{(*,4,3^{a-1},2,1)} \in \operatorname{res}^{n-2}_{n-3} D^{(*,4,3^a,1)}$. Furthermore, the last node in the row of length 4 in $(*, 4, 3^a, 1)$ is the third normal 0-node from the bottom. If it is 0-good, then $3D^{(*,3^{a+1},1)} \in \operatorname{res}^{n-2}_{n-3} D^{(*,4,3^a,1)}$ by Theorem 2.4(iii). If it is not good, then the 0-good node is above it and $\varepsilon_0(\lambda) \geq 4$, in which case we get $4D^{(*,4,3^a,1)} \in \operatorname{res}^{n-2}_{n-3} D^{(*,4,3^a,1)}$, where by the first $(*, 4, 3^a, 1)$ we understand a partition obtained from the second $(*, 4, 3^a, 1)$ by removing a box from a row of length greater than 4. Thus we have

$$2^{a(\lambda)+1} D^{(*,4,3^{a-1},2,1)} + 3 \cdot 2^{a(\lambda)} D^{(*,3^{a+1},1)} \in \operatorname{res}_{n-3} D^{\lambda}$$

or

$$2^{a(\lambda)+1} D^{(*,4,3^{a-1},2,1)} + 2^{a(\lambda)} D^{(*,3^{a+1},1)} + 4 \cdot 2^{a(\lambda)} D^{(*,4,3^a,1)} \in \operatorname{res}_{n-3} D^{\lambda}.$$

The second case is much easier so we continue just with the first one. On restriction to $n-4$, we now get

$$2^{a(\lambda)+1} D^{(*,4,3^{a-1},2)} + 6 \cdot 2^{a(\lambda)} D^{(*,3^a,2,1)} \in \mathrm{res}_{n-4} D^\lambda$$

Note that $a(\lambda) + a((*, 4, 3^{a-1}, 2)) = 1$, so we further get

$$4D^{(*,4,3^{a-1},1)} + 6 \cdot 2^{a(\lambda)} D^{(*,3^a,2)} \in \mathrm{res}_{n-5} D^\lambda.$$

Now consider $\mathrm{res}^{n-5}_{n-6} 4D^{(*,4,3^{a-1},1)}$. Note that $\varepsilon_0((*, 4, 3^{a-1}, 1)) \geq 3$, so removal of the 0-good node yields a contribution of at least 12 composition factors, none of which is isomorphic to a basic or a second basic module. Finally $\mathrm{res}^{n-5}_{n-6} 6 \cdot 2^{a(\lambda)} D^{(*,3^a,2)}$ yields $12D^{(*,3^a,1)}$, which again cannot be basic or second basic, since here $*$ stands for some parts of length greater than 4. The restriction $\mathrm{res}^n_{n-6} D^{(*,5,4,2,1)}$ is treated similarly.

Now, let $p = 5$. Using Lemma 4.4 and the assumptions $n > 16$ and $\lambda \neq \alpha_n, \beta_n$, we arrive at the following six possibilities for λ:

$$(*, 5, 4, 3, 2, 1),\ (*, 6, 4, 3, 2, 1),\ (*, 7, 3, 2, 1),$$
$$(*, 6, 5^a, 4, 1),\ (*, 7, 6, 4, 1),\ (*, 9, 6, 4, 1),$$

with $a \geq 1$ and $* \neq \varnothing$, except possibly in the last two cases. Now we use Theorem 2.4 to show that:

- $\mathrm{res}_{n-6} D^{(*,5,4,3,2,1)}$ contains $48D^{(*,5,3,2)}$ or $20D^{(*,5,3,1)} + 4D^{(*,4,3,2)}$ or $20D^{(*,5,3,1)} + 12D^{(*,4,3,2,1)}$.
- $\mathrm{res}_{n-6} D^{(*,6,4,3,2,1)} \ni 4D^{(*,6,4)} + 20D^{(*,6,3,1)}$.
- $\mathrm{res}_{n-6} D^{(*,7,3,2,1)} \ni 20D^{(*,6,1)} + 10D^{(*,5,2)}$.
- $\mathrm{res}_{n-6} D^{(*,6,5^a,4,1)}$ has at least 4 composition factors of the form $D^{(*,6,5^{a-1},4)}$ and either 20 composition factors of the form $D^{(*,5^a,4,1)}$, or 12 composition factors of the form $D^{(*,5^a,4,1)}$ and 16 composition factors of the form $D^{(*,6,5^{a-1},4,1)}$.
- In the case $* = \varnothing$ we get the exception (c), while in the case $* \neq \varnothing$ we get $\mathrm{res}_{n-6} D^{(*,7,6,4,1)} \ni 20D^{(*,7,4,1)} + 4D^{(*,6,5,1)}$.
- $20D^{(*,9,4,1)} + 4D^{(*,8,5,1)} \in \mathrm{res}_{n-6} D^{(*,9,6,4,1)}$.

Finally, let $p \geq 7$. Using Lemma 4.4 and the assumptions $n > 10$ and $\lambda \neq \alpha_n, \beta_n$ we arrive at the following possibilities for λ (with $a \geq 0$):

$$(*, 4, 3, 2, 1),\ (*, p-3, 3, 2, 1),\ (*, p-1, p-2, 2, 1),\ (*, p+2, p-2, 2, 1),$$
$$(*, p+2, p+1, p^a, p-1, 1),\ (*, 2p-1, p+1, p^a, p-1, 1).$$

If $\lambda = (*, 4, 3, 2, 1)$ then $* \neq \varnothing$ as $n > 10$. In this case we get

$$4D^{(*,4)} + 20D^{(*,3,1)} \in \operatorname{res}_{n-6} D^{\lambda}.$$

If $\lambda = (*, p-3, 3, 2, 1)$, we may assume that $p > 7$ (otherwise we are in the previous case). If $* = \varnothing$, we are in the exceptional case (a), and Theorem 2.4 yields the composition factors of the restriction as claimed in the theorem. If $* \neq \varnothing$, we get similar composition factors but with partitions starting with '$*$', and such composition factors are neither basic nor second basic.

If $\lambda = (*, p-1, p-2, 2, 1)$, we have

$$12D^{(*,p-1,p-5)} + 12D^{(*,p-2,p-4)} \in \operatorname{res}_{n-6} D^{\lambda}.$$

Let $\lambda = (*, p+2, p-2, 2, 1)$. If $* = \varnothing$, then $a(\lambda) = 1$, and using Theorem 2.4, we get $16D^{(p+2,p-5)} + 8D^{(p+1,p-5,1)} \in \operatorname{res}_{n-6} D^{\lambda}$. Otherwise, we get

$$16D^{(*,p+2,p-5)} + 20D^{(*,p+1,p-4)} \in \operatorname{res}_{n-6} D^{\lambda}.$$

If $\lambda = (*, p+2, p+1, p^a, p-1, 1)$, then

$$4D^{(*,p+2,p+1,p^a,p-6)} + 16D^{(*,p+2,p^{a+1},p-5)} + 20D^{(*,p+1,p^{a+1},p-4)} + 4D^{(*,p^{a+2},p-3)} \in \operatorname{res}_{n-6} D^{\lambda}.$$

If $* \neq \varnothing$, all of these composition factors are neither basic nor second basic. Otherwise we are in the exceptional case (b).

The case $\lambda = (*, 2p-1, p+1, p^a, p-1, 1)$ is similar to the case

$$\lambda = (*, p+2, p+1, p^a, p-1, 1). \qquad \square$$

We will also need the following result on JS(0)-modules:

Lemma 4.11. *Let* $\lambda \in \mathcal{RP}_p(n)$ *for* $n \geq 12$. *Assume* $\lambda \in \mathrm{JS}(0)$ *and* $\lambda \neq \alpha_n, \beta_n$. *Then either*

(a) $d_3(\lambda) \geq 3$, *or*

(b) $d_3(\lambda) = 2$, $p \geq 5$, *and* $n = mp + 1$ *for some* $m \geq 2$.

Proof. Applying Lemma 4.5 to $V := D^{\lambda}$ we have $\operatorname{res}_{n-1} V = U = D^{\mu}$ with $\mu \in \mathrm{JS}(1)$. Assume $d_3(V) \leq 2$ so that $d_2(U) \leq 2$. Now we can apply Lemma 4.5 to $\mu \in \mathrm{JS}(1)$ and arrive at one of the three cases (ii)–(iv) described in Lemma 4.5. In the case (ii) (so $p = 3$), the condition $d_2(U) \leq 2$ implies by Lemma 4.8 that $\mu = \alpha_{n-1}$ or β_{n-1}. In the case (iii) (and $p > 3$), then since $n \geq 12$ by Lemma 4.7 either we have $\mu = \alpha_{n-1}$ or we arrive at (b). Similarly, in the case (iv) by Lemma 4.6 either we have $\mu = \beta_{n-1}$ or we arrive at (b).

Assuming furthermore that (b) does not hold for V, we conclude that $\mu \in \{\alpha_{n-1}, \beta_{n-1}\}$. Since $\lambda \neq \alpha_n, \beta_n$, by Theorem 3.6 we must have $\lambda = \gamma_n$. But then $\lambda \notin \mathrm{JS}(0)$ by Lemma 3.8. □

5. The case $\sum \varepsilon_i(\lambda) = 2$

5A. *The subcase where all $\varepsilon_i(\lambda) \leq 1$.*

Lemma 5.1. *Let $\lambda \in \mathcal{RP}_p(n)$. If there exist $i \neq j$ with $\varepsilon_i(\lambda) = \varepsilon_j(\lambda) = 1$ and $\varepsilon_k(\lambda) = 0$ for all $k \neq i, j$, then at least one of $\tilde{e}_i\lambda$, $\tilde{e}_j\lambda$ is not* JS.

Proof. Assume that $\tilde{e}_i\lambda, \tilde{e}_j\lambda \in \mathrm{JS}$. Then by Theorem 2.4, we have

$$\mathrm{res}_{n-1} D^\lambda \cong n_1 D^{\tilde{e}_i\lambda} \oplus n_2 D^{\tilde{e}_j\lambda}$$

and

$$\mathrm{res}_{n-2} D^\lambda = n_1 m_1 D^{\tilde{e}_j\tilde{e}_i\lambda} \oplus n_2 m_2 D^{\tilde{e}_i\tilde{e}_j\lambda},$$

for some $n_1, n_2, m_1, m_2 \in \{1, 2\}$. Moreover, by Lemma 2.1, we have $\tilde{e}_i\tilde{e}_j\lambda = \tilde{e}_j\tilde{e}_i\lambda$. It follows that the restrictions $\mathrm{res}_{n-2} D^{\tilde{e}_i\lambda}$ and $\mathrm{res}_{n-2} D^{\tilde{e}_j\lambda}$ are both homogeneous with the same composition factor $D^{\tilde{e}_i\tilde{e}_j\lambda}$. So, if $p > 3$, we get a contradiction with Lemma 2.5.

Let $p = 3$. Then we may assume that $i = 0$ and $j = 1$. Note that by the assumption $\varepsilon_0(\lambda) = \varepsilon_1(\lambda) = 1$, each weight appearing in D^λ ends on $1, 0$ or on $0, 1$, and both of these occur. After application of $\tilde{e}_1$ to D^λ only the weights of the form $(*, 0, 1)$ survive and yield weights of the form $(*, 0)$. Since $\tilde{e}_1\lambda \in \mathrm{JS}(0)$, we conclude that $\varepsilon_0(\tilde{e}_1\lambda) = 1$, and so all weights of $D^{\tilde{e}_1\lambda}$ are of the form $(*, 1, 0)$. Similarly all weights of $D^{\tilde{e}_0\lambda}$ are of the form $(*, 0, 1)$. Thus the weights of D^λ are actually of the form $(*, 0, 1, 0)$ and $(*, 1, 0, 1)$. However, by the "Serre relations" [Kleshchev 2005, Lemma 20.4.2 and Lemma 22.3.8], the existence of a weight $(*, 1, 0, 1)$ implies the existence of $(*, 1, 1, 0)$ or $(*, 0, 1, 1)$, which now leads to a contradiction. □

Lemma 5.2. *Let $\lambda \in \mathcal{RP}_p(n) \setminus \{\alpha_n, \beta_n, \gamma_n, \delta_n\}$. Suppose that $\varepsilon_i(\lambda) = \varepsilon_j(\lambda) = 1$ for some $i \neq j$ in $I \setminus \{0\}$, and $\varepsilon_k(\lambda) = 0$ for all $k \neq i, j$. Then*:

(i) $\mathrm{res}_{n-1} D^\lambda \cong 2^{a(\lambda)} D^{\tilde{e}_i\lambda} \oplus 2^{a(\lambda)} D^{\tilde{e}_j\lambda}$. *Moreover, $\tilde{e}_i\lambda$ and $\tilde{e}_j\lambda$ are not both JS, and $\tilde{e}_i\lambda, \tilde{e}_j\lambda \neq \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}$. In particular, $d_1(\lambda) \geq 2$.*

(ii) $d_2(\lambda) \geq 5$.

Proof. By Theorem 2.4, we have $\mathrm{res}_{n-1} D^\lambda \cong 2^{a(\lambda)} D^{\tilde{e}_i\lambda} \oplus 2^{a(\lambda)} D^{\tilde{e}_j\lambda}$. In view of Lemma 5.1, we now have (i).

By Lemma 2.1, $\varepsilon_i(\tilde{e}_j\lambda) > 0$ and $\varepsilon_j(\tilde{e}_i\lambda) > 0$, so

$$2^{a(\lambda)} 2^{a(\tilde{e}_i\lambda)} D^{\tilde{e}_j\tilde{e}_i\lambda} + 2^{a(\lambda)} 2^{a(\tilde{e}_j\lambda)} D^{\tilde{e}_i\tilde{e}_j\lambda} = 2 D^{\tilde{e}_j\tilde{e}_i\lambda} + 2 D^{\tilde{e}_i\tilde{e}_j\lambda} \in \mathrm{res}_{n-2} D^\lambda$$

(it might happen that $\tilde{e}_i\tilde{e}_j\lambda = \tilde{e}_j\tilde{e}_i\lambda$, in which case the above formula is interpreted as $4D^{\tilde{e}_i\tilde{e}_j\lambda} \in \mathrm{res}_{n-2} D^\lambda$). Moreover, since not both $\tilde{e}_i\lambda$ and $\tilde{e}_j\lambda$ are JS, we may assume without loss of generality that $\tilde{e}_i\lambda$ is not JS, i.e., $\sum_k \varepsilon_k(\tilde{e}_i\lambda) > 1$. Therefore $\varepsilon_j(\tilde{e}_i\lambda) \geq 2$ or there exists $k \neq i, j$ with $\varepsilon_k(\tilde{e}_i\lambda) > 0$. In the first case, we conclude that actually $4D^{\tilde{e}_j\tilde{e}_i\lambda} + 2D^{\tilde{e}_i\tilde{e}_j\lambda} \in \mathrm{res}_{n-2} D^\lambda$, whence $d_2(\lambda) \geq 6$. In the second case we get $2D^{\tilde{e}_j\tilde{e}_i\lambda} + 2D^{\tilde{e}_i\tilde{e}_j\lambda} + 2^{a(\lambda)}D^{\tilde{e}_k\tilde{e}_i\lambda} \in \mathrm{res}_{n-2} D^\lambda$, so $d_2(\lambda) \geq 5$. □

Lemma 5.3. *Let $\lambda \in \mathcal{RP}_p(n) \setminus \{\alpha_n, \beta_n, \gamma_n, \delta_n\}$. Suppose that $\varepsilon_i(\lambda) = \varepsilon_0(\lambda) = 1$ for some i in $I \setminus \{0\}$, and $\varepsilon_k(\lambda) = 0$ for all $k \neq i, 0$. Then*:

(i) $\mathrm{res}_{n-1} D^\lambda \cong 2^{a(\lambda)}D^{\tilde{e}_i\lambda} \oplus D^{\tilde{e}_0\lambda}$. *Moreover, $\tilde{e}_i\lambda$ and $\tilde{e}_0\lambda$ are not both JS, and $\tilde{e}_i\lambda, \tilde{e}_j\lambda \neq \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}$. In particular, $d_1(\lambda) \geq 2$.*

(ii) $d_2(\lambda) \geq 3$.

Proof. By Theorem 2.4, $\mathrm{res}_{n-1} D^\lambda \cong 2^{a(\lambda)}D^{\tilde{e}_i\lambda} \oplus D^{\tilde{e}_0\lambda}$. In view of Lemma 5.1, we now have (i). By Lemma 2.1, $\varepsilon_i(\tilde{e}_0\lambda) > 0$ and $\varepsilon_0(\tilde{e}_i\lambda) > 0$, so

$$2^{a(\lambda)}D^{\tilde{e}_0\tilde{e}_i\lambda} + 2^{a(\tilde{e}_0\lambda)}D^{\tilde{e}_i\tilde{e}_0\lambda} = 2^{a(\lambda)}(D^{\tilde{e}_0\tilde{e}_i\lambda} + D^{\tilde{e}_i\tilde{e}_0\lambda}) \in \mathrm{res}_{n-2} D^\lambda.$$

Moreover, from (i), not both $\tilde{e}_i\lambda$ and $\tilde{e}_0\lambda$ are JS. Assume that $\tilde{e}_i\lambda \notin$ JS. Then $\varepsilon_0(\tilde{e}_i\lambda) \geq 2$ or there exists $k \neq i, 0$ with $\varepsilon_k(\tilde{e}_i\lambda) > 0$. In the first case, we conclude that actually $2 \cdot 2^{a(\lambda)}D^{\tilde{e}_0\tilde{e}_i\lambda} + 2^{a(\lambda)}D^{\tilde{e}_i\tilde{e}_0\lambda} \in \mathrm{res}_{n-2} D^\lambda$, whence $d_2(\lambda) \geq 3$. In the second case we get $2^{a(\lambda)}(D^{\tilde{e}_0\tilde{e}_i\lambda} + D^{\tilde{e}_i\tilde{e}_0\lambda}) + 2D^{\tilde{e}_k\tilde{e}_i\lambda} \in \mathrm{res}_{n-2} D^\lambda$, so $d_2(\lambda) \geq 4$. The case $\tilde{e}_0\lambda \notin$ JS is considered similarly. □

Corollary 5.4. *Let $\lambda \in \mathcal{RP}_p(n) \setminus \{\alpha_n, \beta_n, \gamma_n, \delta_n\}$, and $i \neq j$ be elements of I such that $\varepsilon_i(\lambda) \neq 0$, $\varepsilon_j(\lambda) \neq 0$, and $\varepsilon_k(\lambda) = 0$ for all $k \in I \setminus \{i, j\}$. Then* $\mathrm{res}_{n-2} e_i(D^\lambda)$ *or* $\mathrm{res}_{n-2} e_j(D^\lambda)$ *is reducible.*

Proof. If $\varepsilon_i(\lambda) \geq 2$, then by Lemma 2.1, we have $\varepsilon_i(\tilde{e}_j\lambda) \geq 2$. Since $D^{\tilde{e}_j\lambda} \in e_j(D^\lambda)$ by Theorem 2.4, we conclude that $\mathrm{res}_{n-2} e_j(D^\lambda)$ is reducible. So we may assume that $\varepsilon_i(\lambda) = 1$ and similarly $\varepsilon_j(\lambda) = 1$. If both i, j are not 0, we can now use Lemma 5.2(i). If one of i, j is 0 use Lemma 5.3(i) instead. □

5B. *The subcase where some $\varepsilon_i(\lambda) = 2$.*

Lemma 5.5. *Let $\lambda \in \mathcal{RP}_p(n) \setminus \{\alpha_n, \beta_n, \gamma_n, \delta_n\}$. Suppose that $\varepsilon_i(\lambda) = 2$ for some $i \in I$, and $\varepsilon_k(\lambda) = 0$ for all $k \neq i$. If $\tilde{e}_i\lambda \in$ JS, then $i \neq 0$ and*

$$2^{a(\lambda)}(2D^{\tilde{e}_i\lambda} + D^\mu) \in \mathrm{res}_{n-1} D^\lambda,$$

where $\tilde{e}_i\lambda \neq \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}$ and $\mu \neq \alpha_{n-1}$.

Proof. First of all, by Lemma 4.3(i), we have $i \neq 0$. By Theorem 2.4,

$$\mathrm{res}_{n-1} D^\lambda \cong 2^{a(\lambda)}e_i(D^\lambda),$$

and $2D^{\tilde{e}_i\lambda} \in e_i(D^\lambda)$. Since $\lambda \neq \alpha_n, \beta_n, \gamma_n$, we get $\tilde{e}_i\lambda \neq \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}$. It remains to prove that $e_i(D^\lambda)$ has another composition factor which is not basic spin.

The partition λ has two i-normal nodes. Denote them by A and B, and assume that A is above B. Then A is good and $\tilde{e}_i\lambda = \lambda_A$. Moreover, since the bottom removable node of λ is always normal, we know that B is in the last row.

Assume first that $\lambda_B \in \mathcal{RP}_p(n-1)$. In this case $D^{\lambda_B} \in \mathrm{res}_{n-1} D^\lambda$ by condition (iv) in the conclusion of Theorem 2.4. Assume that $\lambda_B = \alpha_{n-1}$. Inspecting the formulas for the partitions α_{n-1} and taking into account the assumption $\lambda \neq \alpha_n, \beta_n, \gamma_n$, we see that B must be of content 0 which contradicts the assumption $i \neq 0$.

Assume finally that $\lambda_B \notin \mathcal{RP}_p(n-1)$. In this case λ is of the form $\lambda = (*, k+p, k)$, and A is in the second row from the bottom, i.e., $\lambda_A = (*, k+p-1, k)$. Since $\lambda_A \in \mathrm{JS}(i)$, B should be the only normal node of λ_A. In particular the node C immediately to the left of A should not be normal in λ_A. It follows that $k = (p+1)/2$ and $i = \ell$.

Note that D^λ has a weight of the form

$$(i_1, \dots, i_{n-3}, \ell-1, \ell, \ell)$$

since $\varepsilon_\ell(\lambda) = 2$. By [Kleshchev 2005, Lemma 20.4.2 and Lemma 22.3.8],

$$(i_1, \dots, i_{n-3}, \ell, \ell-1, \ell)$$

is also a weight of D^λ. Therefore $e_{\ell-1}(e_\ell(D^\lambda)) \neq 0$. Since $e_{\ell-1}(D^{\tilde{e}_\ell\lambda}) = 0$, this shows that there is a composition factor D^μ of $e_\ell(D^\lambda)$ not isomorphic to $D^{\tilde{e}_\ell\lambda}$, and containing the weight $(i_1, \dots, i_{n-3}, \ell, \ell-1)$.

If $\mu = \alpha_{n-1}$ for all such composition factors, then it follows that all the weights $(i_1, \dots, i_{n-3}, \ell, \ell-1)$ are the same and are equal to

$$(\mathrm{cont}_p\, 0, \mathrm{cont}_p\, 1, \dots, \mathrm{cont}_p(n-1)),$$

see Lemma 3.4. Hence the only weights appearing in D^λ are of the form

$$(\mathrm{cont}_p\, 0, \mathrm{cont}_p\, 1, \dots, \mathrm{cont}_p(n-3), \ell-1, \ell, \ell)$$

or

$$(\mathrm{cont}_p\, 0, \mathrm{cont}_p\, 1, \dots, \mathrm{cont}_p(n-3), \ell, \ell-1, \ell).$$

Hence $D^{\alpha_{n-3}}$ is the only composition factor of $\mathrm{res}_{n-3} D^\lambda$. So $D^{\alpha_{n-2}}$ or $D^{\beta_{n-2}}$ are the only modules which appear in the socle of $\mathrm{res}_{n-2} D^\lambda$. Therefore $D^{\alpha_{n-1}}$, $D^{\beta_{n-1}}$ or $D^{\gamma_{n-1}}$ are the only modules which appear in the socle of $\mathrm{res}_{n-1} D^\lambda$, whence $\lambda \in \{\alpha_n, \beta_n, \gamma_n, \delta_n\}$, giving a contradiction. □

Lemma 5.6. *Let $\lambda \in \mathcal{RP}_p(n) \setminus \{\alpha_n, \beta_n, \gamma_n, \delta_n\}$. Suppose that $\varepsilon_i(\lambda) = 2$ for some $i \in I$, and $\varepsilon_k(\lambda) = 0$ for all $k \neq i$. Then $d_2(\lambda) \geq 3$.*

Proof. By Theorem 2.4, we have $2^{1-\delta_{i,0}} \cdot 2D^{\tilde{e}_i^2\lambda} \in \mathrm{res}_{n-2} D^\lambda$, so we may assume that $i = 0$. Then by Lemma 4.3, $\tilde{e}_0\lambda$ is not JS, and hence $\varepsilon_1(\tilde{e}_0\lambda) > 0$. So $D^{\tilde{e}_1\tilde{e}_0\lambda}$ is also a composition factor of $\mathrm{res}_{n-2} D^\lambda$. □

Lemma 5.7. *Let* $\lambda \in \mathcal{RP}_p(n) \setminus \{\alpha_n, \beta_n, \gamma_n\}$. *If* $d_2(\lambda) \le 2$, *then* $\lambda \in \mathrm{JS}(0)$, *or* $\lambda = \delta_n$ *and one of the conclusions* (i)–(viii) *of Lemma 3.9 holds.*

Proof. By Lemma 3.9, we may assume that $\lambda \neq \delta_n$. Further, it is clear that we may assume that $\sum_i \varepsilon_i(\lambda) \le 2$. If $\lambda \in \mathrm{JS}(i)$, then it follows from Lemmas 4.5, 4.6, 4.7, and 4.8 that $i = 0$. Finally, suppose that $\sum_i \varepsilon_i(\lambda) = 2$. These cases follow from Lemmas 5.2, 5.3, and 5.6. □

6. Proof of the Main Theorem

6A. ***Preliminary remarks.*** We denote

$$a_n := \dim A_n = 2^{\lfloor \frac{n-\kappa_n}{2} \rfloor},$$
$$b_n := \dim B_n = 2^{\lfloor \frac{n-1-\kappa_{n-1}}{2} \rfloor}(n-2-\kappa_n-2\kappa_{n-1}).$$

Define the following nondecreasing functions (of n):

$$f(n) := 2b_n = 2^{\lfloor \frac{n+1-\kappa_{n-1}}{2} \rfloor}(n-2-\kappa_n-2\kappa_{n-1}),$$
$$f^*(n) := \frac{4b_n}{2^{a(\beta_n)}} = 2^{\lfloor \frac{n+2-\kappa_{n-1}}{2} \rfloor}(n-2-\kappa_n-2\kappa_{n-1}).$$

Clearly, $f^*(n) \ge f(n)$.

We say that an irreducible $\mathcal{T}_n$-supermodule V is *large*, if it is neither a basic, nor a second basic module. We also denote by $d(p,n)$ the smallest dimension of large irreducible $\mathcal{T}_n$-supermodules. By Lemma 3.7(iii), the sequence $d(p,n)$ is nondecreasing for $n \ge 8$ (and p fixed).

Lemma 6.1. *The Main Theorem is equivalent to the following statement*: *If an irreducible* $\mathcal{T}_n$*-supermodule* V *satisfies at least one of the two conditions*

(i) $\dim V < f(n)$,

(ii) $\dim V < f^*(n)$ *and* $a(V) = 1$,

then V *is either* A_n *or* B_n.

Proof. Let W be a faithful irreducible $\mathbb{F}G$-module, where $G = \hat{\mathsf{A}}_n$ or $\hat{\mathsf{S}}_n$, and consider an irreducible $\mathcal{T}_n$-supermodule V such that W is an irreducible constituent of V considered as an $\mathbb{F}G$-module. If $G = \hat{\mathsf{A}}_n$, then $\dim V = 2(\dim W)$, and the bound stated in the Main Theorem for $G = \hat{\mathsf{A}}_n$ is precisely $f(n)/2$. Consider the case $G = \hat{\mathsf{S}}_n$. Then $\dim V = 2^{a(V)}(\dim W)$, and the bound specified in the Main Theorem for $G = \hat{\mathsf{S}}_n$ is $f^*(n)/2$.

Assume the Main Theorem holds. If $\dim V$ satisfies (i), then taking $G = \hat{\mathsf{A}}_n$ we see that $\dim W < f(n)/2$ and so W is a basic or second basic representation. If V satisfies (ii), then taking $G = \hat{\mathsf{S}}_n$ we see that $\dim W < f^*(n)/2$ and so W is again a basic or second basic representation. In either case, we can conclude that V is either A_n or B_n.

In the other direction, let $\dim W$ satisfy any of the bounds stated in the Main Theorem. Then $\dim V$ satisfies (i) if $G = \hat{\mathsf{A}}_n$ or if $G = \hat{\mathsf{S}}_n$ but $a(V) = 0$, and $\dim V$ satisfies (ii) if $G = \hat{\mathsf{S}}_n$ and $a(V) = 1$. By our assumption, V is either A_n or B_n, whence W is a basic or a second basic representation. □

Set $\pi_n := \lfloor (n - \kappa_n)/2 \rfloor$. Then $(n-2)/2 \le \pi_n \le n/2$, and so for $m \le n$ we have

$$(n-m)/2 - 1 \le \pi_n - \pi_m \le (n-m)/2 + 1.$$

In particular, $0 \le \pi_n - \pi_{n-1} \le 1$, and so the sequence $\{\pi_n\}_{n=1}^{\infty}$ is nondecreasing; also, $\pi_{n-1} - \pi_{n-3} \le 2$.

6B. ***Induction base:*** $\mathbf{11 \le n \le 15}$**.** We will prove the Main Theorem by induction on $n \ge 11$. First, we establish the induction base:

Lemma 6.2. *The statement of the Main Theorem holds true if* $12 \le n \le 15$, *or if* $n = 11$ *but* $(n, p, G) \ne (11, 3, \hat{\mathsf{A}}_{11})$.

Proof. If $11 \le n \le 13$ then one can use [Conway et al. 1985; Jansen et al. 1995; Breuer et al.] to verify the Main Theorem. Also observe that

$$d(p,13) = \begin{cases} 3456, & p = 0, 3, 7, \text{ or } > 13, \\ 2240, & p = 5, \\ 1664, & p = 11, \\ 2816, & p = 13. \end{cases} \tag{6-1}$$

Now assume that $n = 14$ or 15. By Lemma 6.1, it suffices to show that $\dim V \ge f^*(n)$ for any large irreducible $\mathcal{T}_n$-supermodule $V = D^\lambda$. By Lemma 3.7(iii), $\mathrm{res}_{13} V$ has a large composition factor, and so $\dim V \ge d(p, 13)$. Direct computation using (6-1) shows that $d(p, 13) \ge f^*(n)$, unless $n = 14$ and $p = 5, 11$, or $n = 15$ and $p = 5, 11, 13$. To treat these exceptions, we observe that

$$d(p,12) = \begin{cases} 1408, & p = 11 \text{ or } \ge 13, \\ 1344, & p = 5; \end{cases} \tag{6-2}$$

in particular, $3d(p, 12) > f^*(15)$. So we may assume that $d_2(V) \le 2$, $\dim V < f^*(n)$, and apply Lemma 5.7 to V. Moreover, since $d(p, 13) > f(14)$, we may also assume $a(V) = 1$ for $n = 14$. Furthermore, for $n = 15$ we may assume $V \notin \mathrm{JS}(0)$ as otherwise $\dim V \ge 3d(p, 12)$ by Lemma 4.11. Now we will rule out the remaining exceptions case by case.

• $(n, p) = (14, 11)$. Under this condition, γ_{14} does not exist, so either $\lambda = \delta_{14}$ or $V \in \mathrm{JS}(0)$. In the former case, by Lemma 3.9 we must have $\delta_{14} = (11, 2, 1)$ and

$$\dim V \geq 2(\dim D^{\gamma_{13}}) + \dim D^{\alpha_{13}} > 2 \cdot 1664 > 2 \cdot 1536 = f^*(14).$$

In the latter case, $\mathrm{res}_{13} V = D^{\mu}$ with $\mu \in \mathrm{JS}(1)$ and $a(D^{\mu}) = a(V) = 1$ by Lemma 4.5. It then follows that $\mathrm{res}_{12} V = 2W$ for some faithful irreducible $\mathcal{T}_{12}$-supermodule W. By our assumption,

$$1664 = d(p, 13) \leq \dim V = \dim D^{\mu} < f^*(14) = 3072,$$

and $\dim D^{\mu}$ is twice the dimension of some irreducible $\hat{\mathrm{A}}_{13}$-module. Inspecting [Breuer et al.] we see that $\dim D^{\mu} = 1664$, whence $\dim W = 832$. However, $\hat{\mathrm{A}}_{12}$ does not have any faithful irreducible representation of degree 416; see [Jansen et al. 1995].

• $(n, p) = (14, 5)$. Under this condition, δ_{14} does not exist, so either $\lambda = \gamma_{14}$ or $V \in \mathrm{JS}(0)$. In the former case, by Lemma 3.8 we have

$$\dim V \geq 2(\dim D^{\beta_{13}}) + \dim D^{\delta_{13}} > 2(2 \cdot 352 + 1120) > 2 \cdot 1536 = f^*(14).$$

In the latter case, as before we can write $\mathrm{res}_{13} V = D^{\mu}$ with $\mu \in \mathrm{JS}(1)$ and $a(D^{\mu}) = a(V) = 1$, and $\mathrm{res}_{12} V = 2W$ for some faithful irreducible $\mathcal{T}_{12}$-supermodule W. By our assumption,

$$2240 = d(p, 13) \leq \dim V = \dim D^{\mu} < f^*(14) = 3072.$$

Inspecting [Breuer et al.] we see that $\dim D^{\mu} \in \{2240, 2752\}$, so $\dim W \in \{1120, 1376\}$. However, $\hat{\mathrm{A}}_{12}$ does not have any faithful irreducible representation of degree 560 or 688; see [Jansen et al. 1995].

• $(n, p) = (15, 5)$. Under this condition γ_{15} does not exist, so we need to consider only $\lambda = \delta_{15}$. Now by Lemma 3.9 we have $\lambda = (7, 5, 3)$ and

$$\dim V \geq 2(\dim D^{\delta_{13}}) + 4(\dim D^{\beta_{13}}) > 6B_{13} = 4224 > 2 \cdot 1536 = f^*(15).$$

• $(n, p) = (15, 11)$. Here δ_{15} does not exist, so we may assume $\lambda = \gamma_{15}$. By Lemmas 3.7(iii) and 3.8 we have

$$\dim V \geq 4(\dim D^{\beta_{14}}) + d(p, 13) = 4736 > 2 \cdot 1664 = f^*(15).$$

• $(n, p) = (15, 13)$. By Lemma 3.9 we may assume $\lambda \neq \delta_{15}$ and so $\lambda = \gamma_{15}$. Now by Lemma 3.8 we have

$$\dim V \geq \dim D^{\beta_{14}} + \dim D^{\gamma_{14}} \geq B_{14} + d(p, 13) = 3456 > 2 \cdot 1664 = f^*(15). \quad \square$$

6C. ***The third basic representations D^{γ_n}.*** The following result will be fed into the inductive step in the proof of the Main Theorem:

Proposition 6.3. *Let $n \geq 12$ and $V = D^{\gamma_n}$. Assume in addition that the dimension of any large irreducible $\mathcal{T}_m$-supermodule is at least $f(m)$ whenever $12 \leq m \leq n-1$. Then* $\dim V \geq f^*(n)$. *If moreover V satisfies the additional conditions*

$$n \geq 15 \text{ is odd}, \quad p \nmid (n-1), \quad \text{and} \quad d_1(V) \geq 2, \tag{6-3}$$

then $\dim V \geq f^*(n+1)/2$.

Proof. We will proceed by induction on $n \geq 12$ according to the cases in Lemma 3.8.

(i) First we consider the case where $p = 0$ or $p > n$. Then $\gamma_n = (n-2, 2)$. By the dimension formula given in [Hoffman and Humphreys 1992] we have

$$\dim V = 2^{\lfloor \frac{n-3}{2} \rfloor}(n-1)(n-4).$$

In particular, $\dim V > 4b_n \geq f^*(n)$. Also, $\dim V > f^*(n+1)/2$ if $n \geq 15$ is odd.

(ii) Next assume that $n = p+1$. By Lemma 3.8(ii),

$$\dim D^{\gamma_n} \geq a_{n-1} + 2b_{n-1} = \frac{a_n}{2} + 2b_n. \tag{6-4}$$

Since $f^*(n) = 2b_n$ in this case, we get $\dim V > f^*(n)$.

(iii) Assume we are in the case (iii) of Lemma 3.8; in particular $n \geq 13$. In this case we have

$$\frac{\dim D^{\gamma_n}}{2^{\sigma(n)}} \geq 2b_{n-1} + \dim D^{\delta_{n-1}} \geq 4b_{n-1} = 4b_n. \tag{6-5}$$

It follows that $\dim V \geq 4b_n = 2f(n) \geq f^*(n)$.

(iv) Consider the case (iv) of Lemma 3.8. If $n = 12$, then $p = 5$, and $\dim V \geq 1344 > 1280 = f^*(12)$. Assume now that $n \geq 13$ and $a \geq 2$. By Lemma 3.8(iv) and (6-5),

$$\dim V \geq 2^{\sigma(n-1)}b_{n-1} + \dim D^{\gamma_{n-1}} \geq 2^{\sigma(n-1)} \cdot 5b_{n-1} = 2^{\lfloor \frac{n-3}{2} \rfloor + \sigma(n-1)}(5n-25). \tag{6-6}$$

On the other hand,

$$f^*(n) = 2^{\lfloor \frac{n+2}{2} \rfloor}(n-2) = 2^{\lfloor \frac{n+1}{2} \rfloor + \sigma(n-1)}(n-2).$$

Hence $\dim(V) \geq f^*(n)$ if $n \geq 17$. If $n = 16$, then $p = 7$. In this case, instead of (6-5) we use the stronger estimate

$$\frac{\dim D^{\gamma_{15}}}{2^{\sigma(15)}} \geq 2b_{14} + \dim D^{\delta_{14}} \geq 2b_{14} + d(p, 13) = 4864,$$

yielding $\dim V \geq 11136 > 7168 = f^*(16)$. If $n = 14$, then $p = 3$, and $\dim V \geq d(p, 13) = 3456 > 3072 = f^*(14)$. The cases $n = 13, 15$ cannot occur since $n = ap + 2$ with $a \geq 2$. If moreover V satisfies (6-3), then since $\mathrm{res}_{n-1} V$ contains an additional large composition factor in addition to $D^{\gamma_{n-1}}$, instead of (6-6) we now have

$$\begin{aligned}\dim V &\geq 2^{\sigma(n-1)} b_{n-1} + \dim D^{\gamma_{n-1}} + f(n-1) \\ &= 2^{(n-3)/2}(7n - 35) > 2^{(n+1)/2}(n-1) \geq f^*(n+1)/2.\end{aligned}$$

Next suppose that $n = p + 2 \geq 15$. By Lemma 3.7(iii), $\mathrm{res}_{n-2} D^{\gamma_{n-1}}$ must contain a large composition factor Y, and $\dim Y \geq f(n-2) = 2b_{n-2}$ by our assumption. It follows by Lemma 3.8(ii) that $\dim D^{\gamma_{n-1}} \geq a_{n-2} + 4b_{n-2}$. Applying Lemma 3.8(iv), we obtain

$$\dim V \geq b_{n-1} + \dim D^{\gamma_{n-1}} \geq b_{n-1} + (a_{n-2} + 4b_{n-2}) = 2^{\frac{n-3}{2}}(5n - 24). \quad (6\text{-}7)$$

Since $f^*(n) = 2^{(n+1)/2} \cdot (n-2)$, we are done if $n \geq 16$. If $n = 15$, then $p = 13$ and by (6-1) we have

$$\dim V \geq b_{14} + \dim D^{\gamma_{14}} \geq b_{14} + d(p, 13) = 3456 > 3328 = f^*(15).$$

If $n = 13$, then $p = 11$ and $\dim V \geq d(p, 13) = 1664 > 1408 = f^*(13)$ by (6-1). If moreover V satisfies (6-3), then since $\mathrm{res}_{n-1} V$ contains an additional large composition factor in addition to $D^{\gamma_{n-1}}$, instead of (6-7) we now have

$$\begin{aligned}\dim V &\geq b_{n-1} + \dim D^{\gamma_{n-1}} + f(n-1) = 2^{(n-3)/2}(7n - 34) \\ &> 2^{(n+1)/2}(n-1) \geq f^*(n+1)/2.\end{aligned}$$

(v) Now we consider the case $n = p + 4$ and $p \geq 11$. Again by Lemma 3.7(iii), $\mathrm{res}_{n-1} D^{\gamma_n}$ must contain a large composition factor X, and $\dim X \geq f(n-1)$ by our assumption. In fact, since γ_n has exactly one good node (a 1-good node) with two 1-normal nodes and $a(\gamma_n) = 1$, by Theorem 2.4 we see that $\mathrm{res}_{n-1} D^{\gamma_n} = 2W$, where the $\mathcal{T}_{n-1}$-supermodule W has $D^{\beta_{n-1}}$ as head and socle and X as one of the composition factors in between. Thus X has multiplicity at least 2 in $\mathrm{res}_{n-1} D^{\gamma_{n-1}}$. Hence by Lemma 3.8(v) we have

$$\dim D^{\gamma_n} \geq 4b_{n-1} + 2(\dim X) \geq 8b_{n-1} = 2^{\frac{n-3}{2}}(8n - 24). \quad (6\text{-}8)$$

Since $f^*(n) = 2^{(n+1)/2}(n-2)$ and $f^*(n+1) \leq 2^{(n+3)/2}(n-1)$ in this case, we get $\dim V > \max\{f^*(n), f^*(n+1)/2\}$.

(vi) Assume we are in the case (vi) of Lemma 3.8; in particular, $n \geq 14$. Suppose first that $2 \mid n$. By Theorem 3.6, $D^{\gamma_{n-2}}$ appears in $\mathrm{soc}(\mathrm{res}_{n-2} D^{\delta_{n-1}})$; furthermore, $d_1(D^{\delta_{n-1}}) \geq 2$ by Lemma 3.9. Thus $\mathrm{res}_{n-2} D^{\delta_{n-1}}$ has at least two large composition factors: $D^{\gamma_{n-2}}$ and another one, say, Y. According to (iv), $\dim D^{\gamma_{n-2}} \geq f^*(n-2)$.

On the other hand, $\dim Y \geq f(n-2)$ by our assumption. It follows that $\dim D^{\delta_{n-1}} \geq f^*(n-2) + f(n-2)$. Hence Lemma 3.8(vi) implies

$$\dim D^{\gamma_n} \geq 2b_{n-1} + \dim D^{\delta_{n-1}} \geq 2b_{n-1} + f^*(n-2) + f(n-2) = 2^{\frac{n-2}{2}}(5n-18).$$

Since $f^*(n) = 2^{(n+2)/2}(n-2)$, we obtain $\dim V > f^*(n)$.

Now let n be odd. Then Lemma 3.8(vi) implies that

$$\dim D^{\gamma_n} \geq 4b_{n-1} + 2(\dim D^{\delta_{n-1}}) \geq 8b_{n-1} = 2^{\frac{n-3}{2}}(8n-24). \tag{6-9}$$

Also, $f^*(n) = 2^{(n+1)/2}(n-2)$ and $f^*(n+1) \leq 2^{(n+3)/2}(n-1)$ in this case, so $\dim V > \max\{f^*(n), f^*(n+1)/2\}$.

(vii) Finally, we consider the case (vii) of Lemma 3.8; in particular, $p \geq 7$ and $n \geq 12$. If $n = 12$, then $p = 7$, and so by [Breuer et al.] we have $\dim V \geq 1408 > 1280 = f^*(12)$. Now we may assume that $n \geq 13$.

Suppose in addition that n is odd, so that $\sigma(a+b) = 1$. According to (v) and (vi), $\dim D^{\gamma_{n-1}} \geq f^*(n-1) = 4b_{n-1}$. Hence by Lemma 3.8(vii) we have

$$\dim D^{\gamma_n} \geq 2(b_{n-1} + \dim D^{\gamma_{n-1}}) \geq 10b_{n-1} = 2^{\frac{n-3}{2}}(10n-30). \tag{6-10}$$

Since $f^*(n) = 2^{(n+1)/2}(n-2)$ and $f^*(n+1) \leq 2^{(n+3)/2}(n-1)$, we are done.

Assume now that n is even. If $b = 5$, then $\dim D^{\gamma_{n-1}} \geq 8b_{n-2}$ by (6-8) and (6-9). On the other hand, if $b > 5$, then $\dim D^{\gamma_{n-1}} \geq 10b_{n-2}$ by (6-10). Thus in either case we have $\dim D^{\gamma_{n-1}} \geq 8b_{n-2}$. Now Lemma 3.8(vii) implies that

$$\dim V \geq b_{n-1} + \dim D^{\gamma_{n-1}} \geq b_{n-1} + 8b_{n-2} = 2^{\frac{n-4}{2}}(10n-38).$$

Since $f^*(n) = 2^{(n+2)/2}(n-2)$, we again have $\dim(V) > f^*(n)$. □

Proposition 6.4. *Let* $n \geq 14$, *and let* $V = D^\lambda$ *be a large irreducible* $\mathcal{T}_n$*-supermodule. Assume in addition that the dimension of any large irreducible* $\mathcal{T}_m$*-supermodule is at least* $f(m)$ *whenever* $12 \leq m \leq n-1$. *Then one of the following holds.*

(i) $d_2(\lambda) \geq 3$.

(ii) $\lambda \in \mathrm{JS}(0)$.

(iii) $\lambda = \gamma_n$, $\lambda \notin \mathrm{JS}$, *and* $\dim V \geq f^*(n)$.

(iv) $\lambda = \delta_n$, $n \equiv 0, 3, 6 \pmod{p}$, *one of the conclusions* (*iv*)–(*viii*) *of Lemma 3.9 holds*, *and* $\dim V \geq f^*(n)$.

Proof. (1) Assume that $\lambda \notin \mathrm{JS}(0)$ and $d_2(\lambda) \leq 2$. Then we can apply Lemma 5.7. If $\lambda = \gamma_n$, then $\lambda \notin \mathrm{JS}$ (see e.g. Lemma 3.8), and $\dim V \geq f^*(n)$ by Proposition 6.3. We may now assume that $\lambda = \delta_n$, in particular, one of the cases (iv)–(viii) of Lemma 3.9 occurs. By Proposition 6.3 and our assumptions, $\dim D^{\gamma_m} \geq f^*(m)$ for $m = n-1$ and $m = n-2$.

(2) Here we consider the case $n = p+3$ (so that $p \geq 11$). By Lemma 3.7(iii), $\mathrm{res}_{n-3}\, D^{\gamma_{n-2}}$ must have some large composition factor Z, and $\dim Z \geq f(n-3) = 2b_{n-3}$ by the assumptions. Applying items (ii) and (iv) of Lemma 3.8 we get

$$\dim D^{\gamma_{n-2}} \geq a_{n-3} + 2b_{n-3} + \dim Z,\ \dim D^{\gamma_{n-1}} \geq b_{n-2} + \dim D^{\gamma_{n-2}}. \quad (6\text{-}11)$$

Together with Lemma 3.9(iv), this implies

$$\dim V \geq a_{n-1} + 2(\dim D^{\gamma_{n-1}}) \geq a_{n-1} + 2(a_{n-3} + 4b_{n-3} + b_{n-2}) = 2^{\frac{n-2}{2}}(5n-28).$$

Since $f^*(n) = 2^{(n+2)/2}(n-2)$, we are done if $n \geq 20$. Suppose that $n \leq 19$, so that $n = p+3 = 16$ or $n = 14$. If $n = 16$, then $\dim Z \geq d(p, 13) = 2816$, and so (6-11) implies

$$\dim D^{\gamma_{14}} \geq 4160,\ \dim D^{\gamma_{15}} \geq 4800.$$

It follows that $\dim V \geq 9728 > 7168 = f^*(16)$. If $n = 14$, then $\dim D^{\gamma_{13}} \geq d(p, 13) = 1664$, so

$$\dim V \geq a_{13} + 2(\dim D^{\gamma_{13}}) = 3392 > 3072 = f^*(14).$$

(3) Next suppose that $n = mp+3$ with $p > 3$ and $m \geq 2$. By items (iii) and (iv) Lemma 3.8 we have

$$\dim D^{\gamma_{n-2}} \geq 2^{\sigma(n)}(2b_{n-3} + \dim D^{\delta_{n-3}}),\ \dim D^{\gamma_{n-1}} \geq 2^{\sigma(n)} b_{n-2} + \dim D^{\gamma_{n-2}}. \quad (6\text{-}12)$$

By our assumptions, $\dim D^{\delta_{n-3}} \geq f(n-3) = 2b_{n-3}$. Together with Lemma 3.9(v), this implies

$$\dim V \geq 2(\dim D^{\gamma_{n-1}}) \geq 2^{1+\sigma(n)}(b_{n-2} + 4b_{n-3}) = 2^{\sigma(n)+\lfloor\frac{n-2}{2}\rfloor}(5n-30). \quad (6\text{-}13)$$

Since $f^*(n) = 2^{\lfloor (n-2)/2\rfloor}(4n-8)$, we are done unless $2 \mid n \leq 20$. In the remaining case, $(n, p) = (18, 5)$. Then $d_1(\delta_{15}) \geq 2$ by Lemma 3.9, and so $\dim D^{\delta_{15}} \geq 2d(p, 13) = 4480$. Thus (6-12) implies that

$$\dim D^{\gamma_{16}} \geq 7552,\ \dim D^{\gamma_{17}} \geq 9088,$$

whence $\dim V \geq 18176 > 16384 = f^*(18)$.

(4) If $p > 5$ and $n = p+6$, then since $\dim D^{\gamma_{n-2}} \geq f(n-2) = 2b_{n-2}$, by Lemma 3.9(vi) we have

$$\dim V \geq 6b_{n-2} = 2^{(n-3)/2}(6n-24) > 2^{(n+1)/2} \cdot (n-2) = f^*(n). \quad (6\text{-}14)$$

If $p = 3 \mid n$, then since $\dim D^{\gamma_{n-1}} \geq f^*(n-1)$, by Lemma 3.9(vii) we have

$$\dim V \geq 2f^*(n-1) \geq 2^{\lfloor\frac{n+1}{2}\rfloor}(2n-6) \geq 2^{\lfloor\frac{n+2}{2}\rfloor}(n-3) = f^*(n).$$

If $5 < p \mid n$, then using $\dim D^{\gamma_{n-2}} \geq f^*(n-2)$ and Lemma 3.9(viii) we obtain

$$\dim V \geq 2b_{n-2} + 2f^*(n-2) \geq 2^{\lfloor \frac{n-2}{2} \rfloor}(5n-20) > 2^{\lfloor \frac{n+2}{2} \rfloor}(n-3) = f^*(n).$$

If $p = 5 \mid n$ and n is odd, then Lemma 3.9(viii) and our assumptions imply

$$\dim V \geq 4b_{n-2} + 2f(n-2) = 2^{\frac{n+3}{2}}(n-4) > 2^{\frac{n+1}{2}}(n-3) = f^*(n).$$

Finally, assume that $p = 5 \mid n$ and $n \geq 20$ is even. By Lemma 3.9, $d_1(\delta_{n-2}) \geq 2$, whence $\dim D^{\delta_{n-2}} \geq 2f(n-3)$ by our assumptions. Hence Lemma 3.9(viii) yields

$$\begin{aligned} \dim V &\geq 4b_{n-2} + 2(\dim D^{\delta_{n-2}}) \geq 4b_{n-2} + 4f(n-3) \\ &= 2^{n/2}(3n-14) > 2^{(n+2)/2}(n-3) = f^*(n). \end{aligned}$$ □

6D. ***The case*** $V \in$ **JS.**

Lemma 6.5. *If* $n \geq 23$ *and* $(n, p) \neq (24, 17)$, *then* $f^*(n) \leq 24f(n-6)$.

Proof. First assume that $p \mid (n-7)$. Then $f(n-6) = 2^{\lfloor (n-6)/2 \rfloor}(n-10)$. In particular, $f^*(n) \leq 24f(n-6)$ if $n \geq 26$. If $n = 25$, then $p = 3$, $f^*(25) = 2^{13} \cdot 21 < 24 \cdot (2^9 \cdot 15) = 24f(19)$. If $n = 24$, then $p = 17$. If $n = 23$, then $p > 2$ cannot divide $n-7$.

Next assume that $p \nmid (n-7)$. Then $f(n-6) \geq 2^{\lfloor (n-5)/2 \rfloor}(n-9)$, and so $f^*(n) \leq 24f(n-6)$ if $n \geq 23$. □

Proposition 6.6. *Let* $n \geq 16$ *and* $V \in \mathrm{JS}(0)$ *be a large irreducible* $\mathcal{T}_n$*-supermodule. Assume in addition that, if* $m := n-6 \geq 12$, *then the dimension of any large irreducible* $\mathcal{T}_m$*-supermodule is at least* $f(m)$. *Then* $\dim V \geq f^*(n)$.

Proof. Using the fact that γ_n is never in JS(0) (see Lemma 3.8, for instance), we may assume that $V = D^\lambda$ and $\lambda \neq \gamma_n$.

(i) First we claim that if $p = 17$ then the dimension of any large irreducible $\mathcal{T}_{16}$-supermodule $Y = D^\mu$ is at least $3d(p, 13) = 10368$. This is certainly true if $d_j(Y) \geq 3$ for any $j \leq 3$. Otherwise $d_2(Y) \leq 2$, and so by Lemma 5.7 either $\mu \in \mathrm{JS}(0)$, or $\mu = \delta_{16}, \gamma_{16}$. In the former case $d_3(Y) \geq 3$ by Lemma 4.11. Also $d_2(\delta_{16}) \geq 3$ by Lemma 3.9. So we may assume $\mu = \gamma_{16}$. Applying Lemma 3.8(i) three times, we see that

$$\mathrm{res}_{13} Y \cong 2D^{\gamma_{13}} + 2b_{13} + 2b_{14} + b_{15}.$$

Since $\dim D^{\gamma_{13}} \geq d(p, 13)$, we also have $\dim Y > 3d(p, 13)$ in this case.

By Lemma 3.7(iii), any large irreducible $\mathcal{T}_{18}$-supermodule X has dimension at least 10368.

(ii) Now we consider the case $n \geq 23$ and apply Proposition 4.10 to λ. In particular, $d_6(\lambda) \geq 20$; more precisely, either $d_6(\lambda) \geq 24$, or

$$\dim V \geq 20 f(n-6) + 20 b_{n-6} + 4 a_{n-6} > 30 f(n-6).$$

Thus we always have $\dim V \geq 24 f(n-6)$. If furthermore $(n, p) \neq (24, 17)$, then the last inequality implies $\dim V \geq f^*(n)$ by Lemma 6.5. Assume now that $(n, p) = (24, 17)$. Then by the result of (i) we have

$$\dim V \geq 20 \cdot 10368 > 2^{13} \cdot 22 = f^*(24).$$

(iii) The rest of the proof is to handle the cases $16 \leq n \leq 22$.

• Consider the case $n = 16, 17$. First suppose that $p \neq 5, 11$. By Lemma 4.11, $d_3(\lambda) \geq 3$, hence

$$\dim V \geq 3d(p, 13) \geq 8448 > 7680 \geq f^*(n)$$

by (6-1). If $(n, p) = (16, 5)$, then $d_2(\lambda) \geq 2$ by Lemma 4.11, whence

$$\dim V \geq 2d(p, 13) \geq 4480 > 3072 = f^*(16)$$

by (6-1). On the other hand, the proof of Proposition 4.10 shows that if $(n, p) = (16, 11)$ then λ can be only $(6, 4, 3, 2, 1)$ which however does not belong to JS(0). If $n = 17$ and $p = 5$ or $p = 11$, then $d_6(\lambda) \geq 24$ by Proposition 4.10, whence

$$\dim V \geq 24d(p, 11) \geq 24 \cdot 864 > 7680 = f^*(17).$$

• Let $n = 18$. By Proposition 4.10, $d_6(\lambda) \geq 24$ if $p \neq 5$ and $d_6(\lambda) \geq 20$ if $p = 5$. Now if $p \neq 3$, then

$$\dim V \geq 20d(p, 12) \geq 20 \cdot 1344 > 16384 \geq f^*(18).$$

If $p = 3$, then

$$\dim V \geq 24d(p, 12) = 24 \cdot 640 = 15360 = f^*(18).$$

• Suppose $19 \leq n \leq 21$. By Proposition 4.10, $d_6(\lambda) \geq 24$ if $(n, p) \neq (20, 17)$ and $d_6(\lambda) \geq 20$ otherwise. Now if $(n, p) \neq (20, 17)$, then

$$\dim V \geq 24d(p, 13) \geq 24 \cdot 1664 > 38912 \geq f^*(n).$$

If $(n, p) = (20, 17)$, then

$$\dim V \geq 20d(p, 13) = 20 \cdot 3456 > 36864 = f^*(20).$$

• Finally, let $n = 22$. By Proposition 4.10, $d_6(\lambda) \geq 24$ if $p \neq 19$ and $d_6(\lambda) \geq 20$ if $p = 19$. By the assumptions, the dimension of any large irreducible $\mathcal{T}_{16}$-module Y is at least $f(16) = 3584$ if $p \neq 5$. We claim that $\dim Y > 3584$ also for $p = 5$.

(Indeed, by Lemmas 5.7, 4.11, and 3.9, either $d_j(Y) \geq 2$ for some $j \in \{2,3\}$, or $Y \cong D^{\gamma_{16}}$. In the former case, $\dim Y \geq 2d(p,13) = 4480$. In the latter case, by p. (iii) of the proof of Proposition 6.3, $\dim Y \geq 4b_{15} = 6144$.) Now if $p \neq 19$, then

$$\dim V \geq 24 \cdot 3584 > 81920 \geq f^*(22).$$

If $p = 19$, then by Proposition 4.10 we have

$$\dim V \geq \min\{20f(16) + 20b_{16}, 24f(16)\} = 24f(16) = 24 \cdot 3584 > f^*(22). \quad \square$$

Proposition 6.7. *Let $n \geq 16$ and V be a large irreducible $\mathcal{T}_n$-supermodule. Assume that*:

(i) $\mathrm{res}_{n-1} V$ *is irreducible but* $V \notin \mathrm{JS}(0)$;

(ii) *the dimension of any large irreducible $\mathcal{T}_m$-supermodule is at least $f(m)$ for $12 \leq m \leq n-1$.*

Then $a(V) = 0$ and $\dim V \geq f(n)$.

Proof. The assumptions in (i) imply that $V \in \mathrm{JS}(i)$ for some $i > 0$ and that $a(V) = 0$. By Proposition 6.4 we may assume that $d_2(V) \geq 3$ (as otherwise $\dim V \geq f^*(n)$); i.e., $\mathrm{res}_{n-2} V$ contains at least three large composition factors W_j, $1 \leq j \leq 3$. Applying the hypothesis of (ii) to $m = n-2$, we get $\dim W_j \geq f(n-2)$ and so $\dim V \geq 3f(n-2)$. Assume in addition that $\pi_{n-1} - \pi_{n-3} \leq 1$. Then

$$3f(n-2) \geq 2^{\pi_{n-3}}(6n-36) \geq 2^{\pi_{n-1}-1}(6n-36) \geq 2^{\pi_{n-1}+1} \cdot (n-2) \geq f(n),$$

and we are done.

Next we consider the case $(n,p) = (17,7)$. Then $\mathrm{res}_{13} W_j$ contains a large composition factor. Hence, by (6-1) we have $\dim W_j \geq d(p,13) = 3456$, whence $\dim V \geq 3 \cdot 3456 > 7680 = f(17)$, and we are done again.

So we may assume that $\pi_{n-1} - \pi_{n-3} \geq 2$; equivalently, n is odd and $p \mid (n-3)$. Since we have already considered the case $(n,p) = (17,7)$, we may assume that $n \geq 21$. It suffices to show that $\dim W_j \geq f(n)/3$ for $1 \leq j \leq 3$. There are the following four possibilities for W_j.

• $W_j \cong D^{\gamma_{n-2}}$. By Proposition 6.3 we have

$$\dim W_j \geq f^*(n-2) = 2^{\frac{n-1}{2}}(n-6) > 2^{\frac{n+1}{2}}(n-2)/3 = f(n)/3.$$

• $\mathrm{res}_{n-3} W_j$ is reducible but $W_j \not\cong D^{\gamma_{n-2}}$. Since W_j is large, it must have a large composition factor by Lemma 3.7(iii); furthermore, $\mathrm{res}_{n-3} W_j$ can contain neither A_{n-3} nor B_{n-3} in its socle. It follows that $d_1(W_j) \geq 2$, and so, applying the hypothesis of (ii) to $m = n-3$ we get

$$\dim W_j \geq 2f(n-3) = 2^{\frac{n-1}{2}}(n-6) > 2^{\frac{n+1}{2}}(n-2)/3 = f(n)/3.$$

• $W_j \in \mathrm{JS}(0)$. Applying Proposition 4.10 to W_j and the hypothesis of (ii) to $m = n-8$ we get

$$\dim W_j \geq 24 f(n-8) \geq 24 \cdot 2^{\frac{n-9}{2}}(n-12) \geq 2^{\frac{n+1}{2}}(n-2)/3 = f(n)/3.$$

• $W_j \in \mathrm{JS}(k)$ for some $k > 0$. Then $d_2(W_j) \geq 3$ by Proposition 6.4 (note that the conclusion (iv) of Proposition 6.4 cannot hold since $p \mid (n-3)$). Applying the hypothesis of (ii) to $m = n-4$ we get

$$\dim W_j \geq 3 f(n-4) = 3 \cdot 2^{\frac{n-3}{2}}(n-6) \geq 2^{\frac{n+1}{2}}(n-2)/3 = f(n)/3.$$

The proposition is proved. □

Proposition 6.8. *Let $n \geq 16$ and V be a large irreducible $\mathcal{T}_n$-supermodule. Assume that*:

(i) *$V \in \mathrm{JS}(i)$ for some $i \neq 0$ and $a(V) = 1$;*

(ii) *for $12 \leq m \leq n-1$, the dimension of any large irreducible $\mathcal{T}_m$-supermodule X is at least $f(m)$ if $a(X) = 0$, and at least $f^*(m)$ if $a(X) = 1$.*

Then $\dim V \geq f^*(n)$.

Proof. (1) The assumptions imply that $\mathrm{res}_{n-1} V = 2U$, where U is a large irreducible $\mathcal{T}_{n-1}$-supermodule with $a(U) = 0$. By Proposition 6.4, $d_1(U) = d_2(V)/2 > 1$ (as otherwise $\dim V \geq f^*(n)$); in particular, $U \notin \mathrm{JS}(0)$. Applying Proposition 6.4 to U we see that either $U \cong D^{\gamma_{n-1}}$, or $p \mid (n-1)(n-4)(n-7)$ and $U \cong D^{\delta_{n-1}}$, or $d_2(U) \geq 3$.

(2) Assume we are in the first case: $U \cong D^{\gamma_{n-1}}$. Then by Theorem 3.6, either $V \cong D^{\gamma_n}$ or $V \cong D^{\delta_n}$. The first possibility is ruled out since $V \in \mathrm{JS}$. If the second possibility occurs, then Lemma 4.1 implies that $n = mp$ for some $m \geq 2$, $p > 3$, and $\delta_n = (p+2, p^{m-2}, p-2)$, which means that δ_n satisfies the conclusion (viii) of Lemma 3.9. In this case, part (4) of the proof of Proposition 6.4 shows that $\dim V \geq f^*(n)$.

(3) Consider the second case: $U \cong D^{\delta_{n-1}}$ but $d_2(U) \leq 2$. Then $\dim U \geq f^*(n-1)$ by Proposition 6.4. Now if $p \mid (n-1)$, then

$$\dim V \geq 2 f^*(n-1) = 2^{\lfloor (n+3)/2 \rfloor}(n-4) > 2^{\lfloor (n+1)/2 \rfloor}(n-4) = f^*(n).$$

Likewise, if $5 \leq p \mid (n-4)$ and n is odd then

$$\dim V \geq 2 f^*(n-1) = 2^{\frac{n+3}{2}}(n-3) > 2^{\frac{n+1}{2}}(n-2) = f^*(n).$$

Suppose that $5 \leq p \mid (n-4)$ and $2 \mid n$; in particular, we are in the case (v) of Lemma 3.9. Then (6-13) implies that

$$\dim V \geq 2^{\frac{n}{2}}(5n-35) > 2^{\frac{n+2}{2}}(n-2) = f^*(n).$$

Suppose that $n = p + 7 \geq 16$; in particular, we are in the case (vi) of Lemma 3.9. Then (6-14) implies that

$$\dim V \geq 2^{\frac{n}{2}}(3n - 15) > 2^{\frac{n+2}{2}}(n-2) = f^*(n).$$

(4) From now on we may assume that $d_2(U) \geq 3$ and so $\mathrm{res}_{n-3} U$ contains at least three large composition factors T_j, $1 \leq j \leq 3$. Applying the hypothesis of (ii) to $m = n - 3$, we get $\dim T_j \geq f(n-3)$ and so $\dim V \geq 6f(n-3)$. Assume in addition that either n is odd, or $2 \mid n \geq 18$ and $p \nmid (n-4)$. Then

$$\dim V \geq 6f(n-3) \geq 6 \cdot 2^{\lfloor \frac{n-2}{2} \rfloor}(n-7) \geq 2^{\lfloor \frac{n+2}{2} \rfloor}(n-2) \geq f^*(n).$$

If $n = 16$, then $\dim T_j \geq d(p, 13) \geq 1664$ by (6-1), whence

$$\dim V \geq 6 \cdot 1664 = 9984 > 7168 \geq f^*(16).$$

If $n \in \{18, 20\}$ and $p \mid (n-4)$, then $(n, p) = (18, 7)$, in which case $\dim T_j \geq d(p, 13) \geq 3456$ by (6-1) and so

$$\dim V \geq 6 \cdot 3456 = 20736 > 16384 = f^*(18).$$

(5) It remains to consider the case where $n \geq 22$ is even, $p \mid (n-4)$, and $\dim U < f^*(n)/2$. Recall that U is large, $a(U) = 0$, $d_1(U) \geq 2$ and $U \not\cong D^{\gamma_{n-1}}$. Thus $\mathrm{res}_{n-2} U$ cannot contain A_{n-2} or B_{n-2} in its socle. Also, since

$$f(n-2) = 2^{(n-2)/2}(n-4) > f^*(n)/5,$$

we have that $\dim U < (5/2)f(n-2)$ and so $d_1(U) \leq 2$ by the hypothesis in (ii) for $m = n-2$. It follows that $d_1(U) = 2$, i.e., $\mathrm{res}_{n-2} U$ contains exactly two large composition factors W_j, $j = 1, 2$. Assume in addition that some W_j has $a(W_j) = 1$. By the hypothesis in (ii) for $m = n-2$, in this case we have

$$\dim U \geq f(n-2) + f^*(n-2) = 2^{(n-2)/2}(3n-12) > 2^{n/2}(n-2) \geq f^*(n)/2,$$

and we are done again.

We conclude by Theorem 2.4 that $\mathrm{res}_{n-2} U = e_0(U)$ is reducible, with a large irreducible $\mathcal{T}_{n-2}$-supermodule $W \cong W_1 \cong W_2$ as its socle and head. Furthermore, if $p = 3$, then by the hypothesis in (ii) for $m = n-1$ we have

$$\dim U \geq f(n-1) = 2^{(n-2)/2}(n-4) = f^*(n)/2.$$

So we may assume $p > 3$. We will distinguish the following three subcases according to Proposition 6.4 applied to W (note that $n - 2 \equiv 2 (\bmod\ p)$ and so the conclusion (iv) of Proposition 6.4 cannot hold) and show that $\dim W \geq f^*(n)/4$, which contradicts the assumption $\dim U < f^*(n)/2$.

• $d_2(W) \geq 3$. Applying the hypothesis of (ii) to $m = n-4$ we get

$$\dim W \geq 3f(n-4) = 3 \cdot 2^{(n-4)/2}(n-7) > 2^{(n-2)/2}(n-2) = f^*(n)/4$$

as $n \geq 22$, and so we are done.

• $W \in \mathrm{JS}(0)$. Since $n \geq 22$, we can apply Proposition 4.10 to W and the hypothesis of (ii) to $m = n-8$ to get

$$\dim W \geq 24f(n-8) \geq 24 \cdot 2^{(n-8)/2}(n-12) > 2^{(n-2)/2}(n-2) = f^*(n)/4.$$

• $W \cong D^{\gamma_{n-2}}$. Recall that $2p \mid (n-4)$. Hence by Proposition 6.3 we have

$$\dim W \geq f^*(n-2) = 2^{n/2}(n-4) > 2^{(n-2)/2}(n-2) = f^*(n)/4. \qquad \square$$

6E. ***Inductive step of the proof of the main theorem.*** As a consequence of the results proved in Sections 6A–6D we obtain the following:

Corollary 6.9. *For the induction step of the proof of the Main Theorem, it suffices to prove that, if* $V = D^\lambda$ *is any irreducible* $\mathcal{T}_n$*-supermodule satisfying all the following conditions*

(i) $n \geq 16$, $\lambda \neq \alpha_n, \beta_n, \gamma_n$;

(ii) $V \notin \mathrm{JS}$, $d_1(V) \geq 2$, $d_2(V) \geq 3$, *and all the simple summands of the head and the socle of* $\mathrm{res}_{n-1} V$ *are large*

then $\dim V \geq f(n)$, *and, furthermore,* $\dim V \geq f^*(n)$ *when* $a(V) = 1$.

Proof. By the induction hypothesis, the dimension of any irreducible $\mathcal{T}_m$-supermodule X is at least $f(m)$ if $a(X) = 0$ and at least $f^*(m)$ if $a(X) = 1$ for $12 \leq m \leq n-1$. By Lemma 6.2 and Propositions 6.3, 6.6 we may now assume that $n \geq 16$, $\lambda \neq \alpha_n, \beta_n, \gamma_n$ and $V \notin \mathrm{JS}(0)$. Now, if $\mathrm{res}_{n-1} V$ is irreducible, then $V \in \mathrm{JS}(i)$ for some $i > 0$ and $a(V) = 0$, in which case we also have $\dim V \geq f(n)$ by Proposition 6.7. The case $V \in \mathrm{JS}(i)$ with $a(V) = 1$ is treated in Proposition 6.8. So we may assume that $V \notin \mathrm{JS}$. Since $\lambda \neq \alpha_n, \beta_n, \gamma_n$, $\mathrm{res}_{n-1} V$ cannot contain A_{n-1} or B_{n-1} in the socle or in the head. It now follows that $d_1(V) \geq 2$. Also, if $d_2(V) \leq 2$, then we may assume $\dim V \geq f^*(n)$ by Proposition 6.4. $\square$

Now we will complete the induction step of the proof of the Main Theorem. Arguing by contradiction, we will assume that the irreducible $\mathcal{T}_n$-supermodule V satisfies the conditions listed in Corollary 6.9, but

$$\dim V < \begin{cases} f(n) & \text{if } a(V) = 0, \\ f^*(n) & \text{if } a(V) = 1. \end{cases}$$

The condition $d_1(V) \geq 2$ implies that $\mathrm{res}_{n-1} V$ contains at least two large composition factors U_j, $j = 1, 2$, and $\dim U_j \geq f(n-1)$ by the induction hypothesis, whence $\dim V \geq 2f(n-1)$. Similarly, the condition $d_2(V) \geq 3$ implies that $\dim V \geq 3f(n-2)$.

We distinguish between the following three cases.

6E.1. *Case I:* $\pi_{n-1} - \pi_{n-3} = 2$. This case happens precisely when n is odd and $p \mid (n-3)$, whence

$$f^*(n) = f(n) = 2^{\frac{n+1}{2}}(n-2-\kappa_n), \;\; f(n-1) = 2^{\frac{n-1}{2}}(n-3) = \frac{f^*(n-1)}{2}.$$

In particular, if $p = 3$ then $f^*(n) = 2f(n-1) \leq \dim V$. So we may assume $p > 3$. Then

$$\dim V - 2f(n-1) < f(n) - 2f(n-1) = 2^{(n+1)/2} = 2a_{n-1} < b_{n-1} < f(n-1).$$

It follows that $d_1(V) = 2$, and aside from U_1, U_2, $\mathrm{res}_{n-1} V$ can have at most one more composition factor which is then isomorphic to A_{n-1}. Also, if $a(U_j) = 1$ for some j, then by the induction hypothesis, $\dim U_j \geq f^*(n-1) = 2f(n-1)$, and so we would have $\dim V \geq 3f(n-1) > f(n)$. Thus $a(U_j) = 0$ for $j = 1, 2$.

Suppose that $a(V) = 0$. The above conditions on $\mathrm{res}_{n-1} V$ imply by Theorem 2.4 that $\mathrm{res}_{n-1} V = e_0(V)$ has socle and head both isomorphic to $U \cong U_1 \cong U_2$. Since $d_2(V) \geq 3$ (and all composition factors of $\mathrm{res}_{n-2} A_{n-1}$ are isomorphic to A_{n-2}), we see that $d_1(U) \geq 2$; in particular, $U \notin \mathrm{JS}(0)$. Also, $\dim U \leq (\dim V)/2 < f^*(n-1)$. Hence Proposition 6.4 applied to U yields $d_2(U) \geq 3$. It follows that

$$\dim V \geq 2(\dim U) \geq 6f(n-3) = 2^{\frac{n-3}{2}}(6n-36) > 2^{\frac{n+1}{2}}(n-2) = f(n).$$

Next suppose that $a(V) = 1$. Then the above conditions on $\mathrm{res}_{n-1} V$ imply by Theorem 2.4 that $\mathrm{res}_{n-1} V = 2e_i(V) = 2U$ with $U \cong U_1 \cong U_2$ and $i > 0$. Since $d_2(V) \geq 3$ we see that $d_1(U) \geq 2$ and so $U \notin \mathrm{JS}(0)$. Also, $\dim U \leq (\dim V)/2 < f^*(n-1)$. Hence Proposition 6.4 applied to U again yields $d_2(U) \geq 3$ and $\dim V \geq 6f(n-3) > f(n)$. In either case we have reached a contradiction.

6E.2. *Case II:* $\pi_{n-1} - \pi_{n-2} = 0$. This case happens precisely when either $p \mid (n-1)$, or $p \nmid (n-1)(n-2)$ and $2 \mid n$. In the former case,

$$f^*(n) = 2^{\lfloor \frac{n+1}{2} \rfloor}(n-4) \leq 2^{1+\lfloor \frac{n}{2} \rfloor}(n-4) = 2f(n-1) \leq \dim V$$

a contradiction. Likewise, in the latter case,

$$f(n) = 2^{\frac{n}{2}}(n-2-\kappa_n) \leq 2^{1+\frac{n}{2}}(n-3) = 2f(n-1) \leq \dim V.$$

If in addition $p \mid n$, then

$$f^*(n) = 2^{1+\frac{n}{2}}(n-3) = 2f(n-1) \leq \dim V.$$

Hence we may assume that $p \nmid n(n-1)(n-2)$, $2 \mid n$, and $a(V) = 1$. In this case

$$\dim V - 2f(n-1) < f^*(n) - 2f(n-1) = 2^{(n+2)/2} = 4a_{n-1} < b_{n-1} < f(n-1).$$

It follows that $d_1(V) = 2$, and aside from U_1, U_2, all other composition factors of $\mathrm{res}_{n-1} V$ (if any) must be isomorphic to A_{n-1}.

Suppose in addition that $e_i(V) \neq 0$ for some $i > 0$. Then we may assume that U_1 is in $\mathrm{soc}(e_i(V))$. As $a(V) = 1$, $2e_i(V)$ is a direct summand of $\mathrm{res}_{n-1} V$. In particular, if there is some $k \neq i$ such that $e_k(V) \neq 0$, then $\mathrm{soc}(e_k(V))$ must be A_{n-1}, contrary to our hypotheses. Thus $\mathrm{res}_{n-1} V = 2e_i(V)$ in this case. Now $e_i(V)$ has a composition factor U_1 with multiplicity one and all other composition factors (if any) are isomorphic to A_{n-1}. By our hypotheses, $\mathrm{soc}(e_i(V)) = U_1$. It follows that $\varepsilon_i(\lambda) = 1$, and so $e_i(V) = U_1$ is irreducible by Theorem 2.4(v). Thus $V \in \mathrm{JS}(i)$, a contradiction.

We have shown that $\mathrm{res}_{n-1} V = e_0(V)$, with

$$U := U_1 = \mathrm{soc}(e_0(V)) \cong \mathrm{head}(e_0(V)) = U_2,$$

$\varepsilon_0(\lambda) = 2$, and $a(U) = a(V) = 1$. Now $d_1(U) = d_2(V)/2 > 1$; in particular, $U \notin \mathrm{JS}(0)$. Thus we can apply Proposition 6.4 and distinguish the following subcases.

(a) *Suppose* $d_2(U) \geq 3$ *and* $p \nmid (n-4)$. Then

$$\dim V \geq 2(\dim U) \geq 6f(n-3) \geq 2^{(n-2)/2}(6n-36) > 2^{(n+2)/2}(n-2) = f^*(n).$$

(b) *Suppose* $p \mid (n-4)$ *and* $U \ncong D^{\gamma_{n-1}}$. Recall that $d_1(U) \geq 2$. If $d_1(U) \geq 3$, or if some large composition factor X of $\mathrm{res}_{n-2} U$ has $a(X) = 1$, then since $f^*(n-2) = 2f(n-2)$, the induction hypothesis implies

$$\dim V \geq 2(\dim U) \geq 6f(n-2) \geq 2^{(n-2)/2}(6n-24) > 2^{(n+2)/2}(n-2) = f^*(n).$$

Thus $d_1(U) = 2$ and every large composition factor W of $\mathrm{res}_{n-2} U$ has $a(W) = 0$. Moreover, the socle and head of $\mathrm{res}_{n-2} U$ can contain neither A_{n-2} nor B_{n-2}. It follows by Theorem 2.4 that $\mathrm{res}_{n-2} U = 2e_i(U) = 2W$ for some $i > 0$ and some irreducible $\mathcal{T}_{n-2}$-supermodule W. In particular, $U \in \mathrm{JS}(i)$. We have shown that $\varepsilon_k(\lambda) = 2\delta_{k,0}$ and $\tilde{e}_0\lambda = U \in \mathrm{JS}$. Furthermore, $\lambda \neq \gamma_n$ by our assumption. Hence, by Lemma 5.5 we must have $\lambda = \delta_n$. But in this case $D^{\gamma_{n-1}}$ appears in the socle of $\mathrm{res}_{n-1} V$ by Theorem 3.6(v). Thus $U \cong D^{\gamma_{n-1}}$, contrary to our assumption.

(c) *Suppose* $p \nmid (n-4)$, $d_2(U) \leq 2$ *and* $U \ncong D^{\gamma_{n-1}}$. Since $p \nmid (n-1)$ and $U \notin \mathrm{JS}(0)$, by Proposition 6.4 this can happen only when $n = p+7$ (so that $p \geq 11$), and $U = D^{\delta_{n-1}}$ as specified in Lemma 3.9(vi). Applying Lemma 3.9(vi) and Proposition 6.3, we obtain

$$\dim V \geq 2(\dim U) \geq 4f^*(n-2) \geq 2^{n/2}(4n-16) > 2^{(n+2)/2}(n-2) = f^*(n).$$

(d) *Suppose* $U \cong D^{\gamma_{n-1}}$. In this case γ_{n-1} satisfies the condition (6-3). Hence $\dim U \geq f^*(n)/2$ by Proposition 6.3, yielding a contradiction again.

6E.3. *Case III:* $\pi_{n-1}-\pi_{n-2}=\pi_{n-1}-\pi_{n-3}=1$. This case arises precisely when either $p \mid (n-2)$, or $p \nmid (n-1)(n-2)(n-3)$ and $2 \nmid n$. In particular,

$$\dim V \geq 3f(n-2) \geq 2^{\lfloor \frac{n-1}{2} \rfloor}(3n-15) > 2^{\lfloor \frac{n+1}{2} \rfloor}(n-2) \geq f(n).$$

Thus we get a contradiction if $a(V)=0$, or if $f^*(n)=f(n)$.

Hence $a(V)=1$ and $f^*(n)>f(n)$, i.e., n is even and $p \mid (n-2)$; in particular, $f^*(n)=2^{(n+2)/2}(n-2)$. If $n=16$ then $p=7$. In this case, since $d_3(V) \geq d_2(V) \geq 3$, by (6-1) we must have

$$\dim V \geq 3d(p,13) \geq 10368 > 7168 = f^*(16),$$

a contradiction.

So we may assume that $n \geq 20$. We will show that each of the large composition factors U_j of $\mathrm{res}_{n-1} V$ has dimension at least $f^*(n)/2 = 2^{n/2}(n-2)$, leading to the contradiction that $\dim V \geq f^*(n)$. Since $n-1 \equiv 1 (\bmod\ p)$, by Proposition 6.4 we need to consider the following three possibilities for U_j.

(a) $d_2(U_j) \geq 3$. Applying the induction hypothesis to the large composition factors of $\mathrm{res}_{n-3} U_j$ we get

$$\dim U_j \geq 3f(n-3) = 2^{(n-2)/2}(3n-15) \geq f^*(n)/2.$$

(b) $U_j \cong D^{\gamma_{n-1}}$. Recall that $2p \mid (n-2)$ (in particular $n \geq 2p+2$), hence using (6-5) we have

$$\dim U_j \geq 8b_{n-2} = 2^{n/2}(2n-10) > f^*(n)/2.$$

(c) $U_j \in \mathrm{JS}(0)$. Applying Proposition 4.10 and the induction hypothesis to the large composition factors of $\mathrm{res}_{n-7} U_j$ we get

$$\dim U_j \geq 24f(n-7) \geq 24 \cdot 2^{(n-8)/2}(n-11) \geq 2^{n/2}(n-2) = f^*(n)/2$$

if $n \geq 29$. Also, if $p \neq 3$, then

$$\dim U_j \geq 24f(n-7) \geq 24 \cdot 2^{(n-6)/2}(n-10) \geq 2^{n/2}(n-2) = f^*(n)/2.$$

It remains to rule out the cases where $16 \leq n \leq 28$ and $2p = 6 \mid (n-2)$, i.e., $n=20$ or $n=26$. If $n=20$, then by Proposition 4.10 and (6-1) we have

$$\dim U_j \geq 24 \cdot d(p,13) \geq 24 \cdot 3456 > 18432 = f^*(20)/2.$$

Finally, assume $(n,p)=(26,3)$. We claim that any large irreducible $\mathcal{T}_{19}$-supermodule X has dimension at least $3d(p,13)=10368$. (Indeed, this is certainly true if $d_2(X) \geq 3$ or $d_3(X) \geq 3$. If $d_2(X), d_3(X) \leq 2$, then $X \cong D^{\gamma_{19}}$ by

Proposition 6.4 and Lemma 4.11. In this case $\dim X \geq f^*(19) = 15360$ by Proposition 6.3.) Now applying Proposition 4.10 to U_j we get

$$\dim U_j \geq 24 \cdot 10368 = 248832 > 196608 = f^*(n)/2.$$

We have completed the proof of the Main Theorem.

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klesh@uoregon.edu *Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, United States*

tiep@math.arizona.edu *Department of Mathematics, The University of Arizona, 617 North Santa Rita Avenue, P.O. Box 210089, Tucson, AZ 85721-0089, United States*

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Secant varieties of Segre–Veronese varieties

Claudiu Raicu

We prove that the ideal of the variety of secant lines to a Segre–Veronese variety is generated in degree three by minors of flattenings. In the special case of a Segre variety this was conjectured by Garcia, Stillman and Sturmfels, inspired by work on algebraic statistics, as well as by Pachter and Sturmfels, inspired by work on phylogenetic inference. In addition, we describe the decomposition of the coordinate ring of the secant line variety of a Segre–Veronese variety into a sum of irreducible representations under the natural action of a product of general linear groups.

1. Introduction

Spaces of matrices (or 2-tensors) are stratified according to rank by the secant varieties of Segre products of two projective spaces. The defining ideals of these secant varieties are known to be generated by minors of generic matrices. It is an important problem, with applications in algebraic statistics, biology, signal processing, complexity theory etc., to understand (border) rank varieties of higher order tensors. These are (upon taking closure) the classical secant varieties to Segre varieties, whose equations are far from being understood. To get an idea about the boundary of our knowledge, note that the Salmon problem [Allman 2007], which asks for the generators of the ideal of $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, the variety of secant 3-planes to the Segre product of three projective 3-spaces, is still unsolved (although its set-theoretic version has been recently resolved in [Friedland 2010; Friedland and Gross 2012]; see also [Bates and Oeding 2011]).

Flattenings (see Section 2D) provide an easy tool for obtaining some equations for secant varieties of Segre products, but they are not sufficient in general, as can be seen for example in the case of the Salmon problem. Inspired by the study of Bayesian networks, Garcia, Stillman and Sturmfels conjectured [Garcia et al. 2005, Conjecture 21] that flattenings give all the equations of the first secant variety of

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the Segre variety. This conjecture also appeared at the same time in a biological context, namely in work of Pachter and Sturmfels on phylogenetic inference [2004, Conjecture 13].

Conjecture 1.1 (Garcia–Stillman–Sturmfels). *The ideal of the secant line variety of a Segre product of projective spaces is generated by* 3×3 *minors of flattenings.*

The set-theoretic version of this conjecture was obtained by Landsberg and Manivel [2004], as well as the case of a 3-factor Segre product. The 2-factor case is classical, while the 4-factor case was resolved by Landsberg and Weyman [2007]. The 5-factor case was proved by Allman and Rhodes [2008]. We prove the GSS conjecture in Corollary 4.2 as a consequence of our main result, Theorem 4.1, which is the corresponding statement for Segre–Veronese varieties.

It is a general fact that for a subvariety X in projective space which is not contained in a hyperplane, the ideal of the variety $\sigma_k(X)$ of secant $(k-1)$-planes to X has no equations in degree less than $k+1$. If $X = G/P$ is a rational homogeneous variety, a theorem of Kostant (see [Landsberg 2012, Chapter 16] or the remark preceding [Landsberg and Manivel 2004, Proposition 3.3]) states that the ideal of X is generated in the smallest possible degree (that is, in degree two), and Landsberg and Manivel [2004] asked whether this is also true for the first secant variety of X. It turns out that when X is the D_7-spinor variety, there are in fact no cubics in the ideal of $\sigma_2(X)$ (see [Landsberg and Weyman 2009; Manivel 2009]). In Theorem 4.1, we provide a class of G/P's, the Segre–Veronese varieties for which the answer to the question of Landsberg and Manivel is positive. This generalizes a result of Kanev [1999] stating that the ideal of the secant line variety of a Veronese variety is generated in degree three. We obtain furthermore an explicit decomposition into irreducible representations of the homogeneous coordinate ring of the secant line variety of a Segre–Veronese variety, thus making it possible to compute the Hilbert function for this class of varieties. This can be regarded as a generalization of the computation of the degree of these secant varieties in [Cox and Sidman 2007].

Before stating the main theorem, we establish some notation. For a vector space V, V^* denotes its dual, and $\mathbb{P}V$ denotes the projective space of lines in V. If $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ is a partition, S_μ denotes the corresponding Schur functor (if $\mu_2 = 0$ we get symmetric powers, whereas if all $\mu_i = 1$, we get exterior powers). For positive integers $d_1, \dots, d_n$, $SV_{d_1,\dots,d_n}$ denotes the Segre–Veronese embedding of a product of n projective spaces via the complete linear system of the ample line bundle $\mathbb{O}(d_1, \dots, d_n)$. $\sigma_2(X)$ denotes the variety of secant lines to X.

Theorem 4.1. *Let* $X = SV_{d_1,\dots,d_n}(\mathbb{P}V_1^* \times \mathbb{P}V_2^* \times \cdots \times \mathbb{P}V_n^*)$ *be a Segre–Veronese variety, where each* V_i *is a vector space of dimension at least* 2 *over a field* K *of characteristic zero. The ideal of* $\sigma_2(X)$ *is generated by* 3×3 *minors of flattenings, and moreover, for every nonnegative integer* r *we have the decomposition of the* *degree r part of its homogeneous coordinate ring*

$$K[\sigma_2(X)]_r = \bigoplus_{\substack{\lambda=(\lambda^1,\dots,\lambda^n)\\ \lambda^i \vdash rd_i}} (S_{\lambda^1}V_1 \otimes \cdots \otimes S_{\lambda^n}V_n)^{m_\lambda},$$

where m_λ is obtained as follows. Set

$$f_\lambda = \max_{i=1,\dots,n} \left\lceil \frac{\lambda_2^i}{d_i} \right\rceil, \quad e_\lambda = \lambda_2^1 + \cdots + \lambda_2^n.$$

If some partition λ^i has more than two parts, or if $e_\lambda < 2f_\lambda$, then $m_\lambda = 0$. If $e_\lambda \geq r-1$, then $m_\lambda = \lfloor r/2 \rfloor - f_\lambda + 1$, unless e_λ is odd and r is even, in which case $m_\lambda = \lfloor r/2 \rfloor - f_\lambda$. If $e_\lambda < r-1$ and $e_\lambda \geq 2f_\lambda$, then $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda + 1$, unless e_λ is odd, in which case $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda$.

The ideal of the Segre–Veronese variety itself was proved to be generated by 2×2 minors of flattenings by Bernardi [2008], generalizing previously known results on the Segre and Veronese varieties. The corresponding result for a Segre variety was obtained by Grone [1977] in the set-theoretic version and proved by Hà [2002] ideal-theoretically. The set-theoretic version of the result for the Veronese variety goes back to Wakeford [1919], while the ideal-theoretic version was only obtained much later by Pucci [1998]. Even though the higher secant varieties are not always generated by minors of flattenings, Catalisano, Geramita and Gimigliano [Catalisano et al. 2008] describe a large class of examples where this is in fact the case. In their examples the k-th secant variety of a Segre (or Segre–Veronese) variety is cut out by the $(k+1)$-minors of a single matrix of flattenings.

Theorem 4.1 above has further consequences to deriving certain plethystic formulas for decomposing (in special cases) symmetric powers of triple tensor products (Corollary 4.3a) and Schur functors applied to tensor products of two vector spaces (Corollary 4.3b), or even symmetric plethysm (Corollary 4.4).

The main technique introduced in our work does not seek to employ the particularities of specific instances of Segre–Veronese varieties, but instead tries to capture only the essential features that are shared between all these varieties. We work in some sense with spaces of "generic tensors", and rather concentrate on their "generic equations". The latter are representations of products of symmetric groups, which can be defined abstractly with no relation to spaces of tensors (although what led us to them was their realization as zero-weight spaces of particular tensor representations). The main point is that the generic equations yield, by a process of *specialization*, the equations of any specific secant variety of a Segre–Veronese variety. One can also go back, via *polarization*, from the equations of a specific secant variety to (a subset of) the generic equations. The main tools that we employ in analyzing the generic equations of the varieties of secant lines are combinatorial:

graph theory and tableau combinatorics. We hope that similar methods, particularly replacing graph theory with the theory of simplicial complexes, could be used to give an analogous picture for higher secant varieties. The main goal of our work is to set up a general framework that would help understand arbitrary secant varieties, and illustrate how insights from combinatorics occur naturally in this framework, providing new results in the case of the varieties of secant lines.

The general representation theoretic approach to the study of projective varieties with a group action has been successful in providing algorithms for computing the equations and homogeneous coordinate rings of these varieties degree by degree. The main caveat of these algorithms was their failure to provide a good stopping condition that would allow one to decide when the full set of minimal generators of the ideal of a variety has been computed. Our combinatorial methods try to fill this gap by allowing one to identify a specific structure that characterizes the vanishing of a (generic) polynomial on the variety. This structure has the property that is inherited as the degree of the polynomial increases, and also that its presence in any high degree is manifestly a consequence of the inheritance from a finite set of small degrees. Concretely, for the ideal of the secant line variety of a Segre–Veronese variety, we will see that (generic) polynomials of degree r are represented by graphs (Section 4B) with r vertices. The structure that makes a polynomial the equation of the secant variety is the presence in the associated graph of a complete subgraph on three vertices (a triangle; see Remark 4.5). This is clearly a structure that is inherited by adding new vertices and edges to the graph, and also it is a structure that's inherited from degree three — the smallest degree where a complete graph on three vertices could exist. In [Oeding and Raicu 2011], we employ similar ideas to give a proof of a conjecture of Landsberg and Weyman regarding the generators of the ideal of the tangential variety of a Segre variety. The structure that makes a polynomial the equation of the tangential variety turns out to be more involved, represented by a finite list of subgraphs on two, three or four vertices [ibid., Section 3.5].

Finding equations for higher secant varieties of Segre–Veronese varieties turns out to be a delicate task, even in the case of two factors ($n = 2$) with not too positive embeddings (small d_1, d_2). Recent progress in this direction has been obtained by Cartwright, Erman and Oeding [Cartwright et al. 2012].

Since finding precise descriptions of the equations, and more generally syzygies, of secant varieties to Segre–Veronese varieties constitutes such an intricate project, much of the current effort is directed to finding more qualitative statements. Draisma and Kuttler [2011] prove that for each k, there is an uniform bound $d(k)$ such that the $(k-1)$-st secant variety of any Segre variety is cut out (set-theoretically) by equations of degree at most $d(k)$. Theorem 4.1 implies that $d(2) = 3$, even ideal-theoretically.

For higher syzygies, Snowden [2010] proves that all the syzygies of Segre varieties are obtained from a finite amount of data via an iterative process. It would be interesting to know if the same result holds for the secant varieties. This would generalize the result of Draisma and Kuttler. For Veronese varieties, the asymptotic picture of the Betti tables is described in work of Ein and Lazarsfeld [2012]. Again, it would be desirable to have analogous results for secant varieties.

The structure of the paper is as follows. In Section 2 we give the basic definitions for secant varieties and Segre–Veronese varieties. We introduce the basic notions from representation theory that are used throughout the work, and describe the process of flattening a tensor, which leads to the notion of a flattening matrix. Section 3 builds the framework for analyzing the equations and homogeneous coordinate rings of arbitrary secant varieties of Segre–Veronese varieties. Even though we were only able to work out the details of this analysis in the case of the first secant variety, we believe that the general method of approach may be used to shed some light on the case of higher secant varieties. In particular, the new insight of concentrating on the "generic equations" is presented in detail and in the generality needed to deal with arbitrary secant varieties. Section 4 is inspired by a conjecture of Garcia, Stillman and Sturmfels, describing the generators of the ideal of the variety of secant lines to a Segre variety. We prove more generally that this description holds for the first secant variety of a Segre–Veronese variety. We also give a representation theoretic decomposition of the coordinate ring of this variety, which allows us to deduce certain plethystic formulas based on known computations of dimensions of secant varieties of Segre varieties.

2. Preliminaries

Throughout this work, K denotes a field of characteristic 0. All the varieties we study are of finite type over K, and are reduced and irreducible. $\mathbb{P}^N$ denotes the N-dimensional projective space over K. We write $\mathbb{P}W$ for $\mathbb{P}^N$ when we think of $\mathbb{P}^N$ as the space of 1-dimensional subspaces (lines) in a vector space W of dimension $N+1$ over K. Given a nonzero vector $w \in W$, we denote by $[w]$ the corresponding line. The coordinate ring of $\mathbb{P}W$ is $\mathrm{Sym}(W^*)$, the symmetric algebra on the vector space W^* of linear functionals on W.

2A. *Secant varieties.*

Definition 2.1. Given a subvariety $X \subset \mathbb{P}^N$, the $(k-1)$-*st secant variety* of X, denoted $\sigma_k(X)$, is the closure of the union of linear subspaces spanned by k points on X:

$$\sigma_k(X) = \overline{\bigcup_{x_1,\dots,x_k \in X} \mathbb{P}_{x_1,\dots,x_k}}.$$

Alternatively, if we write $\mathbb{P}^N = \mathbb{P}W$ for some vector space W, and let $\hat{X} \subset W$ denote the cone over X, then we can define $\sigma_k(X)$ by specifying its cone $\widehat{\sigma_k(X)}$. This is the closure of the image of the map

$$s : \hat{X} \times \cdots \times \hat{X} \longrightarrow W, \quad \text{defined by } s(x_1, \ldots, x_k) = x_1 + \cdots + x_k.$$

The main problem we are concerned with is this:

Problem. *Given* (*the equations of*) X, *determine* (*the equations of*) $\sigma_k(X)$.

More precisely, given the homogeneous ideal $I(X)$ of the subvariety $X \subset \mathbb{P}W$, we would like to describe the generators of $I(\sigma_k(X))$. Alternatively, we would like to understand the homogeneous coordinate ring of $\sigma_k(X)$, which we denote by $K[\sigma_k(X)]$. As we will see, this is a difficult problem even in the case when X is simple, that is, isomorphic to a projective space (or a product of such). There is thus little hope of giving an uniform satisfactory answer in the generality with which we posed the problem. However, the following observation provides a general approach to the problem, which we exploit in the future sections.

The ideal/homogeneous coordinate ring of a subvariety $Y \subset \mathbb{P}W$ coincides with the ideal/affine coordinate ring of its cone $\hat{Y} \subset W$, hence our problem is equivalent to understanding $I(\widehat{\sigma_k(X)})$ and $K[\widehat{\sigma_k(X)}]$. The morphism s of affine varieties defined above corresponds to a ring map

$$s^{\#} : \mathrm{Sym}(W^*) \to K[\hat{X} \times \cdots \times \hat{X}] = K[\hat{X}] \otimes \cdots \otimes K[\hat{X}].$$

We have that $I(\widehat{\sigma_k(X)})$ and $K[\widehat{\sigma_k(X)}]$ are the kernel and image respectively of $s^{\#}$. The main focus for us will be on the case when X is a Segre–Veronese variety (described in the following section), and $k = 2$.

2B. ***Segre–Veronese varieties.*** Consider vector spaces $V_1, \ldots, V_n$ of dimensions $m_1, \ldots, m_n \geq 2$, respectively, with duals $V_1^*, \ldots, V_n^*$, and fix positive integers $d_1, \ldots, d_n$. We let

$$X = \mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*$$

and think of it as a subvariety in projective space via the embedding determined by the line bundle $\mathcal{O}_X(d_1, \ldots, d_n)$. Explicitly, X is the image of the map

$$SV_{d_1,\ldots,d_n} : \mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^* \to \mathbb{P}(\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n^*)$$

given by

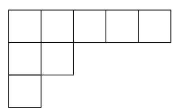
$$([e_1], \ldots, [e_n]) \mapsto [e_1^{d_1} \otimes \cdots \otimes e_n^{d_n}].$$

We call X a *Segre–Veronese variety*.

For such X we prove that $I(\sigma_2(X))$ is generated in degree 3 and we describe the decomposition of $K[\sigma_2(X)]$ into a sum of irreducible representations of the product of general linear groups $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$ (Theorem 4.1).

When $n = 1$ we set $d = d_1$, $V = V_1$. The image of SV_d is the d-th Veronese embedding, or d-uple embedding of the projective space $\mathbb{P}V^*$, which we denote by $\mathrm{Ver}_d(\mathbb{P}V^*)$. When $d_1 = \cdots = d_n = 1$, the image of $SV_{1,1,\ldots,1}$ is the Segre variety $\mathrm{Seg}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)$. An element of $\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n^*$ is called a *(partially symmetric) tensor*. The points in the cone over the Segre–Veronese variety are called *pure tensors*.

2C. ***Representation theory.*** We refer the reader to [Fulton and Harris 1991] for the basic representation theory of symmetric and general linear groups. Given a positive integer r, a partition μ of r is a nonincreasing sequence of nonnegative integers $\mu_1 \geq \mu_2 \geq \cdots$ with $r = \sum \mu_i$. We write $\mu = (\mu_1, \mu_2, \ldots)$. Alternatively, if μ is a partition having i_j parts equal to μ_j for all j, then we write $\mu = (\mu_1^{i_1} \mu_2^{i_2} \cdots)$. To a partition $\mu = (\mu_1, \mu_2, \ldots)$ we associate a *Young diagram* which consists of left-justified rows of boxes, with μ_i boxes in the i-th row. For $\mu = (5, 2, 1)$, the corresponding Young diagram is



For a vector space W, a positive integer r and a partition μ of r, we denote by $S_\mu W$ the corresponding irreducible representation of $\mathrm{GL}(W)$: S_μ are commonly known as *Schur functors*, and we make the convention that $S_{(d)}$ denotes the symmetric power functor, while $S_{(1^d)}$ denotes the exterior power functor. We write S_r for the symmetric group on r letters, and $[\mu]$ for the irreducible S_r-representation corresponding to μ: $[(d)]$ denotes the trivial representation and $[(1^d)]$ denotes the sign representation.

Given a positive integer n and a sequence of nonnegative integers $\underline{r} = (r_1, \ldots, r_n)$, we define an *$n$-partition* of $\underline{r}$ to be an n-tuple of partitions $\lambda = (\lambda^1, \ldots, \lambda^n)$, with λ^j a partition of r_j, $j = 1, \ldots, n$. We write $\lambda^j \vdash r_j$ and $\lambda \vdash^n \underline{r}$. Given vector spaces $V_1, \ldots, V_n$ as above, we often write $\mathrm{GL}(V)$ for $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$. We write $S_\lambda V$ for the irreducible $\mathrm{GL}(V)$-representation $S_{\lambda^1} V_1 \otimes \cdots \otimes S_{\lambda^n} V_n$. Similarly, we write $[\lambda]$ for the irreducible representation $[\lambda^1] \otimes \cdots \otimes [\lambda^n]$ of the n-fold product of symmetric groups $S_{\underline{r}} = S_{r_1} \times \cdots \times S_{r_n}$. We have:

Lemma 2.2 (Schur–Weyl duality).

$$V_1^{\otimes r_1} \otimes \cdots \otimes V_n^{\otimes r_n} = \bigoplus_{\lambda \vdash^n \underline{r}} [\lambda] \otimes S_\lambda V.$$

Most of the group actions we consider are left actions, denoted by $\cdot$. We use the symbol $*$ for right actions, to distinguish them from left actions.

For a subgroup $H \subset G$ and representations U of H and W of G, we write

$$\operatorname{Ind}_H^G(U) = K[G] \otimes_{K[H]} U \quad \text{and} \quad \operatorname{Res}_H^G(W) = W_H,$$

for the *induced representation* of U and *restricted representation* of W, where $K[M]$ denotes the group algebra of a group M, and W_H is just W, regarded as an H-module. We write W^G for the G*-invariants* of the representation W, that is,

$$W^G = \operatorname{Hom}_G(\mathbf{1}, W) \subset \operatorname{Hom}_K(\mathbf{1}, W) = W,$$

where $\mathbf{1}$ denotes the trivial representation of G.

Remark 2.3. If G is finite, let

$$s_G = \sum_{g \in G} g \in K[G].$$

We can realize W^G as the image of the map $W \to W$ given by $w \mapsto s_G \cdot w$. Assume furthermore that $H \subset G$ is a subgroup, and let s_H denote the corresponding element in $K[H]$. We have a natural inclusion of the trivial representation of H

$$\mathbf{1} \hookrightarrow K[H], \quad 1 \mapsto s_H,$$

which after tensoring with $K[G]$ becomes

$$\operatorname{Ind}_H^G(\mathbf{1}) = K[G] \otimes_{K[H]} \mathbf{1} \hookrightarrow K[G] \otimes_{K[H]} K[H] \simeq K[G],$$

so that we can identify $\operatorname{Ind}_H^G(\mathbf{1})$ with $K[G] \cdot s_H$.

Lemma 2.4 (Frobenius reciprocity).

$$W^H = \operatorname{Hom}_H(\mathbf{1}, \operatorname{Res}_H^G(W)) = \operatorname{Hom}_G(\operatorname{Ind}_H^G(\mathbf{1}), W).$$

Given an n-partition $\lambda = (\lambda^1, \dots, \lambda^n)$ of $\underline{r}$, we define an n*-tableau* of shape λ to be an n-tuple $T = (T^1, \dots, T^n)$, which we usually write as $T^1 \otimes \cdots \otimes T^n$, where each T^i is a tableau of shape λ^i. A tableau is *canonical* if its entries index its boxes consecutively from left to right, and top to bottom. We say that T is canonical if each T^i is, in which case we write T_λ for T. If $T = (\lambda^1, \lambda^2)$, with $\lambda^1 = (3, 2)$, $\lambda^2 = (3, 1, 1)$, then the canonical 2-tableau of shape λ is

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2} \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & & \\ \cline{1-1} 5 & & \\ \cline{1-1} \end{array}.$$

We consider the subgroups of $S_{\underline{r}}$ given by

$$R_\lambda = \{g \in S_{\underline{r}} : g \text{ preserves each row of } T_\lambda\},$$
$$C_\lambda = \{g \in S_{\underline{r}} : g \text{ preserves each column of } T_\lambda\}$$

and define the symmetrizers

$$a_\lambda = \sum_{g\in R_\lambda} g, \quad b_\lambda = \sum_{g\in C_\lambda} \operatorname{sgn}(g)\cdot g, \quad c_\lambda = a_\lambda \cdot b_\lambda,$$

with $\operatorname{sgn}(g) = \prod_i \operatorname{sgn}(g_i)$ for $g = (g_1, \dots, g_n) \in S_{\underline{r}}$, where $\operatorname{sgn}(g_i)$ denotes the signature of the permutation g_i.

The $\mathrm{GL}(V)$- (or $S_{\underline{r}}$-) representations W that we consider decompose as a direct sum of $S_\lambda V$'s (or $[\lambda]$'s) with $\lambda \vdash^n \underline{r}$. We write

$$W = \bigoplus_\lambda W_\lambda,$$

where $W_\lambda \simeq (S_\lambda V)^{m_\lambda}$ (or $W_\lambda \simeq [\lambda]^{m_\lambda}$) for some nonnegative integer $m_\lambda = m_\lambda(W)$, called the *multiplicity* of $S_\lambda V$ (or $[\lambda]$) in W. We call W_λ the λ*-part* of the representation W.

Recall that m_j denotes the dimension of V_j, $j = 1, \dots, n$. We fix bases

$$\mathcal{B}_j = \{x_{ij} : i = 1, \dots, m_j\}$$

for V_j ordered by $x_{ij} > x_{i+1,j}$. We choose the *maximal torus* $T = T_1 \times \cdots \times T_n \subset \mathrm{GL}(V)$, with T_j the set of diagonal matrices with respect to $\mathcal{B}_j$. We choose the *Borel subgroup* of $\mathrm{GL}(V)$ to be $B = B_1 \times \cdots \times B_n$, where B_j is the subgroup of upper triangular matrices in $\mathrm{GL}(V_j)$ with respect to $\mathcal{B}_j$. Given a $\mathrm{GL}(V)$-representation W, a *weight vector* w with *weight* $a = (a_1, \dots, a_n)$, $a_i \in T_i^*$, is a nonzero vector in W with the property that for any $t = (t_1, \dots, t_n) \in T$,

$$t \cdot w = a_1(t_1) \cdots a_n(t_n) w.$$

The vectors with this property form a vector space called the *a-weight space* of W, which we denote by $\mathrm{wt}_a(W)$.

A *highest weight vector* of a $\mathrm{GL}(V)$-representation W is an element $w \in W$ invariant under B. $W = S_\lambda V$ has a unique (up to scaling) highest weight vector w with corresponding weight $\lambda = (\lambda^1, \dots, \lambda^n)$. In general, we define the λ*-highest weight space* of a $\mathrm{GL}(V)$-representation W to be the set of highest weight vectors in W with weight λ, and denote it by $\mathrm{hwt}_\lambda(W)$. If W is an $S_{\underline{r}}$-representation, the λ*-highest weight space* of W is the vector space $\mathrm{hwt}_\lambda(W) = c_\lambda \cdot W \subset W$, where c_λ is the Young symmetrizer defined above. In both cases, $\mathrm{hwt}_\lambda(W)$ is a vector space of dimension $m_\lambda(W)$.

2D. *Flattenings.* Given decompositions $d_i = a_i + b_i$, with $a_i, b_i \geq 0$, $i = 1, \dots, n$, we let $A = (a_1, \dots, a_n)$, $B = (b_1, \dots, b_n)$, so that $\underline{d} = (d_1, \dots, d_n) = A + B$, and embed

$$\mathrm{Sym}^{d_1} V_1^* \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n^* \hookrightarrow V_A^* \otimes V_B^*$$

in the usual way, where

$$V_A = \operatorname{Sym}^{a_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{a_n} V_n, \qquad V_B = \operatorname{Sym}^{b_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{b_n} V_n.$$

This embedding allows us to *flatten* any tensor in $\operatorname{Sym}^{d_1} V_1^* \otimes \cdots \otimes \operatorname{Sym}^{d_n} V_n^*$ to a 2-tensor, that is, a matrix, in $V_A^* \otimes V_B^*$. We call such a matrix an (A, B)-*flattening* of our tensor. If $|A| = a_1 + \cdots + a_n$ then we also say that this matrix is an $|A|$-flattening, or a $|B|$-flattening, by symmetry.

We obtain an inclusion

$$SV_{d_1,\ldots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*) \hookrightarrow \operatorname{Seg}(\mathbb{P}V_A^* \times \mathbb{P}V_B^*),$$

and consequently

$$\sigma_k(SV_{d_1,\ldots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)) \hookrightarrow \sigma_k(\operatorname{Seg}(\mathbb{P}V_A^* \times \mathbb{P}V_B^*)),$$

where the latter secant variety coincides with (the projectivization of) the set of matrices of rank at most k in $V_A^* \otimes V_B^*$. This set is cut out by the $(k+1) \times (k+1)$ minors of the generic matrix in $V_A^* \otimes V_B^*$. This observation yields equations for the secant varieties of Segre–Veronese varieties (see also [Landsberg 2012, Chapter 7]).

Lemma 2.5. *For any decomposition* $\underline{d} = A + B$ *and any* $k \geq 1$, *the ideal of* $(k+1) \times (k+1)$ *minors of the generic matrix given by the* (A, B)*-flattening of* $\operatorname{Sym}^{d_1} V_1^* \otimes \cdots \otimes \operatorname{Sym}^{d_n} V_n^*$ *is contained in the ideal of*

$$\sigma_k(SV_{d_1,\ldots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)).$$

Definition 2.6. We write $F_{A,B}^{k+1,r}(V) = F_{A,B}^{k+1,r}(V_1, \ldots, V_n)$ for the degree r part of the ideal of $(k+1) \times (k+1)$ minors of the (A, B)-flattening. We call the elements of $F_{A,B}^{k+1,r}(V)$ *flattening equations*.

Note that the invariant way of writing the generators of the ideal of $(k+1)\times(k+1)$ minors of the (A, B)-flattening in the preceding lemma ($F_{A,B}^{k+1,k+1}(V)$) is as the image of the composition

$$\bigwedge^{k+1} V_A \otimes \bigwedge^{k+1} V_B \hookrightarrow \operatorname{Sym}^{k+1}(V_A \otimes V_B) \longrightarrow \operatorname{Sym}^{k+1}(\operatorname{Sym}^{d_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{d_n} V_n),$$

where the first map is the usual inclusion map, while the last one is induced by the multiplication maps $\operatorname{Sym}^{a_i} V_i \otimes \operatorname{Sym}^{b_i} V_i \to \operatorname{Sym}^{d_i} V_i$.

2E. ***The ideal and coordinate ring of a Segre–Veronese variety.*** If

$$X = SV_{d_1,\ldots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*),$$

then the ideal $I(X)$ is generated by 2×2 minors of flattenings [Bernardi 2008], that is, when $k = 1$ the equations described in Lemma 2.5 are sufficient to generate the ideal of the corresponding variety. As for the homogeneous coordinate ring of a Segre–Veronese variety, we have the decomposition

$$K[X] = \bigoplus_{r\geq 0}(\operatorname{Sym}^{rd_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{rd_n} V_n). \tag{*}$$

This decomposition will turn out to be useful in the next section, in conjunction with the map $s^{\#}$ defined in Section 2A. In Section 4 we give a description of $K[\sigma_2(X)]$ analogous to (*), and prove that the 3×3 minors of flattenings generate the homogeneous ideal of $\sigma_2(X)$.

The statements above regarding the ideal and coordinate ring of a Segre–Veronese variety hold more generally for rational homogeneous varieties (G/P), and have been obtained in unpublished work by Kostant; see [Landsberg 2012, Chapter 16].

3. Equations of the secant varieties of a Segre–Veronese variety

This section introduces the main new tool for understanding the equations and coordinate rings of the secant varieties of Segre–Veronese varieties, from a representation theoretic/combinatorial perspective. All the subsequent work is based on the ideas described here. The usual method for analyzing the secant varieties of Segre–Veronese varieties is based on the representation theory of general linear groups. We review some of its basic ideas, including the notion of *inheritance*, in Section 3A. The new insight of restricting the analysis to special equations of the secant varieties, the "generic equations", is presented in Section 3B. More precisely, we use Schur–Weyl duality to translate questions about representations of general linear groups into questions about representations of symmetric groups and tableau combinatorics. The relationship between the two situations is made precise in Section 3C. One should think of the "generic equations" as a set of equations that give rise by specialization to all the equations of the secant varieties of Segre–Veronese varieties. Similarly, we have the "generic flattening equations" which by specialization yield the usual flattening equations.

3A. ***Multiprolongations and inheritance.*** In this section $V_1, \dots, V_n$ are (as always) vector spaces over a field K of characteristic zero. We switch from the Sym^d notation to the more compact Schur functor notation $S_{(d)}$ described in Section 2C. The homogeneous coordinate ring of $\mathbb{P}(S_{(d_1)}V_1^* \otimes \cdots \otimes S_{(d_n)}V_n^*)$ is

$$S = \operatorname{Sym}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n),$$

the symmetric algebra of the vector space $S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n$. This vector space has a natural basis $\mathcal{B} = \mathcal{B}_{d_1,\dots,d_n}$ consisting of tensor products of monomials in the elements of the bases $\mathcal{B}_1, \dots, \mathcal{B}_n$ of $V_1, \dots, V_n$. We write this basis, suggestively, as $\mathcal{B} = \operatorname{Sym}^{d_1} \mathcal{B}_1 \otimes \cdots \otimes \operatorname{Sym}^{d_n} \mathcal{B}_n$. We can index the elements of $\mathcal{B}$ by n-tuples

$\alpha = (\alpha_1, \ldots, \alpha_n)$ of multisets α_i of size d_i with entries in $\{1, \ldots, m_i = \dim(V_i)\}$, as follows. The α-th element of the basis $\mathcal{B}$ is

$$z_\alpha = \Bigl(\prod_{i_1 \in \alpha_1} x_{i_1,1}\Bigr) \otimes \cdots \otimes \Bigl(\prod_{i_n \in \alpha_n} x_{i_n,n}\Bigr),$$

and we think of z_α as a linear form in S.

We can therefore identify S with the polynomial ring $K[z_\alpha]$. One would like to have a precise description of the ideal $I \subset S$ of polynomials vanishing on $\sigma_k(SV_{d_1,\ldots,d_n}(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*))$, but this is a very difficult problem, as mentioned in the introduction. We obtain such a description for the case $k = 2$ in Theorem 4.1. The case $k = 1$ was already known, as described in Section 2E.

Given a positive integer r and a partition $\mu = (\mu_1, \ldots, \mu_t) \vdash r$, we consider the set $\mathcal{P}_\mu$ of all (unordered) partitions of $\{1, \ldots, r\}$ of shape μ, that is,

$$\mathcal{P}_\mu = \Bigl\{A = \{A_1, \ldots, A_t\} : |A_i| = \mu_i \text{ and } \bigsqcup_{i=1}^{t} A_i = \{1, \ldots, r\}\Bigr\},$$

as opposed to the set of ordered partitions where we take instead $A = (A_1, \ldots, A_t)$.

Definition 3.1. For a partition $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$ of r, we consider the map

$$\pi_\mu : S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n) \longrightarrow \bigotimes_{j=1}^{s} S_{(i_j)}(S_{(\mu_j d_1)}V_1 \otimes \cdots \otimes S_{(\mu_j d_n)}V_n),$$

given by

$$z_1 \cdots z_r \mapsto \sum_{A \in \mathcal{P}_\mu} \bigotimes_{j=1}^{s} \prod_{\substack{B \in A \\ |B| = \mu_j}} m(z_i : i \in B),$$

where $m : (S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)^{\otimes \mu_j} \to S_{(\mu_j d_1)}V_1 \otimes \cdots \otimes S_{(\mu_j d_n)}V_n$ denotes the usual componentwise multiplication map.

We write $\pi_\mu(V)$ or $\pi_\mu(V_1, \ldots, V_n)$ for the map π_μ just defined, when we want to distinguish it from its generic version (Definition 3.11). We also write

$$U_r^d(V) = U_r^d(V_1, \ldots, V_n) \quad \text{and} \quad U_\mu^d(V) = U_\mu^d(V_1, \ldots, V_n)$$

for the source and target of $\pi_\mu(V)$, respectively (see Definitions 3.7 and 3.10 for the generic versions of these spaces).

A more invariant way of stating Definition 3.1 is as follows. If $\mu = (\mu_1, \ldots, \mu_t)$, then the map π_μ is the composition between the usual inclusion

$$\begin{aligned} S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n) &\hookrightarrow (S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)^{\otimes r} \\ &= (S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)^{\otimes \mu_1} \otimes \cdots \otimes (S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)^{\otimes \mu_t}, \end{aligned}$$

and the tensor product of the natural multiplication maps

$$m : (S_{(d_i)} V_i)^{\otimes \mu_j} \longrightarrow S_{(\mu_j d_i)} V_i.$$

Example 3.2. Let $n = 2$, $d_1 = 2$, $d_2 = 1$, $r = 4$, $\mu = (2, 2) = (2^2)$, $\dim(V_1) = 2$, $\dim(V_2) = 3$. Take

$$z_1 = z_{(\{1,2\},\{1\})}, \quad z_2 = z_{(\{1,1\},\{3\})}, \quad z_3 = z_{(\{1,1\},\{1\})}, \quad z_4 = z_{(\{2,2\},\{2\})}.$$

We have

$$\begin{aligned}\pi_\mu(z_1 \cdot z_2 \cdot z_3 \cdot z_4) &= m(z_1, z_2) \cdot m(z_3, z_4) + m(z_1, z_3) \cdot m(z_2, z_4) + m(z_1, z_4) \cdot m(z_2, z_3) \\ &= z_{(\{1,1,1,2\},\{1,3\})} \cdot z_{(\{1,1,2,2\},\{1,2\})} + z_{(\{1,1,1,2\},\{1,1\})} \cdot z_{(\{1,1,2,2\},\{2,3\})} \\ &\qquad + z_{(\{1,2,2,2\},\{1,2\})} \cdot z_{(\{1,1,1,1\},\{1,3\})}.\end{aligned}$$

A more "visual" way of representing the monomials in

$$\mathrm{Sym}(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_n} V_n) = K[z_\alpha]$$

and the maps π_μ is as follows. We identify each z_α with an $1 \times n$ block with entries the multisets α_i:

$$z_\alpha = \boxed{\alpha_1 \,|\, \alpha_2 \,|\, \cdots \,|\, \alpha_n}\,.$$

We represent a monomial $m = z_{\alpha^1} \cdots z_{\alpha^r}$ of degree r as an $r \times n$ block M, whose rows correspond to the variables z_{α^i} in the way described above:

$$m \equiv M = \begin{array}{|c|c|c|c|}\hline \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_n^1 \\ \hline \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \alpha_1^r & \alpha_2^r & \cdots & \alpha_n^r \\ \hline \end{array}\,.$$

The order of the rows is irrelevant, since the z_{α^i} commute. The way π_μ acts on an $r \times n$ block M is as follows: it partitions in all possible ways the set of rows of M into subsets of sizes equal to the parts of μ, collapses the elements of each subset into a single row, and takes the sum of all blocks obtained in this way. Here by collapsing a set of rows we mean taking the columnwise union of the entries of the rows. More precisely, if M is the $r \times n$ block corresponding to $z_{\alpha^1} \cdots z_{\alpha^r}$ and $\mu = (\mu_1, \ldots, \mu_t)$, then

$$\pi_\mu M = \sum_{\substack{A \in \mathcal{P}_\mu \\ A = \{A_1, \ldots, A_t\}}} \begin{array}{|c|c|c|}\hline \cdots & \bigcup_{i \in A_1} \alpha_k^i & \cdots \\ \hline \cdots & \bigcup_{i \in A_2} \alpha_k^i & \cdots \\ \hline \vdots & \ddots & \vdots \\ \hline \cdots & \bigcup_{i \in A_t} \alpha_k^i & \cdots \\ \hline \end{array}\,.$$

Note that if two A_i have the same cardinality, then the variables corresponding to their rows commute, so we can harmlessly interchange them.

Example 3.3. With these conventions, we can rewrite Example 3.2 as

$$\begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 1,1 & 3 \\ \hline 1,1 & 1 \\ \hline 2,2 & 2 \\ \hline \end{array} \xrightarrow{\pi_{(2,2)}} \begin{array}{|c|c|}\hline 1,1,1,2 & 1,3 \\ \hline 1,1,2,2 & 1,2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,1,1,2 & 1,1 \\ \hline 1,1,2,2 & 2,3 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,2,2,2 & 1,2 \\ \hline 1,1,1,1 & 1,3 \\ \hline \end{array}.$$

Proposition 3.4 (multiprolongations [Landsberg 2012, Section 7.5]). *For a positive integer r, the polynomials of degree r vanishing on*

$$\sigma_k(SV_{d_1,\dots,d_n}(\mathbb{P}V_1^*\times\cdots\times\mathbb{P}V_n^*))$$

are precisely the elements of $S_{(r)}(S_{(d_1)}V_1\otimes\cdots\otimes S_{(d_n)}V_n)$ in the intersection of the kernels of the maps π_μ, where μ ranges over all partitions of r with (at most) k parts.

Proof. Let X denote the Segre–Veronese variety $SV_{d_1,\dots,d_n}(\mathbb{P}V_1^*\times\cdots\times\mathbb{P}V_n^*)$. As in Section 2A, there exists a map ($s^\#$, which we now denote π)

$$\pi:\mathrm{Sym}(S_{(d_1)}V_1\otimes\cdots\otimes S_{(d_n)}V_n)\longrightarrow K[X]^{\otimes k},$$

whose kernel and image coincide with the ideal and homogeneous coordinate ring respectively, of $\sigma_k(X)$. Using the description of $K[X]$ given in Section 2E, we obtain that the degree r part of the target of π is

$$(K[X]^{\otimes k})_r = \bigoplus_{\mu_1+\cdots+\mu_k=r}(S_{(\mu_1 d_1)}V_1\otimes\cdots\otimes S_{(\mu_1 d_n)}V_n)\otimes\cdots\otimes(S_{(\mu_k d_1)}V_1\otimes\cdots\otimes S_{(\mu_k d_n)}V_n).$$

The degree r component of π, which we call π_r, is then a direct sum of maps π_μ as in Definition 3.1, where μ ranges over partitions of r with at most k parts. The conclusion of the proposition now follows. To see that it's enough to only consider partitions with exactly k parts, note that if μ has fewer than k parts, and $\widehat{\mu}$ is a partition obtained by subdividing μ (splitting some of the parts of μ into smaller pieces), then π_μ factors through (up to a multiplicative factor) $\pi_{\widehat{\mu}}$, hence $\ker(\pi_\mu)\supset\ker(\pi_{\widehat{\mu}})$, so the contribution of $\ker(\pi_\mu)$ to the intersection of kernels is superfluous. □

Definition 3.5 (multiprolongations). We write $I_\mu(V)=I_\mu^d(V)$ for the kernel of the map $\pi_\mu(V)$, and $I_r(V)=I_r^d(V)$ for the intersection of the kernels of the maps $\pi_\mu(V)$ as μ ranges over partitions of r with k parts. that is, $I_r(V)$ is the degree r part of the ideal of $\sigma_k(SV_{d_1,\dots,d_n}(\mathbb{P}V_1^*\times\cdots\times\mathbb{P}V_n^*))$.

Given the description of the ideal of $\sigma_k(X)$ as the kernel of the $\mathrm{GL}(V)$-equivariant map π, we now proceed to analyze π irreducible representation by irreducible representation. That is, we fix a positive integer r and an n-partition $\lambda = (\lambda^1, \dots, \lambda^n)$ of $(rd_1, \dots, rd_n)$, and we restrict π to the λ-parts of its source and target. The map π depends functorially on the vector spaces $V_1, \dots, V_n$, and its kernel and image stabilize from a representation theoretic point of view as the dimensions of the V_i increase. More precisely, we have the following

Proposition 3.6 (inheritance [Landsberg 2012, Section 7.4]). *Fix an n-partition $\lambda \vdash^n r \cdot (d_1, \dots, d_n)$. Let l_j denote the number of parts of λ^j, for $j = 1, \dots, n$. Then the multiplicities of $S_\lambda V$ in the kernel and image respectively of π are independent of the dimensions m_j of the V_j, as long as $m_j \geq l_j$. Moreover, if some l_j is larger than k, then $S_\lambda V$ doesn't occur as a representation in the image of π.*

Proof. The last statement follows from the representation theoretic description of the coordinate ring of a Segre–Veronese variety, and Pieri's rule: every irreducible representation $S_\lambda V$ occurring in $K[X]^{\otimes k}$ must have the property that each λ^j has at most k parts.

As for the first part, note that π is completely determined by what it does on the λ-highest weight vectors, and that the λ-highest weight vector of an irreducible representation $S_\lambda V$ only depends on the first l_j elements of the basis $\mathcal{B}_j$, for $j = 1, \dots, n$. □

We just saw in the previous proposition that (the λ-part of) π is essentially insensitive to expanding or shrinking the vector spaces V_i, as long as their dimensions remain larger than l_i. Also, the last part of the proposition allows us to concentrate on n-partitions λ where each λ^i has at most k parts. To understand π, we thus have the freedom to pick the dimensions of the V_i to be positive integers at least equal to k. It might seem natural then to pick these dimensions as small as possible (equal to k), and understand the kernel and image of π in that situation. However, we choose not to do so, and instead we fix a positive degree r and concentrate our attention on the map π_r, the degree r part of π. We assume that

$$\dim(V_i) = r \cdot d_i, \quad i = 1, \dots, n.$$

The reason for this assumption is that now the $\mathfrak{sl}$ zero-weight spaces of the source and target of π_r are nonempty and generate the corresponding representations. Therefore π_r is determined by its restriction to these zero-weight spaces, which suddenly makes our problem combinatorial: the zero-weight spaces are modules over the Weyl group, which is just the product of symmetric groups $S_{rd_1} \times \cdots \times S_{rd_n}$, allowing us to use the representation theory of the symmetric groups to analyze the map π_r. We call this reduction the "generic case", because the $\mathfrak{sl}$ zero-weight subspace of $S_{(r)}(S_{(d_1)} V_1 \otimes \cdots \otimes S_{(d_n)} V_n)$ is the subspace containing the most generic tensors.

3B. ***The "generic case".***

3B.1. *Generic multiprolongations.* We let $\underline{d}, \underline{r}$ denote the sequences of numbers $(d_1, \dots, d_n)$ and $r \cdot \underline{d} = (rd_1, \dots, rd_n)$ respectively. We let $S_{\underline{r}}$ denote the product of symmetric groups $S_{rd_1} \times \cdots \times S_{rd_n}$, the Weyl group of the Lie algebra $\mathfrak{sl}(V)$ of $\mathrm{GL}(V)$ (recall that $\dim(V_j) = m_j = rd_j$ for $j = 1, \dots, n$).

Definition 3.7. We denote by $U_r^{\underline{d}}$ the $\mathfrak{sl}(V)$ zero-weight space of the representation $S_{(r)}(S_{(d_1)}V_1 \otimes \cdots \otimes S_{(d_n)}V_n)$. $U_r^{\underline{d}}$ has a basis consisting of monomials $m = z_{\alpha^1} \cdots z_{\alpha^r}$, where for each j, the elements of $\{\alpha_j^1, \dots, \alpha_j^r\}$ form a partition of the set $\{1, \dots, rd_j\}$, with $|\alpha_j^i| = d_j$. Alternatively, $U_r^{\underline{d}}$ has a basis consisting of $r \times n$ blocks M, where each column of M yields a partition of the set $\{1, \dots, rd_j\}$ with r equal parts.

Example 3.8. For $n = 2$, $d_1 = 2$, $d_2 = 1$, $r = 4$, a typical element of $U_r^{\underline{d}}$ is

$$M = \begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 2,3 & 4 \\ \hline 4,5 & 2 \\ \hline 7,8 & 3 \\ \hline\end{array} = z_{(\{1,6\},\{1\})} \cdot z_{(\{2,3\},\{4\})} \cdot z_{(\{4,5\},\{2\})} \cdot z_{(\{7,8\},\{3\})} = m.$$

$S_{\underline{r}}$ acts on $U_r^{\underline{d}}$ by letting its j-th factor S_{rd_j} act on the j-th columns of the blocks M described above. As an abstract representation, we have

$$U_r^{\underline{d}} \simeq \mathrm{Ind}_{(S_{d_1} \times \cdots \times S_{d_n})^r \wr S_r}^{S_{\underline{r}}}(\mathbf{1}),$$

where $\wr$ denotes the *wreath product* of $(S_{d_1} \times \cdots \times S_{d_n})^r$ with S_r, and $\mathbf{1}$ denotes the trivial representation (we will say more about this identification in Section 3C). For now, recall that for a group H and positive integer r, the *wreath product* $H^r \wr S_r$ of H^r with the symmetric group S_r is just the semidirect product $H^r \rtimes S_r$, where S_r acts on H^r by permuting the r copies of H. The dimension of the space $U_r^{\underline{d}}$ is

$$N = N_r^{\underline{d}} = \frac{(rd_1)!(rd_2)! \cdots (rd_n)!}{(d_1!d_2! \cdots d_n!)^r \cdot r!}.$$

Example 3.9. Continuing Example 3.8, let $\sigma = (\sigma_1, \sigma_2) \in S_8 \times S_4$, where, in cycle notation, $\sigma_1 = (1, 2)(5, 3, 7)$, $\sigma_2 = (1, 4, 3)$. Then

$$\sigma \cdot M = \begin{array}{|c|c|}\hline 2,6 & 4 \\ \hline 1,7 & 3 \\ \hline 4,3 & 2 \\ \hline 5,8 & 1 \\ \hline\end{array}, \quad \text{or } \sigma \cdot m = z_{(\{2,6\},\{4\})} \cdot z_{(\{1,7\},\{3\})} \cdot z_{(\{4,3\},\{2\})} \cdot z_{(\{5,8\},\{1\})}.$$

Definition 3.10. For a partition μ written in multiplicative notation $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$ as in Definition 3.1, we define the space $U_\mu^{\underline{d}}$ to be the $\mathfrak{sl}$ zero-weight space of the representation

$$\bigotimes_{j=1}^{s} S_{(i_j)}(S_{(\mu_j d_1)}V_1 \otimes \cdots \otimes S_{(\mu_j d_n)}V_n).$$

Writing $\mu = (\mu_1, \ldots, \mu_t)$ we can realize $U_\mu^{\underline{d}}$ as the vector space with a basis consisting of $t \times n$ blocks M with the entry in row i and column j consisting of $\mu_i \cdot d_j$ elements from the set $\{1, \ldots, rd_j\}$, in such a way that each column of M represents a partition of $\{1, \ldots, rd_j\}$. As usual, we identify two blocks if they differ by permutations of rows of the same size, that is, corresponding to equal parts of μ. Note that when $\mu = (1^r)$ we get $U_\mu^{\underline{d}} = U_r^{\underline{d}}$, recovering Definition 3.7.

We can now define the generic version of the map π_μ from Definition 3.1:

Definition 3.11. For a partition $\mu \vdash r$ as in Definition 3.1, we define the map

$$\pi_\mu : U_r^{\underline{d}} \longrightarrow U_\mu^{\underline{d}},$$

to be the restriction of the map from Definition 3.1 to the $\mathfrak{sl}$ zero-weight spaces of the source and target.

Example 3.12. The generic analogue of Example 3.3 is:

$$\begin{array}{|c|c|}\hline 1,6&1\\\hline 2,3&4\\\hline 4,5&2\\\hline 7,8&3\\\hline\end{array} \xrightarrow{\pi_{(2,2)}} \begin{array}{|c|c|}\hline 1,2,3,6&1,4\\\hline 4,5,7,8&2,3\\\hline\end{array} + \begin{array}{|c|c|}\hline 1,4,5,6&1,2\\\hline 2,3,7,8&3,4\\\hline\end{array} + \begin{array}{|c|c|}\hline 1,6,7,8&1,3\\\hline 2,3,4,5&2,4\\\hline\end{array}.$$

If instead of the partition $(2, 2)$ we take $\mu = (2, 1, 1) = (1^2 2)$, then we have

$$\begin{array}{|c|c|}\hline 1,6&1\\\hline 2,3&4\\\hline 4,5&2\\\hline 7,8&3\\\hline\end{array} \xrightarrow{\pi_{(2,1,1)}} \begin{array}{|c|c|}\hline 1,2,3,6&1,4\\\hline 4,5&2\\\hline 7,8&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 1,4,5,6&1,2\\\hline 2,3&4\\\hline 7,8&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 1,6,7,8&1,3\\\hline 2,3&4\\\hline 4,5&2\\\hline\end{array}$$
$$+ \begin{array}{|c|c|}\hline 2,3,4,5&2,4\\\hline 1,6&1\\\hline 7,8&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 2,3,7,8&3,4\\\hline 1,6&1\\\hline 4,5&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 4,5,7,8&2,3\\\hline 1,6&1\\\hline 2,3&4\\\hline\end{array}.$$

Note that if we compose $\pi_{(2,1,1)}$ with the multiplication map that collapses together the last two rows of a block in $U_{(2,1,1)}^{(2,1)}$, then we obtain the map $2 \cdot \pi_{(2,2)}$.

Definition 3.13 (generic multiprolongations). We write $I_\mu = I_\mu^{\underline{d}}$ for the kernel of the map π_μ (introduced in Definition 3.11), and $I_r = I_r^{\underline{d}}$ for the intersection of I_μ, as μ ranges over partitions of r with at most (exactly) k parts. We refer to I_r as the set of *generic equations* for $\sigma_k(SV_{\underline{d}}(\mathbb{P}V_1^* \otimes \cdots \otimes \mathbb{P}V_n^*))$, or *generic multiprolongations* (see Proposition 3.4).

3B.2. *Tableaux.* The maps π_μ, for various partitions μ, are $S_{\underline{r}}$-equivariant, so to understand them it suffices to analyze them irreducible representation by irreducible representation. Recall that irreducible $S_{\underline{r}}$-representations are classified by n-partitions $\lambda \vdash^n \underline{r}$, so we fix one such. This gives rise to a Young symmetrizer c_λ as explained in Section 2C, and all the data of π_μ (concerning the λ-parts of its kernel and image) is contained in its restriction to the λ-highest weight spaces of the source and target, that is, in the map

$$\pi_\mu = \pi_\mu(\lambda) : c_\lambda \cdot U_r^{\underline{d}} \longrightarrow c_\lambda \cdot U_\mu^{\underline{d}}.$$

We now introduce the tableau formalism that's fundamental for the proof of our main results, giving a combinatorial perspective on the analysis of the kernels and images of the maps π_μ, which are the main objects we're after.

The representations $U_\mu^{\underline{d}}$ are spanned by blocks M as in Definition 3.10, hence the vector spaces $c_\lambda \cdot U_\mu^{\underline{d}}$ are spanned by elements of the form $c_\lambda \cdot M$, which we shall represent as n-tableaux, according to the following definition.

Definition 3.14. Given a partition $\mu = (\mu_1, \dots, \mu_t) \vdash r$, an n-partition $\lambda \vdash^n \underline{r}$ and a block $M \in U_\mu^{\underline{d}}$, we associate to the element $c_\lambda \cdot M \in c_\lambda \cdot U_\mu^{\underline{d}}$ the n-tableau

$$T = (T^1, \dots, T^n) = T^1 \otimes \cdots \otimes T^n$$

of shape λ, obtained as follows. Suppose that the block M has the set α_j^i in its i-th row and j-th column. Then we set equal to i the entries in the boxes of T^j indexed by elements of α_j^i (recall from Section 2C that the boxes of a tableau are indexed canonically: from left to right and top to bottom). Note that each tableau T^j has entries $1, \dots, t$, with i appearing exactly $\mu_i \cdot d_j$ times.

Note also that in order to construct the n-tableau T we have made a choice of the ordering of the rows of M: interchanging rows i and i' when $\mu_i = \mu_{i'}$ should yield the same element $M \in U_\mu^{\underline{d}}$, therefore we identify the corresponding n-tableaux that differ by interchanging the entries equal to i and i'.

Example 3.15. We let $n = 2$, $\underline{d} = (2, 1)$, $r = 4$, $\mu = (2, 2)$ as in Example 3.2, and consider the 2-partition $\lambda = (\lambda^1, \lambda^2)$, with $\lambda^1 = (5, 3)$, $\lambda^2 = (2, 1, 1)$. We have the situation depicted in Figure 1.

Let's write down the action of the map π_μ on the tableaux of Figure 1:

$$\pi_\mu\left(\begin{array}{ccccc} 1&2&2&3&3\\ 1&4&4&& \end{array} \otimes \begin{array}{cc} 1&3\\ 4&\\ 2& \end{array}\right)$$

$$= \begin{array}{ccccc} 1&1&1&2&2\\ 1&2&2&& \end{array} \otimes \begin{array}{cc} 1&2\\ 2&\\ 1& \end{array} + \begin{array}{ccccc} 1&2&2&1&1\\ 1&2&2&& \end{array} \otimes \begin{array}{cc} 1&1\\ 2&\\ 2& \end{array} + \begin{array}{ccccc} 1&2&2&2&2\\ 1&1&1&& \end{array} \otimes \begin{array}{cc} 1&2\\ 1&\\ 2& \end{array}.$$

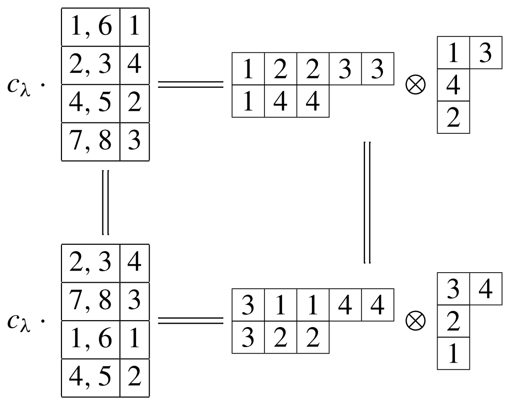


Figure 1. (See Example 3.15.)

We collect in the following lemma the basic relations that n-tableaux satisfy.

Lemma 3.16. *Fix an n partition $\lambda \vdash^n \underline{r}$, and let T be an n-tableau of shape λ.*

(1) *If σ is a permutation of the entries of T that preserves the set of entries in each column of T, then $\sigma(T) = \operatorname{sgn}(\sigma) \cdot T$. In particular, if T has repeated entries in a column, then $T = 0$.*

(2) *If σ is a permutation of the entries of T that interchanges columns of the same size of some tableau T^j, then $\sigma(T) = T$.*

(3) *Assume that one of the tableaux of T, say T^j has a column C of size t with entries $a_1, a_2, \dots, a_t$, and that b is an entry of T^j to the right of C. Let σ_i denote the transposition that interchanges a_i with b. We have*

$$T = \sum_{i=1}^{t} \sigma_i(T).$$

We write this as

$$\begin{array}{|c|c|}\hline a_1 & b \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_i & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_t & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \sum_{i=1}^{t} \begin{array}{|c|c|}\hline a_1 & a_i \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} b & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} a_t & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} ,$$

disregarding the entries of T that don't get perturbed.

Proof. (1) follows from the fact that if $\sigma \in C_\lambda$ is a column permutation, then $b_\lambda \cdot \sigma = -b_\lambda$.

(2) follows from the fact that if σ permutes columns of the same size, then $\sigma \in R_\lambda$ is a permutation that preserves the rows of the canonical n-tableau of shape λ (so in particular $a_\lambda \cdot \sigma = a_\lambda$), and σ commutes with b_λ. It follows that

$$c_\lambda \cdot \sigma = a_\lambda \cdot (b_\lambda \cdot \sigma) = a_\lambda \cdot (\sigma \cdot b_\lambda) = (a_\lambda \cdot \sigma) \cdot b_\lambda = a_\lambda \cdot b_\lambda = c_\lambda.$$

(3) follows from Corollary 3.22 (note the rest of the proof uses the formalism of Section 3B.3 below). Let us assume first that all entries $a_1, \dots, a_t, b$ are distinct. If $\tilde{T}$ is the n-tableau obtained by circling the entries $a_1, \dots, a_t, b$, then

$$\tilde{T} = \begin{array}{|c|c|}\hline \textcircled{a_1} & b \\ \hline \vdots & \\ \cline{1-1} \textcircled{a_i} & \\ \cline{1-1} \vdots & \\ \cline{1-1} \textcircled{a_t} & \\ \hline \end{array} \quad - \sum_{i=1}^{t} \begin{array}{|c|c|}\hline \textcircled{a_1} & a_i \\ \hline \vdots & \\ \cline{1-1} \textcircled{b} & \\ \cline{1-1} \vdots & \\ \cline{1-1} \textcircled{a_t} & \\ \hline \end{array} \quad .$$

By skew-symmetry on columns (part (1)), the effect of circling t entries in the same column of a tableau T is precisely multiplying T by $t!$. It follows that we can rewrite the relation above as

$$\tilde{T} = t! \cdot \left(\begin{array}{|c|c|}\hline a_1 & b \\ \hline \vdots & \\ \cline{1-1} a_i & \\ \cline{1-1} \vdots & \\ \cline{1-1} a_t & \\ \hline \end{array} \quad - \sum_{i=1}^{t} \begin{array}{|c|c|}\hline a_1 & a_i \\ \hline \vdots & \\ \cline{1-1} b & \\ \cline{1-1} \vdots & \\ \cline{1-1} a_t & \\ \hline \end{array} \right).$$

By Corollary 3.22, $\tilde{T} = 0$, which combined with the preceding equality yields the desired relation.

Now if $a_1, \dots, a_t, b$ are not distinct, then either $a_i = a_j$ for some $i \neq j$, or $b = a_i$ for some i. If $a_i = a_j$, then T and $\sigma_k(T)$, $k \neq i, j$, have repeated entries in the column C, hence they are zero. Relation (3) becomes then $0 = \sigma_i(T) + \sigma_j(T)$. But this is true by part (1), because $\sigma_i(T)$ and $\sigma_j(T)$ differ by a column transposition.

Assume now that $b = a_i$ for some i. Then $\sigma_j(T)$ has repeated entries in the column C for $j \neq i$, thus relation (3) becomes $T = \sigma_i(T)$, which is true because $a_i = b$. □

There is one last ingredient that we need to introduce in the generic setting, namely *generic flattenings*.

3B.3. *Generic flattenings.*

Definition 3.17 (generic flattenings). For a decomposition $\underline{d} = A + B$, where $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$ (so $d_i = a_i + b_i$ for $i = 1, \dots, n$), we define a *generic* (A, B)*-flattening* to be an $n \times n$ matrix whose (i, j)-entry is $z_{\alpha^i \cup \beta^j}$, for

$\alpha^i = (\alpha^i_1, \dots, \alpha^i_n)$, $\beta^i = (\beta^i_1, \dots, \beta^i_n)$, with $|\alpha^i_j| = a_j$, $|\beta^i_j| = b_j$, and such that for fixed j, the sets α^i_j, β^i_j form a partition of the set $\{1, \dots, rd_j\}$.

We write $F^{k,r}_{A,B}$ for the subspace of $U^{\underline{d}}_r$ spanned by expressions of the form

$$[\alpha^1, \dots, \alpha^k \,|\, \beta^1, \dots, \beta^k] \cdot z_{\gamma^{k+1}} \cdots z_{\gamma^r},$$

where $[\alpha^1, \dots, \alpha^k \,|\, \beta^1, \dots, \beta^k] = \det(z_{\alpha^i \cup \beta^j})$, $\alpha^i = (\alpha^i_1, \dots, \alpha^i_n)$, $\beta^i = (\beta^i_1, \dots, \beta^i_n)$, $\gamma^i = (\gamma^i_1, \dots, \gamma^i_n)$, with $|\alpha^i_j| = a_j$, $|\beta^i_j| = b_j$ and $|\gamma^i_j| = d_j$, and such that for fixed j, the sets α^i_j, β^i_j, γ^i_j form a partition of the set $\{1, \dots, rd_j\}$. We refer to the elements of $F^{k,r}_{A,B}$ as *generic flattening equations.*

Example 3.18. Take $n = 2$, $\underline{d} = (2, 1)$ and $r = 4$, as usual. Take $A = (1, 1)$, $B = (1, 0)$ and $k = 3$. A typical element of $F^{3,4}_{A,B}$ looks like

$$\begin{aligned} D &= [(\{1\},\{1\}),(\{3\},\{4\}),(\{7\},\{3\}) | (\{6\},\{\}),(\{2\},\{\}),(\{8\},\{\})] \cdot z_{(\{4,5\},\{2\})} \\ &= \det \begin{bmatrix} z_{(\{1,6\},\{1\})} & z_{(\{1,2\},\{1\})} & z_{(\{1,8\},\{1\})} \\ z_{(\{3,6\},\{4\})} & z_{(\{3,2\},\{4\})} & z_{(\{3,8\},\{4\})} \\ z_{(\{7,6\},\{3\})} & z_{(\{7,2\},\{3\})} & z_{(\{7,8\},\{3\})} \end{bmatrix} \cdot z_{(\{4,5\},\{2\})}. \end{aligned}$$

Expanding the determinant, we obtain

$$D = \begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 3,2 & 4 \\ \hline 7,8 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 3,6 & 4 \\ \hline 7,8 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1,8 & 1 \\ \hline 3,2 & 4 \\ \hline 7,6 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 3,8 & 4 \\ \hline 7,2 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,8 & 1 \\ \hline 3,6 & 4 \\ \hline 7,2 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 3,8 & 4 \\ \hline 7,6 & 3 \\ \hline 4,5 & 2 \\ \hline \end{array}.$$

Notice that all the blocks in this expansion coincide, except in the entries 2, 6, 8 that get permuted in all possible ways. Let's multiply now D with the Young symmetrizer c_λ for $\lambda = (\lambda^1, \lambda^2)$, $\lambda^1 = (5, 3)$ and $\lambda^2 = (2, 1, 1)$. We get

$$\begin{aligned} c_\lambda \cdot D = {} & \begin{array}{|c|c|c|c|c|}\hline 1&2&2&4&4 \\ \hline 1&3&3 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1&4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|c|c|c|}\hline 1&1&2&4&4 \\ \hline 2&3&3 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1&4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \\ & - \begin{array}{|c|c|c|c|c|}\hline 1&2&2&4&4 \\ \hline 3&3&1 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1&4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|c|c|c|}\hline 1&3&2&4&4 \\ \hline 1&3&2 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1&4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \\ & + \begin{array}{|c|c|c|c|c|}\hline 1&3&2&4&4 \\ \hline 2&3&1 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1&4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&1&2&4&4 \\ \hline 3&3&2 \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|}\hline 1&4 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}. \end{aligned}$$

Note that all the 2-tableaux in this expression coincide, except in the 2nd, 6th and 8th box of their first tableau, which get permuted in all possible ways. We represent $c_\lambda \cdot D$ by a 2-tableau with the entries in boxes 2, 6 and 8 of its first tableau circled (see also Definition 3.19 below):

$$c_\lambda \cdot D = \begin{array}{ccccc} 1 & \textcircled{2} & 2 & 4 & 4 \\ \textcircled{1} & 3 & \textcircled{3} & & \end{array} \otimes \begin{array}{cc} 1 & 4 \\ 3 & \\ 2 & \end{array}.$$

To reformulate this one last time, we write

$$\begin{array}{ccccc} 1 & \textcircled{2} & 2 & 4 & 4 \\ \textcircled{1} & 3 & \textcircled{3} & & \end{array} \otimes \begin{array}{cc} 1 & 4 \\ 3 & \\ 2 & \end{array} = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \cdot \sigma \left(\begin{array}{ccccc} 1 & 2 & 2 & 4 & 4 \\ 1 & 3 & 3 & & \end{array} \otimes \begin{array}{cc} 1 & 4 \\ 3 & \\ 2 & \end{array} \right),$$

where $S_3 = S_{\{1,2,3\}}$ is the symmetric group on the circled entries.

Definition 3.19. Let A, B and $F^{k,r}_{A,B}$ as in Definition 3.17, let

$$D = [\alpha^1, \dots, \alpha^k \,|\, \beta^1, \dots, \beta^k] \cdot z_{\gamma^{k+1}} \cdots z_{\gamma^r} \in F^{k,r}_{A,B},$$

and let $\lambda \vdash^n \underline{r} = (rd_1, \dots, rd_n)$. We let $\gamma^i = \alpha^i \cup \beta^i$ for $i = 1, \dots, k$, and consider $T = c_\lambda \cdot m$ the n-tableau corresponding to the monomial $m = z_{\gamma^1} \cdots z_{\gamma^r}$. We represent $c_\lambda \cdot D \in \operatorname{hwt}_\lambda(F^{k,r}_{A,B})$ as the n-tableau T with the entries in the boxes corresponding to the elements of $\alpha^1, \dots, \alpha^k$ circled. Alternatively, we can circle the entries in the boxes corresponding to the elements of $\beta^1, \dots, \beta^k$.

It follows that a spanning set for $\operatorname{hwt}_\lambda(F^{k,r}_{A,B})$ can be obtained as follows: take all the subsets $\mathscr{C} \subset \{1, \dots, r\}$ of size k, and consider all the n-tableaux T with a_j (alternatively b_j) of each of the elements of $\mathscr{C}$ circled in T^j. Of course, because of the symmetry of the alphabet $\{1, \dots, r\}$, it's enough to only consider $\mathscr{C} = \{1, \dots, k\}$, so that the only entries we ever circle are $1, 2, \dots, k$.

Continuing with Example 3.18, we have

$$c_\lambda \cdot D = \begin{array}{ccccc} 1 & \textcircled{2} & 2 & 4 & 4 \\ \textcircled{1} & 3 & \textcircled{3} & & \end{array} \otimes \begin{array}{cc} 1 & 4 \\ 3 & \\ 2 & \end{array} = \begin{array}{ccccc} \textcircled{1} & 2 & \textcircled{2} & 4 & 4 \\ 1 & \textcircled{3} & 3 & & \end{array} \otimes \begin{array}{cc} \textcircled{1} & 4 \\ \textcircled{3} & \\ \textcircled{2} & \end{array}.$$

Our goal is to reduce the statement of Theorem 4.1 to an equivalent statement that holds in the generic setting, and thus transform our problem into a combinatorial one. More precisely, we would like to say that the space of generic flattening equations coincides with the intersection of the kernels of the (generic) maps π_μ, and that this is enough to conclude the same about the nongeneric case. One issue that arises is that we don't know at this point (although it seems very tempting to assert) that the zero-weight space of the space of flattening equations coincides with the space of generic flattening equations. Section 3C will show how to take care of this issue, and how to reduce all our questions to the generic setting.

3B.4. *1-flattenings.* In this section we focus on the space of generic 1-flattening equations, $F_1 = F_1^{k,r}$, defined as the subspace of $U_{\underline{r}}^{\underline{d}}$ given by

$$F_1^{k,r} = \sum_{\substack{A+B=\underline{d} \\ |A|=1}} F_{A,B}^{k,r}.$$

We shall see that F_1 has a very simple representation theoretic description, which by the results of the next section will carry over to the nongeneric case.

Proposition 3.20. *With the notations above, we have*

$$F_1 = \bigoplus_{\substack{\lambda \vdash^n \underline{r} \\ \lambda_k \neq 0}} (U_{\underline{r}}^{\underline{d}})_\lambda,$$

where $(U_{\underline{r}}^{\underline{d}})_\lambda$ *denotes the* λ*-part of the representation* $U_{\underline{r}}^{\underline{d}}$, *and* $\lambda_k \neq 0$ *means* $\lambda_k^j \neq 0$ *for some* $j = 1, \ldots, n$, *that is, some partition* λ^j *has at least k parts.*

Proof. We divide the proof into two parts:

a) If $\lambda \vdash^n \underline{r}$ is an n-partition with some λ^j having at least k parts, and T is an n-tableau of shape λ, then $T \in F_1$.

b) If $\lambda \vdash^n \underline{r}$ is an n-partition with all λ^j having less than k parts, then $c_\lambda \cdot F_1 = 0$.

Let us start by proving part a). We assume that λ^j has at least k parts and consider T an n-tableau of shape λ. If T^j has repeated entries in its first column, then $T = 0$. Otherwise, we may assume that the first column of T^j has entries $1, 2, \ldots, t$ in this order, where t is the number of parts of λ^j, $t \geq k$. We consider the n-tableau $\tilde{T}$ obtained from T by circling the entries $1, 2, \ldots, k$ in the first column of T^j. We have

$$\tilde{T} = T^1 \otimes \cdots \otimes \begin{array}{|c|c|} \hline \textcircled{1} & \cdots \\ \hline \textcircled{2} & \cdots \\ \hline \vdots & \vdots \\ \hline \textcircled{k} & \cdots \\ \hline k+1 & \cdots \\ \hline \vdots & \\ \cline{1-1} \end{array} \otimes \cdots \otimes T^n,$$

that is,

$$\tilde{T} = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \sigma(T),$$

where S_k denotes the symmetric group on the circled entries. Since $\sigma(T)$ differs from T by the column permutation σ, it follows by the skew-symmetry of tableaux that $\sigma(T) = \mathrm{sgn}(\sigma) \cdot T$. This shows that

$$\tilde{T} = k! \cdot T \iff T = \frac{1}{k!} \cdot \tilde{T} \in F_1,$$

proving a).

To prove b), let

$$D = [\alpha^1, \dots, \alpha^k | \beta^1, \dots, \beta^k] \cdot z_{\gamma^{k+1}} \cdots z_{\gamma^r} \in F^{k,r}_{A,B},$$

for some $A + B = \underline{d}$ with $|A| = 1$. We have that $b_\lambda \cdot D$ is a linear combination of terms that look like D, so in order to prove that $c_\lambda = a_\lambda \cdot b_\lambda$ annihilates D, it suffices to show that $a_\lambda \cdot D = 0$.

We have $A = (a_1, \dots, a_n)$ with $a_j = 1$ for some j and $a_i = 0$ for $i \neq j$. We can thus think of each of $\alpha^1, \dots, \alpha^k$ as specifying a box in the partition λ^j. Since λ^j has less than k parts, it means that two of these boxes, say p and q, lie in the same row of λ^j. Let $\sigma = (p, q)$ be the transposition of the two boxes. σ is an element in the group R_λ of permutations that preserve the rows of the canonical n-tableau of shape λ (Section 2C), which means that $a_\lambda \cdot \sigma = a_\lambda$. However,

$$\begin{aligned}\sigma \cdot [\alpha^1, \dots, \alpha^p, \dots, \alpha^q, \dots, \alpha^k | \beta^1, \dots, \beta^k] \\ &= [\alpha^1, \dots, \alpha^q, \dots, \alpha^p, \dots, \alpha^k | \beta^1, \dots, \beta^k] \\ &= -[\alpha^1, \dots, \alpha^p, \dots, \alpha^q, \dots, \alpha^k | \beta^1, \dots, \beta^k],\end{aligned}$$

since interchanging two rows/columns of a matrix changes the sign of its determinant. We get

$$a_\lambda \cdot D = (a_\lambda \cdot \sigma) \cdot D = a_\lambda \cdot (\sigma \cdot D) = a_\lambda \cdot (-D) = -a_\lambda \cdot D,$$

hence $a_\lambda \cdot D = 0$, as desired. □

Remark 3.21. The nongeneric 1-flattening equations give the equations of the so-called *subspace varieties* (see [Landsberg 2012, Section 7.1; Weyman 2003, Proposition 7.1.2]), and in fact this statement is essentially equivalent to our Proposition 3.20 via the results of the next section, namely Proposition 3.27.

Corollary 3.22. *Let $\mathscr{C} \subset \{1, \dots, r\}$ be a subset of size k. If λ is an n-partition with each λ^j having less than k parts, and $\tilde{T}$ is an n-tableau of shape λ, with one of each entries of $\mathscr{C}$ in $\tilde{T}^j$ circled, then $\tilde{T} = 0$. More generally, with no assumptions on λ, if the circled entries in $\tilde{T}^j$ all lie in columns of size less than k, then $\tilde{T} = 0$.*

Proof. The first part follows directly from Proposition 3.20, since $\tilde{T}$ is a 1-flattening equation, and the space of 1-flattening equations doesn't have nonzero λ-parts when λ is such that each of its partitions have less than k parts.

For the more general statement, we can apply the argument for part b) of the proof of the previous proposition. If

$$D = [\alpha^1, \dots, \alpha^k \,|\, \beta^1, \dots, \beta^k] \cdot z_{\gamma^{k+1}} \cdots z_{\gamma^r} \in F_1$$

is such that each α^i corresponds to a box of $\tilde{T}^j$ situated in a column of size less than k, then since column permutations don't change the columns of the boxes corresponding to the α^i, it follows that $b_\lambda \cdot D$ is a combination of expressions D' with the same properties as D. To show that $c_\lambda \cdot D = 0$, it is thus enough to prove that $a_\lambda \cdot D = 0$. The proof of this statement is identical to the one in the preceding proposition. □

Many of the classical results on the representation theory connected to secant varieties of Segre–Veronese varieties can be recovered from the generic perspective. For some of them, including the Cauchy formula or Strassen's equations, and their generalization by Landsberg and Manivel [2008], the reader may consult [Raicu 2011, Chapter 5].

3C. *Polarization and specialization.* In this section $V_1, \dots, V_n$ are again vector spaces of arbitrary dimensions, $\dim(V_j) = m_j$, $j = 1, \dots, n$. Let $\underline{r} = (r_1, \dots, r_n)$ be a sequence of positive integers, and let

$$W = V_1^{\otimes r_1} \otimes \cdots \otimes V_n^{\otimes r_n}.$$

Let $S_{\underline{r}}$ denote the product of symmetric groups $S_{r_1} \times \cdots \times S_{r_n}$, and let $G \subset S_{\underline{r}}$ be a subgroup. Consider the natural (right) action of $S_{\underline{r}}$ on W obtained by letting S_{r_i} act by permuting the factors of $V_i^{\otimes r_i}$. More precisely, we write the pure tensors in W as

$$v = \bigotimes_{i,j} v_{ij}, \quad \text{with } v_{ij} \in V_j,\ j = 1, \dots, n,\ i = 1, \dots, r_j,$$

and for an element $\sigma = (\sigma^1, \dots, \sigma^n) \in S_{\underline{r}}$, we let

$$v * \sigma = \bigotimes_{i,j} v_{\sigma^j(i)j}.$$

This action commutes with the (left) action of $\mathrm{GL}(V)$ on W, and restricts to an action of G on W. It follows that W^G is a $\mathrm{GL}(V)$-subrepresentation of W.

Proposition 3.23. *Continuing with the notation above, let $U = W^G$, $U' = \mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1})$. Let $\lambda \vdash^n \underline{r}$ be an n-partition with λ^j having at most m_j parts. The multiplicity of $S_\lambda V$ in U is the same with that of $[\lambda]$ in U'.*

Moreover, there exist polarization and specialization maps

$$P_\lambda : \mathrm{wt}_\lambda(U) \longrightarrow U', \qquad Q_\lambda : U' \longrightarrow \mathrm{wt}_\lambda(U),$$

with the following properties:

(1) *Q_λ is surjective.*

(2) *P_λ is a section of Q_λ.*

(3) *P_λ and Q_λ restrict to maps between $\mathrm{hwt}_\lambda(U)$ and $\mathrm{hwt}_\lambda(U')$ which are inverse to each other.*

Proof. The first part is a consequence of Schur–Weyl duality (Lemma 2.2) and Frobenius reciprocity (Lemma 2.4). We start with the identification

$$U = W^G = \mathrm{Hom}_G(\mathbf{1}, \mathrm{Res}_G^{S_{\underline{r}}}(W)).$$

Using Schur–Weyl duality we get

$$W = V_1^{\otimes r_1} \otimes \cdots \otimes V_n^{\otimes r_n} = \bigoplus_{\lambda \vdash^n \underline{r}} [\lambda] \otimes S_\lambda V;$$

therefore the previous equality becomes

$$U = \bigoplus_{\lambda \vdash^n \underline{r}} \mathrm{Hom}_G(\mathbf{1}, \mathrm{Res}_G^{S_{\underline{r}}}([\lambda])) \otimes S_\lambda V.$$

Frobenius reciprocity now yields

$$\mathrm{Hom}_G(\mathbf{1}, \mathrm{Res}_G^{S_{\underline{r}}}([\lambda])) = \mathrm{Hom}_{S_{\underline{r}}}(\mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1}), [\lambda]) = \mathrm{Hom}_{S_{\underline{r}}}(U', [\lambda]).$$

We get

$$U = \bigoplus_{\lambda \vdash^n \underline{r}} \mathrm{Hom}_{S_{\underline{r}}}(U', [\lambda]) \otimes S_\lambda V,$$

hence the multiplicity of $S_\lambda V$ in U coincides with that of $[\lambda]$ in U', as long as $S_\lambda V \neq 0$, that is, as long as m_j is at least as large as the number of parts of the partition λ^j.

It follows that the vector spaces $\mathrm{hwt}_\lambda(U)$ and $\mathrm{hwt}_\lambda(U')$ have the same dimension, equal to the multiplicity of $S_\lambda V$ and $[\lambda]$ in U and U' respectively. We next construct explicit maps P_λ, Q_λ inducing isomorphisms of vector spaces between the two spaces.

We identify an element $\sigma = (\sigma^1, \ldots, \sigma^n) \in S_{\underline{r}}$ with the "tensor"

$$\bigotimes_{i,j} \sigma^j(i),$$

and consider the (regular) representation of $S_{\underline{r}}$ on the vector space R with basis consisting of the tensors σ for $\sigma \in S_{\underline{r}}$. The left action of $S_{\underline{r}}$ on R is given by

$$\sigma \cdot \bigotimes_{i,j} a_{ij} = \bigotimes_{i,j} \sigma^j(a_{ij}),$$

while the right action is given by

$$\bigotimes_{i,j} a_{ij} * \sigma = \bigotimes_{i,j} a_{\sigma^j(i)j}.$$

We consider the vector space map $Q_\lambda : R \to W$ given by

$$\bigotimes_{i,j} a_{ij} \longrightarrow \bigotimes_{i,j} g_j(a_{ij}),$$

where $g_j : \{1, \dots, r_j\} \to \mathcal{B}_j$ is the map sending a to x_{ij} if the a-th box of λ^j is contained in the i-th row of λ^j (or equivalently if $\lambda_1^j + \cdots + \lambda_{i-1}^j < a \le \lambda_1^j + \cdots + \lambda_i^j$). The image of Q_λ is $\mathrm{wt}_\lambda(W)$. It is clear that if $a = \bigotimes_{i,j} a_{ij}$ and $b = \bigotimes_{i,j} b_{ij}$, then $Q_\lambda(a) = Q_\lambda(b)$ if and only if $a = \sigma \cdot b$ for $\sigma \in S_{\underline{r}}$ a permutation that preserves the rows of the canonical n-tableau of shape λ. It follows that we can define $P_\lambda : \mathrm{wt}_\lambda(W) \to R$ by

$$P_\lambda(Q_\lambda(a)) = \frac{1}{\lambda!} a_\lambda \cdot a,$$

where a_λ is the row symmetrizer defined in Section 2C, hence P_λ is a section of Q_λ.

Notice that P_λ and Q_λ are maps of right $S_{\underline{r}}$-modules, that is, they respect the $*$-action of $S_{\underline{r}}$ on R and $\mathrm{wt}_\lambda(W)$ respectively.

Let us prove now that P_λ and Q_λ restrict to inverse isomorphisms between $\mathrm{hwt}_\lambda(R) = c_\lambda \cdot R$ (recall from Section 2C that c_λ denotes the Young symmetrizer corresponding to λ) and $\mathrm{hwt}_\lambda(W)$. The two spaces certainly have the same dimension (take $G = \{e\}$ to be the trivial subgroup of $S_{\underline{r}}$ and apply the first part of the proposition), so it's enough to prove that for $a' \in \mathrm{hwt}_\lambda(R)$

a) $Q_\lambda(a') \in \mathrm{hwt}_\lambda(W)$, and

b) $P_\lambda(Q_\lambda(a')) = a'$.

To see why part b) is true, note that

$$P_\lambda(Q_\lambda(a_\lambda \cdot a)) = \frac{1}{\lambda!} \cdot a_\lambda^2 \cdot a = a_\lambda \cdot a,$$

that is, $P_\lambda \circ Q_\lambda$ fixes $a_\lambda \cdot R$. Since $\mathrm{hwt}_\lambda(R) = c_\lambda \cdot R \subset a_\lambda \cdot R$, it follows that $P_\lambda(Q_\lambda(a')) = a'$. To prove a) we need to show that $Q_\lambda(a')$ is fixed by the Borel (recall the definition of the Borel subgroup from 2C). It's enough to do this when

$$a' = c_\lambda \cdot a, \qquad a = \bigotimes_{i,j} a_{ij}.$$

The pure tensor a corresponds to an element $\sigma \in S_{\underline{r}}$, so we can write $a = e * \sigma$, where

$$e = \bigotimes_{i,j} e_{ij}$$

is the “identity” tensor, $e_{ij} = i$ for all i, j. It follows that

$$Q_\lambda(a') = Q_\lambda(c_\lambda \cdot a) = Q_\lambda(a_\lambda \cdot b_\lambda \cdot e * \sigma) = \lambda! \cdot Q_\lambda(b_\lambda \cdot e) * \sigma.$$

Since the $*$-action commutes with the action of the Borel, it is then enough to prove that $Q_\lambda(b_\lambda \cdot e)$ is fixed by the Borel. But this is a direct computation:

$$Q_\lambda(b_\lambda \cdot e) = \bigotimes_{i,j} x_{1j} \wedge \cdots \wedge x_{(\lambda^j)'_i j},$$

where $(\lambda^j)'$ denotes the conjugate partition of λ^j, so that in fact $(\lambda^j)'_i$ denotes the number of entries in the i-th column of λ^j. In any case, it is clear from the formula of $Q_\lambda(b_\lambda \cdot e)$ that it is invariant under the Borel, proving the claim that P_λ and Q_λ restrict to inverse isomorphisms between $\mathrm{hwt}_\lambda(W)$ and $\mathrm{hwt}_\lambda(R)$.

To finish the proof of the proposition, it suffices to notice that, by Remark 2.3, we have the identities

$$U = W^G = W * s \quad \text{and} \quad U' = \mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1}) = R * s, \quad \text{where } s = \sum_{g \in G} g.$$

Now since P_λ, Q_λ respect the $*$-action, it follows that they restrict to inverse isomorphisms between

$$\mathrm{hwt}_\lambda(W) * s = \mathrm{hwt}_\lambda(W * s) = \mathrm{hwt}_\lambda(U) \quad \text{and} \quad \mathrm{hwt}_\lambda(R) * s = \mathrm{hwt}_\lambda(R * s) = \mathrm{hwt}_\lambda(U'),$$

proving the last part of the proposition. □

We shall apply Proposition 3.23 with $\underline{r} = (rd_1, \dots, rd_n)$ and

$$U = U_r^{\underline{d}}(V) = S_{(r)}(S_{(d_1)} V_1 \otimes \cdots \otimes S_{(d_n)} V_n),$$

or more generally

$$U = U_\mu^{\underline{d}}(V) = \bigotimes_{j=1}^{s} S_{(i_j)}(S_{(\mu_j d_1)} V_1 \otimes \cdots \otimes S_{(\mu_j d_n)} V_n),$$

the source and target respectively of the map π_μ in Definition 3.1. W is now the representation

$$W = V_1^{\otimes r d_1} \otimes \cdots \otimes V_n^{\otimes r d_n}.$$

We start with $U = U_r^{\underline{d}}(V)$. We have $U = W^G$, where $G = (S_{d_1} \times \cdots \times S_{d_n})^r \wr S_r$ is the wreath product between $(S_{d_1} \times \cdots \times S_{d_n})^r$ and S_r, and G is regarded as a subgroup of $S_{\underline{r}}$ as follows. First of all, we identify an element $\sigma \in G$ with a collection

$$\sigma = \Big((\sigma_j^k)_{\substack{j=1,\dots,n,\\ k=1,\dots,r}}, \tau \Big), \quad \text{where } \sigma_j^k \in S_{d_j},\ \tau \in S_r.$$

Then, we think of $S_{\underline{r}} = S_{rd_1} \times \cdots \times S_{rd_n}$ as a product of symmetric groups, where S_{rd_j} is the group of permutations of the set $\mathfrak{D}_j = \{1, \ldots, rd_j\}$. Furthermore, we think of an element $\sigma \in G$ as an element of $S_{\underline{r}}$ by letting σ_j^k act as a permutation of

$$\{(\tau(k)-1)\cdot d_j + 1, \ldots, \tau(k)\cdot d_j\} \subset \mathfrak{D}_j.$$

For example, when $d_1 = \cdots = d_n = 1$, G is just the group S_r, diagonally embedded in S_r^n. With this G, we let $U' = \mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1})$.

One can now see why the representation $U_r^{\underline{d}}$, as defined in the previous section, can be identified with U'. Recall that $U_r^{\underline{d}}$ was defined as a space of $r \times n$ blocks with certain identifications. Consider the block

$$M = \begin{array}{|c|c|c|c|}\hline \{1,\ldots,d_1\} & \{1,\ldots,d_2\} & \cdots & \{1,\ldots,d_n\} \\ \hline \{d_1{+}1,\ldots,2d_1\} & \{d_2{+}1,\ldots,2d_2\} & \cdots & \{d_n{+}1,\ldots,2d_n\} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \{(r{-}1)d_1{+}1,\ldots,rd_1\} & \{(r{-}1)d_2{+}1,\ldots,rd_2\} & \cdots & \{(r{-}1)d_n{+}1,\ldots,rd_n\} \\ \hline \end{array}.$$

G acts trivially on M (because each σ_j^k does, and because the effect of τ is just permuting the rows of M), and all the other blocks are obtained from M by the action of some element of $S_{\underline{r}}$. One should think of the span of M thus as the trivial representation $\mathbf{1}$ of G that's induced to $S_{\underline{r}}$.

It is probably best to forget at this point that U' was the zero-weight space of a certain representation, and just think of it abstractly as the induced representation $\mathrm{Ind}_G^{S_{\underline{r}}}(\mathbf{1})$, with its realization as a space of blocks. An important point to notice now is that for any decomposition $\underline{d} = A + B$ and any k, r, we have

$$P_\lambda(\mathrm{wt}_\lambda(F_{A,B}^{k,r}(V))) \subset F_{A,B}^{k,r}, \quad \text{and} \quad Q_\lambda(F_{A,B}^{k,r}) \subset \mathrm{wt}_\lambda(F_{A,B}^{k,r}(V)),$$

where $F_{A,B}^{k,r}$ (Definition 3.17) is the generic version of $F_{A,B}^{k,r}(V)$ (Definition 2.6). This means that on the corresponding λ-highest weight spaces, P_λ and Q_λ restrict to isomorphisms

$$\mathrm{hwt}_\lambda(F_{A,B}^{k,r}(V)) \simeq \mathrm{hwt}_\lambda(F_{A,B}^{k,r}).$$

Example 3.24. Here's an example of *specialization*, involving blocks we're already familiar with. Let $n = 2,\ d_1 = 2,\ d_2 = 1,\ r = 4,\ \lambda^1 = (5, 3),\ \lambda^2 = (2, 1, 1)$. For the specialization map Q_λ we have

$$M = \begin{array}{|c|c|}\hline 1,6 & 1 \\ \hline 2,3 & 4 \\ \hline 4,5 & 2 \\ \hline 7,8 & 3 \\ \hline \end{array} \xrightarrow{Q_\lambda} \begin{array}{|c|c|}\hline 1,2 & 1 \\ \hline 1,1 & 3 \\ \hline 1,1 & 1 \\ \hline 2,2 & 2 \\ \hline \end{array} = M'.$$

Q_λ sends 1, 2, 3, 4, 5 from the first column of M to 1, because boxes 1, 2, 3, 4, 5 of λ^1 lie in the first row of λ^1, and it sends 6, 7, 8 to 2 because boxes 6, 7, 8 of λ^1 lie in its second row. A similar description holds for the second column of M and λ^2.

Although we won't write down explicitly $P_\lambda(M')$ in this example (see the example below for a concrete illustration of the action of P_λ), we will just mention that $P_\lambda(M')$ is the average of the blocks that specialize to M' via the specialization map Q_λ. Of course, M is one such block, but there are many more others.

Example 3.25. Let $n = 3$, $d_1 = d_2 = d_3 = 1$ and $\lambda^1 = \lambda^2 = \lambda^3 = (2, 1)$. If

$$m = z_{(\{1\},\{1\},\{2\})} z_{(\{2\},\{3\},\{1\})} z_{(\{3\},\{2\},\{3\})} \in U',$$

then $Q_\lambda(m) = z_{(\{1\},\{1\},\{1\})} z_{(\{1\},\{2\},\{1\})} z_{(\{2\},\{1\},\{2\})} \in U$ and

$$\begin{aligned} P_\lambda(Q_\lambda(m)) \\ = \tfrac{1}{8}\big(& z_{(\{1\},\{1\},\{2\})} z_{(\{2\},\{3\},\{1\})} z_{(\{3\},\{2\},\{3\})} + z_{(\{2\},\{1\},\{2\})} z_{(\{1\},\{3\},\{1\})} z_{(\{3\},\{2\},\{3\})} \\ &+ z_{(\{1\},\{1\},\{1\})} z_{(\{2\},\{3\},\{2\})} z_{(\{3\},\{2\},\{3\})} + z_{(\{2\},\{1\},\{1\})} z_{(\{1\},\{3\},\{2\})} z_{(\{3\},\{2\},\{3\})} \\ &+ z_{(\{1\},\{2\},\{2\})} z_{(\{2\},\{3\},\{1\})} z_{(\{3\},\{1\},\{3\})} + z_{(\{2\},\{2\},\{2\})} z_{(\{1\},\{3\},\{1\})} z_{(\{3\},\{1\},\{3\})} \\ &+ z_{(\{1\},\{2\},\{1\})} z_{(\{2\},\{3\},\{2\})} z_{(\{3\},\{1\},\{3\})} + z_{(\{2\},\{2\},\{1\})} z_{(\{1\},\{3\},\{2\})} z_{(\{3\},\{1\},\{3\})}\big). \end{aligned}$$

When $U = U_\mu^{\underline{d}}(V)$, with $\mu = (\mu_1^{i_1} \cdots \mu_s^{i_s})$, we get $U = W^G$, where

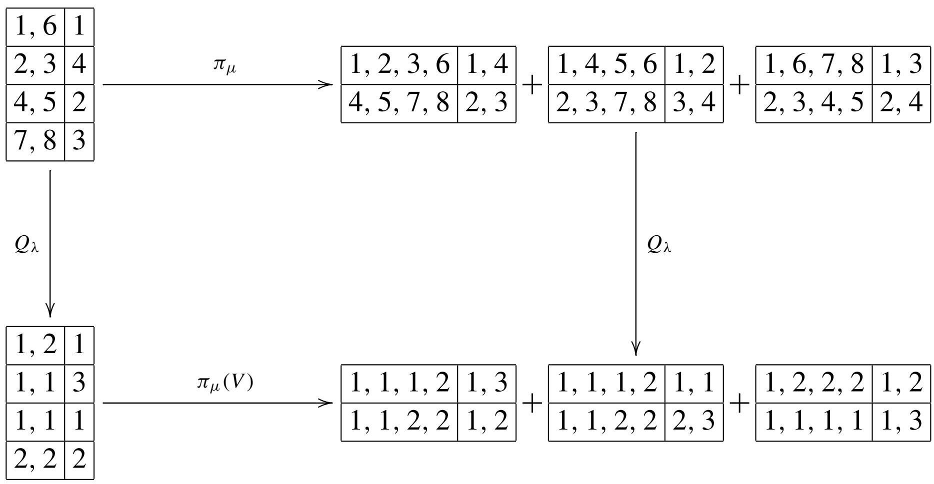
$$G = \bigotimes_{j=1}^{s} \left((S_{\mu_j d_1} \times \cdots \times S_{\mu_j d_n})^{i_j} \wr S_{i_j} \right).$$

It follows that $U' = \operatorname{Ind}_G^{S_r}(\mathbf{1}) = U_\mu^{\underline{d}}$ with the realization as a space of blocks explained in the preceding section.

We note that the maps π_μ and $\pi_\mu(V)$ commute with the polarization and specialization maps P_λ, Q_λ, that is, we have a commutative diagram:

$$\begin{array}{ccc} U_r^{\underline{d}} & \underset{P_\lambda}{\overset{Q_\lambda}{\rightleftarrows}} & \mathrm{wt}_\lambda(U_r^{\underline{d}}(V)) \\ {\scriptstyle \pi_\mu}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\mu(V)} \\ U_\mu^{\underline{d}} & \underset{P_\lambda}{\overset{Q_\lambda}{\rightleftarrows}} & \mathrm{wt}_\lambda(U_\mu^{\underline{d}}(V)) \end{array} \tag{3-1}$$

Example 3.26. Let $\underline{d} = (2, 1)$, $r = 4$, $\mu = (2, 2)$, $\lambda^1 = (5, 3)$, $\lambda^2 = (2, 1, 1)$. We only illustrate the specialization map Q_λ, with the above diagram transposed:



Restricting (3-1) to the λ-highest weight spaces, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{hwt}_\lambda(U^d_r) & \underset{P_\lambda}{\overset{Q_\lambda}{\rightleftarrows}} & \mathrm{hwt}_\lambda(U^d_r(V)) \\ \downarrow \pi_\mu & & \downarrow \pi_\mu(V) \\ \mathrm{hwt}_\lambda(U^d_\mu) & \underset{P_\lambda}{\overset{Q_\lambda}{\rightleftarrows}} & \mathrm{hwt}_\lambda(U^d_\mu(V)) \end{array}$$

where all the horizontal maps are isomorphisms. This shows that the λ-highest weight spaces of the kernels of π_μ and $\pi_\mu(V)$ get identified via the polarization and specialization maps, and therefore the same is true for I^d_r and $I^d_r(V)$: the generic multiprolongations and multiprolongations correspond to each other via polarization and specialization. We summarize the conclusions of this section:

Proposition 3.27. *The polarization and specialization maps P_λ and Q_λ restrict to maps between generic flattening equations and flattening equations, inducing inverse isomorphisms*

$$\mathrm{hwt}_\lambda(F^{k,r}_{A,B}) \simeq \mathrm{hwt}_\lambda(F^{k,r}_{A,B}(V)).$$

They also restrict to maps between the kernels of the generic π_μ and the nongeneric ones, inducing inverse isomorphisms

$$\mathrm{hwt}_\lambda(\ker(\pi_\mu)) \simeq \mathrm{hwt}_\lambda(\ker(\pi_\mu(V))).$$

As a consequence, P_λ and Q_λ yield inverse isomorphisms between the λ-highest weight spaces of generic and nongeneric multiprolongations

$$\mathrm{hwt}_\lambda(I^d_r) \simeq \mathrm{hwt}_\lambda(I^d_r(V)).$$

It follows that in order to show that flattening equations coincide with multiprolongations for the variety of secant lines to a Segre–Veronese variety (Theorem 4.1), it suffices to prove their equality in the generic setting.

4. The secant line variety of a Segre–Veronese variety

This section is based on the techniques developed in the preceding one. We use the reduction to the "generic" situation to work out the analysis of the equations and coordinate rings of secant varieties of Segre–Veronese varieties in the first new interesting case, that of varieties of secant lines. We show how in the case of the secant line variety $\sigma_2(X)$ of a Segre–Veronese variety X, the combinatorics of tableaux can be used to show that the "generic equations" coincide with the 3×3 minors of "generic flattenings". In particular, we confirm a conjecture of Garcia, Stillman and Sturmfels, which constitutes the special case when X is a Segre variety. We also obtain the representation-theoretic description of the homogeneous coordinate ring of $\sigma_2(X)$, which in particular can be used to compute the Hilbert function of $\sigma_2(X)$. In the special cases when $\sigma_2(X)$ coincides with the ambient space, we obtain the decomposition into irreducible representations of certain plethystic compositions. Section 4A describes the statements of our results, while Section 4B contains the details of the proofs.

4A. *Main result and consequences.* The main result of this work is the description of the generators of the ideal of the variety of secant lines to a Segre–Veronese variety, together with the decomposition of its coordinate ring as a sum of irreducible representations.

Theorem 4.1. *Let $X = SV_{d_1,\dots,d_n}(\mathbb{P}V_1^* \times \mathbb{P}V_2^* \times \cdots \times \mathbb{P}V_n^*)$ be a Segre–Veronese variety, where each V_i is a vector space of dimension at least 2 over a field K of characteristic zero. The ideal of $\sigma_2(X)$ is generated by 3×3 minors of flattenings, and moreover, for every nonnegative integer r we have the decomposition of the degree r part of its homogeneous coordinate ring*

$$K[\sigma_2(X)]_r = \bigoplus_{\substack{\lambda=(\lambda^1,\dots,\lambda^n)\\ \lambda^i \vdash r d_i}} (S_{\lambda^1}V_1 \otimes \cdots \otimes S_{\lambda^n}V_n)^{m_\lambda},$$

where m_λ is obtained as follows. Set

$$f_\lambda = \max_{i=1,\dots,n} \left\lceil \frac{\lambda_2^i}{d_i} \right\rceil, \qquad e_\lambda = \lambda_2^1 + \cdots + \lambda_2^n.$$

If some partition λ^i has more than two parts, or if $e_\lambda < 2f_\lambda$, then $m_\lambda = 0$. If $e_\lambda \geq r-1$, then $m_\lambda = \lfloor r/2 \rfloor - f_\lambda + 1$, unless e_λ is odd and r is even, in which case

$m_\lambda = \lfloor r/2 \rfloor - f_\lambda$. If $e_\lambda < r-1$ and $e_\lambda \geq 2f_\lambda$, then $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda + 1$, unless e_λ is odd, in which case $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda$.

As a consequence, we derive the conjecture by Garcia, Stillman and Sturmfels concerning the equations of the secant line variety of a Segre variety.

Corollary 4.2. *The GSS conjecture (Conjecture 1.1) holds: namely, the ideal of the variety of secant lines to a Segre product of projective spaces is generated by* 3×3 *minors of flattenings.*

Proof. This is the first part of Theorem 4.1 when $d_1 = d_2 = \cdots = d_n = 1$. □

Combining Theorem 4.1 with known dimension calculations for secant varieties of Segre and Veronese varieties, we obtain two interesting plethystic formulas. We do not claim that these formulas are new: since all the vector spaces involved have dimension two, the representation theory of $\mathfrak{sl}_2$ can be also used to deduce them. However, we hope that the simple idea we present, together with a generalization of the last part of Theorem 4.1 to higher secant varieties, would yield new plethystic formulas for decomposing Schur functors applied to tensor products of representations.

Corollary 4.3. a) *Let V_1, V_2, V_3 be vector spaces of dimension two over a field K of characteristic zero, and let r be a positive integer. We have the decomposition*

$$\mathrm{Sym}^r(V_1 \otimes V_2 \otimes V_3) = \bigoplus_{\substack{\lambda=(\lambda^1,\lambda^2,\lambda^3)\\ \lambda^i \vdash r}} (S_{\lambda^1}V_1 \otimes S_{\lambda^2}V_2 \otimes S_{\lambda^3}V_3)^{m_\lambda},$$

where m_λ is obtained as follows. Set $f_\lambda = \max\{\lambda^1_2, \lambda^2_2, \lambda^3_2\}$, $e_\lambda = \lambda^1_2 + \lambda^2_2 + \lambda^3_2$. If some partition λ^i has more than two parts, or if $e_\lambda < 2f_\lambda$, then $m_\lambda = 0$. If $e_\lambda \geq r-1$, then $m_\lambda = \lfloor r/2 \rfloor - f_\lambda + 1$, unless e_λ is odd and r is even, in which case $m_\lambda = \lfloor r/2 \rfloor - f_\lambda$. If $e_\lambda < r-1$ and $e_\lambda \geq 2f_\lambda$, then $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda + 1$, unless e_λ is odd, in which case $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda$.

b) *Let V_1, V_2 be vector spaces of dimension two over a field K of characteristic zero, let r be a positive integer and let $\mu = (\mu_1, \mu_2)$ be a partition of r with at most two parts. We have the decomposition*

$$S_\mu(V_1 \otimes V_2) = \bigoplus_{\substack{\lambda=(\lambda^1,\lambda^2)\\ \lambda^i \vdash r}} (S_{\lambda^1}V_1 \otimes S_{\lambda^2}V_2)^{m_\lambda},$$

with $m_\lambda = m_{(\lambda^1,\lambda^2,\mu)}$, where $m_{(\lambda^1,\lambda^2,\mu)}$ is as defined in part a).

Proof. Part a) follows from the fact that the secant line variety of a 3-factor Segre variety X has the expected dimension, namely $2 \cdot \dim(X) + 1$. In the case we are interested in, $X = \mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ has dimension 3 and is a subvariety of

$\mathbb{P}^{2\cdot 2\cdot 2-1} = \mathbb{P}^7$, so $\sigma_2(X)$ fills in the whole space. This means that the coordinate ring of $\sigma_2(X)$ and $\mathbb{P}^7$ coincide, that is,

$$K[\sigma_2(X)] = \mathrm{Sym}(V_1 \otimes V_2 \otimes V_3),$$

and therefore we can use the description of Theorem 4.1 to compute

$$K[\sigma_2(X)]_r = \mathrm{Sym}^r(V_1 \otimes V_2 \otimes V_3).$$

As for part b), let V_3 be another vector space of dimension two. Part a) tells us how to decompose $\mathrm{Sym}^r(V_1 \otimes V_2 \otimes V_3)$ in general. On the other hand, regarding $V_1 \otimes V_2 \otimes V_3$ as the tensor product between the vector spaces $V_1 \otimes V_2$ and V_3, we can use Cauchy's formula to obtain

$$\mathrm{Sym}^r(V_1 \otimes V_2 \otimes V_3) = \bigoplus_{\mu \vdash r} S_\mu(V_1 \otimes V_2) \otimes S_\mu V_3.$$

Now the desired formula for the multiplicity of the irreducible representations occurring in $S_\mu(V_1 \otimes V_2)$ follows by combining the formula from part a) with the Cauchy formula depicted above. □

Corollary 4.4. *Let V be a vector space of dimension two over a field K of characteristic zero. We have the decomposition*

$$\mathrm{Sym}^r(\mathrm{Sym}^3(V)) = \bigoplus_{\lambda \vdash 3r} (S_\lambda V)^{m_\lambda},$$

where m_λ is obtained as follows. Set

$$f_\lambda = \left\lceil \frac{\lambda_2}{3} \right\rceil, \qquad e_\lambda = \lambda_2.$$

If λ has more than two parts, or if $e_\lambda < 2f_\lambda$ (that is, $\lambda_2 = 1$), then $m_\lambda = 0$. If $e_\lambda \geq r-1$, then $m_\lambda = \lfloor r/2 \rfloor - f_\lambda + 1$, unless e_λ is odd and r is even, in which case $m_\lambda = \lfloor r/2 \rfloor - f_\lambda$. If $e_\lambda < r-1$ and $e_\lambda \geq 2f_\lambda$, then $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda + 1$, unless e_λ is odd, in which case $m_\lambda = \lfloor (e_\lambda+1)/2 \rfloor - f_\lambda$.

Proof. This follows from the fact that $\sigma_2(\mathrm{Ver}_3(\mathbb{P}^1))$, the secant line variety of the twisted cubic, fills in the space, hence its coordinate ring is $\mathrm{Sym}(\mathrm{Sym}^3(V))$. Using the description in Theorem 4.1 with $n = 1$, $d_1 = 3$ and $V = V_1$ of dimension 2, we obtain the desired formula. □

4B. *Proof of the main result.*

Proof of Theorem 4.1. We start by outlining the main steps of the proof. We fix a sequence of positive integers $\underline{d} = (d_1, \dots, d_n)$ and a positive degree r, and let

$\underline{r} = (rd_1, \dots, rd_n)$. By Proposition 3.27, it suffices to prove the generic version of the theorem. More precisely, we let

$$F = \sum_{A+B=\underline{d}} F^{3,r}_{A,B} \subset U^{\underline{d}}_r$$

be the set of all generic flattening equations, and let F_i denote the subspace spanned by those generic flattening equations with $|A| = i$. (As the rest of the proof will imply, we have $F = F_1 + F_2 + F_3$; see [Raicu 2010; Raicu 2011, Chapter 6] for more precise results in this direction in the case $n = 1$ of the Veronese variety.)

Recall that $I = I^{\underline{d}}_r$ denotes the space of generic multiprolongations of degree r (Definition 3.13), that is, I is the kernel of the map

$$\pi = \bigoplus_{\mu=(\mu_1,\mu_2)\vdash r} \pi_\mu : U = U^{\underline{d}}_r \longrightarrow \bigoplus_{\mu=(\mu_1,\mu_2)\vdash r} U^{\underline{d}}_\mu.$$

We have $F \subset I$, by combining Lemma 2.5 with Proposition 3.27. We will show that $F = I$ and that the image of π decomposes into irreducible $S_{\underline{r}}$-representations as

$$\pi(U) = \bigoplus_{\lambda \vdash^n \underline{r}} [\lambda]^{m_\lambda},$$

where m_λ is as defined in the statement of the theorem.

We list the main steps below. The details will occupy the rest of the section.

Step 0: If λ is an n-partition with some λ^i having at least three parts, then $\mathrm{hwt}_\lambda(U) = \mathrm{hwt}_\lambda(F)$ (Proposition 3.20), hence $\mathrm{hwt}_\lambda(F) = \mathrm{hwt}_\lambda(I)$ since $F \subset I \subset U$. Moreover, this also shows that $m_\lambda = 0$.

Step 1: We fix an n-partition λ of $\underline{r}$ with each λ^i having at most two parts. We identify each tableau T with a certain graph G. We show that graphs containing odd cycles are contained in F.

Step 2: We show that the λ-highest weight space of U/F is spanned by bipartite graphs that are as connected as possible, that is, that are either connected, or a union of a tree and some isolated nodes.

Step 3: We introduce the notion of *type* associated to a graph G as in *Step 2*, encoding the sizes of the sets in the bipartition of the maximal component of G. We show that if G_1, G_2 have the same type, then $G_1 = \pm G_2$ (modulo F).

Step 4: If we let

$$\pi = \bigoplus_{\substack{\mu\vdash r \\ \mu=(a\geq b)}} \pi_\mu : U \longrightarrow \bigoplus_{\substack{\mu\vdash r \\ \mu=(a\geq b)}} U^n_\mu,$$

and if G_i are graphs of distinct types (not contained in F), then the elements $\pi(G_i)$ are linearly independent. This suffices to prove that F and the kernel of π are the same, that is, that $F = I$. The formulas for the multiplicities m_λ follow from counting the number of G_i, that is, the number of possible types.

Remark 4.5. A careful look at the details of the proof will show that, in fact, in Steps 2 and 3, instead of working modulo F it is enough to work modulo the subspace of U spanned by graphs containing odd cycles, and that moreover, any graph containing an odd cycle is a linear combination of graphs containing a triangle. This implies that in fact F, and hence I, is spanned by graphs containing a triangle. This is the structure alluded to in the introduction that makes a graph (that is, a generic polynomial) "vanish" on the secant line variety of a Segre–Veronese variety.

Step 1. We fix an n-partition λ of $\underline{r}$ with $\lambda^i = (\lambda^i_1 \geq \lambda^i_2 \geq 0)$, for $i = 1, \dots, n$. For each n-tableau T of shape λ we construct a graph G with r vertices labeled by the elements of the alphabet $\mathcal{A} = \{1, \dots, r\}$ as follows. For each tableau T^i of T and column

$$\begin{array}{|c|}\hline x \\ \hline y \\ \hline\end{array}$$

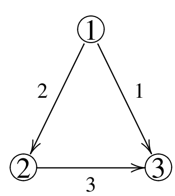
of T^i of length 2, G has an oriented edge (x, y) which we label by the index i. We will often refer to the labels of the edges of G as colors. Note that we allow G to have multiple edges between two vertices (some call such G a multigraph), but at any given vertex there can be at most d_i incident edges of color i. Since we think of two n-tableaux as being the same if they differ by a permutation of $\mathcal{A}$, we shall also identify two graphs if they differ by a relabeling of their nodes. Note that a graph G determines an element in $\mathrm{hwt}_\lambda(U)$, by considering a tableau T with columns

$$\begin{array}{|c|}\hline x \\ \hline y \\ \hline\end{array}$$

for each edge (x, y) of G. The order of the columns of T is not determined by G, but part (2) of Lemma 3.16 states that any such T yields the same element of $\mathrm{hwt}_\lambda(U)$. The orientation of the edges of our graphs will be mostly irrelevant: reversing the orientation of an edge of $G = T$ will correspond to changing G to $-G$ (see part (1) of Lemma 3.16). When we talk about connectedness and cycles, we don't take into account the orientation of the edges.

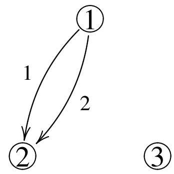
Example 4.6. The graph

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} =$$



is connected and has a cycle of length 3, while

$$\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1}\end{array} =$$



is disconnected and has a cycle of length 2.

From now on we work modulo F, and more precisely, inside the λ-highest weight space of (U/F). This space is generated by the graphs described above. The main result of *Step 1* is this:

Proposition 4.7. *If G has an odd cycle, then $G = 0$ (that is, G is in F).*

We first need to establish some fundamental relations, that will be used throughout the rest of the proof.

Lemma 4.8. *The following relations between tableaux/graphs hold (see the interpretation below):*

a) $\begin{array}{|c|}\hline x\\\hline y\\\hline\end{array} = -\begin{array}{|c|}\hline y\\\hline x\\\hline\end{array}$; *in particular,* $\begin{array}{|c|}\hline x\\\hline x\\\hline\end{array} = 0$.

b) $\begin{array}{|c|c|}\hline x & z\\\hline y \\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline x & y\\\hline z \\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline z & x\\\hline y \\\cline{1-1}\end{array}$.

c) $\begin{array}{|c|c|}\hline x & z\\\hline y \\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z \\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline x & y\\\hline z \\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & z\\\hline y \\\cline{1-1}\end{array}$.

d) $\begin{array}{|c|c|}\hline x & z\\\hline y \\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & z\\\hline y \\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z \\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline x & z\\\hline y \\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z \\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z \\\cline{1-1}\end{array}$.

Interpretation: For an expression $E = \sum_T a_T \cdot T$, where the T are n-tableaux of shape λ, we say that $E = 0$ if

$$\sum_T a_T \cdot T \in F \subset U.$$

If all the n tableaux occurring in the expression E contain the same n-subtableau S, then we suppress S entirely from the notation (see also the comment in part (3) of Lemma 3.16).

Example 4.9. One interpretation of part b) of Lemma 4.8 could be that

$$\begin{array}{|c|c|}\hline a & b\\\hline c & d\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline x & z & t\\\hline y \\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline a & b\\\hline c & d\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline x & y & t\\\hline z \\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline a & b\\\hline c & d\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline z & x & t\\\hline y \\\cline{1-1}\end{array},$$

for any $\{a, b, c, d\} = \{x, y, z, t\} = \{1, 2, 3, 4\}$. The 2-subtableau S is in this case

$$S = \begin{array}{|c|c|}\hline a & b\\\hline c & d\\\hline\end{array} \otimes \begin{array}{|c|}\hline t\\\hline\end{array}.$$

Proof of Lemma 4.8. a) is part (1) of Lemma 3.16.

b) follows from part (3) of the same lemma (since all columns of our tableaux have size at most two).

c) We have

$$\begin{array}{|c|c|}\hline \textcircled{x} & \textcircled{y}\\\hline \textcircled{z}\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline \textcircled{x} & \textcircled{z}\\\hline \textcircled{y}\\\cline{1-1}\end{array} = \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\cdot\sigma\left(\begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array}\right) = 0,$$

(because the left hand side is contained in F_2). Using parts a) and b) repeatedly, we can express everything in terms of

$$\begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array} \quad\text{and}\quad \begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array},$$

and after simplifications, the preceding equation becomes

$$3\cdot\left(\begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array} - \begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array}\right) = 0.$$

d) Part c) states that any tensor expression in

$$a = \begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array} \quad\text{and}\quad b = \begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array}$$

does not depend on the order in which a and b appear, so we can think of the pure tensors in a, b as commuting monomials in a, b. Writing

$$\begin{array}{|c|c|}\hline y & x\\\hline z\\\cline{1-1}\end{array} = b - a,$$

we can translate

$$\begin{array}{|c|c|}\hline \textcircled{x} & \textcircled{z}\\\hline \textcircled{y}\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline \textcircled{x} & \textcircled{z}\\\hline \textcircled{y}\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline \textcircled{x} & \textcircled{y}\\\hline \textcircled{z}\\\cline{1-1}\end{array} = \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\cdot\sigma\left(\begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array}\right) = 0$$

into

$$a^2b - a^2(b-a) + (a-b)^2b - b^2a - (b-a)^2a + b^2(a-b) = 0,$$

which simplifies to $3(a^2b - ab^2) = 0$, that is, $a^2b = ab^2$, or

$$\begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline x & z\\\hline y\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline x & y\\\hline z\\\cline{1-1}\end{array}.$$ □

Corollary 4.10. *If G is a graph having a connected component H consisting of two nodes joined by an odd number of edges, then $G = 0$.*

Proof. Interchanging the labels of the two nodes of H preserves G, but by part a) of Lemma 4.8, it also transforms G into $(-1)^e G$, where e is the number of edges in H. Since e is odd, $G = 0$. □

Corollary 4.11. *If G is a graph containing cycles of length* 1 *or* 3, *then* $G = 0$.

Proof. If G has a cycle of length 1, this follows from part a) of Lemma 4.8. If G has a cycle of length 3, we may assume this cycle is $C = \textcircled{1} \to \textcircled{2} \to \textcircled{3} \to \textcircled{1}$. We have several cases to analyze, depending on the colors of the edges in this cycle.

If the edges in C have distinct colors, we need to prove that

$$\ytableaushort{13,2} \otimes \ytableaushort{21,3} \otimes \ytableaushort{12,3} = 0.$$

We have by part b) of Lemma 4.8 applied to the middle tableau that

$$\ytableaushort{13,2} \otimes \ytableaushort{21,3} \otimes \ytableaushort{12,3} = \ytableaushort{13,2} \otimes \ytableaushort{12,3} \otimes \ytableaushort{12,3} - \ytableaushort{13,2} \otimes \ytableaushort{13,2} \otimes \ytableaushort{12,3} = 0,$$

where the last equality is part d) of the same lemma.

If the edges of C have the same color, we need to prove that

$$\ytableaushort{112,233} = 0.$$

We have

$$0 = \ytableaushort{{\textcircled{1}}{\textcircled{1}}{\textcircled{2}},{\textcircled{2}}{\textcircled{3}}{\textcircled{3}}} = \ytableaushort{112,233} - \ytableaushort{221,133} - \ytableaushort{332,211} - \ytableaushort{113,322} + \ytableaushort{223,311} + \ytableaushort{331,122}$$
$$= \ytableaushort{112,233} + \ytableaushort{121,233} + \ytableaushort{211,332} + \ytableaushort{112,323} + \ytableaushort{211,323} + \ytableaushort{121,332} = 6 \cdot \ytableaushort{112,233},$$

where the penultimate equality follows from skew-symmetry on rows, while the last one follows from part (2) of Lemma 3.16.

Finally, suppose that the edges of C have two colors, say $(1, 2)$ and $(1, 3)$ have the same color. We need to prove that

$$\ytableaushort{1123,23} \otimes \ytableaushort{21,3} = 0.$$

As in the preceding case,

$$\begin{aligned} 0 &= \ytableaushort{{\textcircled{1}}{\textcircled{1}}{\textcircled{2}}{\textcircled{3}},{\textcircled{2}}{\textcircled{3}}} \otimes \ytableaushort{{\textcircled{2}}{\textcircled{1}},{\textcircled{3}}} \\ &= \ytableaushort{1123,23} \otimes \ytableaushort{21,3} - \ytableaushort{2213,13} \otimes \ytableaushort{12,3} - \ytableaushort{3321,21} \otimes \ytableaushort{23,1} \\ &\quad - \ytableaushort{1132,32} \otimes \ytableaushort{31,2} + \ytableaushort{2231,31} \otimes \ytableaushort{32,1} + \ytableaushort{3312,12} \otimes \ytableaushort{13,2} \\ &= 6 \cdot \ytableaushort{1123,23} \otimes \ytableaushort{21,3}, \end{aligned}$$

where the last equality follows by utilizing repeatedly parts a) and c) of Lemma 4.8.

For example, we have for the second term that

$$\begin{array}{|c|c|c|c|}\hline 2&2&1&3\\\hline 1&3\\\cline{1-2}\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&2&1&3\\\hline 2&3\\\cline{1-2}\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = -\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3\\\cline{1-2}\end{array} \otimes \begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array},$$

where the last equality follows by applying part c) of Lemma 4.8 in the form

$$\begin{array}{|c|c|}\hline y&z\\\hline x\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline z&y\\\hline x\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline z&y\\\hline x\\\cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline y&z\\\hline x\\\cline{1-1}\end{array},$$

with

$$\begin{array}{|c|c|}\hline y&z\\\hline x\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|}\hline z&y\\\hline x\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}. \qquad \square$$

Corollary 4.12. *If an n-tableau T contains the columns*

$$C_1 = \begin{array}{|c|}\hline x\\\hline y\\\hline\end{array} \quad \textit{and} \quad C_2 = \begin{array}{|c|}\hline x\\\hline z\\\hline\end{array},$$

and T' is obtained from T by interchanging two boxes $\boxed{y}$ and $\boxed{z}$ from the same tableau T^i of T, and not contained in any of C_1, C_2, then $T = T'$ (modulo F).

Proof. If

$$\begin{array}{|c|}\hline y\\\hline z\\\hline\end{array}$$

is a column of T^i then T contains a triangle, hence $T = 0$. Since interchanging y and z transforms T into $T' = -T = 0$, it follows that $T = T'$. We can assume then that y and z don't lie in the same column of T^i. If they both belong to columns of size one of T^i, then interchanging them preserves T (see part (2) of Lemma 3.16). Otherwise we may assume that y belongs to a column of size two in T^i, hence we have the relation

$$\begin{array}{|c|c|}\hline y&z\\\hline *\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline y&*\\\hline z\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline z&y\\\hline *\\\cline{1-1}\end{array} = \begin{array}{|c|c|}\hline z&y\\\hline *\\\cline{1-1}\end{array},$$

where the last equality follows from the fact that any tableau containing C_1, C_2 and

$$\begin{array}{|c|}\hline y\\\hline z\\\hline\end{array}$$

is a graph containing a triangle, that is, it is zero (Corollary 4.11). $\square$

Proof of Proposition 4.7. We show that a graph G (with corresponding tableau T) containing an odd cycle of length at least 5 is a linear combination of graphs with shorter odd cycles. The conclusion then follows by induction from Corollary 4.11. Suppose that $C : \textcircled{1} \to \textcircled{2} \to \cdots \to \textcircled{k} \to \textcircled{1}$ is an odd cycle in G, with $k \geq 5$. We denote by E_i the edge $(i, i+1)$ ($E_k = (k, 1)$).

Let's assume first that there are two consecutive edges of C of the same color: say E_1 and E_2 have color 1. If not all edges of C have color 1, we may assume that E_3 has color 2, so that T contains the subtableau

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}.$$

Since E_1, E_2 have color 1, it follows that $d_1 \geq 2$, hence there are at least two 4's in T^1. One of them is thus not contained in E_5, and therefore in none of the edges of C. We apply Corollary 4.12 with C_1, C_2 the columns corresponding to E_2, E_3, $y = 2$ and $z = 4$. We can thus interchange the 2 in E_1 with a $4 \in T^1$ not in any E_i, obtaining an n-tableau $T' = T$, with T' containing the cycle ① → ④ → ⑤ → ⋯ → ⓚ → ① of length $k - 2$.

If all the E_i have color 1, T contains the subtableau

$$S = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & \cdots & k \\ \hline 2 & 3 & 4 & 5 & \cdots & 1 \\ \hline \end{array} \subset T^1.$$

If there is an edge $(3, 4)$ of G with color different from 1, then we can replace E_3 by that edge and apply the previous case. If $d_1 > 2$ then T^1 has a 4 not contained in any E_i, so we can again use the argument from the previous paragraph. Suppose now that $d_1 = 2$. The proof of Corollary 4.12 shows that we can interchange 3 with 4 in all T^i $(i \neq 1)$, modulo tableaux containing S and an edge $(3, 4)$ of color different from 1. But these we know are zero (modulo F) by the argument above, so we can write $T = T'$ where T' is obtained from T by interchanging all 3's and 4's in T^i for $i \geq 2$. We now use the relation

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 5 & 4 & 3 \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 3 \\ \hline 2 & 4 & 4 & 5 \\ \hline \end{array},$$

to write

$$T' = T'' + T''',$$

where T'' contains the cycle ① → ② → ⑤ → ⋯ → ⓚ → ① of length $k - 2$, and T''' differs from T by interchanging all the 3's and 4's in T, and doing a column transposition in the column of E_3. This shows that $T = T' = 0 - T$, hence $T = 0$.

Finally, we assume that no two consecutive edges have the same color. Since the cycle is odd, we can find three consecutive edges with distinct colors, say E_1, E_2 and E_3, with colors $1, 2$ and 3 respectively. By Corollary 4.12, we have

$$T = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline \end{array}.$$

If the edge E_4 in C doesn't have color 1, then it survives after interchanging 2 and 4 as above, hence T is equal with a graph containing the odd cycle

① → ④ → ⑤ → ⋯ → ⓚ → ①

of length $k-2$.

Suppose now that E_4 has color 1. If the edge E_5 doesn't have color 2, then we may repeat the argument above replacing the edges E_1, E_2 and E_3 with E_2, E_3 and E_4 respectively. Otherwise, T contains the subtableau (with $*=6$ if $k>5$ and $*=1$ if $k=5$),

$$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 2&5\\\hline 3&*\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}=\begin{array}{|c|c|}\hline 1&2\\\hline 4&5\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 2&5\\\hline 3&*\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 3&5\\\hline 4\\\cline{1-1}\end{array}=\begin{array}{|c|c|}\hline 1&2\\\hline 4&5\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 2&5\\\hline 3&*\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 5&3\\\hline 4\\\cline{1-1}\end{array},$$

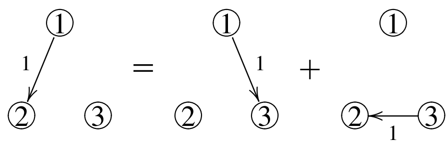
where the first equality follows by interchanging 2 and 4 in the first factor, while the last one follows by interchanging 3 and 5 in the last factor (in both cases we apply Corollary 4.12). It follows that T is equal to a graph containing the odd cycle ① → ④ → ⑤ → $\cdots$ → ⓚ → ① of length $k-2$, concluding the proof. □

Step 2. We first translate the relations in part b) of Lemma 4.8 into *basic operations* on graphs. We start with the following:

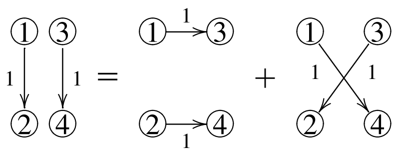
Definition 4.13. A node ⓙ is said to be i-saturated if there are d_i edges of color i incident to ⓙ.

Remark 4.14 (basic operations). Let G be a graph containing an edge $(1,2)$ of color 1. The following relations hold:

(1) If the vertex ③ is not 1-saturated, then $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}=\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}+\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array}$ becomes



(2) If G has an edge $(3,4)$ of color 1, then $\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}=\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline 4&2\\\hline\end{array}$ becomes



Proposition 4.15. *Let* λ *be as before, and let*

$$e_\lambda=\sum_{i=1}^{n}\lambda_2^i.$$

If $e_\lambda\geq r-1$, *then* $\mathrm{hwt}_\lambda(U/F)$ *is spanned by connected graphs. If* $e_\lambda<r-1$, *then* $\mathrm{hwt}_\lambda(U/F)$ *is spanned by graphs G that consist of a tree, together with a collection of isolated nodes.*

Proof. We first show that if G has two connected components H_1, H_2 with H_1 containing a cycle, then we can write $G = G_1 + G_2$, where G_1 and G_2 are graphs obtained from G by joining the components H_1, H_2 together.

Consider an edge $(1, 2)$ contained in a cycle of H_1, having say color 1. Consider a node ③ of H_2 and suppose first it is not 1-saturated. Using the first basic operation of Remark 4.14, we get that $G = G_1 + G_2$, where G_1, G_2 are obtained from G by connecting H_2 to H_1 via an edge of color 1. If ③ is 1-saturated, then in particular there exists at least one edge, say $(3, 4)$, of color 1 in H_2. The second basic operation of Remark 4.14 yields $G = G_1 + G_2$, where G_1, G_2 are obtained from G by connecting H_1 and H_2 via two edges of color 1.

If $e_\lambda \geq r - 1$, then G will contain cycles as long as it is not connected, so iterating the procedure above, we can write G as a linear combination of connected graphs.

If $e_\lambda < r - 1$, the argument above reduces the problem to the case when G is a union of trees, some of which may be isolated nodes. We show that if G has at least two components that are not nodes, then $G = G_1 + G_2$, where G_1, G_2 are unions of trees, and the sizes of the largest components of G_1, G_2 are strictly larger than the size of the largest component of G. Induction on the size of the largest component of G then concludes the proof of the proposition.

Let H_1 be the largest component of G, and let H_2 be another component which isn't a node. If H_2 has only one edge, then $G = 0$ by Corollary 4.10. Consider a leaf of H_1, say ③, and assume first that all edges in H_2 have the same color, say 1. Since H_2 has more than one edge and is connected, it must have a vertex with at least two incident edges of color 1, that is, $d_1 \geq 2$. This means that ③ is not 1-saturated. Let $(1, 2)$ be an edge of H_2 (of color 1). The first basic operation of Remark 4.14 shows that $G = G_1 + G_2$, where G_1, G_2 are obtained from G by expanding its largest component.

Assume now that the edges in H_2 have at least two colors, and that the edge incident to ③ has color 2. Let $(1, 2)$ be an edge of H_2 of color different from 2, say 1. ③ is not 1-saturated, thus we can use the first basic operation of Remark 4.14 as in the preceding case. □

Step 3. Combining *Step 1* with *Step 2* we get that, depending on the n-partition λ, $\mathrm{hwt}_\lambda(U/F)$ is spanned either by connected graphs without odd cycles, or by graphs consisting of a tree and some isolated nodes. These graphs are going to be important for the rest of the proof, so we make the following

Definition 4.16 (MCB graphs). A *maximally connected bipartite (MCB) graph* is either a connected graph without odd cycles, or a graph consisting of a tree together with a collection of isolated nodes.

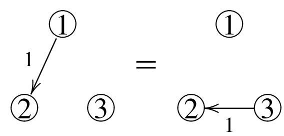
For an MCB graph G, the maximal connected component admits an essentially unique *bipartition* of its vertex set into subsets A, B of sizes $a \geq b$ (that is, vertices in the same subset A or B are not connected by an edge). We say that G has *type* $(a, b; \lambda)$ (or just (a, b) when λ is understood), and that it is *canonically oriented* if all the edges have source in A and target in B (when $a = b$, there are two canonical orientations). We have the following:

Proposition 4.17. *If G_1, G_2 are canonically oriented MCB graphs of type (a, b), then $G_1 = G_2$.*

We first need to refine the relations of Remark 4.14:

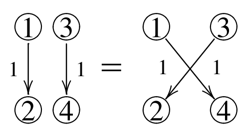
Remark 4.18 (refined basic operations). Suppose that G is an MCB graph with vertex bipartition $A \sqcup B$ as in Definition 4.16.

(1) Assume that ③ is not 1-saturated, $(1, 2)$ is an edge of color 1, and ①, ③ belong to A. If ①, ③ are contained in the same connected component of the graph obtained from G by removing the edge $(1, 2)$, then



This follows from the fact that the conditions above guarantee that the term that was left out from the first basic operation of Remark 4.14 has an odd cycle, and hence equals 0 by Proposition 4.7.

(2) Assume that $(1, 2)$ and $(3, 4)$ are edges of color 1, ①, ③ $\in A$ and ②, ④ $\in B$, and either ① and ③, or ② and ④ are in the same connected component of the graph obtained from G by removing the edges $(1, 2)$ and $(3, 4)$. Then



As above, the missing term from the second basic operation has an odd cycle, and hence equals 0.

Proof of Proposition 4.17. We prove by induction on e_λ (the number of "edges" of λ), that it is possible to get from G_1 to G_2 via a series of refined basic operations. If $e_\lambda = 0$, there is nothing to prove. Suppose now that $e_\lambda > 0$.

We call an edge E of an MCB graph G *nondisconnecting* if the graph obtained from G by removing E is still an MCB graph. More explicitly, if $e_\lambda \geq r$, then E must be contained in a cycle of G, and if $e_\lambda < r$, then one of the endpoints of E must be a leaf of G.

The idea of proof is to reduce to the case when G_1, G_2 have nondisconnecting edges E_1, E_2 of the same color, such that $G'_1 = G_1 \setminus E_1$ and $G'_2 = G_2 \setminus E_2$ are canonically oriented MCB graphs of the same type. Once this is done, we remove E_1, E_2 from G_1, G_2 and apply induction to conclude that G_2' can be obtained from G_1' via a series of refined basic operations (as in Remark 4.18). We then put back the edges E_1, E_2 and lift the sequence of operations to the original graphs. The main difficulty lies in creating the nondisconnecting edges E_1, E_2, which is especially laborious when the graphs contain no cycles. For the convenience of the reader, this case is illustrated by Example 4.19 below.

Inductive step. We will prove later that for any nondisconnecting edge E_2 of G_2 of color c, there exist a sequence of refined basic operations which transforms G_1 into a new graph $\hat{G}_1$ having a nondisconnecting edge E_1 of color c, such that the graphs G_1' and G_2' obtained from $\hat{G}_1$ and G_2 by removing the edges E_1 and E_2 have the same type. Assuming this, by induction we can find a series of refined basic operations that transform G_1' into G_2'. We lift this sequence of operations to $\hat{G}_1$ as follows: the refined basic operations of type (2) are performed just as if the edge E_1 was not contained in $\hat{G}_1$, as well as the operations of type (1) that don't transform an edge E' of color c into one that's incident to E_1; the operations of type (1) involving an edge E' of color c that gets transformed into an edge incident to E_1 are replaced by operations of type (2) involving E' and E_1. It is clear that E_1 remains nondisconnecting along the process, so we end up with the graphs G_1'' and G_2 that coincide after removing the nondisconnecting edges E_1 and E_2 of color c. At most two more refined operations of type (2) (that correspond to correcting the positions of the endpoints of E_1) are then sufficient to transform G_1'' into G_2, concluding the proof.

Creating a nondisconnecting edge when the graphs contain cycles. We show that if $e_\lambda \geq r$ and G_1 has an edge E_1 of color c, then we can find a refined basic operation that makes E_1 nondisconnecting. Suppose that E_1 is disconnecting, and let H_1, H_2 be the connected components of the graph obtained from G_1 by removing the edge E_1. One of H_1, H_2 must contain a cycle, say H_1, and let O, Y be consecutive edges of this cycle, of colors o(*range*) and y(*ellow*) (note that o might coincide with y). If H_2 has a node N that is not o-saturated or not y-saturated, then a refined operation of type (1) involving the node N (as ③) and one of the edges O, Y (as the edge $(1,2)$) will make E_1 a nondisconnecting edge. Otherwise, if every vertex of H_2 is both o- and y-saturated, then there exists a cycle in H_2 consisting of edges of colors o and y (if $o = y$, then since O, Y are incident edges of color o, it means that $d_o \geq 2$, in particular any o-saturated node has at least two incident edges of color o; if $o \neq y$, then any o- and y-saturated node has at least one o-incident and one y- incident edge; in both cases, the nodes in H_2 have at least two incident edges, so we can find a cycle as stated). A refined basic operation of type (2) involving an o-edge on this cycle and O (or an y-edge and Y) will make E_1 nondisconnecting.

Creating a nondisconnecting edge when the graphs have no cycles. If $e_\lambda < r$ and G_2 has a nondisconnecting edge E_2 of color c, then we prove that we can find a sequence of refined basic operations that transforms G_1 into a graph $\hat{G}_1$ containing a nondisconnecting edge E_1 of color c, and moreover $\hat{G}_1 - E_1$ and $G_2 - E_2$ have the same type. We may assume that $e_\lambda = r - 1$, by removing the isolated nodes of G_1 and G_2. Suppose that the graphs G_i have vertex bipartitions $A_i \sqcup B_i$, with $|A_i| = a$, $|B_i| = b$, and that $E_2 = (x, y)$, with ⓨ $\in B_2$ a leaf of G_2. This means that the graph G_2, and hence also G_1, has at most $(b-1)\cdot d_c + 1$ edges of color c, and for any color $c' \neq c$, it has at most $(b-1)\cdot d_{c'}$ edges of color c'. In particular, for any color $c' \neq c$, there exists a node in B_1 which is not c'-saturated. Consider an edge $E = (u, v)$ of color c in G_1, with ⓤ $\in A_1$, ⓥ $\in B_1$. Let H_1, H_2 be the connected components of $G_1 - E$ containing u and v respectively. We prove by descending induction on the size of H_2 that we can make E nondisconnecting, with its endpoint in B_1 being a leaf.

If $H_2 = \{$ⓥ$\}$ then E is nondisconnecting. More generally, if $H_2 \cap B_1 = \{$ⓥ$\}$, then we may assume that all the edges in H_2 have color c. If E' is an edge of H_2 of color $c' \neq c$ (see the second transformation in Example 4.19 below), then there are at most $(b-1)\cdot d_{c'} - 1$ edges of color c' in H_1, so that we can find a vertex C' in H_1 that is not c'-saturated. A refined basic operation of type (1) involving E' and C' decreases the size of H_2 by one, so we can conclude by induction. Assume now that the edges in H_2 have color c. Together with the edge E, we get at least two edges of color c outside H_1, which means that H_1 has at most $(b-1)\cdot d_c - 1$ edges of color c, that is, it has a vertex that is not c-saturated. We now do a refined basic operation of type (1) as before, involving that vertex and an edge of H_2, and conclude by induction.

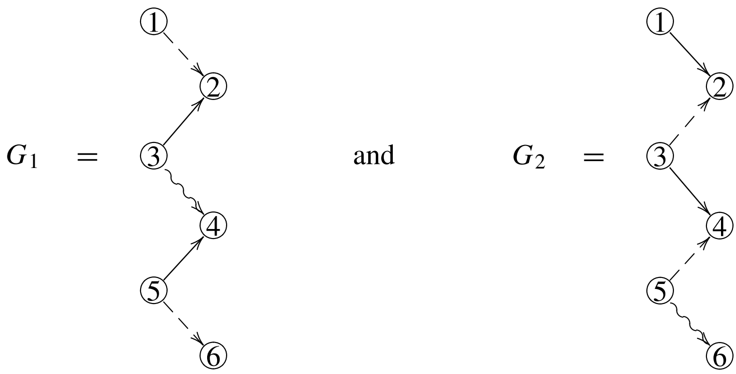
We may now assume $|H_2 \cap B_1| > 1$ (see the first transformation in Example 4.19 below). Therefore there exist distinct edges $Y = (u', v)$ of color y(*ellow*) and $O = (u', v')$ of color o(*range*) in H_2 (y and o might coincide). If $o = c$ then we replace E with (u', v'), which decreases the size of H_2, so that we can conclude by induction. If there exists a vertex $W \in H_1 \cap B_1$ that is not y-saturated, then the refined basic operation involving Y and W decreases the size of H_2. Likewise, if there exists a vertex $W \in H_1 \cap A_1$ that is not o-saturated, then the refined basic operation involving O and W also decreases the size of H_2. We may therefore assume that all nodes in $B_1 \cap H_1$ are y-saturated, and those in $A_1 \cap H_1$ are o-saturated, and show that this leads to a contradiction. If $y = o$, then since u' has two incident edges of color o, we must have $d_o \geq 2$. All the nodes of H_1 being saturated implies that they have degree at least $d_o \geq 2$, so H_1 contains a cycle, which is a contradiction. If $y \neq o$, then H_1 must contain at least $|H_1 \cap A_1|$ edges of color o (since each vertex in $H_1 \cap A_1$ is o-saturated) and at least $|H_1 \cap B_1|$ edges of color y, that is, H_1 contains at least $|H_1|$ edges, hence it can't be a tree. □

Example 4.19. Consider the 3-tableaux

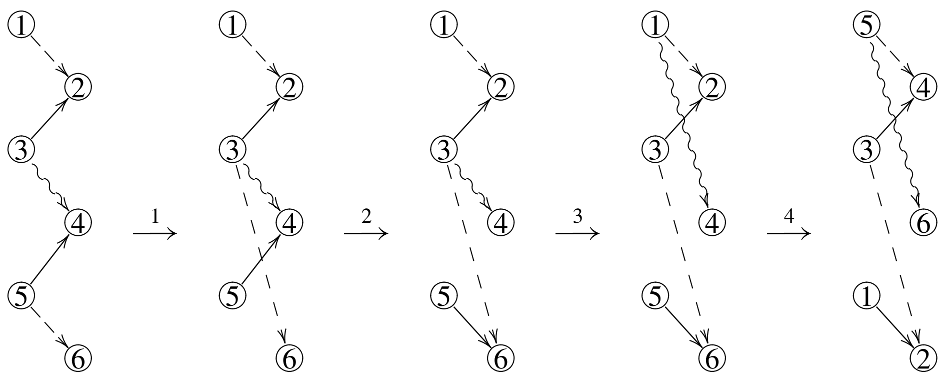
$$T_1 = \begin{array}{|c|c|c|c|}\hline 3&5&1&6\\ \hline 2&4 \\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 3&1&2&5&6\\ \hline 4 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1&5&3&4\\ \hline 2&6 \\ \cline{1-2}\end{array},$$

$$T_2 = \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\ \hline 2&4 \\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 5&1&2&3&4\\ \hline 6 \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 3&5&1&6\\ \hline 2&4 \\ \cline{1-2}\end{array},$$

with corresponding graphs

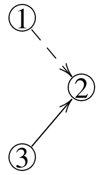
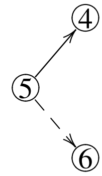


where color 1 corresponds to $\longrightarrow$, color 2 to $\rightsquigarrow$, and color 3 to $\dashrightarrow$. G_1 and G_2 are MCB of the same type (Definition 4.16), and in fact $G_1 = 0$, since it is the same as the graph obtained by reversing the orientation of its 5 edges (an odd number), and this equals $-G_1$ by part a) of Lemma 4.8. However, it is unclear a priori that G_2 is also equal to 0. We use the algorithm described in the proof of Proposition 4.17 to get a sequence of refined basic operations that transforms G_1 into G_2. We first make the edge of G_1 of color 2 nondisconnecting, and then adjust its position (the third step) and relabel the nodes (last step) to get G_2:



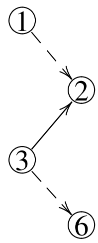
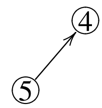
With the notation in the last paragraph of the proof of Proposition 4.17, we have $E = (3, 4)$ a disconnecting edge, $A_1 = \{1, 3, 5\}$, $B_1 = \{2, 4, 6\}$ a bipartition of the vertex set of G_1. We'd like to make E nondisconnecting, with its endpoint in B_1

being a leaf. We have

H_1 =  and H_2 = 

We also have $Y = (5, 4)$ of color $y = \longrightarrow$ and $O = (5, 6)$ of color $o = - - \to$. The unique vertex ② in $H_1 \cap B_1$ is y-saturated, and ① $\in H_1 \cap A_1$ is o-saturated, but $W =$ ③ is not o-saturated. The refined basic operation involving W and O yields the first transformation.

We now have

H_1 =  and H_2 = 

We are in the case $H_2 \cap B_1 = \{ⓥ\} = \{④\}$. The edge $E' = (5, 4)$ has color $c' = \longrightarrow$, different from $c = \rightsquigarrow$. $W =$ ⑥ is a vertex in $H_1 \cap B_1$ which is not c'-saturated, so we can use the refined basic operation involving E' and W as our second transformation, making E a nondisconnecting edge as desired.

We next adjust the position of E, in order to get the graph G_2. We use the refined operation involving the vertex ① and the edge $(3, 4)$. The last transformation involves relabeling the nodes ⑤, ⑥, ②, ① and ④ by ①, ②, ④, ⑤ and ⑥ respectively.

Corollary 4.20. *If G is a canonically oriented MCB graph of type (a, a), having an odd number of edges, then $G = 0$.*

Proof. Changing the orientation of all the edges of G, we obtain a canonically oriented MCB graph G' of the same type as G. It follows from Proposition 4.17 that $G = G'$. On the other hand, we get by part a) of Lemma 4.8 that $G' = -G$, hence $G = 0$. □

Step 4. The preceding steps yield:

Corollary 4.21. *For e_λ, f_λ as in Theorem 4.1, the space $(U/F)_\lambda$ is spanned by MCB graphs $G_{\mu'}$ of type $\mu' = (a' \geq b')$ (see Definition 4.16 for the type of an MCB graph), with $a' + b' = \min(e_\lambda + 1, r)$ and $b' \geq f_\lambda$. Moreover, $G_{\mu'} = 0$ if $a' = b'$ and e_λ is odd.*

Proof. The last statement is the content of Corollary 4.20. We know that $(U/F)_\lambda$ is spanned by MCB graphs (Proposition 4.15), and the condition $b' \geq f_\lambda$ follows from the fact that any graph G has at least λ_2^i/d_i vertices incident to edges of color i, and any edge is incident to one vertex in each of the two sets of the bipartition. The number of vertices in the maximal connected component of an MCB graph of type μ' is $a' + b' = \min(e_\lambda + 1, r)$.

It remains to show that if $\mu' = (a' \geq b')$, $a' + b' = \min(e_\lambda + 1, r)$ and $b' \geq f_\lambda$, then there exists an MCB graph $G_{\mu'}$ of type μ'. Consider A' and B' disjoint sets consisting of a' and b' vertices in $\{①, \dots, ⓡ\}$ respectively. For every $i = 1, \dots, n$ we draw λ_2^i edges of color i joining pairs of elements in A' and B', in such a way that no vertex has more than d_i incident edges of color i. This is possible since $\lambda_2^i/d_i \leq f_\lambda \leq b' \leq a'$. If the bipartite graph G (with vertex set $A' \cup B'$) obtained in this way is connected, then we get an MCB graph $G_{\mu'}$ by adding to G the isolated nodes outside $A' \cup B'$. If G is not connected, then it has an edge E of color c contained in a cycle, and a vertex v outside the connected component of E. If v is not c-saturated, we can move E to make it incident to v, and preserve the bipartition of G (as in the refined basic operations of type (1), Remark 4.18), thus obtaining a graph with fewer components. If v is c-saturated, let E' be an incident edge of color c. We move E and E' as in a refined basic operation of type (2), connecting the components of E and v. Repeating this procedure will eventually yield a connected graph G and an MCB graph $G_{\mu'}$ as above. □

Lemma 4.22. *Consider canonically oriented graphs $G_{\mu'}$ as above, one for each type $\mu' = (a', b')$, with $a' \neq b'$ when e_λ is odd. If*

$$\pi = \bigoplus_{\substack{\mu \vdash r \\ \mu = (a \geq b)}} \pi_\mu : U \longrightarrow \bigoplus_{\substack{\mu \vdash r \\ \mu = (a \geq b)}} U_\mu^{\underline{d}},$$

then the set $\{\pi(G_{\mu'})\}_{\mu'}$ is linearly independent. In particular, $F = I$ and the graphs $G_{\mu'}$ give a basis of $(U/F)_\lambda$. This shows that $\dim((U/I)_\lambda) = m_\lambda$, *where m_λ is as described in Theorem 4.1, concluding the proof of our main result.*

Proof. Note that the number of $G_{\mu'}$ is precisely m_λ, so the last statement follows once we prove the independence of the $G_{\mu'}$. This is a consequence of the linear independence of $\{\pi(G_{\mu'})\}_{\mu'}$, which in turn follows once we show that for $\mu = (a, b)$, $\mu' = (a', b')$, we have

(1) $\pi_\mu(G_{\mu'}) = 0$ if $b < b'$, and

(2) $\pi_\mu(G_{\mu'}) \neq 0$ if $b = b'$.

Recall that $G_{\mu'} = T_{\mu'}$, for some n-tableau $T_{\mu'}$. We have

$$\pi_\mu(T_{\mu'}) = \sum T_i, \tag{*}$$

where each T_i is an n-tableau with entries $1, 2$, obtained from a partition $A \sqcup B = \{1, \dots, r\}$ by setting equal to 1 and 2 the entries of $T_{\mu'}$ from A and B respectively.

To prove (1), note that since $|B| = b < b'$, for each i the endpoints of some edge in $G_{\mu'}$ have to be set to the same value, so T_i has repeated entries in some column, that is, $T_i = 0$. It follows that $\pi_\mu(G_{\mu'}) = \sum T_i = 0$.

To prove (2), let $A' \sqcup B'$ be the bipartition of the maximal connected component of $G_{\mu'}$, and take

$$\mu = (a, b) = (d - b', b').$$

The only n-tableau(x) T_i in (*) without repeated entries in some column is (are) the n-tableau T_1 obtained from setting the entries of $A = \{1, \dots, r\} - B'$ to 1, and the entries of $B = B'$ to 2 (and if $|A'| = |B'|$, the n-tableau T_2 obtained by setting the entries of $A = \{1, \dots, r\} - A'$ to 1 and the entries of $B = A'$ to 2). Since in the latter case e_λ must be even, we get in fact that $T_1 = T_2$, since T_1 and T_2 differ by an even number of transpositions within columns, and by permutations of columns of size 1. It follows that it's enough to prove that $T_1 \neq 0$.

Up to permutations within columns, and permutations of columns of the same size, we may assume that

$$T_1 = c_\lambda \cdot m = c_\lambda \cdot z_{(A,\dots,A)} \cdot z_{(B,\dots,B)},$$

where $A = \{1, \dots, a\}$ and $B = \{a+1, \dots, a+b\}$, that is, $T_1 = T_1^1 \otimes \cdots \otimes T_1^n$, with

$$T_1^i = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 2 & 2 & \cdots \\ \hline 2 & 2 & \cdots & 2 \\ \cline{1-4}\end{array}.$$

If $a > b$ and $\sigma = \tau \cdot \tau'$, with τ a row permutation and τ' a column permutation of the canonical n-tableau T_λ of shape λ, then $\sigma \cdot m \neq m$, unless $\tau' = \mathrm{id}$. This shows that the coefficient of m in T_1 is a positive number, hence $T_1 \neq 0$. If $a = b$, $\sigma \cdot m = m$ and $\tau' \neq \mathrm{id}$, then τ' must transpose all the pairs $(1, 2)$ in the columns of T_1 of size 2. Since T_1 has e_λ (an even number) of such columns, the signature of τ' must be $+1$. It follows again that the coefficient of m in T_1 is positive and therefore $T_1 \neq 0$. □ □

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craicu@math.princeton.edu *Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, United States*

Institute of Mathematics "Simion Stoilow", Romanian Academy of Sciences, 21 Calea Grivitei Street, 010702 Bucharest, Romania

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