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# Galois representations associated with unitary groups over $\mathbb{Q}$

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We show that a cuspidal automorphic representation $\pi = \bigotimes_{\ell \le \infty} \pi_\ell$ of a unitary similitude group $\mathrm{GU}(a, b)_{/\mathbb{Q}}$ with archimedean component $\pi_\infty$ in a regular discrete series has an associated $(a+b)$-dimensional $p$-adic Galois representation with Frobenius eigenvalues given by the local base change parameters for all primes $\ell$ such that $\pi_\ell$ and $\mathrm{GU}(a, b)$ are unramified.

## 1. Introduction

In this paper we explain how results of Morel [2010] on the cohomology of the noncompact Shimura varieties associated to unitary similitude groups over $\mathbb{Q}$ can be combined with results of Shin [2011] on the cohomology of certain compact Shimura varieties and with certain analytic results — most notably the stability of the gamma factors arising from the doubling method for unitary groups [Lapid and Rallis 2005; Brenner 2008] — to prove that a cuspidal automorphic representation $\pi$ of $\mathrm{GU}(a, b)_{/\mathbb{Q}}$ with archimedean component in a discrete series and regular (in a sense made precise below) has an associated $(a+b)$-dimensional $p$-adic Galois representation with Frobenius eigenvalues given by the local base change parameters for all primes $\ell$ such that $\pi$ and $\mathrm{GU}(a, b)$ are unramified. Our motivation for this is the use in [Skinner and Urban 2010] of these $p$-adic Galois representations in the case $(a, b) = (2, 2)$ to prove the Iwasawa–Greenberg main conjecture for a large class of modular forms. The main results include Theorems A and B below, whose proofs are intertwined.

Let $K$ be an imaginary quadratic field of discriminant $d_K$. Let $n = a + b$ be a partition of a positive integer $n$ as the sum of two nonnegative integers $a$ and $b$. Then

$$J_{a,b} := \begin{pmatrix} 1_a & \\ & -1_b \end{pmatrix}$$

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defines an Hermitian pairing on the space $V := K^n$. Let $G := \mathrm{GU}(a, b)_{/\mathbb{Q}}$ denote the unitary similitude group over $\mathbb{Q}$ of the Hermitian pair $(V, J_{a,b})$. The $L$-packets of discrete series representations of $G(\mathbb{R})$ are naturally indexed by the irreducible algebraic representations of $G_{/K}$ (see Section 4.1). By a *regular* discrete series representation of $G(\mathbb{R})$ we will mean one belonging to an $L$-packet indexed by a representation with regular highest weight.

Let $H := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m \times \mathrm{GL}_n)$. For any $\mathbb{Q}$-algebra $R$, let $(x, g) \mapsto (\bar{x}, \bar{g})$ be the involution of $H(R) = (R \otimes K)^\times \times \mathrm{GL}_n(R \otimes K)$ induced by the nontrivial automorphism of $K$, and let $\theta$ be the involution defined by $\theta((x, g)) = (\bar{x}, \bar{x}\,{}^t\bar{g}^{-1})$. Note that an irreducible admissible representation $\sigma$ of $H(\mathbb{A}_\mathbb{Q})$ is given by a pair $(\psi, \tau)$ consisting of an admissible character $\psi$ of $\mathbb{A}_K^\times$ and an irreducible admissible representation $\tau$ of $\mathrm{GL}_n(\mathbb{A}_K)$ and that $\sigma = (\psi, \tau)$ is $\theta$-stable (that is, $\sigma^\theta \cong \sigma$) if and only if $\tau^\vee \cong \tau^c$ and $\psi = \psi^c \chi_\tau^c$, where $\chi_\tau$ is the central character of $\tau$ and the superscripts '$\vee$' and '$c$' denote, respectively, the contragredient and composition with the involution induced by the nontrivial automorphism of $K$. Let $\mathrm{BC} : {}^L G \to {}^L H$ be the base change morphism (see Section 2.3).

**Theorem A** (weak base change). *Let $\pi$ be an irreducible cuspidal representation of $G(\mathbb{A}_\mathbb{Q})$ and let $\chi_\pi$ be its central character* (*a character of $\mathbb{A}_K^\times$*). *Let $\Sigma(\pi)$ be the finite set of primes $\ell$ such that either $\pi_\ell$ is ramified or $\ell | d_K$. Suppose $ab \neq 0$ and $\pi_\infty$ is a regular discrete series belonging to an $L$-packet indexed by a representation $\xi$. There exists an automorphic representation $\sigma = (\psi, \tau)$ of $H(\mathbb{A}_\mathbb{Q})$ such that*:

(a) $\sigma^\theta \cong \sigma$, $\psi = \chi_\pi^c$ *and* $\chi_\tau = \chi_\pi^c / \chi_\pi$.

(b) *For a prime $\ell \notin \Sigma(\pi)$, $\sigma_\ell$ is unramified, and if $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G$ is the Langlands parameter of $\pi_\ell$ then*

$$\psi_{\sigma_\ell} := \mathrm{BC} \circ \psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L H$$

*is the Langlands parameter of $\sigma_\ell$. In particular, for any idèle class character $\chi$ of $\mathbb{A}_K^\times$ there is equality of twisted standard $L$-functions*

$$L_{\Sigma(\pi)}(s, \pi \times \chi) = L_{\Sigma(\pi)}(s, \tau \times \chi).$$

(c) *$\sigma_\infty$ has the same infinitesimal character as $\xi \otimes \xi^\theta$.*

There is a natural identification of $G_{/K}$ with $\mathbb{G}_m \times \mathrm{GL}_n$ (see Section 2.2) and hence of $G(\mathbb{R} \otimes K)$ with $H(\mathbb{R})$, which then identifies $\xi$, and hence $\xi^\theta$, as a representation of $H(\mathbb{R})$. The (partial) standard $L$-function of $\pi$ is as defined as in [Li 1992, §3].

Let $\overline{K}$ be an algebraic closure of $K$ and let $G_K := \mathrm{Gal}(\overline{K}/K)$. For each finite place $v$ of $K$ let $\overline{K}_v$ be an algebraic closure of $K_v$ and fix an embedding $\overline{K} \hookrightarrow \overline{K}_v$. The latter identifies $G_{K_v} := \mathrm{Gal}(\overline{K}_v/K_v)$ with a decomposition group for $v$ in $G_K$ and hence the Weil group $W_{K_v} \subset G_{K_v}$ with a subgroup of $G_K$.

Let $p$ be a prime and $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$. Let $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ be an isomorphism. Our conventions for Galois representations are geometric.

**Theorem B** (Galois representations). *Let $\pi$ be an irreducible cuspidal representation of $G(\mathbb{A}_{\mathbb{Q}})$ and let $\chi_\pi$ be its central character. Let $\Sigma(\pi)$ be the finite set of primes $\ell$ such that either $\pi_\ell$ is ramified or $\ell | d_K$. Suppose $ab \neq 0$ and $\pi_\infty$ is a regular discrete series belonging to an $L$-packet indexed by the representation $\xi$. Let $\sigma = (\psi, \tau)$ be as in Theorem A. There exists a continuous, semisimple representation $\rho_\pi = \rho_{\pi,\iota} : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that*:

(a) $\rho_\pi^c \simeq \rho_\pi^\vee \otimes \rho_{\chi_\pi^{1+c}} \epsilon^{1-n}$.

(b) *$\rho_\pi$ is unramified at all finite places not above primes in $\Sigma_p(\pi) := \Sigma(\pi) \cup \{p\}$, and for such a place $w$*

$$(\rho_\pi |_{W_{K_w}})^{ss} = \iota \mathrm{Rec}_w \big(\tau_w \otimes \psi_w | \cdot |_w^{(1-n)/2}\big).$$

*In particular,*

$$L_{\Sigma_p(\pi)}(s, \rho_\pi) = L_{\Sigma_p(\pi)}\Big(s + \frac{1-n}{2}, \tau \times \psi\Big).$$

(c) *For $v|p$, $\rho_\pi |_{G_{K_v}}$ is potentially semistable of Hodge–Tate-type $\xi$.*

(d) *If $p \notin \Sigma(\pi)$ then*

(d) *If $p \notin \Sigma(\pi)$ then for any $v|p$, $\rho_\pi|_{G_{K_v}}$ is crystalline. Moreover, for any $j$ in $\mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$ the eigenvalues of the action of the $[K_v : \mathbb{Q}_p]$-th power of the crystalline Frobenius on*

$$D_{\mathrm{cris}}(\rho_\pi|_{G_{K_v}}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, j} \overline{\mathbb{Q}}_p$$

*are the eigenvalues of the action of Frobenius on $\iota\mathrm{Rec}_v\big(\tau_v \otimes \psi_v | \cdot |_v^{(1-n)/2}\big)$.*

For any irreducible admissible representation $\alpha$ of $\mathrm{GL}_n(K_w)$, $\mathrm{Rec}_w(\alpha)$ is the Weil–Deligne representation over $\mathbb{C}$ associated by the local Langlands correspondence, and $\iota\mathrm{Rec}_w(\alpha)$ is the representation over $\overline{\mathbb{Q}}_p$ obtained by change of scalars via $\iota$. For $\rho_\pi|_{G_{K_v}}$ to be of Hodge–Tate type $\xi$ means that the Hodge–Tate weights can be read off from $\xi$ in a prescribed way (see Section 4.4).

As the proof of Theorem A shows, there is a partition $n = m_1 + \cdots + m_r$ such that the representation $\tau$ in Theorem A is of the form $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ with $\tau_i$ a cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_K)$ such that $\tau_i^c \cong \tau_i^\vee$ and $\sigma_i := \tau_i \otimes |\cdot|^{(m_i-n)/2}$ is regular algebraic in the sense of [Clozel 1990]. Then the representation $\rho_\pi$ of Theorem B is just $\rho_\psi \otimes \big(\bigoplus_{i=1}^r \rho_{\sigma_i,\iota}\big)$, where $\rho_{\sigma_i,\iota}$ is the $m_i$-dimensional $p$-adic Galois representation associated to $\sigma_i$ ($\rho_{\sigma_i,\iota}$ is obtained from [Shin 2011]).

The theory of pseudorepresentations in combination with congruences between automorphic forms allows the weakening of some of the hypotheses of Theorem B —

cases where $ab = 0$ or where $\xi$ is not regular can be allowed. But we do not include this here.

If $\mathbb{Q}$ is replaced by a totally real field of degree greater than one, then the analogs of Theorems A and B are known, the weak base change having been proved by Labesse [2011]. Furthermore, versions of these theorems have been proved by Morel [2010], who proves Theorem A but with $\Sigma(\pi)$ replaced by an indeterminate set of primes, and by Harris and Labesse [2004], who require additional conditions at some finite primes. The work of Morel is the starting point of our proofs.

Our proofs of Theorems A and B proceed essentially as follows. By results of Morel, an automorphic representation $\sigma = (\psi, \tau)$ of $H(\mathbb{A}_{\mathbb{Q}})$ as in Theorem A exists but with $\Sigma(\pi)$ replaced by an indeterminate set $S \supseteq \Sigma(\pi)$. Furthermore, $\tau$ is a subquotient of an induced representation $\mathrm{Ind}_P^{\mathrm{GL}_n}\left(\bigotimes_{i=1}^r \tau_i\right)$ with $P \subset \mathrm{GL}_n$ the standard parabolic associated with a partition $n = m_1 + \cdots + m_r$ and each $\tau_i$ a discrete representation of $\mathrm{GL}_{m_i}(\mathbb{A}_{\mathbb{Q}})$ such that $\tau_i^c \cong \tau_i^\vee$. By considering absolute values of Satake parameters, it follows from the work of Mœglin and Waldspurger characterizing the discrete series representations of $\mathrm{GL}_{m_i}(\mathbb{A}_{\mathbb{Q}})$ that each $\tau_i$ is cuspidal, and a consideration of infinitesimal characters yields that $\sigma_i := \tau_i \otimes |\cdot|^{(n_i-n)/2}$ is algebraic with the same infinitesimal character as a regular irreducible representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_i}$. The regularity of $\xi$ is used in both these arguments. Then $\rho_{\pi,\iota} := \rho_\psi \otimes \left(\bigoplus_{i=1}^r \rho_{\sigma_i,\iota}\right)$, with $\rho_{\sigma_i,\iota}$ being the representation deduced from the work of Shin, satisfies conclusions (a), (b), and (c) of Theorem B with $\Sigma(\pi)$ replaced by $S$. It then remains to show that (b) of Theorem A also holds for $\ell \in S$ but $\ell \notin \Sigma(\pi)$, for then (b) and (d) of Theorem B follow from the corresponding results for the $\rho_{\sigma_i,\iota}$. To obtain (b) of Theorem A for such an $\ell$ we first observe that the representation $\bigwedge^a \rho_{\pi,\iota}$ is unramified at the places $w|\ell$. This is because Morel has essentially shown that this representation appears in the intersection cohomology of a Shimura variety associated to $\pi$ that has good reduction at $w|\ell$ (some argument is required to reduce to the nonendoscopic case); this is another point at which the regularity of $\xi$ is used. Then the local-global compatibility satisfied by the $\rho_{\sigma_i,\iota}$ implies that there is a finite order character $\chi_\ell$ of $K_\ell^\times$ such that each $\tau_{i,w} \otimes \chi_w$ is unramified, and hence a principal series representation of $\mathrm{GL}_{m_i}(K_w)$ with Satake parameters all having the same absolute values (again using regularity of $\xi$). This information is then combined with that coming from the $\gamma$-factors of the standard $L$-functions. Lapid and Rallis have defined local $\gamma$-factors $\gamma(s, \pi_v \times \chi_v)$ for the standard $L$-function of $\pi$ such that

$$L_S(s, \pi \times \chi) = \prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \chi_v) \times L_S(1-s, \pi^\vee \times \chi^{-1}),$$

and Brenner has proved stability for these $\gamma$-factors at nonarchimedean places. Comparing this with the functional equation for $L_S(s, \tau \times \chi)$ and choosing a global character $\chi$ with $\ell$-component $\chi_\ell$ and with sufficiently ramified $q$-components $\chi_q$ for $\ell \neq q \in S$ yields an equality between $\gamma$-factors for $\pi$ and

$$\tau : \gamma(s, \pi_\ell \times \chi_\ell) = \gamma(s, \tau_\ell \times \chi_\ell).$$

Comparing the definitions of these gamma factors and exploiting some freedom in the choice of $\chi_\ell$ and $\chi$ then yields conclusion (b) of Theorem A.

After some preliminary remarks fixing notation for unitary and related groups in Section 2, in Section 3 we give the analytic arguments involving $L$-functions and $\gamma$-factors. In Section 4 we then recall the results of Morel and Shin and explain how Theorems A and B follow.

## 2. Preliminaries

We adopt the following notation and conventions.

**2.1.** ***Galois groups and representations.*** Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field of discriminant $d_K$. For $F = \mathbb{Q}$ or $K$, let $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Let $W_F$ be a Weil group of $F$; this comes with a homomorphism to $G_F$. For each place $v$ of $F$ fix an algebraic closure $\overline{F}_v$ of $F_v$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{F}_v$. The latter identifies $G_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$ with a decomposition group in $G_F$. Let $W_{F_v}$ be the Weil group of $F_v$; for a finite place $v$, $W_{F_v}$ is a subgroup of $G_{F_v}$ and so is identified with a subgroup of $G_F$. Fix a homomorphism $W_{F_v} \to W_F$ compatible with the fixed inclusion $G_{F_v} \subset G_F$. We denote the action on $K$ of the nontrivial automorphism in $\mathrm{Gal}(K/\mathbb{Q})$ by $x \mapsto \bar{x}$. For simplicity, we also fix an embedding $K \hookrightarrow \mathbb{C}$ (equivalently, an isomorphism $\overline{K}_\infty \cong \mathbb{C}$).

Let $p$ be fixed prime and $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ a fixed isomorphism. Our conventions for $p$-adic Galois representations are geometric: $L$-functions of representations of $G_F$ or $G_{F_v}$ are defined by taking characteristic polynomials of *geometric* Frobenius elements.

For an algebraic Hecke character of $\mathbb{A}_F^\times$ (so $\chi_\infty(x) = \mathrm{sgn}(x)^r x^t$ if $F = \mathbb{Q}$ and $\chi_\infty(x) = x^r \bar{x}^t$ if $F = K$, for some $r, t \in \mathbb{Z}$) let

$$\rho_\chi = \rho_{\chi,\iota} : G_F \to \overline{\mathbb{Q}}_p^\times$$

be the $p$-adic Galois character such that $L_{\{p\}}(s, \rho_\chi) = L_{\{p\}}(s, \chi)$. Then $\epsilon : G_F \to \mathbb{Z}_p^\times$ is the $p$-adic character associated to the norm $|\cdot|_F$ character of $\mathbb{A}_F^\times$; this is the $p$-adic cyclotomic character: for a geometric Frobenius $\mathrm{frob}_v$, $v \nmid p\infty$,

$$\epsilon(\mathrm{frob}_v) = \mathrm{Norm}(v)^{-1}.$$

**2.2.** ***The groups: G, $G_0$, H, and $H_0$.*** Let $n_1, \ldots, n_k$ be positive integers and $n := n_1 + \cdots + n_k$. For each $i = 1, \ldots, k$ let $n_i = a_i + b_i$ be a partition of $n_i$ as a sum of two nonnegative integers. Let

$$J_i = J_{a_i,b_i} := \begin{pmatrix} 1_{a_i} & \\ & -1_{b_i} \end{pmatrix}.$$

Then $J_i$ defines a Hermitian pairing on $K^{n_i}$. Let

$$G = G(U(a_1, b_1) \times \cdots \times U(a_k, b_k))_{/\mathbb{Q}}$$

and let $\mu : G \to \mathbb{G}_m$ be its similitude character. That is, for any $\mathbb{Q}$-algebra $R$,

$$G(R) = \left\{ g = (g_1, \ldots, g_k) \in \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R \otimes K) : \exists \lambda \in R^\times \text{ such that } g_i J_i^t \bar{g}_i = \lambda J_i \right\}$$

and $\mu(g) = \lambda$. Here $g \mapsto \bar{g}$ is the involution of $\mathrm{GL}_m(R \otimes K)$ defined by the action of the nontrivial automorphism of $K$. Let $G_0 := U(a_1, b_1) \times \cdots \times U(a_k, b_k)$ be the kernel of $\mu$.

For any $K$-algebra $R$ there is a natural isomorphism $R \otimes K \xrightarrow{\sim} R \times R$, $r \otimes x \mapsto (rx, r\bar{x})$. Using this, we identify $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{GL}_{n_i}$:

$$g = (g_i', g_i'') \in G(R) \subset \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R \otimes K) = \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R) \times \mathrm{GL}_{n_i}(R)$$

is identified with $(\mu(g), (g_i')) \in R^\times \times \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R)$. Then $G_{0/K}$ is identified with the subgroup $\prod_{i=1}^{k} \mathrm{GL}_{n_i}$.

Let $H := \mathrm{Res}_{K/\mathbb{Q}} G_{/K}$. Then $H_{/K}$ is identified with $G_{/K} \times G_{/K}$. The identification of $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{GL}_{n_i}$ identifies $H$ with

$$\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_{n_i}.$$

Let $\theta$ be the involution of $H$ defined by

$$\theta(x, (g_i)) = (\bar{x}, (\bar{x}^t \bar{g}_i^{-1})).$$

Let $H_0 := \mathrm{Res}_{K/\mathbb{Q}} G_0$. Note that $\theta$ also defines an involution $(g_i) \mapsto ({}^t\bar{g}_i^{-1})$ of $H_0$. An irreducible admissible representation of $H(\mathbb{A}_\mathbb{Q})$ is given by a tuple $(\psi, (\tau_i))$ with $\psi$ an admissible character of $\mathbb{A}_K^\times$ and each $\tau_i$ an irreducible admissible representation of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$.

**2.3. Dual groups and $L$-groups.** The identification of $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{GL}_{n_i}$ also identifies the dual group $\widehat{G}$ with $\mathbb{C}^\times \times \prod_{i=1}^{k} \mathrm{GL}_{n_i}(\mathbb{C})$, with $G_\mathbb{Q}$ acting through the quotient $\mathrm{Gal}(K/\mathbb{Q})$ and the nontrivial automorphism $c \in \mathrm{Gal}(K/\mathbb{Q})$ acting by

$$c(x, (g_i)) = \left( x \prod_{i=1}^{k} \det g_i, (\Phi_{n_i}^{-1t} g_i^{-1} \Phi_{n_i}) \right),$$

where $\Phi_m := (\Phi_{m,ij}) = ((-1)^{i+1}\delta_{i,m-j+1})$. Put ${}^LG := \widehat{G} \rtimes W_{\mathbb{Q}}$. Similarly, $\widehat{G}_0 = \prod_{i=1}^k \mathrm{GL}_{n_i}(\mathbb{C})$ with the same action of $G_{\mathbb{Q}}$; let ${}^LG_0 := \widehat{G}_0 \rtimes W_{\mathbb{Q}}$. The $L$-homomorphism corresponding to taking an irreducible admissible $G_0(\mathbb{A}_{\mathbb{Q}})$-constituent of an irreducible admissible $G(\mathbb{A}_{\mathbb{Q}})$ representation is the projection

$$ {}^LG \to {}^LG_0,\ (x, (g_i)) \rtimes w \mapsto (g_i) \rtimes w. $$

Since $H_{/K} = G_{/K} \times G_{/K}$, $\widehat{H} = \widehat{G} \times \widehat{G}$ with the action of $G_{\mathbb{Q}}$ again factoring through $\mathrm{Gal}(K/\mathbb{Q})$ and with $c(x, y) = (c(y), c(x))$. Similarly, $\widehat{H}_0 = \widehat{G}_0 \times \widehat{G}_0$ with the same action of $G_{\mathbb{Q}}$. Put ${}^LH := \widehat{H} \rtimes W_{\mathbb{Q}}$ and ${}^LH_0 := \widehat{H}_0 \rtimes W_{\mathbb{Q}}$. The diagonal embedding $\widehat{G} \hookrightarrow \widehat{H} = \widehat{G} \times \widehat{G}$ is $G_{\mathbb{Q}}$-equivariant; its extension to $L$-groups

$$ \mathrm{BC} : {}^LG \to {}^LH $$

is the base change map. Let $\mathrm{BC} : {}^LG_0 \to {}^LH_0$ be the similarly defined map.

## 3. *L*-functions and *γ*-factors

In this section we prove the key analytic ingredient of our proof of Theorems A and B. We assume in the argument that $G_0 = U(a, b)$ (that is, $k$=1).

Let $\pi$ be a cuspidal automorphic representation of $G_0(\mathbb{A}_{\mathbb{Q}})$. Let $\Sigma(\pi)$ be the finite set of primes $\ell$ such that either $\pi_\ell$ is ramified or $\ell | d_K$. By the principle of functoriality for the $L$-group homomorphism $\mathrm{BC} : {}^LG_0 \to {}^LH_0$ it is expected — at the very least — that there should be a *weak base change* of $\pi$ to $H_0(\mathbb{A}_{\mathbb{Q}})$. That is, there should exist an automorphic representation $\tau$ of $H_0(\mathbb{A}_{\mathbb{Q}})$ (equivalently, of $\mathrm{GL}_n(\mathbb{A}_K)$) such that for $\ell \notin \Sigma(\pi)$, the Langlands parameter $\psi_{\tau_\ell} : W_{\mathbb{Q}_\ell} \to {}^LH_0$ of $\tau_\ell$ is just $\mathrm{BC} \circ \psi_{\pi_\ell}$, with $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^LG_0$ the Langlands parameter of $\pi_\ell$. We say that $\tau$ is a *very weak base change* of $\pi$ if there is *some* set $S \supset \Sigma(\pi)$ such that this relation between Langlands parameters holds for all $\ell \notin S$.

**Proposition 1.** *Let $\pi$ be a cuspidal automorphic representation of $G_0(\mathbb{A}_{\mathbb{Q}})$. Assume that there exists a very weak base change $\tau$ of $\pi$ to $H_0(\mathbb{A}_{\mathbb{Q}})$. If $\tau$ is a tempered principal series at every finite place $\ell \notin \Sigma(\pi)$, then $\tau$ is a weak base change of $\pi$.*

We deduce the conclusion of this proposition by comparing $L$-functions and $\gamma$-factors. Let $R := \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$. Then $\widehat{R} = \mathbb{C}^\times \times \mathbb{C}^\times$ with $G_{\mathbb{Q}}$ acting through $\mathrm{Gal}(K/\mathbb{Q})$ and the nontrivial automorphism $c$ of $K$ acting as $c(x_1, x_2) = (x_2, x_1)$. Let ${}^LR := \widehat{R} \rtimes W_{\mathbb{Q}}$. Let $\omega$ be a Hecke character of $\mathbb{A}_K$. Then $\omega$ is an irreducible admissible representation of $R(\mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_K^\times$; we let $\psi_{\omega_\ell} : W_{\mathbb{Q}_\ell} \to {}^LR$ be the Langlands parameter associated with $\omega_\ell := \bigotimes_{v|\ell} \omega_v$ (coming from class field theory). The $L$-groups of $G_0 \times R$ and $H_0 \times R$ are ${}^L(G_0 \times R) = {}^LG_0 \times_{W_{\mathbb{Q}}} {}^LR = (\widehat{G}_0 \times \widehat{R}) \rtimes W_{\mathbb{Q}}$ and ${}^L(H_0 \times R) = {}^LH_0 \times_{W_{\mathbb{Q}}} {}^LR = (\widehat{H}_0 \times \widehat{R}) \rtimes W_{\mathbb{Q}}$, with $W_{\mathbb{Q}}$ acting on each factor.

Let $\pi$ and $\tau$ be as in the proposition. The unramified local $L$-factors $L(s, \pi_\ell \times \omega_\ell)$ of the standard $L$-function of $\pi \times \omega$ are the $L$-factors associated with the representation $r_{st} : {}^L(G_0 \times R) \to \mathrm{GL}_{2n}(\mathbb{C})$,

$$r_{st}((g,(x_1,x_2)) \rtimes 1) = \begin{pmatrix} x_1 g & \\ & x_2 \Phi_n^{-1t} g^{-1} \Phi_n \end{pmatrix} \qquad r_{st}(1 \rtimes c) = \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}.$$

If $\ell \nmid d_K$ and $\pi_\ell$ and $\omega_\ell$ are unramified, then

$$L(s, \pi_\ell \times \omega_\ell) = \det(1 - \ell^{-s} r_{st}(\psi_{\pi_\ell}(\mathrm{frob}_\ell), \psi_{\omega_\ell}(\mathrm{frob}_\ell)))^{-1}.$$

Similarly, the local unramified $L$-factors $L(s, \tau_\ell \times \omega_\ell) := \prod_{v|\ell} L(s, \tau_v \times \omega_v)$ are the $L$-factors associated with the homomorphism $r'_{st} : {}^L(H_0 \times R) \to \mathrm{GL}_{2n}(\mathbb{C})$,

$$r'_{st}(((g_1,g_2),(x_1,x_2)) \rtimes 1) = \begin{pmatrix} x_1 g_1 & \\ & x_2 \Phi_n^{-1t} g_2^{-1} \Phi_n \end{pmatrix} \qquad r'_{st}(1 \rtimes c) = \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}.$$

In particular, $r_{st} = r'_{st} \circ (\mathrm{BC} \times id)$, so $L(s, \pi_\ell \times \omega_\ell) = L(s, \tau_\ell \times \omega_\ell)$ if $\ell \nmid d_K$ and $\pi_\ell$, $\tau_\ell$, and $\omega_\ell$ are unramified and $\psi_{\tau_\ell} := \mathrm{BC} \circ \psi_{\pi_\ell}$ (so for all $\ell \notin S$).

**Lemma 2.** *Suppose $\ell \nmid d_K$ and $\pi_\ell$ are $\tau_\ell$ are unramified. If*

$$L(s, \pi_\ell \times \omega_\ell) = L(s, \tau_\ell \times \omega_\ell)$$

*for all unramified $\omega_\ell$, then $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$.*

*Proof.* Let

$$\psi_{\pi_\ell}(\mathrm{frob}_\ell) = t \rtimes \mathrm{frob}_\ell, \quad t = \mathrm{diag}(t_1, \ldots, t_n), \quad \text{and } \psi_{\tau_\ell}(\mathrm{frob}_\ell) = (h,h) \rtimes \mathrm{frob}_\ell,$$

$h = \mathrm{diag}(h_1, \ldots, h_n)$ ($\psi_{\tau_\ell}$ must be of this form as $\tau_\ell^c \cong \tau_\ell^\vee$). Suppose first that $\ell$ does not split in $K$. As $\mathrm{frob}_\ell = c$ in $\mathrm{Gal}(K/\mathbb{Q})$, the condition that $L(s, \pi_\ell \times \omega_\ell) = L(s, \tau_\ell \times \omega_\ell)$ is just that $t_i/t_{n-i} = h_i/h_{n-i}$ (after possibly reordering the $h_i$). That is, $t = zh$ for some $z \in \mathbb{C}^\times$, and so $(z,1)\psi_{\tau_\ell}(\mathrm{frob}_\ell)(z^{-1},1) = \mathrm{BC} \circ \psi_{\pi_\ell}(\mathrm{frob}_\ell)$. Hence, $\psi_{\tau_\ell}$ is equivalent to $\mathrm{BC} \circ \psi_{\pi_\ell}$.

Suppose that $\ell$ splits in $K$. Let $\psi_{\omega_\ell}(\mathrm{frob}_\ell) = (\alpha, \beta) \rtimes \mathrm{frob}_\ell$. As $\mathrm{frob}_\ell = 1$ in $\mathrm{Gal}(K/\mathbb{Q})$, the equality $L(s, \pi_\ell \times \omega_\ell) = L(s, \tau_\ell \times \omega_\ell)$ means that

$$\mathrm{diag}(\alpha t, \beta \Phi_n^{-1} t^{-1} \Phi_n) \in \mathrm{GL}_{2n}(\mathbb{C}) \quad \text{and} \quad \mathrm{diag}(\alpha h, \beta \Phi_n^{-1} h^{-1} \Phi_n) \in \mathrm{GL}_{2n}(\mathbb{C})$$

are equivalent. As $\alpha$ and $\beta$ can be arbitrary, it follows that $t$ and $h$ are equivalent, so $\mathrm{BC} \circ \psi_{\pi_\ell}$ is equivalent to $\psi_{\tau_\ell}$. □

Let $S \supset \Sigma(\pi)$ be any finite set of primes such that $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$ for all $\ell \notin S$. The (partial) standard $L$-functions $L_S(s, \pi \times \omega)$ and $L_S(s, \tau \times \omega)$, given by the Euler products

$$L_S(s, \pi \times \omega) = \prod_{\ell \notin S} L(s, \pi_s \times \omega_\ell) \quad \text{and} \quad L_S(s, \tau \times \omega) = \prod_{\ell \notin S} L(s, \tau_\ell \times \omega_\ell)$$

for $\mathrm{Re}(s) \gg 0$, satisfy

$$L_S(s, \pi \times \omega) = L_S(s, \tau \times \omega).$$

The doubling method of Piatetski-Shapiro and Rallis provides an integral representation of $L_S(s, \pi \times \omega)$ as well as local $\gamma$-factors at all places; see [Gelbart et al. 1987, Part A] and especially [Lapid and Rallis 2005]. In particular, for each place $v$ of $\mathbb{Q}$, Lapid and Rallis have defined local $\gamma$-factors $\gamma(s, \pi_v \times \omega_v) := \gamma_v(s, \pi_v \times \omega_v, \psi_v)$, $\psi_v$ being the standard additive character of $K_v$ and proved that the local $\gamma$-factors $\gamma(s, \pi_v \times \omega_v)$ are compatible with parabolic induction and are as expected in the unramified cases. The functional equation for $L_S(s, \pi \times \omega)$ is then

$$L_S(s, \pi \times \omega) = \prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \omega_v) \times L_S(1-s, \pi^\vee \times \omega^{-1}).$$

Comparing this with the usual functional equation for the standard $\mathrm{GL}_n$ $L$-function $L_S(s, \tau \times \omega)$ we find that

$$\prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \omega_v) = \prod_{v \in S \cup \{\infty\}} \prod_{w \mid v} \gamma(s, \tau_w \times \omega_w), \tag{3.1}$$

where $w$ is a place of $K$ and $\gamma(s, \tau_w \times \omega_w)$ is the $\gamma$-factor defined by Godement and Jacquet (again using the standard additive characters). For a place $v$ of $\mathbb{Q}$, set

$$\gamma(s, \tau_v \times \omega_v) := \prod_{w \mid v} \gamma(s, \tau_w \times \omega_w).$$

We exploit *stability* of $\gamma$-factors. This says that if $\pi_1$ and $\pi_2$ are two irreducible admissible representations of $G_0(\mathbb{Q}_\ell)$, then for $\chi$ a sufficiently ramified character of $K_\ell^\times$, $\gamma(s, \pi_1 \times \chi) = \gamma(s, \pi_2 \times \chi)$. This has been proved by Brenner [2008]. Stability is also known for the Godement–Jacquet $\gamma$-factors for $\mathrm{GL}_n$. Taking $\pi_1 = \pi_\ell$ and $\pi_2$ to be an unramified tempered principal series, we see that if $\omega_\ell$ is sufficiently ramified then

$$\gamma(s, \pi_\ell \times \omega_\ell) = \gamma(s, \pi_2 \times \omega_\ell) = \gamma(s, \tau_2 \times \omega_\ell) = \gamma(s, \tau_\ell \times \omega_\ell), \tag{3.2}$$

where $\tau_2$ is the representation of $H_0(\mathbb{Q}_\ell) = \mathrm{GL}_n(K_\ell)$ having Langlands parameter equal to the composition with BC of the parameter of $\pi_2$; $\tau_2$ is also an unramified tempered principal series. The first and last equalities in (3.2) come from stability, and the middle comes from [Lapid and Rallis 2005, Theorem 4]: part 1 of this theorem, together with the hypothesis that $\pi_2$ is a principal series, reduces the equality to the minimal cases — the anisotropic cases, which are part 7 of the theorem, and the isotropic cases, which are part 8 — plus the analog of part 2 for the Godement–Jacquet $\gamma$-factors (compatibility with parabolic induction).

It is easy to see that given any finite set of primes $S'$ it is possible to find a set $S'' \supset S \cup S'$ and a finite order Hecke character $\omega$ of $\mathbb{A}_K^\times$ such that $\omega_\ell$ is arbitrary for all $\ell \in S'$, and $\omega_\ell$ is sufficiently ramified at all primes $\ell \in S'' - S'$ and unramified at all primes not in $S''$. Taking $S' = \varnothing$, we deduce from (3.1) and (3.2) that $\gamma(s, \pi_\infty \times \omega_\infty) = \gamma(s, \tau_\infty \times \omega_\infty)$. Taking $S' = \{\ell\}$, any prime $\ell$, we then deduce from (3.1) and (3.2) that

$$\gamma(s, \pi_\ell \times \omega_\ell) = \gamma(s, \tau_\ell \times \omega_\ell) \tag{3.3}$$

always.

Suppose now that $\ell \notin \Sigma(\pi)$. By hypothesis, $\tau_v$ is a tempered principal series for $v|\ell$. Suppose first that $\ell$ is inert in $K$. Then $\tau_\ell \cong \pi(\mu_1, \dots, \mu_n)$ with $|\mu_i(x)| = 1$ for all $x \in K_\ell^\times$. Fix $j$ between 1 and $n$ and choose $\omega_\ell$ so that $\mu_j\omega_\ell$ is unramified. Let $I \subset \{1, \dots, n\}$ be the set of indices such that $\mu_i\omega_\ell$ is unramified. Then

$$\gamma(s, \tau_\ell \times \omega_\ell) = \prod_{i \in I} \frac{1 - \mu_i\omega_\ell(\ell)\ell^{-2s}}{1 - \mu_i^{-1}\omega_\ell^{-1}(\ell)\ell^{2s-2}} \times \prod_{i \notin I} \gamma(s, \mu_i\omega_\ell).$$

As $\mu_i\omega_\ell$ is ramified for $i \notin I$, $\gamma(s, \mu_i\omega_\ell)$ is holomorphic with no zeros. Furthermore, the temperedness of $\tau_\ell$ ensures that there is no cancellation between the numerators and denominators of the factors coming from the $i \in I$. Therefore, $\gamma(s, \tau_\ell \times \omega_\ell)$ has $|I| \geq 1$ poles. However, if $\omega_\ell$ is ramified, then, since $\pi_\ell$ is unramified, it follows from combining parts 1, 7, and 8 of [Lapid and Rallis 2005, Theorem 4] that $\gamma(s, \pi_\ell \times \omega_\ell)$ is holomorphic. So it must be that $\omega_\ell$ — and hence $\mu_j$ — is unramified. But $j$ was arbitrary, so each $\mu_i$ is unramified: $\tau_\ell$ is an unramified principal series. Therefore, by (3.3),

$$\frac{L(1-s, \pi_\ell^\vee)}{L(s, \pi_\ell)} = \gamma(s, \pi_\ell) = \gamma(s, \tau_\ell) = \frac{L(1-s, \tau_\ell^\vee)}{L(s, \tau_\ell)}$$

(for the first equality, see part 3 of [Lapid and Rallis 2005, Thm. 4]). As $\tau_\ell$ is tempered, the zeros of the right-hand side are those of $L(s, \tau_\ell)^{-1}$, while those of the left-hand side are *a priori* a subset of those of $L(s, \pi_\ell)^{-1}$. This means that $L(s, \tau_\ell)/L(s, \pi_\ell)$ is holomorphic. But each of $L(s, \tau_\ell)^{-1}$ and $L(s, \pi_\ell)^{-1}$ is a polynomial of degree $n$ in $\ell^{-2s}$ with constant term 1, and so they must be equal. That is, $L(s, \pi_\ell) = L(s, \tau_\ell)$. Since an unramified $\omega_\ell$ equals $|\cdot|_\ell^t$ for some $t \in \mathbb{C}$, it follows that $L(s, \pi_\ell \otimes \omega_\ell) = L(s + t, \pi_\ell) = L(s + t, \tau_\ell) = L(s, \tau_\ell \otimes \omega_\ell)$, which implies — by Lemma 2 — that $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$.

Suppose that $\ell = v\bar{v}$ splits in $K$. Viewing $\mathbb{Q}_\ell$ as a $K$-algebra via the embedding that induces $v$, $G_0(\mathbb{Q}_\ell)$ is identified with $\mathrm{GL}_n(K_v) = \mathrm{GL}_n(\mathbb{Q}_\ell)$ and $\pi_\ell$ with a representation $\pi_v$ of $\mathrm{GL}_n(\mathbb{Q}_\ell)$. Let $\pi_{\bar{v}} = \pi_v^\vee$. Then

$$\begin{aligned}\gamma(s, \pi_v \times \omega_v)\gamma(s, \pi_{\bar{v}} \times \omega_{\bar{v}}) &= \gamma(s, \pi_\ell \times \omega_\ell) \\ &= \gamma(s, \tau_\ell \times \omega_\ell) = \gamma(s, \tau_v \times \omega_v)\gamma(s, \tau_{\bar{v}} \times \omega_{\bar{v}}).\end{aligned}$$

The first equality follows from part 8 of [Lapid and Rallis 2005, Theorem 4]. By choosing $\omega_\ell$ so that $\omega_{\bar{v}}$ is sufficiently ramified but $\omega_v$ is unramified, $\gamma(s, \pi_{\bar{v}} \times \omega_{\bar{v}})$ and $\gamma(s, \tau_{\bar{v}} \times \omega_{\bar{v}})$ can be assumed to be holomorphic with no zeros. Arguing as in the nonsplit case then yields that $\tau_v$ is unramified and $L(s, \tau_v) = L(s, \pi_v)$ (recall that $\tau_v$ and $\tau_{\bar{v}}$ are assumed to be principal series and tempered). Reversing the role of $\omega_v$ and $\omega_{\bar{v}}$ then yields that $\tau_{\bar{v}}$ is unramified and $L(s, \tau_{\bar{v}}) = L(s, \pi_{\bar{v}})$. As $L(s, \pi_\ell) = L(s, \pi_v)L(s, \pi_{\bar{v}})$, it follows that $L(s, \pi_\ell \otimes \omega_\ell) = L(s, \tau_\ell \otimes \omega_\ell)$ for all unramified $\omega_\ell$, which — by Lemma 2 again — implies that $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$. This completes the proof of Proposition 1.

## 4. $\sigma$ and $\rho_\pi$

In this section, $k$ is arbitrary.

**4.1.** ***Algebraic representations and discrete series for $G(\mathbb{R})$.*** Let $T \subset G$ be the subgroup of diagonal elements. Then $T_{/K}$ is identified with the diagonal subgroup

$$\mathbb{G}_m^{1+n} = \mathbb{G}_m^{1+n_1+\cdots+n_k} \subset \mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{GL}_{n_i},$$

and the character group $X(T)$ is identified with $\mathbb{Z}^{1+n}$: to $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k) \in \mathbb{Z}^{1+n}$, $\underline{c}_i \in \mathbb{Z}^{n_i}$, corresponds the character

$$(t_0, (\mathrm{diag}(t_{i,1}, \ldots, t_{i,n_i})) \mapsto t_0^{c_0} \prod_{i=1}^{n} \prod_{j=1}^{n_i} t_{i,j}^{c_{i,j}}.$$

We take the dominant characters to be those that are dominant with respect to the upper-triangular Borel $B$; this is equivalent to $c_{i,1} \geq c_{i,2} \geq \cdots \geq c_{i,n_i}$. Regular dominant characters are those where the inequalities are strict. The (regular) irreducible algebraic representations of $G_{/K}$ are indexed by the (regular) dominant characters in $X(T)$: to the representation $\xi$ corresponds its highest weight with respect to the pair $(T, B)$.

The $L$-packets of discrete series representations of $G(\mathbb{R})$ are indexed by equivalence classes of elliptic Langlands parameters $\psi : W_\mathbb{R} \to {}^L G$. The restriction to $W_\mathbb{C} = \mathbb{C}^\times$ of such a $\psi$ is equivalent to a representation of the form

$$z \mapsto ((z/\bar{z})^{p_0}, (\mathrm{diag}((z/\bar{z})^{p_{i,1}}, \ldots, (z/\bar{z})^{p_{i,r_i}}))) \rtimes z$$

with $p_0 \in \mathbb{Z}$ and $p_{i,j} \in (n_i - 1)/2 + \mathbb{Z}$; the ordering can be chosen so that $p_{i,1} > \cdots > p_{i,r_i}$. Let $c_{i,j} := p_{i,j} - (n_i - 2i + 1)/2$. Then $c_{i,1} \geq \cdots \geq c_{i,r_i}$, and $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k)$, $c_0 := p_0$ and $\underline{c}_i := (c_{i,1}, \ldots, c_{i,r_i})$, is a dominant character of $X(T)$ and so corresponds to an irreducible algebraic representation $\xi$ of $G_{/K}$ of highest weight $\underline{c}$. This gives a parametrization of the discrete series $L$-packets by the irreducible algebraic representations of $G_{/K}$; we denote the $L$-packet indexed by $\xi$ by $\Pi_d(\xi)$. By a regular discrete series we will mean one belonging to an $L$-packet $\Pi_d(\xi)$ with $\xi$ having regular highest weight.

**4.2. $\sigma$.** Suppose $a_i b_i \neq 0$ for all $i$. Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ with $\pi_\infty \in \Pi_d(\xi)$ for some regular algebraic representation $\xi$ of $G_{/K}$. Let $\chi_\pi$ be the character of the scalar torus $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \subset G$ determined by $\pi$ (an algebraic Hecke character of $\mathbb{A}_K^\times$). Let $\Sigma(\pi)$ be the finite set comprising the primes $\ell$ such that either $\pi_\ell$ is ramified or $\ell | d_K$. Let $\underline{c} \in X(T)$ be the (regular) highest weight of $\xi$. Put $i(\underline{c}) := (c_0', -\underline{c}_1', \ldots, -\underline{c}_k')$, where if $\underline{c}_i = (c_{i,1}, \ldots, c_{i,n_i})$ then $\underline{c}_i' := (c_{i,n_i}, \ldots, c_{i,1})$ and $c_0' := c_0 + \sum_{i=1}^k \sum_{j=1}^{n_i} c_{i,j}$. Then $i(\underline{c})$ is also a regular dominant character in $X(T)$.

The weight of an irreducible algebraic representation of $G_{/K}$ is the integer $m$ such that the action of the central torus $\mathbb{G}_m \subset G$ is given by $x \mapsto x^m$; the weight of the representation $\xi$ with highest weight $\underline{c} \in X(T)$ is $c_0 + c_0'$.

It follows from the proofs of Corollary 8.5.3 and Lemma 8.5.6 in [Morel 2010] — see especially the top paragraph on page 156 there — that there exist partitions $n_i = m_{i,1} + \cdots + m_{i,r_i}$ with each $m_{i,j} > 0$, irreducible automorphic representations $\tau_{i,j}$ of $\mathrm{GL}_{m_{i,j}}(\mathbb{A}_K)$, and a finite set of primes $S \supset \Sigma(\pi)$ satisfying the following conditions:

- $\tau_{i,j}$ is discrete.
- $\tau_{i,j}^c = \tau_{i,j}^\vee$.
- For $\ell \notin S$ and $v|\ell$, each $\tau_{i,j,v}$ is unramified.
- Let $\ell \notin S$, $v|\ell$, and let $\tau_{i,v}$ be the unramified irreducible subquotient of $\mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}}\big(\bigotimes_j \tau_{i,j,v}\big)$ and $\sigma_\ell$ the irreducible representation of $H(\mathbb{Q}_\ell)$ defined by the tuple $\big(\bigotimes_{v|\ell} \chi_\pi^c, \big(\bigotimes_{v|\ell} \tau_{i,v}\big)\big)$. If $\psi_{\pi_\ell}$ is the Langlands parameter of $\pi_\ell$, then $\mathrm{BC} \circ \psi_\ell$ is the Langlands parameter of $\sigma_\ell$.
- The infinitesimal character of $\tau_i := \mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}}\big(\bigotimes_j \tau_{i,j}\big)$ is the same as that of the absolutely irreducible algebraic character of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$ of highest weight $(\underline{c}_i, -\underline{c}_i')$; $\chi_\pi^c(z) = z^{c_0}\bar{z}^{c_0'}$.

Here, $P_i \subset \mathrm{GL}_{n_i}$ is the standard parabolic associated with the partition $n_i = m_{i,1} + \cdots + m_{i,r_i}$.

Recall that the infinitesimal character of an admissible representation of $\mathrm{GL}_m(\mathbb{C})$ is an element of $\mathfrak{a}_{m,\mathbb{C}}^\vee$ modulo the action of the Weyl group $W(\mathfrak{gl}_{m,\mathbb{C}}, \mathfrak{a}_{m,\mathbb{C}})$, where $\mathfrak{gl}_m := \mathrm{Lie}(\mathrm{GL}_m(\mathbb{C}))$ and $\mathfrak{a}_m := \mathrm{Lie}(A_m(\mathbb{C}))$ with $A_m := \mathbb{G}_m^m \subset \mathrm{GL}_m$ the diagonal torus. Identifying $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C} \times \mathbb{C}$ via $z \otimes w \mapsto (zw, \bar{z}w)$ and $\mathbb{C} = \mathrm{Lie}(\mathbb{C}^\times)$ (in the usual way, so the exponential map is $z \mapsto e^z$) identifies $\mathfrak{a}_{m,\mathbb{C}}$ with $\mathbb{C}^m \times \mathbb{C}^m$, and hence $\mathfrak{a}_{m,\mathbb{C}}^\vee := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}_{m,\mathbb{C}}, \mathbb{C}) = \mathbb{C}^m \times \mathbb{C}^m$ (using the dual basis); $W(\mathfrak{gl}_{m,\mathbb{C}}), \mathfrak{a}_{m,\mathbb{C}}^\vee)$ is then identified with $\mathfrak{S}_m \times \mathfrak{S}_m$. An absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ corresponds to its highest weight with respect to

$$(\mathrm{Res}_{K/\mathbb{Q}}A_m, \mathrm{Res}_{K/\mathbb{Q}}B_m),$$

$B_m \subset \mathrm{GL}_m$ being the upper-triangular Borel; this is an element of

$$X(\mathrm{Res}_{K/\mathbb{Q}}A_m) = X(A_m) \times X(A_m)$$

(the identification being via $\mathrm{Res}_{K/\mathbb{Q}}A_{m\,/K} = A_m \times A_m$) given by a pair of dominant characters of $X(A_m) = \mathbb{Z}^m$ (the last identification is the usual one: $\underline{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m$ corresponds to the character $\mathrm{diag}(t_1, \dots, t_m) \mapsto t_1^{c_1} \cdots t_m^{c_m}$; dominant characters satisfy $c_1 \geq \cdots \geq c_m$, and regular dominant characters are those where the inequalities are strict). The infinitesimal character of the irreducible representation of highest weight $(\underline{c}_1, \underline{c}_2)$ is $(\underline{c}_1, \underline{c}_2) + \rho_{\mathrm{GL}_m} \in \mathfrak{a}^\vee_{m,\mathbb{C}}$, where $\rho_{\mathrm{GL}_m} := ((m-1)/2, (m-3)/2, \dots, (3-m)/2, (1-m)/2)$ is half the sum of the usual positive roots in $\mathfrak{gl}_m$.

As $\xi$ is regular, if the weight of $\xi$ is zero (that is, $c_0 + c_0' = 0$) then by [Morel 2010, Theorem 7.3.1], the Satake parameters of $\pi_\ell$, $\ell \notin S$, all have absolute value 1. The same is then true of the Satake parameters of $\tau_{i,j,v}$ for any $v|\ell$ as $\psi_{\sigma_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$. For $\xi$ having general weight $m \in \mathbb{Z}$, let $\pi'$ and $\xi'$ be the twists of $\pi$ and $\xi$, respectively, by the character $\mu(\cdot)^{-m}$; then $\xi'$ is regular of weight 0 and $\pi'_\infty \in \Pi_d(\xi')$. The representations of the $\mathrm{GL}_{n_i}(\mathbb{A}_K)$ associated to $\pi'$ as above are the same as those associated to $\pi$: this can be seen by the relation between Langlands parameters at $\ell \notin S$. The case of general weight then follows immediately from that of weight zero. Therefore, we also have that

- for $\ell \notin S$, $v|\ell$, the Satake parameters of $\tau_{i,j,v}$ all have absolute value 1 - $\tau_{i,j,v}$ is tempered; furthermore, $\tau_{i,v} = \mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}} \left(\bigotimes_j \tau_{i,j,v}\right)$ and is a tempered principal series.

**Lemma 3.** *Each $\tau_{i,j}$ is cuspidal, and $\sigma_{i,j} := \tau_{i,j} \otimes |\cdot|^{(m_{i,j}-n_i)/2}$ is algebraic and has the same infinitesimal character as a regular absolutely irreducible algebraic representation $\xi_{i,j}$ of* $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$.

Here $\sigma_{i,j}$ being algebraic automorphic representation of $\mathrm{GL}_{m_{i,j}}(\mathbb{A}_K)$ is as in [Clozel 1990, 1.2.3]: the infinitesimal character $\underline{b}_{i,j} \in \mathfrak{a}^\vee_{m_{i,j},\mathbb{C}} = \mathbb{C}^{m_{i,j}} \times \mathbb{C}^{m_{i,j}}$ of $\sigma_{i,\infty}$ satisfies $\underline{b}_{i,j} + (1-m_{i,j})/2 \in \mathbb{Z}^{m_{i,j}} \times \mathbb{Z}^{m_{i,j}}$.

*Proof.* As $\tau_{i,j}$ is discrete, by the main results of [Mœglin and Waldspurger 1989] there is a factorization $m_{i,j} = s_{i,j} r_{i,j}$ and an irreducible cuspidal automorphic representation $\alpha_{i,j}$ of $\mathrm{GL}_{s_{i,j}}(\mathbb{A}_K)$ such that $\tau_{i,j}$ is the unique irreducible quotient of

$$\mathrm{Ind}_{P_{i,j}}^{\mathrm{GL}_{m_{i,j}}} \beta_{i,j} \quad \beta_{i,j} = (\alpha_{i,j} \otimes |\cdot|^{(1-r_{i,j})/2}) \otimes \cdots \otimes (\alpha_{i,j} \otimes |\cdot|^{(r_{i,j}-1)/2}),$$

where $P_{i,j} \subset \mathrm{GL}_{m_{i,j}}$ is the standard parabolic associated with the partition $m_{i,j} = s_{i,j} + \cdots + s_{i,j}$ ($r_{i,j}$ summands). Since for all but finitely many $v$ the Satake parameters of $\tau_{i,j,v}$ all have the same absolute value, it must then be that $r_{i,j} = 1$, and so $\tau_{i,j} = \alpha_{i,j}$ is cuspidal.[1]

Let $\underline{a}_{i,j} \in \mathfrak{a}^\vee_{m_{i,j},\mathbb{C}}$ be the infinitesimal character of $\tau_{i,j,\infty}$. Then the infinitesimal character of $\tau_{i,\infty}$ is $\underline{a}_i := (\underline{a}_{i,1}, \ldots, \underline{a}_{i,r_i}) \in \mathfrak{a}^\vee_{n_i,\mathbb{C}}$. In particular, there exist $L', L'' \subset \{1, \ldots, n_i\}$ of cardinality $m = m_{i,j}$ such that $\underline{a} = \underline{a}_{i,j} = (\underline{a}', \underline{a}'') \in \mathbb{C}^m \times \mathbb{C}^m$ with $\underline{a}'$ and $\underline{a}''$ equal to $(c_{i,\ell} + (n_i - 2\ell + 1)/2)_{\ell \in L'}$ and $(-c_{i,\ell} + (2\ell - n_i + 1)/2)_{\ell \in L''}$, respectively. Suppose $L' = \{\ell'_1, \ldots, \ell'_m\}$ with $\ell'_1 < \ell'_2 < \cdots < \ell'_m$ and $L'' = \{\ell''_1, \ldots, \ell''_m\}$ with $\ell''_1 > \ell''_2 > \cdots > \ell''_m$. Then the infinitesimal character $\underline{b} = \underline{b}_{i,j}$ of $\sigma_{i,j}$ is given by $\underline{b} = \underline{a} + (m - n_i)/2 = (\underline{d}', \underline{d}'') + \rho_{\mathrm{GL}_m}$, where

$$\underline{d}' = (c_{i,\ell'_k} + k - \ell'_k)_{1 \le k \le m} \quad \text{and} \quad \underline{d}'' = (-c_{i,\ell''_k} + \ell''_k - n_i + k)_{1 \le k \le m}.$$

As $\rho_{\mathrm{GL}_m} + (1-m)/2 \in \mathbb{Z}^m$, it follows that $\underline{b} + (1-m)/2 \in \mathbb{Z}^m \times \mathbb{Z}^m$, so $\sigma_{i,j}$ is algebraic. Also,

$$c_{i,\ell'_k} + k - \ell'_k - c_{i,\ell'_{k+1}} - k - 1 + \ell'_{k+1} = c_{i,\ell'_k} - c_{i,\ell'_{k+1}} - 1 + \ell'_{k+1} - \ell'_k \ge 1$$
$$-c_{i,\ell''_k} + \ell''_k - n_i + k + c_{i,\ell''_{k+1}} - \ell''_{k+1} + n_i - k - 1 = c_{i,\ell''_{k+1}} - c_{i,\ell''_k} + \ell''_k - \ell''_{k+1} - 1 \ge 1,$$

so $\underline{d}'$ and $\underline{d}''$ are both regular and dominant. Therefore,

$$\underline{d} := (\underline{d}', \underline{d}'') \in X(A_m) \times X(A_m)$$

corresponds to a regular absolutely irreducible algebraic representation $\xi_{i,j}$ of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ with infinitesimal character $\underline{d} + \rho_{\mathrm{GL}_m} = \underline{b}$. □

**Corollary 4.** *The cuspidal representations $\tau_{i,j}$ are tempered at all finite places. Furthermore, each $\tau_i$ is irreducible and tempered at all finite places.*

*Proof.* Choose an algebraic Hecke character $\chi$ of $\mathbb{A}_K^\times$ such that $\chi\chi^c = |\cdot|^{n_i - m_{i,j}}$. Then $\sigma_{i,j} \otimes \chi$ is a conjugate self-dual algebraic cuspidal representation with infinitesimal character that of a regular absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$. Therefore, $\sigma_{i,j} \otimes \chi$ is tempered at all finite places by [Shin 2011, Corollary 1.3]. The claims about $\tau_{i,j}$ and $\tau_i$ follow easily from this. □

Put

$$\psi := \chi_\pi^c \quad \text{and} \quad \sigma := (\psi, (\tau_i)). \tag{4.4}$$

Then $\sigma$ is identified with an irreducible automorphic representation of $H(\mathbb{A}_\mathbb{Q})$. This is a very weak base change of $\pi$ in the sense that the Langlands parameter $\psi_{\sigma_\ell}$ of $\sigma_\ell$ is $\mathrm{BC} \circ \psi_{\pi_\ell}$ for all $\ell \notin S$, $\psi_{\pi_\ell}$ being the Langlands parameter of $\pi_\ell$.

[1]This can also be seen by considering infinitesimal characters.

**Remark 5.** Suppose $k=1$. Let $\pi_0$ be an irreducible automorphic constituent of the restriction of $\pi$ to $G_0(\mathbb{A}_\mathbb{Q})$. Then $\tau=\tau_1$ is a very weak base change of $\pi_0$ to $H_0(\mathbb{A}_\mathbb{Q})$ that is tempered at all finite places. By Proposition 1, to complete the proof of Theorem A it suffices to show that $\tau_v$ is a principal series for all $v|\ell$, $\ell\notin\Sigma(\pi)$. This is done in the following by analyzing certain Galois representations associated with $\tau$.

**4.3. $\rho_\pi$.** Let $\rho: G_K\to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ be a continuous representation. Let $\xi$ be an absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ with highest weight $(\underline{c}_1,\underline{c}_2)\in X(A_m)\times X(A_m)=\mathbb{Z}^m\times\mathbb{Z}^m$. Let $v|p$ be a place of $K$. Recall that $\rho_v:=\rho|_{G_{\mathbb{Q}_v}}$ being Hodge–Tate means that the graded $(\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}K_v)$-module $D_{\mathrm{HT},v}(\rho_v):=(\rho_v\otimes B_{\mathrm{HT},v})^{G_{K_v}}$, $B_{\mathrm{HT},v}:=\bigoplus_{t\in\mathbb{Z}}\widehat{\overline{K}_v(t)}$, is a free $(\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}K_v)$-module of rank $m$. By $\rho_v$ being of Hodge–Tate type $\xi$ we mean that for any $j\in\mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v,\overline{\mathbb{Q}}_p)$, the graded $\overline{\mathbb{Q}}_p$-module $D_{\mathrm{HT}}(\rho_v)\otimes_{\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}K_v,j}\overline{\mathbb{Q}}_p$ is nonzero in degrees $i-1-c_{1,i}$, $i=1,..,m$, if the restriction of $j$ to $K$ is the fixed embedding $K\hookrightarrow\overline{\mathbb{Q}}_p\cong\mathbb{C}$ and otherwise is nonzero in degrees $i-1-c_{2,i}$, $i=1,\dots,m$.

Let $\sigma_{i,j}$ be as in Lemma 3. From [Shin 2011] we conclude that there exist representations $\rho_{i,j}=\rho_{\sigma_{i,j},\iota}:G_K\to\mathrm{GL}_{m_{i,j}}(\overline{\mathbb{Q}}_p)$ such that

- $\rho_{i,j}$ is continuous and semisimple,
- for $v\nmid p$, $\mathrm{WD}(\rho_{i,j}|_{G_{K_v}})^{\mathrm{Fr-ss}}=\iota\mathrm{Rec}_v\big(\sigma_{i,j,v}\otimes|\cdot|_v^{(1-m_{i,j})/2}\big)$,
- $\rho_{i,j}^c\cong\rho_{i,j}^\vee\otimes\epsilon^{1-n_i}$,
- for each $v|p$, $\rho_{i,j}|_{G_{K_v}}$ is potentially semistable of Hodge–Tate type $\xi_{i,j}$,
- for $v|p$, if $\sigma_{i,j,v}$ is unramified then $\rho_{i,j}|_{G_{K_v}}$ is crystalline and the eigenvalues of the $[K_v:\mathbb{Q}_p]$-th power of the crystalline Frobenius on
$$D_{\mathrm{cris}}(\rho_{i,j}|_{G_{K_v}})\otimes_{\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}K_v^0,\lambda}\overline{\mathbb{Q}}_p,\quad \text{any }\lambda\in\mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v^0,\overline{\mathbb{Q}}_p),$$
are the Frobenius eigenvalues of $\iota\mathrm{Rec}_v\big(\sigma_{i,j,v}\otimes|\cdot|_v^{(1-m_{i,j})/2)}\big)$, where $K_v^0\subset K_v$ is the maximal absolutely unramified extension.

Here $\mathrm{WD}(\rho_{i,j}|_{G_{K_v}})^{\mathrm{Fr-ss}}$ is the Frobenius semisimple Weil–Deligne representation associated to the $\rho_{i,j}|_{G_{K_v}}$.

The existence of $\rho_{i,j}$ follows from [Shin 2011, Theorem 1.2]: As in the proof of Corollary 4, choose an algebraic Hecke character $\chi$ of $\mathbb{A}_K$ such that $\sigma_{i,j}\otimes\chi$ is conjugate self-dual; such a character can be chosen to be unramified at any given finite set of finite places. Then [ibid., Theorem 1.2] applies to $\sigma_{i,j}\otimes\chi$ and we set $\rho_{i,j}:=R_{p,\iota}(\sigma_{i,j}^\vee\otimes\chi^{-1})\otimes\rho_{\chi,\iota}^\vee$ in Shin's notation (the contragredients are here because of the normalization of the local Langlands correspondence in [Shin 2011]). By varying the set of primes at which $\chi$ is unramified we obtain the compatibility with the local Langlands correspondence at all $v\nmid p$. A comparison between the eigenvalues of the $[K_v : \mathbb{Q}_p]$th-power of the crystalline Frobenius eigenvalues and the Frobenius eigenvalues of the Weil–Deligne representation is not stated explicitly in [ibid.] but can be obtained by appealing to the comparison theorem in [Katz and Messing 1974]: the arguments in [Shin 2011, §7] and especially [Taylor and Yoshida 2007, §2] explain that there is a solvable CM-extension $L/K$ in which all places of $K$ above $p$ split and such that $\mathrm{BC}_{L/K}(\sigma_{i,j}^\vee \otimes \chi^{-1})$ is cuspidal and an algebraic Hecke character $\psi$ of $\mathbb{A}_L^\times$ unramified at all primes above $p$ such that some multiple of the $p$-adic $G_L$-representation $R_{p,\iota}(\mathrm{BC}_{L/K}(\sigma_{i,j}^\vee \otimes \chi^{-1})) \otimes \rho_{\psi,\iota}$ is cut out by correspondences acting on the cohomology with constant coefficients of a self-product of the universal abelian variety over a compact Shimura variety (with good reduction at $v$ if $\sigma_{i,j,v} \otimes \chi_v$ is unramified). Here $\mathrm{BC}_{L/K}(\cdot)$ denotes the base change lift to $\mathrm{GL}_n(\mathbb{A}_L)$.

Put

$$\rho_i := \textstyle\bigoplus_{j=1}^{r_i} \rho_{i,j}, \quad i = 1, \dots, k,$$

and

$$\rho_\pi := \rho_\psi \otimes \big(\textstyle\bigoplus_{i=1}^{k} \rho_i\big). \tag{4.5}$$

**Remark 6.** Suppose $k = 1$. Then $\rho_\pi$ satisfies the conclusions of Theorem B, but with $S$ replacing $\Sigma(\pi)$ and with the additional condition that $p \notin S$ for part (d); the definition of $\rho_\pi$ being of "Hodge–Tate type $\xi$" is given after Theorem 10 below.

**Proposition 7.** *For $v|\ell$, $\ell \notin \Sigma(\pi)$, the representations $\tau_{i,j,v}$ and $\tau_{i,v}$ are tempered principal series.*

Our proof of this proposition will come from an understanding of the ramification at $v|\ell$, $\ell \notin \Sigma_p(\pi)$, of the representation

$$r_\pi := \rho_\psi \otimes \bigotimes_{i=1}^{k} \textstyle\bigwedge^{a_i} \rho_i.$$

First, we explain what it means for $\pi$ to be an endoscopic lift. This means that each $n_i$ has a partition $n_i = n_i^+ + n_i^-$ as a sum of nonnegative integers with some $n_j^+ n_j^- \neq 0$ and such that $\sum_{i=1}^k n_i^-$ is even, and that there is a cuspidal automorphic representation $\gamma$ of $G'(\mathbb{A}_\mathbb{Q})$, with

$$G' := G(U(a_1^+, b_1^+) \times U(a_1^-, b_1^-) \times \cdots \times U(a_k^+, b_k^+) \times U(a_k^-, b_k^-))$$

and

$$(a_i^\pm, b_i^\pm) = \Big(\Big\lfloor \frac{n_i^\pm + 1}{2} \Big\rfloor, \Big\lceil \frac{n_i^\pm - 1}{2} \Big\rceil\Big),$$

such that $\gamma$ is unramified at each prime $\ell \notin \Sigma(\pi)$, and for each $\ell \notin \Sigma(\pi)$ the Langlands parameter $\psi_{\pi_\ell}$ of $\pi_\ell$ is the composition of the Langlands parameter $\psi_{\gamma_\ell}$

of $\gamma_\ell$ with the endoscopic $L$-group homomorphism

$$\mathrm{End} : {}^LG' \to {}^LG,$$

is defined as follows (see also [Morel 2010, Proposition 2.3.2]. Let $\epsilon_{K/\mathbb{Q}}:W_\mathbb{Q}\to\{\pm1\}$ be the nontrivial quadratic character factoring through $\mathrm{Gal}(K/\mathbb{Q})$; by class field theory this determines a quadratic character $\omega_{K/\mathbb{Q}} : \mathbb{A}^\times/\mathbb{Q}^\times \to \{\pm1\}$. Fix a finite-order Hecke character $\omega_K$ of $\mathbb{A}_K^\times$ such that $\omega_K|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$ and let $\mu : W_K \to \mathbb{C}^\times$ be the character corresponding via class field theory. Let $c \in W_\mathbb{Q}$ be a lift of the nontrivial automorphism of $K$. Define $\varphi : W_\mathbb{Q} \to {}^LG$ by

$$\varphi(c) = \left(1, \left(\begin{pmatrix} \Phi_{n_i^+} & \\ & (-1)^{n_i^+}\Phi_{n_i^-} \end{pmatrix} \Phi_{n_i}^{-1}\right)\right) \rtimes c,$$

$$\varphi(w) = \left(1, \left(\begin{pmatrix} \mu^{n_i^-}(w) I_{n_i^+} & \\ & \mu^{-n_i^+}(w) I_{n_i^-} \end{pmatrix}\right)\right) \rtimes w, \quad w \in W_K.$$

The endoscopic map is then

$$\mathrm{End}((\lambda, (g_i^+, g_i^-)) \rtimes w) = \left(\lambda, \begin{pmatrix} g_i^+ & \\ & g_i^- \end{pmatrix}\right)\varphi(w).$$

Here $((\lambda, (g_i^+, g_i^-)) \in \widehat{G}' = \mathbb{C}^\times \times \prod_{i=1}^k \mathrm{GL}_{n_i^+}(\mathbb{C}) \times \mathrm{GL}_{n_i^-}(\mathbb{C})$.

**Lemma 8.** *Either $\pi$ is an endoscopic lift of some $\gamma$ with $\gamma_\infty$ a regular discrete series or the representation $r_\pi$ is unramified at all $v|\ell$, $\ell \notin \Sigma_p(\pi)$.*

*Proof.* By [Morel 2010, Theorem 7.2.2] (see also the proof of [ibid., Theorem 7.3.1]), either $\pi$ is an endoscopic lift of some $\gamma$ with $\gamma_\infty$ a regular discrete series indexed by a representation with the same weight as $\xi$ (see [ibid., Lem. 7.3.4]) or (some multiple of) $r_\pi^\vee$ occurs[2] in the middle degree intersection cohomology of a Shimura variety associated with $G$, $\xi$, and $\pi$. By [Lan 2008], this Shimura variety is known to have good reduction at all $v|\ell$, $\ell \notin \Sigma_p(\pi)$, so the representation $r_\pi$ is unramified at such $v$. □

*Proof of Proposition 7.* Let $v|\ell$, $\ell \notin \Sigma(\pi)$. Suppose $\pi$ is the endoscopic lift of some $\gamma$ with $\gamma_\infty$ a regular discrete series. Let $\sigma' = (\psi', (\tau_i^+, \tau_i^-))$ be the very weak base change of $\gamma$ as in Section 4.2 (so $\tau_i^\pm$ is an irreducible automorphic representation of $\mathrm{GL}_{n_i^\pm}(\mathbb{A}_K)$). From the definition of $\pi$ being an endoscopic lift of $\gamma$, it follows that

$$\tau_i = (\tau_i^+ \otimes \omega_K^{-n_i^-}) \boxplus (\tau_i^- \otimes \omega_K^{n_i^+})$$

[2]Theorem 7.2.2 of [Morel 2010] only applies to the case $k = 1$ as stated, but it is asserted at the start of [ibid., 7.2] that the results and proofs "would work the same way" for general $k$. Indeed, the result for the case $k > 1$ is stated and used in the proof of [ibid., Theorem 7.3.1].

(as $\tau_i^{\pm}$ is tempered by Corollary 4). We may therefore reduce to the case where $\pi$ is not endoscopic, and hence, by Lemma 8, to the case where $r_\pi$ is unramified at $v$.

Suppose that $r_\pi$ is unramified at $v$. Consider the isogeny

$$G_1 := \mathrm{GL}_1 \times \prod_{i=1}^{k} \prod_{j=1}^{r_i} \mathrm{GL}_{m_{i,j}} \to G_2 := \mathrm{GL}_1 \otimes \bigotimes_{i=1}^{k} \mathrm{GL}_{\binom{n_i}{a_i}},$$

$$(\lambda, (g_{i,j})) \mapsto \lambda \otimes \bigotimes_{i=1}^{k} \textstyle\bigwedge^{a_i} (\mathrm{diag}(g_{i,1}, \ldots, g_{i,r_i})).$$

The kernel of this isogeny is central. As $r_\pi$ is the composition of

$$\rho := \rho_\psi \oplus \bigoplus_{i=1}^{k} \rho_i : G_K \to G_1(\overline{\mathbb{Q}}_p)$$

with this isogeny, it then follows that since $r_\pi$ is unramified at $v$, the image of inertia at $v$ under $\rho$ is contained in the center of $G_1(\overline{\mathbb{Q}}_p)$, and so the image of inertia at $v$ under each $\rho_{i,j}$ is central. So some finite-order twist of each $\rho_{i,j}$ is unramified at $v$, which — by compatibility of $\rho_{i,j}$ with the local Langlands correspondence — implies that a finite-order twist of each $\sigma_{i,j,v}$, and hence of each $\tau_{i,j,v}$, is unramified. By Corollary 4, $\tau_{i,j,v}$ is also tempered. It follows that each $\tau_{i,j,v}$ is a tempered principal series, so each $\tau_{i,v}$ must also be a tempered principal series. □

**4.4.** ***The main results.*** We can now state our main results, of which Theorems A and B are special cases, and complete their proofs.

**Theorem 9.** *Let $\pi$ be an irreducible cuspidal representation of $G(\mathbb{A}_\mathbb{Q})$ and let $\chi_\pi$ be the character of the scalar torus $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \subset G$ determined by $\pi$ (a character of $\mathbb{A}_K^\times$). Let $\Sigma(\pi)$ be the finite set of primes $\ell$ such that either $\pi_\ell$ is ramified or $\ell | d_K$. Suppose $a_i b_i \neq 0$, $i = 1, \ldots, k$, and $\pi_\infty$ is a regular discrete series belonging to an $L$-packet $\Pi_d(\xi)$. There exists an automorphic representation $\sigma = (\psi, (\tau_i))$ of $H(\mathbb{A}_\mathbb{Q})$ such that*:

(a) $\sigma^\theta \cong \sigma$; $\psi = \chi_\pi^c$.

(b) *For a prime $\ell \notin \Sigma(\pi)$, $\sigma_\ell$ is unramified, and if $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G$ is the Langlands parameter of $\pi_\ell$ then*

$$\psi_{\sigma_\ell} := \mathrm{BC} \circ \psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L H$$

*is the Langlands parameter of $\sigma_\ell$.*

(c) *$\sigma_\infty$ has the same infinitesimal character as $\xi \otimes \xi^\theta$.*

*Proof.* Let $\sigma = (\psi, (\tau_i))$ be as in (4.4). Then part (a) holds. Furthermore, there exists a finite set of primes $S \supset \Sigma(\pi)$ such that part (b) holds with $S$ replacing $\Sigma(\pi)$.

Let $\pi_0 \subset \pi$ be an irreducible automorphic representation of $G_0(\mathbb{A}_\mathbb{Q})$. Then $\pi_0$ is given by a tuple $(\pi_{0,i})_{1 \le i \le k}$ with $\pi_{0,i}$ an automorphic representation of $U(a_i, b_i)$.

For $\ell \notin \Sigma(\pi)$, each $\pi_{0,i,\ell}$ is unramified and the Langlands parameter $\psi_{\pi_{0,i,\ell}}$ of $\pi_{0,i,\ell}$ is given by composing $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G$ with the projection

$$ {}^L G \to {}^L G_0 \to {}^L U(a_i, b_i) = \mathrm{GL}_{n_i}(\mathbb{C}) \rtimes W_{\mathbb{Q}}. $$

From part (b) holding for $\ell \notin S$ it then follows that for such $\ell$ the Langlands parameter of $\tau_{i,\ell}$ is $\mathrm{BC} \circ \psi_{\pi_{0,i,\ell}}$; that is, $\tau_i$ is a very weak base change of $\pi_{0,i}$. But by Proposition 7, $\tau_{i,v}$ is a tempered principal series for each $v|\ell$, $\ell \notin \Sigma(\pi)$, so it follows from Proposition 1 that $\tau_i$ is a weak base change of $\pi_{0,i}$. That (b) holds is then immediate from the relation between the Langlands parameters of $\pi_\ell$ and of the $\pi_{0,i,\ell}$.

To see that part (c) holds, we first recall that the infinitesimal character of an admissible representation of $H(\mathbb{R})$ is an element of $\mathfrak{s}_{\mathbb{C}}^\vee$ up to action of the Weyl group $W(\mathfrak{h}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}})$, where $\mathfrak{h} := \mathrm{Lie}(H(\mathbb{R}))$ and $\mathfrak{s} := \mathrm{Lie}(S(\mathbb{R}))$ with $S := \mathrm{Res}_{K/\mathbb{Q}} T_{/K} \subset G$ the group of diagonal matrices. Then $S_{/K} = T_{/K} \times T_{/K}$ and $X(S) = X(T) \times X(T)$. The irreducible algebraic representations of $H_{/K}$ correspond to pairs of dominant characters of $X(T)$ — the highest weight of the representation with respect to $S$ and the upper-triangular Borel. In particular, the representation $\xi \otimes \xi^\theta$ corresponds to $(\underline{c}, i(\underline{c}))$ and has infinitesimal character $(\underline{c}, i(\underline{c})) + \rho_H$, where $\rho_H := (0, (\rho_{\mathrm{GL}_{n_i}}))$. On the other hand, $S(\mathbb{R}) = \mathbb{C}^\times \times \prod_{i=1}^k A_{n_i}$ so

$$ \mathfrak{s}_{\mathbb{C}}^\vee = \mathbb{C}^2 \oplus \mathfrak{a}_{n_1,\mathbb{C}}^\vee \oplus \cdots \oplus \mathfrak{a}_{n_j,\mathbb{C}}^\vee = \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}, $$

and the infinitesimal character of $\sigma_\infty$ is $(c_0, c_0') \bigoplus_{i=1}^k$(infinitesimal character of $\tau_i$). Since the infinitesimal character of $\tau_i$ is $(\underline{c}_i, -\underline{c}_i') + \rho_{\mathrm{GL}_{n_i}}$, the infinitesimal character of $\sigma_\infty$ is $((c_0, (\underline{c}_i + \rho_{\mathrm{GL}_{n_i}})), (c_0', (-\underline{c}_i' + \rho_{\mathrm{GL}_{n_i}}))) = (\underline{c}, i(\underline{c})) + \rho_H$. □

**Theorem 10.** *Let $\pi$ be an irreducible cuspidal representation of $G(\mathbb{A}_{\mathbb{Q}})$ and let $\chi_\pi$ be the character of the scalar torus $\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m \subset G$ determined by $\pi$ (a character of $\mathbb{A}_K^\times / K^\times$). Let $\Sigma(\pi)$ be the finite set of primes $\ell$ such that either $\pi_\ell$ is ramified or $\ell | d_K$. Suppose $a_i b_i \neq 0$, $i = 1, \ldots, k$, and $\pi_\infty$ is a regular discrete series belonging to an $L$-packet $\Pi_d(\xi)$. Let $\sigma = (\psi, (\tau_i))$ be as in Theorem 9. There exists a continuous, semisimple representation*

$$ \rho_\pi = \rho_{\pi,\iota} : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p) $$

*such that:*

(1) *$\rho_\pi$ is unramified at all finite places not above primes in $\Sigma_p(\pi) := \Sigma(\pi) \cup \{p\}$, and for such a place $w$*

$$ (\rho_\pi|_{W_{K_w}})^{ss} = \bigoplus_{i=1}^k \iota \mathrm{Rec}_w\big(\tau_{i,w} \otimes \psi_w |\cdot|_w^{(1-n_i)/2}\big). $$

(b) *For $v|p$, $\rho_\pi|_{G_{K_v}}$ is potentially semistable of Hodge–Tate-type $\xi$.*

(c) *If $p \notin \Sigma(\pi)$ then for any $v|p$, $\rho_\pi|_{G_{K_v}}$ is crystalline; for any*

$$j \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$$

*the eigenvalues of the action of the $[K_v : \mathbb{Q}_p]$-th power of the crystalline Frobenius on*

$$D_{\mathrm{cris}}(\rho_\pi|_{G_{K_v}}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, j} \overline{\mathbb{Q}}_p$$

*are the eigenvalues of the action of Frobenius on*

$$\bigoplus_{i=1}^{k} \iota\mathrm{Rec}_v\big(\tau_{i,v} \otimes \psi_v |\cdot|_v^{(1-n_i)/2}\big).$$

Let $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k) \in X(T)$ be the highest weight of $\xi$. By $\rho_\pi|_{G_{K_v}}$ being of Hodge–Tate type $\xi$, we mean that $\rho_\pi$ is of Hodge–Tate type $(c_0 + \underline{c}, \underline{c}'_0 + \underline{c}')$.

*Proof.* If we take $\rho_\pi$ to be as in (4.5), then (a) is immediate from Theorem 9(b) and the definition of $\rho_\pi$ as being the twist by $\rho_\psi$ of the sum of the $\rho_{i,j}$. From the proof of Lemma 3, the character $\xi_{i,j}$ has highest weights

$$(c_{i,\ell'_t} + t - \ell'_t, -c_{i,\ell''_t} + \ell''_t - n_i + t)_{1 \le t \le m_{i,j}},$$

and so for $v|p$,

$$D_{\mathrm{HT},v}(\rho_{i,j}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, \zeta} \overline{\mathbb{Q}}_p$$

is nonzero in degrees $\ell'_t - 1 - c_{i,\ell'_t}$ if $\zeta \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$ induces the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}_p \cong \mathbb{C}$, and otherwise is nonzero in degrees $n_i - \ell''_t - 1 + c_{i,\ell''_t}$. That $\rho_\pi|_{G_{K_v}}$ is of Hodge–Tate type $\xi$ then follows from this and the fact that $\psi_\infty(z) = z^{c_0}\bar{z}^{c'_0}$ and so $\rho_\psi$ is of Hodge–Tate type $(c_0, c'_0)$. That $\rho_\pi|_{G_{K_v}}$, $v|p$, is potentially semistable and even crystalline with the prescribed Frobenius eigenvalues if $v|p$ follows from the corresponding facts for $\rho_\psi$ and the $\rho_{i,j}$. □

Theorems A and B are just the special cases where $k = 1$.

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