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Galois representations associated with unitary groups over $\mathbb{Q}$

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We show that a cuspidal automorphic representation $\pi = \bigotimes_{\ell\le\infty}\pi_\ell$ of a unitary similitude group $\mathrm{GU}(a,b)_{/\mathbb{Q}}$ with archimedean component π_∞ in a regular discrete series has an associated $(a+b)$-dimensional p-adic Galois representation with Frobenius eigenvalues given by the local base change parameters for all primes ℓ such that π_ℓ and $\mathrm{GU}(a,b)$ are unramified.

1. Introduction

In this paper we explain how results of Morel [2010] on the cohomology of the noncompact Shimura varieties associated to unitary similitude groups over $\mathbb{Q}$ can be combined with results of Shin [2011] on the cohomology of certain compact Shimura varieties and with certain analytic results — most notably the stability of the gamma factors arising from the doubling method for unitary groups [Lapid and Rallis 2005; Brenner 2008] — to prove that a cuspidal automorphic representation π of $\mathrm{GU}(a,b)_{/\mathbb{Q}}$ with archimedean component in a discrete series and regular (in a sense made precise below) has an associated $(a+b)$-dimensional p-adic Galois representation with Frobenius eigenvalues given by the local base change parameters for all primes ℓ such that π and $\mathrm{GU}(a,b)$ are unramified. Our motivation for this is the use in [Skinner and Urban 2010] of these p-adic Galois representations in the case $(a,b)=(2,2)$ to prove the Iwasawa–Greenberg main conjecture for a large class of modular forms. The main results include Theorems A and B below, whose proofs are intertwined.

Let K be an imaginary quadratic field of discriminant d_K. Let $n=a+b$ be a partition of a positive integer n as the sum of two nonnegative integers a and b. Then

$$J_{a,b} := \begin{pmatrix} 1_a & \\ & -1_b \end{pmatrix}$$

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defines an Hermitian pairing on the space $V := K^n$. Let $G := \mathrm{GU}(a, b)_{/\mathbb{Q}}$ denote the unitary similitude group over $\mathbb{Q}$ of the Hermitian pair $(V, J_{a,b})$. The L-packets of discrete series representations of $G(\mathbb{R})$ are naturally indexed by the irreducible algebraic representations of $G_{/K}$ (see Section 4.1). By a *regular* discrete series representation of $G(\mathbb{R})$ we will mean one belonging to an L-packet indexed by a representation with regular highest weight.

Let $H := \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m \times \mathrm{GL}_n)$. For any $\mathbb{Q}$-algebra R, let $(x, g) \mapsto (\bar{x}, \bar{g})$ be the involution of $H(R) = (R \otimes K)^\times \times \mathrm{GL}_n(R \otimes K)$ induced by the nontrivial automorphism of K, and let θ be the involution defined by $\theta((x, g)) = (\bar{x}, \bar{x}\,{}^t\bar{g}^{-1})$. Note that an irreducible admissible representation σ of $H(\mathbb{A}_\mathbb{Q})$ is given by a pair (ψ, τ) consisting of an admissible character ψ of $\mathbb{A}_K^\times$ and an irreducible admissible representation τ of $\mathrm{GL}_n(\mathbb{A}_K)$ and that $\sigma = (\psi, \tau)$ is θ-stable (that is, $\sigma^\theta \cong \sigma$) if and only if $\tau^\vee \cong \tau^c$ and $\psi = \psi^c \chi_\tau^c$, where χ_τ is the central character of τ and the superscripts '$\vee$' and 'c' denote, respectively, the contragredient and composition with the involution induced by the nontrivial automorphism of K. Let $\mathrm{BC} : {}^L G \to {}^L H$ be the base change morphism (see Section 2.3).

Theorem A (weak base change). *Let π be an irreducible cuspidal representation of $G(\mathbb{A}_\mathbb{Q})$ and let χ_π be its central character (a character of $\mathbb{A}_K^\times$). Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Suppose $ab \neq 0$ and π_∞ is a regular discrete series belonging to an L-packet indexed by a representation ξ. There exists an automorphic representation $\sigma = (\psi, \tau)$ of $H(\mathbb{A}_\mathbb{Q})$ such that*:

(a) $\sigma^\theta \cong \sigma$, $\psi = \chi_\pi^c$ *and* $\chi_\tau = \chi_\pi^c / \chi_\pi$.

(b) *For a prime $\ell \notin \Sigma(\pi)$, σ_ℓ is unramified, and if $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G$ is the Langlands parameter of π_ℓ then*

$$\psi_{\sigma_\ell} := \mathrm{BC} \circ \psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L H$$

is the Langlands parameter of σ_ℓ. In particular, for any idèle class character χ of $\mathbb{A}_K^\times$ there is equality of twisted standard L-functions

$$L_{\Sigma(\pi)}(s, \pi \times \chi) = L_{\Sigma(\pi)}(s, \tau \times \chi).$$

(c) σ_∞ *has the same infinitesimal character as* $\xi \otimes \xi^\theta$.

There is a natural identification of $G_{/K}$ with $\mathbb{G}_m \times \mathrm{GL}_n$ (see Section 2.2) and hence of $G(\mathbb{R} \otimes K)$ with $H(\mathbb{R})$, which then identifies ξ, and hence ξ^θ, as a representation of $H(\mathbb{R})$. The (partial) standard L-function of π is as defined as in [Li 1992, §3].

Let $\overline{K}$ be an algebraic closure of K and let $G_K := \mathrm{Gal}(\overline{K}/K)$. For each finite place v of K let $\overline{K}_v$ be an algebraic closure of K_v and fix an embedding $\overline{K} \hookrightarrow \overline{K}_v$. The latter identifies $G_{K_v} := \mathrm{Gal}(\overline{K}_v/K_v)$ with a decomposition group for v in G_K and hence the Weil group $W_{K_v} \subset G_{K_v}$ with a subgroup of G_K.

Let p be a prime and $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$. Let $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ be an isomorphism. Our conventions for Galois representations are geometric.

Theorem B (Galois representations). *Let π be an irreducible cuspidal representation of $G(\mathbb{A}_{\mathbb{Q}})$ and let χ_π be its central character. Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Suppose $ab \neq 0$ and π_∞ is a regular discrete series belonging to an L-packet indexed by the representation ξ. Let $\sigma = (\psi, \tau)$ be as in Theorem A. There exists a continuous, semisimple representation $\rho_\pi = \rho_{\pi,\iota} : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that:*

(a) $\rho_\pi^c \simeq \rho_\pi^\vee \otimes \rho_{\chi_\pi^{1+c}} \epsilon^{1-n}$.

(b) *ρ_π is unramified at all finite places not above primes in $\Sigma_p(\pi) := \Sigma(\pi) \cup \{p\}$, and for such a place w*

$$(\rho_\pi|_{W_{K_w}})^{ss} = \iota \mathrm{Rec}_w\big(\tau_w \otimes \psi_w |\cdot|_w^{(1-n)/2}\big).$$

In particular,

$$L_{\Sigma_p(\pi)}(s, \rho_\pi) = L_{\Sigma_p(\pi)}\Big(s + \frac{1-n}{2}, \tau \times \psi\Big).$$

(c) *For $v|p$, $\rho_\pi|_{G_{K_v}}$ is potentially semistable of Hodge–Tate-type ξ.*

(d) *If $p \notin \Sigma(\pi)$ then*

(d) *If $p \notin \Sigma(\pi)$ then for any $v|p$, $\rho_\pi|_{G_{K_v}}$ is crystalline. Moreover, for any j in $\mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$ the eigenvalues of the action of the $[K_v : \mathbb{Q}_p]$-th power of the crystalline Frobenius on*

$$D_{\mathrm{cris}}(\rho_\pi|_{G_{K_v}}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, j} \overline{\mathbb{Q}}_p$$

are the eigenvalues of the action of Frobenius on $\iota\mathrm{Rec}_v\big(\tau_v \otimes \psi_v |\cdot|_v^{(1-n)/2}\big)$.

For any irreducible admissible representation α of $\mathrm{GL}_n(K_w)$, $\mathrm{Rec}_w(\alpha)$ is the Weil–Deligne representation over $\mathbb{C}$ associated by the local Langlands correspondence, and $\iota\mathrm{Rec}_w(\alpha)$ is the representation over $\overline{\mathbb{Q}}_p$ obtained by change of scalars via ι. For $\rho_\pi|_{G_{K_v}}$ to be of Hodge–Tate type ξ means that the Hodge–Tate weights can be read off from ξ in a prescribed way (see Section 4.4).

As the proof of Theorem A shows, there is a partition $n = m_1 + \cdots + m_r$ such that the representation τ in Theorem A is of the form $\tau = \tau_1 \boxplus \cdots \boxplus \tau_r$ with τ_i a cuspidal automorphic representation of $\mathrm{GL}_{m_i}(\mathbb{A}_K)$ such that $\tau_i^c \cong \tau_i^\vee$ and $\sigma_i := \tau_i \otimes |\cdot|^{(m_i-n)/2}$ is regular algebraic in the sense of [Clozel 1990]. Then the representation ρ_π of Theorem B is just $\rho_\psi \otimes \big(\bigoplus_{i=1}^r \rho_{\sigma_i,\iota}\big)$, where $\rho_{\sigma_i,\iota}$ is the m_i-dimensional p-adic Galois representation associated to σ_i ($\rho_{\sigma_i,\iota}$ is obtained from [Shin 2011]).

The theory of pseudorepresentations in combination with congruences between automorphic forms allows the weakening of some of the hypotheses of Theorem B —

cases where $ab = 0$ or where ξ is not regular can be allowed. But we do not include this here.

If $\mathbb{Q}$ is replaced by a totally real field of degree greater than one, then the analogs of Theorems A and B are known, the weak base change having been proved by Labesse [2011]. Furthermore, versions of these theorems have been proved by Morel [2010], who proves Theorem A but with $\Sigma(\pi)$ replaced by an indeterminate set of primes, and by Harris and Labesse [2004], who require additional conditions at some finite primes. The work of Morel is the starting point of our proofs.

Our proofs of Theorems A and B proceed essentially as follows. By results of Morel, an automorphic representation $\sigma = (\psi, \tau)$ of $H(\mathbb{A}_{\mathbb{Q}})$ as in Theorem A exists but with $\Sigma(\pi)$ replaced by an indeterminate set $S \supseteq \Sigma(\pi)$. Furthermore, τ is a subquotient of an induced representation $\mathrm{Ind}_P^{\mathrm{GL}_n}\left(\bigotimes_{i=1}^r \tau_i\right)$ with $P \subset \mathrm{GL}_n$ the standard parabolic associated with a partition $n = m_1 + \cdots + m_r$ and each τ_i a discrete representation of $\mathrm{GL}_{m_i}(\mathbb{A}_{\mathbb{Q}})$ such that $\tau_i^c \cong \tau_i^\vee$. By considering absolute values of Satake parameters, it follows from the work of Mœglin and Waldspurger characterizing the discrete series representations of $\mathrm{GL}_{m_i}(\mathbb{A}_{\mathbb{Q}})$ that each τ_i is cuspidal, and a consideration of infinitesimal characters yields that $\sigma_i := \tau_i \otimes |\cdot|^{(n_i-n)/2}$ is algebraic with the same infinitesimal character as a regular irreducible representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_i}$. The regularity of ξ is used in both these arguments. Then $\rho_{\pi,\iota} := \rho_\psi \otimes \left(\bigoplus_{i=1}^r \rho_{\sigma_i,\iota}\right)$, with $\rho_{\sigma_i,\iota}$ being the representation deduced from the work of Shin, satisfies conclusions (a), (b), and (c) of Theorem B with $\Sigma(\pi)$ replaced by S. It then remains to show that (b) of Theorem A also holds for $\ell \in S$ but $\ell \notin \Sigma(\pi)$, for then (b) and (d) of Theorem B follow from the corresponding results for the $\rho_{\sigma_i,\iota}$. To obtain (b) of Theorem A for such an ℓ we first observe that the representation $\bigwedge^a \rho_{\pi,\iota}$ is unramified at the places $w|\ell$. This is because Morel has essentially shown that this representation appears in the intersection cohomology of a Shimura variety associated to π that has good reduction at $w|\ell$ (some argument is required to reduce to the nonendoscopic case); this is another point at which the regularity of ξ is used. Then the local-global compatibility satisfied by the $\rho_{\sigma_i,\iota}$ implies that there is a finite order character χ_ℓ of $K_\ell^\times$ such that each $\tau_{i,w} \otimes \chi_w$ is unramified, and hence a principal series representation of $\mathrm{GL}_{m_i}(K_w)$ with Satake parameters all having the same absolute values (again using regularity of ξ). This information is then combined with that coming from the γ-factors of the standard L-functions. Lapid and Rallis have defined local γ-factors $\gamma(s, \pi_v \times \chi_v)$ for the standard L-function of π such that

$$L_S(s, \pi \times \chi) = \prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \chi_v) \times L_S(1-s, \pi^\vee \times \chi^{-1}),$$

and Brenner has proved stability for these γ-factors at nonarchimedean places. Comparing this with the functional equation for $L_S(s, \tau \times \chi)$ and choosing a global character χ with ℓ-component χ_ℓ and with sufficiently ramified q-components χ_q for $\ell \neq q \in S$ yields an equality between γ-factors for π and

$$\tau : \gamma(s, \pi_\ell \times \chi_\ell) = \gamma(s, \tau_\ell \times \chi_\ell).$$

Comparing the definitions of these gamma factors and exploiting some freedom in the choice of χ_ℓ and χ then yields conclusion (b) of Theorem A.

After some preliminary remarks fixing notation for unitary and related groups in Section 2, in Section 3 we give the analytic arguments involving L-functions and γ-factors. In Section 4 we then recall the results of Morel and Shin and explain how Theorems A and B follow.

2. Preliminaries

We adopt the following notation and conventions.

2.1. ***Galois groups and representations.*** Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and let $K \subset \overline{\mathbb{Q}}$ be an imaginary quadratic field of discriminant d_K. For $F = \mathbb{Q}$ or K, let $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Let W_F be a Weil group of F; this comes with a homomorphism to G_F. For each place v of F fix an algebraic closure $\overline{F}_v$ of F_v and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{F}_v$. The latter identifies $G_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$ with a decomposition group in G_F. Let W_{F_v} be the Weil group of F_v; for a finite place v, W_{F_v} is a subgroup of G_{F_v} and so is identified with a subgroup of G_F. Fix a homomorphism $W_{F_v} \to W_F$ compatible with the fixed inclusion $G_{F_v} \subset G_F$. We denote the action on K of the nontrivial automorphism in $\mathrm{Gal}(K/\mathbb{Q})$ by $x \mapsto \bar{x}$. For simplicity, we also fix an embedding $K \hookrightarrow \mathbb{C}$ (equivalently, an isomorphism $\overline{K}_\infty \cong \mathbb{C}$).

Let p be fixed prime and $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ a fixed isomorphism. Our conventions for p-adic Galois representations are geometric: L-functions of representations of G_F or G_{F_v} are defined by taking characteristic polynomials of *geometric* Frobenius elements.

For an algebraic Hecke character of $\mathbb{A}_F^\times$ (so $\chi_\infty(x) = \mathrm{sgn}(x)^r x^t$ if $F = \mathbb{Q}$ and $\chi_\infty(x) = x^r \bar{x}^t$ if $F = K$, for some $r, t \in \mathbb{Z}$) let

$$\rho_\chi = \rho_{\chi,\iota} : G_F \to \overline{\mathbb{Q}}_p^\times$$

be the p-adic Galois character such that $L_{\{p\}}(s, \rho_\chi) = L_{\{p\}}(s, \chi)$. Then $\epsilon : G_F \to \mathbb{Z}_p^\times$ is the p-adic character associated to the norm $|\cdot|_F$ character of $\mathbb{A}_F^\times$; this is the p-adic cyclotomic character: for a geometric Frobenius frob_v, $v \nmid p\infty$,

$$\epsilon(\mathrm{frob}_v) = \mathrm{Norm}(v)^{-1}.$$

2.2. ***The groups: G, G_0, H, and H_0.*** Let $n_1, \ldots, n_k$ be positive integers and $n := n_1 + \cdots + n_k$. For each $i = 1, \ldots, k$ let $n_i = a_i + b_i$ be a partition of n_i as a sum of two nonnegative integers. Let

$$J_i = J_{a_i,b_i} := \begin{pmatrix} 1_{a_i} & \\ & -1_{b_i} \end{pmatrix}.$$

Then J_i defines a Hermitian pairing on K^{n_i}. Let

$$G = G(U(a_1, b_1) \times \cdots \times U(a_k, b_k))_{/\mathbb{Q}}$$

and let $\mu : G \to \mathbb{G}_m$ be its similitude character. That is, for any $\mathbb{Q}$-algebra R,

$$G(R) = \left\{ g = (g_1, \ldots, g_k) \in \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R \otimes K) : \exists \lambda \in R^\times \text{ such that } g_i J_i^t \bar{g}_i = \lambda J_i \right\}$$

and $\mu(g) = \lambda$. Here $g \mapsto \bar{g}$ is the involution of $\mathrm{GL}_m(R \otimes K)$ defined by the action of the nontrivial automorphism of K. Let $G_0 := U(a_1, b_1) \times \cdots \times U(a_k, b_k)$ be the kernel of μ.

For any K-algebra R there is a natural isomorphism $R \otimes K \xrightarrow{\sim} R \times R$, $r \otimes x \mapsto (rx, r\bar{x})$. Using this, we identify $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^k \mathrm{GL}_{n_i}$:

$$g = (g_i', g_i'') \in G(R) \subset \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R \otimes K) = \prod_{i=1}^{k} \mathrm{GL}_{n_i}(R) \times \mathrm{GL}_{n_i}(R)$$

is identified with $(\mu(g), (g_i')) \in R^\times \times \prod_{i=1}^k \mathrm{GL}_{n_i}(R)$. Then $G_{0/K}$ is identified with the subgroup $\prod_{i=1}^k \mathrm{GL}_{n_i}$.

Let $H := \mathrm{Res}_{K/\mathbb{Q}} G_{/K}$. Then $H_{/K}$ is identified with $G_{/K} \times G_{/K}$. The identification of $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^k \mathrm{GL}_{n_i}$ identifies H with

$$\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_{n_i}.$$

Let θ be the involution of H defined by

$$\theta(x, (g_i)) = (\bar{x}, (\bar{x}^t \bar{g}_i^{-1})).$$

Let $H_0 := \mathrm{Res}_{K/\mathbb{Q}} G_0$. Note that θ also defines an involution $(g_i) \mapsto ({}^t\bar{g}_i^{-1})$ of H_0. An irreducible admissible representation of $H(\mathbb{A}_\mathbb{Q})$ is given by a tuple $(\psi, (\tau_i))$ with ψ an admissible character of $\mathbb{A}_K^\times$ and each τ_i an irreducible admissible representation of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$.

2.3. Dual groups and L-groups. The identification of $G_{/K}$ with $\mathbb{G}_m \times \prod_{i=1}^k \mathrm{GL}_{n_i}$ also identifies the dual group $\widehat{G}$ with $\mathbb{C}^\times \times \prod_{i=1}^k \mathrm{GL}_{n_i}(\mathbb{C})$, with $G_\mathbb{Q}$ acting through the quotient $\mathrm{Gal}(K/\mathbb{Q})$ and the nontrivial automorphism $c \in \mathrm{Gal}(K/\mathbb{Q})$ acting by

$$c(x, (g_i)) = \left(x \prod_{i=1}^{k} \det g_i, (\Phi_{n_i}^{-1t} g_i^{-1} \Phi_{n_i}) \right),$$

where $\Phi_m := (\Phi_{m,ij}) = ((-1)^{i+1}\delta_{i,m-j+1})$. Put ${}^L G := \widehat{G} \rtimes W_{\mathbb{Q}}$. Similarly, $\widehat{G}_0 = \prod_{i=1}^k \mathrm{GL}_{n_i}(\mathbb{C})$ with the same action of $G_{\mathbb{Q}}$; let ${}^L G_0 := \widehat{G}_0 \rtimes W_{\mathbb{Q}}$. The L-homomorphism corresponding to taking an irreducible admissible $G_0(\mathbb{A}_{\mathbb{Q}})$-constituent of an irreducible admissible $G(\mathbb{A}_{\mathbb{Q}})$ representation is the projection

$${}^L G \to {}^L G_0, (x, (g_i)) \rtimes w \mapsto (g_i) \rtimes w.$$

Since $H_{/K} = G_{/K} \times G_{/K}$, $\widehat{H} = \widehat{G} \times \widehat{G}$ with the action of $G_{\mathbb{Q}}$ again factoring through $\mathrm{Gal}(K/\mathbb{Q})$ and with $c(x, y) = (c(y), c(x))$. Similarly, $\widehat{H}_0 = \widehat{G}_0 \times \widehat{G}_0$ with the same action of $G_{\mathbb{Q}}$. Put ${}^L H := \widehat{H} \rtimes W_{\mathbb{Q}}$ and ${}^L H_0 := \widehat{H}_0 \rtimes W_{\mathbb{Q}}$. The diagonal embedding $\widehat{G} \hookrightarrow \widehat{H} = \widehat{G} \times \widehat{G}$ is $G_{\mathbb{Q}}$-equivariant; its extension to L-groups

$$\mathrm{BC} : {}^L G \to {}^L H$$

is the base change map. Let $\mathrm{BC} : {}^L G_0 \to {}^L H_0$ be the similarly defined map.

3. L-functions and γ-factors

In this section we prove the key analytic ingredient of our proof of Theorems A and B. We assume in the argument that $G_0 = U(a, b)$ (that is, k=1).

Let π be a cuspidal automorphic representation of $G_0(\mathbb{A}_{\mathbb{Q}})$. Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. By the principle of functoriality for the L-group homomorphism $\mathrm{BC} : {}^L G_0 \to {}^L H_0$ it is expected — at the very least — that there should be a *weak base change* of π to $H_0(\mathbb{A}_{\mathbb{Q}})$. That is, there should exist an automorphic representation τ of $H_0(\mathbb{A}_{\mathbb{Q}})$ (equivalently, of $\mathrm{GL}_n(\mathbb{A}_K)$) such that for $\ell \notin \Sigma(\pi)$, the Langlands parameter $\psi_{\tau_\ell} : W_{\mathbb{Q}_\ell} \to {}^L H_0$ of τ_ℓ is just $\mathrm{BC} \circ \psi_{\pi_\ell}$, with $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G_0$ the Langlands parameter of π_ℓ. We say that τ is a *very weak base change* of π if there is *some* set $S \supset \Sigma(\pi)$ such that this relation between Langlands parameters holds for all $\ell \notin S$.

Proposition 1. *Let π be a cuspidal automorphic representation of $G_0(\mathbb{A}_{\mathbb{Q}})$. Assume that there exists a very weak base change τ of π to $H_0(\mathbb{A}_{\mathbb{Q}})$. If τ is a tempered principal series at every finite place $\ell \notin \Sigma(\pi)$, then τ is a weak base change of π.*

We deduce the conclusion of this proposition by comparing L-functions and γ-factors. Let $R := \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$. Then $\widehat{R} = \mathbb{C}^\times \times \mathbb{C}^\times$ with $G_{\mathbb{Q}}$ acting through $\mathrm{Gal}(K/\mathbb{Q})$ and the nontrivial automorphism c of K acting as $c(x_1, x_2) = (x_2, x_1)$. Let ${}^L R := \widehat{R} \rtimes W_{\mathbb{Q}}$. Let ω be a Hecke character of $\mathbb{A}_K$. Then ω is an irreducible admissible representation of $R(\mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_K^\times$; we let $\psi_{\omega_\ell} : W_{\mathbb{Q}_\ell} \to {}^L R$ be the Langlands parameter associated with $\omega_\ell := \bigotimes_{v|\ell} \omega_v$ (coming from class field theory). The L-groups of $G_0 \times R$ and $H_0 \times R$ are ${}^L(G_0 \times R) = {}^L G_0 \times_{W_{\mathbb{Q}}} {}^L R = (\widehat{G}_0 \times \widehat{R}) \rtimes W_{\mathbb{Q}}$ and ${}^L(H_0 \times R) = {}^L H_0 \times_{W_{\mathbb{Q}}} {}^L R = (\widehat{H}_0 \times \widehat{R}) \rtimes W_{\mathbb{Q}}$, with $W_{\mathbb{Q}}$ acting on each factor.

Let π and τ be as in the proposition. The unramified local L-factors $L(s, \pi_\ell \times \omega_\ell)$ of the standard L-function of $\pi \times \omega$ are the L-factors associated with the representation $r_{st} : {}^L(G_0 \times R) \to \mathrm{GL}_{2n}(\mathbb{C})$,

$$r_{st}((g, (x_1, x_2)) \rtimes 1) = \begin{pmatrix} x_1 g & \\ & x_2 \Phi_n^{-1t} g^{-1} \Phi_n \end{pmatrix} \qquad r_{st}(1 \rtimes c) = \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}.$$

If $\ell \nmid d_K$ and π_ℓ and ω_ℓ are unramified, then

$$L(s, \pi_\ell \times \omega_\ell) = \det(1 - \ell^{-s} r_{st}(\psi_{\pi_\ell}(\mathrm{frob}_\ell), \psi_{\omega_\ell}(\mathrm{frob}_\ell)))^{-1}.$$

Similarly, the local unramified L-factors $L(s, \tau_\ell \times \omega_\ell) := \prod_{v|\ell} L(s, \tau_v \times \omega_v)$ are the L-factors associated with the homomorphism $r'_{st} : {}^L(H_0 \times R) \to \mathrm{GL}_{2n}(\mathbb{C})$,

$$r'_{st}(((g_1, g_2), (x_1, x_2)) \rtimes 1) = \begin{pmatrix} x_1 g_1 & \\ & x_2 \Phi_n^{-1t} g_2^{-1} \Phi_n \end{pmatrix} \qquad r'_{st}(1 \rtimes c) = \begin{pmatrix} & 1_n \\ 1_n & \end{pmatrix}.$$

In particular, $r_{st} = r'_{st} \circ (\mathrm{BC} \times id)$, so $L(s, \pi_\ell \times \omega_\ell) = L(s, \tau_\ell \times \omega_\ell)$ if $\ell \nmid d_K$ and π_ℓ, τ_ℓ, and ω_ℓ are unramified and $\psi_{\tau_\ell} := \mathrm{BC} \circ \psi_{\pi_\ell}$ (so for all $\ell \notin S$).

Lemma 2. *Suppose $\ell \nmid d_K$ and π_ℓ are τ_ℓ are unramified. If*

$$L(s, \pi_\ell \times \omega_\ell) = L(s, \tau_\ell \times \omega_\ell)$$

for all unramified ω_ℓ, then $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$.

Proof. Let

$$\psi_{\pi_\ell}(\mathrm{frob}_\ell) = t \rtimes \mathrm{frob}_\ell, \quad t = \mathrm{diag}(t_1, \dots, t_n), \quad \text{and } \psi_{\tau_\ell}(\mathrm{frob}_\ell) = (h, h) \rtimes \mathrm{frob}_\ell,$$

$h = \mathrm{diag}(h_1, \dots, h_n)$ (ψ_{τ_ℓ} must be of this form as $\tau_\ell^c \cong \tau_\ell^\vee$). Suppose first that ℓ does not split in K. As $\mathrm{frob}_\ell = c$ in $\mathrm{Gal}(K/\mathbb{Q})$, the condition that $L(s, \pi_\ell \times \omega_\ell) = L(s, \tau_\ell \times \omega_\ell)$ is just that $t_i / t_{n-i} = h_i / h_{n-i}$ (after possibly reordering the h_i). That is, $t = zh$ for some $z \in \mathbb{C}^\times$, and so $(z, 1)\psi_{\tau_\ell}(\mathrm{frob}_\ell)(z^{-1}, 1) = \mathrm{BC} \circ \psi_{\pi_\ell}(\mathrm{frob}_\ell)$. Hence, ψ_{τ_ℓ} is equivalent to $\mathrm{BC} \circ \psi_{\pi_\ell}$.

Suppose that ℓ splits in K. Let $\psi_{\omega_\ell}(\mathrm{frob}_\ell) = (\alpha, \beta) \rtimes \mathrm{frob}_\ell$. As $\mathrm{frob}_\ell = 1$ in $\mathrm{Gal}(K/\mathbb{Q})$, the equality $L(s, \pi_\ell \times \omega_\ell) = L(s, \tau_\ell \times \omega_\ell)$ means that

$$\mathrm{diag}(\alpha t, \beta \Phi_n^{-1} t^{-1} \Phi_n) \in \mathrm{GL}_{2n}(\mathbb{C}) \quad \text{and} \quad \mathrm{diag}(\alpha h, \beta \Phi_n^{-1} h^{-1} \Phi_n) \in \mathrm{GL}_{2n}(\mathbb{C})$$

are equivalent. As α and β can be arbitrary, it follows that t and h are equivalent, so $\mathrm{BC} \circ \psi_{\pi_\ell}$ is equivalent to ψ_{τ_ℓ}. □

Let $S \supset \Sigma(\pi)$ be any finite set of primes such that $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$ for all $\ell \notin S$. The (partial) standard L-functions $L_S(s, \pi \times \omega)$ and $L_S(s, \tau \times \omega)$, given by the Euler products

$$L_S(s, \pi \times \omega) = \prod_{\ell \notin S} L(s, \pi_s \times \omega_\ell) \quad \text{and} \quad L_S(s, \tau \times \omega) = \prod_{\ell \notin S} L(s, \tau_\ell \times \omega_\ell)$$

for $\mathrm{Re}(s) \gg 0$, satisfy

$$L_S(s, \pi \times \omega) = L_S(s, \tau \times \omega).$$

The doubling method of Piatetski-Shapiro and Rallis provides an integral representation of $L_S(s, \pi \times \omega)$ as well as local γ-factors at all places; see [Gelbart et al. 1987, Part A] and especially [Lapid and Rallis 2005]. In particular, for each place v of $\mathbb{Q}$, Lapid and Rallis have defined local γ-factors $\gamma(s, \pi_v \times \omega_v) := \gamma_v(s, \pi_v \times \omega_v, \psi_v)$, ψ_v being the standard additive character of K_v and proved that the local γ-factors $\gamma(s, \pi_v \times \omega_v)$ are compatible with parabolic induction and are as expected in the unramified cases. The functional equation for $L_S(s, \pi \times \omega)$ is then

$$L_S(s, \pi \times \omega) = \prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \omega_v) \times L_S(1 - s, \pi^\vee \times \omega^{-1}).$$

Comparing this with the usual functional equation for the standard GL_n L-function $L_S(s, \tau \times \omega)$ we find that

$$\prod_{v \in S \cup \{\infty\}} \gamma(s, \pi_v \times \omega_v) = \prod_{v \in S \cup \{\infty\}} \prod_{w \mid v} \gamma(s, \tau_w \times \omega_w), \tag{3.1}$$

where w is a place of K and $\gamma(s, \tau_w \times \omega_w)$ is the γ-factor defined by Godement and Jacquet (again using the standard additive characters). For a place v of $\mathbb{Q}$, set

$$\gamma(s, \tau_v \times \omega_v) := \prod_{w \mid v} \gamma(s, \tau_w \times \omega_w).$$

We exploit *stability* of γ-factors. This says that if π_1 and π_2 are two irreducible admissible representations of $G_0(\mathbb{Q}_\ell)$, then for χ a sufficiently ramified character of $K_\ell^\times$, $\gamma(s, \pi_1 \times \chi) = \gamma(s, \pi_2 \times \chi)$. This has been proved by Brenner [2008]. Stability is also known for the Godement–Jacquet γ-factors for GL_n. Taking $\pi_1 = \pi_\ell$ and π_2 to be an unramified tempered principal series, we see that if ω_ℓ is sufficiently ramified then

$$\gamma(s, \pi_\ell \times \omega_\ell) = \gamma(s, \pi_2 \times \omega_\ell) = \gamma(s, \tau_2 \times \omega_\ell) = \gamma(s, \tau_\ell \times \omega_\ell), \tag{3.2}$$

where τ_2 is the representation of $H_0(\mathbb{Q}_\ell) = \mathrm{GL}_n(K_\ell)$ having Langlands parameter equal to the composition with BC of the parameter of π_2; τ_2 is also an unramified tempered principal series. The first and last equalities in (3.2) come from stability, and the middle comes from [Lapid and Rallis 2005, Theorem 4]: part 1 of this theorem, together with the hypothesis that π_2 is a principal series, reduces the equality to the minimal cases — the anisotropic cases, which are part 7 of the theorem, and the isotropic cases, which are part 8 — plus the analog of part 2 for the Godement–Jacquet γ-factors (compatibility with parabolic induction).

It is easy to see that given any finite set of primes S' it is possible to find a set $S'' \supset S \cup S'$ and a finite order Hecke character ω of $\mathbb{A}_K^\times$ such that ω_ℓ is arbitrary for all $\ell \in S'$, and ω_ℓ is sufficiently ramified at all primes $\ell \in S'' - S'$ and unramified at all primes not in S''. Taking $S' = \varnothing$, we deduce from (3.1) and (3.2) that $\gamma(s, \pi_\infty \times \omega_\infty) = \gamma(s, \tau_\infty \times \omega_\infty)$. Taking $S' = \{\ell\}$, any prime ℓ, we then deduce from (3.1) and (3.2) that

$$\gamma(s, \pi_\ell \times \omega_\ell) = \gamma(s, \tau_\ell \times \omega_\ell) \tag{3.3}$$

always.

Suppose now that $\ell \notin \Sigma(\pi)$. By hypothesis, τ_v is a tempered principal series for $v|\ell$. Suppose first that ℓ is inert in K. Then $\tau_\ell \cong \pi(\mu_1, \dots, \mu_n)$ with $|\mu_i(x)| = 1$ for all $x \in K_\ell^\times$. Fix j between 1 and n and choose ω_ℓ so that $\mu_j \omega_\ell$ is unramified. Let $I \subset \{1, \dots, n\}$ be the set of indices such that $\mu_i \omega_\ell$ is unramified. Then

$$\gamma(s, \tau_\ell \times \omega_\ell) = \prod_{i \in I} \frac{1 - \mu_i \omega_\ell(\ell) \ell^{-2s}}{1 - \mu_i^{-1} \omega_\ell^{-1}(\ell) \ell^{2s-2}} \times \prod_{i \notin I} \gamma(s, \mu_i \omega_\ell).$$

As $\mu_i \omega_\ell$ is ramified for $i \notin I$, $\gamma(s, \mu_i \omega_\ell)$ is holomorphic with no zeros. Furthermore, the temperedness of τ_ℓ ensures that there is no cancellation between the numerators and denominators of the factors coming from the $i \in I$. Therefore, $\gamma(s, \tau_\ell \times \omega_\ell)$ has $|I| \geq 1$ poles. However, if ω_ℓ is ramified, then, since π_ℓ is unramified, it follows from combining parts 1, 7, and 8 of [Lapid and Rallis 2005, Theorem 4] that $\gamma(s, \pi_\ell \times \omega_\ell)$ is holomorphic. So it must be that ω_ℓ — and hence μ_j — is unramified. But j was arbitrary, so each μ_i is unramified: τ_ℓ is an unramified principal series. Therefore, by (3.3),

$$\frac{L(1-s, \pi_\ell^\vee)}{L(s, \pi_\ell)} = \gamma(s, \pi_\ell) = \gamma(s, \tau_\ell) = \frac{L(1-s, \tau_\ell^\vee)}{L(s, \tau_\ell)}$$

(for the first equality, see part 3 of [Lapid and Rallis 2005, Thm. 4]). As τ_ℓ is tempered, the zeros of the right-hand side are those of $L(s, \tau_\ell)^{-1}$, while those of the left-hand side are *a priori* a subset of those of $L(s, \pi_\ell)^{-1}$. This means that $L(s, \tau_\ell)/L(s, \pi_\ell)$ is holomorphic. But each of $L(s, \tau_\ell)^{-1}$ and $L(s, \pi_\ell)^{-1}$ is a polynomial of degree n in ℓ^{-2s} with constant term 1, and so they must be equal. That is, $L(s, \pi_\ell) = L(s, \tau_\ell)$. Since an unramified ω_ℓ equals $|\cdot|_\ell^t$ for some $t \in \mathbb{C}$, it follows that $L(s, \pi_\ell \otimes \omega_\ell) = L(s+t, \pi_\ell) = L(s+t, \tau_\ell) = L(s, \tau_\ell \otimes \omega_\ell)$, which implies — by Lemma 2 — that $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$.

Suppose that $\ell = v\bar{v}$ splits in K. Viewing $\mathbb{Q}_\ell$ as a K-algebra via the embedding that induces v, $G_0(\mathbb{Q}_\ell)$ is identified with $\mathrm{GL}_n(K_v) = \mathrm{GL}_n(\mathbb{Q}_\ell)$ and π_ℓ with a representation π_v of $\mathrm{GL}_n(\mathbb{Q}_\ell)$. Let $\pi_{\bar{v}} = \pi_v^\vee$. Then

$$\begin{aligned}\gamma(s, \pi_v \times \omega_v)\gamma(s, \pi_{\bar{v}} \times \omega_{\bar{v}}) &= \gamma(s, \pi_\ell \times \omega_\ell) \\ &= \gamma(s, \tau_\ell \times \omega_\ell) = \gamma(s, \tau_v \times \omega_v)\gamma(s, \tau_{\bar{v}} \times \omega_{\bar{v}}).\end{aligned}$$

The first equality follows from part 8 of [Lapid and Rallis 2005, Theorem 4]. By choosing ω_ℓ so that $\omega_{\bar{v}}$ is sufficiently ramified but ω_v is unramified, $\gamma(s, \pi_{\bar{v}} \times \omega_{\bar{v}})$ and $\gamma(s, \tau_{\bar{v}} \times \omega_{\bar{v}})$ can be assumed to be holomorphic with no zeros. Arguing as in the nonsplit case then yields that τ_v is unramified and $L(s, \tau_v) = L(s, \pi_v)$ (recall that τ_v and $\tau_{\bar{v}}$ are assumed to be principal series and tempered). Reversing the role of ω_v and $\omega_{\bar{v}}$ then yields that $\tau_{\bar{v}}$ is unramified and $L(s, \tau_{\bar{v}}) = L(s, \pi_{\bar{v}})$. As $L(s, \pi_\ell) = L(s, \pi_v)L(s, \pi_{\bar{v}})$, it follows that $L(s, \pi_\ell \otimes \omega_\ell) = L(s, \tau_\ell \otimes \omega_\ell)$ for all unramified ω_ℓ, which — by Lemma 2 again — implies that $\psi_{\tau_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$. This completes the proof of Proposition 1.

4. σ and ρ_π

In this section, k is arbitrary.

4.1. ***Algebraic representations and discrete series for $G(\mathbb{R})$.*** Let $T \subset G$ be the subgroup of diagonal elements. Then $T_{/K}$ is identified with the diagonal subgroup

$$\mathbb{G}_m^{1+n} = \mathbb{G}_m^{1+n_1+\cdots+n_k} \subset \mathbb{G}_m \times \prod_{i=1}^{k} \mathrm{GL}_{n_i},$$

and the character group $X(T)$ is identified with $\mathbb{Z}^{1+n}$: to $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k) \in \mathbb{Z}^{1+n}$, $\underline{c}_i \in \mathbb{Z}^{n_i}$, corresponds the character

$$(t_0, (\mathrm{diag}(t_{i,1}, \ldots, t_{i,n_i})) \mapsto t_0^{c_0} \prod_{i=1}^{n} \prod_{j=1}^{n_i} t_{i,j}^{c_{i,j}}.$$

We take the dominant characters to be those that are dominant with respect to the upper-triangular Borel B; this is equivalent to $c_{i,1} \geq c_{i,2} \geq \cdots \geq c_{i,n_i}$. Regular dominant characters are those where the inequalities are strict. The (regular) irreducible algebraic representations of $G_{/K}$ are indexed by the (regular) dominant characters in $X(T)$: to the representation ξ corresponds its highest weight with respect to the pair (T, B).

The L-packets of discrete series representations of $G(\mathbb{R})$ are indexed by equivalence classes of elliptic Langlands parameters $\psi : W_{\mathbb{R}} \to {}^L G$. The restriction to $W_{\mathbb{C}} = \mathbb{C}^\times$ of such a ψ is equivalent to a representation of the form

$$z \mapsto ((z/\bar{z})^{p_0}, (\mathrm{diag}((z/\bar{z})^{p_{i,1}}, \ldots, (z/\bar{z})^{p_{i,r_i}}))) \rtimes z$$

with $p_0 \in \mathbb{Z}$ and $p_{i,j} \in (n_i - 1)/2 + \mathbb{Z}$; the ordering can be chosen so that $p_{i,1} > \cdots > p_{i,r_i}$. Let $c_{i,j} := p_{i,j} - (n_i - 2i + 1)/2$. Then $c_{i,1} \geq \cdots \geq c_{i,r_i}$, and $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k)$, $c_0 := p_0$ and $\underline{c}_i := (c_{i,1}, \ldots, c_{i,r_i})$, is a dominant character of $X(T)$ and so corresponds to an irreducible algebraic representation ξ of $G_{/K}$ of highest weight $\underline{c}$. This gives a parametrization of the discrete series L-packets by the irreducible algebraic representations of $G_{/K}$; we denote the L-packet indexed by ξ by $\Pi_d(\xi)$. By a regular discrete series we will mean one belonging to an L-packet $\Pi_d(\xi)$ with ξ having regular highest weight.

4.2. $\boldsymbol{\sigma}$**.** Suppose $a_i b_i \neq 0$ for all i. Let π be a cuspidal automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ with $\pi_\infty \in \Pi_d(\xi)$ for some regular algebraic representation ξ of $G_{/K}$. Let χ_π be the character of the scalar torus $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \subset G$ determined by π (an algebraic Hecke character of $\mathbb{A}_K^\times$). Let $\Sigma(\pi)$ be the finite set comprising the primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Let $\underline{c} \in X(T)$ be the (regular) highest weight of ξ. Put $i(\underline{c}) := (c_0', -\underline{c}_1', \dots, -\underline{c}_k')$, where if $\underline{c}_i = (c_{i,1}, \dots, c_{i,n_i})$ then $\underline{c}_i' := (c_{i,n_i}, \dots, c_{i,1})$ and $c_0' := c_0 + \sum_{i=1}^k \sum_{j=1}^{n_i} c_{i,j}$. Then $i(\underline{c})$ is also a regular dominant character in $X(T)$.

The weight of an irreducible algebraic representation of $G_{/K}$ is the integer m such that the action of the central torus $\mathbb{G}_m \subset G$ is given by $x \mapsto x^m$; the weight of the representation ξ with highest weight $\underline{c} \in X(T)$ is $c_0 + c_0'$.

It follows from the proofs of Corollary 8.5.3 and Lemma 8.5.6 in [Morel 2010] — see especially the top paragraph on page 156 there — that there exist partitions $n_i = m_{i,1} + \cdots + m_{i,r_i}$ with each $m_{i,j} > 0$, irreducible automorphic representations $\tau_{i,j}$ of $\mathrm{GL}_{m_{i,j}}(\mathbb{A}_K)$, and a finite set of primes $S \supset \Sigma(\pi)$ satisfying the following conditions:

- $\tau_{i,j}$ is discrete.
- $\tau_{i,j}^c = \tau_{i,j}^\vee$.
- For $\ell \notin S$ and $v | \ell$, each $\tau_{i,j,v}$ is unramified.
- Let $\ell \notin S$, $v | \ell$, and let $\tau_{i,v}$ be the unramified irreducible subquotient of $\mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}}\big(\bigotimes_j \tau_{i,j,v}\big)$ and σ_ℓ the irreducible representation of $H(\mathbb{Q}_\ell)$ defined by the tuple $\big(\bigotimes_{v|\ell} \chi_\pi^c, \big(\bigotimes_{v|\ell} \tau_{i,v}\big)\big)$. If ψ_{π_ℓ} is the Langlands parameter of π_ℓ, then $\mathrm{BC} \circ \psi_\ell$ is the Langlands parameter of σ_ℓ.
- The infinitesimal character of $\tau_i := \mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}}\big(\bigotimes_j \tau_{i,j}\big)$ is the same as that of the absolutely irreducible algebraic character of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$ of highest weight $(\underline{c}_i, -\underline{c}_i')$; $\chi_\pi^c(z) = z^{c_0}\bar{z}^{c_0'}$.

Here, $P_i \subset \mathrm{GL}_{n_i}$ is the standard parabolic associated with the partition $n_i = m_{i,1} + \cdots + m_{i,r_i}$.

Recall that the infinitesimal character of an admissible representation of $\mathrm{GL}_m(\mathbb{C})$ is an element of $\mathfrak{a}_{m,\mathbb{C}}^\vee$ modulo the action of the Weyl group $W(\mathfrak{gl}_{m,\mathbb{C}}, \mathfrak{a}_{m,\mathbb{C}})$, where $\mathfrak{gl}_m := \mathrm{Lie}(\mathrm{GL}_m(\mathbb{C}))$ and $\mathfrak{a}_m := \mathrm{Lie}(A_m(\mathbb{C}))$ with $A_m := \mathbb{G}_m^m \subset \mathrm{GL}_m$ the diagonal torus. Identifying $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C} \times \mathbb{C}$ via $z \otimes w \mapsto (zw, \bar{z}w)$ and $\mathbb{C} = \mathrm{Lie}(\mathbb{C}^\times)$ (in the usual way, so the exponential map is $z \mapsto e^z$) identifies $\mathfrak{a}_{m,\mathbb{C}}$ with $\mathbb{C}^m \times \mathbb{C}^m$, and hence $\mathfrak{a}_{m,\mathbb{C}}^\vee := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{a}_{m,\mathbb{C}}, \mathbb{C}) = \mathbb{C}^m \times \mathbb{C}^m$ (using the dual basis); $W(\mathfrak{gl}_{m,\mathbb{C}}), \mathfrak{a}_{m,\mathbb{C}}^\vee)$ is then identified with $\mathfrak{S}_m \times \mathfrak{S}_m$. An absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ corresponds to its highest weight with respect to

$$(\mathrm{Res}_{K/\mathbb{Q}}A_m, \mathrm{Res}_{K/\mathbb{Q}}B_m),$$

$B_m \subset \mathrm{GL}_m$ being the upper-triangular Borel; this is an element of

$$X(\mathrm{Res}_{K/\mathbb{Q}}A_m) = X(A_m) \times X(A_m)$$

(the identification being via $\mathrm{Res}_{K/\mathbb{Q}}A_{m/K} = A_m \times A_m$) given by a pair of dominant characters of $X(A_m) = \mathbb{Z}^m$ (the last identification is the usual one: $\underline{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m$ corresponds to the character $\mathrm{diag}(t_1, \dots, t_m) \mapsto t_1^{c_1} \cdots t_m^{c_m}$; dominant characters satisfy $c_1 \geq \cdots \geq c_m$, and regular dominant characters are those where the inequalities are strict). The infinitesimal character of the irreducible representation of highest weight $(\underline{c}_1, \underline{c}_2)$ is $(\underline{c}_1, \underline{c}_2) + \rho_{\mathrm{GL}_m} \in \mathfrak{a}^\vee_{m,\mathbb{C}}$, where $\rho_{\mathrm{GL}_m} := ((m-1)/2, (m-3)/2, \dots, (3-m)/2, (1-m)/2)$ is half the sum of the usual positive roots in $\mathfrak{gl}_m$.

As ξ is regular, if the weight of ξ is zero (that is, $c_0 + c_0' = 0$) then by [Morel 2010, Theorem 7.3.1], the Satake parameters of π_ℓ, $\ell \notin S$, all have absolute value 1. The same is then true of the Satake parameters of $\tau_{i,j,v}$ for any $v|\ell$ as $\psi_{\sigma_\ell} = \mathrm{BC} \circ \psi_{\pi_\ell}$. For ξ having general weight $m \in \mathbb{Z}$, let π' and ξ' be the twists of π and ξ, respectively, by the character $\mu(\cdot)^{-m}$; then ξ' is regular of weight 0 and $\pi'_\infty \in \Pi_d(\xi')$. The representations of the $\mathrm{GL}_{n_i}(\mathbb{A}_K)$ associated to π' as above are the same as those associated to π: this can be seen by the relation between Langlands parameters at $\ell \notin S$. The case of general weight then follows immediately from that of weight zero. Therefore, we also have that

- for $\ell \notin S$, $v|\ell$, the Satake parameters of $\tau_{i,j,v}$ all have absolute value 1 - $\tau_{i,j,v}$ is tempered; furthermore, $\tau_{i,v} = \mathrm{Ind}_{P_i}^{\mathrm{GL}_{n_i}}\left(\bigotimes_j \tau_{i,j,v}\right)$ and is a tempered principal series.

Lemma 3. *Each $\tau_{i,j}$ is cuspidal, and $\sigma_{i,j} := \tau_{i,j} \otimes |\cdot|^{(m_{i,j}-n_i)/2}$ is algebraic and has the same infinitesimal character as a regular absolutely irreducible algebraic representation $\xi_{i,j}$ of* $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$.

Here $\sigma_{i,j}$ being algebraic automorphic representation of $\mathrm{GL}_{m_{i,j}}(\mathbb{A}_K)$ is as in [Clozel 1990, 1.2.3]: the infinitesimal character $\underline{b}_{i,j} \in \mathfrak{a}^\vee_{m_{i,j},\mathbb{C}} = \mathbb{C}^{m_{i,j}} \times \mathbb{C}^{m_{i,j}}$ of $\sigma_{i,\infty}$ satisfies $\underline{b}_{i,j} + (1 - m_{i,j})/2 \in \mathbb{Z}^{m_{i,j}} \times \mathbb{Z}^{m_{i,j}}$.

Proof. As $\tau_{i,j}$ is discrete, by the main results of [Mœglin and Waldspurger 1989] there is a factorization $m_{i,j} = s_{i,j}r_{i,j}$ and an irreducible cuspidal automorphic representation $\alpha_{i,j}$ of $\mathrm{GL}_{s_{i,j}}(\mathbb{A}_K)$ such that $\tau_{i,j}$ is the unique irreducible quotient of

$$\mathrm{Ind}_{P_{i,j}}^{\mathrm{GL}_{m_{i,j}}} \beta_{i,j} \quad \beta_{i,j} = (\alpha_{i,j} \otimes |\cdot|^{(1-r_{i,j})/2}) \otimes \cdots \otimes (\alpha_{i,j} \otimes |\cdot|^{(r_{i,j}-1)/2}),$$

where $P_{i,j} \subset \mathrm{GL}_{m_{i,j}}$ is the standard parabolic associated with the partition $m_{i,j} = s_{i,j} + \cdots + s_{i,j}$ ($r_{i,j}$ summands). Since for all but finitely many v the Satake parameters of $\tau_{i,j,v}$ all have the same absolute value, it must then be that $r_{i,j} = 1$, and so $\tau_{i,j} = \alpha_{i,j}$ is cuspidal.[1]

Let $\underline{a}_{i,j} \in \mathfrak{a}^\vee_{m_{i,j},\mathbb{C}}$ be the infinitesimal character of $\tau_{i,j,\infty}$. Then the infinitesimal character of $\tau_{i,\infty}$ is $\underline{a}_i := (\underline{a}_{i,1}, \ldots, \underline{a}_{i,r_i}) \in \mathfrak{a}^\vee_{n_i,\mathbb{C}}$. In particular, there exist $L', L'' \subset \{1, \ldots, n_i\}$ of cardinality $m = m_{i,j}$ such that $\underline{a} = \underline{a}_{i,j} = (\underline{a}', \underline{a}'') \in \mathbb{C}^m \times \mathbb{C}^m$ with $\underline{a}'$ and $\underline{a}''$ equal to $(c_{i,\ell} + (n_i - 2\ell + 1)/2)_{\ell \in L'}$ and $(-c_{i,\ell} + (2\ell - n_i + 1)/2)_{\ell \in L''}$, respectively. Suppose $L' = \{\ell'_1, \ldots, \ell'_m\}$ with $\ell'_1 < \ell'_2 < \cdots < \ell'_m$ and $L'' = \{\ell''_1, \ldots, \ell''_m\}$ with $\ell''_1 > \ell''_2 > \cdots > \ell''_m$. Then the infinitesimal character $\underline{b} = \underline{b}_{i,j}$ of $\sigma_{i,j}$ is given by $\underline{b} = \underline{a} + (m - n_i)/2 = (\underline{d}', \underline{d}'') + \rho_{\mathrm{GL}_m}$, where

$$\underline{d}' = (c_{i,\ell'_k} + k - \ell'_k)_{1 \le k \le m} \quad \text{and} \quad \underline{d}'' = (-c_{i,\ell''_k} + \ell''_k - n_i + k)_{1 \le k \le m}.$$

As $\rho_{\mathrm{GL}_m} + (1 - m)/2 \in \mathbb{Z}^m$, it follows that $\underline{b} + (1 - m)/2 \in \mathbb{Z}^m \times \mathbb{Z}^m$, so $\sigma_{i,j}$ is algebraic. Also,

$$c_{i,\ell'_k} + k - \ell'_k - c_{i,\ell'_{k+1}} - k - 1 + \ell'_{k+1} = c_{i,\ell'_k} - c_{i,\ell'_{k+1}} - 1 + \ell'_{k+1} - \ell'_k \ge 1$$
$$-c_{i,\ell''_k} + \ell''_k - n_i + k + c_{i,\ell''_{k+1}} - \ell''_{k+1} + n_i - k - 1 = c_{i,\ell''_{k+1}} - c_{i,\ell''_k} + \ell''_k - \ell''_{k+1} - 1 \ge 1,$$

so $\underline{d}'$ and $\underline{d}''$ are both regular and dominant. Therefore,

$$\underline{d} := (\underline{d}', \underline{d}'') \in X(A_m) \times X(A_m)$$

corresponds to a regular absolutely irreducible algebraic representation $\xi_{i,j}$ of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ with infinitesimal character $\underline{d} + \rho_{\mathrm{GL}_m} = \underline{b}$. □

Corollary 4. *The cuspidal representations $\tau_{i,j}$ are tempered at all finite places. Furthermore, each τ_i is irreducible and tempered at all finite places.*

Proof. Choose an algebraic Hecke character χ of $\mathbb{A}_K^\times$ such that $\chi\chi^c = |\cdot|^{n_i - m_{i,j}}$. Then $\sigma_{i,j} \otimes \chi$ is a conjugate self-dual algebraic cuspidal representation with infinitesimal character that of a regular absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_{m_{i,j}}$. Therefore, $\sigma_{i,j} \otimes \chi$ is tempered at all finite places by [Shin 2011, Corollary 1.3]. The claims about $\tau_{i,j}$ and τ_i follow easily from this. □

Put

$$\psi := \chi_\pi^c \quad \text{and} \quad \sigma := (\psi, (\tau_i)). \tag{4.4}$$

Then σ is identified with an irreducible automorphic representation of $H(\mathbb{A}_\mathbb{Q})$. This is a very weak base change of π in the sense that the Langlands parameter ψ_{σ_ℓ} of σ_ℓ is $\mathrm{BC} \circ \psi_{\pi_\ell}$ for all $\ell \notin S$, ψ_{π_ℓ} being the Langlands parameter of π_ℓ.

[1] This can also be seen by considering infinitesimal characters.

Remark 5. Suppose $k = 1$. Let π_0 be an irreducible automorphic constituent of the restriction of π to $G_0(\mathbb{A}_{\mathbb{Q}})$. Then $\tau = \tau_1$ is a very weak base change of π_0 to $H_0(\mathbb{A}_{\mathbb{Q}})$ that is tempered at all finite places. By Proposition 1, to complete the proof of Theorem A it suffices to show that τ_v is a principal series for all $v|\ell$, $\ell \notin \Sigma(\pi)$. This is done in the following by analyzing certain Galois representations associated with τ.

4.3. $\boldsymbol{\rho_\pi}$. Let $\rho : G_K \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ be a continuous representation. Let ξ be an absolutely irreducible algebraic representation of $\mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_m$ with highest weight $(\underline{c}_1, \underline{c}_2) \in X(A_m) \times X(A_m) = \mathbb{Z}^m \times \mathbb{Z}^m$. Let $v|p$ be a place of K. Recall that $\rho_v := \rho|_{G_{\mathbb{Q}_v}}$ being Hodge–Tate means that the graded $(\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v)$-module $D_{\mathrm{HT},v}(\rho_v) := (\rho_v \otimes B_{\mathrm{HT},v})^{G_{K_v}}$, $B_{\mathrm{HT},v} := \bigoplus_{t\in\mathbb{Z}} \widehat{\overline{K}}_v(t)$, is a free $(\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v)$-module of rank m. By ρ_v being of Hodge–Tate type ξ we mean that for any $j \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$, the graded $\overline{\mathbb{Q}}_p$-module $D_{\mathrm{HT}}(\rho_v) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, j} \overline{\mathbb{Q}}_p$ is nonzero in degrees $i - 1 - c_{1,i}$, $i = 1, .., m$, if the restriction of j to K is the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and otherwise is nonzero in degrees $i - 1 - c_{2,i}$, $i = 1, \ldots, m$.

Let $\sigma_{i,j}$ be as in Lemma 3. From [Shin 2011] we conclude that there exist representations $\rho_{i,j} = \rho_{\sigma_{i,j},\iota} : G_K \to \mathrm{GL}_{m_{i,j}}(\overline{\mathbb{Q}}_p)$ such that

- $\rho_{i,j}$ is continuous and semisimple,
- for $v \nmid p$, $\mathrm{WD}(\rho_{i,j}|_{G_{K_v}})^{\mathrm{Fr-ss}} = \iota\mathrm{Rec}_v\big(\sigma_{i,j,v} \otimes |\cdot|_v^{(1-m_{i,j})/2}\big)$,
- $\rho^c_{i,j} \cong \rho^\vee_{i,j} \otimes \epsilon^{1-n_i}$,
- for each $v|p$, $\rho_{i,j}|_{G_{K_v}}$ is potentially semistable of Hodge–Tate type $\xi_{i,j}$,
- for $v|p$, if $\sigma_{i,j,v}$ is unramified then $\rho_{i,j}|_{G_{K_v}}$ is crystalline and the eigenvalues of the $[K_v : \mathbb{Q}_p]$-th power of the crystalline Frobenius on
$$D_{\mathrm{cris}}(\rho_{i,j}|_{G_{K_v}}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v^0, \lambda} \overline{\mathbb{Q}}_p, \quad \text{any } \lambda \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v^0, \overline{\mathbb{Q}}_p),$$
are the Frobenius eigenvalues of $\iota\mathrm{Rec}_v\big(\sigma_{i,j,v} \otimes |\cdot|_v^{(1-m_{i,j})/2)}\big)$, where $K_v^0 \subset K_v$ is the maximal absolutely unramified extension.

Here $\mathrm{WD}(\rho_{i,j}|_{G_{K_v}})^{\mathrm{Fr-ss}}$ is the Frobenius semisimple Weil–Deligne representation associated to the $\rho_{i,j}|_{G_{K_v}}$.

The existence of $\rho_{i,j}$ follows from [Shin 2011, Theorem 1.2]: As in the proof of Corollary 4, choose an algebraic Hecke character χ of $\mathbb{A}_K$ such that $\sigma_{i,j} \otimes \chi$ is conjugate self-dual; such a character can be chosen to be unramified at any given finite set of finite places. Then [ibid., Theorem 1.2] applies to $\sigma_{i,j} \otimes \chi$ and we set $\rho_{i,j} := R_{p,\iota}(\sigma^\vee_{i,j} \otimes \chi^{-1}) \otimes \rho^\vee_{\chi,\iota}$ in Shin's notation (the contragredients are here because of the normalization of the local Langlands correspondence in [Shin 2011]). By varying the set of primes at which χ is unramified we obtain the compatibility with the local Langlands correspondence at all $v \nmid p$. A comparison between the eigenvalues of the $[K_v:\mathbb{Q}_p]$th-power of the crystalline Frobenius eigenvalues and the Frobenius eigenvalues of the Weil–Deligne representation is not stated explicitly in [ibid.] but can be obtained by appealing to the comparison theorem in [Katz and Messing 1974]: the arguments in [Shin 2011, §7] and especially [Taylor and Yoshida 2007, §2] explain that there is a solvable CM-extension L/K in which all places of K above p split and such that $\mathrm{BC}_{L/K}(\sigma_{i,j}^\vee\otimes\chi^{-1})$ is cuspidal and an algebraic Hecke character ψ of $\mathbb{A}_L^\times$ unramified at all primes above p such that some multiple of the p-adic G_L-representation $R_{p,\iota}(\mathrm{BC}_{L/K}(\sigma_{i,j}^\vee\otimes\chi^{-1}))\otimes\rho_{\psi,\iota}$ is cut out by correspondences acting on the cohomology with constant coefficients of a self-product of the universal abelian variety over a compact Shimura variety (with good reduction at v if $\sigma_{i,j,v}\otimes\chi_v$ is unramified). Here $\mathrm{BC}_{L/K}(\cdot)$ denotes the base change lift to $\mathrm{GL}_n(\mathbb{A}_L)$.

Put

$$\rho_i := \textstyle\bigoplus_{j=1}^{r_i}\rho_{i,j},\qquad i=1,\dots,k,$$

and

$$\rho_\pi := \rho_\psi\otimes\big(\textstyle\bigoplus_{i=1}^{k}\rho_i\big). \tag{4.5}$$

Remark 6. Suppose $k=1$. Then ρ_π satisfies the conclusions of Theorem B, but with S replacing $\Sigma(\pi)$ and with the additional condition that $p\notin S$ for part (d); the definition of ρ_π being of "Hodge–Tate type ξ" is given after Theorem 10 below.

Proposition 7. *For $v|\ell$, $\ell\notin\Sigma(\pi)$, the representations $\tau_{i,j,v}$ and $\tau_{i,v}$ are tempered principal series.*

Our proof of this proposition will come from an understanding of the ramification at $v|\ell$, $\ell\notin\Sigma_p(\pi)$, of the representation

$$r_\pi := \rho_\psi\otimes\bigotimes_{i=1}^{k}\textstyle\bigwedge^{a_i}\rho_i.$$

First, we explain what it means for π to be an endoscopic lift. This means that each n_i has a partition $n_i=n_i^+ + n_i^-$ as a sum of nonnegative integers with some $n_j^+n_j^-\neq 0$ and such that $\sum_{i=1}^k n_i^-$ is even, and that there is a cuspidal automorphic representation γ of $G'(\mathbb{A}_\mathbb{Q})$, with

$$G' := G(U(a_1^+,b_1^+)\times U(a_1^-,b_1^-)\times\cdots\times U(a_k^+,b_k^+)\times U(a_k^-,b_k^-))$$

and

$$(a_i^\pm,b_i^\pm)=\Big(\Big\lfloor\frac{n_i^\pm+1}{2}\Big\rfloor,\Big\lceil\frac{n_i^\pm-1}{2}\Big\rceil\Big),$$

such that γ is unramified at each prime $\ell\notin\Sigma(\pi)$, and for each $\ell\notin\Sigma(\pi)$ the Langlands parameter ψ_{π_ℓ} of π_ℓ is the composition of the Langlands parameter ψ_{γ_ℓ}

of γ_ℓ with the endoscopic L-group homomorphism

$$\mathrm{End} : {}^L G' \to {}^L G,$$

is defined as follows (see also [Morel 2010, Proposition 2.3.2]. Let $\epsilon_{K/\mathbb{Q}}:W_{\mathbb{Q}}\to\{\pm 1\}$ be the nontrivial quadratic character factoring through $\mathrm{Gal}(K/\mathbb{Q})$; by class field theory this determines a quadratic character $\omega_{K/\mathbb{Q}} : \mathbb{A}^\times/\mathbb{Q}^\times \to \{\pm 1\}$. Fix a finite-order Hecke character ω_K of $\mathbb{A}_K^\times$ such that $\omega_K|_{\mathbb{A}^\times} = \omega_{K/\mathbb{Q}}$ and let $\mu : W_K \to \mathbb{C}^\times$ be the character corresponding via class field theory. Let $c \in W_{\mathbb{Q}}$ be a lift of the nontrivial automorphism of K. Define $\varphi : W_{\mathbb{Q}} \to {}^L G$ by

$$\varphi(c) = \left(1, \left(\begin{pmatrix} \Phi_{n_i^+} & \\ & (-1)^{n_i^+}\Phi_{n_i^-} \end{pmatrix} \Phi_{n_i}^{-1}\right)\right) \rtimes c,$$

$$\varphi(w) = \left(1, \left(\begin{pmatrix} \mu^{n_i^-}(w) I_{n_i^+} & \\ & \mu^{-n_i^+}(w) I_{n_i^-} \end{pmatrix}\right)\right) \rtimes w, \qquad w \in W_K.$$

The endoscopic map is then

$$\mathrm{End}((\lambda, (g_i^+, g_i^-)) \rtimes w) = \left(\lambda, \begin{pmatrix} g_i^+ & \\ & g_i^- \end{pmatrix}\right) \varphi(w).$$

Here $((\lambda, (g_i^+, g_i^-)) \in \widehat{G}' = \mathbb{C}^\times \times \prod_{i=1}^k \mathrm{GL}_{n_i^+}(\mathbb{C}) \times \mathrm{GL}_{n_i^-}(\mathbb{C})$.

Lemma 8. *Either π is an endoscopic lift of some γ with γ_∞ a regular discrete series or the representation r_π is unramified at all $v|\ell$, $\ell \notin \Sigma_p(\pi)$.*

Proof. By [Morel 2010, Theorem 7.2.2] (see also the proof of [ibid., Theorem 7.3.1]), either π is an endoscopic lift of some γ with γ_∞ a regular discrete series indexed by a representation with the same weight as ξ (see [ibid., Lem. 7.3.4]) or (some multiple of) $r_\pi^\vee$ occurs[2] in the middle degree intersection cohomology of a Shimura variety associated with G, ξ, and π. By [Lan 2008], this Shimura variety is known to have good reduction at all $v|\ell$, $\ell \notin \Sigma_p(\pi)$, so the representation r_π is unramified at such v. □

Proof of Proposition 7. Let $v|\ell$, $\ell \notin \Sigma(\pi)$. Suppose π is the endoscopic lift of some γ with γ_∞ a regular discrete series. Let $\sigma' = (\psi', (\tau_i^+, \tau_i^-))$ be the very weak base change of γ as in Section 4.2 (so $\tau_i^\pm$ is an irreducible automorphic representation of $\mathrm{GL}_{n_i^\pm}(\mathbb{A}_K)$). From the definition of π being an endoscopic lift of γ, it follows that

$$\tau_i = (\tau_i^+ \otimes \omega_K^{-n_i^-}) \boxplus (\tau_i^- \otimes \omega_K^{n_i^+})$$

[2]Theorem 7.2.2 of [Morel 2010] only applies to the case $k = 1$ as stated, but it is asserted at the start of [ibid., 7.2] that the results and proofs “would work the same way” for general k. Indeed, the result for the case $k > 1$ is stated and used in the proof of [ibid., Theorem 7.3.1].

(as $\tau_i^{\pm}$ is tempered by Corollary 4). We may therefore reduce to the case where π is not endoscopic, and hence, by Lemma 8, to the case where r_π is unramified at v.

Suppose that r_π is unramified at v. Consider the isogeny

$$G_1 := \mathrm{GL}_1 \times \prod_{i=1}^{k} \prod_{j=1}^{r_i} \mathrm{GL}_{m_{i,j}} \to G_2 := \mathrm{GL}_1 \otimes \bigotimes_{i=1}^{k} \mathrm{GL}_{\binom{n_i}{a_i}},$$

$$(\lambda, (g_{i,j})) \mapsto \lambda \otimes \bigotimes_{i=1}^{k} \bigwedge\nolimits^{a_i} (\mathrm{diag}(g_{i,1}, \ldots, g_{i,r_i})).$$

The kernel of this isogeny is central. As r_π is the composition of

$$\rho := \rho_\psi \oplus \bigoplus_{i=1}^{k} \rho_i : G_K \to G_1(\overline{\mathbb{Q}}_p)$$

with this isogeny, it then follows that since r_π is unramified at v, the image of inertia at v under ρ is contained in the center of $G_1(\overline{\mathbb{Q}}_p)$, and so the image of inertia at v under each $\rho_{i,j}$ is central. So some finite-order twist of each $\rho_{i,j}$ is unramified at v, which — by compatibility of $\rho_{i,j}$ with the local Langlands correspondence — implies that a finite-order twist of each $\sigma_{i,j,v}$, and hence of each $\tau_{i,j,v}$, is unramified. By Corollary 4, $\tau_{i,j,v}$ is also tempered. It follows that each $\tau_{i,j,v}$ is a tempered principal series, so each $\tau_{i,v}$ must also be a tempered principal series. □

4.4. ***The main results.*** We can now state our main results, of which Theorems A and B are special cases, and complete their proofs.

Theorem 9. *Let π be an irreducible cuspidal representation of $G(\mathbb{A}_\mathbb{Q})$ and let χ_π be the character of the scalar torus* $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \subset G$ *determined by π (a character of $\mathbb{A}_K^\times$). Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Suppose $a_i b_i \neq 0$, $i = 1, \ldots, k$, and π_∞ is a regular discrete series belonging to an L-packet $\Pi_d(\xi)$. There exists an automorphic representation $\sigma = (\psi, (\tau_i))$ of $H(\mathbb{A}_\mathbb{Q})$ such that*:

(a) $\sigma^\theta \cong \sigma$; $\psi = \chi_\pi^c$.

(b) *For a prime $\ell \notin \Sigma(\pi)$, σ_ℓ is unramified, and if $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G$ is the Langlands parameter of π_ℓ then*

$$\psi_{\sigma_\ell} := \mathrm{BC} \circ \psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L H$$

is the Langlands parameter of σ_ℓ.

(c) *σ_∞ has the same infinitesimal character as $\xi \otimes \xi^\theta$.*

Proof. Let $\sigma = (\psi, (\tau_i))$ be as in (4.4). Then part (a) holds. Furthermore, there exists a finite set of primes $S \supset \Sigma(\pi)$ such that part (b) holds with S replacing $\Sigma(\pi)$.

Let $\pi_0 \subset \pi$ be an irreducible automorphic representation of $G_0(\mathbb{A}_\mathbb{Q})$. Then π_0 is given by a tuple $(\pi_{0,i})_{1 \le i \le k}$ with $\pi_{0,i}$ an automorphic representation of $U(a_i, b_i)$.

For $\ell \notin \Sigma(\pi)$, each $\pi_{0,i,\ell}$ is unramified and the Langlands parameter $\psi_{\pi_{0,i,\ell}}$ of $\pi_{0,i,\ell}$ is given by composing $\psi_{\pi_\ell} : W_{\mathbb{Q}_\ell} \to {}^L G$ with the projection

$$ {}^L G \to {}^L G_0 \to {}^L U(a_i, b_i) = \mathrm{GL}_{n_i}(\mathbb{C}) \rtimes W_{\mathbb{Q}}. $$

From part (b) holding for $\ell \notin S$ it then follows that for such ℓ the Langlands parameter of $\tau_{i,\ell}$ is $\mathrm{BC} \circ \psi_{\pi_{0,i,\ell}}$; that is, τ_i is a very weak base change of $\pi_{0,i}$. But by Proposition 7, $\tau_{i,v}$ is a tempered principal series for each $v|\ell$, $\ell \notin \Sigma(\pi)$, so it follows from Proposition 1 that τ_i is a weak base change of $\pi_{0,i}$. That (b) holds is then immediate from the relation between the Langlands parameters of π_ℓ and of the $\pi_{0,i,\ell}$.

To see that part (c) holds, we first recall that the infinitesimal character of an admissible representation of $H(\mathbb{R})$ is an element of $\mathfrak{s}_{\mathbb{C}}^\vee$ up to action of the Weyl group $W(\mathfrak{h}_{\mathbb{C}}, \mathfrak{s}_{\mathbb{C}})$, where $\mathfrak{h} := \mathrm{Lie}(H(\mathbb{R}))$ and $\mathfrak{s} := \mathrm{Lie}(S(\mathbb{R}))$ with $S := \mathrm{Res}_{K/\mathbb{Q}} T_{/K} \subset G$ the group of diagonal matrices. Then $S_{/K} = T_{/K} \times T_{/K}$ and $X(S) = X(T) \times X(T)$. The irreducible algebraic representations of $H_{/K}$ correspond to pairs of dominant characters of $X(T)$ — the highest weight of the representation with respect to S and the upper-triangular Borel. In particular, the representation $\xi \otimes \xi^\theta$ corresponds to $(\underline{c}, i(\underline{c}))$ and has infinitesimal character $(\underline{c}, i(\underline{c})) + \rho_H$, where $\rho_H := (0, (\rho_{\mathrm{GL}_{n_i}}))$. On the other hand, $S(\mathbb{R}) = \mathbb{C}^\times \times \prod_{i=1}^k A_{n_i}$ so

$$ \mathfrak{s}_{\mathbb{C}}^\vee = \mathbb{C}^2 \oplus \mathfrak{a}_{n_1,\mathbb{C}}^\vee \oplus \cdots \oplus \mathfrak{a}_{n_j,\mathbb{C}}^\vee = \mathbb{C}^{1+n} \times \mathbb{C}^{1+n}, $$

and the infinitesimal character of σ_∞ is $(c_0, c_0') \bigoplus_{i=1}^k$(infinitesimal character of τ_i). Since the infinitesimal character of τ_i is $(\underline{c}_i, -\underline{c}_i') + \rho_{\mathrm{GL}_{n_i}}$, the infinitesimal character of σ_∞ is $((c_0, (\underline{c}_i + \rho_{\mathrm{GL}_{n_i}})), (c_0', (-\underline{c}_i' + \rho_{\mathrm{GL}_{n_i}}))) = (\underline{c}, i(\underline{c})) + \rho_H$. $\square$

Theorem 10. *Let π be an irreducible cuspidal representation of $G(\mathbb{A}_{\mathbb{Q}})$ and let χ_π be the character of the scalar torus $\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m \subset G$ determined by π (a character of $\mathbb{A}_K^\times / K^\times$). Let $\Sigma(\pi)$ be the finite set of primes ℓ such that either π_ℓ is ramified or $\ell | d_K$. Suppose $a_i b_i \neq 0$, $i = 1, \ldots, k$, and π_∞ is a regular discrete series belonging to an L-packet $\Pi_d(\xi)$. Let $\sigma = (\psi, (\tau_i))$ be as in Theorem 9. There exists a continuous, semisimple representation*

$$ \rho_\pi = \rho_{\pi,\iota} : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p) $$

such that:

(1) *ρ_π is unramified at all finite places not above primes in $\Sigma_p(\pi) := \Sigma(\pi) \cup \{p\}$, and for such a place w*

$$ (\rho_\pi|_{W_{K_w}})^{ss} = \bigoplus_{i=1}^k \iota \mathrm{Rec}_w\big(\tau_{i,w} \otimes \psi_w |\cdot|_w^{(1-n_i)/2}\big). $$

(b) *For $v|p$, $\rho_\pi|_{G_{K_v}}$ is potentially semistable of Hodge–Tate-type ξ.*

(c) *If $p \notin \Sigma(\pi)$ then for any $v|p$, $\rho_\pi|_{G_{K_v}}$ is crystalline; for any*

$$j \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$$

the eigenvalues of the action of the $[K_v : \mathbb{Q}_p]$-th power of the crystalline Frobenius on

$$D_{\mathrm{cris}}(\rho_\pi|_{G_{K_v}}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, j} \overline{\mathbb{Q}}_p$$

are the eigenvalues of the action of Frobenius on

$$\bigoplus_{i=1}^{k} \iota\mathrm{Rec}_v\big(\tau_{i,v} \otimes \psi_v|\cdot|_v^{(1-n_i)/2}\big).$$

Let $\underline{c} = (c_0, \underline{c}_1, \ldots, \underline{c}_k) \in X(T)$ be the highest weight of ξ. By $\rho_\pi|_{G_{K_v}}$ being of Hodge–Tate type ξ, we mean that ρ_π is of Hodge–Tate type $(c_0 + \underline{c}, \underline{c}'_0 + \underline{c}')$.

Proof. If we take ρ_π to be as in (4.5), then (a) is immediate from Theorem 9(b) and the definition of ρ_π as being the twist by ρ_ψ of the sum of the $\rho_{i,j}$. From the proof of Lemma 3, the character $\xi_{i,j}$ has highest weights

$$(c_{i,\ell'_t} + t - \ell'_t, -c_{i,\ell''_t} + \ell''_t - n_i + t)_{1 \le t \le m_{i,j}},$$

and so for $v|p$,

$$D_{\mathrm{HT},v}(\rho_{i,j}) \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} K_v, \zeta} \overline{\mathbb{Q}}_p$$

is nonzero in degrees $\ell'_t - 1 - c_{i,\ell'_t}$ if $\zeta \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(K_v, \overline{\mathbb{Q}}_p)$ induces the fixed embedding $K \hookrightarrow \overline{\mathbb{Q}}_p \cong \mathbb{C}$, and otherwise is nonzero in degrees $n_i - \ell''_t - 1 + c_{i,\ell''_t}$. That $\rho_\pi|_{G_{K_v}}$ is of Hodge–Tate type ξ then follows from this and the fact that $\psi_\infty(z) = z^{c_0}\bar{z}^{c'_0}$ and so ρ_ψ is of Hodge–Tate type (c_0, c'_0). That $\rho_\pi|_{G_{K_v}}$, $v|p$, is potentially semistable and even crystalline with the prescribed Frobenius eigenvalues if $v|p$ follows from the corresponding facts for ρ_ψ and the $\rho_{i,j}$. □

Theorems A and B are just the special cases where $k = 1$.

References

[Brenner 2008] E. Brenner, "Stability of the local gamma factor in the unitary case", *J. Number Theory* **128**:5 (2008), 1358–1375. MR 2009a:22011 Zbl 1138.22011

[Clozel 1990] L. Clozel, "Motifs et formes automorphes: Applications du principe de fonctorialité", pp. 77–159 in *Automorphic forms, Shimura varieties, and L-functions* (Ann Arbor, MI, 1988), vol. 1, edited by L. Clozel and J. S. Milne, Perspectives in Mathematics **10**, Academic Press, Boston, MA, 1990. MR 91k:11042 Zbl 0705.11029

[Gelbart et al. 1987] S. Gelbart, I. Piatetski-Shapiro, and S. Rallis, *Explicit constructions of automorphic L-functions*, Lecture Notes in Mathematics **1254**, Springer, Berlin, 1987. MR 89k:11038 Zbl 0612.10022

[Harris and Labesse 2004] M.Harris and J.-P.Labesse, "Conditional base change for unitary groups", *Asian J. Math.* **8**:4 (2004), 653–683. MR 2006g:11098 Zbl 1071.22025

[Katz and Messing 1974] N. M. Katz and W. Messing, "Some consequences of the Riemann hypothesis for varieties over finite fields", *Invent. Math.* **23** (1974), 73–77. MR 48 #11117 Zbl 0275.14011

[Labesse 2011] J.-P. Labesse, "Changement de base CM et séries discrètes", pp. 429–470 in *On the stabilization of the trace formula*, edited by L. Clozel et al., International Press, Somerville, MA, 2011. MR 2856380

[Lan 2008] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, thesis, Harvard University, 2008, available at http://www.math.umn.edu/~kwlan/articles/cpt-PEL-type-thesis.pdf. MR 2711676

[Lapid and Rallis 2005] E. M. Lapid and S. Rallis, "On the local factors of representations of classical groups", pp. 309–359 in *Automorphic representations, L-functions and applications: Progress and prospects*, edited by J. W. Cogdell et al., Ohio State Univ. Math. Res. Inst. Publ. **11**, de Gruyter, Berlin, 2005. MR 2006j:11071 Zbl 1188.11023

[Li 1992] J.-S. Li, "Nonvanishing theorems for the cohomology of certain arithmetic quotients", *J. Reine Angew. Math.* **428** (1992), 177–217. MR 93e:11067 Zbl 0749.11032

[Mœglin and Waldspurger 1989] C. Mœglin and J.-L. Waldspurger, "Le spectre résiduel de GL(n)", *Ann. Sci. École Norm. Sup.* (4) **22**:4 (1989), 605–674. MR 91b:22028 Zbl 0696.10023

[Morel 2010] S. Morel, *On the cohomology of certain noncompact Shimura varieties*, Annals of Mathematics Studies **173**, Princeton University Press, 2010. MR 2011b:11073

[Shin 2011] S. W. Shin, "Galois representations arising from some compact Shimura varieties", *Ann. of Math.* (2) **173**:3 (2011), 1645–1741. MR 2800722 Zbl 05960691

[Skinner and Urban 2010] C. Skinner and E. Urban, "The Iwasawa main conjectures for GL_2", preprint, 2010, available at http://www.math.columbia.edu/~urban/eurp/MC.pdf.

[Taylor and Yoshida 2007] R. Taylor and T. Yoshida, "Compatibility of local and global Langlands correspondences", *J. Amer. Math. Soc.* **20**:2 (2007), 467–493. MR 2007k:11193 Zbl 1210.11118

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