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Two ways to degenerate the Jacobian are the same

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We provide sufficient conditions for the line bundle locus in a family of compact moduli spaces of pure sheaves to be isomorphic to the Néron model. The result applies to moduli spaces constructed by Eduardo Esteves and Carlos Simpson, extending results of Busonero, Caporaso, Melo, Oda, Seshadri, and Viviani.

1. Introduction

1.1. ***Background.*** This paper relates two different approaches to extending families of Jacobian varieties. Recall that if X_0 is a smooth projective curve of genus g, then the associated Jacobian variety is a g-dimensional smooth projective variety J_0 that can be described in two different ways: as the universal abelian variety that contains X_0 (the Albanese variety), and as the moduli space of degree 0 line bundles on X_0 (the Picard variety). If $X_U \to U$ is a family of smooth, projective curves, then the Jacobians of the fibers fit together to form a family $J_U \to U$. In this paper, U will be an open subset of a smooth curve B (or, more generally, a Dedekind scheme), and we will be interested in extending J_U to a family over B. Corresponding to the two different ways of describing the Jacobian (Albanese vs. Picard) are two different approaches to extending the family $J_U \to U$.

Viewing the Jacobian as the Albanese variety, it is natural to try to extend $J_U \to U$ by extending it to a family of group varieties over B. Néron [1964] showed that this can be done in a canonical way. He worked with an arbitrary family of abelian varieties $A_U \to U$ and proved that there is a unique B-smooth group scheme $\mathrm{N} := \mathrm{N}(A_U) \to B$ extending $A_U \to U$ which is universal with respect to a natural mapping property. This scheme is called the Néron model. Arithmetic geometry has seen the use of the Néron model in a number of important results, e.g., [Mazur 1972; 1977; Mazur and Wiles 1984; Gross 1990]. The Néron model of a Jacobian variety plays a particularly prominent role, and an alternative description

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of this scheme in terms of the relative Picard functor was given by Raynaud [1970]. We primarily work with Raynaud's description, which is recalled in Section 2.

An alternative approach, suggested by viewing the Jacobian as the Picard variety, is to extend $J_U \to U$ as a family of moduli spaces of sheaves. This approach was first proposed by Mayer and Mumford [1964]. Typically, one first extends $X_U \to U$ to a family of curves $X \to B$ and then extends J_U to a family $\bar{J} \to B$ with the property that the fiber over a point $b \in B$ is a moduli space of sheaves on X_b parametrizing certain line bundles, together with their degenerations. In this paper, we show that the line bundle locus J in $\bar{J}$ is canonically isomorphic to the Néron model for some schemes $\bar{J}$ constructed in the literature.

To state this more precisely, we need to specify which schemes $\bar{J}$ we consider. The problem of constructing such a family of moduli spaces has been studied by many mathematicians, and they have constructed a number of different compactifications; see for example [Ishida 1978; D'Souza 1979; Oda and Seshadri 1979; Altman and Kleiman 1980; Caporaso 1994; Simpson 1994; Pandharipande 1996; Jarvis 2000; Esteves 2001]. Many of the difficulties to performing such a construction arise from the fact that, when X_b is reducible, the degree 0 line bundles on a fiber X_b do not form a bounded family.

For simplicity, assume the residue field $k(b)$ is algebraically closed and X_b is reduced with components labeled $X_1, \dots, X_n$. Given a line bundle $\mathcal{M}$ of degree 0 on X_b, the sequence $(\deg(\mathcal{M}|_{X_1}), \dots, \deg(\mathcal{M}|_{X_n}))$ is called the multidegree of $\mathcal{M}$. This sequence must sum to 0, but may otherwise be arbitrary, which implies unboundedness. A bounded family can be obtained by fixing the multidegree, and typically the scheme $\bar{J}$ is defined so that it parametrizes (possibly coarsely) line bundles (and their degenerations) that satisfy a numerical condition on the multidegree. This paper focuses on constructions given by Simpson [1994] and by Esteves [2001], which we now describe in more detail.

For the Simpson moduli space, the numerical condition imposed on line bundles is slope semistabilty with respect to an auxiliary ample line bundle. This condition arises from the method of construction: the moduli space is constructed using geometric invariant theory (GIT), and slope stability corresponds to GIT stability. In general, the Simpson moduli space is a coarse moduli space in the sense that nonisomorphic sheaves may correspond to the same point of the space, but it contains an open subscheme (the stable locus) that is a fine moduli space, and we will work exclusively with this locus. Families of moduli spaces of semistable sheaves have been constructed for arbitrary families of projective schemes, but we will only be concerned with the case of families of curves.

The moduli spaces of Esteves parametrize sheaves that are σ-quasistable. Like slope stability, σ-quasistabillity is a numerical conditions on the multidegree, but it is defined in terms of an auxiliary vector bundle $\mathcal{E}$ and a section σ, rather than an ample line bundle. The moduli spaces are constructed for families over an arbitrary locally noetherian base, but strong conditions are required of the fibers: They must be geometrically reduced. The space is constructed as a closed subspace of a (nonnoetherian, nonseparated) algebraic space that was constructed in [Altman and Kleiman 1980]. For nodal curves, Melo and Viviani [2012] describe the relation between the Esteves moduli spaces and the Simpson moduli spaces. However, here we treat these moduli spaces separately.

For a discussion of the relation between these moduli spaces and other moduli spaces constructed in the literature, the reader is directed to [Alexeev 2004; Casalaina-Martin et al. 2011, Section 2]. The reader familiar with the work of Caporaso is warned of one potential point of confusion. In [Caporaso 1994], the compactified Jacobian associated to a stable curve X parametrizes pairs (Y, L) consisting of a line bundle L on a nodal curve Y stably equivalent to X that satisfies certain conditions. The line bundle locus J that we study corresponds to the locus parametrizing pairs (Y, L) with $X = Y$.

1.2. ***Main result.*** The main result of this paper relates the line bundle locus in a proper family of moduli spaces of sheaves to the Néron model of the Jacobian:

Theorem 1. *Fix a Dedekind scheme B. Let $f : X \to B$ be a family of geometrically reduced curves with regular total space X and smooth generic fiber X_η. Let $J \subset \bar{J}$ the locus of line bundles in one of the following moduli spaces*:

- *the Esteves compactified Jacobian $\bar{J}^{\sigma}_{\mathscr{E}}$;*
- *the Simpson compactified Jacobian $\bar{J}^{0}_{\mathscr{L}}$ associated to an f-ample line bundle $\mathscr{L}$ such that slope semistability coincides with slope stability.*

Then J is the Néron model of its generic fiber.

Theorem 1 is the combination of Corollaries 4.2 and 4.5, which themselves are consequences of Theorem 3.9. Theorem 3.9 is quite general, and we expect that it applies to many other fine moduli spaces of sheaves (but *not* coarse ones). In particular, Theorem 3.9 applies to families of curves with possibly nonreduced fibers, though then general results asserting the existence of a suitable moduli space are unknown (but see Section 4.3 for some simple examples).

The arithmetically inclined reader should note Theorem 1 and the results later in this paper do not place any hypotheses on the base Dedekind scheme B. In particular, we do not assume that the residue fields are perfect. This surprised the author initially as there is a body of work (e.g., [Liu et al. 2004; Raynaud 1970]) showing that various pathologies can arise when $k(b)$ fails to be perfect.

Theorem 1 has interesting consequences for both the Néron model and the compactified Jacobian. One consequence of the theorem is that Néron models of Jacobians can often be constructed over high-dimensional bases. The Néron model of an abelian variety is only defined over a (regular) 1-dimensional base B, but no such dimensional hypotheses are needed to apply the existence results from [Simpson 1994; Esteves 2001]. At the end of Section 4.3, we examine a family $J \to \mathbb{P}^2$ over the plane with the property that a dense, open subset of $\mathbb{P}^2$ is covered by lines C such that the restriction J_C of J is the Néron model of its generic fiber. Surprisingly, while the Néron models fit into a 2-dimensional family, their group structure does not.

Theorem 1 also has interesting consequences for the moduli spaces of Esteves and Simpson. Indeed, if $f : X \to B$ is a family of curves satisfying the hypotheses of the theorem, then both the Esteves Jacobians $J^{\sigma}_{\mathscr{E}}$ and the Simpson Jacobians $J^{0}_{\mathscr{L}}$ (for $\mathscr{L}$ as in the hypothesis) are independent of the particular polarizations, and every such Simpson Jacobian is isomorphic to every Esteves Jacobian. This is not immediate from the definitions. At the end of Section 4.1, we discuss this fact in greater detail and pose a related question.

1.3. ***Past results.*** Certain cases of Theorem 1 were already known. In his (unpublished) thesis, Simone Busonero [2008] established Theorem 1 for certain Esteves Jacobians. A different proof using similar techniques that extends the result to the Simpson moduli spaces is due to Melo and Viviani [2012, Theorem 3.1]. They prove Theorem 1 when the fibers of f are nodal and X is regular. We do not discuss the Caporaso universal compactified Jacobian here, but the relation between that scheme and the Néron model was described by Caporaso [2008a; 2008b; 2012, especially Theorem 2.9]. Earlier still, Oda and Seshadri related their ϕ-semistable compactified Jacobians, also not discussed here, to Néron models [Oda and Seshadri 1979, Corollary 14.4]. In each of those papers, an important step in the proof is a combinatorial argument establishing that, for example, the natural map from the set of σ-quasistable multidegrees to the degree class group is a bijection.

The proof given here does not use any combinatorics, and the idea can be described succinctly. Consider the special case where $B := \operatorname{Spec}(\mathbb{C}[\![t]\!])$, which is a strict henselian discrete valuation ring with algebraically closed residue field. There is a natural map $J \to \mathrm{N}$ to the Néron model coming from the universal property of N, and an application of Zariski's main theorem shows that this morphism is an open immersion. Thus the only issue is set-theoretic surjectivity. Because B is henselian, every point on the special fiber of N is the specialization of a section, so surjectivity is equivalent to the surjectivity of the map $J(\mathbb{C}[\![t]\!]) \to J(\operatorname{Frac} \mathbb{C}[\![t]\!])$ that sends a section to its restriction to the generic fiber. A given point $p \in J(\operatorname{Frac} \mathbb{C}[\![t]\!])$ may be extended to a section $\sigma \in \bar{J}(\mathbb{C}[\![t]\!])$ of $\bar{J}$ by the valuative criteria. As $\bar{J}$ is a fine moduli space, σ corresponds to a family of rank 1, torsion-free sheaves, which in fact must be a family of line bundles because X is factorial. We may conclude that $\sigma \in J(\mathbb{C}[\![t]\!])$, yielding the result.

1.4. *Questions.* It would be interesting to know when a Simpson Jacobian $J^0_{\mathscr{L}}$ satisfying the hypotheses of Theorem 1 exists; that is, given a family $f : X \to B$, does there exist an ample line bundle $\mathscr{L}$ such that every $\mathscr{L}$-slope semistable sheaf of degree 0 is stable? We briefly survey the literature on this question at the end of Section 4.2.

More generally, given a family $f : X \to B$, it would be desirable to have a description of the maximal subfunctors of the degree 0 relative Picard functor P^0 representable by a separated B-scheme. We discuss this question in Section 4.3, where we analyze the simple case of genus 1 curves.

1.5. *Organization.* We end this introduction with a few technical remarks about the paper. The moduli spaces of sheaves that we consider are moduli spaces of pure sheaves. On a curve, a coherent sheaf is pure if and only if it is Cohen–Macaulay. This condition is also equivalent to the condition of being torsion-free in the sense of elementary algebra when the curve is integral, and the term "torsion-free" is sometimes used in place of "pure".

The term "family of curves" only to refers to families with geometrically irreducible generic fibers. This is done to avoid notational complications concerning multidegrees. Families of curves are required to be proper, but not projective. A family of curves over a Dedekind scheme can fail to be projective (e.g., [Altman and Kleiman 1980, 8.10]), but projectivity is automatic if the local rings of the total space are factorial, which is the main case of interest. (See Proposition 4.1.)

We prove the main results for families over a base scheme S that is the spectrum of a strict henselian discrete valuation ring rather than a more general Dedekind scheme. Doing so lets us make sectionwise arguments because a smooth family of a henselian base admits many sections. Furthermore, this is not a real restriction: Results over a general Dedekind base can be deduced by passing to the strict henselization.

The body of the paper is organized as follows. In Section 2, we review Rayanud's construction of the maximal separated quotient. We then relate this scheme to a general moduli space of line bundles satisfying some axioms in Section 3. Finally, we describe some schemes that satisfy these axioms in the final section, Section 4.

Conventions

1.1. The symbol X_T denotes the fiber product $X \times_S T$.

1.2. The letter S denotes the spectrum of a strict henselian discrete valuation ring with special point 0 and generic point η.

1.3. A curve over a field k is a proper k-scheme $f_0 : X_0 \to \mathrm{Spec}(k)$ that is geometrically connected and of pure dimension 1.

1.4. If B is a scheme, then a family of curves over B is a proper, flat morphism $f : X \to B$ whose fibers are curves and whose geometric generic fibers are irreducible.

1.5. If $f : Y \to B$ is a finitely presented morphism, then we write $Y^{\mathrm{sm}} \subset Y$ for the smooth locus of f.

1.6. A coherent module I_0 on a noetherian scheme X_0 is rank 1 if the localization of I_0 at x is isomorphic to $\mathcal{O}_{X_0,x}$ for every generic point x.

1.7. A coherent module I_0 on a noetherian scheme X_0 is pure if the dimension of $\mathrm{Supp}(I_0)$ equals the dimension of $\mathrm{Supp}(J_0)$ for every nonzero subsheaf J_0 of I_0.

1.8. If $X_0 \to \mathrm{Spec}(k)$ is proper, then the degree of a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}$ is defined by $\deg(\mathcal{F}) := \chi(\mathcal{F}) - \chi(\mathcal{O}_X)$.

2. Raynaud's maximal separated quotient

We begin by reviewing Raynaud's construction of the Néron model of a Jacobian and, more generally, the maximal separated quotient of the relative Picard functor [Raynaud 1970]. Much of this material is also treated in [Bosch et al. 1990, Chapter 9].

Let S be a strict henselian discrete valuation ring with generic point η and special point 0. Given a family of curves $f : X \to S$, the *relative Picard functor* P of f is defined to be the étale sheaf $\mathrm{P} : S\text{-Sch} \to \mathrm{Grp}$ associated to the functor

$$T \mapsto \mathrm{Pic}(X_T). \tag{2-1}$$

Here $\mathrm{Pic}(X_T)$ is the set of isomorphism classes of line bundles on X_T. Rayanud actually defines P to be the associated fppf sheaf, but then observes that this is the same as the associated étale sheaf ([Raynaud 1970, 1.2]; see also [Kleiman 2005, Remark 9.2.11]). The fibers of P are representable by group schemes locally of finite type, and P itself is representable by an algebraic space if and only if f is cohomologically flat [Raynaud 1970, Theorem 5.2]. Regardless of its representability properties, P is locally finitely presented and formally S-smooth.

Inside of P, we may consider the functor $\mathrm{E} : S\text{-Sch} \to \mathrm{Grp}$ that is defined to be the scheme-theoretic closure of the identity section. This is the fppf subsheaf of P generated by the elements $g \in \mathrm{P}(T)$, where $T \to S$ is flat and $g_\eta \in \mathrm{P}(T_\eta)$ is the identity element. When P is a scheme, this coincides with the usual notion of closure. The representability properties of E are similar to those of P: The fibers of E are group schemes locally of finite type, and E is representable by an algebraic space precisely when f is cohomologically flat [Raynaud 1970, Proposition 5.2]. When representable, $\mathrm{E} \to S$ is an étale S-group space; in general, the generic fiber of E is the trivial group scheme, and the special fiber is a group scheme of dimension equal to $h^0(\mathcal{O}_{X_0}) - h^0(\mathcal{O}_{X_\eta})$.

When E is not the trivial S-group scheme, P does not satisfy the valuative criteria of separatedness. We can, however, form the *maximal separated quotient* $\mathrm{Q} : S\text{-Sch} \to \mathrm{Grp}$ of P. By definition, this is the fppf quotient sheaf $\mathrm{Q} := \mathrm{P}/\mathrm{E}$. The maximal separated quotient Q is always representable by a scheme that is S-smooth, separated, and locally of finite type [Raynaud 1970, Theorem 4.1.1, Proposition 8.0.1]. Rather than working directly with Q, we shall primarily work with the slightly smaller subfunctor $\mathrm{Q}^{\tau} : S\text{-Sch} \to \mathrm{Grp}$, which we now define.

Suppose generally that B is a scheme and $G : S\text{-Sch} \to \mathrm{Grp}$ is a B-group functor whose fibers are representable by group schemes locally of finite type. For every point $b \in B$, we may form the identity component $G_b^{\mathrm{o}} \subset G_b$ and the component group G_b/G_b^{o}. The subgroup functor $G^{\tau} \subset G$ is defined to the subfunctor whose T-valued points are elements $g \in G(T)$ with the property that, for every $t \in T$ mapping to $b \in B$, the element $g_t \in G_b(k(t))$ maps to a torsion element of $G_b/G_b^{\mathrm{o}}(k(t))$. If we instead require that g_t maps to the identity element, then we obtain the subgroup functor $G^{\mathrm{o}} \subset G$. Let us examine these constructions when B equals S and G equals P or Q.

The functors P^{o} and P^{τ} coincide, and this common functor is the étale sheaf associated to the assignment sending T to the set of isomorphism classes of line bundles on X_T that fiberwise have multidegree 0. From this description, it is easy to see that $\mathrm{P}^{\mathrm{o}} = \mathrm{P}^{\tau} \subset \mathrm{P}$ is an open subfunctor. Another open subfunctor of P is the subfunctor parametrizing line bundles on X_T with fiberwise degree 0, which we denote by P^0. It is typographically difficult to distinguish between P^0 and P^{o}, but we will not make use of P^{o} in this paper, so this should not cause confusion.

The functors Q^{o} and Q^{τ} are different in general. They are, however, both open subfunctors of Q [Grothendieck 1966b, Theorem 1.1(i.i), Corollary 1.7]. In particular, they are both representable by smooth and separated S-group schemes that are locally of finite type. In fact, both schemes are of finite type over S as their fibers are easily seen to have a finite number of connected components. The condition that $\mathrm{Q}^{\tau} \subset \mathrm{Q}$ is a closed subscheme is important, but slightly subtle. A characterization of this condition is given by [Raynaud 1970, Proposition 8.1.2(iii)]; one sufficient (but not necessary) condition for $\mathrm{Q}^{\tau} \subset \mathrm{Q}$ to be closed is that the local rings of X are factorial.

The factoriality condition is also almost sufficient to ensure that Q^{τ} is the Néron model of its generic fiber. Suppose that the generic fiber of f is smooth, so the generic fiber of $\mathrm{Q}^{\tau} \to S$ is an abelian variety, and thus it makes sense to speak of the Néron model $\mathrm{N} := \mathrm{N}(\mathrm{Q}^{\tau}_{\eta})$. By the universal property, there is a unique morphism $\mathrm{Q}^{\tau} \to \mathrm{N}$ that is the identity on the generic fiber. Theorem 8.1.4 of [Raynaud 1970] states that if the local rings of X are factorial, then $\mathrm{Q}^{\tau} \to \mathrm{N}$ is an isomorphism in the cases that $k(0)$ is perfect and that a certain invariant δ is coprime to the residual characteristic.

The proof uses the characterization of the Néron model in terms of the weak Néron mapping property. Recall that a S-scheme $Y \to S$ is said to be a weak Néron model of its generic fiber if the natural map $Y(S) \to Y(\eta)$ is bijective. If $G \to S$ is a finite type S-group scheme whose generic fiber is an abelian variety, then G is the Néron model of its generic fiber if and only if it satisfies the weak Néron mapping property [Bosch et al. 1990, Section 7.1, Theorem 1].

3. The main theorem

Here we derive the main results for families over a strict henselian discrete valuation ring S with generic point η and special point 0. Specifically, we provide sufficient conditions for the maximal separated quotient Q^{τ} of the Picard functor to be the Néron model and we relate Q^{τ} to a fine moduli space of line bundles that satisfies certain axioms. These moduli spaces are, by definition, subfunctors of a (large) functor that we now define.

Definition 3.1. If T is a S-scheme, then we define $\mathrm{Sheaf}(X_T)$ to be the set of isomorphism classes of $\mathcal{O}_T$-flat, finitely presented $\mathcal{O}_{X_T}$-modules $\mathcal{I}$ on X_T that are fiberwise pure, rank 1, and of degree 0.

The functor $\mathrm{Sh} = \mathrm{Sh}_{X/S} : S\text{-Sch} \to \mathrm{Sets}$ is defined to be the étale sheaf associated to the functor

$$T \mapsto \mathrm{Sheaf}(X_T). \tag{3-1}$$

There is a tautological transformation $\mathrm{P}^0 \to \mathrm{Sh}$ that realizes P^0 as a subfunctor of Sh.

Lemma 3.2. *The subfunctor* $\mathrm{P}^0 \subset \mathrm{Sh}$ *is open.*

Proof. Given a S-scheme T and a morphism $g : T \to \mathrm{Sh}$, we must show that $T \times_{\mathrm{Sh}} \mathrm{P}^0$ is representable by a scheme and that $T \times_{\mathrm{Sh}} \mathrm{P}^0 \to T$ is an open immersion. Thus, let g be given.

By definition, there exists an étale surjection $p : T' \to T$ and a sheaf $\mathcal{I}' \in \mathrm{Sheaf}(X_{T'})$ that represents $g \circ p : T' \to \mathrm{Sh}$. Consider the subset $U' \subset T'$ of points $t \in T'$ with the property that the restriction of $\mathcal{I}$ to the fiber X_t is a line bundle. This locus is open by [Altman and Kleiman 1980, Lemma 5.12(a)], and one may easily show that U' represents $T' \times_{\mathrm{Sh}} \mathrm{P}^0$. A descent argument establishes the analogous property for the image U of U' under $T' \to T$. This completes the proof. □

A remark about topologies: We work with the étale sheaf associated to (3-1), but one could instead work with the associated fppf sheaf. When f is projective, it is a theorem of Altman and Kleiman [1980, Theorem 7.4] that the subfunctor of Sh parametrizing simple sheaves can be represented by a quasiseparated, locally finitely presented S-algebraic space, and hence is an fppf sheaf. We do not know if Sh is an fppf sheaf in general. Here Sh is just used as a tool for keeping track of representable functors, and certainly any representable subfunctor of Sh is an fppf sheaf.

One reason for working with the étale topology instead of the fppf topology is that it makes the following fact easy to prove.

Fact 3.3. The natural map $\mathrm{Sheaf}(X) \to \mathrm{Sh}(S)$ is surjective.

Proof. Let $g \in \mathrm{Sh}(S)$ be given. By definition, there is an étale morphism $S' \to S$ and an element $\mathcal{I}' \in \mathrm{Sheaf}(X_{S'})$ that maps to $g_{S'} \in \mathrm{Sh}(S')$. But S is strict henselian, so $S' \to S$ may be taken to be an isomorphism $S \to S$ [Grothendieck 1967, Proposition 18.8.1(c)], in which case the result is obvious. □

The following two facts about separably closed fields are standard, but they will be used so frequently that it is convenient to record them.

Fact 3.4. If $k(0)$ is a separably closed field and $f_0 : Y_0 \to \mathrm{Spec}(k(0))$ is smooth of relative dimension n, then the closed points of Y_0 with residue field $k(0)$ are dense.

Proof. This is [Bosch et al. 1990, Corollary 13]. The scheme Y_0 can be covered by affine opens U_0 that admit an étale morphism $p : U_0 \to \mathbb{A}^n_{k(0)}$. Certainly, the closed points with residue field $k(0)$ are dense in the image of p. If $v_0 \in \mathbb{A}^n_{k(0)}$ is one such point, then $p^{-1}(v_0)$ is a finite, étale $k(0)$-algebra, and hence a disjoint union of closed points defined over $k(0)$. Density follows. □

Fact 3.4 is typically used in conjunction with the following fact to assert that a smooth morphism has many sections.

Fact 3.5. Let $Y \to S$ be a smooth morphism over strict henselian discrete valuation ring. Then $Y(S) \to Y(k(0))$ is surjective.

Proof. This is [Grothendieck 1967, Corollary 17.17.3], or [Bosch et al. 1990, Proposition 14]. If U and X' are as in the statement of the former, then we must have $U = S$ and $X' \to U$ may be taken to be an isomorphism (again, by [Grothendieck 1967, Proposition 18.8.1(c)]). □

We now prove the main results of the paper.

Proposition 3.6. *Let $f : X \to S$ be a family of curves and $J \subset \mathrm{P}^0$ a subfunctor such that the generic fibers $J_\eta = \mathrm{P}^0_\eta$ coincide. Assume J is represented by a smooth, finitely presented S-scheme.*

If J is S-separated, then $J \to \mathrm{Q}$ is an open immersion. Furthermore, the image is contained in Q^τ provided $\mathrm{Q}^\tau \subset \mathrm{Q}$ is closed (e.g., the local rings of X are factorial).

Proof. This is an application of Zariski's main theorem. We begin by showing that the induced map $J \to \mathrm{Q}$ is injective on closed points. It is enough to verify this after extending base S so that $k(0)$ is algebraically closed. Thus, we will temporarily assume $k := k(0)$ is algebraically closed and work with k-valued points instead of closed points. Given $q \in \mathrm{Q}(k)$, there is nothing to show when the fiber over q is empty. If nonempty, pick $p \in J(k)$ mapping to q. We may invoke Fact 3.5 to assert that there exists a section $\sigma \in J(S)$ with $\sigma(0) = p$.

The fiber of $\mathrm{P} \to \mathrm{Q}$ over q is the set of elements of the form $p + e$ with $e \in \mathrm{E}(k)$ or, equivalently, the elements of $(\sigma + \mathrm{E})(k)$ [Raynaud 1970, Corollary 4.1.2]. Restricting to J, we see that the fiber of q under $J \to \mathrm{Q}$ is the set of k-valued points of $(\sigma + \mathrm{E}) \cap J$. But $(\sigma + \mathrm{E}) \cap J$ is the scheme-theoretic closure of σ in J (by [Grothendieck 1965, 2.8.5]), which is just the image of σ by separatedness. In particular, the preimage of q under $J \to \mathrm{Q}$ must be the singleton set $\{p\}$. This proves that the map is injective on closed points. We now return to the case where S is a henselian discrete valuation ring (so $k(0)$ is no longer assumed to be algebraically closed).

It follows that the set-theoretic fibers of $J \to \mathrm{Q}$ are finite sets. Indeed, if $Z \subset J$ is the locus of points $x \in J$ with the property that x lies in a positive dimensional fiber, then Z is closed by Chevalley's theorem [Grothendieck 1965, 13.1.3]. Furthermore, Z is contained in the special fiber J_0 and contains no closed points. This is only possible if Z is the empty scheme. In other words, the set-theoretic fibers of $J \to \mathrm{Q}$ are 0-dimensional, and hence finite (by [Grothendieck 1964, 14.1.9]).

It follows immediately from Zariski's main theorem [Grothendieck 1961, 4.4.9] that $J \to \mathrm{Q}$ is an open immersion. This proves the first part of the theorem. To complete the proof, observe that flatness implies that the generic fiber of J_η is dense in J [Grothendieck 1965, 2.8.5]. In particular, J is contained in the closure of J_η in Q. The generic fiber of J coincides with the generic fiber of Q^τ, so the closure of this common scheme is contained in Q^τ when $\mathrm{Q}^\tau \subset \mathrm{Q}$ is closed. This completes the proof. □

Remark 3.7. In Proposition 3.6, we do *not* assume that $J \subset \mathrm{P}^0$ is an open subfunctor, but this condition holds in most cases of interest. When open, J is automatically formally smooth and locally of finite presentation. Thus, the key hypothesis in the proposition is that J is represented by a S-separated scheme. A similar remark holds for Theorem 3.9; there the key hypotheses are that $\bar{J}$ satisfies the valuative criteria of properness and that J is representable. Indeed, we do not even need to assume that $\bar{J}$ is representable.

Under stronger assumptions, we can actually show that the natural map $J \to \mathrm{Q}^\tau$ is an isomorphism. The essential point is to prove that J satisfies the weak Néron mapping property. When J can be embedded in a S-proper moduli space $\bar{J}$, this property holds provided that the local rings of X are factorial. The content of this claim is that a line bundle on the generic fiber can specialize only to a line bundle on the special fiber. By localizing, the claim is equivalent to the following lemma, which is based on a proof from [Altman and Kleiman 1979, p. 27 after Step XII].

Lemma 3.8. *Suppose that* (R, π) *is a discrete valuation ring and* $R \to \mathbb{O}$ *a local, flat algebra extension with* $\mathbb{O}$ *noetherian. Let* M *be a* R*-flat, finite* $\mathbb{O}$*-module with the property that* $M[\pi^{-1}]$ *is free of rank* 1 *and* $M/\pi M$ *is a rank* 1, *pure module. If* $\mathbb{O}$ *is factorial, then* M *is free of rank* 1.

Proof. We can certainly assume $\mathbb{O}$ is not the zero ring. To ease notation, we write $\overline{M} := M/\pi M$ and $\overline{\mathbb{O}} := \mathbb{O}/\pi\mathbb{O}$. It is enough to prove that M is isomorphic to a height 1 ideal. Indeed, such an ideal is principal by the factoriality assumption.

We argue by first showing that M is isomorphic to an ideal of $\mathbb{O}$. Let $\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_n$ be the minimal primes of $\overline{\mathbb{O}}$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ the corresponding primes of $\mathbb{O}$. We have assumed that the stalk $\overline{M} \otimes k(\bar{\mathfrak{p}}_i)$ is 1-dimensional. This stalk coincides with the stalk $M \otimes k(\mathfrak{p}_i)$, so we can conclude that the localization $M_{\mathfrak{p}_i}$ is free of rank 1 for $i = 1, \dots, n$.

We can also conclude that the same holds for the localizations of the dual module $M^\vee := \mathrm{Hom}(M, \mathbb{O})$. An application of the prime avoidance lemma shows that there exists an element $\phi \in M^\vee$ that maps to a generator of $M^\vee_{\mathfrak{p}_i}$ for all i. We will show that $\phi : M \to \mathbb{O}$ realizes M as a R-flat family of ideals (i.e., ϕ is injective with R-flat cokernel).

It is enough to show that the reduction $\bar{\phi} : \overline{M} \to \overline{\mathbb{O}}$ is injective. An element of the kernel of this map is also in the kernel of the composition

$$\overline{M} \to \bigoplus \overline{M}_{\bar{\mathfrak{p}}_i} \to \bigoplus \overline{\mathbb{O}}_{\bar{\mathfrak{p}}_i}.$$

The kernel of the leftmost map is a submodule whose support does not contain any of the primes $\bar{\mathfrak{p}}_i$, and thus must be zero by pureness. Furthermore, the rightmost map is an isomorphism by construction. This proves injectivity.

Consider the ideal $I[\pi^{-1}]$ given by the image of $\phi[\pi^{-1}] : M[\pi^{-1}] \to \mathbb{O}[\pi^{-1}]$. This is a principal ideal, and hence is either the unit ideal or an ideal of height at most 1 (Hauptidealsatz!). By flatness, the same is true of the image I of ϕ. In fact, I cannot be a height zero ideal: The only such prime is the zero ideal, which does not satisfy the hypotheses. Thus, I is either the unit ideal or a height 1 ideal. In either case, I must be principal, and the proof is complete. □

We record the factorial condition as a hypothesis.

Hypothesis 1. We say a family of curves $f : X \to B$ over a Dedekind scheme satisfies Hypothesis 1 if the generic fiber X_η is smooth and the local rings of X_S are factorial for every strict henselization $S \to B$.

Hypothesis 1 is satisfied when X is regular and X_η is smooth. We now prove the main theorem of this paper.

Theorem 3.9. *Let $f : X \to S$ be a family of curves and $\bar{J}$ a subfunctor of* Sh *such that the generic fibers $\bar{J}_\eta = \mathrm{Sh}_\eta$ coincide. Assume the line bundle locus $J \subset \bar{J}$ is represented by a smooth and finitely presented S-scheme.*

If $\bar{J}$ satisfies the valuative criteria of properness and f satisfies Hypothesis 1, then Q^τ is the Néron model and

$$J \subset \mathrm{Q}^\tau = \mathrm{N}$$

is an open subscheme that contains all the $k(0)$-valued points of Q^τ. Furthermore,

$$J = \mathrm{Q}^\tau = \mathrm{N}$$

provided one of the following conditions hold:

(1) *$k(0)$ is algebraically closed*;

(2) *J is stabilized by the identity component Q^{o}.*

Proof. By Proposition 3.6, the natural map $J \to \mathrm{Q}$ is an open immersion with image contained in Q^τ. Using this fact, we can prove that Q^τ is the Néron model of its generic fiber. Indeed, it is enough to prove that Q^τ satisfies the weak Néron mapping property. The open subscheme $J \subset \mathrm{Q}^\tau$, in fact, already satisfies this property. Let $\sigma_\eta \in \mathrm{Q}^\tau(\eta) = J(\eta)$ be given. By properness, we can extend σ_η to a section $\sigma \in \bar{J}(S)$, and this element can be represented by a family $\mathscr{I}$ of pure, rank 1 sheaves (by Fact 3.3). But every such family is a family of line bundles (Lemma 3.8), and hence σ lies in $J(S) \subset \bar{J}(S)$. In other words, J satisfies the weak Néron mapping property.

The weak Néron mapping property of J also implies that the image of J contains all the $k(0)$-valued points of Q^τ. Indeed, every $k(0)$-valued point of Q^τ is the specialization of a section by Fact 3.5. If $k(0)$ is algebraically closed, then we have shown that J contains every $k(0)$-valued point of Q^τ, hence every closed point. Thus, $J = \mathrm{Q}^\tau$, and there is nothing more to show.

Let us now turn our attention to the case where $k(0)$ is only separably closed, but J is stabilized by Q^{o}. Our goal is to show $J = \mathrm{Q}^\tau$, and to show this, we pass to the special fiber $J_0 \to \mathrm{Q}^\tau_0$ and argue with points. Let x be a $\bar{k}(0)$-valued point of Q^τ, where $\bar{k}(0)$ is the algebraic closure of the residue field. By density (Fact 3.4), there exists a $\bar{k}(0)$-valued point y in the image of $J_0 \to \mathrm{Q}^\tau_0$ that lies in the same connected component as x. We have $x = y + (x - y)$, which expresses x as the sum of a point of $\mathrm{Q}^{\mathrm{o}}_0$ and a point of J_0. The point x must lie in J_0 by assumption. This shows that the image of J contains all of Q^τ, completing the proof. □

Remark 3.10. The hypothesis that J is stabilized by the identity component Q^{o} is perhaps unexpected, but it is often satisfied in practice. The moduli space $\bar{J}$ is typically constructed by imposing numerical conditions on the multidegree of a sheaf, and the multidegree is invariant under the action of Q^{o} (because the action is given by tensoring with a multidegree 0 line bundle).

In the next section, we will show that certain moduli spaces constructed in the literature satisfy the hypotheses of Theorem 3.9. There are, however, families of curves $f : X \to S$ with factorial local rings $\mathcal{O}_{X,x}$ such that there does not exist a S-scheme $\bar{J}$ satisfying the conditions of the theorem. Indeed, the family $f : X \to S$ in [Raynaud 1970, Example 9.2.3] is a family of genus 1 curves such that the local rings of X are factorial (even regular), but the natural map $\mathrm{Q}^{\tau} \to \mathrm{N}$ is not an isomorphism. In particular, no $\bar{J}$ satisfying the hypotheses of Theorem 3.9 can exist.

4. Applications

Here we apply Theorem 3.9 to some families of moduli spaces from the literature and then deduce consequences. The two moduli spaces that we are interested in are the Esteves moduli space of quasistable sheaves (Section 4.1) and the Simpson moduli space of slope stable sheaves (Section 4.2). In Section 4.3, we discuss the special case of families of genus 1 curves, where suitable moduli spaces can be constructed explicitly.

The moduli spaces we study are associated to a relatively projective family of curves. We are primarily interested in families over a Dedekind scheme with locally factorial total space, in which case projectivity is automatic. This fact is a consequence of the generalized Chevalley Conjecture when the Dedekind scheme is defined over a field, but we do not know a reference. For completeness, we prove:

Proposition 4.1. *Let $f : X \to B$ be a family of curves over a Dedekind scheme. If the local rings of X are factorial, then f is projective.*

Proof. This proof was explained to the author by Steven Kleiman. Fix a closed point $b_0 \in B$. Given any component $F \subset X_{b_0}$, I claim that we can find a line bundle $\mathcal{L}$ on X that has nonnegative degree on every component of every fiber and strictly positive degree on F.

Pick a closed point $x \in F$ and an open affine neighborhood $U \subset X$ of that point. By the prime avoidance lemma, we can find a regular function $r \in H^0(U, \mathcal{O}_X)$ that does not vanish on any component of X_{b_0} but does vanish at x. Pick a component D of the closure of $\{r = 0\} \subset U$ in X. Then D is a Cartier divisor (by the factoriality assumption) that does not contain any component of any fiber X_b (by construction).

Furthermore, D has nontrivial intersection with F. The associated line bundle $\mathcal{L} := \mathcal{O}_X(D)$ has the desired positivity property.

Now construct one such line bundle for every irreducible component F of X_{b_0} and define $\mathcal{M}$ to be their tensor product. The line bundle $\mathcal{M}$ is nef on every fiber and ample on X_{b_0}. Ampleness is an open condition, so $\mathcal{M}$ is in fact ample on all but finitely many fibers of f. After repeating the construction for each such fiber and forming the tensor product, we have constructed a f-relatively ample line bundle on X. This completes the proof. □

We now turn our attention to the moduli spaces.

4.1. ***Esteves Jacobians.*** We first discuss the Esteves moduli space of quasistable sheaves. This moduli space fits very naturally into the framework of the previous section.

Suppose B be a locally noetherian scheme and $f : X \to B$ a projective family of curves whose fibers are geometrically reduced. Quasistability is defined in terms of a section $\sigma : B \to X^{\text{sm}}$ and a vector bundle $\mathcal{E}$ on X with fiberwise integral slope $\deg(\mathcal{E}_b)/\operatorname{rank}(\mathcal{E}_b)$, which we assume is constant as a function of $b \in B$. Given σ and $\mathcal{E}$, σ-quasistability is a numerical condition on the multidegree of a rank 1, torsion-free sheaf of degree

$$d(\mathcal{E}) = d := -\chi(\mathcal{O}_{X_b}) - \deg(\mathcal{E}_b)/\operatorname{rank}(\mathcal{E}_b).$$

For the definitions (which we will not use), we direct the reader to [Esteves 2001, p. 3051] (for a single sheaf) and [ibid., p. 3054] (for a family). The basic existence theorem is [ibid., Theorem A on p. 3047], which is proved in [ibid., Section 4]). It states that if $\text{Sheaf}_{\mathcal{E}}^{\sigma} : S\text{-Sch} \to \text{Sets}$ is the functor defined by setting $\text{Sheaf}_{\mathcal{E}}^{\sigma}(T)$ equal to the set of isomorphism classes of $\mathcal{O}_T$-flat, finitely presented $\mathcal{O}_{X_T}$-modules that are fiberwise σ-quasistable, then there is a B-proper algebraic space $\bar{J}_{\mathcal{E}}^{\sigma} \to B$ of finite type that represents the étale sheaf associated to $\text{Sheaf}_{\mathcal{E}}^{\sigma}$.

Strictly speaking, our definition differs from the one given in [ibid.] in two ways. First, Esteves does not work with isomorphism classes of sheaves but rather with equivalence classes under the relation given by identifying two sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ on X_T when $\mathcal{I}_1$ is isomorphic to $\mathcal{I}_2 \otimes f^*(\mathcal{L})$ for some line bundle $\mathcal{L}$ on T. Zariski locally on T, the sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ are isomorphic, and it follows that the étale sheaf associated to $\text{Sheaf}_{\mathcal{E}}^{\sigma}$ is canonically isomorphic to the sheaf considered by Esteves. Second, Esteves only defines his functor as a functor from locally noetherian schemes to sets. However, the functor $\text{Sheaf}_{\mathcal{E}}^{\sigma}$ and its associated étale sheaf are easily seen to be locally finitely presented. It follows that $\bar{J}_{\mathcal{E}}^{\sigma}$ represents the étale sheaf associated to $\text{Sheaf}_{\mathcal{E}}^{\sigma}$, rather than just the restriction of this sheaf to locally noetherian schemes.

If f satisfies stronger conditions, then the space $\bar{J}^{\sigma}_{\mathscr{E}}$ is actually a scheme. This is the content of [Esteves 2001, Theorem B, p. 3048], proved on [ibid., p. 3086]. The theorem states that if there exist sections $\sigma_1, \dots, \sigma_n : B \to X^{\mathrm{sm}}$ of f with the property that every irreducible component of a fiber X_b is geometrically integral and contains one of the points $\sigma_1(b), \dots, \sigma_n(b)$, then $\bar{J}^{\sigma}_{\mathscr{E}}$ is a scheme.

In the special case where $B = S$ is a strict henselian discrete valuation ring with generic point η and special point 0, the hypotheses of Theorem B are automatically satisfied. Indeed, the locus of $k(0)$-valued points is dense in the smooth locus X_0^{sm} (Fact 3.4), which in turn is dense in X_0 as X_0 is geometrically reduced. We may conclude that the irreducible components of X_0 are geometrically integral. Finally, every $k(0)$-valued point of X_0 extends to a section $\sigma : S \to X$ (Fact 3.5), so the hypotheses of Theorem B are certainly satisfied.

We call $\bar{J}^{\sigma}_{\mathscr{E}}$ the *Esteves compactified Jacobian*. Inside of the Esteves compactified Jacobian, we can consider the open subscheme parametrizing line bundles. This scheme is called the *Esteves Jacobian* and denoted by $J^{\sigma}_{\mathscr{E}}$. While the scheme $\bar{J}^{\sigma}_{\mathscr{E}}$ parametrizes sheaves, it is not naturally a subfunctor of Sh because it does not parametrize degree 0 sheaves. We can, however, define a natural transformation $\bar{J}^{\sigma}_{\mathscr{E}} \to \mathrm{Sh}$ by the rule

$$\mathscr{I} \mapsto \mathscr{I}(-d \cdot \sigma)$$

Both Proposition 3.6 and Theorem 3.9 apply to $\bar{J}^{\sigma}_{\mathscr{E}}$.

Corollary 4.2. *Fix a Dedekind scheme B. Let $f : X \to B$ be a projective family of geometrically reduced curves. Let $\sigma : B \to X^{sm}$ be a section and $\mathscr{E}$ a vector bundle on X with fiberwise integral slope.*

Then the natural map $J^{\sigma}_{\mathscr{E}} \to \mathrm{Q}$ is an open immersion.

Assume further that f satisfies Hypothesis 1. Then $J^{\sigma}_{\mathscr{E}} = \mathrm{Q}^{\tau}$, and this scheme is the Néron model.

Proof. By localizing, we can assume that $B = S$ is a strict henselian discrete valuation ring, in which case we are reduced to proving that the hypotheses of Proposition 3.6 and Theorem 3.9 hold. The scheme $J^{\sigma}_{\mathscr{E}}$ is easily seen to be formally S-smooth. Indeed, σ-quasistability is a condition on fibers, so the formal smoothness of P^0 implies the formal smoothness of $J^{\sigma}_{\mathscr{E}}$. The remaining hypotheses of Proposition 3.6 are explicitly assumed, so we can deduce the first part of the theorem. To complete the proof, it is enough to show that $J^{\sigma}_{\mathscr{E}}$ is stabilized by Q^{o}. But the action of Q^{o} on $J^{\sigma}_{\mathscr{E}}$ is given by the tensor product against a multidegree 0 line bundle, so this action preserves multidegree and hence σ-quasistability. □

Corollary 4.2 implies that $J^{\sigma}_{\mathscr{E}}$ is a scheme with (unique) B-group scheme structure that extends the group scheme structure of the generic fiber. It is not immediate from the definition that $J^{\sigma}_{\mathscr{E}}$ admits such structure, and Example 4.9 shows that the group structure is special to the case of families over a 1-dimensional base. The result also implies uniqueness results for the Esteves Jacobian; if $\sigma' : B \to X^{\text{sm}}$ is a second section and $\mathscr{E}'$ a second vector bundle on X, then $J^{\sigma'}_{\mathscr{E}'}$ is canonically isomorphic to $J^{\sigma}_{\mathscr{E}}$. In the next section, we will define the Simpson stable Jacobian $J^{0}_{\mathscr{L}}(X)$, and this scheme is also isomorphic to $J^{\sigma}_{\mathscr{E}}$ provided every slope semistable sheaf is stable. It would be interesting to know if these isomorphisms extend to the compactifications. Important results along these lines can be found in [Melo and Viviani 2012; Esteves 2009], but many basic question remain unanswered. Currently, there is no example of a curve $X_0 \to \operatorname{Spec}(k)$ such that two Esteves compactified Jacobians associated to X_0 are nonisomorphic.

4.2. ***Simpson Jacobians.*** The hypotheses to Proposition 3.6 and Theorem 3.9 are satisfied by certain moduli spaces of stable sheaves, which we call Simpson Jacobians. Here we recall Simpson's construction, along with later work of Langer and Maruyama, and then apply results from Section 3. We restrict our attention to families of reduced curves (but see Remark 4.4, and the discussion preceding Example 4.9).

We work over a scheme B that is finitely generated over a universally Japanese ring R (e.g., $R = \mathbb{C}, \mathbb{F}_p, \mathbb{Z}, \dots$). Let $f : X \to B$ a family of curves with f-relatively ample line bundle $\mathscr{L}$, and assume the Euler–Poincaré characteristics $\chi(\mathbb{O}_{X_b})$ and $\chi(\mathscr{L}_b)$ are constant as functions of the base B. Set P_d equal to the polynomial

$$P_d(t) := \deg(\mathscr{L}_b) \cdot t + d + \chi, \tag{4-1}$$

where χ is the Euler–Poincaré characteristic of a fiber of f and $\deg(\mathscr{L}_b)$ is the degree of the restriction of $\mathscr{L}$ to a fiber. This is the Hilbert polynomial of a degree d line bundle.

Given this data, Simpson constructed an associated moduli space in the case that $R = \mathbb{C}$. The Simpson moduli space $\mathrm{M}(\mathbb{O}_X, P_d)$ parametrizes slope semistable sheaves with Hilbert polynomial P_d. (See [Simpson 1994, pp. 54–56] for the definition of semistability). To be precise, define $\mathrm{M}^{\sharp}(\mathbb{O}_X, P_d)$ to be the functor whose T-valued points are isomorphism classes of $\mathbb{O}_T$-flat, finitely presented $\mathbb{O}_{X_T}$-modules whose fibers are $\mathscr{L}$-slope semistable sheaves with Hilbert polynomial P_d. The main existence result [Simpson 1994, Theorem 1.21] asserts that there is a projective scheme $\mathrm{M}(\mathbb{O}_X, P_d)$ that corepresents $\mathrm{M}^{\sharp}(\mathbb{O}_X, P_d)$. Inside of $\mathrm{M}(\mathbb{O}_X, P_d)$, we may consider the open subscheme $\mathrm{M}^{\text{st}}(\mathbb{O}_X, P_d)$ parametrizing $\mathscr{L}$-slope stable sheaves. The stable locus is a fine moduli space: Its $\mathbb{C}$-valued points are in natural bijection with the isomorphism classes of $\mathscr{L}$-slope stable sheaves with Hilbert polynomial P_d, and étale locally on $\mathrm{M}^{\text{st}}(\mathbb{O}_X, P_d)$, the product $X \times_B \mathrm{M}^{\text{st}}(\mathbb{O}_X, P_d)$ admits a universal family of sheaves. The reader may check that these conditions are equivalent to the condition that $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ represents the étale sheaf associated the functor parametrizing stable sheaves. While Simpson only considers the case $R = \mathbb{C}$, later work of Langer [2004a, Theorem 4.1; 2004b, Theorem 0.2] and Maruyama [1996] extends these results to the case where R is an arbitrary universally Japanese ring.

Let us now specialize to the case where B is a Dedekind scheme. When f has reducible fibers, $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ may contain points corresponding to sheaves that are not rank 1; see [López-Martín 2005, Example 2.2]. This is potentially a major source of confusion: The term "rank" is used in a different way in [Simpson 1994], and the sheaves parametrized by $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ are rank 1 in Simpson's sense but not necessary in the sense used here.

We avoid these sheaves. Define the *Simpson stable Jacobian* $J^d_{\mathcal{L}}$ of degree d to be the locus of stable line bundles in $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ (which is an open subscheme by [Altman and Kleiman 1980, Lemma 5.12(a)]). We define the *Simpson stable compactified Jacobian* $\bar{J}^d_{\mathcal{L}}$ to be the subset of the stable locus $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ that corresponds to pure, rank 1 sheaves. (Warning: The compactified Jacobian is a B-proper scheme when every semistable pure sheaf with Hilbert polynomial P_d is stable but not in general!)

When the fibers of $X \to B$ are *geometrically reduced*, a minor modification of the proof of [Pandharipande 1996, Lemma 8.1.1] shows that the subset $\bar{J}^d_{\mathcal{L}} \subset \mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ is closed and open, and hence has a natural scheme structure:

Lemma 4.3. *Assume the fibers of $f : X \to B$ are geometrically reduced. Then the subset $\bar{J}^d_{\mathcal{L}}$ is closed and open in* $\mathrm{M}^{st}(\mathcal{O}_X, P_d)$.

Proof. The main point to prove is that a 1-parameter family of line bundles cannot specialize to a pure sheaf that fails to have rank 1, and this is shown by examining the leading term of the Hilbert polynomial. To begin, we may cover $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ by étale morphisms $\mathrm{M} \to \mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ with the property that a universal family $\mathcal{I}_{\text{uni.}}$ on $\mathrm{M} \times_B X$ exists. It is enough to verify the claim after passing from $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ to an arbitrary such scheme, and so for the remainder of the proof we work with M in place of $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$. We will also abuse notation by denoting the pullback of $\bar{J}^d_{\mathcal{L}}$ under $\mathrm{M} \to \mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ by the same symbol $\bar{J}^d_{\mathcal{L}}$.

We first need to check that $\bar{J}^d_{\mathcal{L}} \subset \mathrm{M}$ is constructible, so that we can make use of the valuative criteria. Given $m \in M$ mapping to $b \in B$, the condition that the fiber I_m is rank 1 is just the condition that the restriction of I_m to X^{sm}_b is a line bundle. Constructibility thus follows from [Grothendieck 1966a, 9.4.7] applied to $\mathrm{M} \times_B X^{\mathrm{sm}} \to \mathrm{M}$.

To finish, it is enough to prove that $\bar{J}^d_{\mathcal{L}}$ is closed under specialization and generalization. Thus, we pass from M to a discrete valuation ring T mapping to M. If

$\mathcal{I}$ is the sheaf on X_T given by the pullback of the universal family, then we need to show that the generic fiber of I_η is rank 1 if and only if the special fiber I_0 is.

To prove this, we turn our attention to the Hilbert polynomial P_d of a fiber of I. This polynomial is defined so that the leading term is $\deg(\mathcal{L}_b)$, and we can express this number in terms of components of a fiber of $X_T \to T$ as follows. If x is generic point of the special fiber X_0, then we define $\deg_x(\mathcal{L}_0)$ to be the degree of the restriction of $\mathcal{L}_0$ to the irreducible component corresponding to x. (Give the component the reduced subscheme structure.) For any generic point y of X_η, we define $\deg_y(\mathcal{L}_\eta)$ in the analogous manner. If $x_1, \dots, x_n$ are all the generic points of X_0 and $y_1, \dots, y_m$ all the generic points of X_η, then we have

$$\begin{aligned}\deg(\mathcal{L}_b) &= \deg_{x_1}(\mathcal{L}_0) + \cdots + \deg_{x_n}(\mathcal{L}_0)\\ &= \deg_{y_1}(\mathcal{L}_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\end{aligned}$$

by, say, [Altman and Kleiman 1979, 2.5.1]. The terms $\deg_{x_i}(\mathcal{L}_0)$ and $\deg_{y_j}(\mathcal{L}_\eta)$ in the equation above are each strictly positive as $\mathcal{L}$ is relatively ample.

We can also express $\deg(\mathcal{L}_b)$ in terms of the generic rank of a fiber of I. Using [Altman and Kleiman 1979, 2.5.1] again, we have

$$\begin{aligned}\deg(\mathcal{L}_b) &= \deg_{x_1}(\mathcal{L}_0)\cdot \ell_{x_1}(I_0) + \cdots + \deg_{x_n}(\mathcal{L}_0)\cdot \ell_{x_n}(I_0)\\ &= \deg_{y_1}(\mathcal{L}_\eta)\cdot \ell_{y_1}(I_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\cdot \ell_{y_m}(I_\eta).\end{aligned}$$

Here $\ell_{x_i}(I_0)$ denotes the length of the localization of I_0 at x_i and similarly for $\ell_{y_j}(I_\eta)$. The fibers of $X_T \to T$ are reduced, so such a length is equal to the minimal number of generators. In particular, these numbers are upper semicontinuous. In other words, if y_j specializes to x_i, then we have $\ell_{y_j}(I_\eta) \le \ell_{x_i}(I_0)$ (by Nakayama's lemma).

The desired result now follows. Suppose first that I_0 is rank 1. Then we have $\ell_{y_i}(I_\eta) \le 1$ for all i by semicontinuity. If some inequality was strict, say $\ell_{y_1}(I_\eta) = 0$, then we would have

$$\begin{aligned}\deg(\mathcal{L}_b) &= \deg_{y_1}(\mathcal{L}_\eta) + \deg_{y_2}(\mathcal{L}_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\\ &> \qquad\qquad\quad \deg_{y_2}(\mathcal{L}_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\\ &\ge \deg_{y_1}(\mathcal{L}_\eta)\cdot \ell_{y_1}(I_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\cdot \ell_{y_m}(I_\eta)\\ &= \deg(\mathcal{L}_b).\end{aligned}$$

This is absurd! Thus, we must have $\ell_{y_i}(I_\eta) = 1$ for all y_i and I_η is rank 1. Similar reasoning shows that if I_η is rank 1, then I_0 is rank 1. □

Remark 4.4. The hypothesis that the fibers of f are geometrically reduced is necessary. Indeed, the moduli space $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ was described in [Chen and Kass 2011] in the case that X is a nonreduced curve whose reduced subscheme X_{red}

is smooth and whose nilradical $\mathcal{N}$ is square-zero (i.e., X is a ribbon). Using that description it is easy to produce examples where $\bar{J}^d_{\mathcal{L}} \subset \mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ is not closed (e.g., take d equal to 0, X to have even genus, and X_{red} to have genus 1). The points of the complement in the closure correspond to stable rank 2 vector bundles on X_{red}.

We now apply Proposition 3.6 and Theorem 3.9 to the Simpson Jacobians.

Corollary 4.5. *Fix a Dedekind scheme B that is finitely generated over a universally Japanese ring. Let $f : X \to B$ be a family of geometrically reduced curves. Let $\mathcal{L}$ be f-relatively ample line bundle.*

Then the natural map $J^0_{\mathcal{L}}(X) \to \mathrm{Q}^\tau$ is an open immersion. Assume further that both of the following conditions hold:

- *Every $\mathcal{L}$-slope semistable rank* 1, *torsion free sheaf of degree* 0 *is $\mathcal{L}$-slope stable.*
- *f satisfies Hypothesis 1.*

Then $J^0_{\mathcal{L}}(X) = \mathrm{Q}^\tau$, and this scheme is the Néron model.

Proof. The local existence of a universal family [Simpson 1994, Theorem 2.1(4)] implies that there is a natural transformation $\bar{J}_{\mathcal{L}}(X) \to \mathrm{Sh}$ with the property that $J_{\mathcal{L}}(X)$ is the preimage of $\mathrm{P}^0 \subset \mathrm{Sh}$. Furthermore, the slope stability condition is a fiberwise condition, so a modification of the argument given in Corollary 4.2 completes the proof. □

Remark 4.6. A minor generalization of Corollary 4.5 can be obtained by allowing for moduli spaces of degree d lines bundles, with $d \neq 0$. If we are given a line bundle $\mathcal{M}$ on X with fiberwise degree d, then there is an associated map $J^d_{\mathcal{L}}(X) \to \mathrm{Q}$ that extends the map on the generic fiber given by tensoring with $\mathcal{M}^{-1}$. With only notational changes the previous corollary generalizes to a statement about this map.

Corollary 4.5 is, of course, only of interest when there exists an $\mathcal{L}$ such that $\mathcal{L}$-slope stability coincides with $\mathcal{L}$-slope semistability. Thus, we ask, When does such an $\mathcal{L}$ exist? A comprehensive discussion of this question would require a digression on stability conditions, so we limit ourselves to reviewing known results about a single curve X_0 over an algebraically closed field. When X_0 is integral, the stability condition is vacuous, so every ample $\mathcal{L}_0$ has the desired property. If X_0 is reducible of genus $g \neq 1$ with only nodes as singularities, then Melo and Viviani have proven the existence of a suitable $\mathcal{L}_0$ [2012, Proposition 6.4]. Stability conditions on reduced, genus 1 curves were analyzed by López-Martín [2005]. She exhibits curves X_0 with the property that there is no $\mathcal{L}_0$ such that every $\mathcal{L}_0$-slope semistable, pure, rank 1 sheaf degree 0 is stable, but a suitable $\mathcal{L}_0$ always exists if one considers sheaves of fixed degree $d \neq 0$. Finally, stability conditions for a ribbon were analyzed in [Chen and Kass 2011]. On a ribbon, the stability condition is independent of $\mathscr{L}_0$, and for rational ribbons, slope stability coincides with slope semistability precisely when the genus g is even. It would be desirable to have a general result asserting (non)existence of a suitable $\mathscr{L}_0$.

4.3. *Genus* 1 *curves*. The Néron model of the Jacobian of a genus 1 curve can be quite complicated (see for example [Liu et al. 2004]), but these complications do not arise if the family admits a section. Suppose B is a Dedekind scheme and $f : X \to B$ is a family of curves such that the total space X is regular and the generic fiber X_η is smooth. If $\sigma : B \to X^{\text{sm}}$ is a section contained in the smooth locus, then there is a canonical identification of the smooth locus X^{sm} with the Néron model N of the Jacobian of X_η. Here we examine how this fact fits into the preceding framework.

Definition 4.7. Let $f : X \to B$ be a family of genus 1 curves over a Dedekind scheme and $\sigma : B \to X^{\text{sm}}$ a section contained in the smooth locus. We define a sheaf $\mathscr{I}_{\text{uni.}}$ on $X \times_B X$ by the formula

$$\mathscr{I}_{\text{uni.}} := \mathscr{I}_\Delta(\pi_1^*(\sigma) + \pi_2^*(\sigma)). \tag{4-2}$$

Here $\mathscr{I}_\Delta$ is the ideal sheaf of the diagonal, and $\pi_1, \pi_2 : X \times_B X \to X$ are the projection maps.

The sheaf $\mathscr{I}_{\text{uni.}}$ determines a transformation $X \to \text{Sh}$ that realizes X as a moduli space of sheaves over itself. Proposition 3.6 and Theorem 3.9 apply to this moduli space.

Corollary 4.8. *Fix a Dedekind scheme B. Let $f : X \to B$ be a family of genus* 1 *curves. Let $\sigma : B \to X^{\text{sm}}$ be a section.*

Then the natural map $X^{\text{sm}} \to \text{Q}$ is an open immersion.

Assume further that f satisfies Hypothesis 1. Then $X^{\text{sm}} = \text{Q}^\tau$, and this scheme is the Néron model.

Let us consider the special case where B is a discrete valuation ring, X is a minimal regular surface, and the residue field $k(0)$ is algebraically closed. The possibilities for the special fiber X_0 are given by the Kodaria–Néron classification ([Kodaira 1960; Néron 1964]; see [Silverman 1994, pp. 353–354] for a recent exposition). The reduced curves appearing in the classification are the reduction types I_n, II, III, and IV. In these cases, one may show that the induced morphism $X \to \text{Sh}$ identifies X with the Esteves compactified Jacobian $\bar{J}^{\sigma}_{\mathscr{O}}$.

In every remaining case (reduction type I_n^*, II^*, III^*, or IV^*) the morphism $X \to \text{Sh}$ is not a special case of the fine moduli spaces discussed in the previous two sections. Indeed, the special fiber X_0 is nonreduced, so the Esteves Jacobian of X is not defined. In Section 4.2, we reviewed Simpson's moduli space $\text{M}^{\text{st}}(\mathscr{O}_X, P_d)$ of stable sheaves, but the image of $X \to \mathrm{Sh}$ cannot be described as a closed subscheme of that space. The reason is that slope stable sheaves are simple, but some fibers of $\mathcal{I}_{\text{uni.}}$ are not simple. Specifically, if $p_0 \in X_0$ lies on the intersection of two components, then the fiber of $\mathcal{I}_{\text{uni}}$ of p_0 fails to be simple. This can be seen as follows. This fiber is the sheaf $\mathcal{I}_{p_0}(+\sigma(0))$, where $\mathcal{I}_{p_0}$ is the ideal of p_0. If $\nu : X_0' \to X_0$ is the blow-up of X_0 at p_0, then one may show that $H^0(X_0', \mathcal{O}_{X_0'})$ is canonically isomorphic to the endomorphism ring of $\mathcal{I}_{p_0}(+\sigma(0))$. An inspection of the Kodaria–Néron table shows that X_0' is disconnected, so $H^0(X_0', \mathcal{O}_{X_0'})$ does not equal $k(0)$ and $\mathcal{I}_{p_0}(+\sigma(0))$ is not simple.

Corollary 4.8 provides a partial answer to a question posed in the introduction: What are the maximal subfunctors J of P^0 represented by a separated B-scheme? When X is, say, regular, a strong result one could hope for is that there is always a subfunctor $\bar{J}$ of Sh satisfying the hypotheses of Theorem 3.9. The line bundle locus $J \subset \bar{J}$ in such a functor has the property that $J \to \mathrm{Q}^{\tau}$ is an isomorphism, and hence J is maximal. Corollary 4.8 shows that such a $\bar{J}$ exists when $f : X \to S$ is a family of genus 1 curves that f admits a section. Similarly, the Esteves compactified Jacobian represents a suitable subfunctor when f has geometrically reduced fibers and admits a section. In general, however, the hope is too optimistic: Raynaud's family, mentioned at the end of Section 3, has that property that no such $\bar{J}$ can exist.

The question of describing maximal subfunctors J is most interesting when f has nonreduced fibers. The slope stable line bundles form a subfunctor $J \subset \mathrm{P}^0$ represented by a S-separated scheme, but our discussion of genus 1 families together with Remark 4.4 suggest that we should consider other methods for constructing a suitable J when f has nonreduced fibers.

In a different direction, one nice property of the moduli spaces described by Corollary 4.8 is that their geometry is very simple. We use these spaces to provide an example showing that a family $J \to B$ of Esteves Jacobians over a regular 2-dimensional base may not have group scheme structure.

Example 4.9 (Néron models in 2-dimensional families). We will construct a 2-dimensional family $f : X \to B$ of plane cubics and an associated Esteves Jacobian $J \to B$ with the property that the group law on the locus $J_U \to U$ parametrizing nonsingular cubics does not extend over all of B. Furthermore, the family is constructed so that a dense open subset of B is covered by nonsingular curves C with the property that the restriction X_C of X to C is regular, so J_C is the Néron model of its generic fiber (and in particular admits group scheme structure that extends the group scheme structure over $C \cap U$). Thus, the Néron models fit into a 2-dimensional family, but their group scheme structure does not.

The idea is as follows. The family we construct has a reducible element $X_{b_0} \to b_0$ with the property that, for every nonsingular curve $C \subset B$ passing through b_0

such that X_C is regular, the restriction of the Esteves Jacobian J_C is the Néron model of its generic fiber. The fiber J_{b_0} inherits a group law from this Néron model, and we show explicitly that this group law depends on the particular choice of C. But, if the group law on J_U extended to J, then all the different group laws on J_{b_0} coming from the different curves C would be the restriction of one common group law on J, which is absurd. We now construct the family.

We work over an algebraically closed field k. The family $X \to B$ will be a net of plane cubics. Let $X_0 \subset \mathbb{P}^2_k$ be a reducible plane cubics that is the union of a smooth quadric Q_0 and a line L_0 that meet in two distinct points. (See Figure 1.) Fix two general points $p_1, p_2 \in L_0(k)$ on the line and one general point $q_1 \in Q_0(k)$ on the quadric. Say that $F \in H^0(\mathbb{P}^2_k, \mathcal{O}(3))$ is an equation for X_0 and $G, H \in H^0(\mathbb{P}^2_k, \mathcal{O}(3))$ are two general cubic equations that vanish on all of the points p_1, p_2, q_1. We will work with the net $V := \langle F, G, H\rangle \subset H^0(\mathbb{P}^2_k, \mathcal{O}(3))$ and the associated family of curves

$$X := \{(p, [r, s, t]) : r \cdot F(p) + s \cdot G(p) + t \cdot H(p) = 0\} \subset \mathbb{P}^2_k \times \mathbb{P}^2_k. \tag{4-3}$$

There are two obvious morphisms $e, f : X \to \mathbb{P}^2_k$ given by the two projections. If we set $B := \mathbb{P}^2_k$ equal to the plane, then the second morphism $f : X \to B$ realizes X as a family of genus 1 curves with $X_0 = f^{-1}(b_0)$, where $b_0 := [1, 0, 0]$. Corresponding to the points $p_1, p_2, q_1 \in X_0(k)$ are three section $\sigma_1, \sigma_2, \tau_1 : B \to X^{\mathrm{sm}}$, which lie in the smooth locus by the generality assumption.

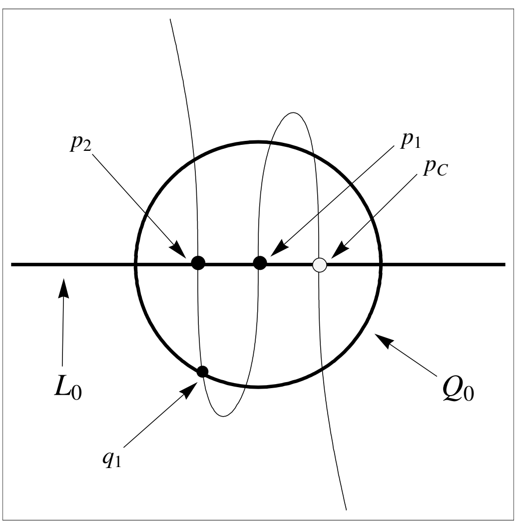

Figure 1. The pencil X_C.

Another application of the generality assumption shows that the fibers of f are reduced, so we can form the Esteves Jacobian $J := J^{\sigma_1}_{\mathscr{E}}$, where $\mathscr{E} = \mathbb{O}_X$. The quasistability condition on a line bundle $\mathscr{L}_0$ on X_0 is the condition that the bidegree $(\deg(\mathscr{L}_{L_0}), \deg(\mathscr{L}_{Q_0}))$ equals $(0, 0)$ or $(1, -1)$. Now we assume $J \to B$ is a group scheme and derive a contradiction.

Suppose that we are given a general line $C \subset B$ in the plane that contains b_0. Such a line corresponds to a 2-dimensional linear subspace of the form $W := \langle F, G_C \rangle \subset V$ for some $G_C \in V$. Invoking generality again, the base locus

$$\{p \in \mathbb{P}^2_k : F(p) = G_C(p) = 0\} \tag{4-4}$$

consists of 9 distinct points. The first projection map $e : X \to \mathbb{P}^2_k$ realizes X_C as the blow-up of the plane at these points, so X_C is regular, and thus J_C is the Néron model of its generic fiber. We now study the group of sections of $J_C \to C$.

The base locus (4-4) includes the points p_1, p_2, q_1. In addition to the points p_1, p_2, a unique third point of the base locus must lie on the line L_0. Let us label that point p_C and write $\sigma_C : C \to X_C$ for the corresponding section.

Now consider the following line bundles on X_C:

$$\mathscr{L}_1 := \mathbb{O}(\sigma_1 - \tau_1), \qquad \mathscr{L}_C := \mathbb{O}(\sigma_C - \tau_1),$$
$$\mathscr{L}_2 := \mathbb{O}(\sigma_2 - \tau_1), \qquad \mathscr{M} := \mathbb{O}(1) \otimes \mathbb{O}(-3 \cdot \tau_1).$$

These lines bundles are all σ_1-quasistable. If we let $g_1, g_2, g_C, h \in J_C(C)$ respectively correspond to $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_C, \mathscr{M}$, then I claim we have

$$g_1 + g_2 + g_C = h. \tag{4-5}$$

Indeed, it is enough to verify the claim after passing to the generic fiber of $J_C \to C$, where the equation is just the statement that the points p_1, p_2, p_C all lie on a line (the line L_0). Now suppose that $J \to \mathbb{P}^2_k$ admits a group law extending the group law of the generic fiber. Then the specialization of (4-5) to J_{b_0} holds for all C simultaneously. In particular, the isomorphism class of the line bundle $\mathbb{O}_{X_{b_0}}(p_C - q_1)$ is independent of the particular line $C \subset \mathbb{P}^2_k$ chosen. But this is absurd: For distinct general lines C_1, C_2, the points p_{C_1} and p_{C_2} (and hence the associated line bundles) are distinct! This completes our discussion of this example.

This example is particularly interesting in light of [Oda and Seshadri 1979]. The authors of that paper consider the case of a family of nodal curves $f : X \to B$ over a suitable Dedekind scheme with the property that X is regular. Let J_η be the Jacobian of the generic fiber. Given a closed point $0 \in B$, they prove that the special fiber N_0 of the Néron model of J_η depends only on the curve X_0 and not the particular family f [ibid., Corollary 14.4]. This result must be interpreted with care: In our example, the group law depends on a particular choice of family, but any two such group laws define isomorphic group schemes.

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