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Zeros of real irreducible characters of finite groups

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We prove that if all real-valued irreducible characters of a finite group G with Frobenius–Schur indicator 1 are nonzero at all 2-elements of G, then G has a normal Sylow 2-subgroup. This result generalizes the celebrated Ito–Michler theorem (for the prime 2 and real, absolutely irreducible, representations), as well as several recent results on nonvanishing elements of finite groups.

1. Introduction

Suppose that G is a finite group. Let $\mathrm{Irr}(G)$ be the set of the irreducible complex characters of G, and let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Write $\mathrm{Irr}_{\mathbb{F}}(G)$ for the set of those $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) \in \mathbb{F}$ for all $g \in G$. Hence $\mathrm{Irr}_{\mathbb{R}}(G)$ is the set of *real-valued* (or *real*) irreducible characters of G.

As shown in recent papers [Dolfi et al. 2008; Navarro et al. 2009; Navarro and Tiep 2010], several fundamental results on characters of finite groups admit a version in which $\mathrm{Irr}(G)$ is replaced by $\mathrm{Irr}_{\mathbb{F}}(G)$ for a suitable field $\mathbb{F}$. For instance, S. Dolfi, G. Navarro and P. H. Tiep proved in [Dolfi et al. 2008] that if all $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ have odd degree, then a Sylow 2-subgroup of G is normal in G (therefore, providing a strong version of the celebrated Ito–Michler theorem for the prime $p = 2$).

In this paper, we turn our attention to the *nonvanishing* elements of a finite group G. These elements, introduced by M. Isaacs, G. Navarro and T. R. Wolf in [Isaacs et al. 1999], are the $x \in G$ such that $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}(G)$. Since their definition, there has been an increasing interest in the set of the nonvanishing elements of finite groups. See for instance [Dolfi et al. 2009; Dolfi et al. 2010c; Dolfi et al. 2010d; Dolfi et al. 2010a; Dolfi et al. 2010b]. One of most relevant results in this area was obtained by S. Dolfi, E. Pacifici, L. Sanus and P. Spiga in [Dolfi et al. 2009], where they proved that if all the p-elements of a finite group G are nonvanishing, then G has a normal Sylow p-subgroup. Since characters of degree not divisible by p cannot vanish on any p-element (by an elementary

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argument involving roots of unity — see for instance Lemma 5.1), this result is again an extension of the Ito–Michler theorem.

Recall that the Frobenius–Schur indicator of $\chi \in \mathrm{Irr}(G)$ is 0 if χ is nonreal, ± 1 if χ is real; moreover it is 1 precisely when χ is afforded by a real representation of G.

Our main result in this paper is the following.

Theorem A. *Let G be a finite group. If $\chi(x) \neq 0$ for all real-valued irreducible characters χ of G with Frobenius–Schur indicator* 1 *and all* 2*-elements $x \in G$, then G has a normal Sylow* 2*-subgroup.*

Since odd degree characters do not vanish on 2-elements, Theorem A above provides at the same time a generalization of [Dolfi et al. 2008, Theorem A] and of the $p = 2$ case of [Dolfi et al. 2009, Theorem A]. As an immediate consequence of Theorem A, we obtain the following refinement of the Ito–Michler theorem for the prime 2 and real, absolutely irreducible, representations:

Theorem B. *Let G be a finite group. If $\chi(1)$ is odd for all real-valued irreducible characters χ of G with Frobenius–Schur indicator* 1*, then G has a normal Sylow* 2*-subgroup.*

A few remarks are in order here. First of all, the hypotheses of our Theorem A here are strictly more general than those of [Dolfi et al. 2008, Theorem A]. In Section 5 below, we will describe an interesting family of examples of groups G, having real irreducible characters of even degree, such that all 2-elements of G are nonvanishing. We also mention that in order to obtain the solvable part of Theorem A, we will prove a result guaranteeing the existence of real 2-defect zero characters, which might be of independent interest; see Theorem 2.4.

2. Regular orbits and characters of 2-defect zero

We will need the following result, showing that real characters are remarkably well-behaved across odd sections. As usual, if N is a normal subgroup of a group G and $\theta \in \mathrm{Irr}(N)$, we denote by $I_G(\theta)$ the inertia subgroup of θ in G and by $\mathrm{Irr}(G|\theta)$ the set of the irreducible characters of G that lie over θ. For brevity, we call $\chi \in \mathrm{Irr}(G)$ *strongly real* if the Frobenius–Schur indicator of χ equals 1, and let $\mathrm{Irr}_+(G)$ denote the set of all strongly real irreducible characters of G. Certainly, if $H \leq G$ and $\chi = \lambda^G \in \mathrm{Irr}(G)$ for some $\lambda \in \mathrm{Irr}_+(H)$, then $\chi \in \mathrm{Irr}_+(G)$.

Lemma 2.1. *Let G be a finite group and let $N \lhd G$ with G/N of odd order.*

(i) *If $\theta \in \mathrm{Irr}_{\mathbb{R}}(N)$, then there exists a unique $\chi \in \mathrm{Irr}_{\mathbb{R}}(G|\theta)$.*

(ii) *If $\theta \in \mathrm{Irr}_+(N)$, then there exists a unique $\chi \in \mathrm{Irr}_+(G|\theta)$.*

Proof. Part (i) is [Navarro and Tiep 2008, Corollary 2.2].

For (ii), let $T = I_G(\theta)$. Since $|T/N|$ is odd, θ extends to a real character λ of T by [Navarro and Tiep 2008, Lemma 2.1]. As $\lambda_N = \theta$ is strongly real, the same holds for λ. Now $\chi = \lambda^G$ is irreducible and strongly real. The uniqueness of χ follows from (i). $\square$

Lemma 2.2. *Let $G = N\langle j\rangle$ be a split extension of a normal subgroup N by a subgroup $\langle j\rangle$ of order 2. Suppose that $\alpha \in \mathrm{Irr}(N)$ is of odd degree, and that $\alpha^j = \bar{\alpha} \neq \alpha$. Then α^G is irreducible and strongly real.*

Proof. The irreducibility of α^G is obvious. Let α be afforded by a representation $\Phi : N \to \mathrm{GL}_n(\mathbb{C})$, so that $n = \alpha(1)$ is odd. Then the representations $\Phi^j : x \mapsto \Phi(jxj^{-1})$ and $\Phi^* : x \mapsto {}^t\Phi(x^{-1})$ afford the same character $\bar{\alpha}$, whence $\Phi(jxj^{-1}) = A\,{}^t\Phi(x^{-1})A^{-1}$ for some $A \in \mathrm{GL}_n(\mathbb{C})$. Conjugating by j once more, we see that $A \cdot {}^tA^{-1}$ commutes with $\Phi(x)$ for all $x \in N$. By Schur's lemma, ${}^tA = \kappa A$ for some $\kappa \in \mathbb{C}$. Transposing once more, we get $\kappa^2 = 1$. But $A \in \mathrm{GL}_n(\mathbb{C})$ and n is odd, so $\kappa = 1$, that is, $A = {}^tA$. Now we define $\Psi : G \to \mathrm{GL}_{2n}(\mathbb{C})$ by

$$\Psi(x) = \begin{pmatrix} \Phi(x) & 0 \\ 0 & A\,{}^t\Phi(x^{-1})A^{-1} \end{pmatrix}, \qquad \Psi(xj) = \Psi(x) \cdot \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

for all $x \in N$; in particular, $\Psi(xj) = \Psi(j)\Psi(jxj^{-1})$. It is straightforward to check that Ψ is a group homomorphism, and that ${}^t\Psi(g) \cdot M\Psi(g) = M$ for all $g \in G$ and with

$$M := \begin{pmatrix} 0 & A^{-1} \\ {}^tA^{-1} & 0 \end{pmatrix}.$$

Thus the $\mathbb{C}G$-module corresponding to Ψ supports a G-invariant symmetric bilinear form (with Gram matrix M) and affords the character α^G, whence α^G is strongly real. $\square$

Note that the examples with $(G, N, \alpha(1)) = (2\mathsf{S}_7, 2\mathsf{A}_7, 4)$ and with $(G, N, \alpha(1)) = (Q_8, C_4, 1)$ show that one cannot remove any of the assumptions of Lemma 2.2.

A character $\chi \in \mathrm{Irr}(G)$ is said to be of *p-defect zero* for a given prime p if p does not divide $|G|/\chi(1)$. By a fundamental result of R. Brauer [Isaacs 1976, Theorem 8.17], if $\chi \in \mathrm{Irr}(G)$ is a character of p-defect zero, then $\chi(g) = 0$ for every element $g \in G$ such that p divides the order $o(g)$ of g. Next we recall the following result of G. R. Robinson:

Lemma 2.3 [Robinson 1989, Remark 2, p. 254]. *Let G be a finite group and let $\chi \in \mathrm{Irr}(G)$ be a real character of 2-defect zero. Then χ is strongly real.*

Let G be a finite group and let U be a faithful G-module. We recall that a G-orbit $\{u^g \mid g \in G\}$ of G on U is a *regular orbit* if its cardinality is equal to $|G|$ or, equivalently, if $\mathbf{C}_G(u) = 1$.

Theorem 2.4. *Let G be a finite group. Assume that $\mathbf{O}_2(G) = 1$ and that G has a nilpotent normal 2-complement M. Let P be a Sylow 2-subgroup of G and assume that whenever U is a faithful $\mathbb{F}_q[P]$-module, P has a regular orbit on U, where q is a prime dividing $|M|$. Then there exists a strongly real irreducible character $\chi \in \mathrm{Irr}_+(G)$ of 2-defect zero.*

If P is an abelian 2-group, then P has a regular orbit in every faithful action on a module of coprime characteristic. In fact, this is an application of Brodkey's theorem [1963].

We observe, for completeness, that a 2-group P acting faithfully on a module U of characteristic $q \neq 2$ has no regular orbit on U only if q is either a Mersenne or Fermat prime, and P involves a section isomorphic to the dihedral group D_8. This follows from [Manz and Wolf 1993, Theorems 4.4 and 4.8], using Maschke's theorem and standard arguments for passing from irreducible to completely reducible modules.

Theorem 2.4 will be derived from the following result, whose somewhat more technical statement will be needed in the proof of Theorem A.

Theorem 2.5. *Let G be a finite group with a nontrivial Sylow 2-subgroup P. Assume that $\mathbf{O}_2(G) = 1$ and that G has a nilpotent normal 2-complement M. Assume in addition that, whenever U is a faithful $\mathbb{F}_q[P]$-module, P has a regular orbit on U, where q is a prime dividing $|M|$. Then there exist a character $\theta \in \mathrm{Irr}(M)$ and an element $z \in P$, such that $\theta^G \in \mathrm{Irr}(G)$ and $\theta^z = \bar{\theta}$.*

Proof. Let $P \in \mathrm{Syl}_2(G)$. Since $\mathbf{O}_2(G) = 1$, P acts faithfully on M. By coprimality, P acts faithfully on $M/\Phi(M)$, as well. So, by factoring out $\Phi(M)$, we can assume that

$$M = L_1 \times L_2 \times \cdots \times L_k,$$

where each L_i is an irreducible $\mathbb{F}_{q_i}[P]$-module for some prime $q_i \neq 2$. We define $W_i = \mathbf{C}_P(L_i)$ for any $i = 1, \ldots, k$. Observe that W_i is a normal subgroup of P for each i, and that $\bigcap_{i=1}^k W_i = 1$, since P acts faithfully on M.

Now, let $\mathcal{B}$ be a subset of $\{W_1, \ldots, W_k\}$ minimal such that

$$\bigcap_{W \in \mathcal{B}} W = 1.$$

We can assume that $\mathcal{B} = \{W_1, \ldots, W_n\}$ for some $n \leq k$. Thus P acts faithfully on $U = L_1 \times \cdots \times L_n$.

By assumption, for all $i \in \{1, \ldots, n\}$, there exists an element $u_i \in L_i$ such that $\mathbf{C}_P(u_i) = W_i$. So, if we set $u = (u_1, \ldots, u_n) \in U$, it follows that $\mathbf{C}_P(u) = 1$. Now, we consider the dual group $\hat{U} = \mathrm{Irr}(U)$. Since $|U|$ is odd, by [Isaacs 1976, Theorem 13.24], U and $\hat{U}$ are isomorphic as P-modules. Hence there exists $\mu \in \hat{U}$

such that $I_P(\mu) = 1$, where $I_P(\mu)$ is the inertia group of μ in P. Consider now, for $1 \leq j \leq n$, the subgroup

$$H_j = \bigcap_{\substack{1 \leq t \leq n \\ t \neq j}} W_t.$$

Note that H_j is a normal subgroup of P and that H_j is not contained in W_j, by the minimality of $\mathscr{B}$. Furthermore, $H_j \cap W_j = 1$ for each j. Now, let $z_j \in \mathbf{Z}(P) \cap H_j$ be an involution; such an element certainly exists, as H_j is a nontrivial normal subgroup of P. So, $\mathbf{C}_{L_j}(z_j)$ is a P-submodule of L_j and $\mathbf{C}_{L_j}(z_j) < L_j$ as $z_j \notin W_j$. As L_j is irreducible, it follows that $\mathbf{C}_{L_j}(z_j) = 1$. Hence z_j inverts every element of L_j; see, for instance, [Huppert 1998, Theorem 16.9(e)]. Moreover, as $z_j \in H_j$, z_j centralizes L_i for every $i \neq j$, $1 \leq i, j \leq n$.

Let $z = z_1 \cdots z_n$, and observe that z inverts every element of U. By the isomorphism of P-modules $U \cong \hat{U}$, then z inverts every irreducible character of U. In particular, $\mu^z = \mu^{-1} = \bar{\mu}$. Now, we can write $M = U \times N$, where N is P-invariant. Let $\theta = \mu \times 1_N \in \operatorname{Irr}(M)$. Then, we have $\theta^z = \overline{\theta}$ and $I_P(\theta) = I_P(\mu) = 1$. Thus, by Clifford theory $\theta^G \in \operatorname{Irr}(G)$ and the proof is complete. □

Proof of Theorem 2.4. Clearly, we may assume $P \neq 1$. So, by Theorem 2.5 there exists a character $\theta \in \operatorname{Irr}(M)$ such that $\chi = \theta^G \in \operatorname{Irr}(G)$ and an element $z \in P$ such that $\theta^z = \overline{\theta}$. Hence,

$$\overline{\chi} = \overline{\theta^G} = \overline{\theta}^G = (\theta^z)^G = \theta^G = \chi,$$

so $\chi \in \operatorname{Irr}_{\mathbb{R}}(G)$. Next, since $\chi(1) = |G:M|\,\theta(1) = |P|\,\theta(1)$, χ is a character of 2-defect zero of G. Hence χ is strongly real by Lemma 2.3. □

3. Proof of Theorem A

We will need the following deep result concerning the existence of suitable strongly real characters of almost simple groups. We state it here and prove it in Section 4.

Theorem 3.1. *Let S be any finite nonabelian simple group. For any H with $S \leq H \leq \operatorname{Aut}(S)$, there exist a character $\theta \in \operatorname{Irr}(S)$ and a 2-element $x \in S$, such that both the following conditions apply*:

(i) $\theta(x^\sigma) = 0$ *for all* $\sigma \in \operatorname{Aut}(S)$.

(ii) *There exists a subgroup J with $I_H(\theta) \leq J \leq H$ and a strongly real character $\alpha \in \operatorname{Irr}(J|\theta)$.*

We can now proceed to prove Theorem A, which we restate below.

Theorem 3.2. *Let G be a finite group and $P \in \operatorname{Syl}_2(G)$. Suppose that $\chi(x) \neq 0$ for all $\chi \in \operatorname{Irr}_+(G)$ and for all 2-elements $x \in G$. Then $P \lhd G$.*

Proof. Let G be a minimal counterexample to the statement; in particular, $P \neq 1$. Let $M \neq 1$ be a minimal normal subgroup of G.

1. Observe that $\mathrm{Irr}_+(G/M) \subseteq \mathrm{Irr}_+(G)$ and 2-elements of G/M lift to 2-elements of G. Hence, $PM \lhd G$ by minimality of G. If M_1 is another minimal normal subgroup of G with $M_1 \neq M$, then $G/M \times G/M_1$ has a normal Sylow 2-subgroup, as both G/M and G/M_1 do. Since $M \cap M_1 = 1$, G is isomorphic to a subgroup of $G/M \times G/M_1$ and hence G has a normal Sylow 2-subgroup, a contradiction. So, M is the only minimal normal subgroup of G.

2. Suppose first that 2 divides $|M|$. If M is solvable, then M is a 2-group and so $P = PM \lhd G$, a contradiction. Hence M is not solvable. Thus $M = S_1 \times \cdots \times S_t$, where $S_i \cong S$, a nonabelian simple group. Write $S := S_1$, $H := \mathbf{N}_G(S)$ and $C := \mathbf{C}_G(S)$. Thus, H/C is isomorphic to a subgroup $\overline{H}$ of $\mathrm{Aut}(S)$, with $S \leq \overline{H} \leq \mathrm{Aut}(S)$. By Theorem 3.1, there exists a character $\theta \in \mathrm{Irr}(S)$ and a 2-element $x \in S$ such that $\theta(x^\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(S)$. Moreover, there exists a subgroup J with $I_{\overline{H}}(\theta) \leq J \leq \overline{H}$ and a strongly real character $\alpha \in \mathrm{Irr}(J|\theta)$. By the Clifford correspondence [Isaacs 1976, Theorem 6.11], $\alpha = \lambda^J$ for a suitable character $\lambda \in \mathrm{Irr}(I_{\overline{H}}(\theta)|\theta)$. Therefore, $\beta := \lambda^{\overline{H}}$ is an irreducible character of $\overline{H}$. Furthermore, β is strongly real as $\beta = (\lambda^J)^{\overline{H}} = \alpha^{\overline{H}}$, and β lies over θ.

Let now $\psi := \theta \times 1_S \times \cdots \times 1_S \in \mathrm{Irr}(M)$. Note that $C \lhd I_G(\psi) \leq H$ and that $I_G(\psi)/C$ is isomorphic to $I_{\overline{H}}(\theta)$. Hence, by lifting characters from the corresponding factor groups modulo C, we can view $\lambda \in \mathrm{Irr}(I_G(\psi)|\psi)$ and $\lambda^H = \beta \in \mathrm{Irr}_+(H)$.

Define $\chi = \lambda^G$. By the Clifford correspondence, χ is an irreducible character of G and, since $\chi = \beta^G$, we have $\chi \in \mathrm{Irr}_+(G)$. We will show that χ vanishes on the 2-element $g = (x, x, \ldots, x) \in M$. In fact, χ lies over ψ and hence the restriction χ_M is a sum of conjugates ψ^y, with $y \in G$. Now, each conjugate ψ^y is of the form

$$\psi^y = 1_S \times \cdots \times 1_S \times \theta^\sigma \times 1_S \times \cdots \times 1_S,$$

for a suitable $\sigma \in \mathrm{Aut}(S)$. Thus $\psi^y(g) = \theta(x^{\sigma^{-1}}) = 0$ for all $y \in G$. It follows that $\chi(g) = 0$, against our assumptions.

3. We have shown that M is an elementary abelian q-group for some prime $q \neq 2$.

Let $Z := \Omega_1(\mathbf{Z}(P))$ so that $Z \neq 1$. Since $|M|$ is odd, $ZM/M = \Omega_1(\mathbf{Z}(PM/M))$ and so $ZM \lhd G$. Observe also that M is a normal nilpotent 2-complement of ZM and that Z is a Sylow 2-subgroup of ZM. Moreover, $\mathbf{O}_2(ZM) = 1$, as $\mathbf{O}_2(ZM)$ is normal in G and M is the unique minimal normal subgroup of G. Finally, since Z is abelian, Z has a regular orbit on every faithful Z-module of odd characteristic. Thus, by Theorem 2.5 there exist $\theta \in \mathrm{Irr}(M)$ and $z \in Z$, such that $\theta^z = \overline{\theta}$ and $\theta^{ZM} \in \mathrm{Irr}(ZM)$. Since $Z \neq 1$ and $q \neq 2$, we must have that $\overline{\theta} \neq \theta$ and $z \neq 1$; in fact z is an involution.

Let $T = I_G(\theta) \cap PM = I_{PM}(\theta)$. Since $q \neq 2$, θ has a *canonical extension* $\gamma \in \mathrm{Irr}(T)$; see [Isaacs 1976, Corollary 8.16]. By uniqueness of the canonical extension of $\overline{\theta}$, it follows that $\gamma^z = \overline{\gamma}$. Let $\delta = \gamma^{PM}$. By Clifford theory δ is irreducible. Since $\theta^z = \overline{\theta} \neq \theta$, we see that $z \notin T$ but z normalizes T. It follows that $K := T\langle j \rangle$ is a split extension of T by $\langle j \rangle$. We have already shown that $\gamma^z = \overline{\gamma}$. Also, $\gamma_M = \theta$ is nonreal. Hence γ is nonreal and has degree 1. Applying Lemma 2.2 to the character γ of T, we see that γ^K is strongly real. Consequently, $\delta = (\gamma^K)^{PM}$ is strongly real.

Recalling that PM is a normal subgroup of odd index in G, by Lemma 2.1(ii) there exists a character $\chi \in \mathrm{Irr}_+(G)$ that lies over δ.

Now, we show that $\delta(g) = 0$ for every $g \in ZM \smallsetminus M$. In fact, as $\theta^{ZM} \in \mathrm{Irr}(ZM)$, by Clifford theory $T \cap ZM = I_{ZM}(\theta) = M$. As both M and ZM are normal in G, we get that for all $x \in G$, $T^x \cap ZM = (T \cap ZM)^x = M^x = M$. So,

$$ZM \cap \Big(\bigcup_{x \in G} T^x\Big) = M.$$

As $\delta = \gamma^{PM}$ with $\gamma \in \mathrm{Irr}(T)$, the formula of character induction yields that $\delta(g) = 0$ for all $g \in ZM \smallsetminus M$.

Note now that, because $ZM > M$, there exists a 2-element $g_0 \in ZM \smallsetminus M$. Finally, observe that χ_{PM} is a sum of conjugates δ^y of δ in G and that $\delta^y(g_0) = \delta(g_0^{y^{-1}}) = 0$, since $g_0^{y^{-1}} \in ZM \smallsetminus M$ for all $y \in G$. Therefore, we conclude that $\chi(g_0) = 0$, with $\chi \in \mathrm{Irr}_+(G)$ and g_0 a 2-element of G, the final contradiction. □

4. Almost simple groups

This section is devoted to proving Theorem 3.1. First we handle some easy cases:

Lemma 4.1. *Theorem 3.1 holds if S is an alternating group, a sporadic simple group, or a simple group of Lie type in characteristic* 2.

Proof. The cases of A_5, A_6, and all the sporadic groups can be verified directly using [Conway et al. 1985]. Assume $S = \mathsf{A}_n$ with $n \geq 7$; in particular $\mathrm{Aut}(S) \cong \mathsf{S}_n$. As shown in [Dolfi et al. 2009, Proposition 2.4], one can find a character θ satisfying the conditions described in Theorem 3.1, which extends to a strongly real character α of $\mathrm{Aut}(S)$.

If $S = {}^2F_4(2)'$, then we can choose $J = H$ and $\theta \in \mathrm{Irr}(H)$ of degree 2048 if $H = S$ and of degree 4096 if $H = \mathrm{Aut}(S)$ (and $x \neq 1$ any 2-element in S); see [Conway et al. 1985]. The case $\mathrm{Sp}_4(2)' \cong \mathsf{A}_6$ has been considered above. For all other simple groups of Lie type in characteristic 2, we choose θ to be the Steinberg character St and $1 \neq x \in S$ to be any 2-element: it is well-known [Feit 1993] that St vanishes at any 2-singular element and extends to the character of a rational representation of H. □

Lemma 4.2. *Let G be a finite group with a normal subgroup N, and $\chi \in \mathrm{Irr}(G)$. Then χ_N is irreducible (over N) if and only if the characters $\chi\alpha$, where $\alpha \in \mathrm{Irr}(G/N)$, are all irreducible and pairwise distinct. Moreover, in this case the irreducible characters of G that lie above χ_N are precisely the characters $\chi\alpha$, where $\alpha \in \mathrm{Irr}(G/N)$.*

Proof. The "only if" direction is Gallagher's theorem [Isaacs 1976, Theorem 6.17]. For the "if" direction, observe that the hypothesis implies

$$(\chi_N)^G = \chi \cdot (1_N)^G = \sum_{\alpha \in \mathrm{Irr}(G/N)} \alpha(1)\chi\alpha$$

contains χ with multiplicity 1, and so $[\chi_N, \chi_N]_N = [\chi, (\chi_N)^G]_G = 1$, as stated. □

In the rest of this section, let S be a simple group of Lie type in characteristic $p > 2$. We can find a simple algebraic group $\mathcal{G}$ of adjoint type defined over a field of characteristic p and a Frobenius morphism $F : \mathcal{G} \to \mathcal{G}$ such that $S = [G, G]$ for $G := \mathcal{G}^F$. We refer to [Carter 1985; Digne and Michel 1991] for basic facts on the Deligne–Lusztig theory of complex representations of finite groups of Lie type. In particular, irreducible characters of G are partitioned into (rational) Lusztig series that are labeled by conjugacy classes of semisimple elements s in the dual group L, where the pair $(\mathcal{L}, F^*)$ is dual to $(\mathcal{G}, F)$ and $L = \mathcal{L}^{F^*}$. Since $\mathcal{L}$ is simply connected, $\mathbf{C}_{\mathcal{L}}(s)$ is connected for any semisimple element $s \in \mathcal{L}$; see [Carter 1985, Theorem 3.5.6]. Hence the L-conjugacy class s^L corresponds to a (unique) irreducible (semisimple) character χ_s of G of degree $[L : \mathbf{C}_L(s)]_{p'}$; see [Digne and Michel 1991, §14]. Since χ_s belongs to the Lusztig series defined by s^L, two semisimple characters χ_s and χ_t are equal precisely when s and t are conjugate in L.

The structure of $\mathrm{Aut}(S)$ is described in [Gorenstein et al. 1994, Theorem 2.5.12]; in particular, it is a split extension of G by an abelian group (of field and graph automorphisms), which we denote by $A(S)$.

Our arguments will rely on the following proposition, which is of independent interest:

Proposition 4.3. *In the notation above, assume that $s \in L$ is a semisimple element of order coprime to $|\mathbf{Z}(L)|$. Then the following statements hold.*

(i) *If s is real in L then χ_s is strongly real.*

(ii) *Let σ be a bijective morphism of the algebraic group $\mathcal{G}$ commuting with F and let σ^* be the corresponding morphism of $\mathcal{L}$. Assume that χ_s is σ-invariant. Then s and $\sigma^*(s)$ are L-conjugate. Moreover, if σ is a Frobenius morphism, then s is $\mathcal{L}$-conjugate to some element in $\mathcal{L}^{\sigma^*}$; in particular, $|s|$ divides $|\mathcal{L}^{\sigma^*}|$.*

(iii) *$\theta := (\chi_s)_S$ is irreducible (over S).*

(iv) *Let σ be a bijective morphism of $\mathcal{G}$ commuting with F (and so fixing G and S). Suppose that σ fixes θ. Then σ fixes χ_s.*

Proof. Part (ii) and the statement about χ_s being real in (i) can be proved exactly in the same way as [Dolfi et al. 2008, Lemma 2.5] using [Navarro et al. 2008, Corollary 2.5]. Assume now that s is real. Since $\mathcal{G}$ is of adjoint type, $\mathbf{Z}(\mathcal{G}) = 1$; in particular, it is connected, and $\mathbf{Z}(G) = \mathbf{Z}(\mathcal{G})^F = 1$ by [Carter 1985, Proposition 3.6.8]. Hence, by [Vinroot 2010, Theorem 4.2], the Frobenius–Schur indicator of χ_s is 1, as stated in (i).

For (iii), by [Digne and Michel 1991, Proposition 13.30] and its proof, the characters $\alpha \in \mathrm{Irr}(G/S)$ are exactly the semisimple characters χ_z with $z \in \mathbf{Z}(L)$; moreover, $\chi_{sz} = \chi_s\chi_z$. Observe that sz and st are not L-conjugate if $z, t \in \mathbf{Z}(L)$ are distinct. (Indeed, suppose $g(sz)g^{-1} = st$ for some $g \in L$. Then since $|s|$ is coprime to $|\mathbf{Z}(L)|$, we have

$$|s| = |gsg^{-1}| = |s \cdot (tz^{-1})| = |s| \cdot |tz^{-1}|,$$

and so $z = t$.) It follows that the characters χ_{sz} are all irreducible and pairwise distinct. By Lemma 4.2, $\theta = (\chi_s)_S$ is irreducible, and $\mathrm{Irr}(G|\theta) = \{\chi_{sz} \mid z \in \mathbf{Z}(L)\}$.

Suppose now that σ fixes θ as in (iv). Since σ fixes G, it now fixes $\mathrm{Irr}(G|\theta)$ and so it sends χ_s to χ_{sz} for some $z \in \mathbf{Z}(L)$. Let σ^* be the morphism of $\mathcal{L}$ corresponding to σ. By [Navarro et al. 2008, Corollary 2.5], sz and $\sigma^*(s)$ are L-conjugate. In particular, $|s| = |\sigma^*(s)| = |sz| = |s| \cdot |z|$, and so $z = 1$ as stated. □

Proposition 4.4. *Theorem 3.1 holds if S is one of the following simple groups in characteristic $p > 2$: $G_2(q)$, ${}^2G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, or $E_8(q)$, where $q = p^f$.*

Proof. Notice that in each of these cases, $\mathrm{Out}(S) = A(S)$ is cyclic, of order $2f$ if $S = G_2(q)$ and $p = 3$, of order $3f$ if $S = {}^3D_4(q)$, and of order f otherwise; see for instance [Gorenstein et al. 1994, Theorem 2.5.12]. Furthermore, $S = G \cong L$; see [Carter 1985, p. 120]. Choose the integer m to be 6, 12, 12, or 30, if $S = {}^2G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, or $E_8(q)$, respectively. If $S = G_2(q)$, we choose $m = 3$ if $q = 3^f$ with f odd, and $m = 6$ otherwise. By [Zsigmondy 1892], there exists a *primitive prime divisor* (p.p.d.) $r = r(p, mf)$ of $p^{mf} - 1$, that is, a prime divisor of $p^{mf} - 1$ that does not divide $\prod_{i=1}^{mf-1}(p^i - 1)$. According to [Moretó and Tiep 2008, Lemma 2.3], L contains a semisimple element s of order r for which $\mathbf{C}_L(s)$ is a torus of order dividing $\Phi_m(q)$ if $\Phi_m(t)$ is the m-th cyclotomic polynomial in t; in particular, s is regular. It is well-known [Tiep and Zalesski 2005, Proposition 3.1] that every semisimple element $s \in L$ is real. It then follows by Proposition 4.3(i) that $\theta := \chi_s$ is strongly real.

We claim that χ_s is not stable under any nontrivial outer automorphism σ of S. Indeed, since $\mathrm{Aut}(S)$ is a split extension of S by the cyclic group $\mathrm{Out}(S)$ in the cases under consideration [Gorenstein et al. 1994, Theorem 2.5.12] we can choose σ to be a power σ_0^e of some canonical outer automorphism σ_0 of S.

(a) If $S = {}^2G_2(q)$, $G_2(q)$ (with $p \neq 3$), $F_4(q)$, or $E_8(q)$, then σ_0 is induced by the field automorphism $x \mapsto x^p$, and so we can choose e such that $1 \le e < f$ and $e \mid f$. In this case, $|\mathscr{L}^{\sigma^*}|$ is equal to the order of $|L|$, but with q replaced by p^e, and hence is not divisible by r by the choice of r.

(b) Now suppose that $S = G_2(q)$ and $p = 3$. Then $1 \le e < 2f$, $e \mid 2f$, and $|\mathscr{L}^{\sigma^*}|$ equals $|G_2(p^{e/2})|$ if $2 \mid e$ and $|{}^2G_2(p^e)|$ if e is odd. In either case, r is coprime to $|\mathscr{L}^{\sigma^*}|$ by the choice of r.

(c) Finally, consider the case $S = {}^3D_4(q)$. Then we can choose e so that $1 \le e < 3f$ and $e \mid 3f$. Now $|\mathscr{L}^{\sigma^*}|$ equals $|D_4(p^e)|$ and so r is coprime to $|\mathscr{L}^{\sigma^*}|$ by the choice of r.

We have shown that $I_{\mathrm{Aut}(S)}(\chi_s) = S$, whence $I_H(\chi_s) = S$. Since $\chi_s(1) = [L : \mathbf{C}_L(s)]_{p'}$ and $|\mathbf{C}_L(s)|$, being a divisor of $\Phi_m(q)$, is odd, we see that χ_s has 2-defect zero and so χ_s vanishes at any 2-element $1 \neq x \in S$. Hence we are done by taking $J = S$ and $\alpha = \chi_s$. □

Proposition 4.5. *Theorem 3.1 holds if S is any of the following simple groups of Lie type in characteristic $p > 2$: $\mathrm{PSL}_2(q)$ with $q \ge 5$; $\mathrm{PSp}_{2n}(q)$ with $n \ge 2$; $\Omega_{2n+1}(q)$ with $n \ge 3$; $\mathrm{P}\Omega^{\epsilon}_{2n}(q)$ with $2 \mid n$, $n \ge 6$ for $\epsilon = +$, and $n \ge 4$ for $\epsilon = -$; or $E_7(q)$.*

Proof. **1.** Recall that $L = \mathrm{SL}_2(q)$, respectively $\mathrm{Spin}_{2n+1}(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{Spin}^{\epsilon}_{2n}(q)$, or $E_7(q)_{sc}$ in the described cases; in particular, $\mathbf{Z}(L)$ is a 2-group. We write $q = p^f$ as usual. By [Tiep and Zalesski 2005, Proposition 3.1], any semisimple element in L is real. Now we choose a semisimple element $s \in L$ of (odd) order r, where r is selected as follows.

(i) If $L = \mathrm{SL}_2(q)$, then $r = (q+\epsilon)/2$ if $\epsilon = \pm 1$ is chosen so that $q \equiv \epsilon \pmod 4$.

(ii) Next, $r = r(p, 2nf)$ is a p.p.d. of $p^{2nf} - 1$ in the other classical cases, unless $L = \mathrm{Spin}^+_{2n}(q)$.

(iii) In the case $L = E_7(q)$, $r = r(p, 18f)$.

(iv) In the remaining case, $L = \mathrm{Spin}^+_{2n}(q)$ contains a central product

$$C = \mathrm{Spin}^-_4(q) * \mathrm{Spin}^-_{2n-4}(q),$$

and we choose $s = s_1 s_2 \in C$ where $s_1 \in \mathrm{Spin}^-_4(q) \cong \mathrm{SL}_2(q^2)$ has order $(q^2+1)/2$ and $s_2 \in \mathrm{Spin}^-_{2n-4}(q)$ has order $(q^{n-2}+1)/2$. More precisely, if β and γ denote some elements in $\bar{\mathbb{F}}_q^\times$ of orders $(q^2+1)/2$ and $(q^{n-2}+1)/2$, respectively, then we can choose s to act on the natural $\mathscr{L}$-module $\bar{\mathbb{F}}_q^{2n}$ with spectrum $\{\beta^i, \beta^{-i} \mid i = 1, q\} \cup \{\gamma^{q^j}, \gamma^{-q^j} \mid 0 \le j \le n-3\}$.

In these cases, it is straightforward to check (see for instance [Moretó and Tiep 2008, Lemmas 2.3 and 2.4]) that s is a regular semisimple element, and $T^* := \mathbf{C}_L(s)$ is a maximal torus of order $q+\epsilon$, q^n+1, $\Phi_2(q)\Phi_{18}(q)$, or $(q^2+1)(q^{n-2}+1)$, respectively. Hence, χ_s is a strongly real irreducible character of G, and in fact $\chi_s = \pm R^G_{T,\vartheta}$ is a Deligne–Lusztig character corresponding to some maximal torus T of G in duality with T^*; in particular, $|T| = |T^*|$. (Indeed, since $\mathcal{T}^* := \mathbf{C}_{\mathcal{L}}(s)$ is a torus, this is the unique torus containing s. Now if $(\mathcal{T}, \vartheta)$ is in duality with $(\mathcal{T}^*, s)$, then $T = \mathcal{T}^F$ and $\chi_s = \pm R^G_{T,\vartheta}$.) Now the character formula [Carter 1985, Theorem 7.2.8] shows that $\chi_s(x) = 0$ for any semisimple element $x \in G$ with $|\mathbf{C}_G(x)|$ *not* divisible by $|T|$.

2. By Proposition 4.3(iii), $\theta := (\chi_s)_S$ is irreducible and strongly real. Furthermore, when $S = \mathrm{PSL}_2(q)$, we have $\theta(1) = q - \epsilon$ and so θ has 2-defect 0, whence it vanishes at any nontrivial 2-element $x \in S$. In the remaining cases, we will find an involution $x \in S$ such that $|\mathbf{C}_G(x)|$ is not divisible by $|T|$. If $S = \mathrm{PSp}_{2n}(q)$, choose x to be an involution with centralizer of type $\mathrm{Sp}_2(q) \times \mathrm{Sp}_{2n-2}(q)$ (in $\mathrm{Sp}_{2n}(q)$). If $S = \Omega_{2n+1}(q)$, choose x to be an involution with centralizer of type $\mathrm{GO}_4^+(q) \times \mathrm{GO}_{2n-3}(q)$ (in $\mathrm{GO}_{2n+1}(q)$). For $S = \mathrm{P}\Omega^\epsilon_{2n}(q)$, choose x to be an involution with centralizer of type $\mathrm{GO}_4^+(q) \times \mathrm{GO}^\epsilon_{2n-4}(q)$ (in $\mathrm{GO}^\epsilon_{2n}(q)$). Finally, for S of type $E_7(q)$, choose x to be an involution with centralizer of type $\mathrm{SL}_2(q) * \mathrm{Spin}_{16}(q)$; see [Gorenstein et al. 1994, Table 4.5.1]. It is straightforward to check that $|\mathbf{C}_G(x)|$ is not divisible by $|T|$ for the chosen element x. Then for any $\sigma \in \mathrm{Aut}(S)$, $|\mathbf{C}_G(x^\sigma)| = |\mathbf{C}_G(x)|$ (as $G \lhd \mathrm{Aut}(S)$), whence $\theta(x^\sigma) = 0$.

3. Next we show that any automorphism $\sigma \in \mathrm{Aut}(S)$ that fixes θ must belong to G. Since $\mathrm{Aut}(S) = G : A(S)$ and G fixes $\theta = (\chi_s)_S$, we may assume $\sigma \in A(S)$. Recall that $|A(S)| = 2f$ if $S = \mathrm{P}\Omega^\epsilon_{2n}(q)$ and $|A(S)| = f$ otherwise. Let $\sigma_0 \in A(S)$ denote the automorphism of S (and of G, $\mathcal{G}$) induced by the field automorphism $y \mapsto y^p$. If $S = \mathrm{P}\Omega^\epsilon_{2n}(q)$, we denote by $\tau \in A(S)$ the nontrivial graph automorphism commuting with F (otherwise set $\tau = 1_S$). Notice that $G = \mathcal{G}^F$ with $F = \sigma_0^f$, unless $S = \mathrm{P}\Omega^-_{2n}(q)$ in which case $F = \tau\sigma_0^f$. Then $A(S)$ is generated by σ_0 and τ. It follows that σ can be extended to a Frobenius morphism of $\mathcal{G}$, which commutes with F, unless $\sigma = \tau$ and $S = \mathrm{P}\Omega^+_{2n}(q)$. Replacing σ by $\tau\sigma_0^f$ in the latter case, we again see that σ extends to a Frobenius morphism of $\mathcal{G}$ that commutes with F. Since σ fixes θ, σ fixes χ_s by Proposition 4.3(iv), which in turn implies that $|s|$ divides $|\mathcal{L}^{\sigma^*}|$ by Proposition 4.3(ii).

3a. First consider the case $\sigma = \sigma_0^e$. Then $|\mathcal{L}^{\sigma^*}|$ equals the order of L but with q replaced by p^e; denote it by $|L(p^e)|$. Suppose $S = \mathrm{PSL}_2(q)$; in particular $A(S) = \langle\sigma_0\rangle \cong C_f$ and so we may choose $e \mid f$. If $q \equiv 1 \pmod 4$, we get $|s| = (p^f+1)/2$ divides $p^{2e}-1$, which is possible only when $e = f$. If $q \equiv -1 \pmod 4$, then f is odd; hence $|s| = (p^f-1)/2$ can divide $p^{2e}-1$ only when

$e = f$. Next suppose $S = \mathrm{PSp}_{2n}(q)$ or $\Omega_{2n+1}(q)$. Then $|s| = r(p, 2nf)$ can divide $|L(p^e)|$ only when $f \mid e$. If S is of type $E_7(q)$, then $|s| = r(p, 18f)$ can divide $|L(p^e)|$ only when $f \mid e$. In all these cases, $A(S) = \langle\sigma_0\rangle \cong C_f$, so we conclude that $\sigma_S = 1_S$. Consider the case $S = \mathrm{P}\Omega^-_{2n}(q)$. Since $|s| = r(p, 2nf)$ divides $|L(p^e)|$, we get $f \mid e$. Recall that $A(S) = \langle\sigma_0\rangle \cong C_{2f}$ for $S = \mathrm{P}\Omega^-_{2n}(q)$, so we may assume $e \mid (2f)$. Now if $2f \mid e$ then $\sigma_S = 1_S$. On the other hand, if $f = e$, then $|s| = r(p, 2nf)$ does not divide $|L(p^e)|$.

Assume now that $S = \mathrm{P}\Omega^+_{2n}(q)$. Since the order of $(\sigma_0)_S$ is f, we may assume that $0 \le e \le f/2$. Observe that $|s|$ is divisible by some p.p.d. $r_1 = r(p, (2n-4)f)$. Now since r_1 divides $|L(p^e)|$, we get $(2n-4)f \mid je$ for some j, $1 \le j \le 2n-2$. But then $je \le (n-1)f < (2n-4)f$, so $e = 0$ and $\sigma_S = 1_S$.

3b. It remains to consider the case σ is not contained in $\langle\sigma_0\rangle$. This can happen only when $S = \mathrm{P}\Omega^+_{2n}(q)$. Since $(\sigma_0)^f$ acts trivially on S, we can write $\sigma = \tau\sigma_0^e$ with $1 \le e \le f$. Moreover, replacing σ by $\sigma^{-1} = \tau\sigma_0^{f-e}$ (while acting on S), we may in fact assume that $1 \le e \le f/2$ or $e = f$. Now $r_1 = r(p, (2n-4)f)$ divides $|s|$ and $|s|$ divides $|\mathcal{L}^{\sigma^*}|$, and so we get $(2n-4)f \mid 2je$ for some j with $1 \le j \le n$. It follows that $e \ge (n-2)f/n > f/2$ as $n \ge 6$. We have shown that $e = f$, and so by Proposition 4.3(ii), s is $\mathcal{L}$-conjugate to some element $s' \in \mathcal{L}^{\sigma^*} = \mathrm{Spin}^-_{2n}(q)$. Certainly, $|s'| = |s|$ is divisible by $r_1 = r(p, (2n-4)f)$. Observe that the r_1-part of s' has centralizer of type $\mathrm{GO}^+_4(q) \times \mathrm{GO}^-_{2n-4}(q)$ (in $\mathrm{GO}^-_{2n}(q)$). Hence, the action of s' on the natural $\mathcal{L}$-module $V = \bar{\mathbb{F}}_q^{2n}$ is induced by $\mathrm{diag}(A, B)$ with $A \in \mathrm{GO}^+_4(q)$ and $B \in \mathrm{GO}^-_{2n-4}(q)$. But in this case, the spectrum of s' and s on V cannot have the shape indicated in (iv) above.

We have shown that $I_{\mathrm{Aut}(S)}(\theta) = G$; in particular, if $S \le H \le \mathrm{Aut}(S)$, we have $I_H(\theta) = G \cap H$. Choosing $J = G \cap H$ and $\alpha = (\chi_s)_J$, we are done. □

Lemma 4.6. *Let $\mathcal{L}$ be a simple simply connected algebraic group of type A_n with $n \ge 2$, D_n with $n \ge 3$ odd, or E_6, $F : \mathcal{L} \to \mathcal{L}$ a Frobenius morphism, and let $L := \mathcal{L}^F$. Let $\varphi \in \mathrm{Aut}(L)$ be a (nontrivial) graph automorphism of L (modulo inner-diagonal automorphisms). Then $\varphi(s)$ and s^{-1} are conjugate in L for any semisimple element $s \in L$.*

Proof. It is well-known [Steinberg 1968, §10] that such an automorphism φ lifts to an automorphism $\varphi = \psi\tau$ of $\mathcal{L}$, where ψ is inner: $\psi(x) = gxg^{-1}$ for some $g \in \mathcal{L}$, and τ acts as the inversion $t \mapsto t^{-1}$ on some maximal torus $\mathcal{T}$ of $\mathcal{L}$. Since s is semisimple, $s = hth^{-1}$ for some $t \in \mathcal{T}$ and $h \in \mathcal{L}$. Thus

$$\varphi(s) = \psi\tau(hth^{-1}) = g\tau(h)t^{-1}\tau(h)^{-1}g^{-1} = zs^{-1}z^{-1},$$

where $z := g\tau(h)h^{-1} \in \mathcal{L}$. Since s and $\varphi(s)$ are F-stable, we see $z^{-1}F(z) \in \mathbf{C}_{\mathcal{L}}(s)$. But $\mathcal{L}$ is simply connected; hence $\mathbf{C}_{\mathcal{L}}(s)$ is connected and F-stable. Therefore, by the Lang–Steinberg theorem, there is $c \in \mathbf{C}_{\mathcal{L}}(s)$ such that $z^{-1}F(z) = c^{-1}F(c)$, that is, $u := zc^{-1} \in L$. It follows that $\varphi(s) = zs^{-1}z^{-1} = us^{-1}u^{-1}$ is L-conjugate to s^{-1}. $\square$

Proposition 4.7. *Theorem 3.1 holds if S is any of the following simple groups of Lie type in characteristic $p > 2$: $\mathrm{PSL}_n(q)$ with $n \geq 3$, $\mathrm{PSU}_n(q)$ with $n \geq 5$ odd, $\mathrm{P}\Omega^{\epsilon}_{2n}(q)$ with $n \geq 5$ odd, $E_6(q)$, or ${}^2E_6(q)$.*

Proof. Recall that $L = \mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$, $\mathrm{Spin}^{\epsilon}_{2n}(q)$, $E_6(q)_{sc}$, or ${}^2E_6(q)_{sc}$ in the described cases, respectively, and we write $q = p^f$ as usual. In all these cases, S, G, and L have an outer automorphism that lifts to an involutive graph automorphism τ of $\mathscr{L}$ mentioned in the proof of Lemma 4.6. In particular, $\tau(X) = {}^tX^{-1}$ in the SL and SU cases, and τ acts on $S = \mathrm{P}\Omega^{\epsilon}_{2n}(q)$ as a conjugation by some element $X \in \mathrm{GO}^{\epsilon}_{2n}(q) \smallsetminus \mathrm{SO}^{\epsilon}_{2n}(q)$. Also recall that $G \lhd \mathrm{Aut}(S)$ and τG generates a subgroup of order 2 in $\mathrm{Aut}(S)/G \cong A(S)$. Our proof will be divided in two cases according to whether the subgroup GH of $\mathrm{Aut}(S)$ contains $\langle G, \tau\rangle$ or not. In the former case, we will choose $\theta = \chi_S$ with $\chi \in \mathrm{Irr}(G)$ being nonreal and use τ to produce a real character for some subgroup $J > I_H(\theta)$. In the latter case, we choose $\theta = \chi_S$ with $\chi \in \mathrm{Irr}(G)$ being real and with $I_H(\theta) \leq G$. In fact, we also consider $\mathrm{PSU}_n(q)$ with $n \geq 4$ even in all parts of this proof, except in part 6 below. Moreover, even though the case of $\mathrm{PSU}_n(q)$ with $n \geq 5$ odd is also handled in Proposition 4.8 (below) using a different method, we also treat it here, since the character χ constructed here in this case will be used in some of our other works.

Case I (GH does not contain $\langle G, \tau\rangle$). **1.** We will construct $\chi \in \mathrm{Irr}(G)$, $\theta = \chi_S$, and $x \in S$ as follows.

Case Ia. Suppose $S = \mathrm{PSL}_3(q)$ and f is odd. Then $A(S) \cong C_{2f}$ contains a unique involution τ and $|H/(H \cap G)|$, $|(H \cap G)/S|$ are odd. In this case, we choose χ to be the unipotent (Weil) character of G of degree $q(q+1)$ and $x \in S$ to be *any* element of order $(q^2-1)_2$. Note that $\chi + 1_G$ is just the permutation character of G acting on the 1-spaces of the natural $\mathrm{GL}_3(q)$-module $\mathbb{F}_q^3$. It follows that χ is strongly real, $\theta = \chi_S$ is irreducible, and $\chi(x) = 0$. By Lemma 2.1(ii), θ extends to a strongly real character of $J := I_H(\theta)$.

Now we may assume that we are not in the case (Ia), and choose a semisimple element $s \in L$ of order r as follows.

Case Ib. Suppose that $S = \mathrm{PSL}^{\epsilon}_n(q)$, where either $n \geq 4$, or $(n, \epsilon) = (3, +)$ and $2|f$. Choose $m \in \{n, n-1\}$ to be *even* and $r = r(p, mf)$ a p.p.d. of $p^{mf} - 1$. Note that our hypothesis on n and f guarantees that r exists, and furthermore, r is coprime to $|\mathbf{Z}(L)| = \gcd(n,\ q-\epsilon)$. Embed $\mathrm{Sp}_m(q)$ in $L = \mathrm{SL}^{\epsilon}_n(q)$ and choose $s \in \mathrm{Sp}_m(q)$ of order r. One can check that $|\mathbf{C}_L(s)| = (q^{m/2}+1)^2(q-\epsilon)^{n-m-1}$ if $\epsilon = -$ and $m \equiv 2 \pmod 4$, and $|\mathbf{C}_L(s)| = (q^m - 1)(q-\epsilon)^{n-m-1}$ otherwise.

Case Ic. Next suppose that $S = \mathrm{P}\Omega^{\epsilon}_{2n}(q)$. Then choose $r = r(p, (2n-2)f) > 2$, a p.p.d. of $p^{(2n-2)f} - 1$, and $s \in \mathrm{Spin}^{-}_{2n-2}(q) < L$ of order r. By [Moretó and Tiep 2008, Lemma 2.4], $|\mathbf{C}_L(s)| = (q^{n-1}+1)(q+\epsilon)$.

Case Id. In the case $L = E^{\epsilon}_6(q)_{sc}$ (where $\epsilon = +$ for E_6 and $\epsilon = -$ for ${}^2E_6(q)$), we choose $s \in F_4(q) < L$ of order $r = r(p, 12f) \geq 13$. By [Moretó and Tiep 2008, Lemma 2.3], $|\mathbf{C}_L(s)| = \Phi_{12}(q) \cdot (q^2 + q\epsilon + 1)$.

In Cases Ib–Id, it is straightforward to check that s is a regular semisimple element; furthermore, s is real by [Tiep and Zalesski 2005, Proposition 3.1]. Hence, $\chi = \chi_s$ is a strongly real irreducible character of G, and, arguing as in part (1) of the proof of Proposition 4.5, we see that $\chi_s(x^{\sigma}) = 0$ for all $\sigma \in \mathrm{Aut}(S)$, whenever $x \in G$ is any semisimple element with $|\mathbf{C}_G(x)|$ *not* divisible by $|T| = |\mathbf{C}_L(s)|$. Also, by Proposition 4.3(iii), $\theta := (\chi_s)_S$ is irreducible.

2. Observe that, when $L = E^{\epsilon}_6(q)$, χ and θ have 2-defect 0, whence they vanish at any 2-element $1 \neq x \in S$. In the remaining cases, we now find a 2-element $x \in S$ such that $|\mathbf{C}_G(x)|$ is not divisible by $|T|$. If $L = \mathrm{SL}^{\epsilon}_n(q)$, we choose x represented by $\mathrm{diag}(x_1, I_{n-2}) \in \mathrm{SL}_2(q) \times \mathrm{SL}^{\epsilon}_{n-2}(q)$ with $|x_1| = 4$. One can then check that $|\mathbf{C}_G(x)| = |\mathrm{GL}^{\epsilon}_{n-2}(q)| \cdot (q-\alpha)$ with $\alpha = \pm 1$ chosen such that $4 \mid (q-\alpha)$, whence $|\mathbf{C}_G(x)|$ is not divisible by $|T|$. Finally, if $L = \mathrm{Spin}^{\epsilon}_{2n}(q)$, then we choose x to be an involution with centralizer of type $\mathrm{GO}^{+}_4(q) \times \mathrm{GO}^{\epsilon}_{2n-4}(q)$ (in $\mathrm{GO}^{\epsilon}_{2n}(q)$). It is easy to see that $|\mathbf{C}_G(x)|$ is not divisible by r for the chosen element x. Thus for all $\sigma \in \mathrm{Aut}(S)$, $\theta(x^{\sigma}) = 0$, as required in Theorem 3.1(i).

3. It remains to show that $I_H(\theta) \leq G$ and so θ extends to the strongly real character $\alpha = \chi_J$ of $J = I_H(\theta) = G \cap H$. Since G fixes $\theta = \chi_S$ and GH does not contain $\langle G, \tau \rangle$, it suffices to show that $I_{\mathrm{Aut}(S)}(\theta) = \langle G, \tau \rangle$. Consider any automorphism $\sigma \in \mathrm{Aut}(S)$ that fixes θ. Since $\mathrm{Aut}(S) = G : A(S)$, we may assume $\sigma \in A(S)$, and so in the notation of the proof of Proposition 4.5, we may write $\sigma = \tau^i(\sigma_0)^e$ with $i, e \geq 0$. Since σ fixes θ, σ fixes χ_s by Proposition 4.3(iv), which in turn implies that s and $\sigma^*(s)$ are L-conjugate by Proposition 4.3(ii). But s is real, and $\tau(s)$ is L-conjugate to s^{-1} by Lemma 4.6. Hence, replacing σ by σ^{-1} if necessary, we may assume that $\sigma = (\sigma_0)^e$, where $0 \leq e \leq f/2$ in the (untwisted) cases of SL, Spin^+, and E_6. In the (twisted) cases of SU, Spin^-, and ${}^2E_6$, since τ acts on S as σ_0^f, replacing σ by σ^{-1} we may assume that $\sigma = (\sigma_0)^e$ with $0 \leq e \leq 2f/3$. Also, $r = |s|$ divides $|\mathscr{L}^{\sigma^*}|$ by Proposition 4.3(ii). In either case, we can now check that this can happen only when $e = 0$, that is, $\sigma \in \langle G, \tau \rangle$.

Case II (GH contains $\langle G, \tau \rangle$). **4.** In this case, we will choose a semisimple element $s \in L$ of order r as follows.

Case IIa. Suppose that $S = \mathrm{PSL}_n(q)$. Choose $m \in \{n, n-1\}$ to be *odd* (so $m \geq 3$) and $r_1 = r(p, mf)$ a p.p.d. of $p^{mf} - 1$. Furthermore, choose $r_2 = 1$ if f is odd, and $r_2 = r(p, mf/2)$ a p.p.d. of $p^{mf/2} - 1$ if $2 \mid f$. Then $r = r_1 r_2$ is coprime to $|\mathbf{Z}(L)| = \gcd(n,\ q-1)$. Since $L \geq \mathrm{SL}_m(q)$ contains a cyclic subgroup of order $(q^m - 1)/(q-1)$, we can find a semisimple element $s \in L$ of order r. One can check that $|\mathbf{C}_L(s)| = (q^m - 1)(q-1)^{n-m-1}$.

Case IIb. Suppose that $S = \mathrm{PSU}_n(q)$ with $n \geq 3$. Choose $m \in \{n, n-1\}$ to be *odd* (so $m \geq 3$) and $r = r(p, 2mf)$ a p.p.d. of $p^{2mf} - 1$; in particular, r is coprime to $|\mathbf{Z}(L)| = \gcd(n,\ q+1)$. Now we can find a semisimple element $s \in L$ of order r, with $|\mathbf{C}_L(s)| = (q^m + 1)(q+1)^{n-m-1}$.

Case IIc. Suppose that $S = \mathrm{P}\Omega^+_{2n}(q)$. Choose $r_1 = r(p, nf)$ to be a p.p.d. of $p^{nf} - 1$. Furthermore, choose $r_2 = 1$ if f is odd, and $r_2 = r(p, nf/2)$ a p.p.d. of $p^{nf/2} - 1$ if $2 \mid f$, and set $r = r_1 r_2$. Since $\mathrm{SO}^+_{2n}(q) > \mathrm{GL}_n(q)$ contains a cyclic subgroup of order $q^n - 1$, we can find a semisimple element $s \in L$ of (odd) order r. One can check that $|\mathbf{C}_L(s)| = q^n - 1$.

Case IId. Assume now that $S = \mathrm{P}\Omega^-_{2n}(q)$. Then choose $r = r(p, 2nf)$ to be a p.p.d. of $p^{2nf} - 1$. Since n is odd, $\mathrm{GO}^-_{2n}(q) > \mathrm{GU}_n(q)$ contains a cyclic subgroup of order $q^n + 1$, and so we can find a semisimple element $s \in L$ of order r, with $|\mathbf{C}_L(s)| = q^n + 1$.

Case IIe. Next suppose that $L = E_6(q)_{sc}$. Then choose $r_1 = r(p, 9f)$, a p.p.d. of $p^{9f} - 1$, and choose $r_2 = 1$ if f is odd, and $r_2 = r(p, 9f/2)$, a p.p.d. of $p^{9f/2} - 1$ if $2 \mid f$. Then $r = r_1 r_2$ is coprime to $|\mathbf{Z}(L)| = (3, q-1)$. We claim that there is a regular semisimple element $s \in L$ with $T^* = \mathbf{C}_L(s)$ of order $\Phi_9(q)$. Indeed, by [Moretó and Tiep 2008, Lemma 2.3], there is a regular semisimple element $s_1 \in L$ of order r_1 with $T^* := \mathbf{C}_L(s_1)$ of order $\Phi_9(q)$. If f is odd, set $s = s_1$. Assume $2 \mid f$. Then $\Phi_9(q) = (q^9 - 1)/\Phi_1(q)\Phi_3(q)$ is divisible by r_2, so T^* contains an element s_2 of order r_2. Now set $s = s_1 s_2$.

Case IIf. In the case $L = {}^2E_6(q)_{sc}$, we choose $s \in L$ of order $r = r(p, 18f) \geq 19$. By [Moretó and Tiep 2008, Lemma 2.3], we have $|\mathbf{C}_L(s)| = \Phi_{18}(q)$.

In all these cases, it is straightforward to check that s is a regular semisimple element. Hence, as above, $\chi = \chi_s \in \mathrm{Irr}(G)$, and $\chi_s(x^\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(S)$, whenever $x \in G$ is any semisimple element with $|\mathbf{C}_G(x)|$ *not* divisible by $|T| = |\mathbf{C}_L(s)|$. Also, by Proposition 4.3(iii), $\theta := (\chi_s)_S$ is irreducible.

5. Observe that, when $L = E_6^\epsilon(q)$, both χ and θ have 2-defect 0, whence they vanish at any 2-element $1 \neq x \in S$. In the remaining cases, one easily checks that the 2-element $x \in S$ constructed in part 2 of this proof has the property that $|\mathbf{C}_G(x)|$ is not divisible by $|T^*|$. Thus $\theta(x^\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(S)$, as required in Theorem 3.1(i).

Next we claim that s is *not* real in L. Assume the contrary: $gsg^{-1} = s^{-1}$ for some $g \in L$. Then g normalizes $T^* = \mathbf{C}_L(s)$ and $g^2 \in T^*$. But (using for instance

[Fleischmann et al. 1998, §5 and Theorem 5.7]) one can see that $N_L(T^*)/T^*$ has odd order (indeed it is C_m in IIa and IIb, C_n in IIc and IId, and C_9 in IIe and IIf). It follows that $g \in T^*$ and so $s^2 = 1$, a contradiction.

Now we show that $I_{\mathrm{Aut}(S)}(\chi) = G$. Consider any automorphism $\sigma \in \mathrm{Aut}(S)$ that fixes χ. As in 3), we may write $\sigma = \tau^i(\sigma_0)^e$ with $e \geq 0$ and $i = 0, 1$. Moreover, $\sigma \notin G\tau$ (otherwise $\chi^\sigma = (\chi_s)^\tau = \chi_{s^{-1}} \neq \chi$ as s is not real), so $e > 0$ if $i = 1$. Hence, $r = |s|$ must divide $|\mathscr{L}^{\sigma^*}|$ by Proposition 4.3(ii).

First we consider the twisted cases: $L = \mathrm{SU}_n(q)$, $\mathrm{Spin}^-_{2n}(q)$, or ${}^2E_6(q)_{sc}$. Then $A(S) = \langle \sigma_0 \rangle \cong C_{2f}$ and $(\sigma_0)^f = \tau$ on S. Replacing σ by σ^{-1} if necessary, we may assume that $0 \leq e < f$ and $i = 0$. The condition $r = |s|$ divides $|\mathscr{L}^{\sigma^*}|$ now implies that $e = 0$.

Finally we consider the untwisted cases: $L = \mathrm{SL}_n(q)$, $\mathrm{Spin}^+_{2n}(q)$, or $E_6(q)_{sc}$. Then $A(S) = \langle \sigma_0 \rangle \times \langle \tau \rangle \cong C_f \times C_2$. Replacing σ by σ^{-1} if necessary, we may assume that $0 \leq e \leq f/2$. If $i = 0$, then the condition $r = |s|$ divides $|\mathscr{L}^{\sigma^*}|$ now implies that $e = 0$, that is, $\sigma \in G$. Next assume that $i = 1$ (and so $0 < e \leq f/2$), and $L = \mathrm{SL}_n(q)$ for instance. Then r divides $|\mathscr{L}^{\sigma^*}| = |\,\mathrm{SU}_n(p^e)|$, and so $r_1 = r(p, mf)$ divides $p^{je} - (-1)^j$ for some j, $1 \leq j \leq n$. If j is even, then

$$(n-1)f \leq mf \mid je \leq nf/2,$$

a contradiction as $n \geq 3$. Hence j is odd. Recall that $m \in \{n, n-1\}$ is chosen to be odd and $1 \leq j \leq n$, so $j \leq m$. Now $mf \mid 2je \leq mf$ implies that $e = f/2$. In this case we have that $r_2 = r(p, mf/2)$ divides $p^{ke} - (-1)^k$ for some k, $1 \leq k \leq n$. In particular, $me \mid 2ke$ and so $m \mid 2k$, which implies $m \mid k$ because m is odd. Since $1 \leq k \leq n$ and $m \geq n-1$, we obtain that $k = m$ and so k is odd. But in this case $r_2 = r(p, ke)$ cannot divide $p^{ke} + 1$, a contradiction. The same argument shows that $r = |s|$ cannot divide $|\mathscr{L}^{\sigma^*}|$ if $i = 1$ and $L = \mathrm{Spin}^+_{2n}(q)$ or $E_6(q)_{sc}$.

6. We have shown that $I_{\mathrm{Aut}(S)}(\chi) = G$. Hence, $I_H(\theta) = H \cap G$ by Proposition 4.3(iv). Since

$$H/(G \cap H) \cong GH/G \geq \langle G, \tau \rangle / G \cong C_2$$

by the main hypothesis in Case II, we can find $\varphi \in H \smallsetminus G$ such that φ induces τ modulo G and $\varphi^2 \in G \cap H$. Now set $J = \langle G \cap H, \varphi \rangle$ and $\alpha = (\chi_{G \cap H})^J$. Then by Lemma 4.6 and [Navarro et al. 2008, Corollary 2.5] we have

$$\chi^\varphi = \chi^\tau = (\chi_s)^\tau = \chi_{\tau(s)} = \chi_{s^{-1}} = \bar{\chi},$$

in particular, $\theta^\varphi = \bar{\theta}$, but $\theta^\varphi \neq \theta$ as $\varphi \notin G \cap H = I_H(\theta)$. Since $S \lhd J$, this implies that $\alpha \in \mathrm{Irr}(J|\theta)$. Also, α equals $\chi + \bar{\chi}$ on $G \cap H$ and 0 on $J \smallsetminus (G \cap H)$, whence it is real.

Under the extra assumption that $S \ncong \mathrm{PSU}_n(q)$ with $n \geq 4$ *even*, we now show that α is strongly real. Indeed, setting $K := \langle G, \varphi \rangle = \langle G, \tau \rangle$ and $\vartheta := \chi^K$ (as

$K \cap H = J$), we see that $\vartheta_J = \alpha$, so $\vartheta \in \mathrm{Irr}(K)$. Also, ϑ equals $\chi + \bar{\chi}$ on G and 0 on $K \smallsetminus G$, so ϑ is real.

- Now, if $S = \mathrm{PSL}_n(q)$, then K is a quotient of $\langle \mathrm{GL}_n(q), \tau \rangle$, and so ϑ is strongly real by [Gow 1983, Theorem 2].
- Suppose $S = \mathrm{P}\Omega^{\epsilon}_{2n}(q)$. Then $\langle S, \tau \rangle \le R := \mathrm{PGO}^{\epsilon}_{2n}(q) < K$. Since $\theta^{\tau} = \theta^{\varphi} = \bar{\theta} \ne \theta$ and $\vartheta_S = \theta + \bar{\theta}$, we see that ϑ_R is irreducible. By the main result of [Gow 1985], ϑ_R, as an irreducible character of $\mathrm{GO}^{\epsilon}_{2n}(q)$, is strongly real. In turn, this implies that ϑ is strongly real.
- Suppose that $S = \mathrm{PSU}_n(q)$ with $n \ge 3$ *odd* or $S = E^{\epsilon}_6(q)$. Then $|\mathbf{C}_L(s)| = (q^n+1)/(q+1)$, $\Phi_9(q)$ or $\Phi_{18}(q)$, respectively, and is odd; hence χ and ϑ are of 2-defect zero. Since ϑ is real of 2-defect 0, it is strongly real by Lemma 2.3.

Thus in all cases ϑ is strongly real, and so is $\alpha = \vartheta_J$, as claimed. □

Proposition 4.8. *Theorem 3.1 holds if* $S = \mathrm{PSU}_n(q)$ *where* $n \ge 3$ *and* q *is odd.*

Proof. Keep all the notation of the proof of Proposition 4.7.

1. First we consider the case $S = \mathrm{PSU}_3(q)$. When $q = 3$, one can check using [Conway et al. 1985] that $\mathrm{Irr}(S)$ contains a character θ of degree 14, which extends to a strongly real character of $\mathrm{Aut}(S)$ and vanishes at all elements of order 8 in S. Furthermore, the case where GH contains $\langle G, \tau \rangle$ has already been considered in Case II of the proof of Proposition 4.7. So we may assume that $q \ge 5$ and that GH does not contain $\langle G, \tau \rangle$.

In the notation of [Geck 1990, Table 3.1], consider the irreducible character $\theta = \chi^{(u)}_{q^3+1}$ of degree q^3+1 of $L = \mathrm{SU}_3(q)$, with $u := q+1$. Since θ is trivial at $\mathbf{Z}(L)$, we will view it as an irreducible character θ of $S = L/\mathbf{Z}(L)$. Using the character values listed in [Geck 1990, Table 3.1], one checks that θ is real and τ-invariant (indeed,

$$\chi^{(u)}_{q^3+1} = \chi^{(-u)}_{q^3+1} = \chi^{(uq)}_{q^3+1}$$

by our choice of u). Next, the largest degree of irreducible characters of $G = \mathrm{PGU}_3(q)$ is $(q+1)(q^2-1)$, which is less than $3\theta(1)$. Since G/S has order 1 or 3, it follows that θ is G-invariant. Hence, by [Navarro and Tiep 2008, Lemma 2.1], θ extends to a unique $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$. Viewing χ as a real irreducible character of $\mathrm{GU}_3(q)$, we conclude by [Ohmori 1981, Theorem 7(ii)] that χ is strongly real.

Next we show that $I_{\mathrm{Aut}(S)}(\theta) = \langle G, \tau \rangle$. Since θ is invariant under G and τ and $\mathrm{Aut}(G) = G : A(S)$, it suffices to show that the only nontrivial element $\sigma = (\sigma_0)^e \in A(S)$ that fixes θ is $\tau = (\sigma_0)^f$. So assume that $1 \le e \le f$ for such a σ. Consider an element y belonging to the conjugacy class $C^{(1)}_7$ in [Geck 1990, Table 1.1], so that y^{σ} belongs to the class $C^{(p^e)}_7$. Then the condition $\theta(y) = \theta(y^{\sigma})$ implies that

$$\delta + \delta^{-1} = \delta^{p^e} + \delta^{-p^e}$$

for a fixed $(q-1)$-st primitive root δ of unity in $\mathbb{C}$. Since $1 \le e \le f$ and $q \ge 5$, it follows that $e = f$, as claimed.

We have shown that θ extends to the strongly real character $\alpha = \chi_J$ of $J = H \cap G = I_H(\theta)$. It remains to find an element x satisfying the condition (i) of Theorem 3.1. Suppose first that $q \equiv 3 \pmod 4$. Then we choose $x \in S$ to be any element of order 4 that affords eigenvalue 1 on the natural module $\mathbb{F}_{q^2}^3$ for L. Observe that $x^{\mathrm{Aut}(S)}$ is just the conjugacy class $C_6^{(0,(q+1)/4,3(q+1)/4)}$ in [Geck 1990, Table 1.1], and so $\theta(x) = 0$; cf. [ibid., Table 3.1].

Assume now that $q \equiv 1 \pmod 4$. Then we choose $x \in S$ to be any element of order 8. Any $\mathrm{Aut}(S)$-conjugate x^σ of such an x belongs to the conjugacy class $C_7^{k(q^2-1)/8}$ in [ibid., Table 1.1] for some odd integer k. Hence

$$\theta(x^\sigma) = \delta^{k(q^2-1)/8} + \delta^{-k(q^2-1)/8} = 0$$

since k is odd and $|\delta| = q-1$.

2. From now on we may assume that $S = \mathrm{PSU}_n(q)$ with $n \ge 4$. Then it was shown in parts 2. and 5. of the proof of [Dolfi et al. 2008, Theorem 2.1] that there is a permutation character ρ of $\mathrm{Aut}(S)$ such that $\rho_S = 1_S + \varphi + \psi$ is the sum of three irreducible (unipotent) characters of S, all of distinct degrees, and with exactly one, call it θ, of even degree. In fact, ρ_S is just the permutation character of the action of S on the singular 1-spaces of the natural L-module $V = \mathbb{F}_{q^2}^n$, and

$$\varphi(1) = \frac{(q^n-(-1)^n)(q^{n-1}+(-1)^n q^2)}{(q+1)(q^2-1)}, \quad \psi(1) = \frac{(q^n+(-1)^n q)(q^n-(-1)^n q^2)}{(q+1)(q^2-1)}.$$

Since $S \lhd \mathrm{Aut}(S)$, it follows that the same is true for ρ, and so θ extends to a strongly real character of even degree of $\mathrm{Aut}(S)$. Note that $\theta = \varphi$ if $n \equiv 0, 3 \pmod 4$ and $\theta = \psi$ if $n \equiv 1, 2 \pmod 4$.

3. It remains to find a 2-element $h \in S$ such that $\theta(h^\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(S)$. It suffices to show that $\theta(h) = 0$ since θ is $\mathrm{Aut}(S)$-invariant. To this end, we will use the technique of *dual pairs*; see for instance [Liebeck et al. 2010; Tiep 2010]. We consider the dual pair $X * Y$ inside $\Gamma := \mathrm{GU}_{2n}(q)$, where $X = \mathrm{GU}_2(q)$ and $Y = \mathrm{GU}_n(q)$. More precisely, we view X as $\mathrm{GU}(U)$, where $U = \mathbb{F}_{q^2}^2$ is endowed with a nondegenerate Hermitian form $(\cdot,\cdot)_U$, and Y is meant to be $\mathrm{GU}(V)$, where $V = \mathbb{F}_{q^2}^n$ is endowed with a nondegenerate Hermitian form $(\cdot,\cdot)_V$. Now we consider $W = U \otimes_{\mathbb{F}_{q^2}} V$ with the Hermitian form $(\cdot,\cdot)$ defined via $(u \otimes v, u' \otimes v') = (u,u')_U \cdot (v,v')_V$ for $u \in U$ and $v \in V$. The action of $X \times Y$ on V induces a homomorphism $X \times Y \to \Gamma := \mathrm{GU}(W)$. Recall (see [Tiep and Zalesskii 1997, §4]) that for any $m \ge 1$, the class function

$$\zeta_{m,q}(g) = (-1)^m (-q)^{\dim_{\mathbb{F}_{q^2}} \mathrm{Ker}(g-1)} \tag{4-1}$$

is a (reducible) *Weil character* of $\mathrm{GU}_m(q)$ of degree q^m, where $\mathrm{Ker}(g-1)$ is the fixed point subspace of $g \in \mathrm{GU}_m(q)$ on the natural module $(\overline{\mathbb{F}}_{q^2})^m$ for $\mathrm{GU}_m(q)$. By [Liebeck et al. 2010, Proposition 6.3], the restriction of $\zeta := \zeta_{2n,q}$ to $X \times Y$ decomposes as

$$\zeta_{X\times Y} = \sum_{\alpha \in \mathrm{Irr}(X)} \alpha \otimes \mathsf{D}_\alpha, \tag{4-2}$$

where the Y-characters $\mathsf{D}^\circ_\alpha := \mathsf{D}_\alpha - k_\alpha \cdot 1_Y$ are all irreducible and distinct, for some $k_\alpha \in \{0, 1\}$. Furthermore, $k_\alpha = 1$ precisely when $\alpha = 1_X$ or α is the Steinberg character St of X. Also, D_α can be computed explicitly using the formula

$$\mathsf{D}_\alpha(g) = \frac{1}{|X|}\sum_{x\in X} \overline{\alpha(x)}\zeta(xg). \tag{4-3}$$

In particular, one can show (see [Liebeck et al. 2010, Table III]) that $\mathsf{D}^\circ_{1_X}$ is the only irreducible constituent of ζ_Y of degree $\varphi(1)$, and $\mathsf{D}^\circ_{\mathsf{St}}$ is the only irreducible constituent of ζ_Y of degree $\psi(1)$. On the other hand, (4-1) and (4-2) imply that

$$\zeta_Y = \sum_{\alpha \in \mathrm{Irr}(X)} \alpha(1)\cdot \mathsf{D}^\circ_\alpha + (q+1)\cdot 1_Y$$

is just the permutation character of Y on the points of the vector space V, whence ζ_Y contains ρ_Y, the inflation of $\rho_{\mathrm{PGU}_n(q)}$ to $Y = \mathrm{GU}_n(q)$. It follows that

$$\varphi = (\mathsf{D}^\circ_{1_X})_S - 1_S, \quad \psi = (\mathsf{D}^\circ_{\mathsf{St}})_S - 1_S.$$

Together with (4-1) and (4-3), this will allow us to find the desired element h.

4. Among the irreducible characters of $X = \mathrm{GU}_2(q)$, there are $q+1$ distinct characters ζ_2^i, where $0 \le i \le q$, which are known as (irreducible) *Weil characters* of X. They are computed explicitly in [Tiep and Zalesskii 1997, Lemma 4.1]; furthermore, $\zeta_{2,q} = \sum_{i=0}^q \zeta_2^i$ and $\zeta_2^0 = \mathsf{St}$. In particular,

$$[\zeta_{2,q}, \mathsf{St}]_X = 1. \tag{4-4}$$

Let $\mu_{q+1} := \{c \in \mathbb{F}_{q^2} \mid c^{q+1} = 1\}$. Note that, for any $c \in \mu_{q+1}$, $x \mapsto \zeta_{2,q}(cx)$ is a class function on X. Moreover, using the well-known character table of X (see for instance [Ennola 1963]), we can check that

$$\frac{1}{|X|}\sum_{x\in X} \zeta_{2,q}(cx)\zeta_{2,q}(dx)\overline{\mathsf{St}(x)} = 1 \tag{4-5}$$

for any $c, d \in \mu_{q+1}$, and

$$\frac{1}{|X|}\sum_{x\in X} \zeta_{2,q}(cx)^2\zeta_{2,q}(dx) = 1 \tag{4-6}$$

whenever $c, d \in \mu_{q+1}$ and $c \ne d$.

5. Now we are ready to find the desired element h. This will be done according to $n \pmod 8$. Let $N = 2(q^2-1)_2$ denote the 2-part $(q^4-1)_2$ of (q^4-1), and let γ be a fixed N-th primitive root of unity in $\overline{\mathbb{F}}_{q^2}$. Observe that $\mathrm{GU}_4(q)$ has a cyclic maximal torus T of order q^4-1 that contains an element g_4 conjugate to $\mathrm{diag}(\gamma, \gamma^{-q}, \gamma^{q^2}, \gamma^{-q^3})$ over $\overline{\mathbb{F}}_{q^2}$, and set

$$g_8 := \mathrm{diag}(g_4, g_4^{-1}) \in \mathrm{SU}_8(q).$$

Note that no eigenvalue of g_4 and g_8 belongs to $\mathbb{F}_{q^2}$ by the choice of γ. On the other hand, any eigenvalue of any $x \in X = \mathrm{GU}_2(q)$ belongs to $\mathbb{F}_{q^2}^\times$.

Let $8 \mid n$, and choose $h = \mathrm{diag}(g_8, \dots, g_8) \in \mathrm{SU}_n(q)$. Then, for any $x \in X$, no eigenvalue of xh can be equal to 1, whence $\zeta(xh) = 1$ by (4-1). Hence, by (4-3) we have

$$\theta(h) = \mathsf{D}^\circ_{1_X}(h) = \frac{1}{|X|}\sum_{x \in X} \zeta(xh) - 1 = [1_X, 1_X] - 1 = 0.$$

Next, for $n \equiv 1 \pmod 8$ we choose $h = \mathrm{diag}(g_8, \dots, g_8, 1) \in \mathrm{SU}_n(q)$. Then, for any $x \in X$, no eigenvalue of xg_8 can be equal to 1, whence $\zeta(xh) = \zeta_{2,q}(x)$ by (4-1). It then follows by (4-3) and (4-4) that

$$\theta(h) = \mathsf{D}^\circ_{\mathsf{St}}(h) = \frac{1}{|X|}\sum_{x \in X} \zeta_{2,q}(x)\overline{\mathsf{St}(x)} - 1 = [\zeta_{2,q}, \mathsf{St}] - 1 = 0.$$

For $n \equiv 2 \pmod 8$ we choose $h = \mathrm{diag}(g_8, \dots, g_8, 1, 1) \in \mathrm{SU}_n(q)$. Then, $\zeta(xh) = \zeta_{2,q}(x)^2$ for any $x \in X$ by (4-1). By (4-3) and (4-5) applied to $c = d = 1$ we have

$$\theta(h) = \mathsf{D}^\circ_{\mathsf{St}}(h) = \frac{1}{|X|}\sum_{x \in X} \zeta_{2,q}(x)^2\overline{\mathsf{St}(x)} - 1 = 0.$$

For $n \equiv 3 \pmod 8$ we choose $h = \mathrm{diag}(g_8, \dots, g_8, 1, -1, -1) \in \mathrm{SU}_n(q)$. Again, $\zeta(xh) = \zeta_{2,q}(x)\zeta_{2,q}(-x)^2$ for all $x \in X$. By (4-3) and (4-6) applied to $(c, d) = (-1, 1)$ we have

$$\theta(h) = \mathsf{D}^\circ_{1_X}(h) = \frac{1}{|X|}\sum_{x \in X} \zeta_{2,q}(-x)^2\zeta_{2,q}(x) - 1 = 0.$$

Assume that $n \equiv 5 \pmod 8$. Note that $1 \neq c_1 := \det(g_4^{-1}) = \gamma^{(q^4-1)/(q+1)} \in \mu_{q+1}$. Let $h = \mathrm{diag}(g_8, \dots, g_8, g_4, c_1) \in \mathrm{SU}_n(q)$. Then, $\zeta(xh) = \zeta_{2,q}(c_1x)$ for any $x \in X$ by (4-1), and $\mathsf{St}(c_1x) = \mathsf{St}(x)$ since St is trivial at $\mathbf{Z}(X)$. Hence, by (4-3) and (4-4) we have

$$\mathsf{D}^{\circ}_{\mathsf{St}}(h) = \frac{1}{|X|}\sum_{x\in X}\zeta_{2,q}(c_1x)\overline{\mathsf{St}(x)} - 1$$
$$= \frac{1}{|X|}\sum_{x\in X}\zeta_{2,q}(c_1x)\overline{\mathsf{St}(c_1x)} - 1 = [\zeta_{2,q}, \mathsf{St}]_X - 1 = 0.$$

For $n \equiv 6 \pmod 8$ we choose $h = \mathrm{diag}(g_8, \dots, g_8, g_4, 1, c_1) \in \mathrm{SU}_n(q)$. Then, $\zeta(xh) = \zeta_{2,q}(c_1x)\zeta_{2,q}(x)$ for any $x \in X$ by (4-1). By (4-3) and (4-5) applied to $(c, d) = (c_1, 1)$ we have

$$\theta(h) = \mathsf{D}^{\circ}_{\mathsf{St}}(h) = \frac{1}{|X|}\sum_{x\in X}\zeta_{2,q}(c_1x)\zeta_{2,q}(x)\overline{\mathsf{St}(x)} - 1 = 0.$$

For $n \equiv 7 \pmod 8$ we choose $h = \mathrm{diag}(g_8, \dots, g_8, g_4, 1, 1, c_1) \in \mathrm{SU}_n(q)$. Then, $\zeta(xh) = \zeta_{2,q}(x)^2\zeta_{2,q}(c_1x)$ for any $x \in X$ by (4-1). By (4-3) and (4-6) applied to $(c, d) = (1, c_1)$ we have

$$\theta(h) = \mathsf{D}^{\circ}_{1_X}(h) = \frac{1}{|X|}\sum_{x\in X}\zeta_{2,q}(x)^2\zeta_{2,q}(c_1x) - 1 = 0.$$

Finally, assume that $n \equiv 4 \pmod 8$. Then we choose $h_4 \in \mathrm{SU}_4(q)$ to be conjugate to $\mathrm{diag}(\gamma^2, \gamma^{-2q}, 1, \gamma^{2q-2})$ over $\overline{\mathbb{F}}_{q^2}$, and set $h = \mathrm{diag}(g_8, \dots, g_8, h_4) \in \mathrm{SU}_n(q)$. Note that $\gamma^2, \gamma^{-2q} \in \mathbb{F}^{\times}_{q^2}$ and $1 \neq c_2 := \gamma^{2q-2} \in \mu_{q+1}$. Now, for any $x \in X$, we have

$$\zeta(xh) = \zeta_{2,q}(x)\zeta_{2,q}(c_2x)\zeta_{2,q}(\gamma^2x)\zeta_{2,q}(\gamma^{-2q}x)$$

by (4-1). Direct computation using the character table of $\mathrm{GU}_2(q)$ shows that

$$\theta(h) = \mathsf{D}^{\circ}_{1_X}(h) = \frac{1}{|X|}\sum_{x\in X}\zeta_{2,q}(x)\zeta_{2,q}(c_2x)\zeta_{2,q}(\gamma^2x)\zeta_{2,q}(\gamma^{-2q}x) - 1 = 0,$$

and so we are done. $\square$

To complete the proof of Theorem 3.1, we handle the case $S = \mathrm{P}\Omega_8^+(q)$:

Proposition 4.9. *Theorem 3.1 holds in the case $S = \mathrm{P}\Omega_8^+(q)$ with q odd.*

Proof. **1.** Suppose that $q = 3$. Then, according to [Conway et al. 1985], S has a unique irreducible character θ of degree 300, which is strongly real, and a unique conjugacy class ($4E$ in the notation of [ibid.]) of elements x of order 4 with $|\mathbf{C}_S(x)| = 1536$ and $\theta(x) = 0$. It follows that $\theta(x^\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(S)$. Furthermore, one can show (directly, or using [GAP 2004]) that θ extends to a rational character of $\mathrm{Aut}(S) = S \cdot \mathsf{S}_4$. From now on we may assume that $q = p^f \geq 5$.

2. Choose $\epsilon = \pm 1$ such that $q \equiv \epsilon \pmod 4$. Also view S as $L/\mathbf{Z}(L)$, where $L = \mathrm{Spin}_8^+(q)$ and $\mathbf{Z}(L) \cong C_2 \times C_2$. Fix an orthonormal basis $(e_1, \dots, e_4)$ of $\mathbb{R}^4$ and realize the simple roots of the algebraic group $\mathscr{L} = \mathrm{Spin}_8(\bar{\mathbb{F}}_q)$ as

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_3 + e_4$$

as usual. Then the four fundamental weights of $\mathscr{L}$ are given by

$$\varpi_1 = e_1, \quad \varpi_2 = e_1 + e_2, \quad \varpi_3 = \frac{e_1 + e_2 + e_3 - e_4}{2}, \quad \varpi_4 = \frac{e_1 + e_2 + e_3 + e_4}{2}.$$

Let $\Gamma \cong \mathsf{S}_3$ denote the subgroup of $A(S)$ consisting of graph automorphisms. Then Γ permutes the 3 fundamental weights ϖ_1, ϖ_3, ϖ_4 transitively and faithfully, and fixes ϖ_2. Consider the corresponding $\mathscr{L}$-modules $V_i = V(\varpi_i)$ with highest weight ϖ_i, $i = 1, 3, 4$. Then the set of weights of V_i is

$$\begin{array}{ll} \{\pm e_j \mid 1 \le j \le 4\}, & \text{when } i = 1, \\ \left\{\frac{1}{2}\sum_{j=1}^4 a_j e_j \mid a_j = \pm 1,\ \prod_{j=1}^4 a_j = -1\right\}, & \text{when } i = 3 \\ \left\{\frac{1}{2}\sum_{j=1}^4 a_j e_j \mid a_j = \pm 1,\ \prod_{j=1}^4 a_j = 1\right\}, & \text{when } i = 4. \end{array}$$

We can think of V_1 as the natural module for $K := \Omega_8^+(q)$.

3. We will use the description above to show that L contains a regular semisimple element s of (odd) order $N := (q^3 + \epsilon)/2$ if $q \ge 9$ and $N := (q^2 + 1)/2$ if $q = 5, 7$ such that s^σ is not L-conjugate to s for any nontrivial $\sigma \in \Gamma$.

Indeed, assume that $q \ge 9$. Then fix $\delta \in \bar{\mathbb{F}}_q^\times$ of order $(q^3 + \epsilon)/2$ and choose $s \in L$ to be the unique inverse image of odd order of $\bar{s} \in \Omega_2^{-\epsilon}(q) \times \Omega_6^{-\epsilon}(q) < K$ with spectrum $\mathrm{Spec}(s, V_1) = \{\delta^j \mid j \in J_1\}$, where

$$J_1 = \{\pm 1, \pm r, \pm r^2, \pm 2(r^2 + r + 1)\}$$

and $r := -\epsilon q$. Thus we may assume that

$$e_1(s) = \delta, \quad e_2(s) = \delta^r, \quad e_3(s) = \delta^{r^2}, \quad e_4(s) = \delta^{2(r^2+r+1)}.$$

Hence $\mathrm{Spec}(s, V_i) = \{\delta^{(N+j)/2} \mid j \in J_i\}$ for $i = 3, 4$, where

$$J_3 = \{\pm(3r^2 + 3r + 1), \pm(3r^2 + r + 3), \pm(r^2 + 3r + 3), \pm(r^2 + r + 1)\},$$
$$J_4 = \{\pm(3r^2 + r + 1), \pm(r^2 + 3r + 1), \pm(r^2 + r + 3), \pm 3(r^2 + r + 1)\}.$$

Recall that $|\delta| = N \ge 5(r^2 + r + 1)$ and $|r| \ge 9$ since $q \ge 9$. Hence δ belongs to $\mathrm{Spec}(s, V_1)$ but neither to $\mathrm{Spec}(s, V_3)$ nor $\mathrm{Spec}(s, V_4)$, and similarly $\delta^{(N+r^2+r+3)/2}$ belongs to $\mathrm{Spec}(s, V_4)$ but not to $\mathrm{Spec}(s, V_3)$. Thus s has pairwise different spectra on the three modules V_1, V_3 and V_4 permuted faithfully by Γ, whence s and s^σ cannot be L-conjugate for any $1 \ne \sigma \in \Gamma$. Arguing as in the proof of [Moretó and

Tiep 2008, Lemma 2.3], we can view s as an element of $\mathrm{SO}_8^+(q)$ to calculate the order of its centralizer and find that $T^* = \mathbf{C}_L(s)$ is a torus of order $(q+\epsilon)(q^3+\epsilon)$; in particular, s is regular.

Suppose now that $q = 5$ or 7. Then fix $\delta \in \bar{\mathbb{F}}_q^\times$ of order $(q^2+1)/2$ and choose $s \in L$ to be the unique inverse image of odd order of $\bar{s} \in \Omega_4^-(q) \times \Omega_4^-(q) < K$ with spectrum $\mathrm{Spec}(s, V_1) = \{\delta^j \mid j \in J_1\}$, where

$$J_1 = \{\pm 1, \pm q, \pm 2, \pm 2q\}.$$

Thus we may assume that

$$e_1(s) = \delta, \quad e_2(s) = \delta^q, \quad e_3(s) = \delta^2, \quad e_4(s) = \delta^{2q}.$$

Hence $\mathrm{Spec}(s, V_i) = \{\delta^{j/2} \mid j \in J_i\}$ for $i = 3, 4$, where

$$J_3 = \{\pm(3q+1), \pm(q+3), \pm(3q-1), \pm(q-3)\},$$
$$J_4 = \{\pm(q+1), \pm(3q-3), \pm(q-1), \pm(3q+3)\}.$$

One can again check that s has pairwise different spectra on the three modules V_1, V_3 and V_4, and so s and s^σ cannot be L-conjugate for any $1 \neq \sigma \in \Gamma$. Furthermore, s is regular and $T^* = \mathbf{C}_L(s)$ is a torus of order $(q^2+1)^2$.

4. By [Tiep and Zalesski 2005, Proposition 3.1], s is real. It now follows by Proposition 4.3 that χ_s is a strongly real irreducible character of G, and $\theta := (\chi_s)_S$ is irreducible. We claim that $I_H(\theta) \leq G$. Once this is completed, we can take $J = G \cap H$ and $\alpha = (\chi_s)_J$ as usual. As in the proof of Proposition 4.7, it suffices to show that if $\sigma \in A(S) \cong C_f \times \mathrm{S}_3$ fixes χ_s then σ is trivial. Write $\sigma = \tau(\sigma_0)^e$ for some $\tau \in \Gamma$ and $0 \leq e < f$ (and σ_0 is induced by the field automorphism $y \mapsto y^p$ as usual). By the results of 3) we may assume $0 < e < f$; in particular, $f \geq 2$ and so $q \geq 9$. By Proposition 4.3(ii), s^L is σ^*-stable and $N = |s|$ divides $|\mathscr{L}^{\sigma^*}|$.

First assume that $|\tau| = 3$, that is, τ is a triality graph automorphism. Then $N = (q^3+\epsilon)/2$ divides $|\mathscr{L}^{\sigma^*}| = |{}^3D_4(p^e)|$. Using a suitable p.p.d. of N one can now show that $3f \mid 12e$ and so $f \mid 4e$. It follows that s^L is stable under $\sigma^4 = \tau$, contrary to the results of 3).

Now we may assume that $|\tau| = 1$ or 2, and so $N = (q^3+\epsilon)/2$ divides $|\mathscr{L}^{\sigma^*}| = |\mathrm{Spin}_8^\alpha(p^e)|$ with $\alpha = +$ or $-$, respectively. Using a suitable p.p.d. of N we now see that $3f \mid 6e$ or $3f \mid 8e$. If f is odd, then we get that $f \mid e$, a contradiction as $0 < e < f$. Hence f is even, $\epsilon = +$, and $N = (q^3+1)/2$ is divisible by $r = r(p, 6f)$, a p.p.d. of $p^{6f} - 1$. Since r divides $|\mathrm{Spin}_8^\alpha(p^e)|$, we must have $6f \mid 6e$ or $6f \mid 8e$. In the former case we again have $f \mid e$, a contradiction. So $6f \mid 8e$, and $e = 3f/4$ as $0 < e < f$. In this case, s^L is stable under $\sigma^2 = (\sigma_0)^{f/2}$. Repeating the argument above, we see that $r = r(p, 6f)$ divides $|\mathrm{Spin}_8^+(p^{f/2})|$, which is impossible.

5. We have shown that $I_H(\theta) = G \cap H$ and obviously θ extends to the strongly real character $(\chi_s)_{G\cap H}$. It remains to find a 2-element $x \in S$ such that $\theta(x^\sigma) = 0$ for all $\sigma \in \mathrm{Aut}(S)$. Since s is regular, we have $\chi_s = \pm R^G_{T,\vartheta}$ for some maximal torus T of order $|T| = |T^*|$. Recall that $q \equiv \epsilon \pmod 4$, hence we can choose $x \in S$ to be represented by $\mathrm{diag}(-I_2, I_6) \in \Omega^\epsilon_2(q) \times \Omega^\epsilon_6(q) < \Omega^+_8(q)$ with centralizer $\mathrm{GO}^\epsilon_2(q) \times \mathrm{GO}^\epsilon_6(q)$ (in $\mathrm{GO}^+_8(q)$). It is easy to see that $|\mathbf{C}_G(x)|$ is not divisible by $|T|$. Thus, $\theta(x^\sigma) = 0$ for any $\sigma \in \mathrm{Aut}(S)$. □

5. Final remarks

We start with a well-known lemma; see, for instance, [Bubboloni et al. 2009, Lemma 2.1]. We provide a proof for the sake of completeness.

Lemma 5.1. *Let $\chi \in \mathrm{Irr}(G)$ and let g be a p-element of the group G, p a prime. If $\chi(g) = 0$, then p divides $\chi(1)$.*

Proof. Let ω be a primitive p^a-th root of unity, where $p^a = o(g)$, and write $n = \chi(1)$. Then $\chi(g) = \sum_{i=1}^n \omega^{k_i} = 0$ for suitable integers $0 \le k_i \le p^a$, and ω is a root of the polynomial $q(x) = \sum_{i=1}^n x^{k_i}$. Hence, the p^a-th cyclotomic polynomial $\Phi(x)$ divides $q(x)$ (over $\mathbb{Q}$, hence also over $\mathbb{Z}$ by Gauss's lemma). In particular, $\Phi(1) = p$ divides $q(1) = \chi(1)$, as required. □

Using Lemma 5.1, from Theorem A we immediately obtain Theorem B, which in turn implies the following.

Corollary 5.2 [Dolfi et al. 2008, Theorem A]. *Let G be a finite group. If every $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ has odd degree, then G has a normal Sylow 2-subgroup.*

However, the following class of examples shows that it is not possible to deduce our Theorem A from [Dolfi et al. 2008, Theorem A] (even if we require $\chi(x) \neq 0$ for all $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ and all 2-elements $x \in G$).

Example 5.3. For every Mersenne prime $q > 7$ there exists a Frobenius $\{2, q\}$-group G such that

(a) $\chi(g) \neq 0$ for all $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ and every 2-element $g \in G$, and

(b) there exists a $\chi_0 \in \mathrm{Irr}_{\mathbb{R}}(G)$ with $\chi(1)$ even.

Let $q = 2^t - 1$ be a Mersenne prime, with $t > 3$ a prime. Write $n = t - 1$. As shown in [Isaacs 1989, Section 4] and in [Riedl 1999], one can construct a remarkable class of 2-groups (or, in general, p-groups for any prime p) $P_n(2, t)$ as subgroups of the group of units of suitable skew-polynomial rings. We recall that the same class of groups has also been considered in [Hanaki and Okuyama 1997], where they are given as matrix groups.

We mention (see [Riedl 1999]) that the group $P = P_n(2, t)$ has order 2^{tn}, that the upper central series of P coincides with the lower central series of P and that all its factors are elementary abelian groups of order 2^t. Moreover, P has a fixed point free group of automorphisms Q of order q. Hence, the semidirect product $G = PQ$ is a Frobenius $\{2, q\}$-group.

As proved in [Bubboloni et al. 2009, Example 1] (see also [Isaacs et al. 1999, Theorem 5.1]), $\chi(g) \neq 0$ for every $\chi \in \mathrm{Irr}(G)$ and for every element $g \in G$ of 2-power order. So, in particular, (a) is satisfied.

To prove (b), we denote by $\mathrm{Cl}_{\mathbb{R}}(P)$ the set of the P-conjugacy classes of real elements of P. (Observe that they are precisely the classes where every irreducible character of P assumes a real value). As an application of Brauer permutation lemma [Isaacs 1976, (6.32)], we know that $|\mathrm{Irr}_{\mathbb{R}}(P)| = |\mathrm{Cl}_{\mathbb{R}}(P)|$. Let $W = \mathbf{Z}_2(P)$ be the second term of the (upper) central series of G. By [Riedl 1999, part (i) of Corollary 2.12 and Lemma 6.1], we see that $|W| = 2^{2t}$ and that W is elementary abelian, because $t > 3$. Since every involution is a real element of P, it follows that $|\mathrm{Cl}_{\mathbb{R}}(P)| > |\mathbf{Z}(P)| = 2^t$. Therefore, as P has $|P/P'| = 2^t$ linear characters, we conclude that there exists a nonlinear $\psi \in \mathrm{Irr}_{\mathbb{R}}(P)$. So, $\chi = \psi^G \in \mathrm{Irr}_{\mathbb{R}}(G)$ is a real character of even degree of G.

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