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on certain surfaces**

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Let V be a nonsingular projective surface defined over $\mathbb{Q}$ and having at least two elliptic fibrations defined over $\mathbb{Q}$; the most interesting case, though not the only one, is when V is a K3 surface with these properties. We also assume that $V(\mathbb{Q})$ is not empty. The object of this paper is to prove, under a weak hypothesis, the Zariski density of $V(\mathbb{Q})$ and to study the closure of $V(\mathbb{Q})$ under the real and the p-adic topologies. The first object is achieved by the following theorem:

Let V be a nonsingular surface defined over $\mathbb{Q}$ and having at least two distinct elliptic fibrations. There is an explicitly computable Zariski closed proper subset X of V defined over $\mathbb{Q}$ such that if there is a point P_0 of $V(\mathbb{Q})$ not in X then $V(\mathbb{Q})$ is Zariski dense in V.

The methods employed to study the closure of $V(\mathbb{Q})$ in the real or p-adic topology demand an almost complete knowledge of V; a typical example of what they can achieve is as follows. Let V_c be

$$V_c : X_0^4 + cX_1^4 = X_2^4 + cX_3^4 \quad \text{for } c = 2, 4 \text{ or } 8;$$

then $V_c(\mathbb{Q})$ is dense in $V_c(\mathbb{Q}_2)$ for $c = 2, 4, 8$.

1. *Introduction.* Let V be a nonsingular projective surface defined over $\mathbb{Q}$ and having at least two elliptic fibrations defined over $\mathbb{Q}$; the most interesting case, though not the only one, is when V is a K3 surface with these properties. (By an elliptic fibration we mean a fibration by curves of genus 1; we do not assume that these curves have distinguished points, so they need not be elliptic curves in the number-theoretic sense.) We also assume that $V(\mathbb{Q})$ is not empty. The object of this paper is to prove, under a weak hypothesis, the Zariski density of $V(\mathbb{Q})$ and to study the closure of $V(\mathbb{Q})$ under the real and the p-adic topologies. These results can be found in Section 2; Sections 3 and 4 contain applications and examples. Note that the geometric terminology in this paper is that of Weil.

For the real and Zariski topologies, such results have been proved for a particular family of such surfaces in [Logan et al. 2010], and for one such surface already in [Swinnerton-Dyer 1968]; see Section 3. I am indebted to the referee for drawing

my attention to [Bogomolov and Tschinkel 1998], where Proposition 2.13 proves that such V have the closely related property of potential density, and to René Pannekoek for pointing out a major error in an earlier draft.

2. *The main theorems.* Let $\mathscr{F}$ denote an elliptic fibration of V defined over $\mathbb{Q}$. Then $\mathscr{F}$ is given by a map $\theta : V \to \mathbb{P}^1$ defined over $\mathbb{Q}$ whose generic fibre $\theta^{-1}(t)$ is a curve of genus 1 defined over $\mathbb{Q}(t)$. The Jacobian of $\theta^{-1}(t)$ is also defined over $\mathbb{Q}(t)$. Let P be generic over $\mathbb{Q}(t)$ on this Jacobian and let W be the locus of P over $\mathbb{Q}$; then $P \mapsto t$ gives a natural map $\phi : W \to \mathbb{P}^1$ whose fibre over t is the Jacobian of $\theta^{-1}(t)$, and for any c such that $C = \theta^{-1}(c)$ is nonsingular the Jacobian of C is $J = \phi^{-1}(c)$. In other words, ϕ is the Jacobian elliptic fibration associated to θ.

In this situation C is an n-covering of J for some n, and by abuse of language we can claim that n only depends on the fibration and not on c. (The abuse of language here is that n is not unique.) However, this property by itself does not determine the covering map uniquely. We shall require that the covering map is the restriction to C of a rational map $\psi : V \to W$ defined over $\mathbb{Q}$. One way of ensuring this is as follows. Let P be generic on V, let C be the fibre through P and J the corresponding fibre of W, let $\mathfrak{a}$ be the section of C by a hyperplane Π defined over $\mathbb{Q}$, and write $n = \deg(\mathfrak{a})$. The divisor $nP - \mathfrak{a}$ on C has degree 0 and is defined over $\mathbb{Q}(P)$; so, it is represented by a point R on J, also defined over $\mathbb{Q}(P)$, and we can take ψ to be the map defined over $\mathbb{Q}$ that sends P to R. This ψ is defined except possibly at points of the singular fibres.

Theorem 1. *Let V be a nonsingular surface defined over $\mathbb{Q}$ and having at least two distinct elliptic fibrations. There is an explicitly computable Zariski closed proper subset X of V defined over $\mathbb{Q}$ such that if there is a point P_0 of $V(\mathbb{Q})$ not in X then $V(\mathbb{Q})$ is Zariski dense in V.*

Remarks. Here *explicitly computable* means that if we are given equations for V and the two fibrations θ_1 and θ_2 we can compute equations for X. If P_0 in $V(\mathbb{Q})$ is known, then in fact it is easy to test whether P_0 is in X without needing to calculate X. A similar theorem holds over any algebraic number field. For applications in this more general case, it will often be helpful to use the sufficient condition due to Lutz for a point on an elliptic curve not to be torsion; see [Silverman 1986, Corollary VIII.7.2].

Proof. Denote by $\mathscr{F}_\nu$ for $\nu = 1, 2$ the two given elliptic fibrations on V. We retain the notation introduced above, except that objects associated with $\mathscr{F}_\nu$ will have ν as a subscript. We take $\mu = 3 - \nu$, so that $\{\mu, \nu\} = \{1, 2\}$.

We start by defining a certain Zariski closed set X_ν on V. Let Y'_ν be the union of all the singular fibres of ϕ_ν. By a theorem of Mazur, if Γ is an elliptic curve defined over $\mathbb{Q}$ and R, also defined over $\mathbb{Q}$, is an m-torsion point of Γ, then $m \leq 10$

or $m = 12$. If t is generic on $\mathbb{P}^1$ with $C_\nu = \theta_\nu^{-1}(t)$ and $J_\nu = \phi_\nu^{-1}(t)$, let $\mathfrak{b}_\nu$ be the union of the m-division points on J_ν for these values of m. Thus $\mathfrak{b}_\nu$ is a positive divisor defined over $\mathbb{Q}(t)$, and there is a (reducible) curve Y_ν'' on W_ν defined over $\mathbb{Q}$ and such that the intersection of Y_ν'' with J_ν is $\mathfrak{b}_\nu$. Now let X_ν be the union of the support of $\psi_\nu^{-1}(Y_\nu' \cup Y_\nu'')$ and those fibres of $\mathscr{F}_\nu$ that lie entirely in $\psi_\mu^{-1}(Y_\mu'')$. Clearly X_ν is Zariski closed. Write $X = X_1 \cap X_2$. The effect of this construction is that if P_0 is a point of $V(\mathbb{Q})$ not in X_ν then the fibre $C_{0\nu}$ through P_0 is nonsingular and not in $\psi_\mu^{-1}(Y_\mu'')$, and $\psi_\nu(P_0)$ is not a torsion point of the corresponding $J_{0\nu}$. In applications we can replace X by the Zariski closure of $X(\mathbb{Q})$, which will usually be much smaller; in practice one would normally choose P_0 in $V(\mathbb{Q})$ and then verify that it has the properties stated in the previous sentence for at least one value of ν.

Now let P_0 be a point of V not in X; without loss of generality we can assume that P_0 is not in X_1. For any integer r, denote by P_r the point obtained from P_0 by translation along C_{01} by $[r]\psi_1(P_0)$, where $[r]$ denotes multiplication by r on J_{01}. By construction, the points P_r are distinct and defined over $\mathbb{Q}$, so they are Zariski dense in C_{01}. Moreover C_{01} is not contained in $\psi_2^{-1}(Y_2'')$ by definition, and it is not in $\psi_2^{-1}(Y_2')$ because a nonsingular fibre of $\mathscr{F}_1$ cannot lie in a fibre of $\mathscr{F}_2$, so only finitely many of the P_r lie in X_2. Let C_{r2} be the fibre of $\mathscr{F}_2$ through P_r. The number of P_r that lie on a given fibre of $\mathscr{F}_2$ is bounded, so there are infinitely many C_{r2}. By an argument like that above for C_{01}, if P_r is not in X_2 then $C_{r2}(\mathbb{Q})$ is Zariski dense in C_{r2}. Hence $V(\mathbb{Q})$, which contains all the $C_{r2}(\mathbb{Q})$, is Zariski dense in V. □

For use in the proof of Theorem 2 below, we note that $J_{01}(\mathbb{Q})$ is not merely Zariski dense in J_{01} but dense in the real topology in that connected component of $J_{01}(\mathbb{R})$ which contains the identity element. Hence $C_{01}(\mathbb{Q})$ is not merely Zariski dense in C_{01} but dense in the real topology in that connected component of $C_{01}(\mathbb{R})$ which contains P_0. A similar remark applies to each C_{r2} for which P_r is not in X_2. Now let Z be the union of the singular points of the singular fibres of $\mathscr{F}_1$ and $\mathscr{F}_2$ and the other points at which the fibres of $\mathscr{F}_1$ and $\mathscr{F}_2$ have intersection multiplicity greater than 1; clearly Z is Zariski closed. Let X be as in Theorem 1 and write $U = V \setminus X$.

Theorem 2. *Let $\mathscr{R}$ denote the closure of $U(\mathbb{Q})$ in $V(\mathbb{R})$ under the real topology. Then the boundary of $\mathscr{R}$ is contained in $Z \cup X$.*

Proof. Let A be a point of $\mathscr{R}$ that is not in $Z \cup X$. The fibres of $\mathscr{F}_1$ and $\mathscr{F}_2$ through A are nonsingular at A and transversal there; by continuity there is a neighbourhood of A in $V(\mathbb{R})$ at every point of which the same properties hold. After contracting this neighbourhood if necessary, we can suppose that the map χ defined by $P \mapsto (\theta_1(P), \theta_2(P))$ is a homeomorphism of this neighbourhood to an

open subset of $\mathbb{P}^1 \times \mathbb{P}^1$ and that the neighbourhood does not meet X; without loss of generality we can suppose that it does not meet X_1. Choose open intervals I_1, I_2 of $\mathbb{P}^1$ such that this open subset contains $I_1 \times I_2$ and $I_1 \times I_2$ contains $\chi(A)$. Since A is in $\mathfrak{R}$, there is a point P_0 of $U(\mathbb{Q})$ in $\mathcal{N} = \chi^{-1}(I_1 \times I_2)$, and P_0 is not in X_1. But if C_{01} is the fibre of $\mathcal{F}_1$ through P_0 then $C_{01} \cap \mathcal{N}$ is homeomorphic to I_2 and hence is connected. By the remark that follows the proof of Theorem 1, $C_{01}(\mathbb{Q})$ is dense in $C_{01} \cap \mathcal{N}$. If P' is any point of $C_{01}(\mathbb{Q}) \cap \mathcal{N}$ other than the finitely many which lie in X_2, and if C'_2 is the fibre of $\mathcal{F}_2$ that passes through P', then by the same remark $C'_2(\mathbb{Q}) \cap \mathcal{N}$ is dense in $C'_2 \cap \mathcal{N}$. Hence $V(\mathbb{Q}) \cap \mathcal{N}$ is dense in $\mathcal{N}$, and so is $U(\mathbb{Q}) \cap \mathcal{N}$ since X is nowhere dense in $\mathcal{N}$. In other words, $\mathfrak{R} \supset \mathcal{N}$, and therefore A is not in the boundary of $\mathfrak{R}$. $\square$

The same ideas can be applied to the p-adic topologies, though the results that they yield are weaker and the calculations needed to make use of them are more complicated. Let $\mathcal{S}$ be a finite nonempty set of finite primes and let $\mathcal{S}^+$ be the union of $\mathcal{S}$ and the infinite place of $\mathbb{Q}$. For the rest of this section we work in the topology associated with $\mathcal{S}$ or $\mathcal{S}^+$; unfortunately I have not been able to prove any density theorems for the adelic topology.

Lemma 1. *Let*

$$E : \eta^2 + a_1 \xi \eta + a_3 \eta = \xi^3 + a_2 \xi^2 + a_4 \xi + a_6$$

be an elliptic curve defined over $\mathbb{Q}_p$, with all the a_i in $p\mathbb{Z}_p$, and suppose further that $4 \mid a_1$ and $4 \mid a_3$ if $p = 2$, and that $9 \mid a_2$ if $p = 3$. Let $\mathbb{Y}_p$ be $\mathbb{Z}_p$ if $p = 2$ or 3 and $p\mathbb{Z}_p$ if $p > 3$, and write $\zeta = \xi/\eta$. Let E' be the union of the point at infinity and the subset of $E(\mathbb{Q}_p)$ for which ξ^{-1} is in $\mathbb{Y}_p$. Then E' is a subgroup of $E(\mathbb{Q}_p)$. Moreover the map

$$E' \to \{\zeta \text{ in } \mathbb{Y}_p\} \tag{1}$$

is one-one and bicontinuous. If $A_0 = (\xi_0, \eta_0)$ is a point of $E'(\mathbb{Q})$ other than the point at infinity, then $E'(\mathbb{Q})$ is dense in that part of $E'(\mathbb{Q}_p)$ in which $v_p(\xi) \leq v_p(\xi(A_0))$.

Remark. A less ugly proof can be given by using the theory of formal groups; compare [Silverman 1986, Chapter IV]. The lemma is a weaker and easier version of a result due to René Pannekoek [2012]. I am indebted to him for communicating his result to me.

Proof. We can reduce to the case $a_1 = a_2 = a_3 = 0$ by an obvious linear transformation, which does not change the description of E'. If (ξ, η) is in E' then

$$3v(\xi) = 2v(\eta) \leq 0 \text{ and therefore } v(\zeta) = v(\xi/\eta) = -\tfrac{1}{2}v(\xi) \geq 0.$$

The map (1) is defined and continuous. Conversely, if we write $u = \xi^{-1}$ then

$$u = \zeta^2(1 + a_4 u^2 + a_6 u^3). \tag{2}$$

For ζ in $\mathbb{Y}_p$ this has just one solution u in $\mathbb{Y}_p$, which is a continuous function of ζ with $v(u) = 2v(\zeta)$. Hence, the map (1) is a homeomorphism. If $\tilde{E}$ is the reduction of E mod p, then the image of E' is $\tilde{E}(\mathbb{F}_p) \setminus (0,0)$ if $p = 2$ or 3 and the point at infinity otherwise. This is a group, so the sum or difference of two points of E' is again a point of E'. Hence E' is a subgroup of E.

$\mathbb{Z}$ acts on E' by $P \mapsto [m]P$. We now show that this map is continuous, so that $\mathbb{Z}_p$ acts continuously on E'. Let $P_i = (\xi_i, \eta_i)$ for $i = 1, 2$ be points of E' and write $\zeta_i = \xi_i/\eta_i$ and $n_i = v(\zeta_i)$. The addition formula on E gives

$$\xi(P_1 + P_2) = \alpha^2 - \xi_1 - \xi_2, \quad \eta(P_1 + P_2) = -\alpha(\xi(P_1 + P_2) - \xi_1) - \eta_1, \quad (3)$$

where in general $\alpha = (\eta_1 - \eta_2)/(\xi_1 - \xi_2)$; if $P_1 = P_2$, then $\alpha = (3\xi_1^2 + a_4)/2\eta_1$.

If $p = 2$ and $P_1 = P_2$ then $v(\alpha) = -n_1 - 1$ so that $\zeta([2]P_1) \equiv 2\zeta_1 \bmod 2^{n_1+2}$. If $p = 3$ and $P_1 = P_2$ then $v(\alpha) \geq 1 - n_1$, so that $\xi([2]P_1) \equiv -2\xi_1 \bmod 3^{2-2n_1}$ and $\eta([2]P_1) \equiv -\eta_1 \bmod 3^{2-3n_1}$; thus $\zeta([2]P_1) \equiv 2\zeta_1 \bmod 3^{2+n_1}$. If instead $P_2 = [2]P_1$ then $\alpha \equiv 2\eta_1/3\xi_1 \bmod 3^{-n_1}$, so that $\xi([3]P_1) \equiv \alpha^2 \bmod 3^{-2n_1}$ and $\zeta([3]P_1) \equiv 3\zeta_1 \bmod 3^{2+n_1}$. If $p > 3$ then $\xi_1 \equiv \zeta_1^{-2} \bmod p^{2-2n_1}$ and $\eta_1 \equiv \zeta_1^{-3} \bmod p^{2-3n_1}$. If $P_1 = P_2$ then $\alpha \equiv \frac{3}{2}\zeta_1^{-1} \bmod p^{2-n_1}$ so that $\xi([2]P_1) \equiv \frac{1}{4}\zeta_1^{-2} \bmod p^{2-2n_1}$ and $\eta([2]P_1) \equiv \frac{1}{8}\zeta_1^{-3} \bmod p^{2-3n_1}$; thus $\zeta([2]P_1) \equiv 2\zeta_1 \bmod p^{2+n_1}$. For $3 \leq m \leq p$ we now take $P_2 = [m-1]P_1$; it follows by induction on m that (as p-adic numbers)

$$\alpha(m^2 - m) \equiv (m^2 - m + 1)\zeta_1^{-1} \bmod mp^{2-n_1},$$

$$m^2\xi([m]P_1) \equiv \zeta_1^{-2} \bmod m^2 p^{2-2n_1},$$

$$m^3\eta([m]P_1) \equiv \zeta_1^{-3} \bmod m^3 p^{2-3n_1},$$

and therefore $\zeta([m]P_1) \equiv m\zeta_1 \bmod mp^{2+n_1}$.

It follows by induction on r that $\zeta([p^r]P_1) \equiv p^r\zeta_1 \bmod p^{r+n_1+1}$ for any p and any $r > 0$. Now suppose that $n_2 > n_1$. By (2),

$$\xi_2 \equiv \zeta_2^{-2} \bmod p^{2n_2+2}, \quad \eta_2 \equiv \zeta_2^{-3} \bmod p^{3n_2+2}$$

and therefore $\alpha \equiv \zeta_2^{-1}(1 - \eta_1/\eta_2 + \xi_1/\xi_2) \bmod p^{3n_2-4n_1}$. Hence (as p-adic numbers)

$$\xi(P_1 + P_2) \equiv \xi_1 - 2\eta_1\zeta_2 \bmod 2p^{2n_2-4n_1} \quad (4)$$

by the first equation of (3). Now $\eta(P_1 + P_2) \equiv \eta_1 \bmod p^{-n_2-2n_1}$ by the second equation of (3). This tells us which square root to take in the equation for E, and combining this fact with (4) gives

$$\eta(P_1 + P_2) \equiv \eta_1 \bmod 2p^{n_2-4n_1}. \quad (5)$$

Now (4) and (5), taken with the first sentence of this paragraph, prove that $\mathbb{Z}$ acts continuously on E'. Hence $\mathbb{Z}_p$ acts on E' too, and this action is continuous. If A_0 is as in the lemma and A satisfies $v(\xi) \leq v(\xi(A_0))$, a successive approximation

argument shows that there is an m in $\mathbb{Z}_p$ such that $A = [m]A_0$. If n in $\mathbb{Z}$ is close to m then $[n]A_0$ will be close to A. $\qquad\square$

Theorem 3. *Let X_1, X_2 be as in the proof of Theorem 1, and let P_0 be a point of $V(\mathbb{Q})$ not in $X_1 \cup X_2$. Then there is a constructable open neighbourhood $\mathscr{D}$ of P_0 in $\prod V(\mathbb{Q}_p)$, where the product is taken over all p in $\mathscr{S}$, such that $V(\mathbb{Q})$ is dense in $\mathscr{D}$. A similar result holds for $\mathscr{S}^+$.*

Proof. In the notation of Lemma 1, we can rescale the equations of W so that J_{01} satisfies the conditions imposed on E for each p in $\mathscr{S}$, where J_{01} is again the Jacobian of the fibre of $\mathscr{F}_1$ through P_0. For each such p, let r_p be the least positive integer r such that $[r]\psi_1(P_0)$ is in the set J_{01}' derived from J_{01} in the way that E' was derived from E and let r^* be a convenient common multiple of the r_p for p in $\mathscr{S}$. Let $(\xi^*, \eta^*) = [r^*]\psi_1(P_0)$ and let m_p be the exact power of p that divides ξ^*/η^*. Let $\mathscr{M}_{p1}(P_0) \subset J_{01}(\mathbb{Q}_p)$ be the subset of J_{01}' which corresponds to the ζ in $m_p\mathbb{Z}_p$ under the isomorphism corresponding to (1), and let $\mathscr{N}_{p1}(P_0) \subset C_{01}(\mathbb{Q}_p)$ consist of the points obtained from P_0 by translation by an element of $\mathscr{M}_{p1}(P_0)$. If we denote by P_r the translation of P_0 by $[r]\psi_1(P_0)$ then by the Chinese remainder theorem the P_r for which r is a multiple of r^* are distinct and dense in $\prod \mathscr{N}_{p1}(P_0)$ where the product is taken over all p in $\mathscr{S}$.

We can apply a similar construction with $\nu = 2$ to obtain a neighbourhood $\mathscr{A}$ of P_0 in $\prod C_{02}(\mathbb{Q}_p)$ in which points of $V(\mathbb{Q})$ are dense. If in the construction of the previous paragraph we replace P_0 by any point P' in $\mathscr{A}$, everything involved in the construction varies continuously with P'; so after reducing $\mathscr{A}$ if necessary, we can assume that the r_p and the m_p are independent of $\mathscr{A}$. One consequence of this is that if $P' = \prod P_p'$ then $\psi_1(P_p')$ cannot be a torsion point; and it is now easy to see that we can choose $\mathscr{D}$ to be the union of the $\prod \mathscr{N}_{p1}(P')$ where the union is taken over all points P' of $\mathscr{A}$.

The corresponding result for $\mathscr{S}^+$ now follows easily. For let $\mathscr{D}_0$, containing P_0, be a small open subset of the $\mathscr{D}$ associated with $\mathscr{S}$; by diminishing $\mathscr{D}_0$ we can further assume that it is disjoint from $X_1 \cup X_2$. Let $\mathscr{R}(\mathscr{D}_0)$ be the closure of the projection of $\phi V(\mathbb{Q}) \cap (\mathscr{D}_0 \times V(\mathbb{R}))$ onto $V(\mathbb{R})$, where ϕ is the obvious diagonal map. Then it is enough to show that there is a nonempty open set contained in $\mathscr{R}(\mathscr{D}_0)$ and independent of $\mathscr{D}_0$. Suppose that r^* is even and divisible by a high enough power of every p in $\mathscr{S}$ and by all the r_p defined in the second paragraph of this proof. Then r^* depends only on $\mathscr{D}_0$, and the translation of P_0 by $[r^*]\psi_1(P_0)$ lies in $\mathscr{D}_0$ and is dense in the connected component containing P_0 of the fibre of $\mathscr{F}_1$ through P_0. There is a corresponding result for $\mathscr{F}_2$. So the argument used to prove Theorem 2 still applies in this situation, and the boundary of $\mathscr{R}(\mathscr{D}_0)$ lies in $Z \cup X$, which is independent of $\mathscr{D}_0$. Thus $\mathscr{R}(\mathscr{D}_0)$ contains that connected component of $V(\mathbb{R}) \setminus (Z \cup X)$ in which P_0 lies. $\qquad\square$

In what circumstances will the machinery of Theorem 3 enable us to prove that $\overline{V(\mathbb{Q})}$, in the topology generated by $\mathscr{S}$, is at least as large as it actually is? (A similar question for $\mathscr{S}^+$ can be addressed in much the same way; the details are left to the reader.) For this purpose we make the temporary assumption that $\overline{V(\mathbb{Q})}$ is open as well as closed — which would certainly hold if for example the entire obstruction to weak approximation consists of finitely many Brauer–Manin conditions.

Suppose that P_0 is in $V(\mathbb{Q}_p)$ and not on a singular fibre of $\mathscr{F}_1$. Let r_p be the least positive r such that $[r]\psi_1(P_0)$ is in J'_{01} in the notation of the last proof; then $\psi_1(P_0)$ can only be a torsion point of J_{01} if $[r_p]\psi_1(P_0)$ is the identity element of J_{01}. Thus the points P_0 in $V(\mathbb{Q}_p)$ not on a singular fibre of $\mathscr{F}_1$ and for which $\psi_1(P_0)$ is torsion lie on a (reducible) curve Z_{1p}. Let $\mathscr{X}$ be a compact open subset of $\prod V(\mathbb{Q}_p)$ such that if $\prod A_p$ is in $\mathscr{X}$ then no A_p lies in Z_{1p} or on a singular fibre of $\mathscr{F}_1$. By compactness, the values of m_p and r_p associated with A_p are bounded as $\prod A_p$ varies in $\mathscr{X}$; thus if P_0 is a point of $\mathscr{X} \cap V(\mathbb{Q})$ the size of the domain $\mathscr{D}$ associated with P_0 in the sense of Theorem 3 is bounded below. This implies that provided $V(\mathbb{Q})$ is actually dense in $\mathscr{X}$ we can prove this by exhibiting finitely many points of $V(\mathbb{Q})$. What we want to prove for any particular V is of course stronger than this. But these considerations do suggest that it will be easiest to prove that $\overline{V(\mathbb{Q})}$ as a subset of $\prod V(\mathbb{Q}_p)$ is as large as it actually is when there are so many elliptic fibrations $\mathscr{F}_\nu$ that $\bigcap X_\nu(\mathbb{Q})$ is empty. Notice that for specific V we can usually modify the argument so as to make $\mathscr{D}$ substantially larger than the $\mathscr{D}$ constructed above. We illustrate this in Section 4.

3. *An example of Theorems 1 and 2.* The best known examples of K3 surfaces with at least two elliptic pencils are given by equations of the form

$$a_0 X_0^4 + a_1 X_1^4 + a_2 X_2^4 + a_3 X_3^4 = 0 \quad \text{where} \quad a_0 a_1 a_2 a_3 = b^2. \tag{6}$$

Theorems on the density of rational points in the real and the Zariski topologies on such a surface, assuming that it contains at least one rational point lying outside a certain subvariety, already appear in [Logan et al. 2010]. The earliest example of such a result, for the real topology and one special V, appears to be [Swinnerton-Dyer 1968]. The arguments in [Logan et al. 2010] do not appear to extend to the most general V considered in this paper, but the methods of this paper do yield, in the case of (6), a simpler proof of their result, which is stated here as Theorem 4. For this, and for later applications, we need to calculate the exceptional sets X and Z for (6), and for this it is convenient to repeat some formulae from [Swinnerton-Dyer 2000], though the reader is warned that there are some errors in that paper.

Denote by V the K3 surface given by (6). The union of the singular fibres of V is the union of the 48 lines on V. There is an obvious map from V to

$$T : a_0 Y_0^2 + a_1 Y_1^2 + a_2 Y_2^2 + a_3 Y_3^2 = 0. \tag{7}$$

Since $a_0a_1a_2a_3$ is a square, each of the two families of lines on T is defined over $\mathbb{Q}$, and the lifts of these families to V give two elliptic fibrations on V. The fibre through $P_0 = (x_0, x_1, x_2, x_3)$ of one of the fibrations is given by four equations of the form

$$d_{il}X_i^2 + d_{jl}X_j^2 + d_{kl}X_k^2 = 0 \tag{8}$$

where i, j, k, l are 0, 1, 2, 3 in some order. Any three of these equations are linearly dependent. A possible set of values of the coefficients is

$$\left.\begin{array}{ll} d_{01} = b(a_2x_2^4 + a_3x_3^4), & d_{23} = (b/a_0a_1)d_{01}, \\ d_{02} = -a_2(a_0a_3x_0^2x_3^2 + bx_1^2x_2^2), & d_{31} = (b/a_0a_2)d_{02}, \\ d_{03} = a_3(a_0a_2x_0^2x_2^2 - bx_1^2x_3^2), & d_{12} = (b/a_0a_3)d_{03}, \end{array}\right\} \tag{9}$$

together with $d_{ji} = -d_{ij}$ for all i, j. The d_{ij} are nonzero since the fibre is absolutely irreducible. In these formulae we can permute the subscripts, provided that for an odd permutation we also change the sign of b. The Jacobian of this fibre can be written in the form

$$\eta^2 = (\xi - c_1)(\xi - c_2)(\xi - c_3) \tag{10}$$

where

$$c_1 - c_2 = d_{03}d_{21}, \quad c_2 - c_3 = d_{01}d_{32}, \quad c_3 - c_1 = d_{02}d_{13}. \tag{11}$$

If temporarily we write $P = (\xi, \eta)$, then

$$\xi([2]P) - c_2 = \frac{((\xi - c_2)^2 - d_{01}d_{03}d_{21}d_{23})^2}{4\eta^2}. \tag{12}$$

The map $\psi : V \to W$, which is effectively the canonical map from each fibre to its Jacobian, can be taken to be

$$\eta = d_{12}d_{23}d_{31}X_1X_2X_3/X_0^3, \quad \xi - c_i = d_{ij}d_{ki}X_i^2/X_0^2 \tag{13}$$

where i, j, k is any permutation of 1, 2, 3. This map has degree 4. There is also an identification of the fibre with its Jacobian which maps P_0 to the identity. This satisfies

$$\xi - c_i = d_{0j}d_{0k}\frac{d_{ji}x_jX_j + d_{ki}x_kX_k - d_{0i}x_0X_0}{d_{i0}x_iX_i + d_{j0}x_jX_j + d_{k0}x_kX_k} \tag{14}$$

where again i, j, k is any permutation of 1, 2, 3. The formula for η is too complicated to be useful. But an immediate deduction from the formula for ξ is that the translation along the fibre of either system by the 2-division point $(c_i, 0)$ is obtained by changing the signs of x_0 and x_i. We shall use this in the proof of Theorem 4. To obtain formulae for the other fibration we need only replace b by $-b$ in all the formulae of this paragraph.

Lemma 2. *Suppose that the point A_0 given by* (13) *with x_i for X_i is a rational torsion point, that $x_0x_1x_2x_3 \neq 0$ and that P_0 does not lie on any of the lines of V. Then*

$$4a_j^2 d_{kl}^2 x_j^4 x_k^2 x_l^2 + d_{jk} d_{jl}(a_i x_i^4 + a_j x_j^4)^2 = 0 \tag{15}$$

for some permutation i, j, k, l of $0, 1, 2, 3$. This condition depends only on the value of i.

Proof. The point A_i that is the translation of A_0 by the 2-division point $(c_i, 0)$ on the Jacobian satisfies

$$\xi - c_i = \frac{d_{0j}d_{k0}x_0^2}{x_i^2}, \quad \xi - c_j = \frac{d_{0k}d_{kj}x_k^2}{x_i^2}, \quad \xi - c_k = \frac{d_{0j}d_{jk}x_j^2}{x_i^2} \tag{16}$$

where again i, j, k is a permutation of $1, 2, 3$. We can restore the symmetry by writing

$$c_i = c_{i0} = c_{0i} = c_{jk} = c_{kj};$$

thus we have $c_{ik} - c_{jk} = d_{ij}d_{kl}$ for any even permutation i, j, k, l of $0, 1, 2, 3$. The point A_i is now given by

$$\xi - c_{ij} = d_{jk}d_{lj}x_j^2/x_i^2 \tag{17}$$

where i, j, k, l is any permutation of $0, 1, 2, 3$; in particular A_i is not a 2-division point. Such a value of ξ corresponds to two distinct rational values of η, so we appear to have constructed twelve torsion points on (10). If these are not all distinct then at least two of the four values of ξ given by (17) must be equal. Suppose for example that the values of ξ given by $i = 0$ and $i = 1$ are equal; then by comparing the two formulae for $\xi - c_1$ we obtain $a_0 x_0^4 + a_1 x_1^4 = 0$, which implies that P_0 lies on one of the lines of V.

Suppose instead that the twelve torsion points exhibited above are all distinct, so that by Mazur's theorem they form a group of order 12. Without loss of generality we can assume that one of the two 3-division points is A_0. The condition that (ξ, η) is a 3-division point of (10) can be written in the form

$$4\eta^2(\xi - c_1) = \{(\xi - c_1)^2 - (c_1 - c_2)(c_1 - c_3)\}^2,$$

which reduces by (10) and (11) to

$$4a_1^2 d_{23}^2 x_1^4 x_2^2 x_3^2 + d_{12} d_{13}(a_0 x_0^4 + a_1 x_1^4)^2 = 0.$$

Symmetry now completes the proof of the lemma.   □

Theorem 4. *Suppose that V is given by* (6). *If there is a rational point $P_0 = (x_0, x_1, x_2, x_3)$ on V with $x_0x_1x_2x_3 \neq 0$ and not lying on one of the lines of V, then $V(\mathbb{Q})$ is Zariski dense on V and dense in $V(\mathbb{R})$.*

Proof. We retain the notation in the proof of Theorem 1. To deduce Zariski density from Theorem 1 we need to show that P_0 does not lie in X. The first step is to show that the fibre of $\mathcal{F}_\nu$ is not contained in $\psi_\mu^{-1}(Y_\mu'')$ — in other words, that if P' is generic on the fibre of $\mathcal{F}_\nu$ through P_0 then $\psi_\mu(P')$ is not a torsion point. Suppose otherwise; then the exact order of this torsion point cannot depend on the choice of P', provided that P' does not lie on a singular fibre of $\mathcal{F}_\mu$. By the proof of Lemma 2 this exact order must be 3 or 6. But by changing the signs of a suitable pair of x_i we can alter this order from 3 to 6 or from 6 to 3, and this is a contradiction.

It is now enough to show that P_0 does not lie in $(\psi_1^{-1}Y_1'')\cap(\psi_2^{-1}Y_2'')$; by Lemma 2 this is equivalent to showing that it does not satisfy (15) for both b and $-b$. Suppose otherwise; without loss of generality we can assume that for $b > 0$ it satisfies (15) with $i = 0$ and that a_0, a_2 are positive and a_1, a_3 negative. In (15) we take $j = 1, k = 2, l = 3$; by (9) we obtain

$$a_0a_2a_3x_0^4x_2^2x_3^2 + a_2bx_0^2x_1^2x_2^4 - a_3bx_0^2x_1^2x_3^4 + 3a_1a_2a_3x_1^4x_2^2x_3^2 = 0. \qquad (18)$$

On the left the first term is negative and the other three are all positive; so $a_0a_2x_0^2x_2^2 > bx_1^2x_3^2$ and $a_0x_0^4 > -3a_1x_1^4$. The first of these, taken with (6), implies that $a_0x_0^4$ and $a_2x_2^4$ lie between $-a_1x_1^4$ and $-a_3x_3^4$; so

$$-a_3x_3^4 > (a_0x_0^4 \text{ and } a_2x_2^4) > -a_1x_1^4. \qquad (19)$$

We can argue similarly for any one of the four equations (15) with $-b$ for b; the only one which yields a condition compatible with (19) is the one with $i = 2$. Remembering that we can cyclically permute the subscripts $1, 2, 3$ in (18), we therefore have

$$a_0a_3a_1x_0^4x_3^2x_1^2 + a_3bx_0^2x_2^2x_3^4 - a_1bx_0^2x_2^2x_1^4 + 3a_2a_3a_1x_2^4x_3^2x_1^2 = 0,$$
$$3a_0a_3a_1x_0^4x_3^2x_1^2 + a_3bx_0^2x_2^2x_3^4 - a_1bx_0^2x_2^2x_1^4 + a_2a_3a_1x_2^4x_3^2x_1^2 = 0.$$

The difference of these is $2a_1a_3x_1^2x_3^2(a_0x_0^4 - a_2x_2^4) = 0$. But now either of the last two displayed equations implies that $a_3x_3^4/a_1x_1^4 = (2 + \sqrt{5})^2$, so that P_0 would not be rational. This completes the proof of Zariski density.

To deduce density in $V(\mathbb{R})$ from Theorem 2 we still need to find the points at which nonsingular fibres of $\mathcal{F}_1$ and $\mathcal{F}_2$ touch. These fibres are the pull-backs of the lines on (7), on which two lines of opposite systems are always transversal; so the points at which fibres of $\mathcal{F}_1$ and $\mathcal{F}_2$ touch are just those which have multiplicity greater than 1 in the pull-back of a point of (7). These are just those points of V at which at least one of the x_i vanishes. Thus a maximum connected component in $V(\mathbb{R})$ of the complement of Z consists of all the points at which the signs of the X_i/X_j take assigned values; and since we can vary the signs of the x_i, if one of

these connected components meets $V(\mathbb{Q})$ then each of them does. Thus the closure of $V(\mathbb{Q})$ contains the whole of $V(\mathbb{R}) \setminus Z$, and therefore the whole of $V(\mathbb{R})$. □

4. *An example of Theorem 3.* In this section we treat two particular examples of Theorem 3, for each of which V has the form

$$a_0 X_0^4 + a_1 X_1^4 = a_0 X_2^4 + a_1 X_3^4 \quad \text{with } a_0, a_1 \text{ positive.} \tag{20}$$

In constructing examples to illustrate the machinery of Theorem 3, we have to choose a suitable $\mathscr{S}$. It is not unreasonable to hope that the only places involved in describing the obstruction to weak approximation on a given V are those at which V has bad reduction and possibly the infinite place. (The corresponding result for nonsingular cubic surfaces is known; see [Swinnerton-Dyer 2001]. An analogous result for the Brauer–Manin obstruction on arbitrary diagonal quartic surfaces defined over $\mathbb{Q}$ is due to Colliot-Thélène and Skorobogatov [2013].) If we want V to have the form (6) and to have bad reduction only at 2, then there are just seven such V which are everywhere locally soluble, and there are just two of them for which the arithmetic part of the Brauer–Manin obstruction is trivial. (See Bright's table [2002, Appendix A].) These can conveniently be written as

$$V_c : \ X_0^4 + c X_1^4 = X_2^4 + c X_3^4 \quad \text{for } c = a_1/a_0 = 2, 4 \text{ or } 8.$$

Here V_2 and V_8 are the same, but by considering both we can confine ourselves to the study of points for which the X_i are integers with X_0, X_2 odd. In Bright's table V_4 is case A75, and Ieronymou, Skorobogatov and Zarhin [Ieronymou et al. 2011] have shown that it has no transcendental Brauer–Manin obstruction either. V_2 and V_8 are instances of Bright's case A104. We shall prove

Theorem 5. $V_c(\mathbb{Q})$ *is dense in* $V_c(\mathbb{Q}_2)$ *for* $c = 2, 4, 8$.

There is a substantial disparity between the numbers of essentially distinct nontrivial solutions of V_2 and of V_4: for example, V_4 has 599 such solutions of height less than 25000, the smallest having height 9, whereas V_2 has 43 such solutions of height less than 25000, the smallest having height 139. This is perhaps accounted for by the fact that the Nèron–Severi group of V_4 has rank 9 whereas that of V_2 has rank 6.

The general surface (20) is case A45 in Bright's table. This surface contains four rational lines. If it has other parametric solutions, which I doubt, even the simplest nonsingular one has degree at least 17. It has three kinds of elliptic pencils of low degree, where the degree means the degree of the general curve of the pencil. Apart from these, which are described below, if there are any other elliptic pencils whose general fibre is nonsingular, they have degree at least 19.

There are four elliptic pencils of degree 3, the curves of such a pencil being those which lie in a plane through one of the rational lines and are residual to that line.

A typical such pencil, which we shall call $\mathscr{F}_3$, consists of the curves given by

$$\left.\begin{array}{l} X_0 - X_2 = \lambda(X_1 - X_3), \\ \lambda a_0(X_0^3 + X_0^2 X_2 + X_0 X_2^2 + X_2^3) + a_1(X_1^3 + X_1^2 X_3 + X_1 X_3^2 + X_3^3) = 0, \end{array}\right\} \quad (21)$$

where λ is a parameter. This curve contains the rational point $(\lambda, 1, -\lambda, -1)$ and can therefore be taken to be its own Jacobian. In general it has no rational torsion points, but if $\lambda = -a_1/a_0\alpha^3$ then it contains the point $(\alpha, 1, \alpha, 1)$, which is a 2-division point. By means of the transformation

$$\left.\begin{array}{ll} X_0 = -a_1 X + \lambda Y, & X_1 = a_0\lambda(a_1^2 - a_0^2\lambda^8)Z + a_0\lambda^3 X + Y, \\ X_2 = -a_1 X - \lambda Y, & X_3 = a_0\lambda(a_1^2 - a_0^2\lambda^8)Z + a_0\lambda^3 X - Y, \end{array}\right\} \quad (22)$$

whose inverse is

$$X = -\frac{X_0 + X_2}{2a_1}, \quad Y = \frac{X_1 - X_3}{2}, \quad Z = \frac{X_1 + X_3 - 2a_0\lambda^3 X}{2a_0\lambda(a_1^2 - a_0^2\lambda^8)}, \quad (23)$$

we can take this curve into the canonical form

$$Y^2 Z = X^3 - 3a_0^2\lambda^6 X^2 Z - 3\lambda^4 a_0^2\theta X Z^2 - a_0^2\lambda^2\theta^2 Z^3 \quad (24)$$

where $\theta = a_1^2 - a_0^2\lambda^8$. If we write $X = (U + a_0^2\lambda^6)Z$ and $Y = WZ$, this becomes

$$W^2 = U^3 - 3a_0^2 a_1^2\lambda^4 U - a_0^2 a_1^2\lambda^2(a_1^2 + \lambda^8 a_0^2). \quad (25)$$

$\mathscr{F}_3$ is the only elliptic pencil on (20) that we shall use in the argument that follows. However, for possible applications elsewhere we record some useful information about the other pencils of low degree on (20).

There are two elliptic pencils of degree 4, which were described for the more general surface (6) in Section 3. There are four elliptic pencils of degree 6; a typical one, parametrized by α, consists of the curves which are the intersections of the cubic surfaces

$$a_0(X_0^3 + X_0^2 X_2 + X_0 X_2^2 + X_2^3) + \alpha(X_0 X_3 - X_1 X_2)(X_1 - X_3) = 0,$$
$$a_1(X_1^3 + X_1^2 X_3 + X_1 X_3^2 + X_3^3) - \alpha(X_0 X_3 - X_1 X_2)(X_0 - X_2) = 0,$$

residual to the three common lines $X_0 = \kappa X_2, X_1 = \kappa X_3$ where $\kappa = -1$ or $\kappa = \pm\sqrt{-1}$. This curve is a 2-covering of its Jacobian, and in canonical form it can be written

$$Y^2 = \alpha^3(a_0 + a_1 X^4) + a_0 a_1(3\alpha^2 - a_0 a_1)X^2$$

where

$$X = \frac{X_1 + X_3}{X_0 + X_2}, \quad Y = \frac{a_1(a_0 + \alpha X^2)(X_1 + X_3)}{X_0 - X_2}.$$

We can recover the original variables by writing

$$X_0 = Y + a_1 X (a_0 + \alpha X^2), \quad X_1 = XY - a_0(\alpha + a_1 X^2),$$
$$X_2 = Y - a_1 X (a_0 + \alpha X^2), \quad X_3 = XY + a_0(\alpha + a_1 X^2).$$

The proof of Theorem 5 consists of a number of steps, each of which is described in a separate lemma. Throughout, v will be the normalized 2-adic valuation on $\mathbb{Q}_2$ and $v(c) = \gamma$. In Lemmas 3 and 4, $P_0 = (x_0, x_1, x_2, x_3)$ will be a point of $V_c(\mathbb{Q}_2)$, with the x_i odd; it then follows from the second equation of (21) that

$$v(\lambda) + v(x_0 + x_2) = \gamma + v(x_1 + x_3). \tag{26}$$

The next two lemmas, taken together, provide for V_c a quantitative version of Theorem 3. Let $\mathscr{F}_3$ be the fibration of degree 3 described by (21).

Lemma 3. *Let P_0 be in $V_c(\mathbb{Q})$ with all x_i odd, and let F_3 be the fibre of $\mathscr{F}_3$ through P_0.*

(i) *If $2 \parallel (x_0 - x_2)$ and $2 \parallel (x_1 - x_3)$, then $F_3(\mathbb{Q})$ is dense in that part of $F_3(\mathbb{Q}_2)$ in which the X_i are odd with $2 \parallel (X_1 - X_3)$ and $v(X_0 + X_2) \geq v(x_0 + x_2)$.*

(ii) *If $2 \parallel (x_0 + x_2)$ and $2 \parallel (x_1 - x_3)$, then $F_3(\mathbb{Q})$ is dense in that part of $F_3(\mathbb{Q}_2)$ in which the X_i are odd.*

Proof. Let $\mathscr{D}$ be that part of $F_3(\mathbb{Q}_2)$ in which we have to prove that $F_3(\mathbb{Q})$ is dense. In the notation of (22) and (23), let x, y, z be the values of X, Y, Z at P_0 and let $\lambda = (x_0 - x_2)/(x_1 - x_3)$ be the value of the parameter in (21) which defines F_3. We can take $a_0 = 1$, $a_1 = c$.

In case (i), we have $v(\lambda) = 0$; it now follows that $v(X_1 + X_3) = v(2a_0\lambda^3 X)$ and $v(Z) > v(X)$. Thus $3v(X/Z) = v(Y/X) \leq v(y/x) = 3v(x/z) < 0$, where the equalities follow from (24). The conclusion now follows from Lemma 1.

In case (ii), we have $v(\lambda) = \gamma + \delta - 1$ by (26), where $\delta = v(x_1 + x_3) \geq 2$; so if we write $\xi = 2^{-2\gamma} X/Z$ and $\eta = 2^{-3\gamma} Y/Z$, the curve F_3 takes the form $\eta^2 = \xi^3 + \cdots$ with coefficients satisfying the conditions of Lemma 1. At P_0 we have $v(x) = -\gamma$, $v(y) = 0$, $v(z) = -3\gamma$, where the last result again follows from (26); so $v(\xi) = 0$. Hence, by Lemma 1, $F_3(\mathbb{Q})$ is dense in that part of $F_3(\mathbb{Q}_2)$ in which $v(\xi) \leq 0$. But in $\mathscr{D}$ we have

$$v(X_0 + X_2) = v(X_1 + X_3) + \gamma - v(\lambda) = v(X_1 + X_3) + 1 - \delta.$$

Hence $v(X_1 + X_3) < v(2\lambda^3 X) = v(X_0 + X_2) + 2\gamma + 3\delta - 3$, so that

$$v(Z) = v(X_1 + X_3) - v(\lambda) - 2\gamma - 1 = v(X_0 + X_2) - 3\gamma - 1$$

whence $v(\xi) = v(X_0 + X_2) - v(Z) - 3\gamma - 1 = 0$. □

Lemma 4. *Suppose there is a point P_0 of $V_c(\mathbb{Q})$ with all x_i odd and $2^2 \parallel (x_1 + x_3)$. Let $\mathcal{D}_1$ be the part of $V_c(\mathbb{Q}_2)$ in which the X_i are odd. Then $V_c(\mathbb{Q})$ is dense in $\mathcal{D}_1$.*

Proof. Let $Q = (u_0, u_1, u_2, u_3)$ be a point of $\mathcal{D}_1$; after changing the signs of u_2 and/or u_3 if necessary, we can assume that $2 \parallel (u_0 - u_2)$ and $2 \parallel (u_1 - u_3)$, and we write $\lambda = (u_0 - u_2)/(u_1 - u_3)$. Similarly we can assume that $2 \parallel (x_0 - x_2)$. Since $v(\lambda) = 0$, the analogue for Q of (26) shows that $v(u_0 + u_2) \geq \gamma + 2$. Now let F_3' be the fibre of $\mathcal{F}_3$ through Q. Write $\mu = (x_0 + x_2)/(x_1 - x_3)$ and let F_3'' be the intersection of V_c with $X_0 + X_2 = \mu(X_1 - X_3)$ residual to the line $X_0 + X_2 = X_1 - X_3 = 0$; thus P_0 lies on F_3''.

Let $R = (y_0, y_1, y_2, y_3)$ be a point of $F_3' \cap F_3''$. If we can ensure that the y_i are odd elements of $\mathbb{Z}_2$ with $2 \parallel (y_0 - y_2)$, $2 \parallel (y_1 - y_3)$ and $v(u_0 + u_2) \geq v(y_0 + y_2)$, we can argue as follows. By Lemma 3(ii) with the sign of X_2 reversed, $F_3''(\mathbb{Q})$ is dense in that part of $F_3''(\mathbb{Q}_2)$ in which all the X_i are odd integers; hence there is a point R' of $F_3''(\mathbb{Q})$ arbitrarily close to R. Let F_3^* be the fibre of $\mathcal{F}_3$ through R', so that it is arbitrarily close to F_3'. By Lemma 3(i) $F_3^*(\mathbb{Q})$ is dense in that part of $F_3^*(\mathbb{Q}_2)$ in which the X_i are odd with $2 \parallel (X_1 - X_3)$ and $v(X_0 + X_2) \geq v(y_0 + y_2)$, and in particular in that part of $F_3^*(\mathbb{Q}_2)$ which is close to Q. Hence there is a point of $V_c(\mathbb{Q})$ close to Q.

It remains to prove the assertions about R. The points of $F_3' \cap F_3''$ are those points (y_0, y_1, y_2, y_3) of V_c at which

$$y_0 - y_2 = \lambda(y_1 - y_3), \quad y_0 + y_2 = \mu(y_1 - y_3), \quad y_1 - y_3 \neq 0,$$

where $v(\lambda) = 0$, $v(\mu) = \gamma + \delta - 1$. Write $y_1 = y_3 + 2\eta$; then $y_0 = (\lambda + \mu)\eta$ and $y_2 = (\mu - \lambda)\eta$, so that the equation for R becomes

$$\lambda\mu(\lambda^2 + \mu^2)\eta^3 + c(y_3 + \eta)(y_3^2 + 2y_3\eta + 2\eta^2) = 0.$$

By Hensel's lemma, this has a solution in $\mathbb{Z}_2$ with y_3, η odd; and for this solution $v(y_3 + \eta) = \delta - 1$ and thus $v(y_1 + y_3) = \delta \leq v(u_1 + u_3)$. $\qquad \square$

Suitable points P_0 exist for each of the three values of c. We can for example take P_0 to be $(849, 653, -969, 167)$ when $c = 2$, $(189, 677, -557, -657)$ when $c = 4$, and $(1197, 177, 499, 707)$ when $c = 8$.

The next two lemmas again provide a quantitative version of Theorem 3. In them we denote by $\mathcal{D}_3$ the part of $V_c(\mathbb{Q}_2)$ in which X_0, X_2 are odd and X_1, X_3 even, with just one of X_1, X_3 divisible by 4; and $P_0 = (x_0, x_1, x_2, x_3)$ will be a point of $\mathcal{D}_3$ defined over $\mathbb{Q}$. As before let x, y, z be the values of X, Y, Z at P_0 and let $\lambda = (x_0 - x_2)/(x_1 - x_3)$ be the value of the parameter in (21) which defines F_3. We can take $a_0 = 1$, $a_1 = c$.

Lemma 5. *Let F_3 be the fibre of $\mathcal{F}_3$ through P_0, where $2 \parallel (x_0 - x_2)$. Then $F_3(\mathbb{Q})$ is dense in $F_3 \cap \mathcal{D}_3$.*

Proof. Since $v(\lambda) = 0$, we have $v(X_0 - X_2) = 1$ at each point of $F_3 \cap \mathscr{D}_3$. Thus at each such point $v(X_0 + X_2) = \gamma + 2$ and $v(X) = 1$, $v(Y) = 0$, $v(Z) = 0$. In the notation of (25) we have $v(U) = v(W) = 0$; and (25) satisfies the conditions of Lemma 1. □

Lemma 6. *Suppose that there is a point P_0 in $\mathscr{D}_3$ defined over $\mathbb{Q}$. Then $V_c(\mathbb{Q})$ is dense in $\mathscr{D}_3$.*

Proof. Let $Q = (u_0, u_1, u_2, u_3)$ be a point of $\mathscr{D}_3$; after changing the sign of x_2 and/or u_2 if necessary, we can assume that $2 \parallel (x_0 - x_2)$ and $2 \parallel (u_0 - u_2)$. Write $\lambda = (u_0 - u_2)/(u_1 - u_3)$ and $\mu = (x_0 - x_2)/(x_1 + x_3)$. Let F_3' be the fibre of $\mathscr{F}_3$ through Q and F_3'' the intersection of V_c with $X_0 - X_2 = \mu(X_1 + X_3)$ residual to the line $X_0 - X_2 = X_1 + X_3 = 0$; thus P_0 lies on F_3''.

Let $R = (y_0, y_1, y_2, y_3)$ be a point of $F_3' \cap F_3''$. If we write $y_0 + y_3 = \chi(y_0 - y_2)$ and use $y_0 - y_2 = \lambda(y_1 - y_3) = \mu(y_1 + y_3)$, we find that χ satisfies

$$\chi(\chi^2 + 1) = c(\lambda^2 + \mu^2)(\lambda\mu)^{-3}.$$

Since $v(\lambda) = v(\mu) = 0$, this equation is satisfied by one value of χ in $\mathbb{Q}_2$ with $v(\chi) = \gamma + 1$, and the corresponding R is in $\mathscr{D}_3$. The lemma now follows by the same argument as in the second paragraph of the proof of Lemma 4. □

Suitable points P_0 exist for each of the three values of c. We can for example take P_0 to be $(489, 684, -577, 662)$ for $c = 2$, $(61, 168, -237, 58)$ for $c = 4$, and $(257, 22, 223, 124)$ for $c = 8$.

Henceforth, for any integer $\beta > 0$ let $\mathscr{D}_4^{\beta}$ consist of those points of $V_c(\mathbb{Q}_2)$ with X_0, X_2 odd and $v(X_1) = v(X_3) = \beta$ and let $\mathscr{D}_5^{\beta}$ consist of those points of $V_c(\mathbb{Q}_2)$ with X_0, X_2 odd and $v(X_1) > v(X_3) = \beta$ or $v(X_3) > v(X_1) = \beta$. Thus $\mathscr{D}_5^1$ is what we have previously called $\mathscr{D}_3$. To complete the proof of Theorem 5 we need to prove that $V_c(\mathbb{Q})$ is dense in each $\mathscr{D}_4^{\beta}$ with $\beta > 0$ and each $\mathscr{D}_5^{\beta}$ with $\beta > 1$.

Lemma 7. *If $V_c(\mathbb{Q})$ is dense in $\mathscr{D}_1$ then it is dense in $\mathscr{D}_5^{\beta}$ for each $\beta > 1$.*

Proof. Let $Q = (u_0, u_1, u_2, u_3)$ be a point of $\mathscr{D}_5^{\beta}$ where $\beta > 1$; after changing the sign of u_2 if necessary, we can assume that $2 \parallel (u_0 - u_2)$ and therefore $v(u_0 + u_2) = 4\beta - 2$. Let $\lambda = (u_0 - u_2)/(u_1 - u_3)$, so that $v(\lambda) = 1 - \beta$. At Q we have $v(X) = 4\beta - 3$, $v(Y) = \beta - 1$ and therefore $v(Z) = 10(\beta - 1)$; hence also $v(U) = 6(1 - \beta)$, $v(W) = 9(1 - \beta)$. If we write $\xi = 2^{4(\beta-1)}U$, $\eta = 2^{6(\beta-1)}W$ in (25), then the equation between ξ and η satisfies the conditions of Lemma 1. In the notation of that lemma, let $R = (y_0, y_1, y_2, y_3)$ be the point of E' which satisfies $\zeta(R) = 2^{1-\beta}\zeta(Q)$; R exists because $v(\zeta(Q)) = \beta - 1$. Thus $v(\xi/\eta) = 0$ at R, and so $v(\xi) = v(\eta) = 0$ since R is in E'. Moreover if we choose the representation of R so that $v(y_0 - y_2) = 1$, we have $v(y_1 - y_3) = 1 - v(\lambda) = \beta$; so $v(Y) = \beta - 1$, $v(Z) = 7(\beta - 1)$, $v(X) = \beta - 1$,

$v(y_0 + y_2) = \beta + \gamma > 1$ at R. The results for $y_0 \pm y_2$ show that y_0 and y_2 are odd integers. If we assume that $v(y_1 + y_3) \leq 0$, we find that

$$\beta + \gamma + 2 = v(y_0^4 - y_2^4) = \gamma + v(y_1^4 - y_3^4) \leq \beta + \gamma - 1,$$

and if instead we assume that $v(y_1 + y_3) \geq 2$, we find that

$$\beta + \gamma + 2 = v(y_0^4 - y_2^4) = \gamma + v(y_1^4 - y_3^4) \geq \beta + \gamma + 4,$$

both of which are absurd. So we must have $v(y_1 + y_3) = 1$, whence R is in $\mathscr{D}_1$. Hence there is a point R' in $V_c(\mathbb{Q})$ arbitrarily close to R. Let $Q' = [2^{\beta-1}]R'$, where the operation is to be carried out on the fibre of $\mathscr{F}_3$ through R'; then Q' is in $V_c(\mathbb{Q})$ and arbitrarily close to Q. □

Lemma 8. *If $V_c(\mathbb{Q})$ is dense in $\mathscr{D}_1$ then it is dense in $\mathscr{D}_4^\beta$ for each $\beta > 0$.*

Proof. Let $Q = (u_0, u_1, u_2, u_3)$ be a point of $\mathscr{D}_4^\beta$ where $\beta > 0$; after changing the signs of u_2 and/or u_3 if necessary, we can assume that $v(u_0 - u_2) = 1$ and $v(u_1 - u_3) = \beta + 1$. Thus $v(u_1 + u_3) = \beta + \delta$ for some $\delta > 1$, whence $v(u_0 + u_2) = 4\beta + \gamma + \delta$. Let $\lambda = (u_0 - u_2)/(u_1 - u_3)$, so that $v(\lambda) = -\beta$. At Q we have $v(X) = 4\beta + \delta - 1$, $v(Y) = \beta$ and therefore $v(Z) > 10\beta + \delta - 1$ and it now follows from (24) that $v(Z) = 10\beta + 3\delta - 3$. Hence $v(U) = -6\beta - 2\delta + 2$ and $v(W) = -9\beta - 3\delta + 3$. If we write $\xi = 2^{4\beta}U$, $\eta = 2^{6\beta}W$ in (25) then the equation between ξ and η satisfies the conditions of Lemma 1. In the notation of that lemma, let $R = (y_0, y_1, y_2, y_3)$ be the point of E' which satisfies $\zeta(R) = 2^{1-\beta-\delta}\zeta(Q)$; R exists because $v(\zeta(Q)) = \beta + \delta - 1$. Thus $v(\xi/\eta) = 0$ at R, and so $v(\xi) = v(\eta) = 0$ since R is in E'. Moreover if we choose the representation of R so that $v(y_0 - y_2) = 1$ we have $v(y_1 - y_3) = 1 - v(\lambda) = 1 + \beta$; so $v(Y) = \beta$, $v(Z) = 7\beta$, $v(X) = \beta$, $v(y_0 + y_2) = \beta + \gamma + 1 > 2$ at R. The results for $y_0 \pm y_2$ show that y_0 and y_2 are odd integers. If we assume that $v(y_1 + y_3) \leq 0$, we find that

$$\beta + \gamma + 3 = v(y_0^4 - y_2^4) = \gamma + v(y_1^4 - y_3^4) \leq \beta + \gamma,$$

and if instead we assume that $v(y_1 + y_3) \geq 2$, we find that

$$\beta + \gamma + 3 = v(y_0^4 - y_2^4) = \gamma + v(y_1^4 - y_3^4) \geq \beta + \gamma + 5,$$

both of which are absurd. So we must have $v(y_1 + y_3) = 1$, whence R is in $\mathscr{D}_1$. Hence there is a point R' in $V_c(\mathbb{Q})$ arbitrarily close to R. Let $Q' = [2^{\beta+\delta-1}]R'$, where the operation is to be carried out on the fibre of $\mathscr{F}_3$ through R'; then Q' is in $V_c(\mathbb{Q})$ and arbitrarily close to Q. □

Proof of Theorem 5. $V_c(\mathbb{Q})$ is dense in $\mathscr{D}_1$ by Lemma 4, and dense in $\mathscr{D}_3$ by Lemma 6. It is dense in the rest of that part of $V_c(\mathbb{Q}_2)$ in which the X_i are integers with X_0, X_2 odd, by Lemmas 7 and 8. This is all we need. □

Very little extra effort is needed to take account also of the infinite place.

Corollary. *The image of $V_c(\mathbb{Q})$ is dense in $V_c(\mathbb{Q}_2) \times V_c(\mathbb{R})$.*

The argument in the last paragraph of the proof of Theorem 3 still works, and it shows that the image of $V_c(\mathbb{Q})$ is dense in at least one of the $V_c(\mathbb{Q}_2) \times \mathcal{R}$, where $\mathcal{R}$ is one of the connected components of $V_c(\mathbb{R}) \setminus (Z \cup X)$. As in the proof of Theorem 4, each of these connected components is given by fixing the signs of the X_i / X_j, and because we can change the sign of any x_i the corollary follows.

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