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Betti diagrams from graphs

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The emergence of Boij–Söderberg theory has given rise to new connections between combinatorics and commutative algebra. Herzog, Sharifan, and Varbaro recently showed that every Betti diagram of an ideal with a k-linear minimal resolution arises from that of the Stanley–Reisner ideal of a simplicial complex. In this paper, we extend their result for the special case of 2-linear resolutions using purely combinatorial methods. Specifically, we show bijective correspondences between Betti diagrams of ideals with 2-linear resolutions, threshold graphs, and anti-lecture-hall compositions. Moreover, we prove that any Betti diagram of a module with a 2-linear resolution is realized by a direct sum of Stanley–Reisner rings associated to threshold graphs. Our key observation is that these objects are the lattice points in a normal reflexive lattice polytope.

1. Introduction

A fundamental problem in commutative algebra is to characterize the coarsely graded Betti numbers of the finitely generated graded modules over a fixed polynomial ring. Originating with Hilbert in the 1890s, this task largely eluded mathematicians until 2006, when Boij and Söderberg introduced the following relaxation: Instead of trying to determine whether or not a table of nonnegative integers is the Betti diagram of a module, one should try to determine if some rational scalar of the table is the Betti diagram of a module. This shifted the viewpoint to studying rays in a rational cone and with this new geometric picture, the subject has seen a great deal of progress over the last six years. In particular, the idea led Boij and Söderberg [2008] to conjecture that every Betti diagram of a module can be decomposed in a specific and predictable way. Eisenbud and Schreyer [2009] proved this for Cohen–Macaulay modules, and Boij and Söderberg [2012] later extended that proof to the general setting.

A natural question that arises from Boij–Söderberg theory is the following: If a module is constructed from a combinatorial object, such as the edge ideal of a graph or the Stanley–Reisner ideal of a simplicial complex, can any of the combinatorial properties of that object be seen in the Boij–Söderberg decomposition of the module? Herzog, Sharifan, and Varbaro [Herzog et al. 2012] recently gave an elegant partial

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answer to this question for the special case of ideals with k-linear resolutions by showing that every Betti diagram of an ideal with a k-linear minimal resolution can be realized by the Stanley–Reisner ideal of a certain simplicial complex. More specifically, they prove that from the coefficients of a Boij–Söderberg decomposition of a k-linear Betti diagram, one obtains an O-sequence which, by a famous result of Eagon and Reiner along with Macaulay's theorem, yields a simplicial complex with the desired properties. Nagel and Sturgeon [2013] employ a similar approach to show that the k-linear Betti diagrams can be realized with hyperedge ideals of k-uniform Ferrers hypergraphs.

In this paper, we restrict our attention to the case of 2-linear resolutions and give an alternate characterization of the Betti diagrams of ideals with 2-linear minimal resolutions using purely combinatorial means. We show that every Betti diagram from an ideal with a 2-linear resolution is realized by a Stanley–Reisner ring constructed from a threshold graph and that this correspondence is a bijection.

Theorem 4.12. *For every* 2*-linear ideal* I *in* S, *there is a unique threshold graph* T *on* $n+1$ *vertices with* $\beta(S/I) = \beta(\Bbbk[T])$.

Moreover, for any such ideal, we give an efficient algorithm for constructing its corresponding threshold graph that avoids expensive computations like Hochster's formula; rather, we can generate all such Betti diagrams recursively with affine transformations, avoiding operators such as Ext and Tor. Even more interesting, we find that these diagrams are the lattice points of a normal reflexive lattice simplex that is combinatorially equivalent to a simplex of anti-lecture-hall compositions and, from this geometric picture, we prove that any Betti diagram of a *module* with a 2-linear resolution arises from a direct sum of Stanley–Reisner rings constructed from threshold graphs.

Theorem 4.16. *For every finitely generated*, *graded* S*-module* M *with* 2*-linear minimal free resolution and* $\beta_{0,0}(M) = m$, *there is a collection of* m *threshold graphs* $\{T_1, \ldots, T_m\}$, *not necessarily distinct*, *such that* $\beta(M) = \beta(\Bbbk[T_1] \oplus \cdots \oplus \Bbbk[T_m])$.

The paper is organized as follows: In Section 2, we give a quick review of the necessary concepts from commutative algebra and Boij–Söderberg theory. In Section 3, we interpret the main theorem of Boij–Söderberg theory in terms of linear algebra for the special case of modules with k-linear minimal resolutions. We prove our main theorems in Section 4 and conclude with some interesting connections to discrete geometry in Section 5.

2. Preliminaries

We begin with a review of the basic definitions and theorems from Boij–Söderberg theory. For a more detailed introduction, we recommend [Fløystad 2012].

Commutative algebra. Let $\Bbbk$ be a field and $S = \Bbbk[x_1, \dots, x_n]$. For any finitely generated graded S-module M, let M_i denote its graded piece of degree i and let $M(d)$ denote the *twisting* of M by d, that is, the module such that $M(d)_i \cong M_{i+d}$. A *minimal graded free resolution* of M is an exact complex

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_l,$$

where each F_i is a graded free S-module of the form

$$\bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$$

such that the number of basis elements is minimal and each map is graded.

The value $\beta_{i,j}$ is called the i-th *graded Betti number* of degree j. These numbers are a refinement of the ordinary Betti numbers $\beta_i = \sum_j \beta_{i,j}$ and are independent of the choice of resolution of M, thus yielding an important numerical invariant of M. We often express the graded Betti numbers in a two-dimensional array called the *Betti diagram* of M, denoted by $\beta(M)$. Since $\beta_{i,j} = 0$ whenever $i > j$, it is customary to write $\beta(M)$ such that $\beta_{i,j}$ is in position $(j-i,\ i)$. That is,

$$\beta(M) = \begin{bmatrix} \beta_{0,0} & \beta_{1,1} & \dots & \beta_{l,l} \\ \beta_{0,1} & \beta_{1,2} & \dots & \beta_{l,l+1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{0,r} & \beta_{1,r+1} & \dots & \beta_{l,l+r} \end{bmatrix}.$$

A Betti diagram is called *pure* if every column has at most one nonzero entry, that is, for each $i \in \{0, \dots, l\}$, $\beta_{i,j} \neq 0$ for at most one $j \in \mathbb{Z}$.

Boij–Söderberg theory. Let $\mathbb{Z}^{n+1}_{\mathrm{deg}}$ denote the set of strictly increasing nonnegative integer sequences $\boldsymbol{d} = (d_0, \dots, d_s)$ with $s \le n$, called *degree sequences*, along with the partial order given by

$$(d_0, \dots, d_s) \ge (e_0, \dots, e_t)$$

whenever $s \le t$ and $d_i \ge e_i$ for all $i \in \{0, \dots, s\}$. To every $\boldsymbol{d} \in \mathbb{Z}^{n+1}_{\mathrm{deg}}$, we associate a pure Betti diagram $\pi(\boldsymbol{d})$ with entries defined as follows:

$$\pi_{i,j}(\boldsymbol{d}) = \begin{cases} \displaystyle\prod_{k \neq 0,i} \left| \frac{d_k - d_0}{d_k - d_i} \right| & i \ge 0,\ j = d_i, \\ 0 & \text{otherwise.} \end{cases}$$

The main theorem of Boij–Söderberg theory states that the Betti diagram of any graded S-module can be written as a positive rational combination of $\pi(\boldsymbol{d})$'s. It was originally conjectured by Boij and Söderberg [2008], proven for Cohen–Macaulay modules by Eisenbud and Schreyer [2009], and then generalized thus:

Theorem 2.1 [Boij and Söderberg 2012]. *For every graded S-module M, there exists a vector $c \in \mathbb{Q}^p_{\geq 0}$ and a chain of degree sequences $\boldsymbol{d}^1 < \boldsymbol{d}^2 < \cdots < \boldsymbol{d}^p$ in $\mathbb{Z}^{n+1}_{\deg}$ such that*

$$\beta(M) = c_1\pi(\boldsymbol{d}^1) + \cdots + c_p\pi(\boldsymbol{d}^p).$$

The combination in Theorem 2.1 is called a *Boij–Söderberg decomposition* of M and the entries of c are called *Boij–Söderberg coefficients*. This decomposition is not unique in general, but there is a simple algorithm for computing a set of coefficients that satisfy the theorem, see [Fløystad 2012].

3. Betti diagrams of 2-linear resolutions

An ideal I in S is called *k-linear* if $\beta_{i,j}(I) = 0$ whenever $j - i \neq k - 1$. If I is 2-linear, then the Betti diagram of $M = S/I$ looks like

$$\beta(M) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_s \end{bmatrix}$$

for some $s \leq n$. Our aim is to translate the statement of Theorem 2.1, for S-modules with 2-linear resolutions, into linear algebraic terms. For this, it will be convenient to consider the *reduced Betti vector* $\omega(M) = [\beta_1, \ldots, \beta_s]$ in place of $\beta(M)$.

If M is a 2-linear S-module, then every $\boldsymbol{d}^l$ in Theorem 2.1 is of the form $(0, 2, \ldots, l+1)$. So, let $\pi^l = \pi(\boldsymbol{d}^l)$, ω^l be the reduced Betti vector corresponding to π^l, and Ω be the lower-diagonal $n \times n$ matrix whose l-th row is ω^l. We leave it to the reader to verify the following:

Lemma 3.1. *The matrix Ω is invertible and has ij-entry $\omega^i_j = j\binom{i+1}{j+1}$. Moreover, the ij-entry of Ω^{-1} is $(-1)^{i-j}\frac{1}{i}\binom{i+1}{j+1}$.*

Since any subset of row vectors in Ω forms a chain in $\mathbb{Z}^{n+1}_{\deg}$, we can replace the vector $c \in \mathbb{Q}^p_{>0}$ in Theorem 2.1 with a vector $c \in \mathbb{Q}^n_{\geq 0}$ such that $\sum_i c_i = \beta_{0,0}(M)$.

Theorem 3.2. *For every 2-linear (graded) S-module M with $\beta_{0,0}(M) = m$,*

$$\beta(M) = c_1\pi^1 + \cdots + c_n\pi^n,$$

where $c = \omega(M)\Omega^{-1} \in \mathbb{Q}^n_{\geq 0}$ and $\sum_i c_i = m$.

Remark 3.3. When $\beta_{0,0}(M) = 1$, Theorem 3.2 asserts that $\omega(M)$ is a lattice point in the $(n-1)$-dimensional simplex spanned by row vectors of Ω.

We conclude this section with some classic examples of 2-linear ideals that arise from graph theory. A *graph* G consists of a finite set $V(G)$, called the *vertex set*, and a subset $E(G)$ of $\binom{V(G)}{2}$, called the *edge set*. To simplify notation, we write uv instead of $\{u, v\}$ for each edge in G. For any subset of vertices $W \subset V(G)$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $E(G) \cap \binom{W}{2}$.

If $W = V(G) \setminus S$ for some $S \subseteq V(G)$, we write $G \setminus S$ instead of $G[W]$. A subgraph C of the form $V(C) = \{v_1, \dots, v_l\}$ and $E(C) = \{v_i v_{i+1} \mid 1 \leq i < l\} \cup \{v_1 v_l\}$ is called a *cycle* of length l. We say G is *chordal* if it has no induced cycles of length greater than three or, equivalently, if $E(C) \subsetneq E(G[C])$ for every cycle of length greater than three. The elements of $E(G[C]) \setminus E(C)$ are called *chords*. Chordal graphs have many interesting properties that are actively studied in graph theory. For a thorough introduction to graph theory, we recommend [Diestel 2010].

Given a graph G with vertex set $[n+1] = \{1, \dots, n+1\}$, where n is the number of indeterminates in S, let $R = \Bbbk[x_1, \dots, x_{n+1}]$, let

$$I^c(G) = \langle x_i x_j \mid ij \notin E(G) \rangle \subseteq R$$

be the ideal generated by the monomials corresponding to nonedges in G, and let $\Bbbk[G]$ be the quotient $R/I^c(G)$. The knowledgeable reader may observe that $I^c(G)$ is the edge ideal of the complement of G and $\Bbbk[G]$ is the Stanley–Reisner ring of the clique complex of G. The following theorem was first proved by Fröberg [1990] and then by Dochtermann and Engström [2009], using topological combinatorics.

Theorem 3.4. *A graph G is chordal if and only if $I^c(G)$ is 2-linear. Whenever this is the case,*

$$\beta_{i,j}(\Bbbk[G]) = \sum_{W \in \binom{V(G)}{j}} (-1 + \#\,\textit{components of}\ G[W])$$

for $i = j - 1 \geq 1$.

Example 3.5. If G consists of $n+1$ isolated vertices, then the $\binom{n+1}{i+1}$ induced subgraphs of G with $i+1$ vertices each have $i+1$ connected components. Thus, $\beta_{i,i+1}(\Bbbk[G]) = i\binom{n+1}{i+1}$ for each $i \geq 1$.

Example 3.6. If G consists of a complete graph on n vertices plus an isolated vertex v, then the $\binom{n}{i}$ induced subgraphs of G with $i+1$ vertices that contain v each have two connected components and the remaining induced subgraphs of G (with $i+1$ vertices) are connected. Thus, $\beta_{i,i+1}(\Bbbk[G]) = \binom{n}{i}$ for each $i \geq 1$.

Remark 3.7. If we apply Theorems 3.2 and 3.4 to $\Bbbk[G]$ for some chordal graph G, we get a formula that takes the number of connected components of induced subgraphs of G as input and yields a vector $c \in \mathbb{Q}^n_{\geq 0}$, namely $\omega(\Bbbk[G])\Omega^{-1}$, whose entries sum to 1. It is natural to ask what this formula says if G is not chordal. If the entries of c fail to be nonnegative or sum to 1, then we get a certificate that G is not chordal. Since measuring how far a graph is from being chordal is nontrivial from the viewpoint of complexity, one is inclined to ask if this procedure characterizes chordal graphs.

Alas, this turns out to not be the case — there are nonchordal graphs that yield admissible c's — but these *false chordal graphs* seem to be few. Examples of false

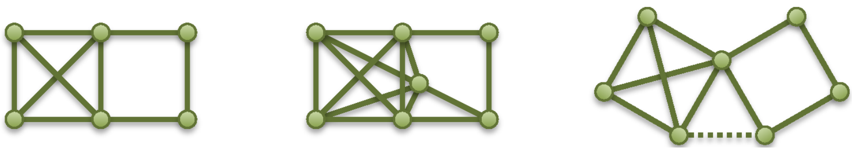

Figure 1. The single false chordal graph on six vertices along with two examples on seven vertices.

chordal graphs on six and seven vertices are illustrated in Figure 1. All other false chordal graphs on seven vertices arise from expanding a (possibly empty) clique of the six-vertex graph or coning over the whole six-vertex graph. We offer some computer-generated statistics on the size of each class of graphs for a given number of vertices:

	1	2	3	4	5	6	7
Chordal	1	2	4	10	27	94	393
False chordal	0	0	0	0	0	1	15
Not chordal	0	0	0	1	7	62	651

4. Betti diagrams from graphs

In this section, we study the Betti diagrams corresponding to a special class of chordal graphs called threshold graphs. We show that threshold graphs on a fixed vertex set have distinct Betti diagrams, that every Betti diagram of a chordal graph is that of a threshold graph on the same number of vertices, that every Betti diagram of an S-algebra with a 2-linear resolution is that of a threshold graph on $n+1$ vertices, and that every Betti diagram of an S-module with a 2-linear resolution is that of a direct sum of Stanley–Reisner rings constructed from threshold graphs on $n+1$ vertices, where n is the number of indeterminates in S.

Betti diagrams from threshold graphs. In a graph G, two vertices are said to be *adjacent* if they are contained in an edge of G. A vertex adjacent to no others is called *isolated* and a vertex adjacent to all others is called *dominating*. For every graph G on n vertices, let G_* be the graph on $n+1$ vertices obtained by adding an isolated vertex to G and, similarly, let G^* be the graph obtained by adding a dominating vertex to G. A graph G is called *threshold* if it can be constructed from a single vertex and a sequence of the operations * and $_*$. It is well known that if G is chordal, then so are G_* and G^*, and thus, all threshold graphs are chordal. We refer to Mahadev and Peled [1995] for a survey that includes the following lemma:

Lemma 4.1. *There are 2^n threshold graphs on $n+1$ vertices. Moreover, every threshold graph is determined by a unique sequence of * and $_*$ operations.*

The Betti diagram of a threshold graph can be constructed recursively in a similar manner to the graph itself. As such, we can quickly calculate the Betti diagram of a threshold graph without the computations in Theorem 3.4.

Proposition 4.2. *If G is a chordal graph on n vertices, then*

$$\omega(\Bbbk[G^*]) = [\omega(\Bbbk[G]) \mid 0] \quad \textit{and} \quad \omega(\Bbbk[G_*]) = \omega(\Bbbk[G])\Lambda + \eta_n, \tag{1}$$

where Λ is the $(n-1)\times n$-matrix whose (i, j) position is 1 if $i = j$ or $j-1$ and 0 otherwise, and η_n is the vector whose i-th entry is $\binom{n}{i}$.

Proof. This is a simple application of Theorem 3.4. For the first part, any subset of vertices containing the dominating vertex in G^* spans a connected graph and therefore, the only nonzero parts of $\omega(\Bbbk[G^*])$ come from $\omega(\Bbbk[G])$. For the second part, we consider whether or not a subset of vertices in G_* contains the isolated vertex v: The induced subgraphs that *do not* contain v contribute $[\omega(\Bbbk[G]) \mid 0]$ to $\omega(\Bbbk[G_*])$ while those that *do* contain v contribute $[0 \mid \omega(\Bbbk[G])] + \eta_n$. □

As a corollary, we find that distinct threshold graphs on a fixed number of vertices have distinct Betti diagrams.

Corollary 4.3. *If T and T' are threshold graphs on the same number of vertices and $\omega(\Bbbk[T]) = \omega(\Bbbk[T'])$, then $T \cong T'$.*

Proof. For any chordal graph G on k vertices, $\omega_{k+1}(\Bbbk[G_*]) \neq \omega_{k+1}(\Bbbk[G^*]) = 0$ by Proposition 4.2. Therefore, since distinct threshold graphs have distinct sequences of $_*$ and * (Lemma 4.1), they must also have distinct Betti diagrams. □

Betti diagrams from chordal graphs. Next, we show that every Betti diagram from a chordal graph arises as the Betti diagram of a threshold graph on the same number of vertices. Moreover, for a given chordal graph, we present an efficient algorithm for constructing its "threshold representative".

Let $\sim_\beta$ be the equivalence relation for graphs on $[n+1]$ defined by

$$G \sim_\beta H \text{ if and only if } \beta(\Bbbk[G]) = \beta(\Bbbk[H])$$

and let $[G]_\beta$ denote the equivalence class of G with respect to $\sim_\beta$. For a chordal graph G on $n+1$ vertices, a threshold graph T (on $n+1$ vertices) is called a *threshold representative* of G if $T \in [G]_\beta$. The next theorem follows from the notion of *algebraic shifting* and can be pieced together from results in [Goodarzi and Yassemi 2012; Klivans 2007; Woodroofe 2011], but we offer a purely graph-theoretic proof instead.

Theorem 4.4. *Every chordal graph G has a unique threshold representative T.*

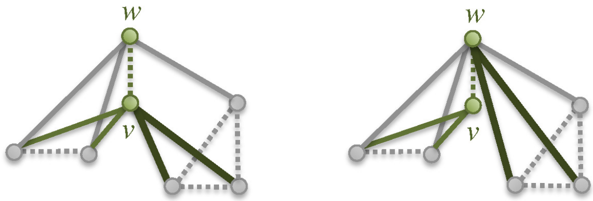

Figure 2. A comparison of a graph G (left) with $G_{v\to w}$ (right).

We proceed with some new machinery. For a graph G with $v, w \in V(G)$, we define a new graph $G_{v\to w}$ on $V(G)$ with

$$E(G_{v\to w}) := \big(E(G) \setminus \{uv \mid u \in N(v; w)\}\big) \cup \big\{uw \mid u \in N(v; w)\big\},$$

where $N(x) = \{y \in V(G) \mid xy \in E(G)\}$ is the *neighborhood* of a vertex x and $N(v; w) = N(v) \setminus \big(\{w\} \cup N(w)\big)$. See Figure 2.

Lemma 4.5. *Let G be a chordal graph.*

(1) *If G is connected with $vw \in E(G)$, then $G_{v\to w}$ is chordal.*

(2) *If G is disconnected with $v, w \in V(G)$ in separate components, then $G_{v\to w}$ is chordal.*

Proof. For each part, we suppose C is a cycle with length $l \geq 4$ in $G' = G_{v\to w}$ and show that C has a chord in G'.

In (1), if $w \notin V(C)$, then $C \subseteq G$ since the only new edges of G' contain w and therefore C has at least one chord in G. If every chord of C in G is removed in G', then they must each contain v and thus $G[V(C \setminus v) \cup w]$ is an induced cycle, which is a contradiction. If $w \in V(C)$, $v \notin V(C)$, and C does not have a chord in G', then $G[V(C) \cup v]$ is an induced cycle since $N(v) \subseteq N(w)$ in G', another contradiction. If $v, w \in V(C)$, then $vw \in E(C)$ and xw is a chord of C in G', where x is the other neighbor of v in C, since $N(v) \subseteq N(w)$ in G'.

In (2), if $w \notin V(C)$, then C contains a chord in $G \setminus w = G' \setminus w \subseteq G'$. So suppose $w \in V(C)$ and C has no chord in G'. Then $G[V(C \setminus w)]$ is contained in the connected component of either v or w in G. If the former is true, then $G[V(C \setminus w) \cup v]$ is an induced cycle and if the latter is true, then C itself is an induced cycle in G, both of which are contradictions. $\square$

For a graph H with $W \subseteq V(H)$, let $\kappa_H(W)$ denote the number of connected components in $H[W]$.

Lemma 4.6. *Let G be a chordal graph.*

(1) *If G is connected with $vw \in E(G)$, then $G_{v\to w} \in [G]_\beta$.*

(2) *If G is disconnected with $v, w \in V(G)$ in separate components, then $G_{v\to w}$ is in $[G]_\beta$.*

Proof. This is a straightforward application of Theorem 3.4 after we make the following calculations. For each part, let $G' = G_{v\to w}$ and $W \subseteq V(G)$.

In (1), if $v, w \notin W$, then $\kappa_G(W) = \kappa_{G'}(W)$ since $G \setminus \{v, w\} = G' \setminus \{v, w\}$ and if $v, w \in W$, then $\kappa_G(W) = \kappa_{G'}(W)$ because the component in $G[W]$ containing v and w spans the same set of vertices as that of $G'[W]$. For the remaining subsets of $V(G)$, we prove that $\kappa_G(W \cup v) + \kappa_G(W \cup w) = \kappa_{G'}(W \cup v) + \kappa_{G'}(W \cup w)$ for every $W \subseteq V(G) \setminus \{v, w\}$. Let $m_\circ(W)$, $m_w(W)$, and $m_v(W)$ denote the number of connected components of $G[W]$ that do not contain any elements of $N(v) \cup N(w)$, $N(v) \setminus N(w)$, and $N(w) \setminus N(v)$, respectively. It is straightforward to check that $\kappa_G(W\cup v) = 1+m_\circ(W)+m_w(W)$, $\kappa_G(W\cup w) = 1+m_\circ(W)+m_v(W)$, $\kappa_{G'}(W\cup v) = 1 + m_\circ(W) + m_v(W) + m_w(W)$, and $\kappa_{G'}(W \cup w) = 1 + m_\circ(W)$.

In (2), we record the difference between $\kappa_G(W)$ and $\kappa_{G'}(W)$. If $v, w \notin W$, then $\kappa_G(W) = \kappa_{G'}(W)$ since $G \setminus \{v, w\} = G' \setminus \{v, w\}$. If $v, w \in W$, then $\kappa_G(W) = \kappa_{G'}(W)$ because every vertex in the component of v in $G[W]$ gets moved to the component of w in $G'[W]$. If $v \in W$ and $w \notin W$, then $\kappa_G(W) = \kappa_{G'}(W) - 1$. If $w \in W$ and $v \notin W$, then $\kappa_G(W) = \kappa_{G'}(W) + 1$. □

Proof of Theorem 4.4. We induct on $|V(G)|$. Let G be a chordal graph on n vertices and fix a vertex $v \in V(G)$. We will apply the operations $_{v\to w}$ or $_{w\to v}$ to G to a get a graph where v is either dominating or isolated.

If G is connected and v is not dominating, then for any vertex $u \in G$ with $d(u, v) = 2$, let $w \in N(v) \cap N(u)$ and replace G with $G_{w\to v}$. Repeat this until v is a dominating vertex, that is, there are no more elements u with $d(v, u) = 2$. The process terminates since G is finite and connected. By Lemma 4.5, the graph G is chordal at every step and by Lemma 4.6, its Betti diagram stays fixed. Since v is dominating and $G \setminus v$ is chordal (being an induced subgraph of a chordal graph), $\beta(\Bbbk[G]) = \beta(\Bbbk[G \setminus v])$. So, by induction, there is a unique (up to isomorphism) threshold graph T such that $\beta(\Bbbk[T^*]) = \beta(\Bbbk[T]) = \beta(\Bbbk[G \setminus v]) = \beta(\Bbbk[G])$.

If G is disconnected, let $w \in V(G)$ be in a separate component in G from v. By Lemmas 4.5 and 4.6, $G_{v\to w}$ is chordal and $\beta(\Bbbk[G]) = \beta(\Bbbk[G_{v\to w}])$; by induction, there exists a unique (up to isomorphism) threshold graph $T \in [G_{v\to w} \setminus v]_\beta$. Thus, $T_* = T \cup \{\alpha\} \in [G]_\beta$ and $\beta(\Bbbk[T_*]) = \beta(\Bbbk[G])$. □

Remark 4.7. The algorithm presented in the proof of Theorem 4.4 is fast. A crude analysis of the complexity is as follows: For each vertex of G, we decompose G into its connected components which takes $O(|V(G)| + |E(G)|)$ and then we repeatedly apply the operations $_{v\to w}$ or $_{w\to v}$; by amortized analysis, this takes only $O(|E(G)|)$ since each edge is moved at most once. Thus, the total complexity is $O(|V(G)|(|V(G)| + |E(G)|)) \approx O(|V(G)|^3)$. The authors suspect that a more thorough analysis would yield a complexity of $O(|V(G)|^2)$, which is the best one could hope for with this problem.

As simple corollaries of Theorem 4.4, we recover two special classes of graphs that are invariant under β.

Corollary 4.8. *If G is a tree on $n+1$ vertices, then $\beta_{i,i+1}(\Bbbk[G]) = i\binom{n}{i+1}$.*

Proof. Since G has exactly n edges and $_{v\to w}$ preserves the number of edges in G, the procedure outlined in the proof of Theorem 4.4 yields a threshold representative T of G that is a star on $n+1$ vertices, that is, a single dominating vertex v and no other edges. Therefore, $T \setminus v$ consists of n isolated points and, by Proposition 4.2 and Example 3.5, $\beta_{i,i+1}(\Bbbk[G]) = \beta_{i,i+1}(\Bbbk[T]) = \beta_{i,i+1}(\Bbbk[T \setminus v]) = (i)\binom{n}{i+1}$. □

The graph from a triangulation of a polygon is called *maximally outerplanar*.

Corollary 4.9. *If G is a maximal outerplanar graph on $n+1$ vertices, then $\beta_{i,i+1}(\Bbbk[G]) = i\binom{n-1}{i+1}$.*

Proof. By Theorem 4.4, the threshold representative T of G consists of a dominating vertex v and a path on $V(T)\setminus v$. In particular, $T \setminus v$ is a tree on n vertices. The result now follows from Proposition 4.2 and Corollary 4.8. □

Betti diagrams of algebras and modules. Here we present the main results of the paper — that every Betti diagram from a 2-linear ideal in S arises from a Stanley–Reisner ring of a threshold graph on $n+1$ vertices and that every Betti diagram from an S-module with a 2-linear resolution arises from a direct sum of Stanley–Reisner rings constructed from threshold graphs on $n+1$ vertices.

To begin, we establish bijections between the set of threshold graphs on $n+1$ vertices, the set of Betti diagrams from 2-linear ideals in S, and the set of anti-lecture-hall compositions of length n bounded above by 1. An integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the form

$$t \geq \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_n}{n} \geq 0$$

is called *anti-lecture-hall composition of length n* bounded above by t. These sequences were introduced in [Corteel and Savage 2003] and are a well-studied variation of the *lecture hall partitions* in [Bousquet-Mélou and Eriksson 1997a; 1997b]. For our purposes, we only need this result of Corteel, Lee, and Savage:

Theorem 4.10 ([Corteel et al. 2005]). *There are $(t+1)^n$ anti-lecture-hall compositions of length n bounded above by t.*

We remark that $\Bbbk[G] = R$ if G is the complete graph on $n+1$ vertices, so we shall ignore that graph for the rest of the paper.

Proposition 4.11. *The set of noncomplete threshold graphs on $n+1$ vertices, the set of Betti diagrams of quotients of S by 2-linear ideals, and the set of anti-lecture-hall compositions of length n with $\lambda_1 = 1$ are in bijective correspondence.*

Proof. By Lemma 4.1 and Corollary 4.3, there are $2^n - 1$ noncomplete threshold graphs on $n+1$ vertices, each of which corresponds to a distinct Betti diagram. It suffices to show that the Betti diagrams of quotients of S by 2-linear ideals inject into the anti-lecture-hall compositions of length n with $\lambda_1 = 1$, since by Theorem 4.10, there are exactly $2^n - 1$ of them.

Let I be a 2-linear ideal in S and let Ψ be the unimodular matrix with ij-entry equal to $\binom{i-1}{j-1}$. Then there exists $\lambda = [\lambda_1, \dots, \lambda_n] \in \mathbb{Z}^n$ such that $\omega(S/I) = \lambda\Psi$. By Theorem 3.2, we have

$$\lambda\Psi\Omega^{-1} = [c_1, \dots, c_n] \in \mathbb{Q}^n_{\geq 0}$$

with $\sum_{i=1}^n c_i = 1$. We leave it to the reader to verify that $\Psi \cdot \Omega^{-1}$ has ij-entry $1/i$ if $i = j$, $-1/i$ if $i = j+1$, and 0 otherwise. Thus, $c_i = \lambda_i/i - \lambda_{i+1}/(i+1)$ for all $i \in [n-1]$ and $c_n = \lambda_n/n$. In particular, we get

$$1 = \sum_{i=1}^n c_i = \frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_n}{n} = c_n \geq 0$$

and hence, λ is an anti-lecture-hall composition with $\lambda_1 = 1$. □

The first part of our main theorem is a simple corollary of Proposition 4.11. In particular, it asserts that the injection in Proposition 4.2 is in fact a bijection.

Theorem 4.12 (Main Theorem, Part 1). *For every* 2*-linear ideal I in S, there is a unique threshold graph T on $n+1$ vertices with $\beta(S/I) = \beta(\Bbbk[T])$.*

Remark 4.13. For a given 2-linear ideal I in S, it is easy to construct the graph T realizing its Betti diagram.

Example 4.14. To illustrate Theorem 4.12 at work, consider the ideal

$$I = \langle x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_1x_5 + x_2x_4, x_4^2\rangle \subseteq S = \Bbbk[x_1, \dots, x_5].$$

Then

$$\beta(S/I) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 7 & 11 & 6 & 1 & 0 \end{bmatrix}.$$

In order to find a threshold graph T on six vertices whose Betti diagram is $\beta(S/I)$, we sequentially apply the inverses of the affine transformations in Proposition 4.2 depending on whether or not the sequences end in 0. (We leave it to the reader to verify that the inverse of Λ in Proposition 4.2 is the $n \times (n-1)$-matrix whose (i, j) position is $(-1)^{i+j}$ if $i \leq j$ and 0 otherwise.)

$$[7, 11, 6, 1, 0] \xrightarrow{-^*} [7, 11, 6, 1] \xrightarrow{-_*} [3, 2, 0] \xrightarrow{-^*} [3, 2] \xrightarrow{-_*} [1] \xrightarrow{-_*} [0]$$

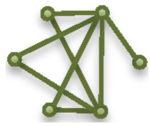

Figure 3. The threshold graph T on six vertices with $\omega(\Bbbk[T]) = [7, 11, 6, 1, 0]$.

From this, we see that $\beta(S/I) = \beta(\Bbbk[T])$, where T is the threshold graph with sequence ${}_{**}{}^{*}{}_{*}{}^{*}$ drawn in Figure 3.

For the rest of the paper, we take a more geometric approach. Specifically, we make use of the fact (Remark 3.3) that the reduced Betti vectors of these diagrams are lattice points in the $(n-1)$-dimensional lattice simplex P_n spanned by the row vectors of Ω. Illustrations of P_1 through P_4, labeled by reduced Betti vectors, Boij–Söderberg coefficients, truncated coordinates (see Section 5), and corresponding chordal graphs are shown in Figures 4 and 5, with the threshold graphs colored dark green. Notice that each P_n contains two copies of P_{n-1}, colored blue and red, corresponding to the first and second equations, respectively, in (1) (see Proposition 4.2).

We continue with some standard definitions from discrete geometry. The integer points $\mathbb{Z}^d \subseteq \mathbb{R}^d$ form a *lattice*. The integer points of a polytope are its *lattice points*

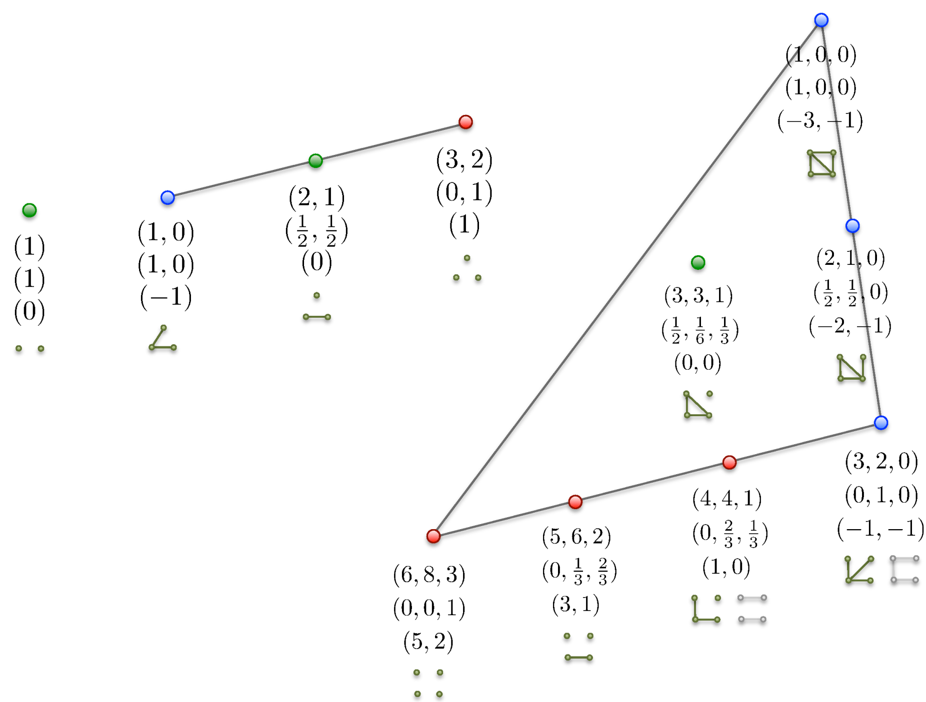

Figure 4. The lattice polytopes P_1 (left), P_2 (middle), and P_3 (right).

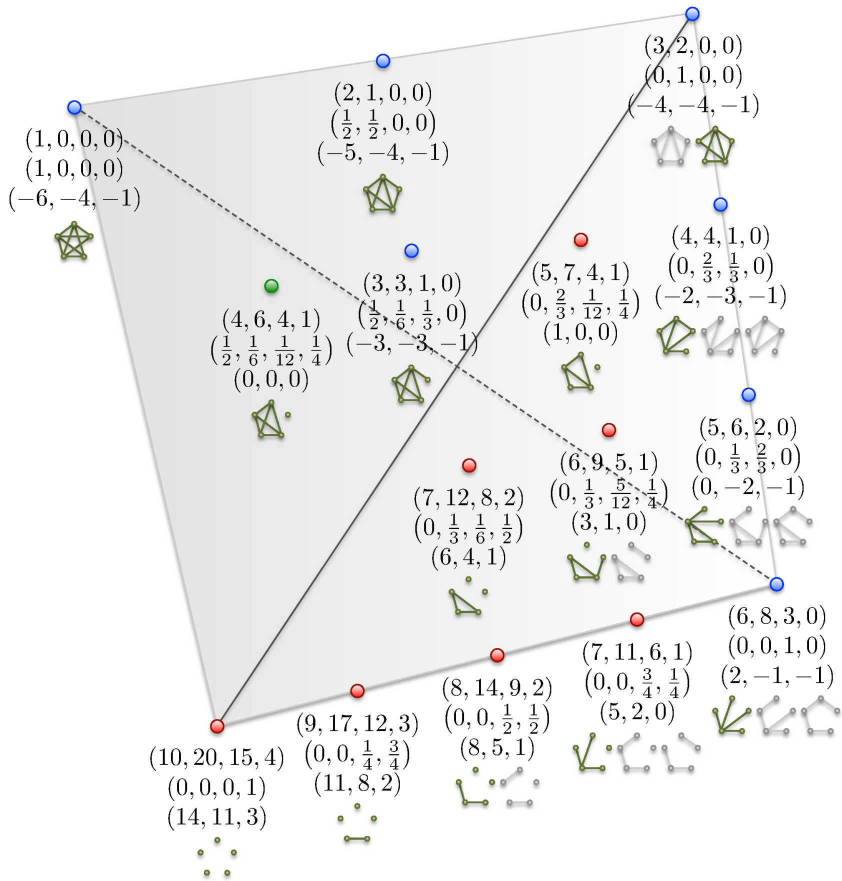

Figure 5. The lattice polytope P_4.

and a polytope is called a *lattice polytope* if all its vertices are lattice points. For a polytope P with vertices $\{v_1, \dots, v_s\}$ and $t \in \mathbb{N}$, let tP denote the t-th *dilation* of P, that is, the polytope attained by taking the convex hull of the points $\{t \cdot v_1, \dots, t \cdot v_s\}$, let $S_P \subseteq \mathbb{Z}^{d+1}$ denote the semigroup generated by $\big\{[1, p_1, \dots, p_d] : (p_1, \dots, p_d) \in P \cap \mathbb{Z}^d\big\}$, and let $\operatorname{gp}(S_P)$ be the smallest group containing S_P, that is the group of differences in S_P. We say P is *normal* if $x \in \operatorname{gp}(S_P)$ such that $s \cdot x \in S_P$ for some $s \in \mathbb{N}$ implies that $x \in S_P$. We refer to [Barvinok 2002; Bruns et al. 1997] for questions on lattice polytopes.

Proposition 4.15. *The lattice simplex P_n is normal for each $n \in \mathbb{N}$.*

Proof. It is straightforward to check that the anti-lecture-hall compositions of length n bounded above by 1 are the lattice points of the n-dimensional lattice simplex spanned by $(0, \dots, 0)$ and the compositions $\lambda^l = (1, 2, \dots, l, 0, \dots, 0)$ for $l \in [n]$. Let Q_n be the facet spanned by the λ^l. Since normality is preserved under unimodular transformations, we prove that Q_n is normal and apply Ψ from the proof of Proposition 4.11.

To begin, we must truncate the coordinates of Q_n since it is an $(n-1)$-dimensional simplex. Removing the first coordinate yields the simplex with vertices $(0, \ldots, 0)$ and $(2, 3, \ldots, l, 0, \ldots, 0)$ for $l \in [n]$. Then S_{Q_n} is the set of all anti-lecture-hall compositions and $\mathrm{gp}(S_{Q_n}) = \mathbb{Z}^n$. From here it is clear that if $\lambda \in \mathbb{Z}^n$ and $s \cdot \lambda \in S_{Q_n}$ for some $s \in \mathbb{N}$, then $\lambda \in S_{Q_n}$. Hence, Q_n is normal. $\square$

A convenient consequence of normality is that every lattice point in the t-th dilation of a normal polytope P can be written as a sum of t, not necessarily distinct, lattice points in P. With that, we can prove the second part of our main theorem.

Theorem 4.16 (Main Theorem, Part 2). *For every finitely generated, graded S-module M with a 2-linear minimal free resolution and $\beta_{0,0}(M) = m$, there is a collection of m threshold graphs $\{T_1, \ldots, T_m\}$, not necessarily distinct, such that $\beta(M) = \beta(\Bbbk[T_1] \oplus \cdots \oplus \Bbbk[T_m])$.*

Proof. By Theorem 3.2, $\omega(M)$ is a lattice point in mP_n and is a sum of m lattice points $p_1, \ldots, p_m$ in P_n, by Proposition 4.15. Applying Theorem 4.12 yields a threshold graph T_i such that $p_i = \omega(\Bbbk[T_i])$ for each $i \in [m]$, and thus,

$$\beta(M) = \beta(T_1) + \cdots + \beta(T_m) = \beta(\Bbbk[T_1] \oplus \cdots \oplus \Bbbk[T_m]). \qquad \square$$

Remark 4.17. The decomposition in Theorem 4.16 is often not unique. So in the more general setting of modules, we do not know how to construct the family of trees representing a given Betti diagram as we do in the special case of algebras, see Theorem 4.12 and Example 4.14.

5. The geometry of P_n and Q_n

In the previous section, we used the geometry of the lattice simplex P_n of reduced Betti vectors of 2-linear ideals in S (or equivalently, the lattice simplex Q_n of nonzero anti-lecture-hall compositions of length n) to prove algebraic statements about Betti diagrams of algebras and modules with 2-linear resolutions, but these polytopes have many other beautiful geometric properties which make them interesting on their own. In this section, we take the opportunity to showcase a few of these properties. Specifically, we remark that P_n has a simple Ehrhart polynomial, by a result from [Corteel et al. 2005], and we prove that P_n is reflexive.

Given a d-dimensional polytope P, let $\mathrm{Ehr}_P(t)$ denote the number of lattice points in tP. It is well known that $\mathrm{Ehr}_P(t)$ is a degree d polynomial in t, called the *Ehrhart polynomial* of P, with constant term 1 and leading coefficient equal to the volume of P, and that Ehrhart polynomials are preserved under unimodular transformations. For an introduction to Ehrhart theory, see [Beck and Robins 2007].

Theorem 5.1. *For every $n, t \in \mathbb{N}$, $\mathrm{Ehr}_{P_n}(t) = \mathrm{Ehr}_{Q_n}(t) = (t+1)^n - t^n$.*

Proof. Since the matrix Ψ^{-1} in the proof of Proposition 4.11 is unimodular, we know that $\mathrm{Ehr}_{P_n}(t) = \mathrm{Ehr}_{Q_n}(t)$. So, let $A_n(t)$ denote the number of anti-lecture-hall compositions of length n with $\lambda_1 \le t$. Theorem 4.10 gives us $A_n(t) = (t+1)^n$. Since every point in the tQ_n satisfies $\lambda_1 = t$, it follows immediately that

$$\mathrm{Ehr}_{P_n}(t) = \mathrm{Ehr}_{Q_n}(t) = A_n(t) - A_n(t-1) = (t+1)^n - t^n. \qquad \square$$

Next, we prove that P_n is reflexive. For this, we need the concept of a dual (or polar) of a polytope, but restrict to the case of simplices, since those are the only polytopes we consider.

Definition 5.2. Let the vertices of a d-simplex P be recorded as the rows of the $d \times (d-1)$ matrix M and let M^* be the $(d-1) \times d$ matrix such that MM^* has value -1 everywhere outside the diagonal. The d-simplex whose vertices are the columns of M^* is the *dual* P^* of P.

If P is a lattice polytope containing 0 as an interior point such that P^* is also a lattice polytope, then P and P^* are called *reflexive*. These polytopes have several interesting properties and characterizations, for instance, a lattice polytope P is reflexive if and only if its only interior lattice point is 0 and if u and v are two lattice points on the boundary of P, then either u and v are on the same facet, or $u+v$ is in P. This is an important concept with interesting connections to geometry and theoretical physics. For an exposition suitable for researchers with a background in discrete mathematics, we refer to Batyrev and Nill [2008].

Because P_n is an $(n-1)$-dimensional simplex with coordinates in $\mathbb{Z}^n$, for each lattice point $p \in P_n$, we define

$$p_t = [p_1, \dots, p_{n-1}] := [p - \eta_n]_{2 \le i \le n} = \left[p_2 - \tbinom{n}{2}, \dots, p_n - \tbinom{n}{n}\right]$$

to be the *truncated coordinates* of p in P_n.

Theorem 5.3. *The simplex P_n realized in the truncated coordinates is a reflexive lattice polytope.*

Proof. We begin by removing the left-most column of Ω to get the $n \times (n-1)$ matrix Ω'_n. Then the truncated coordinates of P_n are the rows of $\Omega_n = \Omega'_n - \eta_n \mathbf{1}_n$. More explicitly, the ij-entry of Ω'_n is $(j+1)\binom{i+1}{j+2}$ and the j entry of η_n is $\binom{n}{j+1}$.

The dual of P_n, in truncated coordinates, is the simplex whose vertices are the columns of the $(n-1) \times (n)$ matrix Ξ_n satisfying that all values of $\Omega_n \Xi_n$ outside the diagonal are -1. If all entries of Ξ_n are integers, then the dual of P_n is a lattice polytope and hence, P_n is reflexive. To show this, we construct Ξ_n explicitly with three $(n-1) \times n$ matrices, Ξ'_n, Ξ''_n, and Ξ'''_n. The ij-entries of Ξ'_n are $-(i+2)(-1)^{i+j}\binom{i}{j-1}$ and the matrices Ξ''_n and Ξ'''_n are all zero, with the exceptions that the first column of Ξ_n'' is $-2(-1)^i$, and the bottom right-most entry of Ξ_n''' is $1-n$. We consider $\Xi_n = \Xi_n' + \Xi_n'' + \Xi_n'''$.

To calculate the product $\Omega_n \Xi_n$, we separate both Ω_n and Ξ_n into the sums above and then multiply them. The matrix multiplications are straightforward applications of elementary combinatorics, so we only record the results:

(1) The matrix $\Omega_n' \Xi_n'$ is the sum of two matrices. The only nonzero elements of the first are the diagonal ii-entries $i(i+1)$ and the only nonzero elements of the second are the first column $i1$-entries $-i(i+1)$.

(2) The matrix $\eta_n \mathbf{1}_n \Xi_n'$ has 1s everywhere, except that the first column is constant with $-2n+1$ and the last column is $n+1$.

(3) The matrix $\Omega_n' \Xi_n''$ has 0s everywhere, except that the first column's $i1$-entry is $i(i+1)-2$.

(4) The matrix $\eta_n \mathbf{1}_n \Xi_n''$ has 0s everywhere, except that the first column is constant with $2n-2$.

(5) The matrix $\Omega_n' \Xi_n'''$ has 0s everywhere, except that the rightmost bottom corner is $-n^2$.

(6) The matrix $\eta_n \mathbf{1}_n \Xi_n'''$ has 0s everywhere, except that the rightmost column is constant $-n$.

Summing up, we conclude that the ij-entry of $\Omega_n \Xi_n = (\Omega_n' - \eta_n \mathbf{1}_n)(\Xi_n' + \Xi_n'' + \Xi_n''')$ is

$$-1 \ \text{ if } i \neq j, \quad i^2 + i - 1 \ \text{ if } i = j < n, \quad n \ \text{ if } i = j = n. \qquad \square$$

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