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# Kernels for products of $L$ -functions

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The Rankin–Cohen bracket of two Eisenstein series provides a kernel yielding products of the periods of Hecke eigenforms at critical values. Extending this idea leads to a new type of Eisenstein series built with a double sum. We develop the properties of these series and their nonholomorphic analogs and show their connection to values of  $L$ -functions outside the critical strip.

## 1. Introduction

Rankin [1952] introduced the fruitful idea of expressing the product of two critical values of the  $L$ -function of a weight- $k$  Hecke eigenform  $f$  for  $\Gamma = \mathrm{SL}(2, \mathbb{Z})$  in terms of the Petersson scalar product of  $f$  and a product of Eisenstein series:

$$\langle E_{k_1} E_{k_2}, f \rangle = (-1)^{k_1/2} 2^{3-k} \frac{k_1 k_2}{B_{k_1} B_{k_2}} L^*(f, 1) L^*(f, k_2) \quad (1-1)$$

for  $k = k_1 + k_2$ , the Bernoulli numbers  $B_j$  and the completed, entire  $L$ -function of  $f$ ,

$$L^*(f, s) := \frac{\Gamma(s)}{(2\pi)^s} \sum_{m=1}^{\infty} \frac{a_f(m)}{m^s} = \int_0^{\infty} f(iy) y^{s-1} dy.$$

Zagier [1977, p. 149] extended (1-1) to get

$$\langle [E_{k_1}, E_{k_2}]_n, f \rangle = (-1)^{k_1/2} (2\pi i)^n 2^{3-k} \binom{k-2}{n} \frac{k_1 k_2}{B_{k_1} B_{k_2}} L^*(f, n+1) L^*(f, n+k_2), \quad (1-2)$$

where  $k = k_1 + k_2 + 2n$  and  $[g_1, g_2]_n$  stands for the Rankin–Cohen bracket of index  $n$  given by

$$[g_1, g_2]_n := \sum_{r=0}^n (-1)^r \binom{k_1+n-1}{n-r} \binom{k_2+n-1}{r} g_1^{(r)} g_2^{(n-r)}. \quad (1-3)$$

The periods of  $f$  in the critical strip are the numbers

$$L^*(f, 1), L^*(f, 2), \dots, L^*(f, k-1). \quad (1-4)$$

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Zagier [1977, §5] and Kohlen and Zagier [1984] proved important results of the Eichler–Shimura–Manin theory on the algebraicity of these critical values using (1-2). We describe this in more depth in Sections 2C and 8A.

On the face of it, the techniques of [Zagier 1977], employing (1-2), apply only to critical values; an extension to noncritical values,  $L^*(f, j)$  for integers  $j \leq 0$  or  $j \geq k$ , would seem to require Rankin–Cohen brackets of negative index  $n$  or holomorphic Eisenstein series of negative weight, neither of which are defined. Analyzing the structure of the Rankin–Cohen bracket of two Eisenstein series in Section 6 reveals a natural construction, which we call a *double Eisenstein series*:<sup>1</sup>

$$\sum_{\substack{\gamma, \delta \in \Gamma_\infty \setminus \Gamma \\ \gamma \delta^{-1} \neq \Gamma_\infty}} (c_{\gamma \delta^{-1}})^l j(\gamma, z)^{-k_1} j(\delta, z)^{-k_2}, \tag{1-5}$$

where, for  $\gamma \in \Gamma$ , we write

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \quad \text{and} \quad j(\gamma, z) := c_\gamma z + d_\gamma.$$

By comparison, the usual holomorphic Eisenstein series is

$$E_k(z) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} j(\gamma, z)^{-k}. \tag{1-6}$$

The double Eisenstein series (1-5) converges to a weight- $(-k_1 + k_2)$  cusp form when  $l < k_1 - 2, k_2 - 2$ . For negative integers  $l$ , it behaves as a Rankin–Cohen bracket of negative index; see Proposition 2.4. This allows us to further generalize (1-1) and (1-2), and in Section 8, we characterize the field containing an arbitrary value of an  $L$ -function in terms of double Eisenstein series and their Fourier coefficients. In the interesting paper [Cohen et al. 1997], Rankin–Cohen brackets are linked to operations on automorphic pseudodifferential operators and may also be reinterpreted in this framework allowing for more general indices.

An extension of Zagier’s kernel formula (1-2) in the nonholomorphic direction is given in Section 9C. There we show that the holomorphic double Eisenstein series have nonholomorphic counterparts:

$$\sum_{\substack{\gamma, \delta \in \Gamma_\infty \setminus \Gamma \\ \gamma \delta^{-1} \neq \Gamma_\infty}} |c_{\gamma \delta^{-1}}|^{-s-s'} \text{Im}(\gamma z)^s \text{Im}(\delta z)^{s'}. \tag{1-7}$$

These weight-0 functions possess analytic continuations and functional equations resembling those for the classical nonholomorphic Eisenstein series. As kernels, they produce products of  $L$ -functions for *Maass cusp forms*; see Theorem 2.9. The main motivation for this construction was its potential use in the rapidly developing

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<sup>1</sup>In the context of multiple zeta functions, the authors in [Gangl et al. 2006] give a different definition of “double Eisenstein series”. See also [Deninger 1995], for example, for distinct “double Eisenstein–Kronecker series”.

study of periods of Maass forms [Bruggeman et al. 2013; Lewis and Zagier 2001; Manin 2010; Mühlenbruch 2006]. In developing the properties of (1-7), we require a certain kernel  $\mathcal{H}(z; s, s')$  as defined in (9-1). It is interesting to note that Diaconu and Goldfeld [2007] needed exactly the same series for their results on second moments of  $L^*(f, s)$ ; see Section 9A.

## 2. Statement of main results

**2A. Preliminaries.** Our notation is as in [Diamantis and O’Sullivan 2010]. In all sections but two,  $\Gamma$  is the modular group  $SL(2, \mathbb{Z})$  acting on the upper half-plane  $\mathbb{H}$ . The definitions we give for double Eisenstein series extend easily to more general groups, so in Section 4, we prove their basic properties for  $\Gamma$  an arbitrary Fuchsian group of the first kind, and in Section 10, we see how some of our main results are valid in this general context.

Let  $S_k(\Gamma)$  be the  $\mathbb{C}$ -vector space of holomorphic, weight- $k$  cusp forms for  $\Gamma$  and  $M_k(\Gamma)$  the space of modular forms. These spaces are acted on by the Hecke operators  $T_m$ ; see (3-6). Let  $\mathcal{B}_k$  be the unique basis of  $S_k$  consisting of Hecke eigenforms normalized to have first Fourier coefficient 1. We assume throughout this paper that  $f \in \mathcal{B}_k$ . Since  $\langle T_m f, f \rangle = \langle f, T_m f \rangle$ , it follows that all the Fourier coefficients of  $f$  are real, and hence,  $\overline{L^*(f, s)} = L^*(f, \bar{s})$ . Also, recall the functional equation

$$L^*(f, k - s) = (-1)^{k/2} L^*(f, s). \tag{2-1}$$

We summarize some standard properties of the nonholomorphic Eisenstein series; see for example [Iwaniec 2002, Chapters 3 and 6]. Throughout this paper, we use the variables  $z = x + iy \in \mathbb{H}$  and  $s = \sigma + it \in \mathbb{C}$ .

**Definition 2.1.** For  $z \in \mathbb{H}$  and  $s \in \mathbb{C}$  with  $\text{Re}(s) > 1$ , the weight-0, *nonholomorphic Eisenstein series* is

$$E(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \text{Im}(\gamma z)^s = \frac{y^s}{2} \sum_{\substack{c, d \in \mathbb{Z} \\ (c, d) = 1}} |cz + d|^{-2s}. \tag{2-2}$$

Let  $\theta(s) := \pi^{-s} \Gamma(s) \zeta(2s)$ . Then  $E(z, s)$  has a Fourier expansion [Iwaniec 2002, Theorem 3.4], which we may write in the form

$$E(z, s) = y^s + \frac{\theta(1 - s)}{\theta(s)} y^{1-s} + \sum_{m \neq 0} \phi(m, s) |m|^{-1/2} W_s(mz), \tag{2-3}$$

where  $W_s(mz) = 2(|m|y)^{1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi imx}$  is the Whittaker function for  $z \in \mathbb{H}$  and also  $\theta(s)\phi(m, s) = \sigma_{2s-1}(|m|) |m|^{1/2-s}$ . As usual,  $\sigma_s(m) := \sum_{d|m} d^s$  is the divisor function.

For the weight- $k$  ( $k \in 2\mathbb{Z}$ ) nonholomorphic Eisenstein series, generalizing (2-2),

set

$$E_k(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \text{Im}(\gamma z)^s \left( \frac{j(\gamma, z)}{|j(\gamma, z)|} \right)^{-k}. \tag{2-4}$$

Then (2-4) converges to an analytic function of  $s \in \mathbb{C}$  and a smooth function of  $z \in \mathbb{H}$  for  $\text{Re}(s) > 1$ . Also  $y^{-k/2} E_k(z, s)$  has weight  $k$  in  $z$ . Define the *completed nonholomorphic Eisenstein series* as

$$E_k^*(z, s) := \theta_k(s) E_k(z, s) \quad \text{for } \theta_k(s) := \pi^{-s} \Gamma(s + |k|/2) \zeta(2s). \tag{2-5}$$

With (2-3), we see that  $E(z, s)$  has a meromorphic continuation to all  $s \in \mathbb{C}$ . The same is true of  $E_k(z, s)$ ; see [Diamantis and O’Sullivan 2010, §2.1] for example. We have the functional equations

$$\theta(s/2) = \theta((1 - s)/2), \tag{2-6}$$

$$E_k^*(z, s) = E_k^*(z, 1 - s). \tag{2-7}$$

**2B. Holomorphic double Eisenstein series.** Define the subgroup

$$B := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} \subset \text{SL}(2, \mathbb{Z}). \tag{2-8}$$

Then  $\Gamma_\infty$ , the subgroup of  $\Gamma = \text{SL}(2, \mathbb{Z})$  fixing  $\infty$ , is  $B \cup -B$ . For  $\gamma \in \Gamma_\infty \backslash \Gamma$ , the quantities  $c_\gamma, d_\gamma$  and  $j(\gamma, z)$  are only defined up to sign (though even powers are well-defined). For  $\gamma \in B \backslash \Gamma$ , there is no ambiguity in the signs of  $c_\gamma, d_\gamma$  and  $j(\gamma, z)$ .

**Definition 2.2.** Let  $z \in \mathbb{H}$  and  $w \in \mathbb{C}$ . For integers  $k_1, k_2 \geq 3$ , we define the *double Eisenstein series*

$$E_{k_1, k_2}(z, w) := \sum_{\substack{\gamma, \delta \in B \backslash \Gamma \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} j(\gamma, z)^{-k_1} j(\delta, z)^{-k_2}. \tag{2-9}$$

This series is well-defined and converges to a holomorphic function of  $z$  that is a weight- $(k = k_1 + k_2)$  cusp form for  $\text{Re}(w) < k_1 - 1, k_2 - 1$ , as we see in Proposition 4.2. It vanishes identically when  $k_1$  and  $k_2$  have different parity.

Let  $k$  be even. To get the most general kernel, with  $s \in \mathbb{C}$  set

$$E_{s, k-s}(z, w) := \sum_{\substack{\gamma, \delta \in B \backslash \Gamma \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \left( \frac{j(\gamma, z)}{j(\delta, z)} \right)^{-s} j(\delta, z)^{-k}. \tag{2-10}$$

In the usual convention, for  $\rho \in \mathbb{C}$  with  $\rho \neq 0$ , write

$$\rho = |\rho| e^{i \arg(\rho)} \quad \text{for } -\pi < \arg(\rho) \leq \pi$$

and

$$\rho^s = |\rho|^s e^{i \arg(\rho)s} \quad \text{for } s \in \mathbb{C}. \tag{2-11}$$

Note that

$$c_{\gamma\delta^{-1}} = \begin{vmatrix} c_\gamma & d_\gamma \\ c_\delta & d_\delta \end{vmatrix} > 0 \implies \frac{j(\gamma, z)}{j(\delta, z)} \in \mathbb{H} \quad \text{for } z \in \mathbb{H},$$

and so  $(j(\gamma, z)/j(\delta, z))^{-s}$  in (2-10) is well-defined and a holomorphic function of  $s \in \mathbb{C}$  and  $z \in \mathbb{H}$ . Proposition 4.2 shows that  $E_{s,k-s}(z, w)$  converges absolutely and uniformly on compact sets for which  $2 < \sigma < k - 2$  and  $\text{Re}(w) < \sigma - 1, k - 1 - \sigma$ .

Define the *completed double Eisenstein series* as

$$E_{s,k-s}^*(z, w) \tag{2-12}$$

$$:= \left[ \frac{e^{s\pi/2} \Gamma(s) \Gamma(k-s) \Gamma(k-w) \zeta(1-w+s) \zeta(1-w+k-s)}{2^{3-w} \pi^{k+1-w} \Gamma(k-1)} \right] E_{s,k-s}(z, w).$$

**Theorem 2.3.** *Let  $k \geq 6$  be even. The series  $E_{s,k-s}^*(z, w)$  has an analytic continuation to all  $s, w \in \mathbb{C}$  and as a function of  $z$  is always in  $S_k(\Gamma)$ . For any  $f$  in  $\mathcal{B}_k$ , we have*

$$\langle E_{s,k-s}^*(\cdot, w), f \rangle = L^*(f, s) L^*(f, w). \tag{2-13}$$

It follows directly from (2-13) and (2-1) that  $E_{s,k-s}^*(z, w)$  satisfies eight functional equations generated by

$$E_{s,k-s}^*(z, w) = E_{w,k-w}^*(z, s), \tag{2-14}$$

$$E_{s,k-s}^*(z, w) = (-1)^{k/2} E_{k-s,s}^*(z, w). \tag{2-15}$$

The next result shows how  $E_{s,k-s}^*$  is a generalization of the Rankin–Cohen bracket  $[E_{k_1}, E_{k_2}]_n$ .

**Proposition 2.4.** *For  $n \in \mathbb{Z}_{\geq 1}$  and even  $k_1, k_2 \geq 4$ ,*

$$n! [E_{k_1}, E_{k_2}]_n = \frac{2(-1)^{k_1/2} \pi^k \Gamma(k-1)}{(2\pi i)^n \zeta(k_1) \zeta(k_2) \Gamma(k_1) \Gamma(k_2) \Gamma(k-n-1)} E_{k_1+n, k_2+n}^*(z, n+1).$$

Another way to understand these double Eisenstein series is through their connections to nonholomorphic Eisenstein series. Any smooth function transforming with weight  $k$  and with polynomial growth as  $y \rightarrow \infty$  may be projected into  $S_k$  with respect to the Petersson scalar product. See [Diamantis and O’Sullivan 2010, §3.2] and the contained references. Denote this holomorphic projection by  $\pi_{\text{hol}}$ .

**Proposition 2.5.** *Let  $k = k_1 + k_2 \geq 6$  for even  $k_1, k_2 \geq 0$ . Then for all  $s, w \in \mathbb{C}$*

$$E_{s,k-s}^*(z, w) = \pi_{\text{hol}} \left[ (-1)^{k_2/2} y^{-k/2} E_{k_1}^*(z, u) E_{k_2}^*(z, v) / (2\pi^{k/2}) \right],$$

where

$$u = (s + w - k + 1)/2 \quad \text{and} \quad v = (-s + w + 1)/2. \tag{2-16}$$

**2C. Values of  $L$ -functions.** For  $f \in \mathcal{B}_k$ , let  $K_f$  be the field obtained by adjoining to  $\mathbb{Q}$  the Fourier coefficients of  $f$ . We will recall Zagier’s proof of the next result in [Section 8A](#).

**Theorem 2.6** (Manin’s periods theorem). *For each  $f \in \mathcal{B}_k$  there exist real numbers  $\omega_+(f), \omega_-(f)$  such that*

$$L^*(f, s)/\omega_+(f), L^*(f, w)/\omega_-(f) \in K_f$$

for all  $s$  and  $w$  with  $1 \leq s, w \leq k - 1$  and  $s$  even and  $w$  odd.

Let  $m \in \mathbb{Z}$  satisfy  $m \leq 0$  or  $m \geq k$ . Then for these values outside the critical strip we have, according to [\[Kontsevich and Zagier 2001, §3.4\]](#) and the references therein,

$$L^*(f, m) \in \mathcal{P}[1/\pi],$$

where  $\mathcal{P}$  is the ring of periods: complex numbers that may be expressed as an integral of an algebraic function over an algebraic domain. In contrast to the periods (1-4), we do not have much more precise information about the algebraic properties of the values  $L^*(f, m)$ . A special case of a theorem by Koblic [\[1975\]](#) shows, for example, that

$$L^*(f, m) \notin \mathbb{Z} \cdot L^*(f, 1) + \mathbb{Z} \cdot L^*(f, 2) + \dots + \mathbb{Z} \cdot L^*(f, k - 1).$$

Let  $K(\mathbf{E}_{s,k-s}^*(\cdot, w))$  be the field obtained by adjoining to  $\mathbb{Q}$  the Fourier coefficients of  $\mathbf{E}_{s,k-s}^*(\cdot, w)$ , and let  $\omega_+(f)$  and  $\omega_-(f)$  be as given in [Theorem 2.6](#). Then we have:

**Theorem 2.7.** *For all  $f \in \mathcal{B}_k$  and  $s \in \mathbb{C}$ ,*

$$L^*(f, s)/\omega_+(f) \in K(\mathbf{E}_{s,k-s}^*(\cdot, k - 1))K_f,$$

$$L^*(f, s)/\omega_-(f) \in K(\mathbf{E}_{k-2,2}^*(\cdot, s))K_f.$$

The above theorem gives the link between Fourier coefficients of double Eisenstein series and arbitrary values of  $L$ -functions. We hope that this interesting connection will help shed light on  $L^*(f, s)$  for  $s$  outside the set  $\{1, 2, \dots, k - 1\}$ . See the further discussion in [Section 8B](#) for the case when  $s \in \mathbb{Z}$ .

In [Section 8C](#), we also prove results analogous to [Theorem 2.7](#) for the completed  $L$ -function of  $f$  twisted by  $e^{2\pi imp/q}$  for  $p/q \in \mathbb{Q}$ :

$$L^*(f, s; p/q) := \frac{\Gamma(s)}{(2\pi)^s} \sum_{m=1}^{\infty} \frac{a_f(m)e^{2\pi imp/q}}{m^s} = \int_0^{\infty} f(iy + p/q)y^{s-1} dy. \tag{2-17}$$

**2D. Nonholomorphic double Eisenstein series.**

**Definition 2.8.** For  $z \in \mathbb{H}$  and  $w, s, s' \in \mathbb{C}$ , we define the *nonholomorphic double Eisenstein series* as

$$\mathcal{E}(z, w; s, s') := \sum_{\substack{\gamma, \delta \in \Gamma_\infty \setminus \Gamma \\ \gamma\delta^{-1} \neq \Gamma_\infty}} \frac{\text{Im}(\gamma z)^s \text{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{2-18}$$

A simple comparison with (2-2) shows it is absolutely and uniformly convergent for  $\text{Re}(s), \text{Re}(s') > 1$  and  $\text{Re}(w) > 0$ . (This domain of convergence is improved in Proposition 4.3.) The most symmetric form of (2-18) is when  $w = s + s'$ . Define

$$\begin{aligned} \mathcal{E}^*(z; s, s') := & 4\pi^{-s-s'} \Gamma(s)\Gamma(s')\zeta(3s+s')\zeta(s+3s')\mathcal{E}(z, s+s'; s, s') \\ & + 2\theta(s)\theta(s')E(z, s+s'). \end{aligned} \tag{2-19}$$

**Theorem 2.9.** *The completed double Eisenstein series  $\mathcal{E}^*(z; s, s')$  has a meromorphic continuation to all  $s, s' \in \mathbb{C}$  and satisfies the functional equations*

$$\begin{aligned} \mathcal{E}^*(z; s, s') &= \mathcal{E}^*(z; s', s), \\ \mathcal{E}^*(z; s, s') &= \mathcal{E}^*(z; 1-s, 1-s'). \end{aligned}$$

For any even Maass Hecke eigenform  $u_j$ ,

$$\langle \mathcal{E}^*(z; s, s'), u_j \rangle = L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2).$$

**3. Further background results and notation**

We need to introduce two more families of modular forms.

**Definition 3.1.** For  $z \in \mathbb{H}$ ,  $k \geq 4$  in  $2\mathbb{Z}$  and  $m \in \mathbb{Z}_{\geq 0}$ , the holomorphic *Poincaré series* is

$$P_k(z; m) := \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \frac{e^{2\pi im\gamma z}}{j(\gamma, z)^k} = \frac{1}{2} \sum_{\gamma \in B \setminus \Gamma} \frac{e^{2\pi im\gamma z}}{j(\gamma, z)^k}. \tag{3-1}$$

For  $m \geq 1$ , the series  $P_k(z; m)$  span  $S_k(\Gamma)$ . The Eisenstein series  $E_k(z) = P_k(z; 0)$  is not a cusp form but is in the space  $M_k(\Gamma)$ . The second family of modular forms is based on a series due to Cohen [1981].

**Definition 3.2.** The *generalized Cohen kernel* is given by

$$\mathbb{C}_k(z, s; p/q) := \frac{1}{2} \sum_{\gamma \in \Gamma} (\gamma z + p/q)^{-s} j(\gamma, z)^{-k} \tag{3-2}$$

for  $p/q \in \mathbb{Q}$  and  $s \in \mathbb{C}$  with  $1 < \text{Re}(s) < k - 1$ .

In [Diamantis and O’Sullivan 2010, §5], we studied  $\mathcal{C}_k(z, s; p/q)$  (the factor  $1/2$  is included to keep the notation consistent with that article, where  $\Gamma = \text{PSL}(2, \mathbb{Z})$ ). We showed that, for each  $s \in \mathbb{C}$  with  $1 < \text{Re}(s) < k - 1$ ,  $\mathcal{C}_k(z, s; p/q)$  converges to an element of  $S_k(\Gamma)$  with a meromorphic continuation to all  $s \in \mathbb{C}$ . From Proposition 5.4 of the same work, we have

$$\langle \mathcal{C}_k(\cdot, s; p/q), f \rangle = 2^{2-k} \pi e^{-s i \pi / 2} \frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)} L^*(f, k-s; p/q), \tag{3-3}$$

which is a generalization of Cohen’s lemma in [Kohnen and Zagier 1984, §1.2]. For simplicity, we write  $\mathcal{C}_k(z, s)$  for  $\mathcal{C}_k(z, s; 0)$ . The twisted  $L$ -functions satisfy

$$\overline{L^*(f, s; p/q)} = L^*(f, \bar{s}; -p/q), \tag{3-4}$$

$$q^s L^*(f, s; p/q) = (-1)^{k/2} q^{k-s} L^*(f, k-s; -p'/q) \tag{3-5}$$

for  $pp' \equiv 1 \pmod q$  as in [Kowalski et al. 2002, Appendix A.3].

Define  $\mathcal{M}_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = n \right\}$ . Thus,  $\mathcal{M}_1 = \Gamma$ . For  $k \in \mathbb{Z}$  and  $g : \mathbb{H} \rightarrow \mathbb{C}$ , set

$$(g|_k \gamma)(z) := \det(\gamma)^{k/2} g(\gamma z) j(\gamma, z)^{-k}$$

for all  $\gamma \in \mathcal{M}_n$ . The weight- $k$  Hecke operator  $T_n$  acts on  $g \in M_k$  by

$$(T_n g)(z) := n^{k/2-1} \sum_{\gamma \in \Gamma \backslash \mathcal{M}_n} (g|_k \gamma)(z) = n^{k-1} \sum_{\substack{ad=n \\ a, d > 0}} d^{-k} \sum_{0 \leq b < d} g\left(\frac{az+b}{d}\right). \tag{3-6}$$

### 4. Basic properties of double Eisenstein series

We work more generally in this section with  $\Gamma$  a Fuchsian group of the first kind containing at least one cusp. Set

$$\varepsilon_\Gamma := \#\{\Gamma \cap \{-I\}\}. \tag{4-1}$$

Label the finite number of inequivalent cusps  $\mathfrak{a}, \mathfrak{b}$ , etc., and let  $\Gamma_{\mathfrak{a}}$  be the subgroup of  $\Gamma$  fixing  $\mathfrak{a}$ . There exists a corresponding scaling matrix  $\sigma_{\mathfrak{a}} \in \text{SL}(2, \mathbb{R})$  such that  $\sigma_{\mathfrak{a}} \infty = \mathfrak{a}$  and

$$\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = \begin{cases} B \cup -B & \text{if } -I \in \Gamma \ (\varepsilon_\Gamma = 1), \\ B & \text{if } -I \notin \Gamma \ (\varepsilon_\Gamma = 0). \end{cases}$$

Also set  $\Gamma_{\mathfrak{a}}^* := \sigma_{\mathfrak{a}} B \sigma_{\mathfrak{a}}^{-1}$ .

We recall some facts about  $E_{k, \mathfrak{a}}(z, s)$ , the nonholomorphic Eisenstein series associated to the cusp  $\mathfrak{a}$ ; see for example [Iwaniec 2002, Chapter 3; Diamantis and O’Sullivan 2010, §2.1]. It is defined as

$$E_{k, \mathfrak{a}}(z, s) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \text{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s \left( \frac{j(\sigma_{\mathfrak{a}}^{-1} \gamma, z)}{|j(\sigma_{\mathfrak{a}}^{-1} \gamma, z)|} \right)^{-k}$$

and absolutely convergent for  $\text{Re}(s) > 1$ . Put  $E_{k,a}^*(z, s) := \theta_k(s)E_{k,a}(z, s)$  as in (2-5). Then we have the expansion

$$E_{0,a}^*(\sigma_b z, s) = \delta_{ab}\theta(s)y^s + \theta(1-s)Y_{ab}(s)y^{1-s} + \sum_{l \neq 0} Y_{ab}(l, s)W_s(lz), \tag{4-2}$$

and

$$E_{k,a}^*(\sigma_b z, s) = \delta_{ab}\theta_k(s)y^s + \theta_k(1-s)Y_{ab}(s)y^{1-s} + O(e^{-2\pi y}) \tag{4-3}$$

as  $y \rightarrow \infty$  for all  $k \in 2\mathbb{Z}$ . Also, its functional equation is

$$E_{k,a}^*(z, 1-s) = \sum_b Y_{ab}(1-s)E_{k,b}^*(z, s). \tag{4-4}$$

We gave the coefficients  $Y_{ab}(s)$  and  $Y_{ab}(l, s)$  explicitly in the case of  $\Gamma = \text{SL}(2, \mathbb{Z})$  following (2-3), and in general, they involve series containing Kloosterman sums; see [Iwaniec 2002, (3.21) and (3.22)].

For the natural generalization of (2-10), we define the *double Eisenstein series associated to the cusp a* as

$$E_{s,k-s,a}(z, w) := \sum_{\substack{\gamma, \delta \in \Gamma_a^* \setminus \Gamma \\ c_{\sigma_a^{-1}\gamma\delta^{-1}\sigma_a} > 0}} (c_{\sigma_a^{-1}\gamma\delta^{-1}\sigma_a})^{w-1} \left( \frac{j(\sigma_a^{-1}\gamma, z)}{j(\sigma_a^{-1}\delta, z)} \right)^{-s} j(\sigma_a^{-1}\delta, z)^{-k} \tag{4-5}$$

so that

$$E_{s,k-s,a}(\sigma_a z, w) = j(\sigma_a, z)^k \sum_{\substack{\gamma, \delta \in B \setminus \Gamma' \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \left( \frac{j(\gamma, z)}{j(\delta, z)} \right)^{-s} j(\delta, z)^{-k} \tag{4-6}$$

for  $\Gamma' = \sigma_a^{-1}\Gamma\sigma_a$ , which is also a Fuchsian group of the first kind. To establish an initial domain of absolute convergence for (4-6), we consider

$$\sum_{\substack{\gamma, \delta \in B \setminus \Gamma' \\ c_{\gamma\delta^{-1}} > 0}} \left| (c_{\gamma\delta^{-1}})^{w-1} \left( \frac{j(\gamma, z)}{j(\delta, z)} \right)^{-s} j(\delta, z)^{-k} \right|. \tag{4-7}$$

Recalling (2-11), we see that

$$|\rho^s| = |\rho|^\sigma e^{-t \arg(\rho)} \ll_t |\rho|^\sigma \quad \text{for } s = \sigma + it \in \mathbb{C}.$$

Therefore, with  $r = \text{Re}(w)$  and  $\text{Im}(\gamma z) = y|j(\gamma, z)|^{-2}$ , we deduce that (4-7) is bounded by a constant depending on  $s$  times

$$y^{-k/2} \sum_{\substack{\gamma, \delta \in \Gamma_\infty \setminus \Gamma' \\ \gamma\delta^{-1} \neq \Gamma_\infty}} |c_{\gamma\delta^{-1}}|^{r-1} \text{Im}(\gamma z)^{\sigma/2} \text{Im}(\delta z)^{(k-\sigma)/2}. \tag{4-8}$$

**Lemma 4.1.** *There exists a constant  $\kappa_\Gamma > 0$  so that for all  $\gamma, \delta \in \Gamma$  with  $c_{\gamma\delta^{-1}} > 0$*

$$\kappa_\Gamma \leq c_{\gamma\delta^{-1}} \leq \text{Im}(\gamma z)^{-1/2} \text{Im}(\delta z)^{-1/2}.$$

*Proof.* The existence of  $\kappa_\Gamma$  is described in [Iwaniec 2002, §2.5 and §2.6; Shimura 1971, Lemma 1.25]. Set  $\varepsilon(\gamma, z) := j(\gamma, z)/|j(\gamma, z)| = e^{i \arg(j(\gamma, z))}$ . It is easy to verify that, for all  $\gamma, \delta \in \Gamma$  and  $z \in \mathbb{H}$ ,

$$\begin{aligned} c_{\gamma\delta^{-1}} &= c_\gamma j(\delta, z) - c_\delta j(\gamma, z) \\ &= \left( \frac{j(\gamma, z) - \overline{j(\gamma, z)}}{2iy} \right) j(\delta, z) - \left( \frac{j(\delta, z) - \overline{j(\delta, z)}}{2iy} \right) j(\gamma, z) \\ &= (\varepsilon(\delta, z)^{-2} - \varepsilon(\gamma, z)^{-2}) j(\gamma, z) j(\delta, z) / (2iy). \end{aligned}$$

Therefore,

$$\begin{aligned} |c_{\gamma\delta^{-1}}| &= \left| \frac{\varepsilon(\gamma, z)}{\varepsilon(\delta, z)} - \frac{\varepsilon(\delta, z)}{\varepsilon(\gamma, z)} \right| \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2} / 2 \\ &= \left| \operatorname{Im} \left( \frac{\varepsilon(\gamma, z)}{\varepsilon(\delta, z)} \right) \right| \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2} \\ &\leq \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2}. \quad \square \end{aligned}$$

It follows that for  $r' = \max(r, 1)$  and  $\gamma\delta^{-1} \notin \Gamma_\infty$

$$|c_{\gamma\delta^{-1}}|^{r-1} \ll \operatorname{Im}(\gamma z)^{(1-r')/2} \operatorname{Im}(\delta z)^{(1-r')/2} \tag{4-9}$$

for an implied constant depending on  $\Gamma$  and  $r$ . Combining (4-8) and (4-9) shows

$$\begin{aligned} \frac{E_{s, k-s, a}(\sigma_a z, w)}{j(\sigma_a, z)^k} &\ll y^{-k/2} \sum_{\substack{\gamma, \delta \in \Gamma_\infty \setminus \Gamma' \\ \gamma\delta^{-1} \notin \Gamma_\infty}} \operatorname{Im}(\gamma z)^{(1-r'+\sigma)/2} \operatorname{Im}(\delta z)^{(1-r'+k-\sigma)/2} \tag{4-10} \\ &= y^{-k/2} \left[ E_a \left( \sigma_a z, \frac{1-r'+\sigma}{2} \right) E_a \left( \sigma_a z, \frac{1-r'+k-\sigma}{2} \right) - E_a \left( \sigma_a z, 1-r'+\frac{k}{2} \right) \right] \end{aligned}$$

on noting that  $\operatorname{Im}(\gamma z) = \operatorname{Im}(\delta z)$  for  $\gamma\delta^{-1} \in \Gamma_\infty$ . Since  $E_a(z, s)$  is absolutely convergent for  $\sigma = \operatorname{Re}(s) > 1$ , we have proved that the series  $E_{s, k-s, a}(\sigma_a z, w)$ , defined in (4-6), is absolutely convergent for  $2 < \sigma < k-2$  and  $\operatorname{Re}(w) < \sigma-1, k-1-\sigma$ . This convergence is uniform for  $z$  in compact sets of  $\mathbb{H}$  and for  $s$  and  $w$  in compact sets in  $\mathbb{C}$  satisfying the above constraints.

We next verify that  $E_{s, k-s, a}(z, w)$  has weight  $k$  in the  $z$  variable. We have

$$f(z) \in M_k(\Gamma) \iff f(\sigma_a z) j(\sigma_a, z)^{-k} \in M_k(\sigma_a^{-1} \Gamma \sigma_a),$$

so with (4-6), we must prove that

$$g(z) := \sum_{\substack{\gamma, \delta \in B \setminus \Gamma' \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \left( \frac{j(\gamma, z)}{j(\delta, z)} \right)^{-s} j(\delta, z)^{-k}$$

is in  $M_k(\Gamma')$ . For all  $\tau \in \Gamma'$ ,

$$\begin{aligned} \frac{g(\tau z)}{j(\tau, z)^k} &= \sum_{\substack{\gamma, \delta \in B \setminus \Gamma' \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \left( \frac{j(\gamma, \tau z)}{j(\delta, \tau z)} \right)^{-s} j(\delta, \tau z)^{-k} j(\tau, z)^{-k} \\ &= \sum_{\substack{\gamma, \delta \in B \setminus \Gamma' \\ c_{(\gamma\tau)(\delta\tau)^{-1}} > 0}} (c_{(\gamma\tau)(\delta\tau)^{-1}})^{w-1} \left( \frac{j(\gamma\tau, z)}{j(\delta\tau, z)} \right)^{-s} j(\delta\tau, z)^{-k} = g(z) \end{aligned}$$

as required.

We finally show that  $E_{s,k-s}$  is a cusp form. By (4-10), replacing  $z$  by  $\sigma_a^{-1}\sigma_b z$  and using (4-3), for any cusp  $\mathfrak{b}$  we obtain

$$\begin{aligned} &\frac{E_{s,k-s,\mathfrak{a}}(\sigma_b z, w)}{j(\sigma_b, z)^k} \\ &\ll y^{-k/2} \left[ E_{\mathfrak{a}}\left(\sigma_b z, \frac{1-r'+\sigma}{2}\right) E_{\mathfrak{a}}\left(\sigma_b z, \frac{1-r'+k-\sigma}{2}\right) - E_{\mathfrak{a}}\left(\sigma_b z, 1-r'+\frac{k}{2}\right) \right] \\ &\ll y^{1+\sigma-k} + y^{1-\sigma} + y^{1+r'-k} + y^{r'-k} \end{aligned}$$

and approaches 0 as  $y \rightarrow \infty$ . Thus, by a standard argument (see for example [Diamantis and O’Sullivan 2010, Proposition 5.3]),  $E_{s,k-s,\mathfrak{a}}(z, w)$  is a cusp form. Assembling these results, we have shown the following:

**Proposition 4.2.** *Let  $z \in \mathbb{H}$  and  $k \in \mathbb{Z}$ , and let  $s, w \in \mathbb{C}$  satisfy  $2 < \sigma < k - 2$  and  $\text{Re}(w) < \sigma - 1, k - 1 - \sigma$ . For  $\Gamma$  a Fuchsian group of the first kind with cusp  $\mathfrak{a}$ , the series  $E_{s,k-s,\mathfrak{a}}(z, w)$  is absolutely and uniformly convergent for  $s, w$  and  $z$  in compact sets satisfying the above constraints. For each such  $s$  and  $w$ , we have  $E_{s,k-s,\mathfrak{a}}(z, w) \in S_k(\Gamma)$  as a function of  $z$ .*

The same techniques prove the next result for the nonholomorphic double Eisenstein series. Generalizing (2-18), we set

$$\mathcal{E}_{\mathfrak{a}}(\sigma_a z, w; s, s') := \sum_{\substack{\gamma, \delta \in \Gamma_{\infty} \setminus \sigma_a^{-1} \Gamma \sigma_a \\ \gamma\delta^{-1} \neq \Gamma_{\infty}}} \frac{\text{Im}(\gamma z)^s \text{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{4-11}$$

**Proposition 4.3.** *Let  $z \in \mathbb{H}$  and  $s, s', w \in \mathbb{C}$  with  $\sigma = \text{Re}(s)$  and  $\sigma' = \text{Re}(s')$ . The series  $\mathcal{E}_{\mathfrak{a}}(z, w; s, s')$  defined in (4-11) is absolutely and uniformly convergent for  $z, w, s$  and  $s'$  in compact sets satisfying*

$$\sigma, \sigma' > 1 \quad \text{and} \quad \text{Re}(w) > 2 - 2\sigma, 2 - 2\sigma'.$$

Unlike  $E_{s,k-s,\mathfrak{a}}(z, w)$ , the series  $\mathcal{E}_{\mathfrak{a}}(z, w; s, s')$  may have polynomial growth at cusps.

**5. Further results on double Eisenstein series**

**5A. Analytic continuation: proof of Theorem 2.3.** Our next task is to prove the meromorphic continuation of  $E_{s,k-s}(z, w)$  in  $s$  and  $w$ . For  $s$  and  $w$  in the initial domain of convergence, we begin with

$$\begin{aligned} & \zeta(1-w+s)\zeta(1-w+k-s)E_{s,k-s}(z, w) \\ &= \sum_{u,v=1}^{\infty} u^{w-1-s}v^{w-1-k+s} \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ (a,b)=(c,d)=1 \\ ad-bc>0}} (ad-bc)^{w-1} \left(\frac{az+b}{cz+d}\right)^{-s} (cz+d)^{-k} \\ &= \sum_{u,v=1}^{\infty} \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ (a,b)=(c,d)=1 \\ ad-bc>0}} (au \cdot dv - bu \cdot cv)^{w-1} \left(\frac{au \cdot z + bu}{cv \cdot z + dv}\right)^{-s} (cv \cdot z + dv)^{-k} \\ &= \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ ad-bc>0}} (ad-bc)^{w-1} \left(\frac{az+b}{cz+d}\right)^{-s} (cz+d)^{-k} \tag{5-1} \end{aligned}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \frac{1}{n^{1-w}} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_n} \left(\frac{az+b}{cz+d}\right)^{-s} (cz+d)^{-k} \\ &= 2 \sum_{n=1}^{\infty} \frac{T_n \mathcal{C}_k(z, s)}{n^{k-w}}, \tag{5-2} \end{aligned}$$

recalling (3-2). With Proposition 4.2, we know  $E_{s,k-s}(z, w) \in S_k(\Gamma)$  so that

$$\begin{aligned} E_{s,k-s}(z, w) &= \sum_{f \in \mathcal{B}_k} \frac{\langle E_{s,k-s}(\cdot, w), f \rangle}{\langle f, f \rangle} f(z) \implies \\ \zeta(1-w+s)\zeta(1-w+k-s)E_{s,k-s}(z, w) &= 2 \sum_{n=1}^{\infty} \frac{1}{n^{k-w}} \sum_{f \in \mathcal{B}_k} \frac{\langle T_n \mathcal{C}_k(\cdot, s), f \rangle}{\langle f, f \rangle} f(z). \end{aligned}$$

Then

$$\langle T_n \mathcal{C}_k(z, s), f \rangle = \langle \mathcal{C}_k(z, s), T_n f \rangle = a_f(n) \langle \mathcal{C}_k(z, s), f \rangle,$$

and with (3-3), we obtain

$$\begin{aligned} & \zeta(1-w+s)\zeta(1-w+k-s)E_{s,k-s}(z, w) \\ &= 2^{3-w} \pi^{k+1-w} e^{-si\pi/2} \frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)} \\ & \quad \times \sum_{f \in \mathcal{B}_k} L^*(f, k-s)L^*(f, k-w) \frac{f(z)}{\langle f, f \rangle}. \tag{5-3} \end{aligned}$$

Define the completed double Eisenstein series  $E^*$  with (2-12). Then (5-3) becomes

$$E_{s,k-s}^*(z, w) = \sum_{f \in \mathcal{B}_k} L^*(f, s)L^*(f, w) \frac{f(z)}{\langle f, f \rangle}. \tag{5-4}$$

We also now see from (5-4) that  $E_{s,k-s}^*(z, w)$  has an analytic continuation to all  $s$  and  $w$  in  $\mathbb{C}$  and satisfies (2-13) and the two functional equations (2-14) and (2-15). The dihedral group  $D_8$  generated by (2-14) and (2-15) is described in [Diamantis and O’Sullivan 2010, §4.4].  $\square$

**5B. Twisted double Eisenstein series.** In this section, we define the *twisted double Eisenstein series* by

$$\begin{aligned} &\zeta(1-w+s)\zeta(1-w+k-s)E_{s,k-s}(z, w; p/q) \\ &:= \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ ad-bc > 0}} (ad-bc)^{w-1} \left( \frac{az+b}{cz+d} + \frac{p}{q} \right)^{-s} (cz+d)^{-k} \end{aligned} \tag{5-5}$$

for  $p/q \in \mathbb{Q}$  with  $q > 0$  and establish its basic required properties. We remark that the above definition of  $E_{s,k-s}(z, w; p/q)$  comes from generalizing (5-1), but it is not clear how it can be extended to general Fuchsian groups.

Writing

$$\begin{aligned} &(ad-bc)^{w-1} \left( \frac{az+b}{cz+d} + \frac{p}{q} \right)^{-s} \\ &= q^{1-w+s} ((aq+cp)d - (bq+dp)c)^{w-1} \left( \frac{(aq+cp)z + (bq+dp)}{cz+d} \right)^{-s}, \end{aligned}$$

we see that (5-5) equals

$$q^{1-w+s} \sum_{\substack{a',b',c,d \in \mathbb{Z} \\ a'd-b'c > 0}} (a'd-b'c)^{w-1} \left( \frac{a'z+b'}{cz+d} \right)^{-s} (cz+d)^{-k}$$

with  $a' \equiv cp \pmod q$  and  $b' \equiv dp \pmod q$ . Hence,  $E_{s,k-s}(z, w; p/q)$  is a subseries of  $E_{s,k-s}(z, w)$  and, in the same domain of initial convergence, is an element of  $S_k$ .

The analog of (5-2) is

$$\zeta(1-w+s)\zeta(1-w+k-s)E_{s,k-s}(z, w; p/q) = 2 \sum_{n=1}^{\infty} \frac{T_n \mathcal{C}_k(z, s; p/q)}{n^{k-w}}. \tag{5-6}$$

Hence, with (3-3),

$$\begin{aligned} & \zeta(1-w+s)\zeta(1-w+k-s)\mathbf{E}_{s,k-s}(z, w; p/q) \\ &= 2^{3-w}\pi^{k+1-w}e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)} \\ & \quad \times \sum_{f \in \mathcal{B}_k} L^*(f, k-s; p/q)L^*(f, k-w)\frac{f(z)}{\langle f, f \rangle}. \end{aligned} \tag{5-7}$$

Define the completed double Eisenstein series  $\mathbf{E}_{s,k-s}^*(z, w; p/q)$  with the same factor as (2-12), and we obtain

$$\langle \mathbf{E}_{s,k-s}^*(\cdot, w; p/q), f \rangle = L^*(f, k-s; p/q)L^*(f, k-w) \tag{5-8}$$

for any  $f$  in  $\mathcal{B}_k$ . Then (5-7) implies  $\mathbf{E}_{s,k-s}^*(z, w; p/q)$  has an analytic continuation to all  $s$  and  $w$  in  $\mathbb{C}$ . It satisfies the two functional equations

$$\begin{aligned} \mathbf{E}_{s,k-s}^*(z, k-w; p/q) &= (-1)^{k/2}\mathbf{E}_{s,k-s}^*(z, w; p/q), \\ q^s \mathbf{E}_{k-s,s}^*(z, w; p/q) &= (-1)^{k/2}q^{k-s}\mathbf{E}_{s,k-s}^*(z, w; -p'/q) \end{aligned}$$

for  $pp' \equiv 1 \pmod q$  using (2-1) and (3-5), respectively.

### 6. Applying the Rankin–Cohen bracket to Poincaré series

The main objective of this section is to show how double Eisenstein series arise naturally when the Rankin–Cohen bracket is applied to the usual Eisenstein series  $E_k$ . Proposition 2.4 will be a consequence of this. In fact, since there is no difficulty in extending these methods, we compute the Rankin–Cohen bracket of two arbitrary Poincaré series

$$[P_{k_1}(z; m_1), P_{k_2}(z; m_2)]_n$$

for  $m_1, m_2 \geq 0$ . The result may be expressed in terms of the *double Poincaré series* defined below. In this way, the action of the Rankin–Cohen brackets on spaces of modular forms can be completely described. See also Corollary 6.5 at the end of this section.

**Definition 6.1.** Let  $z \in \mathbb{H}$ ,  $k_1, k_2 \geq 3$  in  $\mathbb{Z}$  and  $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ . For  $w \in \mathbb{C}$  with  $\text{Re}(w) < k_1 - 1, k_2 - 1$ , we define the *double Poincaré series*

$$P_{k_1, k_2}(z, w; m_1, m_2) := \sum_{\substack{\gamma, \delta \in B \setminus \Gamma \\ c_\gamma \delta^{-1} > 0}} (c_\gamma \delta^{-1})^{w-1} \frac{e^{2\pi i(m_1 \gamma z + m_2 \delta z)}}{j(\gamma, z)^{k_1} j(\delta, z)^{k_2}}. \tag{6-1}$$

The series (6-1) will vanish identically unless  $k_1$  and  $k_2$  have the same parity. Clearly, we have  $\mathbf{E}_{k_1, k_2}(z, w) = P_{k_1, k_2}(z, w; 0, 0)$ . Since  $|e^{2\pi i(m_1 \gamma z + m_2 \delta z)}| \leq 1$ , it is a simple matter to verify that the work in Section 4 proves that  $P_{k_1, k_2}(z, w; m_1, m_2)$  converges absolutely and uniformly on compacta to a cusp form in  $S_{k_1+k_2}(\Gamma)$ .

For  $l \in \mathbb{Z}_{\geq 0}$ , it is convenient to set

$$Q_k(z, l; m) := \begin{cases} P_k(z; m) & \text{if } l = 0, \\ \frac{1}{2} \sum_{\gamma \in B \setminus \Gamma} \frac{e^{2\pi i m \gamma z} (c_\gamma)^l}{j(\gamma, z)^{k+l}} & \text{if } l \geq 1. \end{cases} \quad (6-2)$$

As in the proof of Proposition 4.2,  $Q_k$  is an absolutely convergent series for  $k$  even and at least 4. The next result may be verified by induction.

**Lemma 6.2.** For every  $j \in \mathbb{Z}_{\geq 0}$ , we have the formulas

$$\begin{aligned} \frac{d^j}{dz^j} E_k(z) &= (-1)^j \frac{(k+j-1)!}{(k-1)!} Q_k(z, j; 0), \\ \frac{d^j}{dz^j} P_k(z; m) &= \sum_{l=0}^j (-1)^{l+j} (2\pi i m)^l \frac{j!}{l!} \binom{k+j-1}{k+l-1} Q_{k+2l}(z, j-l; m) \text{ for } m > 0. \end{aligned}$$

Set

$$A_{k_1, k_2}(l, u)_n := \frac{(k_1+n-1)!(k_2+n-1)!}{l! u! (n-l-u)!(k_1+l-1)!(k_2+u-1)!}.$$

**Proposition 6.3.** For  $m_1, m_2 \in \mathbb{Z}_{\geq 1}$ ,

$$\begin{aligned} [P_{k_1}(z; m_1), P_{k_2}(z; m_2)]_n &= \sum_{\substack{l, u \geq 0 \\ l+u \leq n}} A_{k_1, k_2}(l, u)_n (-2\pi i m_1)^l (2\pi i m_2)^u \\ &\quad \times P_{k_1+n+l-u, k_2+n-l+u}(z, n+1-l-u; m_1, m_2)/2 \\ &\quad + P_{k_1+k_2+2n}(z; m_1+m_2) \sum_{\substack{l, u \geq 0 \\ l+u=n}} A_{k_1, k_2}(l, u)_n (-2\pi i m_1)^l (2\pi i m_2)^u. \end{aligned}$$

*Proof.* With Lemma 6.2,

$$\begin{aligned} [P_{k_1}(z; m_1), P_{k_2}(z; m_2)]_n &= \sum_{l=0}^n \sum_{u=0}^n (2\pi i m_1)^l (2\pi i m_2)^u \frac{(k_1+n-1)!(k_2+n-1)!}{l! u! (k_1+l-1)!(k_2+u-1)!} \\ &\quad \times \sum_{r=l}^{n-u} (-1)^{n+l+u+r} \frac{Q_{k_1+2l}(z, r-l; m_1) Q_{k_2+2u}(z, n-r-u; m_2)}{(r-l)!(n-r-u)!}. \quad (6-3) \end{aligned}$$

The inner sum over  $r$  is

$$\begin{aligned} \frac{(-1)^l}{4(n-l-u)!} \sum_{\gamma, \delta \in B \setminus \Gamma} \frac{e^{2\pi i(m_1 \gamma z + m_2 \delta z)}}{j(\gamma, z)^{k_1+2l} j(\delta, z)^{k_2+2u}} \\ \times \sum_{r=l}^{n-u} \binom{n-l-u}{r-l} \left( \frac{c_\gamma}{j(\gamma, z)} \right)^{r-l} \left( \frac{-c_\delta}{j(\delta, z)} \right)^{n-r-u}, \quad (6-4) \end{aligned}$$

and, employing the binomial theorem, (6-4) reduces to

$$\frac{(-1)^l}{4(n-l-u)!} \sum_{\gamma, \delta \in B \setminus \Gamma} \frac{e^{2\pi i(m_1\gamma z + m_2\delta z)}}{j(\gamma, z)^{k_1+n+l-u} j(\delta, z)^{k_2+n-l+u}} (c_\gamma j(\delta, z) - c_\delta j(\gamma, z))^{n-l-u} \tag{6-5}$$

for  $l + u < n$  and

$$\frac{(-1)^l}{4(n-l-u)!} \sum_{\gamma, \delta \in B \setminus \Gamma} \frac{e^{2\pi i(m_1\gamma z + m_2\delta z)}}{j(\gamma, z)^{k_1+n+l-u} j(\delta, z)^{k_2+n-l+u}} \tag{6-6}$$

for  $l + u = n$ . Noting that

$$c_\gamma j(\delta, z) - c_\delta j(\gamma, z) = \begin{vmatrix} c_\gamma & d_\gamma \\ c_\delta & d_\delta \end{vmatrix} = c_\gamma \delta^{-1}$$

means that (6-5) becomes

$$\frac{(-1)^l}{2(n-l-u)!} P_{k_1+n+l-u, k_2+n-l+u}(z, n+1-l-u; m_1, m_2) \tag{6-7}$$

and (6-6) equals

$$\frac{(-1)^l}{(n-l-u)!} \left( \frac{P_{k_1+n+l-u, k_2+n-l+u}(z, n+1-l-u; m_1, m_2)}{2} + P_{k_1+k_2+2n}(z; m_1 + m_2) \right). \tag{6-8}$$

Putting (6-7) and (6-8) into (6-3) finishes the proof. □

In fact, Proposition 6.3 is also valid for  $m_1$  or  $m_2$  equaling 0 provided we agree that  $(-2\pi i m_1)^l = 1$  in the ambiguous case where  $m_1 = l = 0$  and similarly that  $(2\pi i m_2)^u = 1$  when  $m_2 = u = 0$ . With this notational convention, the proof of the last proposition gives:

**Corollary 6.4.** *For  $m > 0$ , we have*

$$\begin{aligned} [E_{k_1}(z), P_{k_2}(z; m)]_n &= \sum_{u=0}^n A_{k_1, k_2}(0, u)_n (2\pi i m)^u \\ &\times \frac{P_{k_1+n-u, k_2+n+u}(z, n+1-u; 0, m)}{2} + P_{k_1+k_2+2n}(z; m) \cdot A_{k_1, k_2}(0, n)_n (2\pi i m)^n, \\ [E_{k_1}(z), E_{k_2}(z)]_n &= A_{k_1, k_2}(0, 0)_n E_{k_1+n, k_2+n}(z, n+1)/2 + E_{k_1+k_2}(z) \cdot \delta_{n,0}. \end{aligned} \tag{6-9}$$

Proposition 2.4 follows directly from (6-9). Combining Proposition 2.4 with Theorem 2.3 gives a new proof of Zagier’s formula (1-2). His original proof in [1977, Proposition 6] employed Poincaré series.

*Proof of Proposition 2.5.* Let  $F_{s,w}(z) = (-1)^{k_2/2} y^{-k/2} E_{k_1}^*(z, u) E_{k_2}^*(z, v) / (2\pi^{k/2})$  with  $u = (s + w - k + 1)/2$  and  $v = (-s + w + 1)/2$  as before in (2-16). Then

$F_{s,w}(z)$  has weight  $k$  and polynomial growth as  $y \rightarrow \infty$ . It is proved in [Diamantis and O’Sullivan 2010, Proposition 2.1] that

$$\langle F_{s,w}, f \rangle = L^*(f, s)L^*(f, w) \tag{6-10}$$

for all  $f \in B_k$ . Comparing (6-10) with (2-13) shows that

$$E_{s,k-s}^*(\cdot, w) = \pi_{\text{hol}}(F_{s,w}),$$

as required. □

A basic property of Rankin–Cohen brackets naturally emerges from Proposition 6.3 and Corollary 6.4.

**Corollary 6.5.** *For  $g_1 \in M_{k_1}(\Gamma)$  and  $g_2 \in M_{k_2}(\Gamma)$ , we have  $[g_1, g_2]_n \in S_{k_1+k_2+2n}(\Gamma)$  for  $n > 0$ .*

*Proof.* The space  $M_{k_1}(\Gamma)$  is spanned by  $E_{k_1}$  and the Poincaré series  $P_{k_1}(z; m)$  for  $m \in \mathbb{Z}_{\geq 1}$ . So we may write  $g_1$ , and similarly  $g_2$ , as a linear combination of Eisenstein and Poincaré series. Hence,  $[g_1, g_2]_n$  is a linear combination of the Rankin–Cohen brackets appearing in Proposition 6.3 and Corollary 6.4. By these results,  $[g_1, g_2]_n$  is a linear combination of double Poincaré and double Eisenstein series, which are in  $S_{k_1+k_2+2n}(\Gamma)$  as we have already shown. □

It would be interesting to know if  $P_{k_1,k_2}(z, w; m_1, m_2)$  has a meromorphic continuation in  $w$ . As a corollary of work in the next section, we establish the continuation of  $P_{k_1,k_2}(z, w; 0, 0)$  to all  $w \in \mathbb{C}$ .

### 7. The Hecke action

The expression (5-2), giving  $E_{s,k-s}$  in terms of  $\mathcal{C}_k$  acted upon by the Hecke operators, can be studied further and yields an interesting relation between  $E_{s,k-s}(z, w)$  and the generalized Cohen kernel  $\mathcal{C}_k(z, s; p/q)$ .

We have

$$\begin{aligned} T_n \mathcal{C}_k(z, s; p/q) &= n^{k-1} \sum_{\rho \in \Gamma \backslash \mathcal{M}_n} \mathcal{C}_k(\rho z, s; p/q) \cdot j(\rho, z)^{-k} \\ &= \frac{1}{2} n^{k-1} \sum_{\gamma \in \mathcal{M}_n} \left( \gamma z + \frac{p}{q} \right)^{-s} j(\gamma, z)^{-k}. \end{aligned}$$

To decompose  $\mathcal{M}_n$  into left  $\Gamma$ -cosets, set

$$\mathcal{H} := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{Z}_{\geq 0}, ad = n, 0 \leq b < a \right\}$$

so that  $\mathcal{M}_n = \bigcup_{\rho \in \mathfrak{H}} \rho\Gamma$ , a disjoint union. Hence,

$$\begin{aligned} T_n \mathcal{C}_k(z, s; p/q) &= \frac{1}{2} n^{k-1} \sum_{\rho \in \mathfrak{H}} \sum_{\gamma \in \Gamma} \left( \rho\gamma z + \frac{p}{q} \right)^{-s} j(\rho, \gamma z)^{-k} j(\gamma, z)^{-k} \\ &= \frac{1}{2} n^{k-1} \sum_{a|n} \left( \frac{n}{a} \right)^{-k} \left( \frac{a^2}{n} \right)^{-s} \sum_{0 \leq b < a} \sum_{\gamma \in \Gamma} \left( \gamma z + \frac{b}{a} + \frac{n}{a^2} \frac{p}{q} \right)^{-s} j(\gamma, z)^{-k} \\ &= n^{s-1} \sum_{a|n} a^{k-2s} \sum_{0 \leq b < a} \mathcal{C}_k \left( z, s; \frac{b}{a} + \frac{n}{a^2} \frac{p}{q} \right). \end{aligned} \tag{7-1}$$

Combining (7-1) in the case  $p/q = 0$ , with (5-2) we find

$$\begin{aligned} \frac{\zeta(1-w+s)\zeta(1-w+k-s)\mathbf{E}_{s,k-s}(z, w)}{2} &= \sum_{n=1}^{\infty} \frac{T_n \mathcal{C}_k(z, s)}{n^{k-w}} \\ &= \sum_{n=1}^{\infty} n^{s+w-k-1} \sum_{a|n} a^{k-2s} \sum_{0 \leq b < a} \mathcal{C}_k \left( z, s; \frac{b}{a} \right) \\ &= \sum_{a=1}^{\infty} a^{k-2s} \sum_{v=1}^{\infty} (av)^{s+w-k-1} \sum_{0 \leq b < a} \mathcal{C}_k \left( z, s; \frac{b}{a} \right) \\ &= \zeta(k+1-s-w) \sum_{a=1}^{\infty} a^{w-s-1} \sum_{0 \leq b < a} \mathcal{C}_k \left( z, s; \frac{b}{a} \right). \end{aligned}$$

Consequently, for  $2 < \sigma < k-2$  and  $\text{Re}(w) < \sigma - 1, k-1-\sigma$ ,

$$\zeta(1-w+s)\mathbf{E}_{s,k-s}(z, w) = 2 \sum_{a=1}^{\infty} a^{w-s-1} \sum_{b=0}^{a-1} \mathcal{C}_k \left( z, s; \frac{b}{a} \right). \tag{7-2}$$

Upon taking the inner product of both sides with  $f \in \mathfrak{B}_k$ , by using (2-13) and (3-3) and then simplifying, we obtain

$$\begin{aligned} \frac{(2\pi)^{k-w}}{\Gamma(k-w)} L^*(f, s) L^*(f, w) &= \zeta(k+1-s-w) \sum_{a=1}^{\infty} a^{w-s-1} \sum_{b=0}^{a-1} L^* \left( f, k-s; \frac{b}{a} \right). \end{aligned} \tag{7-3}$$

Since the eigenforms  $f$  in  $\mathfrak{B}_k$  span  $S_k$ , we may verify (7-2) by giving another proof of (7-3). Note that the right side of (7-3) equals

$$\begin{aligned} \zeta(k+1-s-w) & \frac{\Gamma(k-s)}{(2\pi)^{k-s}} \sum_{a=1}^{\infty} a^{w-s-1} \sum_{b=0}^{a-1} \sum_{m=1}^{\infty} \frac{a_f(m) e^{2\pi i m b/a}}{m^{k-s}} \\ & = \zeta(k+1-s-w) \frac{\Gamma(k-s)}{(2\pi)^{k-s}} \sum_{m=1}^{\infty} \sum_{a|m} a^{w-s} \frac{a_f(m)}{m^{k-s}} \\ & = \zeta(k+1-s-w) \frac{\Gamma(k-s)}{(2\pi)^{k-s}} \sum_{m=1}^{\infty} \frac{a_f(m) \sigma_{w-s}(m)}{m^{k-s}}. \end{aligned}$$

The series

$$L(f \otimes E(\cdot, v), k-s) := \sum_{m=1}^{\infty} \frac{a_f(m) \sigma_{w-s}(m)}{m^{k-s}}$$

is a convolution  $L$ -series involving the Fourier coefficients of  $f(z)$  and  $E(z, v)$  for  $2v = -s + w + 1$  (as in (2-16)) and, recalling [Zagier 1977, (72)] or [Diamantis and O’Sullivan 2010, (2.11)],

$$\zeta(k+1-s-w) \frac{\Gamma(k-s)}{(2\pi)^{k-s}} L(f \otimes E(\cdot, v), k-s) = \frac{(2\pi)^{k-w}}{\Gamma(k-w)} L^*(f, k-s) L^*(f, k-w). \tag{7-4}$$

Applying the functional equation (2-1) confirms that the right side of (7-4) equals the left side of (7-3).

Looking to simplify (7-2) leads to the natural question, what are the relations between the  $\mathcal{C}_k(z, s; p/q)$  for rational  $p/q$  in the interval  $[0, 1)$ ? For example, it is a simple exercise with (3-3) and (3-5) to show that

$$q^{-s} \mathcal{C}_k(z, s; p/q) = e^{-si\pi} q^{-k+s} \mathcal{C}_k(z, k-s; -p'/q)$$

for  $pp' \equiv 1 \pmod q$ . With  $s = k/2$  at the center of the critical strip, we get an even simpler relation:

$$\mathcal{C}_k(z, k/2; p/q) = (-1)^{k/2} \mathcal{C}_k(z, k/2; -p'/q). \tag{7-5}$$

A more interesting, but speculative, possibility would be to argue in the reverse direction in order to derive information about  $L$ -functions twisted by exponentials with *nonrational* exponents. Specifically, if we established, by other means, relations between the  $\mathcal{C}_k(z, s; x)$  for  $x \notin \mathbb{Q}$ , then (7-2) and other results proven here might lead to relations for  $L$ -functions twisted by exponentials with nonrational exponents. That would be important because such  $L$ -functions play a prominent role in Kaczorowski and Perelli’s program of classifying the Selberg class (see, e.g., [Kaczorowski and Perelli 1999]). Relations between these  $L$ -functions seem to be necessary for the extension of Kaczorowski and Perelli’s classification to degree 2, to which  $L$ -functions of  $GL(2)$  cusp forms belong.

### 8. Periods of cusp forms

**8A. Values of  $L$ -functions inside the critical strip.** We first review Zagier’s proof in [1977, §5] of Manin’s periods theorem. This exhibits a general principle of proving algebraicity we will be using in the next sections.

For all  $s, w \in \mathbb{C}$ , it is convenient to define  $H_{s,w} \in S_k$  by the conditions

$$\langle H_{s,w}, f \rangle = L^*(f, s)L^*(f, w) \quad \text{for all } f \in \mathcal{B}_k.$$

We need the following result:

**Lemma 8.1.** *For  $g \in S_k$  with Fourier coefficients in the field  $K_g$  and  $f \in \mathcal{B}_k$  with coefficients in  $K_f$ ,*

$$\langle g, f \rangle / \langle f, f \rangle \in K_g K_f.$$

*Proof.* See the general result of Shimura [1976, Lemma 4]. It is also a simple extension of [Diamantis and O’Sullivan 2010, Lemma 4.3]. □

Let  $K_{\text{critical}}$  be the field obtained by adjoining to  $\mathbb{Q}$  all the Fourier coefficients of

$$\{H_{s,k-1}, H_{k-2,w} \mid 1 \leq s, w \leq k-1, s \text{ even}, w \text{ odd}\}.$$

Thus, with  $f \in \mathcal{B}_k$  and employing Lemma 8.1,

$$L^*(f, k-1)L^*(f, k-2) = \langle H_{k-1,k-2}, f \rangle = c_f \langle f, f \rangle \tag{8-1}$$

for  $c_f \in K_{\text{critical}}K_f$ , and the left side of (8-1) is nonzero because the Euler product for  $L^*(f, s)$  converges for  $\text{Re}(s) > k/2 + 1/2$ . Set

$$\omega_+(f) := \frac{c_f \langle f, f \rangle}{L^*(f, k-1)} \quad \text{and} \quad \omega_-(f) := \frac{\langle f, f \rangle}{L^*(f, k-2)}. \tag{8-2}$$

Then  $\omega_+(f)\omega_-(f) = \langle f, f \rangle$ , and we have:

**Lemma 8.2.** *For each  $f \in \mathcal{B}_k$ ,*

$$L^*(f, s)/\omega_+(f) \quad \text{and} \quad L^*(f, w)/\omega_-(f) \in K_{\text{critical}}K_f$$

for all  $s$  and  $w$  with  $1 \leq s, w \leq k-1, s$  even and  $w$  odd.

*Proof.* For such  $s$  and  $w$ ,

$$\begin{aligned} \frac{L^*(f, s)}{\omega_+(f)} &= \frac{L^*(f, s)L^*(f, k-1)}{c_f \langle f, f \rangle} = \frac{\langle H_{s,k-1}, f \rangle}{c_f \langle f, f \rangle} = \frac{c'_f \langle f, f \rangle}{c_f \langle f, f \rangle} \in K_{\text{critical}}K_f, \\ \frac{L^*(f, w)}{\omega_-(f)} &= \frac{L^*(f, w)L^*(f, k-2)}{c_f \langle f, f \rangle} = \frac{\langle H_{k-2,w}, f \rangle}{c_f \langle f, f \rangle} = \frac{c''_f \langle f, f \rangle}{c_f \langle f, f \rangle} \in K_{\text{critical}}K_f. \quad \square \end{aligned}$$

To deduce Manin’s theorem from [Lemma 8.2](#), we use Zagier’s explicit expression for  $H_{s,w}$ . For  $n \geq 0$ , even  $k_1, k_2 \geq 4$  and  $k = k_1 + k_2 + 2n$ , [\(1-2\)](#) implies

$$(-1)^{k_1/2} 2^{3-k} \frac{k_1 k_2}{B_{k_1} B_{k_2}} \binom{k-2}{n} H_{n+1, n+k_2} = \frac{[E_{k_1}, E_{k_2}]_n}{(2\pi i)^n}. \tag{8-3}$$

The Fourier coefficients of  $E_{k_1}$  and  $E_{k_2}$  are rational, and hence, the right side of [\(8-3\)](#) has rational coefficients. Then  $H_{n+1, n+k_2}$  has Fourier coefficients in  $\mathbb{Q}$  (and also for  $k_1, k_2 = 2$  [[Kohnen and Zagier 1984](#), p. 214]). It follows that  $K_{\text{critical}} = \mathbb{Q}$  and [Lemma 8.2](#) becomes [Theorem 2.6](#), Manin’s periods theorem.

**8B. Arbitrary  $L$ -values.** With the results of the last section, we may now give the proof of [Theorem 2.7](#), restated here:

**Theorem 8.3.** *For all  $f \in \mathcal{B}_k$  and  $s \in \mathbb{C}$ , with  $\omega_+(f)$  and  $\omega_-(f)$  as in Manin’s theorem,*

$$\begin{aligned} L^*(f, s)/\omega_+(f) &\in K(E_{s, k-s}^*(\cdot, k-1))K_f, \\ L^*(f, s)/\omega_-(f) &\in K(E_{k-2, 2}^*(\cdot, s))K_f. \end{aligned}$$

*Proof.* By [Theorem 2.3](#), we have  $H_{s,w}(z) = E_{s, k-s}^*(z, w)$  for all  $s, w \in \mathbb{C}$ . Thus, arguing as in [Lemma 8.2](#) with  $E_{s, k-s}^*(\cdot, k-1) = H_{s, k-1}$  and  $E_{k-2, 2}^*(\cdot, s) = H_{k-2, s}$  yields the theorem.  $\square$

We indicate briefly how the double Eisenstein series Fourier coefficients required to define  $K(E_{s, k-s}^*(\cdot, k-1))$  and  $K(E_{k-2, 2}^*(\cdot, s))$  in [Theorem 2.7](#) may be calculated when  $s \in \mathbb{Z}$ , using a slight extension of the methods in [[Diamantis and O’Sullivan 2010](#), §3]. We wish to find the  $l$ -th Fourier coefficient,  $a_{s,w}(l)$ , of  $H_{s,w}(z) = E_{s, k-s}^*(z, w)$  for  $s$  even and  $w$  odd (and we assume  $s, w \geq k/2 > 1$ ). With [Proposition 2.5](#), this is  $(-1)^{k_2/2}/(2\pi^{k/2})$  times the  $l$ -th Fourier coefficient of

$$\pi_{\text{hol}}[y^{-k/2} E_{k_1}^*(z, u) E_{k_2}^*(z, v)]$$

for  $u = (s + w - k + 1)/2$  and  $v = (-s + w + 1)/2$  both in  $\mathbb{Z}$ . Let

$$\begin{aligned} F(z) := &y^{-k/2} E_{k_1}^*(z, u) E_{k_2}^*(z, v) - \frac{\theta_{k_1}(u)\theta_{k_2}(1-v)}{\theta_k(s+1-k/2)} y^{-k/2} E_k^*(z, s+1-k/2) \\ &- \frac{\theta_{k_1}(u)\theta_{k_2}(v)}{\theta_k(w+1-k/2)} y^{-k/2} E_k^*(z, w+1-k/2). \end{aligned}$$

Then  $\pi_{\text{hol}}(y^{-k/2} E_{k_1}^*(z, u) E_{k_2}^*(z, v)) = \pi_{\text{hol}}(F(z))$  because  $\pi_{\text{hol}}(y^{-k/2} E_k^*(z, s)) = 0$  for every  $s$ . We have constructed  $F$  so that  $F(z) \ll y^{-\varepsilon}$  as  $y \rightarrow \infty$ , and we may use [[Diamantis and O’Sullivan 2010](#), Lemma 3.3] to obtain

$$a_{s,w}(l) = \frac{(-1)^{k_2/2} (4\pi l)^{k-1}}{(2\pi^{k/2}) (k-2)!} \int_0^\infty F_l(y) e^{-2\pi l y} y^{k-2} dy$$

on writing  $F(z) = \sum_{l \in \mathbb{Z}} e^{2\pi i l x} y^{-k/2} F_l(y)$ . The functions  $F_l(y)$  are sums involving the Fourier coefficients of  $E_{k_1}^*(z, u)$  and  $E_{k_2}^*(z, v)$  with  $u, v \in \mathbb{Z}$ . As shown in [Diamantis and O’Sullivan 2010, Theorem 3.1], these coefficients are simply expressed in terms of divisor functions, Bernoulli numbers and a combinatorial part. For  $s$  and  $w$  in the critical strip, this calculation yields an explicit finite formula for  $a_{s,w}(l)$  in [Diamantis and O’Sullivan 2010, Theorem 1.3] (and another proof that  $H_{s,w}$  in (8-3) has rational Fourier coefficients and that  $K_{\text{critical}} = \mathbb{Q}$ ). For  $s$  and  $w$  outside the critical strip, we obtain infinite series representations for  $a_{s,w}(l)$  but again involving nothing more complicated than divisor functions and Bernoulli numbers. Further details of this computation will appear in [O’Sullivan 2013].

**8C. Twisted periods.** There is an analog of Manin’s periods theorem for twisted  $L$ -functions. Let  $p/q \in \mathbb{Q}$ , and let  $u$  be an integer with  $1 \leq u \leq k - 1$ . Manin shows in [1973, (13)] (see also [Lang 1976, Chapter 5]) that  $i^u \int_0^{p/q} f(iy)y^{u-1} dy$  is an integral linear combination of periods  $i^v \int_0^\infty f(iy)y^{v-1} dy$  for  $v = 1, \dots, k - 1$ . With (2-17), this proves

$$i^u q^{k-2} L^*(f, u; p/q) \in \mathbb{Z} \cdot i L^*(f, 1) + \mathbb{Z} \cdot i^2 L^*(f, 2) + \dots + \mathbb{Z} \cdot i^{k-1} L^*(f, k - 1).$$

Therefore, Theorem 2.6 implies the next result.

**Proposition 8.4.** *For all  $f \in \mathfrak{B}_k$ ,  $p/q \in \mathbb{Q}$  and integers  $u$  with  $1 \leq u \leq k - 1$ ,*

$$L^*(f, u; p/q) \in K_f(i)\omega_+(f) + K_f(i)\omega_-(f).$$

Employing (5-8), a similar proof to that of Theorem 2.7 in the last section shows the following:

**Proposition 8.5.** *For all  $f \in \mathfrak{B}_k$ ,  $p/q \in \mathbb{Q}$  and  $s \in \mathbb{C}$  with  $\omega_+(f)$  and  $\omega_-(f)$  as in Manin’s theorem,*

$$L^*(f, s; p/q)/\omega_+(f) \in K(E_{k-s,s}^*(\cdot, 1; p/q))K_f,$$

$$L^*(f, s; p/q)/\omega_-(f) \in K(E_{k-s,s}^*(\cdot, 2; p/q))K_f.$$

### 9. The nonholomorphic case

**9A. Background results and notation.** We will need a nonholomorphic analog of the Cohen kernel  $\mathcal{C}_k(z, s)$ .

**Definition 9.1.** With  $z \in \mathbb{H}$  and  $s, s' \in \mathbb{C}$ , define the nonholomorphic kernel  $\mathcal{H}$  as

$$\mathcal{H}(z; s, s') := \frac{1}{2} \sum_{\gamma \in \Gamma} \frac{\text{Im}(\gamma z)^{s+s'}}{|\gamma z|^{2s}}. \tag{9-1}$$

Following directly from the results in [Diamantis and O’Sullivan 2010, §5.2], it is absolutely convergent, uniformly on compacta, for  $z \in \mathbb{H}$  and  $\text{Re}(s), \text{Re}(s') > 1/2$ .

The kernel  $\mathcal{H}(z; s, s')$  was introduced by Diaconu and Goldfeld [2007, (2.1)] (though they describe it there as a Poincaré series and their kernel is a product of  $\Gamma$  factors). Starting with the identity [Diaconu and Goldfeld 2007, Proposition 3.5]

$$\langle f \cdot \mathcal{H}(\cdot; s, s'), g \rangle = \frac{\Gamma(s + s' + k - 1)}{2^{s+s'+k-1}} \int_{-\infty}^{\infty} \frac{L^*(f, \alpha + i\beta)L^*(g, -s + s' + k - \alpha - i\beta)}{\Gamma(s + \alpha + i\beta)\Gamma(-s + s' + k - \alpha - i\beta)} d\beta$$

for  $f$  and  $g$  in  $\mathcal{B}_k$ , they provide a new method to establish estimates for the second moment of  $L^*(f, s)$  along the critical line  $\text{Re}(s) = k/2$ . They give similar results for  $L^*(u_j, s)$ , the  $L$ -function associated to a Maass form  $u_j$  as defined below.

The spectral decomposition of  $\mathcal{H}(z; s, s')$  and its meromorphic continuation in the  $s$  and  $s'$  variables is shown in [Diaconu and Goldfeld 2007, §5]. We do the same; our treatment is slightly different, and we include it in Section 9B for completeness.

For  $\Gamma = \text{SL}(2, \mathbb{Z})$ , the discrete spectrum of the Laplace operator  $\Delta = -4y^2\partial_z\partial_{\bar{z}}$  is given by  $u_0$ , the constant eigenfunction, and  $u_j$  for  $j \in \mathbb{Z}_{\geq 1}$  an orthogonal system of Maass cusp forms (see, e.g., [Iwaniec 2002, Chapters 4 and 7]) with Fourier expansions

$$u_j(z) = \sum_{n \neq 0} |n|^{-1/2} v_j(n) W_{s_j}(nz),$$

where  $u_j$  has eigenvalue  $s_j(1 - s_j)$  and by Weyl's law [Iwaniec 2002, (11.5)]

$$\#\{j \mid |\text{Im}(s_j)| \leq T\} = T^2/12 + O(T \log T). \tag{9-2}$$

We may assume the  $u_j$  are Hecke eigenforms normalized to have  $v_j(1) = 1$ . Necessarily we have  $v_j(n) \in \mathbb{R}$ . Let  $\iota$  be the antiholomorphic involution  $(u_j)(z) := u_j(-\bar{z})$ . We may also assume each  $u_j$  is an eigenfunction of this operator, necessarily with eigenvalues  $\pm 1$ . If  $u_j = u_j$ , then  $v_j(n) = v_j(-n)$  and  $u_j$  is called *even*. If  $u_j = -u_j$ , then  $v_j(n) = -v_j(-n)$  and  $u_j$  is *odd*.

The  $L$ -function associated to the Maass cusp form  $u_j$  is

$$L(u_j, s) = \sum_{n=1}^{\infty} v_j(n)/n^s,$$

convergent for  $\text{Re}(s) > 3/2$  since  $v_j(n) \ll n^{1/2}$  by [Iwaniec 2002, (8.8)]. The completed  $L$ -function for an even form  $u_j$  is

$$L^*(u_j, s) := \pi^{-s} \Gamma\left(\frac{s + s_j - 1/2}{2}\right) \Gamma\left(\frac{s - s_j + 1/2}{2}\right) L(u_j, s), \tag{9-3}$$

and it satisfies

$$L^*(u_j, 1 - s) = L^*(u_j, s) = \overline{L^*(u_j, \bar{s})}. \tag{9-4}$$

See [Bump 1997, p. 107] for (9-3), (9-4) and the analogous odd case.

To  $E(z, s)$  (recall (2-3)) we associate the  $L$ -function

$$L(E(\cdot, s), w) := \sum_{m=1}^{\infty} \frac{\phi(m, s)}{m^w}.$$

The well-known identity  $\sum_{m=1}^{\infty} \sigma_x(m)/m^w = \zeta(w)\zeta(w-x)$  implies

$$L(E(\cdot, s), w) = \frac{2\pi^s}{\Gamma(s)} \frac{\zeta(w+s-1/2)\zeta(w-s+1/2)}{\zeta(2s)}. \tag{9-5}$$

**9B. The nonholomorphic kernel  $\mathcal{H}$ .** Throughout this section, we use  $s = \sigma + it$  and  $s' = \sigma' + it'$ . Recall  $\mathcal{H}(z; s, s')$  defined in (9-1) for  $\text{Re}(s), \text{Re}(s') > 1/2$ . Our goal is to find the spectral decomposition of  $\mathcal{H}(z; s, s')$  and prove its meromorphic continuation in  $s$  and  $s'$ . See [Diaconu and Goldfeld 2007, §5] and also [Iwaniec 2002, §7.4] for a similar decomposition and continuation of the automorphic Green function.

A routine verification (using [Jorgenson and O’Sullivan 2005, Lemma 9.2], for example) yields

$$\Delta \mathcal{H}(z; s, s') = (s + s')(1 - s - s')\mathcal{H}(z; s, s') + 4s s' \mathcal{H}(z; s + 1, s' + 1). \tag{9-6}$$

Put

$$\xi_{\mathbb{Z}}(z, s) := \sum_{m \in \mathbb{Z}} \frac{1}{|z + m|^{2s}}.$$

Then

$$\mathcal{H}(z; s, s') = \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \text{Im}(\gamma z)^{s+s'} \xi_{\mathbb{Z}}(\gamma z, s). \tag{9-7}$$

Use the Poisson summation formula as in [Iwaniec 2002, §3.4] or [Goldfeld 2006, Theorem 3.1.8] to see that

$$\xi_{\mathbb{Z}}(z, s) = \frac{\pi^{1/2} \Gamma(s-1/2)}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{1/2-s} \sum_{m \neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x} \tag{9-8}$$

for  $\text{Re}(s) > 1/2$ . Set

$$\xi_{\mathbb{Z}}^{\sharp}(z, s) := \sum_{m \neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}. \tag{9-9}$$

Let  $B_{\rho} := \{z \in \mathbb{C} \mid |z| \leq \rho\}$ . Then with [Jorgenson and O’Sullivan 2008, Lemma 6.4],

$$\sqrt{y} K_{s-1/2}(2\pi y) \ll e^{-2\pi y} (y^{\rho+3} + y^{-\rho-3})$$

for all  $s \in B_{\rho}$  and  $\rho, y > 0$  with the implied constant depending only on  $\rho$ . Hence,

$$\xi_{\mathbb{Z}}^{\sharp}(z, s) \ll \sum_{m=1}^{\infty} e^{-2\pi m y} (m^{\rho+\sigma+2} y^{\rho+5/2} + m^{-\rho+\sigma-4} y^{-\rho-7/2}).$$

We also have [Jorgenson and O’Sullivan 2008, Lemma 6.2]

$$\sum_{m=1}^{\infty} m^{\rho} e^{-2m\pi y} \ll e^{-2\pi y} (1 + y^{-\rho-1})$$

for all  $y > 0$  with the implied constant depending only on  $\rho \geq 0$ . Therefore,

$$\xi_{\mathbb{Z}}^{\sharp}(z, s) \ll e^{-2\pi y} (y^{\rho+5/2} + y^{-\rho-9/2}). \tag{9-10}$$

Consider the weight-0 series

$$\mathcal{H}^{\sharp}(z; s, s') := \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} \text{Im}(\gamma z)^{s'+1/2} \xi_{\mathbb{Z}}^{\sharp}(\gamma z, s). \tag{9-11}$$

With (9-10), we have

$$\mathcal{H}^{\sharp}(z; s, s') \ll \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} (\text{Im}(\gamma z)^{\sigma'+\rho+3} + \text{Im}(\gamma z)^{\sigma'-\rho-4}) e^{-2\pi \text{Im}(\gamma z)} \tag{9-12}$$

so that  $\mathcal{H}^{\sharp}(z; s, s')$  is absolutely convergent for  $\text{Re}(s') > \rho + 5$ .

**Proposition 9.2.** *Let  $\rho > 0$  and  $s, s' \in \mathbb{C}$  satisfy  $\text{Re}(s) > 1/2$ ,  $\text{Re}(s') > \rho + 5$  and  $s \in B_{\rho}$ . Then*

$$\mathcal{H}(z; s, s') = \frac{\pi^{1/2} \Gamma(s - 1/2)}{\Gamma(s)} E(z, s' - s + 1) + \frac{2\pi^s}{\Gamma(s)} \mathcal{H}^{\sharp}(z; s, s'), \tag{9-13}$$

and, for an implied constant depending only on  $s$  and  $s'$ ,

$$\mathcal{H}^{\sharp}(z; s, s') \ll y^{5+\rho-\sigma'} \quad \text{as } y \rightarrow \infty. \tag{9-14}$$

*Proof.* It is clear that (9-13) follows from (9-7), (9-8), (9-9) and (9-11) when  $s$  and  $s'$  are in the stated range. With (9-12) and employing (4-3), we deduce that as  $y \rightarrow \infty$ ,

$$\begin{aligned} \mathcal{H}^{\sharp}(z; s, s') &\ll (y^{\sigma'+\rho+3} + y^{\sigma'-\rho-4}) e^{-2\pi y} \\ &\quad + \sum_{\substack{\gamma \in \Gamma_{\infty} \setminus \Gamma \\ \gamma \neq \Gamma_{\infty}}} (\text{Im}(\gamma z)^{\sigma'+\rho+3} + \text{Im}(\gamma z)^{\sigma'-\rho-4}) \\ &\ll e^{-\pi y} + y^{1-(\sigma'+\rho+3)} + y^{1-(\sigma'-\rho-4)} \\ &\ll y^{5+\rho-\sigma'}. \quad \square \end{aligned}$$

Clearly, for  $\text{Re}(s') > \rho + 5$ , (9-13) gives the meromorphic continuation of  $\mathcal{H}(z; s, s')$  to all  $s \in B_{\rho}$ . For these  $s$  and  $s'$ , it follows from (9-14) that  $\mathcal{H}^{\sharp}$ , as a function of  $z$ , is bounded. Also use (9-6) and (9-13) to show that

$$\Delta \mathcal{H}^{\sharp}(z; s, s') = (s + s')(1 - s - s') \mathcal{H}^{\sharp}(z; s, s') + 4\pi^s \mathcal{H}^{\sharp}(z; s + 1, s' + 1),$$

and hence,  $\Delta\mathcal{H}^\sharp$  is also bounded. Therefore, with [Iwaniec 2002, Theorems 4.7 and 7.3],  $\mathcal{H}^\sharp$  has the spectral decomposition

$$\mathcal{H}^\sharp(z; s, s') = \sum_{j=0}^{\infty} \frac{\langle \mathcal{H}^\sharp(\cdot; s, s'), u_j \rangle}{\langle u_j, u_j \rangle} u_j(z) + \frac{1}{4\pi i} \int_{(1/2)} \langle \mathcal{H}^\sharp(\cdot; s, s'), E(\cdot, r) \rangle E(z, r) dr, \tag{9-15}$$

where the integral is from  $1/2 - i\infty$  to  $1/2 + i\infty$  and the convergence of (9-15) is pointwise absolute in  $z$  and uniform on compacta.

**Lemma 9.3.** *For  $s \in B_\rho$  and  $\text{Re}(s') > \rho + 5$ , we have*

$$\langle \mathcal{H}^\sharp(\cdot; s, s'), u_j \rangle = \frac{\pi^{1/2-s}}{4\Gamma(s')} L^*(u_j, s' - s + 1/2) \Gamma\left(\frac{s' + s + s_j - 1}{2}\right) \Gamma\left(\frac{s' + s - s_j}{2}\right)$$

when  $u_j$  is an even Maass cusp form. If  $u_j$  is odd or constant, then the inner product is zero.

*Proof.* Unfolding,

$$\begin{aligned} \langle \mathcal{H}^\sharp(\cdot; s, s'), u_j \rangle &= \int_{\Gamma \backslash \mathbb{H}} \mathcal{H}^\sharp(z; s, s') \overline{u_j(z)} d\mu(z) \\ &= \int_0^\infty \int_0^1 \left( \sum_{m \neq 0} y^{s'+1/2} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi imx} \right) \overline{u_j(z)} \frac{dx dy}{y^2} \\ &= 2 \sum_{m \neq 0} \nu_j(m) |m|^{s-1/2} \int_0^\infty y^{s'} K_{s-1/2}(2\pi|m|y) K_{\bar{s}_j-1/2}(2\pi|m|y) \frac{dy}{y}. \end{aligned}$$

Evaluating the integral [Iwaniec 2002, p. 205] yields

$$\langle \mathcal{H}^\sharp(\cdot; s, s'), u_j \rangle = \frac{L(u_j, s' - s + 1/2)}{4\pi^{s'} \Gamma(s')} \prod \Gamma\left(\frac{s' \pm (s - 1/2) \pm (\bar{s}_j - 1/2)}{2}\right).$$

Using (9-3) and that  $\bar{s}_j = 1 - s_j$  finishes the proof. □

In the same way, when  $\text{Re}(r) = 1/2$ ,

$$\begin{aligned} \langle \mathcal{H}^\sharp(\cdot; s, s'), E(\cdot, r) \rangle &= \frac{L(\overline{E(\cdot, r)}, s' - s + 1/2)}{4\pi^{s'} \Gamma(s')} \prod \Gamma\left(\frac{s' \pm (s - 1/2) \pm (\bar{r} - 1/2)}{2}\right). \end{aligned}$$

Further,  $\overline{E(z, r)} = E(z, \bar{r}) = E(z, 1 - r)$ , and with (9-5) we have shown the following:

**Lemma 9.4.** For  $s \in B_\rho$  and  $\text{Re}(s') > \rho + 5$ ,

$$\langle \mathcal{H}^\sharp(\cdot; s, s'), E(\cdot, r) \rangle = \frac{\pi^{1/2-s}}{2\Gamma(s')\theta(1-r)} \Gamma\left(\frac{s'+s-r}{2}\right) \times \Gamma\left(\frac{s'+s-1+r}{2}\right) \theta\left(\frac{s'-s+r}{2}\right) \theta\left(\frac{s'-s+1-r}{2}\right).$$

Recall that  $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$  as in (2-5). Let

$$\begin{aligned} \mathcal{K}_1(z; s, s') &:= \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} E(z, s'-s+1), \\ \mathcal{K}_2(z; s, s') &:= \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \sum_{\substack{j=1 \\ u_j \text{ even}}}^\infty L^*(u_j, s'-s+1/2) \Gamma\left(\frac{s'+s+s_j-1}{2}\right) \\ &\quad \times \Gamma\left(\frac{s'+s-s_j}{2}\right) \frac{u_j(z)}{\langle u_j, u_j \rangle}, \\ \mathcal{K}_3(z; s, s') &:= \frac{\pi^{1/2}}{\Gamma(s)\Gamma(s')} \frac{1}{4\pi i} \int_{(1/2)} \Gamma\left(\frac{s'+s-r}{2}\right) \Gamma\left(\frac{s'+s-1+r}{2}\right) \\ &\quad \times \theta\left(\frac{s'-s+r}{2}\right) \theta\left(\frac{s'-s+1-r}{2}\right) \frac{E(z, r)}{\theta(1-r)} dr. \end{aligned}$$

Assembling Proposition 9.2, (9-15) and Lemmas 9.3 and 9.4, we have proven the decomposition

$$\mathcal{K}(z; s, s') = \mathcal{K}_1(z; s, s') + \mathcal{K}_2(z; s, s') + \mathcal{K}_3(z; s, s') \tag{9-16}$$

for  $s \in B_\rho$  and  $\text{Re}(s') > \rho + 5$ . This agrees exactly with [Diaconu and Goldfeld 2007, (5.8)].

Clearly  $\mathcal{K}_1(z; s, s')$  is a meromorphic function of  $s$  and  $s'$  in all of  $\mathbb{C}$ . The same is true for  $\mathcal{K}_2(z; s, s')$  since the factors  $L(u_j, s'-s+1/2)u_j(z)/\langle u_j, u_j \rangle$  have at most polynomial growth as  $\text{Im}(s_j) \rightarrow \infty$  while the  $\Gamma$  factors have exponential decay by Stirling’s formula. See (9-2) and [Iwaniec 2002, §7 and §8] for the necessary bounds. The next result was first established in [Diaconu and Goldfeld 2007, §5].

**Theorem 9.5.** *The nonholomorphic kernel  $\mathcal{K}(z; s, s')$  has a meromorphic continuation to all  $s, s' \in \mathbb{C}$ .*

*Proof.* As we have discussed,  $\mathcal{K}_1(z; s, s')$  and  $\mathcal{K}_2(z; s, s')$  are meromorphic functions of  $s, s' \in \mathbb{C}$ . The poles of  $\Gamma(w)$  are at  $w = 0, -1, -2, \dots$ , and  $\theta(w)$  has poles exactly at  $w = 0, 1/2$  (with residues  $-1/2$  and  $1/2$ , respectively). Therefore, the integral in  $\mathcal{K}_3(z; s, s')$  is certainly an analytic function of  $s$  and  $s'$  for  $\sigma' > \sigma + 1/2$  and  $\sigma > 1/2$  since the  $\Gamma$  and  $\theta$  factors have exponential decay as  $|r| \rightarrow \infty$ . Next, consider  $s$  fixed (with  $\sigma > 1/2$ ) and  $s'$  varying. Consider a point  $r_0$  with  $\text{Re}(r_0) = 1/2$ .

Let  $B(r_0)$  be a small disc centered at  $r_0$  and  $B(1 - r_0)$  an identical disc at  $1 - r_0$ . By deforming the path of integration to a new path  $C$  to the left of  $B(r_0)$  and to the right of  $B(1 - r_0)$ , we may, by Cauchy’s theorem, analytically continue  $\mathcal{H}_3(z; s, s')$  to  $s'$  with  $s' - s \in B(r_0)$ . Let  $C_1$  be a clockwise contour around the left side of  $B(r_0)$  and  $C_2$  be a counterclockwise contour around the right side of  $B(1 - r_0)$  so that  $C = (1/2) + C_1 + C_2$ . For  $s' - s$  inside  $C_1$  (and  $1 - (s' - s)$  inside  $C_2$ ), we have

$$\pi^{-1/2}\Gamma(s)\Gamma(s') \cdot \mathcal{H}_3(z; s, s') = \frac{1}{4\pi i} \int_C * = \frac{1}{4\pi i} \int_{(1/2)} * + \frac{1}{4\pi i} \int_{C_1} * + \frac{1}{4\pi i} \int_{C_2} *,$$

where  $*$  denotes the integrand in the definition of  $\mathcal{H}_3$ . Then

$$\begin{aligned} \frac{1}{4\pi i} \int_{C_1} * &= \frac{-2\pi i}{4\pi i} \left( \operatorname{Res}_{r=s'-s} \theta \left( \frac{s'-s+1-r}{2} \right) \right) \\ &\quad \times \Gamma(s)\Gamma(s'-1/2) \frac{\theta(s'-s)}{\theta(1-s'+s)} E(z, s'-s) \\ &= \frac{1}{2} \Gamma(s)\Gamma(s'-1/2) \frac{\theta(s'-s)}{\theta(1-s'+s)} E(z, s'-s) \\ &= \frac{1}{2} \Gamma(s)\Gamma(s'-1/2) E(z, s-s'+1). \end{aligned}$$

We get the same result for  $(1/4\pi i) \int_{C_2}$ , and for all  $s'$  with  $\sigma - 1/2 < \operatorname{Re}(s') < \sigma + 1/2$ , it follows that the continuation of  $\mathcal{H}_3(z; s, s')$  is given by

$$\begin{aligned} \pi^{-1/2}\Gamma(s)\Gamma(s') \cdot \mathcal{H}_3(z; s, s') &= \Gamma(s)\Gamma(s'-1/2) E(z, s-s'+1) + \frac{1}{4\pi i} \int_{(1/2)} *. \quad (9-17) \end{aligned}$$

Similarly, as  $s'$  crosses the line with real part  $\sigma - 1/2$ , the term

$$-\Gamma(s-1/2)\Gamma(s') E(z, s'-s+1)$$

must be added to the right side of (9-17). Thus, for all  $s'$  with  $1/2 < \operatorname{Re}(s') < \sigma - 1/2$ , the continuation of  $\mathcal{H}(z; s, s')$  is

$$\mathcal{H}(z; s, s') = \frac{\pi^{1/2}\Gamma(s'-1/2)}{\Gamma(s')} E(z, s-s'+1) + \mathcal{H}_2(z; s, s') + \mathcal{H}_3(z; s, s'). \quad (9-18)$$

Clearly, with (9-17) and (9-18) we have demonstrated the meromorphic continuation of  $\mathcal{H}(z; s, s')$  to all  $s, s' \in \mathbb{C}$  with  $\operatorname{Re}(s), \operatorname{Re}(s') > 1/2$ . The continuation to all  $s, s' \in \mathbb{C}$  follows in the same way with further terms in the expression for  $\mathcal{H}(z; s, s')$  appearing from the residues of the poles of  $\Gamma((s'+s-r)/2)\Gamma((s'+s-1+r)/2)$  as  $\operatorname{Re}(s'+s) \rightarrow -\infty$ .  $\square$

**Proposition 9.6.** *We have the functional equation*

$$\mathcal{H}(z; s, s') = \mathcal{H}(z; s', s). \quad (9-19)$$

*Proof.* We may verify (9-19) by comparing (9-16) with (9-18) and using that  $\mathfrak{K}_2(z; s, s') = \mathfrak{K}_2(z; s', s)$  by (9-4) and  $\mathfrak{K}_3(z; s, s') = \mathfrak{K}_3(z; s', s)$  by (2-6). There is a second, easier proof: with  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , replace  $\gamma$  in (9-1) by  $S\gamma$ .  $\square$

**Proposition 9.7.** *For all  $s, s' \in \mathbb{C}$  and any even Maass Hecke eigenform  $u_j$ ,*

$$\langle \mathfrak{K}(\cdot; s, s'), u_j \rangle = \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \Gamma\left(\frac{s'+s+s_j-1}{2}\right) \Gamma\left(\frac{s'+s-s_j}{2}\right) L^*(u_j, s'-s+\frac{1}{2}).$$

*Proof.* Since each  $u_j$  is orthogonal to Eisenstein series, we have by (9-16) (for  $s \in B_\rho$  and  $\text{Re}(s') > \rho + 5$ ) that

$$\langle \mathfrak{K}(\cdot; s, s'), u_j \rangle = \langle \mathfrak{K}_2(\cdot; s, s'), u_j \rangle.$$

The result follows, extending to all  $s, s' \in \mathbb{C}$  by analytic continuation.  $\square$

**9C. Nonholomorphic double Eisenstein series.** A similar argument to the proof of (5-2) shows that, for  $\text{Re}(s), \text{Re}(s') > 1$  and  $\text{Re}(w) \geq 0$ ,

$$\zeta(w+2s)\zeta(w+2s')\mathfrak{E}(z, w; s, s') = \frac{1}{2} \sum_{n=1}^{\infty} \frac{T_n \mathfrak{K}(z; s, s')}{n^{w-1/2}}, \tag{9-20}$$

where, in this context [Goldfeld 2006, (3.12.3)], the appropriately normalized Hecke operator acts as

$$T_n \mathfrak{K}(z) = \frac{1}{n^{1/2}} \sum_{\gamma \in \Gamma \backslash \mathcal{M}_n} \mathfrak{K}(\gamma z).$$

For each Maass form, we have  $T_n u_j = v_j(n) u_j$ , and for the Eisenstein series, [Goldfeld 2006, Proposition 3.14.2] implies  $T_n E(z, s) = n^{s-1/2} \sigma_{1-2s}(n) E(z, s)$ . Therefore, as in (9-5),

$$\sum_{n=1}^{\infty} \frac{T_n E(z, s)}{n^{w-1/2}} = E(z, s) \sum_{n=1}^{\infty} \frac{\sigma_{1-2s}(n)}{n^{w-s}} = E(z, s) \zeta(w-s) \zeta(w+s-1).$$

Now choose any  $\rho > 0$ . For  $s \in B_\rho$ ,  $\text{Re}(s) > 1$ ,  $\text{Re}(s') > \rho + 5$  and  $\text{Re}(w) \geq 0$ , we may apply  $T_n$  to both sides of (9-16) and obtain

$$\begin{aligned} & \zeta(w+2s)\zeta(w+2s')\mathfrak{E}(z, w; s, s') \\ &= \frac{\pi^{1/2}\Gamma(s-1/2)}{2\Gamma(s)} \zeta(s'-s+w)\zeta(s-s'+w-1)E(z, s'-s+1) \\ &+ \frac{\pi^{1/2}}{4\Gamma(s)\Gamma(s')} \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s'-s+1/2) \Gamma\left(\frac{s'+s+s_j-1}{2}\right) \Gamma\left(\frac{s'+s-s_j}{2}\right) \\ &\times L(u_j, w-1/2) \frac{u_j(z)}{\langle u_j, u_j \rangle} + \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \frac{1}{4\pi i} \int_{(1/2)} \theta\left(\frac{s'-s+r}{2}\right) \theta\left(\frac{s'-s+1-r}{2}\right) \\ &\times \Gamma\left(\frac{s'+s-r}{2}\right) \Gamma\left(\frac{s'+s-1+r}{2}\right) \zeta(w-r)\zeta(w-1+r) \frac{E(z, r)}{\theta(1-r)} dr. \tag{9-21} \end{aligned}$$

Put

$$\Omega(s, s'; r) := \theta\left(\frac{s' + s - r}{2}\right)\theta\left(\frac{s' + s - 1 + r}{2}\right) \times \theta\left(\frac{s' - s + r}{2}\right)\theta\left(\frac{s' - s + 1 - r}{2}\right) / \theta(1 - r).$$

Define the completed double Eisenstein series as in (2-19) and write

$$U(z; s, s') := \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s + s' - 1/2)L^*(u_j, s' - s + 1/2)\frac{u_j(z)}{\langle u_j, u_j \rangle}.$$

As in the last section,  $\Omega$  and  $U$  have exponential decay as  $|r|, |\text{Im}(s_j)| \rightarrow \infty$ . Specializing (9-21) to  $w = s + s'$ , we have proved the next result.

**Lemma 9.8.** *For  $s \in B_\rho$ ,  $\text{Re}(s) > 1$  and  $\text{Re}(s') > \rho + 5$ ,*

$$\begin{aligned} \mathcal{E}^*(z; s, s') &= 2\theta(s)\theta(s')E(z; s + s') + 2\theta(1 - s)\theta(s')E(z, s' - s + 1) \\ &\quad + U(z; s, s') + \frac{1}{2\pi i} \int_{(1/2)} \Omega(s, s'; r)E(z, r) dr. \end{aligned} \tag{9-22}$$

From this, we show the following:

**Theorem 9.9.** *The completed double Eisenstein series  $\mathcal{E}^*(z; s, s')$  has a meromorphic continuation to all  $s, s' \in \mathbb{C}$ , and we have the functional equations*

$$\mathcal{E}^*(z; s, s') = \mathcal{E}^*(z; s', s), \tag{9-23}$$

$$\mathcal{E}^*(z; s, s') = \mathcal{E}^*(z; 1 - s, 1 - s'). \tag{9-24}$$

*Proof.* First note that (9-22) gives the meromorphic continuation of  $\mathcal{E}^*(z; s, s')$  to all  $s$  and  $s'$  with  $s \in B_\rho$  and  $\text{Re}(s') > \rho + 5$ . As in the proof of Theorem 9.5, we see that the further continuation in  $s'$  is given by (9-22) along with residues that are picked up as the line of integration is crossed; for  $s \in B_\rho$  fixed and  $\text{Re}(s') \rightarrow -\infty$ , the continuation of  $\mathcal{E}^*(z; s, s')$  is given by (9-22) plus each of the following:

$$\begin{aligned} &2\theta(s)\theta(1 - s')E(z, s - s' + 1) \quad \text{when } \text{Re}(s') < \sigma + 1/2, \\ &-2\theta(1 - s)\theta(s')E(z, s' - s + 1) \quad \text{when } \text{Re}(s') < \sigma - 1/2, \\ &2\theta(1 - s)\theta(1 - s')E(z, 2 - s - s') \quad \text{when } \text{Re}(s') < -\sigma + 1/2, \\ &-2\theta(s)\theta(s')E(z, s + s') \quad \text{when } \text{Re}(s') < -\sigma - 1/2. \end{aligned}$$

We have therefore shown the meromorphic continuation of  $\mathcal{E}^*(z; s, s')$  to all  $s \in B_\rho$  and  $s' \in \mathbb{C}$ . Hence, for all  $s'$  with  $\text{Re}(s') < -\rho - 4$ , say, we have

$$\begin{aligned} \mathcal{E}^*(z; s, s') &= 2\theta(1 - s)\theta(1 - s')E(z, 2 - s - s') + 2\theta(s)\theta(1 - s')E(z, s - s' + 1) \\ &\quad + U(z; s, s') + \frac{1}{2\pi i} \int_{(1/2)} \Omega(s, s'; r)E(z, r) dr. \end{aligned} \tag{9-25}$$

The functional Equation (9-24) is a consequence of the easily checked symmetries  $U(z; 1-s, 1-s') = U(z; s, s')$  and  $\Omega(1-s, 1-s'; r) = \Omega(s, s'; r)$  and a comparison of (9-22) and (9-25). The Equation (9-23) has a similar proof or more simply follows from the definition (2-19).  $\square$

**Proposition 9.10.** *For any even Maass Hecke eigenform  $u_j$  (as in Section 9A) and all  $s, s' \in \mathbb{C}$ ,*

$$\langle \mathcal{E}^*(\cdot; s, s'), u_j \rangle = L^*(u_j, s + s' - 1/2)L^*(u_j, s' - s + 1/2).$$

*Proof.* As in Proposition 9.7, only  $U(z; s, s')$  in (9-22) will contribute to the inner product.  $\square$

With Theorem 9.9 and Proposition 9.10, we have proved Theorem 2.9.

### 10. Double Eisenstein series for general groups

We proved in Section 5A that for  $\Gamma = \text{SL}(2, \mathbb{Z})$  the holomorphic double Eisenstein series  $E_{s,k-s}(z, w)$  may be continued to all  $s$  and  $w$  in  $\mathbb{C}$  and satisfies a family of functional equations. That proof does not extend to groups where Hecke operators are not available. To show the continuation of  $E_{s,k-s,\alpha}(z, w)$  for  $\Gamma$  an arbitrary Fuchsian group of the first kind, we first demonstrate a generalization of Proposition 2.5. Recall the definitions of  $u$  and  $v$  in (2-16) and  $\varepsilon_\Gamma$  in (4-1).

**Theorem 10.1.** *For  $s$  and  $w$  in the initial domain of convergence and even  $k_1, k_2 \geq 0$  with  $k = k_1 + k_2$ , we have*

$$E_{s,k-s,\alpha}^*(z, w) = 2^{\varepsilon_\Gamma - 1} \pi_{\text{hol}} \left[ (-1)^{k_2/2} y^{-k/2} E_{k_1,\alpha}^*(\cdot, 1-u) E_{k_2,\alpha}^*(\cdot, 1-v) / (2\pi^{k/2}) \right]. \tag{10-1}$$

*Proof.* Let  $g \in S_k(\Gamma)$ , and set  $\Gamma' = \sigma_\alpha^{-1} \Gamma \sigma_\alpha$ . Then

$$\begin{aligned} \langle E_{s,k-s,\alpha}(\cdot, w), g \rangle &= \int_{\Gamma' \backslash \mathbb{H}} \text{Im}(\sigma_\alpha z)^k \bar{g}(\sigma_\alpha z) E_{s,k-s,\alpha}(\sigma_\alpha z, w) d\mu z \tag{10-2} \\ &= \int_{\Gamma' \backslash \mathbb{H}} y^k \frac{\bar{g}(\sigma_\alpha z)}{\bar{j}(\sigma_\alpha, z)^k} \sum_{\delta \in B \backslash \Gamma'} j(\delta, z)^{-k} \left[ \sum_{\substack{\gamma \in B \backslash \Gamma' \\ c_\gamma \delta^{-1} > 0}} (c_\gamma \delta^{-1})^{w-1} \left( \frac{j(\gamma, z)}{j(\delta, z)} \right)^{-s} \right] d\mu z. \end{aligned}$$

Since  $g(\sigma_\alpha z) j(\sigma_\alpha, z)^{-k} \in S_k(\Gamma')$ , we have

$$y^k \frac{\bar{g}(\sigma_\alpha z)}{\bar{j}(\sigma_\alpha, z)^k j(\delta, z)^k} = \text{Im}(\delta z)^k \frac{\bar{g}(\sigma_\alpha \delta z)}{\bar{j}(\sigma_\alpha, \delta z)^k}.$$

Note also that  $j(\gamma, z)/j(\delta, z) = j(\gamma \delta^{-1}, \delta z)$ . Hence, (10-2) equals

$$2^{\varepsilon_\Gamma} \int_{\Gamma_\infty \backslash \mathbb{H}} y^k \frac{\bar{g}(\sigma_\alpha z)}{\bar{j}(\sigma_\alpha, z)^k} \left[ \sum_{\substack{\gamma \in B \backslash \Gamma' \\ c_\gamma > 0}} (c_\gamma)^{w-1} j(\gamma, z)^{-s} \right] d\mu z. \tag{10-3}$$

Writing

$$\sum_{\substack{\gamma \in B \setminus \Gamma' \\ c_\gamma > 0}} (c_\gamma)^{w-1} j(\gamma, z)^{-s} = \sum_{\substack{\gamma \in B \setminus \Gamma' / B \\ c_\gamma > 0}} (c_\gamma)^{w-1} \sum_{m \in \mathbb{Z}} j(\gamma, z + m)^{-s}$$

and using the Fourier expansion of  $g$  at  $\mathfrak{a}$ ,  $j(\sigma_{\mathfrak{a}}, z)^{-k} g(\sigma_{\mathfrak{a}} z) = \sum_{n=1}^{\infty} a_{g,\mathfrak{a}}(n) e^{2\pi i n z}$ , we get that (10-3) equals

$$\begin{aligned} 2^{\varepsilon_\Gamma} \sum_{n=1}^{\infty} \overline{a_{g,\mathfrak{a}}}(n) \sum_{\substack{\gamma \in B \setminus \Gamma' / B \\ c_\gamma > 0}} \frac{1}{(c_\gamma)^{s+1-w}} \int_0^\infty \int_{-\infty}^\infty y^{k-2} \frac{e^{-2\pi i n x - 2\pi n y}}{(x + d_\gamma / c_\gamma + i y)^s} dx dy \\ = 2^{\varepsilon_\Gamma} I_k(s) \sum_{n=1}^{\infty} \frac{\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s}} \sum_{\substack{\gamma \in B \setminus \Gamma' / B \\ c_\gamma > 0}} \frac{e^{2\pi i n d_\gamma / c_\gamma}}{(c_\gamma)^{s+1-w}} \end{aligned}$$

for

$$I_k(s) := \int_0^\infty \int_{-\infty}^\infty y^{k-2} \frac{e^{-2\pi i x - 2\pi y}}{(x + i y)^s} dx dy.$$

The inner integral over  $x$  may be evaluated with a formula of Laplace [Whittaker and Watson 1927, p. 246]:

$$\int_{-\infty}^\infty \frac{e^{-2\pi i x}}{(x + i y)^s} dx = e^{-2\pi y} \frac{(2\pi)^s}{\Gamma(s) e^{s i \pi / 2}}$$

so that

$$I_k(s) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \frac{(2\pi)^s}{\Gamma(s) e^{s i \pi / 2}}.$$

With (4-2) and, for example, [Iwaniec 2002, Chapter 3], we recognize

$$\sum_{\substack{\gamma \in B \setminus \Gamma' / B \\ c_\gamma > 0}} \frac{e^{2\pi i n d_\gamma / c_\gamma}}{(c_\gamma)^{2s}} = \sum_{\substack{\gamma \in \Gamma_\infty \setminus \Gamma' / \Gamma_\infty \\ c_\gamma > 0}} \frac{e^{2\pi i n d_\gamma / c_\gamma}}{(c_\gamma)^{2s}} = \frac{Y_{\mathfrak{a}\mathfrak{a}}(n, s)}{\zeta(2s) n^{s-1}}.$$

It follows that we have shown

$$\langle \mathbf{E}_{s,k-s,\mathfrak{a}}^*(\cdot, w), g \rangle = 2^{\varepsilon_\Gamma - 1} \frac{\zeta(2-2u) \Gamma(k-s) \Gamma(k-w)}{(2\pi)^{2k-s-w}} \sum_{n=1}^{\infty} \frac{Y_{\mathfrak{a}\mathfrak{a}}(n, 1-v) \overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s-v}}.$$

Reasoning as in the proof of [Diamantis and O'Sullivan 2010, (2.10)], we also find, for all even  $k_1, k_2 \geq 0$  with  $k_1 + k_2 = k$ ,

$$\begin{aligned} \langle (-1)^{k_2/2} y^{-k/2} E_{k_1,\mathfrak{a}}^*(\cdot, 1-u) E_{k_2,\mathfrak{b}}^*(\cdot, 1-v) / (2\pi^{k/2}), g \rangle \\ = \frac{\zeta(2-2u) \Gamma(k-s) \Gamma(k-w)}{(2\pi)^{2k-s-w}} \sum_{n=1}^{\infty} \frac{Y_{\mathfrak{b}\mathfrak{a}}(n, 1-v) \overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s-v}}. \end{aligned}$$

Since  $\mathbf{E}_{s,k-s,\mathfrak{a}}^*(z, w) \in S_k(\Gamma)$  and  $g \in S_k(\Gamma)$  is arbitrary, (10-1) follows.  $\square$

**Corollary 10.2.** *The double Eisenstein series  $E_{s,k-s,\alpha}^*(z, w)$  has a meromorphic continuation to all  $s, w \in \mathbb{C}$  and as a function of  $z$  is always in  $S_k(\Gamma)$ . It satisfies the functional equation*

$$E_{k-s,s,\alpha}^*(z, w) = (-1)^{k/2} E_{s,k-s,\alpha}^*(z, w). \quad (10-4)$$

*Proof.* Since  $E_{k,\alpha}^*(z, s)$  has a well-known continuation to all  $s \in \mathbb{C}$ , due to Selberg, the continuation of  $E_{s,k-s,\alpha}^*(z, w)$  follows from (10-1). The change of variables  $(s, w) \rightarrow (k-s, w)$  corresponds to  $(u, v) \rightarrow (v, u)$ , and so (10-4) is also a consequence of (10-1).  $\square$

If  $\Gamma$  has more than one cusp, then  $E_{s,k-s,\alpha}^*(z, w)$  does not appear to possess a functional equation of the type (2-14) as  $(s, w) \rightarrow (w, s)$ . This corresponds on the right of (10-1) to  $(u, v) \rightarrow (u, 1-v)$ , and the functional equation for  $E_{k_2,\alpha}^*(\cdot, 1-v)$  involves a sum over cusps as in (4-4).

We remark that the functional Equation (10-4) also follows directly from (4-6) if  $-I \in \Gamma$ : replace  $\gamma$  and  $\delta$  in the sum by  $-\delta$  and  $\gamma$ , respectively.

Finally, it would be interesting to find the continuation in  $s$  and  $s'$  of the non-holomorphic double Eisenstein series  $\mathcal{E}_\alpha^*(z; s, s')$  for general groups. We expect that a similar decomposition to (9-22) should be true.

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