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The geometry and combinatorics of cographic toric face rings

Sebastian Casalaina-Martin, Jesse Leo Kass and Filippo Viviani

In this paper, we define and study a ring associated to a graph that we call the cographic toric face ring or simply the cographic ring. The cographic ring is the toric face ring defined by the following equivalent combinatorial structures of a graph: the cographic arrangement of hyperplanes, the Voronoi polytope, and the poset of totally cyclic orientations. We describe the properties of the cographic ring and, in particular, relate the invariants of the ring to the invariants of the corresponding graph.

Our study of the cographic ring fits into a body of work on describing rings constructed from graphs. Among the rings that can be constructed from a graph, cographic rings are particularly interesting because they appear in the study of compactified Jacobians of nodal curves.

Introduction

In this paper, we define and study a ring $R(\Gamma)$ associated to a graph Γ that we call the cographic toric face ring or simply the cographic ring. The cographic ring $R(\Gamma)$ is the toric face ring defined by the following equivalent combinatorial structures of Γ: the cographic arrangement of hyperplanes $\mathscr{C}_\Gamma^\perp$, the Voronoi polytope Vor_Γ, and the poset of totally cyclic orientations $\mathbb{O}\mathscr{P}_\Gamma$. We describe the properties of the cographic ring and, in particular, relate the invariants of the ring to the invariants of the corresponding graph.

Our study of the cographic ring fits into a body of work on describing rings constructed from graphs. Among the rings that can be constructed from a graph, cographic rings are particularly interesting because they appear in the study of compactified Jacobians.

The authors establish the connection between $R(\Gamma)$ and the local geometry of compactified Jacobians in [Casalaina-Martin et al. 2011]. The compactified Jacobian $\bar{J}_X^d$ of a nodal curve X is the coarse moduli space parametrizing sheaves

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on X that are rank-1, semistable, and of fixed degree d. These moduli spaces have been constructed by Oda and Seshadri [1979], Caporaso [1994], Simpson [1994], and Pandharipande [1996], and the different constructions are reviewed in Section 2 of [Casalaina-Martin et al. 2011]. In Theorem A of the same work, it is proved that the completed local ring of $\bar{J}_X^d$ at a point is isomorphic to a power series ring over the completion of $R(\Gamma)$ for a graph Γ constructed from the dual graph of X.

Also in [Casalaina-Martin et al. 2011], we studied the local structure of the universal compactified Jacobian, which is a family of varieties over the moduli space of stable curves whose fibers are closely related to the compactified Jacobians just discussed. (See Section 2 of [loc. cit.] for a discussion of the relation between the compactified Jacobians from the previous paragraph and the fibers of the universal Jacobian). Caporaso [1994] first constructed the universal compactified Jacobian, and Pandharipande [1996] gave an alternative construction. In [Casalaina-Martin et al. 2011, Theorem A] we gave a presentation of the completed local ring of the universal compactified Jacobian at a point, and we will explore the relation between that ring and the affine semigroup ring defined in Section 5A in the upcoming paper [Casalaina-Martin et al. 2012].

Cographic toric face rings are examples of toric face rings. Recall that a toric face ring is constructed from the same combinatorial data that is used to construct a toric variety: a fan. Let $H_{\mathbb{Z}}$ be a free, finite-rank $\mathbb{Z}$-module and $\mathscr{F}$ be a fan that decomposes $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ into (strongly convex rational polyhedral) cones. Consider the free k-vector space with basis given by monomials X^c indexed by elements $c \in H_{\mathbb{Z}}$. If we define a multiplication law on this vector space by setting

$$X^c \cdot X^{c'} = \begin{cases} X^{c+c'} & \text{if } c, c' \in \sigma \text{ for some } \sigma \in \mathscr{F}, \\ 0 & \text{otherwise} \end{cases}$$

and extending by linearity, then the resulting ring $R(\mathscr{F})$ is the toric face ring (over k) that is associated to $\mathscr{F}$.

We define the cographic toric face ring $R(\Gamma)$ of a graph Γ to be toric face ring associated to the fan that is defined by the cographic arrangement $\mathscr{C}_\Gamma^\perp$. The cographic arrangement is an arrangement of hyperplanes in the real vector space $H_{\mathbb{R}}$ associated to the homology group $H_{\mathbb{Z}} := H_1(\Gamma, \mathbb{Z})$ of the graph. Every edge of Γ naturally induces a functional on $H_{\mathbb{R}}$, and the zero locus of this functional is a hyperplane in $H_{\mathbb{R}}$, provided the functional is nonzero. The cographic arrangement is defined to be the collection of all hyperplanes constructed in this manner. The intersections of these hyperplanes define a fan $\mathscr{F}_\Gamma^\perp$, the cographic fan. The toric face ring associated to this fan is $R(\Gamma)$.

We study the fan $\mathscr{F}_\Gamma^\perp$ in Section 3. The main result of that section is Corollary 3.9, which provides two alternative descriptions of $\mathscr{F}_\Gamma^\perp$. First, using a theorem of Amini, we prove that $\mathscr{F}_\Gamma^\perp$ is equal to the normal fan of the Voronoi polytope Vor_Γ. As a consequence, we can conclude that $\mathscr{F}_\Gamma^\perp$, considered as a poset, is isomorphic to the poset of faces of Vor_Γ ordered by reverse inclusion. Using work of Greene and Zaslavsky, we show that this common poset is also isomorphic to the poset $\mathcal{OP}_\Gamma$ of totally cyclic orientations.

The combinatorial definition of $R(\Gamma)$ does not appear in [Casalaina-Martin et al. 2011]. Rather, the rings in that paper appear as invariants under a torus action. The following theorem, proven in Section 6 (Theorem 6.1), shows that the rings in [Casalaina-Martin et al. 2011] are (completed) cographic rings:

Theorem A. *Let Γ be a finite graph with vertices $V(\Gamma)$, oriented edges $\vec{E}(\Gamma)$, and source and target maps $s, t : \vec{E}(\Gamma) \to V(\Gamma)$. Let*

$$T_\Gamma := \prod_{v \in V(\Gamma)} \mathbb{G}_m \quad \textit{and} \quad A(\Gamma) := \frac{k[U_{\vec{e}}, U_{\bar{e}} : e \in E(\Gamma)]}{(U_{\vec{e}} U_{\bar{e}} : e \in E(\Gamma))}.$$

If we make T_Γ act on $A(\Gamma)$ by

$$\lambda \cdot U_{\vec{e}} = \lambda_{s(\vec{e})} U_{\vec{e}} \lambda_{t(\vec{e})}^{-1},$$

then the invariant subring $A(\Gamma)^{T_\Gamma}$ is isomorphic to the cographic ring $R(\Gamma)$.

The cographic ring $R(\Gamma)$ has reasonable geometric properties. Specifically, in Theorem 5.7, we prove that $R(\Gamma)$ is

- of pure dimension $b_1(\Gamma) = \dim_{\mathbb{R}} H_1(\Gamma, \mathbb{R})$,
- Gorenstein,
- seminormal, and
- semi log canonical.

We also compute invariants of $R(\Gamma)$ in terms of the combinatorics of Γ. The invariants we compute are

- a description of $R(\Gamma)$ in terms of oriented subgraphs (Section 5B),
- the number of minimal primes in terms of orientations (Theorem 5.7(i)),
- the embedded dimension of $R(\Gamma)$ in terms of circuits (Theorem 5.7(vi)), and
- the multiplicity of $R(\Gamma)$ (Theorem 5.7(vii)).

Finally, it is natural to ask what information is lost in passing from Γ to $R(\Gamma)$. An answer to this question is given by Theorem 7.1, which states that $R(\Gamma)$ determines Γ up to three-edge connectivization.

Combinatorially defined rings, such as the cographic toric face ring, have long been used in the study of compactified Jacobians and, more generally, degenerate abelian varieties (see, e.g., [Mumford 1972; Oda and Seshadri 1979; Faltings and Chai 1990; Namikawa 1980; Alexeev and Nakamura 1999; Alexeev 2004]).

In particular, the ring $R(\Gamma)$ we study here is a special case of the rings $R_0(c)$ studied by Alexeev and Nakamura [1999, Theorem 3.17]. There the rings appear naturally as a by-product of Mumford's technique for degenerating an abelian variety. Alexeev and Nakamura [1999, Lemma 4.1] proved that $R_0(c)$ satisfies the Gorenstein condition, and the seminormality was established by Alexeev [2002]. In personal correspondence, Alexeev informed the authors that the techniques of those papers can also be used to establish other results in this paper such as the fact that $R(\Gamma)$ is semi log canonical.

In a different direction, the cographic ring is defined by the cographic fan $\mathscr{F}_\Gamma^\perp$, which is the normal fan to the Voronoi polytope Vor_Γ. There is a body of work studying similar polytopes and the algebra-geometric objects defined by these polytopes. Altmann and Hille [1999] define the polytope of flows associated to an oriented graph (or quiver). Associated to this polytope is a toric variety that they relate to a moduli space. There are also a number of recent papers that study the modular/integral flow polytope in $H_1(\Gamma, \mathbb{R})$. This study is motivated by the work of Beck and Zaslavsky [2006] on interpreting graph polynomials in terms of lattice points. Some recent papers on this topic are [Beck and Zaslavsky 2006; Breuer and Dall 2010; Breuer and Sanyal 2012; Chen 2010]. The paper [Breuer and Dall 2010], in particular, studies graph polynomials using tools from commutative algebra. The Voronoi polytope does not equal the modular/integral flow polytope or the polytope of flows of an oriented graph. It would, however, be interesting to further explore the relation between these polytopes. (We thank the anonymous referee for pointing out this literature.)

This paper suggests several other questions for further study. First, in Section 5A, we exhibit a collection of generators V_γ, indexed by oriented circuits γ, for $R(\Gamma \setminus T, \phi)$. What is an explicit set of generators for the ideal of relations between the variables V_γ? This problem is posed as Problem 5.5. Second, in Theorem 5.7, we give a formula for the multiplicity of $R(\Gamma)$ in terms of the subdiagram volume of certain semigroups associated to Γ. Problem 5.8 is to find an expression for this multiplicity in terms of well-known graph theory invariants. Third, we also prove in Theorem 5.7 that $\mathrm{Spec}(R(\Gamma))$ is semi log canonical. In Problem 5.9, we ask: which graphs Γ have the stronger property that $R(\Gamma)$ is semi divisorial log canonical?

1. Preliminaries

In this section, we review the definitions of the graph-theoretic objects considered in this paper. This will provide the reader with enough background to follow the main ideas of the proof of Theorem A (proven in Section 6) as well as the proofs of many of the geometric properties of cographic rings (proven in Section 4).

1A. ***Notation.*** Following notation of Serre [1980, §2.1], a *graph* Γ will consist of the data $(\vec{E} \overset{s}{\underset{t}{\rightrightarrows}} V, \vec{E} \xrightarrow{\iota} \vec{E})$, where V and $\vec{E}$ are sets, ι is a fixed-point free involution, and s and t are maps satisfying $s(\vec{e}) = t(\iota(\vec{e}))$ for all $\vec{e} \in \vec{E}$. The maps s and t are called the *source* and *target* maps, respectively. We call $V =: V(\Gamma)$ the set of *vertices*. We call $\vec{E} =: \vec{E}(\Gamma)$ the set of *oriented edges*. We define the set of (*unoriented*) *edges* to be $E(\Gamma) = E := \vec{E}/\iota$. An *orientation of an edge* $e \in E$ is a representative for e in $\vec{E}$; we use the notation $\vec{e}$ and $\overleftarrow{e}$ for the two possible orientations of e. An *orientation of a graph* Γ is a section $\phi : E \to \vec{E}$ of the quotient map. An *oriented graph* consists of a pair (Γ, ϕ) where Γ is a graph and ϕ is an orientation. Given an oriented graph, we say that $\phi(e)$ is the *positive orientation* of the edge e. Given a subset $S \subseteq E$, we define $\vec{S} \subseteq \vec{E}$ to be the set of all orientations of the edges in S.

1B. ***Homology of a graph.*** Given a ring A, let $C_0(\Gamma, A) = \vec{C}_0(\Gamma, A)$ be the free A-module with basis $V(\Gamma)$ and $\vec{C}_1(\Gamma, A)$ be the A-module generated by $\vec{E}(\Gamma)$ with the relations $\overleftarrow{e} = -\vec{e}$ for every $e \in E(\Gamma)$. If we fix an orientation, then a basis for $\vec{C}_1(\Gamma, A)$ is given by the positively oriented edges; this induces an isomorphism with the usual group of 1-chains of the simplicial complex associated to Γ. These modules may be put into a chain complex. Define a boundary map ∂ by

$$\partial : \vec{C}_1(\Gamma, A) \to \vec{C}_0(\Gamma, A) = C_0(\Gamma, A), \quad \vec{e} \mapsto t(\vec{e}) - s(\vec{e}).$$

We will denote by $H_\bullet(\Gamma, A)$ the groups obtained from the homology of $\vec{C}_\bullet(\Gamma, A)$. The homology groups $H_\bullet(\Gamma, A)$ coincide with the homology groups of the topological space associated to Γ.

1C. ***The bilinear form.*** The vector space $\vec{C}_1(\Gamma, \mathbb{R})$ is endowed with a positive definite bilinear form

$$(\cdot\,,\,\cdot) : \vec{C}_1(\Gamma, \mathbb{R}) \otimes \vec{C}_1(\Gamma, \mathbb{R}) \to \mathbb{R}$$

that is uniquely determined by $(\vec{e}, \vec{e}) = 1$, $(\vec{e}, \overleftarrow{e}) = -1$, and $(\vec{e}, \vec{f}) = 0$ if $\vec{f} \neq \vec{e}, \overleftarrow{e}$. As above, fixing an orientation induces a basis for $\vec{C}_1(\Gamma, \mathbb{R})$, and in terms of such a basis, this is the standard inner product. By restriction, we get a positive definite bilinear form on $H_1(\Gamma, \mathbb{R}) \subseteq \vec{C}_1(\Gamma, \mathbb{R})$. The pairing $(\cdot\,,\,\cdot)$ allows us to form the product $(\vec{e}, v)$ of an oriented edge $\vec{e}$ with a vector $v \in \vec{C}_1(\Gamma, \mathbb{R})$ but not the product (e, v) of v with an unoriented vector. However, we will write $(e, v) = 0$ to mean $(\vec{e}, v) = 0$ for one (equivalently all) orientations of e.

1D. ***Cographic arrangement.*** We review the definition of the *cographic arrangement* $\mathscr{C}_\Gamma^\perp$ of Γ [Greene and Zaslavsky 1983, §8; Novik et al. 2002, §5].[1] To begin,

[1]The name "cographic arrangement" suggests the fact that $\mathscr{C}_\Gamma^\perp$ depends on the cographic matroid associated to Γ. The notation $\mathscr{C}_\Gamma^\perp$ is used in [Novik et al. 2002] while in [Greene and Zaslavsky 1983] the cographic arrangement is denoted by $\mathscr{H}^\perp[\Gamma]$. There is a dual notion, namely that of the graphic arrangement, which depends only on the graphic matroid associated to Γ and is denoted by

let $\mathcal{H}$ be the coordinate hyperplane arrangement in $\vec{C}_1(\Gamma, \mathbb{R})$. More precisely,

$$\mathcal{H} = \bigcup_{e \in E} \{v \in \vec{C}_1(\Gamma, \mathbb{R}) : (v, e) = 0\}.$$

The restriction of this hyperplane arrangement to $H_1(\Gamma, \mathbb{R})$ is called the cographic arrangement $\mathscr{C}_\Gamma^\perp$. More precisely,

$$\mathscr{C}_\Gamma^\perp = \bigcup_{\substack{e \in E \\ H_1(\Gamma, \mathbb{R}) \not\subseteq \ker(\cdot, e)}} \{v \in \vec{C}_1(\Gamma, \mathbb{R}) : (v, e) = 0\}.$$

The cographic arrangement partitions $H_1(\Gamma, \mathbb{R})$ into a finite collection of strongly convex rational polyhedral cones. These cones, together with their faces, form a (complete) fan that is defined to be the *cographic fan* and is denoted $\mathscr{F}_\Gamma^\perp$.[2] We give a more detailed enumeration of the cones of this fan in Section 3, where we discuss the poset of totally cyclic orientations.

Remark 1.1. The following observation used in the proof of Theorem A is proven in Corollary 3.4. We emphasize it here so that the reader may follow the proof of Theorem A having read just Section 1. *Let $c = \sum_{e \in E} a_e \vec{e}$ and $c' = \sum_{e \in E} a'_e \vec{e}$ be cycles in $H_1(\Gamma, \mathbb{Z})$. Then c and c' lie in a common cone of $\mathscr{F}_\Gamma^\perp$ if and only if, for all $e \in E$, $a_e a'_e \geq 0$.* In words, two cycles lie in a common cone if and only if every common edge is oriented in the same direction.

1E. ***Toric face rings.*** We recall the definition of a toric face ring associated to a fan. In [Ichim and Römer 2007, §2; Bruns et al. 2008, §2], the authors define more generally the toric face ring associated to a monoidal complex. The following definition is a special case:

Definition 1.2. Let $H_\mathbb{Z}$ be a free $\mathbb{Z}$-module of finite rank, and let $\mathscr{F}$ be a fan of (strongly convex rational polyhedral) cones in $H_\mathbb{R} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$ with support $\operatorname{Supp} \mathscr{F}$. The *toric face ring* $R_k(\mathscr{F})$ is the k-algebra whose underlying k-vector space has basis $\{X^c : c \in H_\mathbb{Z} \cap \operatorname{Supp} \mathscr{F}\}$ and whose multiplication is defined by

$$X^c \cdot X^{c'} = \begin{cases} X^{c+c'} & \text{if } c, c' \in \sigma \text{ for some } \sigma \in \mathscr{F}, \\ 0 & \text{otherwise.} \end{cases} \tag{1-1}$$

We will write $R(\mathscr{F})$ if we do not need to specify the base field k.

Remark 1.3. It follows from the definition that $R(\mathscr{F})$ is a reduced ring finitely generated over k. See also Section 5, especially (5-4), for more on generators and relations.

$\mathscr{C}_\Gamma$ in [Novik et al. 2002] and $\mathscr{H}[\Gamma]$ in [Greene and Zaslavsky 1983, §7]. The graphic arrangement of hyperplanes is also studied in [Orlik and Terao 1992, §2.4], where it is denoted by $\mathscr{A}(\Gamma)$.

[2] We use the notation $\mathscr{F}_\Gamma^\perp$ and the name "cographic fan" in order to be consistent with the notation $\mathscr{C}_\Gamma^\perp$ used in [Novik et al. 2002] for the cographic arrangement of hyperplanes.

A cographic toric face ring is the toric face ring associated to a cographic fan.

Definition 1.4. Let Γ be a finite graph. The *cographic toric face ring* $R_k(\Gamma)$ is the toric face k-ring $R(\mathcal{F}_\Gamma^\perp)$ associated to the cographic fan $\mathcal{F}_\Gamma^\perp$. We will write $R(\Gamma)$ if we do not need to specify the base field k.

1F. ***The Voronoi polytope.*** Following [Bacher et al. 1997], we define the *Voronoi polytope* of Γ by

$$\mathrm{Vor}_\Gamma := \{v \in H_1(\Gamma, \mathbb{R}) : (v, v) \leq (v - \lambda, v - \lambda) \text{ for all } \lambda \in H_1(\Gamma, \mathbb{Z})\}.$$

The reader familiar with the Voronoi decomposition of $\mathbb{R}^n$ will recognize this polytope as the unique cell containing the origin in the Voronoi decomposition associated with the lattice $H_1(\Gamma, \mathbb{Z})$ endowed with the scalar product defined in Section 1C (see [Erdahl 1999; Alexeev 2004, §2.5] for more details).

To the Voronoi polytope, we can associate its *normal fan* $\mathcal{N}(\mathrm{Vor}_\Gamma)$, which is defined as follows. Given a face δ of Vor_Γ, we define the (strongly convex rational polyhedral) cone C_δ by

$$C_\delta = \{\alpha \in H_1(\Gamma, \mathbb{R}) : (\alpha, r) \geq (\alpha, r') \text{ for all } r \in \delta \text{ and } r' \in \mathrm{Vor}_\Gamma\}.$$

The normal fan $\mathcal{N}(\mathrm{Vor}_\Gamma)$ of Vor_Γ is the fan whose cones are the cones C_δ.

Remark 1.5. In Proposition 3.8, we will prove that the cographic fan $\mathcal{F}_\Gamma^\perp$ is equal to the normal fan of the Voronoi polytope $\mathcal{N}(\mathrm{Vor}_\Gamma)$.

2. Totally cyclic orientations

Here we define and study totally cyclic orientations of a graph. We also define an oriented circuit on a graph and describe the relation between these circuits and totally cyclic orientations.

2A. ***Subgraphs.*** In this subsection, we introduce some special subgraphs that will play an important role throughout the paper.

Given a graph Γ and a collection $S \subset E(\Gamma)$ of edges, we define $\Gamma \setminus S$ to be the graph, called a *spanning subgraph* (see, e.g., [Oda and Seshadri 1979, §4]), obtained from Γ by removing the edges in S and leaving the vertices unmodified. In other words, $\Gamma \setminus S$ consists of the data

$$(\overrightarrow{E(\Gamma) \setminus S} \underset{t}{\overset{s}{\rightrightarrows}} V, \overrightarrow{E(\Gamma) \setminus S} \xrightarrow{\iota} \overrightarrow{E(\Gamma) \setminus S}).$$

Of particular significance is the special case where $S = \{e\}$ consists of a single edge. If $\Gamma \setminus \{e\}$ has more connected components than Γ, then we say that e is a *separating edge*. The *set of all separating edges* is written $E(\Gamma)_{\mathrm{sep}}$.

Given a chain $c \in \vec{C}_1(\Gamma, \mathbb{R})$, we would like to refer to the underlying graph having only those edges in the support of c. More precisely, given $c \in \vec{C}_1(\Gamma, \mathbb{R})$, let $\mathrm{Supp}(c)$ denote the set of all edges e with the property that $(e, c) \neq 0$. We define Γ_c to be the subgraph of Γ with $V(\Gamma_c) := V(\Gamma)$ and $E(\Gamma_c) := \mathrm{Supp}(c)$. There is a distinguished orientation ϕ_c of Γ_c given by setting $\phi_c(e)$ equal to $\vec{e}$ if $(\vec{e}, c) > 0$ and to $\overleftarrow{e}$ otherwise. Using this subgraph, we can write c as

$$c = \sum_{e \in \mathrm{Supp}(c)} m_c(e)\phi_c(e) \tag{2-1}$$

with all $m_c(e) > 0$. Indeed, we have $m_c(e) = (\phi_c(e), c)$.

2B. ***Totally cyclic orientations and oriented circuits.*** Totally cyclic orientations will play a dominant role in what follows. We are going to review their definition and their basic properties.

Definition 2.1. If Γ is connected, then we say that an orientation ϕ of Γ is *totally cyclic* if there does not exist a nonempty proper subset $W \subset V(\Gamma)$ such that every edge e between a vertex in W and a vertex in the complement $V(\Gamma) \setminus W$ is oriented from W to $V \setminus W$ (i.e., the source of $\phi(e)$ lies in W and the target of $\phi(e)$ lies in $V(\Gamma) \setminus W$). If Γ is disconnected, then we say that an orientation of Γ is totally cyclic if the orientation induced on each connected component of Γ is totally cyclic.

Observe that if Γ is a graph with no edges, then the empty orientation of Γ is a totally cyclic orientation. Totally cyclic orientations are closely related to oriented circuits. Recall that a graph Δ is called *cyclic* if it is connected, free from separating edges, and satisfies $b_1(\Delta) = 1$. A cyclic graph together with a totally cyclic orientation is called an *oriented circuit*. A cyclic graph admits exactly two totally cyclic orientations.

Let $\overrightarrow{\mathrm{Cir}}(\Gamma)$ denote the set of all oriented circuits on Γ; that is, $\gamma = (\Delta, \phi_\Delta)$ is an element of $\overrightarrow{\mathrm{Cir}}(\Gamma)$ if Δ is a cyclic subgraph of Γ and ϕ_Δ is a totally cyclic orientation of Δ. We call $E(\Delta)$ the *support* of $\gamma = (\Delta, \phi_\Delta) \in \overrightarrow{\mathrm{Cir}}(\Gamma)$. There is a natural map

$$\begin{gathered}\overrightarrow{\mathrm{Cir}}(\Gamma) \to H_1(\Gamma, A),\\ \gamma = (\Delta, \phi_\Delta) \mapsto [\gamma] = \sum_{e \in E(\Delta)} \phi_\Delta(e).\end{gathered}$$

With respect to the orientation ϕ of Γ, we can consider $\mathrm{Cir}_\phi(\Gamma) \subset \overrightarrow{\mathrm{Cir}}(\Gamma)$, the subset that consists of oriented circuits on Γ of the form $(\Delta, \phi|_\Delta)$ (i.e., oriented circuits whose orientation is compatible with ϕ).

Remark 2.2. The oriented circuits on Γ, i.e., the elements of $\overrightarrow{\mathrm{Cir}}(\Gamma)$, are the (signed) cocircuits of the cographic oriented matroid $M^*(\Gamma)$ of Γ or, equivalently, the (signed) circuits of the oriented graphic matroid $M(\Gamma)$ of Γ [Björner et al. 1999,

§1.1]. Many of the combinatorial results that follow can be naturally stated using this language. We will limit ourselves to pointing out the connection with the theory when relevant.

The next lemma clarifies the relationship between totally cyclic orientations and compatibly oriented circuits. Recall that an oriented path from $w \in V(\Gamma)$ to $v \in V(\Gamma)$ is a collection of oriented edges $\{\vec{e}_1, \dots, \vec{e}_r\} \subset \vec{E}(\Gamma)$ such that $s(\vec{e}_1) = w$, $t(\vec{e}_i) = s(\vec{e}_{i+1})$ for any $i = 1, \dots, r-1$, and $t(\vec{e}_r) = v$. If ϕ is an orientation of Γ, a path compatibly oriented with respect to ϕ is an oriented path as before of the form $\{\phi(e_1), \dots, \phi(e_r)\}$.

Lemma 2.3. *Let Γ be a graph.*

(1) *The graph Γ admits a totally cyclic orientation if and only if $E(\Gamma)_{\text{sep}} = \varnothing$.*

(2) *Fix an orientation ϕ on Γ. The following conditions are equivalent*:

  (a) *The orientation is totally cyclic.*

  (b) *For any distinct $v, w \in V(\Gamma)$ belonging to the same connected component of Γ, there exists a path compatibly oriented with respect to ϕ from w to v.*

  (c) *The cycles $[\gamma]$ associated to the $\gamma \in \mathrm{Cir}_\phi(\Gamma)$ generate $H_1(\Gamma, \mathbb{Z})$, and $E(\Gamma)_{\text{sep}} = \varnothing$.*

  (d) *Every edge $e \in E$ is contained in the support of a compatibly oriented circuit $\gamma \in \mathrm{Cir}_\phi(\Gamma)$.*

Proof. For part (1), see, e.g., [Caporaso and Viviani 2010, Lemma 2.4.3(1)] and the references therein. Part (2) is a reformulation of [Caporaso and Viviani 2010, Lemma 2.4.3(2)]. The only difference is that part (2) is proved in [loc. cit.] under the additional hypothesis that $E(\Gamma)_{\text{sep}} = \varnothing$. Note, however, that each of the conditions (a), (b), and (d) imply that $E(\Gamma)_{\text{sep}} = \varnothing$; hence, we deduce part (2) as stated above. □

The following well-known lemma can be thought of as a modification of (c) above. We no longer require that the oriented circuits on Γ be oriented compatibly. The statement is essentially that any cycle c in $H_1(\Gamma, \mathbb{Z})$ is a positive linear combination of cycles associated to circuits supported on c.

Lemma 2.4. *Let Γ be a graph, and let $c \in \vec{C}_1(\Gamma, \mathbb{Z})$. Then $c \in H_1(\Gamma, \mathbb{Z})$ if and only if c can be expressed as*

$$c = \sum_{\gamma \in \mathrm{Cir}_{\phi_c}(\Gamma_c)} n_c(\gamma)[\gamma] \tag{2-2}$$

for some natural numbers $n_c(\gamma) \in \mathbb{N}$.

Proof. A direct proof follows from the definitions and is left to the reader. Alternatively, one can use the fact that a covector of an oriented matroid can be written as a composition of cocircuits conformal to it [Björner et al. 1999, Proposition 3.7.2] together with Remark 2.2. □

The oriented circuits can be used to define a simplicial complex that will be used in Section 5B.

Definition 2.5. Two oriented circuits $\gamma = (\Delta, \phi)$ and $\gamma' = (\Delta', \phi')$ are said to be *concordant*, written $\gamma \asymp \gamma'$, if for any $e \in E(\Delta) \cap E(\Delta')$ we have $\phi(e) = \phi'(e)$. We write $\gamma \not\asymp \gamma'$ if γ and γ' are not concordant.

Definition 2.6. The *simplicial complex of concordant circuits* $\Delta(\overrightarrow{\mathrm{Cir}}(\Gamma))$ is defined to be the (abstract) simplicial complex whose elements are collections $\sigma \subseteq \overrightarrow{\mathrm{Cir}}(\Gamma)$ of oriented circuits on Γ with the property that any two circuits are concordant (i.e., if $\gamma_1, \gamma_2 \in \sigma$, then $\gamma_1 \asymp \gamma_2$).

2C. ***The poset $\mathbb{O}\mathcal{P}_\Gamma$ of totally cyclic orientations.*** Totally cyclic orientations naturally form a poset. We recall the definition for the sake of completeness.

Definition 2.7 [Caporaso and Viviani 2010, Definition 5.2.1]. The poset $\mathbb{O}\mathcal{P}_\Gamma$ of *totally cyclic orientations* of Γ is the set of pairs (T, ϕ) where $T \subset E(\Gamma)$ and $\phi : E(\Gamma \setminus T) \to \vec{E}(\Gamma \setminus T)$ is a totally cyclic orientation of $\Gamma \setminus T$,[3] endowed with the partial order

$$(T', \phi') \leq (T, \phi) \iff \Gamma \setminus T' \subseteq \Gamma \setminus T \text{ and } \phi' = \phi|_{E(\Gamma \setminus T')}.$$

We call T the *support* of the pair (T, ϕ).

Using Lemma 2.3(2)(d), we get that

$$(T', \phi') \leq (T, \phi) \iff \mathrm{Cir}_{\phi'}(\Gamma \setminus T') \subseteq \mathrm{Cir}_{\phi}(\Gamma \setminus T). \tag{2-3}$$

The set $\mathrm{Cir}_\phi(\Gamma \setminus T)$ is a collection of concordant cycles. Another connection between orientations and totally cyclic orientations is given by the following definition:

Definition 2.8. Let $\sigma \in \Delta(\overrightarrow{\mathrm{Cir}}(\Gamma))$ be a collection of concordant circuits. To σ we associate the pair $(T_\sigma, \phi_\sigma) \in \mathbb{O}\mathcal{P}_\Gamma$, which is defined as follows. Set T_σ equal to the set of all edges that are *not* contained in a circuit $\gamma \in \sigma$. The orientation ϕ_σ of $\Gamma \setminus T_\sigma$ is defined by setting

$$\phi_\sigma(e) := \begin{cases} \vec{e} & \text{if } (\vec{e}, [\gamma]) > 0 \text{ for all } \gamma \in \sigma, \\ \overleftarrow{e} & \text{if } (\overleftarrow{e}, [\gamma]) > 0 \text{ for all } \gamma \in \sigma. \end{cases}$$

Observe that the orientation ϕ_σ on $\Gamma \setminus T_\sigma$ is a totally cyclic orientation by Lemma 2.3(2)(d) and that $\sigma \subseteq \mathrm{Cir}_{\phi_\sigma}(\Gamma \setminus T_\sigma)$. The following lemma, whose proof is left to the reader, will be useful in the sequel:

[3] The choice of orientation on the complement of T, rather than on T itself, has to do with the importance of the notion of spanning subgraphs of Γ, all of which are of this form. In graph theory, it is customary to denote spanning subgraphs in this way, so we follow that convention.

Lemma 2.9. *The maximal elements of the poset* $\mathcal{OP}_\Gamma$ *are given by* $(E(\Gamma)_{\mathrm{sep}}, \phi)$ *as* ϕ *varies among the totally cyclic orientations of* $\Gamma \setminus E(\Gamma)_{\mathrm{sep}}$. □

Remark 2.10. The poset $\mathcal{OP}_\Gamma$ of totally cyclic orientations is isomorphic to the poset of covectors of the cographic oriented matroid $M^*(\Gamma)$ of Γ [Björner et al. 1999, §3.7]. Equivalently, the poset obtained from $\mathcal{OP}_\Gamma$ by adding an element $\underline{1}$ and declaring that $\underline{1} \geq (T, \phi)$ for any $(T, \phi) \in \mathcal{OP}_\Gamma$ is isomorphic to the big face lattice $\mathcal{F}_{\mathrm{big}}(M^*(\Gamma))$ of the cographic oriented matroid $M^*(\Gamma)$ [Björner et al. 1999, §4.1].

3. Comparing posets: the cographic arrangement, the Voronoi polytope, and totally cyclic orientations

In this section, we prove that the poset $\mathcal{OP}_\Gamma$ of totally cyclic orientations of Γ is isomorphic to the poset of cones (ordered by inclusion) of the cographic fan $\mathcal{F}_\Gamma^\perp$, which we also show is the normal fan of the Voronoi polytope Vor_Γ of Γ.

3A. ***Cographic arrangement.*** Let us start by describing the cographic arrangement $\mathcal{C}_\Gamma^\perp$ associated to Γ in the language of totally cyclic orientations.

For every edge $e \in E(\Gamma)$, we can consider the linear subspace of $H_1(\Gamma, \mathbb{R})$

$$\{(\,\cdot\,, e) = 0\} := \{v \in H_1(\Gamma, \mathbb{R}) : (v, e) = 0\}.$$

This subspace is a proper subspace (i.e., a hyperplane) precisely when e is not a separating edge, and the collection of all such hyperplanes is defined to be the cographic arrangement. Similarly, for any oriented edge $\vec{e} \in \vec{E}(\Gamma)$, we set

$$\{(\,\cdot\,, \vec{e}) \geq 0\} := \{v \in H_1(\Gamma, \mathbb{R}) : (v, \vec{e}) \geq 0\}.$$

As mentioned, the elements of the cographic arrangement partition $H_1(\Gamma, \mathbb{R})$ into a finite collection of rational polyhedral cones. These cones, together with their faces, form the cographic fan $\mathcal{F}_\Gamma^\perp$. We can enumerate these cones and make their relation to totally cyclic orientations more explicit by introducing some notation.

Given a collection T of edges and an orientation ϕ of $\Gamma \setminus T$ (not necessarily totally cyclic), we define (possibly empty) cones $\sigma(T, \phi)$ and $\sigma^{\mathrm{o}}(T, \phi)$ by

$$\sigma(T, \phi) := \bigcap_{e \notin T} \{(\,\cdot\,, \phi(e)) \geq 0\} \cap \bigcap_{e \in T} \{(\,\cdot\,, e) = 0\}, \tag{3-1}$$

$$\sigma^{\mathrm{o}}(T, \phi) := \bigcap_{e \notin T} \{(\,\cdot\,, \phi(e)) > 0\} \cap \bigcap_{e \in T} \{(\,\cdot\,, e) = 0\}. \tag{3-2}$$

The cone $\sigma^{\mathrm{o}}(T, \phi)$ is a subcone of $\sigma(T, \phi)$, and it is the relative interior of $\sigma(T, \phi)$ provided $\sigma^{\mathrm{o}}(T, \phi)$ is nonempty. The cone $\sigma(T, \phi)$ is an element of the cographic fan, and every cone in the fan can be written in this form. While every element of $\mathcal{F}_\Gamma^\perp$ can be written as $\sigma(T, \phi)$, the pair (T, ϕ) is *not* uniquely determined by the cone. The pair (T, ϕ) is, however, uniquely determined if we further require that $(T, \phi) \in \mathcal{OP}_\Gamma$. This fact is proven in the following proposition, which is essentially a restatement of some results of Greene and Zaslavsky [1983, §8]:

Proposition 3.1. (i) *Every cone $\sigma \in \mathcal{F}_\Gamma^\perp$ can be written as $\sigma = \sigma(T, \phi)$ for a unique element $(T, \phi) \in \mathcal{OP}_\Gamma$.*

(ii) *For any $(T, \phi) \in \mathcal{OP}_\Gamma$, the linear span of $\sigma(T, \phi)$ is equal to*

$$\langle \sigma(T, \phi) \rangle = \bigcap_{e \in T} \{(\,\cdot\,, e) = 0\} = H_1(\Gamma \setminus T, \mathbb{R})$$

and has dimension $b_1(\Gamma \setminus T)$.

(iii) *For any $(T, \phi) \in \mathcal{OP}_\Gamma$, the extremal rays of $\sigma(T, \phi)$ are the rays generated by the elements $[\gamma]$ for $\gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)$.*

Proof. Part (i) follows from [Greene and Zaslavsky 1983, Lemma 8.2]. Note that in [ibid.] the authors assume that $E(\Gamma)_{\mathrm{sep}} = \varnothing$. However, it is easily checked that the inclusion map $\Gamma \setminus E(\Gamma)_{\mathrm{sep}} \subseteq \Gamma$ induces natural isomorphisms $\mathcal{F}^\perp_{\Gamma \setminus E(\Gamma)_{\mathrm{sep}}} \cong \mathcal{F}_\Gamma^\perp$ and $\mathcal{OP}_{\Gamma \setminus E(\Gamma)_{\mathrm{sep}}} \cong \mathcal{OP}_\Gamma$. Therefore, the general case follows from the special case treated in [ibid.].

Let us now prove part (ii). The linear subspace $\bigcap_{e \in T} \{(\,\cdot\,, e) = 0\} \subseteq H_1(\Gamma, \mathbb{R})$ is generated by all the cycles of Γ that do not contain edges $e \in T$ in their support and is therefore equal to $H_1(\Gamma \setminus T, \mathbb{R})$, which has dimension equal to $b_1(\Gamma \setminus T)$. Now, to complete the proof, let us establish that $\langle \sigma(T, \phi) \rangle = \bigcap_{e \in T} \{(\,\cdot\,, e) = 0\}$. First, if $\sigma(T, \phi)^\circ = \varnothing$, i.e., if $\sigma(T, \phi) = \{0\}$, then $b_1(\Gamma \setminus T) = 0$ by Lemma 2.3(2)(d). But then $\bigcap_{e \in T} \{(\,\cdot\,, e) = 0\} = H_1(\Gamma \setminus T, \mathbb{R}) = 0$, and we are done. On the other hand, if $\sigma(T, \phi)^\circ \neq \varnothing$, then $\sigma^{\mathrm{o}}(T, \phi)$ is the relative interior of $\sigma(T, \phi)$, and hence, the linear span of $\sigma(T, \phi)$ is equal to $\bigcap_{e \in T} \{(\,\cdot\,, e) = 0\}$.

Finally, let us prove part (iii). From [Greene and Zaslavsky 1983, Lemma 8.5], it follows that the extremal rays of $\sigma(T, \phi)$ are among the rays generated by the elements $[\gamma]$ for $\gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)$. We conclude by showing that for any $\gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)$, the ray generated by $[\gamma]$ is extremal for $\sigma(T, \phi)$. By contradiction, suppose that we can write

$$[\gamma] = \sum_{\substack{\gamma' \in \mathrm{Cir}_\phi(\Gamma \setminus T) \\ \gamma' \neq \gamma}} m_{\gamma'} [\gamma'] \tag{3-3}$$

for some $m_{\gamma'} \in \mathbb{R}_{\geq 0}$. Consider a cycle $\gamma_0 \in \mathrm{Cir}_\phi(\Gamma \setminus T) \setminus \{\gamma\}$ such that $m_{\gamma_0} > 0$ (which clearly exists since $[\gamma] \neq 0$). Since γ and γ_0 are concordant and distinct, there should exist an edge $e \in E(\gamma_0) \setminus E(\gamma)$. Now returning to the expression (3-3), on the left-hand side, neither the oriented edge $\vec{e}$ nor $\overleftarrow{e}$ can appear. On the other hand, on the right-hand side, the oriented edge $\phi(e)$ appears with positive multiplicity because it appears with multiplicity $m_{\gamma_0} > 0$ in $m_{\gamma_0}[\gamma_0]$ and all the oriented circuits appearing in the summation are concordant. This is a contradiction. $\square$

Corollary 3.2. *The association*

$$(T, \phi) \mapsto \sigma(T, \phi)$$

defines an isomorphism between the poset of $\mathcal{OP}_\Gamma$ and the poset of cones of $\mathcal{F}_\Gamma^\perp$ ordered by inclusion.[4] *In particular, the number of connected components of the complement of $\mathcal{C}_\Gamma^\perp$ in $H_1(\Gamma, \mathbb{R})$ is equal to the number of totally cyclic orientations on $\Gamma \setminus E(\Gamma)_{\mathrm{sep}}$.*

Proof. According to Proposition 3.1(i), the map in the statement is bijective. We have to show that

$$\sigma(T, \phi) \subseteq \sigma(T', \phi') \iff (T, \phi) \le (T', \phi').$$

The implication $\Longleftarrow$ is clear by the definition (3-1) of $\sigma(T, \phi)$.

Conversely, assume that $\sigma(T, \phi) \subseteq \sigma(T', \phi')$. There is nothing to show if $\sigma(T, \phi) = \{0\}$ is the origin. Otherwise, by Proposition 3.1(ii), the relative interior $\sigma^{\mathrm{o}}(T, \phi)$ of $\sigma(T, \phi)$ is nonempty, so pick $c \in \sigma^{\mathrm{o}}(T, \phi)$. By formula (3-2), for every $e \notin T$, we have that $(c, \phi(e)) > 0$. Since $c \in \sigma(T', \phi')$, by definition (3-1), we must have $e \notin T'$ and $\phi'(e) = \phi(e)$. This shows that $T \supseteq T'$ and that $\phi'_{\Gamma\setminus T} = \phi$ or in other words that $(T, \phi) \le (T', \phi')$.

The last assertion follows from the first one using the fact that the connected components of the complement of $\mathcal{C}_\Gamma^\perp$ in $H_1(\Gamma, \mathbb{R})$ are the maximal cones in $\mathcal{F}_\Gamma^\perp$ and Lemma 2.9. $\square$

Remark 3.3. The last assertion of Corollary 3.2 is due to Green and Zaslavsky [1983, Lemma 8.1]. Moreover, Greene and Zaslavsky [1983, Theorem 8.1] give a formula for the number of totally cyclic orientations of a graph free from separating edges.

The following well-known result plays a crucial role in the proof of Theorem 6.1:

Corollary 3.4. *Let*

$$c = \sum_{e \in E} a_e \vec{e} \quad \textit{and} \quad c' = \sum_{e \in E} a'_e \vec{e}$$

be cycles in $H_1(\Gamma, \mathbb{Z})$. Then there is a cone of $\mathcal{F}_\Gamma^\perp$ containing c and c' if and only if, for all $e \in E$, $a_e a'_e \ge 0$.

Proof. From Proposition 3.1(i), it follows that c and c' belong to the same cone of $\mathcal{F}_\Gamma^\perp$ if and only if there exists $(T, \phi) \in \mathcal{OP}_\Gamma$ such that $c, c' \in \sigma(T, \phi)$. We conclude by looking at the explicit description (3-1). $\square$

[4]Note that the poset of cones of $\mathcal{F}_\Gamma^\perp$ is anti-isomorphic to the face poset $\mathcal{L}(\mathcal{C}_\Gamma^\perp)$ of the arrangement $\mathcal{C}_\Gamma^\perp$ [Orlik and Terao 1992, Definition 2.18].

Remark 3.5. Corollary 3.2 together with Remark 2.10 imply that the cographic oriented matroid $M^*(\Gamma)$ is represented by the cographic hyperplane arrangement $\mathscr{C}_\Gamma^\perp$ in the sense of [Björner et al. 1999, §1.2(c)]. Using this, Corollary 3.4 is a restatement of the fact that two elements of $H_1(\Gamma, \mathbb{Z})$ belong to the same cone of $\mathscr{F}_\Gamma^\perp$ if and only if their associated covectors are conformal [Björner et al. 1999, §3.7].

3B. ***Voronoi polytope.*** The following description of the faces of Vor_Γ is a restatement, in our notation, of a result of Omid Amini [Amini 2010], which gives a positive answer to a conjecture of Caporaso and Viviani [2010, Conjecture 5.2.8(i)]:

Proposition 3.6 (Amini). (i) *Every face of the Voronoi polytope* Vor_Γ *is of the form*

$$F(T,\phi) := \{v \in \mathrm{Vor}_\Gamma : (v, [\gamma]) = \tfrac{1}{2}([\gamma],[\gamma]) \textit{ for any } \gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)\} \tag{3-4}$$

for some uniquely determined element $(T, \phi) \in \mathbb{O}\mathscr{P}_\Gamma$.

(ii) *For any* $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$, *the dimension of the affine span of* $F(T,\phi)$ *is equal to* $b_1(\Gamma(T)) = b_1(\Gamma) - b_1(\Gamma \setminus T)$.

(iii) *For any* $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$, *the codimension-*$1$ *faces of* Vor_Γ *containing* $F(T,\phi)$ *are exactly those of the form* $F(S,\psi)$, *where* $(S,\psi) \leq (T,\phi)$ *and* $b_1(\Gamma \setminus S) = 1$.

Proof. Part (i) follows by combining [Amini 2010, Theorem 1, Lemma 7]. Part (ii) follows from the remark after [Amini 2010, Lemma 10]. Part (iii) follows from [Amini 2010, Lemma 7]. □

Corollary 3.7 (Amini). *The association*

$$(T,\phi) \mapsto F(T,\phi)$$

defines an isomorphism of posets between the poset $\mathbb{O}\mathscr{P}_\Gamma$ *and the poset of faces of* Vor_Γ *ordered by reverse inclusion. In particular, the number of vertices of* Vor_Γ *is equal to the number of totally cyclic orientations on* $\Gamma \setminus E(\Gamma)_{\mathrm{sep}}$.

Proof. The first statement is a reformulation of [Amini 2010, Theorem 1]. The last assertion follows from the first one together with Lemma 2.9. □

We now show that the cographic fan $\mathscr{F}_\Gamma^\perp$ is the *normal fan* $\mathcal{N}(\mathrm{Vor}_\Gamma)$ of the Voronoi polytope Vor_Γ. The cones of the normal fan, ordered by inclusion, form a poset that is clearly isomorphic to the poset of faces of Vor_Γ, ordered by reverse inclusion.

Proposition 3.8. *The cographic fan* $\mathscr{F}_\Gamma^\perp$ *is equal to* $\mathcal{N}(\mathrm{Vor}_\Gamma)$, *the normal fan of the Voronoi polytope* Vor_Γ.

Proof. By Propositions 3.1 and 3.6, it is enough to show that, for any $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$, the normal cone in $\mathcal{N}(\mathrm{Vor}_\Gamma)$ to the face $F(T,\phi) \subset \mathrm{Vor}_\Gamma$ is equal to $\sigma(T,\phi)$. Fix a face $F(T,\phi)$ of Vor_Γ for some $(T,\phi) \in \mathbb{O}\mathscr{P}_\Gamma$. If (T,ϕ) is equal to the minimal element $\underline{0} = (E(\Gamma)_{\mathrm{sep}}, \varnothing)$ of the poset $\mathbb{O}\mathscr{P}_\Gamma$, then $F(\underline{0}) = \mathrm{Vor}_\Gamma$ and its normal cone is equal to the origin in $H_1(\Gamma, \mathbb{R})$, which is equal to $\sigma(\underline{0})$.

Suppose now that $b_1(\Gamma \setminus T) \geq 1$. Denote by $\{(S_i, \psi_i)\}$ all the elements of $\mathbb{O}\mathcal{P}_\Gamma$ such that $(S_i, \psi_i) \leq (T, \phi)$ and $b_1(\Gamma \setminus S_i) = 1$. Let γ_i be the unique oriented circuit of Γ such that $\mathrm{Cir}_{\psi_i}(\Gamma \setminus S_i) = \{\gamma_i\}$. According to Proposition 3.6(iii), the codimension-1 faces of Vor_Γ containing $F(T, \phi)$ are exactly those of the form $F(S_i, \psi_i)$. Therefore, the normal cone of $F(T, \phi)$ is the cone whose extremal rays are the normal cones to the faces $F(S_i, \psi_i)$, which, using (3-4), are equal to $\sigma(S_i, \psi_i) = \mathbb{R}_{\geq 0} \cdot [\gamma_i]$. By Proposition 3.1(iii), the cone whose extremal rays are given by $\mathbb{R}_{\geq 0} \cdot [\gamma_i]$ is equal to $\sigma(T, \phi)$, which completes the proof. $\square$

Combining Corollaries 3.2 and 3.7 and Proposition 3.8, we get the following incarnations of the poset $\mathbb{O}\mathcal{P}_\Gamma$ of totally cyclic orientations:

Corollary 3.9. *The following posets are isomorphic*:

(1) *the poset* $\mathbb{O}\mathcal{P}_\Gamma$ *of totally cyclic orientations*,

(2) *the poset of faces of the Voronoi polytope* Vor_Γ, *ordered by reverse inclusion*,

(3) *the poset of cones in the normal fan* $\mathcal{N}(\mathrm{Vor}_\Gamma)$, *ordered by inclusion*, *and*

(4) *the poset of cones in the cographic fan* $\mathcal{F}_\Gamma^\perp$, *ordered by inclusion.*

Remark 3.10. Corollary 3.9 together with Remark 2.10 imply that the cographic oriented matroid $M^*(\Gamma)$ is represented by the Voronoi polytope Vor_Γ (which is a zonotope; see, e.g., [Erdahl 1999]) in the sense of [Björner et al. 1999, §2.2].

4. Geometry of toric face rings

Let $H_\mathbb{Z}$ be a free $\mathbb{Z}$-module of finite rank b, and let $\mathcal{F}$ be a fan of (strongly convex rational polyhedral) cones in $H_\mathbb{R} = H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}$. The aim of this section is to study the toric face ring $R(\mathcal{F}) = R_k(\mathcal{F})$ associated to $\mathcal{F}$ as in Definition 1.2. We will pay special attention to fans $\mathcal{F}$ that are *complete*, i.e., such that every $x \in H_\mathbb{R}$ is contained in some cone $\sigma \in \mathcal{F}$, or *polytopal*, i.e., the normal fans of rational polytopes in $H_\mathbb{R}^*$. Note that a polytopal fan is complete, but the converse is false if $b \geq 3$ (see [Oda 1988, p. 84] for an example). In the subsequent sections, we will apply the results of this section to the cographic fan $\mathcal{F}_\Gamma^\perp$ of a graph Γ, which is polytopal by Proposition 3.8.

Note that the fan $\mathcal{F}$ is naturally a poset: given $\sigma, \sigma' \in \mathcal{F}$, we say that $\sigma \geq \sigma'$ if $\sigma \supseteq \sigma'$. The poset $(\mathcal{F}, \geq)$ has some nice properties, which we now describe. Recall the following standard concepts from poset theory. A (finite) poset $(P, \leq)$ is called a *meet-semilattice* if every two elements $x, y \in P$ have a meet (i.e., an element, denoted by $x \wedge y$, that is uniquely characterized by conditions $x \wedge y \leq x, y$ and, if $z \in P$ is such that $z \leq x, y$, then $z \leq x \wedge y$). In a meet-semilattice, every finite subset of elements $\{x_1, \dots, x_n\} \subset P$ admits a meet, denoted by $x_1 \wedge \cdots \wedge x_n$. A meet-semilattice is called *bounded* (from below) if it has a minimum element $\underline{0}$. A bounded meet-semilattice is called *graded* if, for every element $x \in P$, all maximal chains from $\underline{0}$ to x have the same length. If this is the case, we define a function, called the *rank function*, $\rho : P \to \mathbb{N}$ by setting $\rho(x)$ equal to the length of any maximal chain from $\underline{0}$ to x. A graded meet-semilattice is said to be *pure* if all the maximal elements have the same rank, and this maximal rank is called the *rank* of the poset and is denoted by $\operatorname{rk} P$. A graded meet-semilattice is said to be *generated in maximal rank* if every element of P can be obtained as the meet of a subset consisting of maximal elements.

Having made these preliminary remarks, we now collect some of the properties of the poset $(\mathscr{F}, \geq)$ that we will need later.

Lemma 4.1. *The poset* $(\mathscr{F}, \geq)$ *has the following properties*:

(i) $(\mathscr{F}, \geq)$ *is a meet-semilattice*, *where the meet of two cones is equal to their intersection.*

(ii) $(\mathscr{F}, \geq)$ *is bounded with minimum element* $\underline{0}$ *given by the zero cone* $\{0\}$.

(iii) $(\mathscr{F}, \geq)$ *is a graded semilattice with rank function given by* $\rho(\sigma) := \dim \sigma$.

(iv) *If* $\mathscr{F}$ *is complete*, *then* $(\mathscr{F}, \geq)$ *is pure of rank* $\operatorname{rk} \mathscr{F} = b$.

(v) *If* $\mathscr{F}$ *is complete*, *then* $(\mathscr{F}, \geq)$ *is generated in maximal rank.* □

We will denote by $\mathscr{F}_{\max}$ the subset of $\mathscr{F}$ consisting of the *maximal cones* of $\mathscr{F}$.

4A. ***Descriptions of $R(\mathscr{F})$ as an inverse limit and as a quotient.*** In this subsection, we give two descriptions of the toric face ring $R(\mathscr{F})$.

The first description of $R(\mathscr{F})$ is as an inverse limit of affine semigroup rings. For any cone $\sigma \in \mathscr{F}$, consider the semigroup

$$C(\sigma) := \sigma \cap H_{\mathbb{Z}} \subset H_{\mathbb{Z}}, \tag{4-1}$$

which, according to Gordan's lemma (e.g., [Bruns and Herzog 1993, Proposition 6.1.2]), is a positive normal affine semigroup, i.e., a finitely generated semigroup isomorphic to a subsemigroup of $\mathbb{Z}^d$ for some $d \in \mathbb{N}$ such that 0 is the unique invertible element and such that if $m \cdot z \in C(\sigma)$ for some $m \in \mathbb{N}$ and $z \in \mathbb{Z}^d$ then $z \in C(\sigma)$.

Definition 4.2. We define $R_k(\sigma) := k[C(\sigma)]$ to be the affine semigroup ring associated to $C(\sigma)$ (in the sense of [Bruns and Herzog 1993, §6.1]), i.e., the k-algebra whose underlying vector space has basis $\{X^c : c \in C(\sigma)\}$ and whose multiplication is defined by $X^c \cdot X^{c'} := X^{c+c'}$. We will write $R(\sigma)$ if we do not need to specify the base field k. If $\mathscr{F}_\sigma$ is the fan induced by σ (consisting of the cones in $\mathscr{F}$ that are faces of σ), then clearly $R(\sigma) = R(\mathscr{F}_\sigma)$.

The following properties are well-known.

Lemma 4.3. $R(\sigma)$ *is a normal*, *Cohen–Macaulay domain of dimension equal to* $\dim \sigma$.

Proof. By definition, we have $R(\sigma) \subset k[H_{\mathbb{Z}}] = k[x_1^{\pm 1}, \dots, x_b^{\pm 1}]$; hence, $R(\sigma)$ is a domain. $R(\sigma)$ is normal by [Bruns and Herzog 1993, Theorem 6.1.4] and Cohen–Macaulay by a theorem of Hochster [loc. cit., Theorem 6.3.5(a)]. Finally, it follows easily from [loc. cit., Proposition 6.1.1] that the (Krull) dimension of $R(\sigma)$ is equal to $\dim \sigma$. □

Given two elements $\sigma, \sigma' \in \mathcal{F}$ such that $\sigma \geq \sigma'$, or equivalently such that $\sigma \supseteq \sigma'$, there exists a natural projection map between the corresponding affine semigroup rings of Definition 4.2

$$r_{\sigma/\sigma'} : R(\sigma) \twoheadrightarrow R(\sigma'), \qquad X^c \mapsto \begin{cases} X^c & \text{if } c \in \sigma' \subseteq \sigma, \\ 0 & \text{if } c \in \sigma \setminus \sigma'. \end{cases}$$

With respect to these maps, the set $\{R(\sigma) : \sigma \in \mathcal{F}\}$ forms an inverse system of rings. From [Bruns et al. 2008, Proposition 2.2], we deduce the following description of $R(\mathcal{F})$:

Proposition 4.4. *Let $\mathcal{F}$ be a fan. We have an isomorphism*

$$R(\mathcal{F}) = \varprojlim_{\sigma \in \mathcal{F}} R(\sigma).$$

We denote by $r_\sigma : R(\mathcal{F}) \to R(\sigma)$ the natural projection maps.

The second description of $R(\mathcal{F})$ is as a quotient of a polynomial ring. For any cone $\sigma \in \mathcal{F}$, the semigroup $C(\sigma) = \sigma \cap H_{\mathbb{Z}}$ has a unique minimal generating set, called the *Hilbert basis* of $C(\sigma)$ and denoted by $\mathcal{H}_\sigma$ [Miller and Sturmfels 2005, Proposition 7.15]. Therefore, we have a surjection

$$\pi_\sigma : k[V_\alpha : \alpha \in \mathcal{H}_\sigma] \twoheadrightarrow R(\sigma), \qquad V_\alpha \mapsto X^\alpha. \tag{4-2}$$

In the terminology of [Sturmfels 1996, Chapter 4], the kernel of π_σ, which we denote by I_σ, is the *toric ideal* associated to the subset $\mathcal{H}_\sigma$. In the terminology of [Miller and Sturmfels 2005, Chapter II.7], I_σ is the *lattice ideal* associated with the kernel of the group homomorphism

$$p_\sigma : \mathbb{Z}^{\mathcal{H}_\sigma} \to H_{\mathbb{Z}}, \quad \underline{u} = \{u_\alpha\}_{\alpha \in \mathcal{H}_\sigma} \mapsto \sum_{\alpha \in \mathcal{H}_\sigma} u_\alpha \alpha.$$

From [Sturmfels 1996, Lemma 4.1] (see also [Miller and Sturmfels 2005, Theorem 7.3]), we get that I_σ is a binomial ideal with the explicit presentation

$$I_\sigma = \langle V^{\underline{u}} - V^{\underline{v}} : \underline{u}, \underline{v} \in \mathbb{N}^{\mathcal{H}_\sigma} \subset \mathbb{Z}^{\mathcal{H}_\sigma} \text{ with } p_\sigma(\underline{u}) = p_\sigma(\underline{v}) \rangle, \tag{4-3}$$

where, for any $\underline{u} = (u_\alpha)_{\alpha \in \mathcal{H}_\sigma} \in \mathbb{N}^{\mathcal{H}_\sigma}$, we set $V^{\underline{u}} := \prod_{\alpha \in \mathcal{H}_\sigma} V_\alpha^{u_\alpha} \in k[V_\alpha : \alpha \in \mathcal{H}_\sigma]$.

If we set $\mathcal{H}_{\mathcal{F}} := \bigcup_{\sigma \in \mathcal{F}} \mathcal{H}_\sigma$, then, from Definition 1.2, it follows that we have a surjection

$$\pi_{\mathcal{F}} : k[V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}] \twoheadrightarrow R(\mathcal{F}), \qquad V_\alpha \mapsto X^\alpha. \tag{4-4}$$

We denote by $I_{\mathcal{F}}$ the kernel of $\pi_{\mathcal{F}}$. In order to describe the ideal $I_{\mathcal{F}}$, we introduce the abstract simplicial complex $\Delta_{\mathcal{F}}$ on the vertex set $\mathcal{H}_{\mathcal{F}}$ whose faces are the collections of elements of $\mathcal{H}_{\mathcal{F}}$ that belong to the same cone of $\mathcal{F}$. The minimal nonfaces of $\Delta_{\mathcal{F}}$ are formed by pairs $\{\alpha, \alpha'\}$ of elements of $\mathcal{H}_{\mathcal{F}}$ such that α and α' do not belong to the same cone of $\mathcal{F}$; hence, $\Delta_{\mathcal{F}}$ is a flag complex [Stanley 1996, Chapter III, §4]. Consider the Stanley–Reisner ring (or face ring)

$$k[\Delta_{\mathcal{F}}] := \frac{k[V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}]}{(V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathcal{F}})}$$

associated to the flag complex $\Delta_{\mathcal{F}}$ (see [Stanley 1996, Chapter II] for an introduction to Stanley–Reisner rings). Observe that if $\{\alpha, \alpha'\} \notin \Delta_{\mathcal{F}}$, then $X^{[\alpha]} \cdot X^{[\alpha']} = 0$ by Definition 1.2. This implies that the surjection $\pi_{\mathcal{F}}$ factors as

$$\pi_{\mathcal{F}} : k[V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}] \twoheadrightarrow \frac{k[V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}]}{(V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathcal{F}})} = k[\Delta_{\mathcal{F}}] \twoheadrightarrow R(\mathcal{F})$$

or in other words that $(V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathcal{F}}) \subseteq I_{\mathcal{F}}$.

Moreover, observe also that the surjection $\pi_{\mathcal{F}}$ of (4-4) is compatible with the surjections π_σ of (4-2) for every $\sigma \in \mathcal{F}$ in the sense that we have a commutative diagram

$$\begin{tikzcd} k[V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}] \arrow[r, two heads, "\pi_{\mathcal{F}}"] \arrow[d, two heads, shift right, "\theta"'] & R(\mathcal{F}) \arrow[d, two heads, "r_\sigma"] \\ k[V_\alpha : \alpha \in \mathcal{H}_\sigma] \arrow[r, two heads, "\pi_\sigma"] \arrow[u, bend right, "s"'] & R(\sigma) \end{tikzcd} \tag{4-5}$$

where θ is the surjective ring homomorphism given by sending $V_\alpha \mapsto V_\alpha$ if $\alpha \in \mathcal{H}_\sigma \subseteq \mathcal{H}_{\mathcal{F}}$ and $V_\alpha \mapsto 0$ if $\alpha \in \mathcal{H}_{\mathcal{F}} \setminus \mathcal{H}_\sigma$. Both the vertical surjections have natural sections: the left map has a section s obtained by sending $V_\alpha \mapsto V_\alpha$ for any $\alpha \in \mathcal{H}_\sigma \subset \mathcal{H}_{\mathcal{F}}$, and the left map has a section obtained by sending X^c into X^c for any $c \in C(\sigma) = \sigma \cap H_{\mathbb{Z}} \subset H_{\mathbb{Z}}$. Therefore, we can regard I_σ as an ideal of $k[V_\alpha : \alpha \in \mathcal{H}_{\mathcal{F}}]$ by extensions of scalars and, by the above commutative diagram, we have that $I_\sigma \subseteq I_{\mathcal{F}}$.

From [Bruns et al. 2008, Propositions 2.3 and 2.6], we get the following description of the ideal $I_{\mathcal{F}}$:

Proposition 4.5. *Let $\mathcal{F}$ be a fan. The kernel $I_{\mathcal{F}}$ of the map $\pi_{\mathcal{F}}$ of (4-4) is given by*

$$I_{\mathcal{F}} = (V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathcal{F}}) + \sum_{\sigma \in \mathcal{F}} I_\sigma = (V_\alpha V_{\alpha'} : \{\alpha, \alpha'\} \notin \Delta_{\mathcal{F}}) + \sum_{\sigma \in \mathcal{F}_{\max}} I_\sigma,$$

where, as usual, $\mathcal{F}_{\max}$ denotes the subset of $\mathcal{F}$ consisting of the maximal cones.

4B. *Prime ideals of $R(\mathcal{F})$*. We now want to describe the prime ideals of the ring $R(\mathcal{F})$. Observe that, from the Definition 1.2, it follows that $R(\mathcal{F})$ has a natural $\mathbb{Z}^b \cong H_{\mathbb{Z}}$-grading.

Recall the following notions for a $\mathbb{Z}^n$-graded ring R (see, e.g., [Uliczka 2009]). A *graded ideal* is an ideal I of R with the property that for any $x \in I$ all homogenous components of x belong to I as well; this is equivalent to I being generated by homogenous elements. For any ideal I of R, the *graded core* I^* of I is defined as the ideal generated by all homogenous elements of I. It is the largest graded ideal contained in I. If $\mathfrak{p}$ is a prime ideal of R, then $\mathfrak{p}^*$ is a prime ideal [Uliczka 2009, Lemma 1.1(ii)].

For any $\sigma \in \mathcal{F}$, the kernel of the natural projection map $r_\sigma : R(\mathcal{F}) \twoheadrightarrow R(\sigma)$, which is explicitly equal to

$$\mathfrak{p}_\sigma := (\{X^c : c \notin \sigma\}), \tag{4-6}$$

is graded since it is generated by homogeneous elements and is prime by Lemma 4.3. From [Ichim and Römer 2007, Lemma 2.1], we deduce the following description of the graded ideals of $R(\mathcal{F})$:

Proposition 4.6. *The assignment $\sigma \mapsto \mathfrak{p}_\sigma$ gives an isomorphism between the poset $(\mathcal{F}, \geq)$ and the poset of graded prime ideals of $R(\mathcal{F})$ ordered by reverse inclusion. In particular, $\mathfrak{m} = \mathfrak{p}_{\{0\}}$ is the unique graded maximal ideal of $R(\mathcal{F})$, which is also a maximal ideal in the usual sense.*

From Proposition 4.6, we can deduce a description of the minimal primes of $R(\mathcal{F})$.

Corollary 4.7. *The minimal primes of $R(\mathcal{F})$ are the primes $\mathfrak{p}_\sigma$ as σ varies among all the maximal cones of $\mathcal{F}$. In particular, if $\mathcal{F}$ is complete, then $R(\mathcal{F})$ is of pure dimension b.*

Proof. Observe that if $\mathfrak{p}$ is a minimal ideal of $R(\mathcal{F})$, then $\mathfrak{p}^* = \mathfrak{p}$ by the minimality of $\mathfrak{p}$; hence, $\mathfrak{p}$ is graded. Conversely, if $\mathfrak{p}$ is a graded ideal of $R(\mathcal{F})$ that is minimal among the graded ideals of $R(\mathcal{F})$, then $\mathfrak{p}$ is also a minimal ideal of $R(\mathcal{F})$: indeed, if $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{q}^* = \mathfrak{p}$ by the minimality properties of $\mathfrak{p}$; hence, $\mathfrak{q} = \mathfrak{p}$.

It is now clear that the first assertion follows from Proposition 4.6. The last assertion follows from the first one together with Lemmas 4.1(iv) and 4.3. $\square$

Definition 4.8. The poset of *strata* of $R(\mathcal{F})$, denoted by $\operatorname{Str}(R(\mathcal{F}))$, is the set of all the ideals of $R(\mathcal{F})$ that are sums of minimal primes with the order relation given by reverse inclusion.

Geometrically, the poset $\operatorname{Str}(R(\mathcal{F}))$ is the collection of all scheme-theoretic intersections of irreducible components of $\operatorname{Spec} R(\mathcal{F})$ ordered by inclusion.

Corollary 4.9. *If $\mathcal{F}$ is complete, then the assignment $\sigma \mapsto \mathfrak{p}_\sigma$ gives an isomorphism between $(\mathcal{F}, \geq)$ and $\operatorname{Str}(R(\mathcal{F}))$.*

Proof. The statement will follow from Proposition 4.6 if we show that the ideals that are sums of minimal primes of $R(\mathcal{F})$ are exactly those of the form $\mathfrak{p}_\sigma$ for some $\sigma \in \mathcal{F}$. Indeed, given minimal primes $\mathfrak{p}_{\sigma_i}$ for $i = 1, \dots, n$ (see Corollary 4.7), we have that $\bigcap_{i=1}^n \sigma_i = \sigma$ for some $\sigma \in \mathcal{F}$ and, from (4-6), it follows that

$$\sum_{i=1}^{n} \mathfrak{p}_{\sigma_i} = \left(X^c : c \notin \bigcap_{i=1}^{n} \sigma_i \right) = \mathfrak{p}_\sigma. \tag{4-7}$$

Conversely, every cone $\sigma \in \mathscr{F}$ is the intersection of the maximal dimensional cones σ_i containing it by Lemma 4.1(v). Therefore, (4-7) shows that $\mathfrak{p}_\sigma \in \mathrm{Str}(R(\mathscr{F}))$. □

4C. *Gorenstein singularities.* The aim of this subsection is to prove the following:

Theorem 4.10. *If $\mathscr{F}$ is a polytopal fan, then $R(\mathscr{F})$ is a Gorenstein ring and its canonical module $\omega_{R(\mathscr{F})}$ is isomorphic to $R(\mathscr{F})$ as a graded module.*

Proof. This is a consequence of two results from [Ichim and Römer 2007]. The first is Theorem 1.1, stating that a toric face ring $R(\mathscr{F})$ is Cohen–Macaulay provided that the fan $\mathscr{F}$ is *shellable* (see p. 252 of that paper for the definition). The second is Theorem 1.4, stating that $R(\mathscr{F})$ is Gorenstein and its canonical module $\omega_{R(\mathscr{F})}$ is isomorphic to $R(\mathscr{F})$ as a graded module provided that $R(\mathscr{F})$ is Cohen–Macaulay and $\mathscr{F}$ is *Eulerian* (see Definition 6.4 in the same paper).

Now it is enough to recall that a polytopal fan is Eulerian (see, e.g., [Stanley 1994, p. 302]) and shellable by the Bruggesser–Manni theorem [Bruns and Herzog 1993, Theorem 5.2.14]. □

4D. *The normalization.* In this subsection, we prove that the toric face ring of any fan is seminormal and we describe its normalization.

Recall that, given a reduced ring R with total quotient ring $Q(R)$, the *normalization* of R, denoted by $\bar{R}$, is the integral closure of R inside $Q(R)$. R is said to be normal if $R = \bar{R}$ (see [Huneke and Swanson 2006, Definition 1.5.1], for example). Moreover, we need the following:

Definition 4.11. Let R be a Mori ring, i.e., a reduced ring such that $\bar{R}$ is finite over R. The *seminormalization* of R, denoted by ${}^{+}R$, is the biggest subring of $\bar{R}$ such that the induced pull-back map $\mathrm{Spec}({}^{+}R) \to \mathrm{Spec}\, R$ is bijective with trivial residue field extension. We say that R is seminormal if ${}^{+}R = R$.

For the basic properties of seminormal rings, we refer to [Greco and Traverso 1980; Swan 1980]. Observe that $R(\mathscr{F})$ is a Mori ring since it is reduced and finitely generated over a field k (see Remark 1.3).

Theorem 4.12. *Let $\mathscr{F}$ be any fan.*

(i) *The normalization of $R(\mathscr{F})$ is equal to*

$$\overline{R(\mathscr{F})} = \prod_{\sigma \in \mathscr{F}_{\max}} R(\sigma),$$

where $\mathscr{F}_{\max}$ is the subset of $\mathscr{F}$ consisting of all the maximal cones of $\mathscr{F}$.

(ii) *$R(\mathscr{F})$ is a seminormal ring.*

Proof. Let us first prove part (i). By [Huneke and Swanson 2006, Corollary 2.1.13] and Corollary 4.7, we get that the normalization of $R(\mathscr{F})$ is equal to

$$\overline{R(\mathscr{F})} = \prod_{\sigma \in \mathscr{F}_{\max}} \overline{R(\sigma)}.$$

We conclude by Lemma 4.3, which says that each domain $R(\sigma)$ is normal.

Let us now prove part (ii). According to Proposition 4.4 and Lemma 4.3, the ring $R(\mathscr{F})$ is an inverse limit of normal domains. Then the seminormality of $R(\mathscr{F})$ follows from [Swan 1980, Corollary 3.3]. □

4E. ***Semi log canonical singularities.*** In this subsection, we prove that Spec $R(\mathscr{F})$ has semi log canonical singularities provided that $\mathscr{F}$ is a polytopal fan.

We first recall the definitions of log canonical and semi log canonical pairs (see [Kollár and Mori 1998] for log canonical pairs and [Abramovich et al. 1992; Fujino 2000] for semi log canonical pairs). For the relevance of slc singularities in the theory of compactifications of moduli spaces, see [Kollár 2010].

Definition 4.13. Let X be an S_2 variety (i.e., such that the local ring $\mathscr{O}_{X,x}$ of X at any (schematic) point $x \in X$ has depth at least $\min\{2, \dim \mathscr{O}_{X,x}\}$) of pure dimension n over a field k and Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

(i) We say that the pair (X, Δ) is log canonical (or *lc* for short) if

- X is smooth in codimension 1 (or equivalently X is normal) and
- there exists a log resolution $f : Y \to X$ of (X, Δ) such that

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i,$$

where E_i are divisors on Y and $a_i \geq -1$ for every i.

We say that X is lc if the pair $(X, 0)$ is lc, where 0 is the zero divisor.

(ii) We say that the pair (X, Δ) is semi log canonical (or *slc* for short) if

- X is nodal in codimension 1 (or equivalently, X is seminormal and Gorenstein in codimension 1) and
- if $\mu : X^\mu \to X$ is the normalization of X and Θ is the $\mathbb{Q}$-Weil divisor on X given by

$$K_{X^\mu} + \Theta = \mu^*(K_X + \Delta), \tag{4-8}$$

then the pair (X^μ, Θ) is lc.

We say that X is slc if the pair $(X, 0)$ is slc, where 0 is the zero divisor.

Theorem 4.14. *If $\mathscr{F}$ is a polytopal fan, then the variety* Spec $R(\mathscr{F})$ *is slc.*

Proof. Observe that Spec $R(\mathscr{F})$ is Gorenstein by Theorem 4.10 and seminormal by Theorem 4.12(ii); hence, in particular, it is S_2 and nodal in codimension 1 [Greco and Traverso 1980, §8]. Moreover, Spec $R(\mathscr{F})$ is of pure dimension $\operatorname{rk}\mathscr{F}$ by Corollary 4.7. Consider now the normalization morphism (see Theorem 4.12(i))

$$\mu : \operatorname{Spec} \overline{R(\mathscr{F})} = \coprod_{\sigma\in\mathscr{F}_{\max}} \operatorname{Spec} R(\sigma) \to \operatorname{Spec} R(\mathscr{F}).$$

If we apply the formula (4-8) to the above morphism μ and we use the fact that $\Delta = 0$ (by hypothesis) and $K_X = 0$ by Theorem 4.10, then we get that the divisor Θ restricted to each connected component Spec $R(\sigma)$ of the normalization Spec $\overline{R(\mathscr{F})}$ is equal to $-K_{\operatorname{Spec} R(\sigma)}$. Therefore, from Definition 4.13(ii), we get that Spec $R(\mathscr{F})$ is slc if and only if the pair (Spec $R(\sigma)$, $-K_{\operatorname{Spec} R(\sigma)}$) is lc for every $\sigma \in \mathscr{F}_{\max}$. Therefore, we conclude using the fact that for any toric variety Z the pair $(Z, -K_Z)$ is lc [Fujino and Sato 2004, Proposition 2.10; Cox et al. 2011, Corollary 11.4.25]. □

4F. *Embedded dimension.* In this subsection, we compute the embedded dimension of $R(\mathscr{F})$ at its unique graded maximal ideal $\mathfrak{m}$. In doing this, we also compute the embedded dimension of the affine semigroup ring $R(\sigma)$ of Definition 4.2 at the maximal ideal $(X^c : c \in C(\sigma)\setminus\{0\})$, which, by a slight abuse of notation, we also denote by $\mathfrak{m}$.

Recall that given a maximal ideal $\mathfrak{m}$ of a ring R with residue field $k := R/\mathfrak{m}$, the embedded dimension of R at $\mathfrak{m}$ is the dimension of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$. Geometrically, the embedded dimension of R at $\mathfrak{m}$ is the dimension of the Zariski tangent space of Spec(R) at the point $\mathfrak{m} \in$ Spec(R).

Theorem 4.15. *Let $\mathscr{F}$ be a fan.*

(i) *The embedded dimension of $R(\sigma)$ at $\mathfrak{m}$ is equal to the cardinality of the Hilbert basis $\mathscr{H}_\sigma$* (see Section 4A).

(ii) *The embedded dimension of $R(\mathscr{F})$ at $\mathfrak{m}$ is equal to the cardinality of $\mathscr{H}_{\mathscr{F}}$ $(= \bigcup_{\sigma\in\mathscr{F}} \mathscr{H}_\sigma)$.*

Proof. Consider the presentation (4-2) of the ring $R(\sigma)$. Since the elements of the Hilbert basis $\mathscr{H}_\sigma$ cannot be written in a nontrivial way as $\mathbb{N}$-linear combinations of elements in the semigroup $C(\sigma)$ [Miller and Sturmfels 2005, proof of Proposition 7.15], we get that the ideal $I_\sigma = \ker \pi_\sigma$ satisfies

$$I_\sigma \subset \mathfrak{n}^2, \tag{4-9}$$

where $\mathfrak{n} := (V_\alpha : \alpha \in \mathscr{H}_\sigma) \subset k[V_\alpha : \alpha \in \mathscr{H}_\sigma]$. Part (i) now follows from (4-2) and (4-9).

In order to prove part (ii), consider the presentation (4-4) of the ring $R(\mathscr{F})$. It is enough to prove that the ideal $I_{\mathscr{F}} = \ker \pi_{\mathscr{F}}$ satisfies

$$I_{\mathscr{F}} \subset \mathfrak{o}^2, \tag{4-10}$$

where $\mathfrak{o} := (V_\alpha : \alpha \in \mathscr{H}_{\mathscr{F}}) \subset k[V_\alpha : \alpha \in \mathscr{H}_{\mathscr{F}}]$. Consider the generators of $I_{\mathscr{F}}$ given in Proposition 4.5. Clearly the generators of the form $V_\alpha V_{\alpha'}$ (for $\{\alpha, \alpha'\} \notin \Delta_{\mathscr{F}}$) belong to $\mathfrak{o}^2$. In order to deal with the other generators of $I_{\mathscr{F}}$, consider the diagram (4-5). As in the discussion that precedes Proposition 4.5, we view I_σ as included in $I_{\mathscr{F}}$ via the section s. By applying the section s to the inclusion (4-9) and using the obvious inclusion $s(\mathfrak{n}^2) \subseteq \mathfrak{o}^2$, we get the desired inclusion (4-10). □

4G. *Multiplicity.* In this subsection, we study the multiplicity $e_{\mathfrak{m}}(R(\mathscr{F}))$ of $R(\mathscr{F})$ at its unique graded maximal ideal $\mathfrak{m}$.

Recall (see, e.g., [Serre 1965, Chapter IIB, Theorem 3]) that the Hilbert–Samuel function

$$n \mapsto \dim_k R(\mathscr{F})/\mathfrak{m}^n$$

is given, for large values of $n \in \mathbb{N}$, by a polynomial (called the Hilbert–Samuel polynomial) that is denoted by $P_{\mathfrak{m}}(R(\mathscr{F}); n)$. The degree of $P_{\mathfrak{m}}(R(\mathscr{F}); n)$ is equal to $\dim R(\mathscr{F})$ [Serre 1965, Chapter IIIB, Theorem 1]. We can therefore write

$$P_{\mathfrak{m}}(R(\mathscr{F}); n) = e_{\mathfrak{m}}(R(\mathscr{F})) \frac{n^{\dim R(\mathscr{F})}}{\dim R(\mathscr{F})!} + O(n^{\dim R(\mathscr{F})-1}),$$

where $O(n^t)$ denotes a polynomial of degree less than or equal to t and $e_{\mathfrak{m}}(R(\mathscr{F}))$ is, by definition, the multiplicity of $R(\mathscr{F})$ at $\mathfrak{m}$ [Serre 1965, Chapter VA]. The following result is a special case of [Matsumura 1989, Theorem 14.7]:

Theorem 4.16. *If $\mathscr{F}$ is a fan of dimension d (i.e., such that the maximum of the dimension of the cones in $\mathscr{F}$ is d) in $\mathbb{R}^b$, then $R(\mathscr{F})$ has dimension d and its multiplicity is equal to*

$$e_{\mathfrak{m}}(R(\mathscr{F})) = \sum_{\dim \sigma = d} e_{\mathfrak{m}}(R(\sigma)),$$

where $\mathfrak{m}$ is the unique graded maximal ideal of the rings in question.

Proof. The theorem is the special case of [Matsumura 1989, Theorem 14.7], where $A = R(\mathscr{F})$ and $\mathfrak{q} = \mathfrak{m}$. Indeed, the rings $R(\sigma)$ are the localizations of $R(\mathscr{F})$ at minimal primes $\mathfrak{q}$ satisfying $\dim R(\mathscr{F})/\mathfrak{q} = d$ by Corollary 4.7. □

The above result reduces the computation of the multiplicity of $R(\mathscr{F})$ at $\mathfrak{m}$ (for a complete fan $\mathscr{F}$) to that of the affine semigroup rings $R(\sigma)$ at $\mathfrak{m}$ for σ a cone of $\mathscr{F}$ of maximal dimension. These latter multiplicities can be computed geometrically

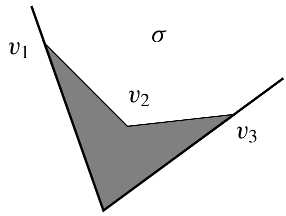

Figure 1. A two-dimensional cone σ whose associated semigroup $C(\sigma)$ has Hilbert basis $\mathscr{H}_\sigma = \{v_1, v_2, v_3\}$. The shaded region is the subdiagram part $K_-(C(\sigma))$ of $C(\sigma)$.

from the affine semigroup $C(\sigma)$ as we now explain following Gel′fand, Kapranov, and Zelevinsky [Gel′fand et al. 1994].

To that aim, we need to recall some definitions. Given a cone $\sigma \in \mathscr{F}$, set $C(\sigma)_{\mathbb{Z}} := \langle\sigma\rangle \cap H_{\mathbb{Z}}$ and $C(\sigma)_{\mathbb{R}} := \langle\sigma\rangle \cap H_{\mathbb{R}}$. We denote by $\mathrm{vol}_{C(\sigma)}$ the unique translation-invariant measure on $C(\sigma)_{\mathbb{R}}$ such that the volume of a standard unimodular simplex Δ (i.e., Δ is the convex hull of a basis of $H_{\mathbb{Z}}$ together with 0) is 1. Following [Gel′fand et al. 1994, p. 184], denote by $K_+(C(\sigma))$ the convex hull of the set $C(\sigma) \setminus \{0\}$ and $K_-(C(\sigma))$ the closure of $\sigma \setminus K_+(C(\sigma))$. The set $K_-(C(\sigma))$ is a bounded (possibly not convex) lattice polyhedron in $C(\sigma)_{\mathbb{R}}$ that is called the *subdiagram part* of $C(\sigma)$.

Definition 4.17 [Gel′fand et al. 1994, Chapter 5, Definition 3.8]. The *subdiagram volume* of $C(\sigma)$ is the natural number

$$u(C(\sigma)) := \mathrm{vol}_{C(\sigma)_{\mathbb{Z}}}(K_-(C(\sigma))).$$

The multiplicity of $R(\sigma)$ at $\mathfrak{m}$ can be computed in terms of the subdiagram volume of $C(\sigma)$ as asserted by the following result, whose proof can be found in [Gel′fand et al. 1994, Chapter 5, Theorem 3.14]:

Theorem 4.18. *The multiplicity of $R(\sigma)$ at $\mathfrak{m}$ is equal to*

$$e_{\mathfrak{m}}(R(\sigma)) = u(C(\sigma)).$$

5. Geometry of cographic rings

The aim of this section is to describe the properties of the cographic ring $R(\Gamma)$ associated to a graph Γ. The main results are Theorem 5.7 and the descriptions of the cographic ring in Section 5B. Recall from Definition 1.4 that $R(\Gamma)$ is the toric face ring associated to the cographic fan $\mathscr{F}_\Gamma^\perp$ in $H_1(\Gamma, \mathbb{R})$, which is a polytopal fan by Proposition 3.8.

According to Proposition 3.1(i), every cone of $\mathscr{F}_\Gamma^\perp$ is of the form

$$\sigma(T, \phi) := \bigcap_{e \notin T} \{(\,\cdot\,, \phi(e)) \geq 0\} \cap \bigcap_{e \in T} \{(\,\cdot\,, e) = 0\}$$

for some uniquely determined element $(T, \phi) \in \mathcal{O}\mathcal{P}_\Gamma$, i.e., a totally cyclic orientation ϕ on $\Gamma \setminus T$. We will denote the positive normal affine semigroup associated to $\sigma(T, \phi)$ as in (4-1) by

$$C(\Gamma \setminus T, \phi) := C(\sigma(T, \phi)) = \sigma(Y, \phi) \cap H_1(\Gamma, \mathbb{Z})$$

and its associated affine semigroup ring (as in Definition 4.2)

$$R(\Gamma \setminus T, \phi) := k[C(\Gamma \setminus T, \phi)].$$

5A. ***Affine semigroup rings $R(\Gamma \setminus T, \phi)$.*** Let us look more closely at the affine semigroup rings $R(\Gamma \setminus T, \phi)$ for a fixed $(T, \phi) \in \mathcal{O}\mathcal{P}_\Gamma$.

The ring $R(\Gamma \setminus T, \phi)$ is a normal, Cohen–Macaulay domain of dimension equal to $\dim \sigma(T, \phi) = b_1(\Gamma \setminus T)$ as follows from Lemma 4.3 and Proposition 3.1(ii). However, the ring $R(\Gamma \setminus T, \phi)$ need not be Gorenstein and indeed not even $\mathbb{Q}$-Gorenstein as the following example shows:

Example 5.1. Consider the totally cyclic oriented graph (Γ, ϕ) depicted in Figure 2.

Consider the pointed rational polyhedral cone $\sigma(\varnothing, \phi) \subset H_1(\Gamma, \mathbb{R})$ and its dual cone $\sigma(\varnothing, \phi)^\vee \subset H_1(\Gamma, \mathbb{R})^\vee$ defined by

$$\sigma(\varnothing, \phi)^\vee := \{\ell \in H_1(\Gamma, \mathbb{R})^\vee : \ell(v) \geq 0 \text{ for every } v \in \sigma(\varnothing, \phi)\}.$$

Since for any edge $e \in E(\Gamma)$, the graph $\Gamma \setminus \{e\}$ with the orientation induced by ϕ is totally cyclic, we get that the cone $\sigma(\varnothing, \phi)$ has five codimension-1 faces defined by the equations $\{(\,\cdot\,, \phi(e_i)) = 0\}$ for $i = 1, \ldots, 5$ (see Corollary 3.2). This implies that the extremal rays of $\sigma(\varnothing, \phi)^\vee$ are the rays generated by $(\,\cdot\,, \phi(e_i))$ for $i = 1, \ldots, 5$.

It follows from [Dais 2002, proof of Theorem 3.12] that $R(\Gamma, \phi)$ is $\mathbb{Q}$-Gorenstein if and only if there exists an element $m \in H_1(\Gamma, \mathbb{Q})$ such that $(m, \phi(e_i)) = 1$ for every

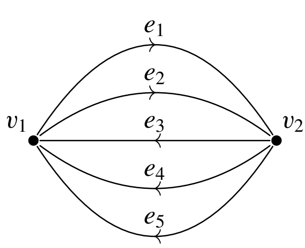

Figure 2. A totally cyclic oriented graph (Γ, ϕ) with $R(\Gamma, \phi)$ not $\mathbb{Q}$-Gorenstein.

$i = 1, \dots, 5$. However, these conditions force m to be equal to $m = \sum_{i=1}^{5} \phi(e_i)$, which is a contradiction since $\partial\bigl(\sum_{i=1}^{5} \phi(e_i)\bigr) = v_1 - v_2 \neq 0$.

Denote by $\mathcal{H}_{(\Gamma \setminus T, \phi)}$ the Hilbert basis (i.e., the minimal generating set) of the positive affine normal semigroup $C(\Gamma \setminus T, \phi)$. From Lemma 2.4, we get the following explicit description of $\mathcal{H}_{(\Gamma \setminus T, \phi)}$:

Proposition 5.2. *The Hilbert basis of $C(\Gamma \setminus T, \phi)$ is equal to*

$$\mathcal{H}_{(\Gamma \setminus T, \phi)} := \{[\gamma] : \gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)\} \subset H_1(\Gamma \setminus T, \mathbb{Z}) \subseteq H_1(\Gamma, \mathbb{Z}).$$

The Hilbert basis $\mathcal{H}_{(\Gamma \setminus T, \phi)}$ of $C(\Gamma \setminus T, \phi)$ enjoys the following remarkable properties:

Lemma 5.3. *Let $(T, \phi) \in \mathcal{OP}_\Gamma$.*

(i) *The group $\mathbb{Z} \cdot \mathcal{H}_{(\Gamma \setminus T, \phi)} \subseteq H_1(\Gamma \setminus T, \mathbb{Z})$ generated by $\mathcal{H}_{(\Gamma \setminus T, \phi)}$ coincides with $H_1(\Gamma \setminus T, \mathbb{Z})$.*

(ii) *The ray $\mathbb{R}_{\geq 0} \cdot [\gamma]$ is extremal for the cone $\sigma(T, \phi) = \mathbb{R}_{\geq 0} \cdot \mathcal{H}_{(\Gamma \setminus T, \phi)}$ for each $[\gamma] \in \mathcal{H}_{(\Gamma \setminus T)}$.*

Proof. Part (i) follows from Lemma 2.3(2)(c). Part (ii) follows from Proposition 3.1(iii). □

We warn the reader that the Hilbert basis $\mathcal{H}_{(\Gamma \setminus T, \phi)}$ need not be unimodular as we show in Example 5.4 below. Recall that a subset $\mathcal{A} \subset \mathbb{Z}^d$ is said to be *unimodular* if $\mathcal{A}$ spans $\mathbb{R}^d$ and, moreover, if we represent the elements of $\mathcal{A}$ as column vectors of a matrix A with respect to a basis of $\mathbb{Z}^d$, then all the nonzero $d \times d$ minors of A have the same absolute value [Sturmfels 1996, p. 70].

Example 5.4. Consider the totally cyclic oriented graph (Γ, ϕ) depicted in Figure 3.

One can check that $b_1(\Gamma) = 4$ and that $\mathcal{H}_{(\Gamma, \phi)}$ consists of the eight elements

$$[\gamma_{ijk}] = \phi(e_i^1) + \phi(e_j^2) + \phi(e_k^3)$$

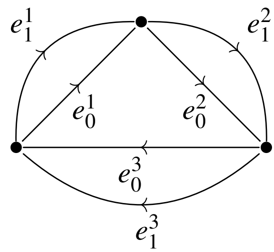

Figure 3. A totally cyclic oriented graph (Γ, ϕ) with $\mathcal{H}_{(\Gamma, \phi)}$ not totally unimodular.

for $i, j, k \in \{0, 1\}$. The elements $\mathcal{B} := \{[\gamma_{000}], [\gamma_{100}], [\gamma_{010}], [\gamma_{001}]\}$ form a basis of $H_1(\Gamma, \mathbb{Z})$. If we order the elements of $\mathcal{H}_{(\Gamma,\phi)}$ as

$$\{[\gamma_{000}], [\gamma_{100}], [\gamma_{010}], [\gamma_{001}], [\gamma_{110}], [\gamma_{101}], [\gamma_{011}], [\gamma_{111}]\},$$

then the elements of $\mathcal{H}_{(\Gamma,\phi)}$, with respect to the basis $\mathcal{B}$, are the column vectors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The minor A_{1234} (i.e., the minor corresponding to the first four columns) is equal to 1 while the minor A_{2348} is equal to 2; hence, $\mathcal{H}_{(\Gamma,\phi)}$ is not unimodular.

According to (4-2) and (4-3), the affine semigroup ring $R(\Gamma \setminus T, \phi)$ admits the presentation

$$R(\Gamma \setminus T, \phi) := \frac{k[V_\gamma : \gamma \in \mathrm{Cir}_\phi(\Gamma \setminus T)]}{I_{(\Gamma \setminus T, \phi)}}, \tag{5-1}$$

where $I_{(\Gamma \setminus T,\phi)} := I_{\sigma(T,\phi)}$ is a binomial ideal, called the toric ideal associated to $\mathcal{H}_{(\Gamma \setminus T,\phi)}$ in the terminology of [Sturmfels 1996, Chapter 4]. The following problem seems interesting:

Problem 5.5. *Find generators for the binomial toric ideal $I_{(\Gamma \setminus T,\phi)}$.*

We warn the reader that the toric ideal $I_{(\Gamma \setminus T,\phi)}$ need not to be homogeneous as shown by the following example:

Example 5.6. Consider the totally cyclicly oriented graph (Γ, ϕ) depicted in Figure 4.

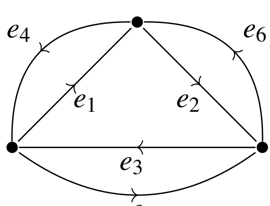

Figure 4. A totally cyclic oriented graph (Γ, ϕ) with $I_{(\Gamma,\phi)}$ not homogeneous.

It is easy to see that $b_1(\Gamma) = 4$ and that $\mathscr{H}_{(\Gamma,\phi)}$ consists of the five elements

$$\begin{aligned}
[\gamma_1] &:= \phi(e_1) + \phi(e_4),\\
[\gamma_2] &:= \phi(e_2) + \phi(e_6),\\
[\gamma_3] &:= \phi(e_3) + \phi(e_5),\\
[\gamma_4] &:= \phi(e_1) + \phi(e_2) + \phi(e_3),\\
[\gamma_5] &:= \phi(e_4) + \phi(e_4) + \phi(e_6).
\end{aligned}$$

The binomial ideal $I_{(\Gamma,\phi)}$ is generated by $V_{\gamma_1} V_{\gamma_2} V_{\gamma_3} - V_{\gamma_4} V_{\gamma_5}$; hence, it is not homogeneous.

5B. ***Descriptions of $R(\Gamma)$ as an inverse limit and as a quotient.*** Using the general results of Section 4A, the ring $R(\Gamma)$ admits two explicit descriptions.

The first description of $R(\Gamma)$ is as an inverse limit of affine semigroup rings (see Proposition 4.4):

$$R(\Gamma) = \varprojlim_{(T,\phi)\in\mathbb{O}\mathscr{P}_\Gamma} R(\Gamma \setminus T, \phi). \tag{5-2}$$

The second description is a presentation of $R(\Gamma)$ as a quotient of a polynomial ring. In order to make this explicit for $R(\Gamma)$, observe first that the union of all the Hilbert bases of the cones $\sigma(T,\phi)$, as (T,ϕ) varies in $\mathbb{O}\mathscr{P}_\Gamma$, is equal to the set of all oriented circuits of Γ, i.e.,

$$\mathscr{H}_{\mathscr{F}_\Gamma^\perp} = \overrightarrow{\mathrm{Cir}}(\Gamma). \tag{5-3}$$

Moreover, Corollary 3.4 implies that the simplicial complex $\Delta_{\mathscr{F}^\perp}$ introduced in Section 4A coincides with the simplicial complex $\Delta(\overrightarrow{\mathrm{Cir}}(\Gamma))$ of concordant circuits as in Definition 2.5, or in symbols,

$$\Delta_{\mathscr{F}^\perp} = \Delta(\overrightarrow{\mathrm{Cir}}(\Gamma)).$$

From (4-4), Proposition 4.5, and Lemma 2.9, we get the presentation of $R(\Gamma)$

$$R(\Gamma) = \frac{k[V_\gamma : \gamma \in \overrightarrow{\mathrm{Cir}}(\Gamma)]}{I_\Gamma}, \tag{5-4}$$

where $I_\Gamma := I_{\mathscr{F}_\Gamma^\perp}$ is explicitly given by

$$I_\Gamma = (V_\gamma V_{\gamma'} : \gamma \not\asymp \gamma') + \sum_{(T,\phi)\in\mathbb{O}\mathscr{P}_\Gamma} I_{(\Gamma\setminus T,\phi)} = (V_\gamma V_{\gamma'} : \gamma \not\asymp \gamma') + \sum_{(E(\Gamma)_{\mathrm{sep}},\phi)\in\mathbb{O}\mathscr{P}_\Gamma} I_{(\Gamma\setminus E(\Gamma)_{\mathrm{sep}},\phi)}. \tag{5-5}$$

From Proposition 4.6, we get that the graded prime ideals of $R(\Gamma)$ are given by

$$\mathfrak{p}_{(T,\phi)} := (\{X^c : c \notin \sigma(T,\phi)\}) \tag{5-6}$$

as (T,ϕ) varies in $\mathbb{O}\mathscr{P}_\Gamma$.

5C. *Singularities of $R(\Gamma)$.* In this subsection, we analyze the singularities of the ring $R(\Gamma)$.

Theorem 5.7. *Let Γ be a graph and $R(\Gamma)$ its associated cographic ring. Then we have the following*:

(i) *$R(\Gamma)$ is a reduced finitely generated k-algebra of pure dimension equal to $b_1(\Gamma)$. The minimal prime ideals of $R(\Gamma)$ are given by $\mathfrak{p}_{(E(\Gamma)_{\mathrm{sep}},\phi)}$ as ϕ varies among all the totally cyclic orientations of $\Gamma \setminus E(\Gamma)_{\mathrm{sep}}$.*

(ii) *$R(\Gamma)$ is Gorenstein, and its canonical module $\omega_{R(\Gamma)}$ is isomorphic to $R(\Gamma)$ as a graded module.*

(iii) *$R(\Gamma)$ is a seminormal ring.*

(iv) *The normalization of $R(\Gamma)$ is equal to*

$$\overline{R(\Gamma)} = \prod_{\phi} R(\Gamma \setminus E(\Gamma)_{\mathrm{sep}}, \phi),$$

where the product is over all the totally cyclic orientations ϕ of $E(\Gamma) \setminus E(\Gamma)_{\mathrm{sep}}$.

(v) *The variety* Spec $R(\Gamma)$ *is slc.*

(vi) *The embedded dimension of $R(\Gamma)$ at $\mathfrak{m}$ is equal to the cardinality of $\overrightarrow{\mathrm{Cir}}(\Gamma)$, the set of oriented circuits on Γ.*

(vii) *The multiplicity of $R(\Gamma)$ at $\mathfrak{m}$ is equal to*

$$e_{\mathfrak{m}}(R(\Gamma)) = \sum_{\phi} e_{\mathfrak{m}}(R(\Gamma \setminus E(\Gamma)_{\mathrm{sep}}, \phi)) = \sum_{\phi} u(C(\Gamma \setminus E(\Gamma)_{\mathrm{sep}}, \phi)),$$

where the sum is over all the totally cyclic orientations ϕ of $\Gamma \setminus E(\Gamma)_{\mathrm{sep}}$ and $\mathfrak{m}$ is the unique graded maximal ideal of the rings in question.

Proof. Part (i) follows from Remark 1.3, Corollary 4.7, and Lemma 2.9. Part (ii) follows Theorem 4.10 using that $\mathscr{F}_{\Gamma}^{\perp}$ is a polytopal fan by Proposition 3.8. Part (iii) follows from Theorem 4.12(ii). Part (iv) follows from Theorem 4.12(i) and Lemma 2.9. Part (v) follows from Theorem 4.14 using that $\mathscr{F}_{\Gamma}^{\perp}$ is polytopal. Part (vi) follows from Theorem 4.15(ii) and (5-3). Part (vii) follows from Theorem 4.16, Theorem 4.18, and Lemma 2.9. □

Problem 5.8. *Express the multiplicity of $R(\Gamma)$ at $\mathfrak{m}$ in terms of well-known graph invariants.*

Problem 5.9. *Characterize the graphs Γ that have the property that* Spec$(R(\Gamma))$ *is semi divisorial log terminal.* (*See* [*Fujino 2000, Definition 1.1*] *for the definition of semi divisorial log terminal.*)

Problem 5.9 is motivated by moduli theory. The singularities of $R(\Gamma)$ are the singularities that appear on compactified Jacobians, and compactified Jacobians arise as limits of abelian varieties. Fujino [2011] shows that, in a suitable sense, it is possible to degenerate an abelian variety to a semi divisorial log terminal variety. If $R(\Gamma)$ is semi divisorial log terminal, then compactified Jacobians are examples of Fujino's degenerations. For a general discussion of singularities and their role in moduli theory, we direct the reader to [Kollár 2010].

Following the proof of Theorem 4.14, Problem 5.9 is equivalent to the following one: characterize the totally cyclic orientations ϕ of a graph Γ that have the property that the pair $(\operatorname{Spec} R(\Gamma,\phi), -K_{R(\Gamma,\phi)})$ is divisorial log terminal (in the sense of [Kollár and Mori 1998]). Note that the pair $(\operatorname{Spec} R(\Gamma,\phi), -K_{R(\Gamma,\phi)})$ does not satisfy the stronger condition of being Kawamata log terminal (and so $\operatorname{Spec} R(\Gamma)$ is not semi Kawamata log terminal) because $-K_{R(\Gamma,\phi)}$ is effective and nonzero.

6. The cographic ring $R(\Gamma)$ as a ring of invariants

In [Casalaina-Martin et al. 2011], the completion of the ring $R(\Gamma)$ with respect to the maximal ideal $\mathfrak{m} = \mathfrak{p}_{\underline{0}}$ appears naturally as a ring of invariants. In this section, we explain this connection. Consider the multiplicative group

$$T_\Gamma := \prod_{v \in V(\Gamma)} \mathbb{G}_m.$$

The elements of $T_\Gamma(S)$ for a k-scheme S can be written as $\lambda = (\lambda_v)_{v\in V(\Gamma)}$ with $\lambda_v \in \mathbb{G}_m(S) = \mathbb{O}_S^*$.

Consider the ring

$$A(\Gamma) := \frac{k[U_{\overleftarrow{e}}, U_{\vec{e}} : e \in E(\Gamma)]}{(U_{\overleftarrow{e}} U_{\vec{e}} : e \in E(\Gamma))}.$$

If we make the group T_Γ act on $A(\Gamma)$ via

$$\lambda \cdot U_{\vec{e}} = \lambda_{s(\vec{e})} U_{\vec{e}} \lambda_{t(\vec{e})}^{-1},$$

then the invariant subring is described by the following theorem:

Theorem 6.1. *The invariant subring $A(\Gamma)^{T_\Gamma}$ is isomorphic to the cographic toric ring $R(\Gamma)$.*

Proof. We prove the theorem by exhibiting a k-basis for the invariant subring that is indexed by $H_1(\Gamma, \mathbb{Z})$ in such a way that multiplication satisfies Equation (1-1). We argue as follows. Grade $A(\Gamma)$ by the $\vec{C}_1(\Gamma, \mathbb{Z})$-grading induced by the obvious grading of $k[U_{\overleftarrow{e}}, U_{\vec{e}} : e \in E(\Gamma)]$ (so the weight of $U_{\vec{e}}$ is $\vec{e}$).

This grading is preserved by the action of T_Γ on $A(\Gamma)$, so the invariant subring is generated by invariant homogeneous elements. Furthermore, given a homogeneous element $M^c = \prod U_{\vec{e}}^{a(\vec{e})}$ of weight $c = \sum a(\vec{e})\vec{e}$, an element $\lambda \in T_\Gamma$ acts as

$$\lambda \cdot M^c = \prod_{\vec{e}} \lambda_{s(\vec{e})} U_{\vec{e}} \lambda_{t(\vec{e})}^{-1} = \Big(\prod_v \lambda_v^{b(v)}\Big) M^c,$$

where $b(v)$ is defined by $\partial(c) = \sum b(v)v$. In particular, we see that M^c is invariant if and only if $\partial(c) = 0$, or in other words, $c \in H_1(\Gamma, \mathbb{Z})$.

We can conclude that the invariant subring is generated by the homogeneous elements M^c whose weight c lies in $H_1(\Gamma, \mathbb{Z})$. In fact, these elements freely generate the invariant subring because distinct elements have distinct weights.

To complete the proof, observe that multiplication satisfies

$$M^c \cdot M^{c'} = \begin{cases} 0 & \text{if } (c, \vec{e}) > 0 \text{ and } (c', \vec{e}) < 0 \text{ for some } \vec{e}, \\ M^{c+c'} & \text{otherwise.} \end{cases} \tag{6-1}$$

The condition that there exists an oriented edge $\vec{e}$ with $(c, \vec{e}) > 0$ and $(c', \vec{e}) < 0$ is equivalent to the condition that c and c' do not lie in a common cone by Corollary 3.4. We can conclude that the rule $X^c \mapsto M^c$ defines an isomorphism between the cographic ring $R(\Gamma)$ and the invariant subring of $A(\Gamma)$. □

7. A Torelli-type result for $R(\Gamma)$

In this section, we investigate when two graphs give rise to the same cographic toric face ring. Before stating the result, we need to briefly recall some operations in graph theory introduced in [Caporaso and Viviani 2010, §2]. Two graphs Γ and Γ' are said to be *cyclic equivalent* (or 2-isomorphic) if there exists a bijection $\epsilon : E(\Gamma) \to E(\Gamma')$ inducing a bijection on the circuits. The cyclic equivalence class of Γ is denoted by $[\Gamma]_{\text{cyc}}$. Given a graph Γ, a 3-*edge connectivization* of Γ is a graph that is obtained from Γ by contracting all the separating edges of Γ and by contracting, for every separating pair of edges, one of the two edges. While a 3-edge connectivization of Γ is not unique (because of the freedom that we have in performing the second operation), its cyclic equivalence class is well-defined; it is called the 3-*edge connected class* of Γ and denoted by $[\Gamma]^3_{\text{cyc}}$.

Theorem 7.1. *Let Γ and Γ' be two graphs. Then $R(\Gamma) \cong R(\Gamma')$ if and only if $[\Gamma]^3_{\text{Cyc}} = [\Gamma']^3_{\text{Cyc}}$.*

Proof. Assume first that $[\Gamma]^3_{\text{Cyc}} = [\Gamma']^3_{\text{Cyc}}$. From [Caporaso and Viviani 2010, proof of Proposition 3.2.3], it follows that $\mathscr{C}_\Gamma^\perp \cong \mathscr{C}_{\Gamma'}^\perp$, i.e., that there exists an $\mathbb{R}$-linear isomorphism $\phi : H_1(\Gamma, \mathbb{R}) \to H_1(\Gamma', \mathbb{R})$ that sends $H_1(\Gamma, \mathbb{Z})$ isomorphically onto $H_1(\Gamma', \mathbb{Z})$ and such that ϕ sends the hyperplanes of $\mathscr{C}_\Gamma^\perp$ bijectively onto the hyperplanes of $\mathscr{C}_{\Gamma'}^\perp$. Since $\mathscr{F}_\Gamma^\perp$ is the fan induced by the arrangement of hyperplanes $\mathscr{C}_\Gamma^\perp$, the above map ϕ will send the cones of $\mathscr{F}_\Gamma^\perp$ bijectively onto the cones of $\mathcal{F}_{\Gamma'}^{\perp}$. Therefore, the map

$$R(\Gamma) \to R(\Gamma'), \quad X^c \mapsto X^{\phi(c)}$$

is an isomorphism of rings.

Conversely, if $R(\Gamma) \cong R(\Gamma')$, then clearly $\mathrm{Str}(R(\Gamma)) \cong \mathrm{Str}(R(\Gamma'))$ (see Definition 4.8). By Corollary 4.9, we deduce that $\mathcal{OP}_\Gamma \cong \mathcal{OP}_{\Gamma'}$, which implies that $[\Gamma]^3_{\mathrm{Cyc}} = [\Gamma']^3_{\mathrm{Cyc}}$ by [Caporaso and Viviani 2010, Theorem 5.3.2]. □

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casa@math.colorado.edu *Department of Mathematics, University of Colorado Boulder, Campus Box 395, Boulder, CO, 80309-0395, United States*
http://math.colorado.edu/~sbc21/

kass@math.uni-hannover.de *Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany*
http://www2.iag.uni-hannover.de/~kass/

filippo.viviani@gmail.com *Dipartimento di Matematica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo 1, 00146 Roma, Italy*
http://ricerca.mat.uniroma3.it/users/viviani/

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Essential p-dimension of algebraic groups whose connected component is a torus

Roland Lötscher, Mark MacDonald, Aurel Meyer and Zinovy Reichstein

Following up on our earlier work and the work of N. Karpenko and A. Merkurjev, we study the essential p-dimension of linear algebraic groups G whose connected component G^0 is a torus.

1. Introduction

Let p be a prime integer and k a base field of characteristic not equal to p. In this paper, we will study the essential p-dimension of linear algebraic k-groups G whose connected component G^0 is an algebraic torus. This is a natural class of groups; for example, normalizers of maximal tori in reductive linear algebraic groups are of this form. This paper is a sequel to [Lötscher et al. 2013], where G was assumed to be of multiplicative type. For background material and further references on the notion of essential dimension, see [Reichstein 2011].

For the purpose of computing $\mathrm{ed}(G; p)$, we may replace the base field k by any field extension whose degree is finite and prime to p. (We will sometimes refer to such field extensions as *prime-to-p extensions*.) In particular, after passing to a suitable prime-to-p extension of k, we may assume that k contains a primitive p-th root of unity ζ_p and that there is a field extension l/k whose degree is a power of p such that (i) the torus $T := G^0$ becomes split and (ii) the étale group G/G^0 becomes constant over l. In this situation, the finite group G/G^0 has a Sylow p-subgroup F defined over k; see [Lötscher et al. 2013, Remark 7.2]. Since G is smooth, we may replace G by the preimage of F without changing its essential p-dimension; see [Meyer and Reichstein 2009, Lemma 4.1]. It is thus natural to restrict our attention to the case where $F := G/G^0$ is a finite p-group. In view of this, we will make the

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following assumptions on k and G for the remainder of this section and throughout much of the rest of the paper:

Notational conventions 1.1. Unless otherwise specified, k will denote a field of characteristic not equal to p containing a primitive p-th root of unity ζ_p, and G will denote an algebraic k-group that fits into an exact sequence

$$1 \to T \to G \xrightarrow{\pi} F \to 1 \tag{1-1}$$

of k-groups, where $T := G^0$ is a torus and $F := G/G^0$ is a finite p-group. Moreover, we will assume that there is a field extension of k of p-power degree over which T becomes split and F becomes constant. Note that F may be twisted (i.e., nonconstant) and T may be nonsplit over k. The extension (1-1) is not assumed to be split (not even over the algebraic closure of k).

To state our main result, we recall that a linear representation $\rho : G \to \mathrm{GL}(V)$ is called *generically free* if there exists a G-invariant dense open subset $U \subseteq V$ such that the scheme-theoretic stabilizer of every point of U is trivial. We will say that ρ is *p-faithful* if $\ker \rho$ is finite of order prime to p. We will say that ρ is *p-generically free* if it is p-faithful and gives rise to a generically free representation of $G/\ker \rho$.

A generically free representation is faithful, but a faithful representation may not be generically free. This phenomenon is not well understood; there is no classification of such representations, and we do not even know for which groups G they occur.[1] It is, however, the source of many of the subtleties we will encounter.

Theorem 1.2. *Let G be an algebraic k-group satisfying Conventions 1.1. Then*

$$\min \dim \rho - \dim G \le \mathrm{ed}(G;\, p) \le \min \dim \mu - \dim G,$$

where the minima are taken respectively over all p-faithful representations ρ of G and p-generically free representations μ of G.

As a simple example, let $k = \mathbb{C}$, $p = 2$ and $G = \mathrm{O}_2 \simeq \mathrm{SO}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ be the group of 2×2 orthogonal matrices, where $G^0 = \mathrm{SO}_2 \simeq \mathbb{G}_m$ is a one-dimensional torus. The natural representation $i : G \hookrightarrow \mathrm{GL}_2$ is faithful but not generically free: if $a^2 + b^2 \neq 0$, then the stabilizer of $v = (a, b) \in \mathbb{C}^2$ is the subgroup of $G = \mathrm{O}_2$ of order 2 generated by the reflection in the line spanned by v. It is easy to see that no two-dimensional representation of O_2 is 2-generically free, but the three-dimensional representation $i \oplus \det$ is generically free. (Here $\det : \mathrm{O}_2 \to \mathrm{GL}_1$ is the determinant.) Theorem 1.2 thus yields $1 \le \mathrm{ed}(\mathrm{O}_2;\, 2) \le 2$. The true value of $\mathrm{ed}(\mathrm{O}_2;\, 2)$ is 2; see [Reichstein 2000, Theorem 10.3].

In general, let us denote the difference between the upper and lower bounds of Theorem 1.2 by $\mathrm{gap}(G;\, p)$. If $G = G^0$ is a torus or $G = F$ is a finite p-group, then

[1] Faithful representations that are not generically free are better understood for connected semisimple groups; see [Vinberg and Popov 1994, Section 7].

$\mathrm{gap}(G;\, p) = 0$ (see [Lötscher et al. 2013, Lemma 2.5; Meyer and Reichstein 2009, Remark 2.1]), and Theorem 1.2 reduces to [Lötscher et al. 2013, Theorems 1.1 and 7.1], respectively. (The case where $G = F$ is a constant finite p-group is due to Karpenko and Merkurjev [2008], whose work was the starting point for both [Lötscher et al. 2013] and the present paper.) We will show that the upper and lower bounds of Theorem 1.2 coincide for a larger class of groups, which we call *tame*; see Definition 7.3 and Corollary 7.4. More generally, we will show:

Theorem 1.3. *Let G be an algebraic k-group satisfying Conventions 1.1. Then* $\mathrm{gap}(G;\, p) \le \dim T - \dim T^{C(F)}$.

Here $C(F)$ is the central p-subgroup of F defined in Section 4, the F-action on T is induced by conjugation in G, and $T^{C(F)} \subseteq T$ denotes the subgroup of elements fixed by $C(F)$.

Our second main result about $\mathrm{gap}(G;\, p)$ is the following "additivity theorem":

Theorem 1.4. *Let G_1 and G_2 be algebraic k-groups satisfying Conventions 1.1. If* $\mathrm{gap}(G_1;\, p) = \mathrm{gap}(G_2;\, p) = 0$, *then* $\mathrm{gap}(G_1 \times G_2;\, p) = 0$, *and* $\mathrm{ed}(G_1 \times G_2;\, p) = \mathrm{ed}(G_1;\, p) + \mathrm{ed}(G_2;\, p)$.

The rest of this paper is structured as follows. In Section 2, we discuss the notion of p-special closure $k^{(p)}$ of a field k and show that passing from k to $k^{(p)}$ does not change the essential p-dimension of any k-group. In Section 3, we show that if $A \to B$ is an isogeny of degree prime to p, then A and B have the same essential p-dimension. Sections 4, 5 and 6 are devoted to the proof of our main Theorem 1.2. In Section 7, we introduce the class of tame groups and show that for these groups the upper and the lower bounds of Theorem 1.2 coincide. In Section 8, we prove Theorem 1.3, and in Section 9, we prove Theorem 1.4. In Section 10, we classify central extensions (1-1) with G of small essential p-dimension.

2. The p-special closure of a field

Let K be an arbitrary field and p be a prime integer. We will denote the algebraic and separable closures of K by K_{alg} and K_{sep}, respectively. Recall that K is called *p-special* if the degree of every finite extension of K is a power of p.

Lemma 2.1. *A field K is p-special if and only if it has no nontrivial prime-to-p extensions.*

Proof. We need to show that if K has no nontrivial prime-to-p extensions, then the degree of every finite extension L/K is a power of p. After passing to the normal closure, we may assume that L is normal over K. Now L/K is generated by a separable extension L_s/K and a purely inseparable extension L_i/K; see [Lang 1965, Proposition VII.7.12]. Hence, it suffices to show that $[L : K]$ is a power of p if (i) L/K is separable or (ii) L/K is purely inseparable.

(i) As above, we may assume that L/K is normal, i.e., Galois. Let Γ_p be a p-Sylow subgroup of $\Gamma = \mathrm{Gal}(L/K)$. Then L^{Γ_p}/K is a prime-to-p extension. Hence, $L^{\Gamma_p} = K$, i.e., $\Gamma = \Gamma_p$, and $[L:K] = |\Gamma|$ is a power of p.

(ii) If $\mathrm{char}(K) \neq p$, a purely inseparable extension L/K is prime-to-p and hence trivial. If $\mathrm{char}(K) = p$, then $[L:K]$ is a power of p. □

By [Elman et al. 2008, Proposition 101.16] for every field K, there exists an algebraic field extension L/K such that L is p-special and every finite subextension of L/K has degree prime to p. Such a field L is called a p-special closure of K and will be denoted by $K^{(p)}$.

The following properties of p-special closures will be important for us in the sequel:

Lemma 2.2. *Let K be a field and K_{alg} an algebraic closure of K containing $K^{(p)}$.*

(a) *$K^{(p)}$ is a direct limit of prime-to-p extensions K_i/K.*

(b) *The field $K^{(p)}$ is perfect if* $\mathrm{char}\, K \neq p$.

(c) *Suppose* $\mathrm{char}\, K \neq p$. *For any prime $q \neq p$, the cohomological q-dimension of $\Psi = \mathrm{Gal}(K_{\mathrm{alg}}/K^{(p)})$ is* $\mathrm{cd}_q(\Psi) = 0$.

Proof. (a) The finite subextensions K'/K of $K^{(p)}/K$ form a direct system with limit $K^{(p)}$. (b) Every finite extension of $K^{(p)}$ has p-power degree and is therefore separable. (c) By construction, Ψ is a profinite p-group. The result follows from [Serre 2002, Corollary 2, I.3]. □

Let l be a base field, Fields $/l$ be the category of field extensions of l and Sets be the category of sets. We call a covariant functor $\mathcal{F}:\text{Fields}/l \to \text{Sets}$ *limit-preserving* if, for any directed system of fields $\{K_i\}$, $\mathcal{F}(\varinjlim K_i) = \varinjlim \mathcal{F}(K_i)$. For example, if A is an algebraic group, the functor $\mathcal{F}(K) = H^1(K, A)$ is limit-preserving; see [Margaux 2007, 2.1].

Lemma 2.3. *Let $\mathcal{F}$ be limit-preserving and $\alpha \in \mathcal{F}(K)$ an object. Denote the image of α in $\mathcal{F}(K^{(p)})$ by $\alpha_{K^{(p)}}$. Then*:

(a) $\mathrm{ed}_{\mathcal{F}}(\alpha; p) = \mathrm{ed}_{\mathcal{F}}(\alpha_{K^{(p)}}; p) = \mathrm{ed}_{\mathcal{F}}(\alpha_{K^{(p)}})$.

(b) $\mathrm{ed}(\mathcal{F}; p) = \mathrm{ed}(\mathcal{F}_{l^{(p)}}; p)$, *where* $\mathcal{F}_{l^{(p)}}:\text{Fields}/l^{(p)} \to \text{Sets}$ *denotes the restriction of $\mathcal{F}$ to* Fields $/l^{(p)}$.

Proof. (a) It is clear that $\mathrm{ed}_{\mathcal{F}}(\alpha; p) \geq \mathrm{ed}_{\mathcal{F}}(\alpha_{K^{(p)}}; p) = \mathrm{ed}_{\mathcal{F}}(\alpha_{K^{(p)}})$ for any functor $\mathcal{F}$. It remains to prove $\mathrm{ed}_{\mathcal{F}}(\alpha; p) \leq \mathrm{ed}_{\mathcal{F}}(\alpha_{K^{(p)}})$. If L/K is finite of degree prime to p,

$$\mathrm{ed}_{\mathcal{F}}(\alpha; p) = \mathrm{ed}_{\mathcal{F}}(\alpha_L; p); \tag{2-1}$$

cf. [Merkurjev 2009, Proposition 1.5] and its proof. For the p-special closure $K^{(p)}$, this is similar and uses (2-1) repeatedly.

Suppose there is a subfield $K_0 \subseteq K^{(p)}$ and $\alpha_{K^{(p)}}$ comes from an element $\beta \in \mathcal{F}(K_0)$ so that $\beta_{K^{(p)}} = \alpha_{K^{(p)}}$. Write $K^{(p)} = \varinjlim \mathcal{L}$, where $\mathcal{L}$ is a direct system of finite prime-to-p extensions of K. Then $K_0 = \varinjlim \mathcal{L}_0$ with $\mathcal{L}_0 = \{L \cap K_0 \mid L \in \mathcal{L}\}$, and by assumption on $\mathcal{F}$, we have $\mathcal{F}(K_0) = \varinjlim_{L' \in \mathcal{L}_0} \mathcal{F}(L')$.

Thus, there is a field $L' = L \cap K_0$ ($L \in \mathcal{L}$) and $\gamma \in \mathcal{F}(L')$ such that $\gamma_{K_0} = \beta$. Since α_L and γ_L become equal over $K^{(p)}$, after possibly passing to a finite extension, we may assume they are equal over L, which is finite of degree prime to p over K. Combining these constructions with (2-1), we see that

$$\mathrm{ed}_{\mathcal{F}}(\alpha;\, p) = \mathrm{ed}_{\mathcal{F}}(\alpha_L;\, p) = \mathrm{ed}_{\mathcal{F}}(\gamma_L;\, p) \leq \mathrm{ed}_{\mathcal{F}}(\gamma_L) \leq \mathrm{trdeg}_l\, K_0.$$

This proves $\mathrm{ed}_{\mathcal{F}}(\alpha;\, p) \leq \mathrm{ed}_{\mathcal{F}}(\alpha_{K^{(p)}})$ since K_0 was an arbitrary field of definition for $\alpha_{K^{(p)}}$.

(b) This follows directly from (a) by taking α of maximal essential p-dimension. □

Proposition 2.4. *Let l be an arbitrary field,*

$$\mathcal{F}, \mathcal{G} : \mathrm{Fields}\,/\,l \to \mathrm{Sets}$$

be limit-preserving functors and $\mathcal{F} \to \mathcal{G}$ be a natural transformation. If the map $\mathcal{F}(K) \to \mathcal{G}(K)$ is bijective or surjective for any p-special field containing l, then, respectively,

$$\mathrm{ed}(\mathcal{F};\, p) = \mathrm{ed}(\mathcal{G};\, p) \quad \textit{or} \quad \mathrm{ed}(\mathcal{F};\, p) \geq \mathrm{ed}(\mathcal{G};\, p).$$

Proof. Assume the maps are surjective. By Lemma 2.2(a), the natural transformation is p-surjective in the terminology of [Merkurjev 2009], so we can apply [Merkurjev 2009, Proposition 1.5] to conclude $\mathrm{ed}(\mathcal{F};\, p) \geq \mathrm{ed}(\mathcal{G};\, p)$.

Now assume the maps are bijective. Let α be in $\mathcal{F}(K)$ for some K/l and β its image in $\mathcal{G}(K)$. We claim that $\mathrm{ed}(\alpha;\, p) = \mathrm{ed}(\beta;\, p)$. First by Lemma 2.3, we may assume that K is p-special. In this situation, it is enough to prove that $\mathrm{ed}(\alpha) \leq \mathrm{ed}(\beta)$ (the opposite inequality is by functoriality).

Assume that β comes from $\beta_0 \in \mathcal{G}(K_0)$ for some field $l \subseteq K_0 \subseteq K$. Let K_0' denote the algebraic closure of K_0 in K. Any finite prime-to-p extension of K_0' is isomorphic (over K_0') to a subfield of K (cf. [Merkurjev 2009, Lemma 6.1]) and hence coincides with K_0'. Thus, K_0' has no nontrivial prime-to-p extensions. By Lemma 2.1, it follows that K_0' is p-special. Since K_0' is an algebraic extension of K_0, we may replace K_0 by K_0' and thus assume that K_0 is p-special. By assumption, $\mathcal{F}(K_0) \to \mathcal{G}(K_0)$ and $\mathcal{F}(K) \to \mathcal{G}(K)$ are bijective; therefore, the unique element $\alpha_0 \in \mathcal{F}(K_0)$ that maps to β_0 must map to α under the natural restriction map. The claim follows.

We obtain $\mathrm{ed}(\mathcal{F};\, p) = \mathrm{ed}(\alpha;\, p) = \mathrm{ed}(\beta;\, p) \leq \mathrm{ed}(\mathcal{G};\, p)$ by taking α of maximal essential p-dimension. □

3. Isogenies

An isogeny of algebraic groups is a surjective morphism $A \to B$ with finite kernel. The degree of an isogeny is the order of its kernel.

Proposition 3.1. *Suppose $A \to B$ is an isogeny of degree prime to p of smooth algebraic groups over a field l of characteristic not equal to p. Then*

(a) *for any p-special field K containing k, the natural map $H^1(K, A) \to H^1(K, B)$ is bijective and*

(b) $\mathrm{ed}(A; p) = \mathrm{ed}(B; p)$.

Example 3.2. Let E_6^{sc} and E_7^{sc} be simply connected simple groups of type E_6 and E_7, respectively. In [Gille and Reichstein 2009, 9.4 and 9.6], it is shown that if k is an algebraically closed field of characteristic not equal to 2 and 3, respectively, then

$$\mathrm{ed}(E_6^{\mathrm{sc}}; 2) = 3 \quad \text{and} \quad \mathrm{ed}(E_7^{\mathrm{sc}}; 3) = 3.$$

For the adjoint groups $E_6^{\mathrm{ad}} = E_6^{\mathrm{sc}}/\mu_3$ and $E_7^{\mathrm{ad}} = E_7^{\mathrm{sc}}/\mu_2$, we therefore have

$$\mathrm{ed}(E_6^{\mathrm{ad}}; 2) = 3 \quad \text{and} \quad \mathrm{ed}(E_7^{\mathrm{ad}}; 3) = 3.$$

For the proof of Proposition 3.1, we will need a lemma.

Lemma 3.3. *Let N be a finite algebraic group over a field l of characteristic not equal to p. The following are equivalent*:

(a) *p does not divide the order of N.*

(b) *p does not divide the order of $N(l_{\mathrm{alg}})$.*

Proof. Let N^0 be the connected component of N and $N^{\text{ét}} = N/N^0$ the étale quotient. Recall that the order of a finite algebraic group N over l is defined as $|N| = \dim_l l[N]$ and $|N| = |N^0||N^{\text{ét}}|$; see, e.g., [Tate 1997]. If $\mathrm{char}\, l = 0$, N^0 is trivial; if $\mathrm{char}\, l = q \neq p$ is positive, $|N^0|$ is a power of q. Hence, N is of order prime to p if and only if the étale algebraic group $N^{\text{ét}}$ is. Since N^0 is connected and finite, $N^0(l_{\mathrm{alg}}) = \{1\}$, so $N(l_{\mathrm{alg}})$ is of order prime to p if and only if the group $N^{\text{ét}}(l_{\mathrm{alg}})$ is. Then $|N^{\text{ét}}| = \dim_l l[N^{\text{ét}}] = |N^{\text{ét}}(l_{\mathrm{alg}})|$; cf. [Bourbaki 1990, V.29 Corollary]. □

Proof of Proposition 3.1. (a) Let N be the kernel of the isogeny $A \to B$ and K be a p-special field over l. Since $K_{\mathrm{sep}} = K_{\mathrm{alg}}$ (see Lemma 2.2(b)), the sequence of K_{sep}-points $1 \to N(K_{\mathrm{sep}}) \to A(K_{\mathrm{sep}}) \to B(K_{\mathrm{sep}}) \to 1$ is exact. By Lemma 3.3, the order of $N(K_{\mathrm{sep}})$ is not divisible by p and therefore coprime to the order of any finite quotient of $\Psi = \mathrm{Gal}(K_{\mathrm{sep}}/K)$. By [Serre 2002, I.5, Exercise 2], this implies that $H^1(K, N) = \{1\}$. Similarly, if ${}_cN$ is the group N twisted by a cocycle $c : \Psi \to A$, then ${}_cN(K_{\mathrm{sep}}) = N(K_{\mathrm{sep}})$ is of order prime to p, and $H^1(K, {}_cN) = \{1\}$. It follows that $H^1(K, A) \to H^1(K, B)$ is injective; cf. [Serre 2002, I.5.5].

Surjectivity is a consequence of [Serre 2002, I, Proposition 46] and the fact that the q-cohomological dimension of Ψ is 0 for any divisor q of $|N(K_{\mathrm{sep}})|$ (Lemma 2.2(c)).

(b) This part follows from (a) and Proposition 2.4. □

4. Proof of the main theorem: an overview

We now assume that Conventions 1.1 are valid. The upper bound in Theorem 1.2 is an easy consequence of Proposition 3.1. Indeed, suppose $\mu : G \to \mathrm{GL}(V)$ is a p-generically free representation. That is, $\ker \mu$ is a finite group of order prime to p, and μ descends to a generically free representation of $G' := G/\ker \mu$. By Proposition 3.1, $\mathrm{ed}(G;\, p) = \mathrm{ed}(G';\, p)$. On the other hand,

$$\mathrm{ed}(G';\, p) \le \mathrm{ed}(G') \le \dim \mu - \dim G' = \dim \mu - \dim G;$$

see [Berhuy and Favi 2003, Lemma 4.11; Merkurjev 2009, Corollary 4.2]. This completes the proof of the upper bound in Theorem 1.2.

The rest of this section will be devoted to outlining a proof of the lower bound of Theorem 1.2. The details (namely, the proofs of Propositions 4.2 and 4.3) will be supplied in the next two sections. The starting point of our argument is [Lötscher et al. 2013, Theorem 3.1], which we reproduce below for the reader's convenience:

Theorem 4.1. *Consider an exact sequence of algebraic groups over a field*

$$1 \to C \to H \to Q \to 1$$

such that C is central in H and is isomorphic to μ_p^r for some $r \ge 0$. Given a character $\chi : C \to \mu_p$, denote by Rep^χ the class of irreducible representations $\phi : H \to \mathrm{GL}(V)$ such that $\phi(c) = \chi(c)\,\mathrm{Id}$ for every $c \in C$.

Assume further that

$$\gcd\{\dim \phi \mid \phi \in \mathrm{Rep}^\chi\} = \min\{\dim \phi \mid \phi \in \mathrm{Rep}^\chi\} \tag{4-1}$$

for every character $\chi : C \to \mu_p$. Then

$$\mathrm{ed}(H;\, p) \ge \min \dim \psi - \dim H,$$

where the minimum is taken over all finite-dimensional representations ψ of H such that $\psi|_C$ is faithful.

To prove the lower bound of Theorem 1.2, we will apply Theorem 4.1 to the exact sequence

$$1 \to C(G) \to G \to Q \to 1, \tag{4-2}$$

where $C(G)$ is a central subgroup of G defined as follows. Recall from [Lötscher et al. 2013, Section 2] that if A is a k-group of multiplicative type, $\mathrm{Split}_k(A)$ is defined as the maximal split k-subgroup of A. That is, if $X(A)$ is the character $\mathrm{Gal}(k_{\mathrm{sep}}/k)$-module of A, then the character module of $\mathrm{Split}_k(A)$ is defined as the largest quotient of $X(A)$ with trivial $\mathrm{Gal}(k_{\mathrm{sep}}/k)$-action.

We denote by $Z(G)[p]$ the p-torsion subgroup of the center $Z(G)$. Note that $Z(G)$ is a commutative group, which is an extension of a p-group by a group of multiplicative type. Since $\mathrm{char}\, k \neq p$, it follows that $Z(G)$ is of multiplicative type. We now define $C(G) := \mathrm{Split}_k(Z(G)[p])$.

In order to show that Theorem 4.1 can be applied to the sequence (4-2), we need to check that condition (4-1) is satisfied. This is a consequence of the following proposition, which will be proved in the next section:

Proposition 4.2. *The dimension of every irreducible representation of G over k is a power of p.*

Applying Theorem 4.1 to the exact sequence (4-2) now yields

$$\mathrm{ed}(G;\, p) \geq \min \dim \rho - \dim G,$$

where the minimum is taken over all representations $\rho : G \to \mathrm{GL}(V)$ such that $\rho|_{C(G)}$ is faithful. This resembles the lower bound of Theorem 1.2; the only difference is that in the statement of Theorem 1.2 we take the minimum over p-faithful representations ρ and here we only ask that $\rho|_{C(G)}$ should be faithful. The following proposition shows that the two bounds are, in fact, the same, thus completing the proof of Theorem 1.2:

Proposition 4.3. *A finite-dimensional representation ρ of G is p-faithful if and only if $\rho|_{C(G)}$ is faithful.*

We will prove Proposition 4.3 in Section 6.

Remark 4.4. The inequality $\min \dim \rho - \dim G \leq \mathrm{ed}(G;\, p)$ of Theorem 1.2, where ρ ranges over all p-faithful representations of G, fails if we take the minimum over just the faithful (rather than p-faithful) representations, even in the case where $G = T$ is a torus.

Indeed, choose T so that the $\mathrm{Gal}(k_{\mathrm{sep}}/k)$-character lattice $X(T)$ of T is a direct summand of a permutation lattice, but $X(T)$ itself is not permutation (see [Colliot-Thélène and Sansuc 1977, 8A] for an example of such a lattice).

In other words, there exists a k-torus T' such that $T \times T'$ is quasisplit (but T is not). This implies that $H^1(K, T \times T') = \{1\}$ and thus $H^1(K, T) = \{1\}$ for any field extension K/k. Consequently, $\mathrm{ed}(T;\, p) = 0$ for every prime p.

On the other hand, we claim that the dimension of the minimal faithful representation of T is strictly bigger than $\dim T$. Assume the contrary. Then there exists a surjective homomorphism $f : P \to X(T)$ of $\mathrm{Gal}(k_{\mathrm{sep}}/k)$-lattices, where P is permutation and $\mathrm{rank}\, P = \dim T$; see, e.g., [Lötscher et al. 2013, Lemma 2.6]. This implies that f has finite kernel and hence is injective. We conclude that f is an isomorphism, so $X(T)$ is a permutation $\mathrm{Gal}(k_{\mathrm{sep}}/k)$-lattice, a contradiction. □

5. Dimensions of irreducible representations

The purpose of this section is to prove Proposition 4.2.

Lemma 5.1. *Let H be a finite p-subgroup of G defined over k. Then H becomes constant after some field extension of k whose degree is a power of p.*

Recall that here G and k are subject to Conventions 1.1.

Proof. After passing to a suitable p-power field extension of k, the torus T becomes split, and F becomes constant. In other words, we may assume that $T \cap H$ is split and the image $\pi(H)$ of H in F is constant. Moreover, after adjoining a primitive root of unity of order $p^m := |T \cap H|$, we may assume that $T \cap H$ is constant (note that $[k(\zeta_{p^m}) : k]$ is a power of p since k is assumed to contain ζ_p). Thus, H is an extension of a constant p-group $\pi(H)$ by a constant p-group $T \cap H$. The group H becomes constant after a p-power field extension if and only if the image of Γ in $\mathrm{Aut}(H(k_{\mathrm{sep}}))$ is a p-group. Thus, it suffices to establish the following claim:

Claim. Let B be a p-group, S a finite subgroup of $\mathrm{Aut}(B)$ and $1 \to A \to B \to C \to 1$ an S-equivariant exact sequence with S acting trivially on A and C. Then S is a p-group.

To prove the claim, assume the contrary. Then S contains a subgroup of prime order $q \neq p$. After replacing S by that subgroup, we may assume without loss of generality that $|S| = q$. Let $b \in B$. Then the image of b in C is fixed under S. Hence, the fiber Ab over this element is S-stable. Since the cardinality of Ab is a power of p and thus is not divisible by q, S has to fix some elements of Ab. Denote one of these elements by b_0. Then $b \in Ab_0$, and since the elements of A are fixed by S, this implies that b is fixed by S as well. This shows that S acts trivially on B, a contradiction. □

The special case of Proposition 4.2, where $T = \{1\}$, i.e., $G = F$ is a finite p-group that becomes constant after a p-power field extension, is established in the course of the proof of [Lötscher et al. 2013, Theorem 7.1]. Our proof of Proposition 4.2 below is based on leveraging this case as follows.

Lemma 5.2. *Let H be a smooth algebraic group defined over a field l and*

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H$$

be an ascending sequence of smooth l-subgroups whose union $\bigcup_{n\geq 1} H_n$ is Zariski dense in H. If $\rho : H \to \mathrm{GL}(V)$ is an irreducible representation of H, then $\rho|_{H_i}$ is irreducible for sufficiently large integers i.

Proof. For each $d = 1, \dots, \dim V - 1$, consider the H-action on the Grassmannian $\mathrm{Gr}(d, V)$ of d-dimensional subspaces of V. Let $X^{(d)} = \mathrm{Gr}(d, V)^H$ and $X_i^{(d)} = \mathrm{Gr}(d, V)^{H_i}$ be the subvarieties of d-dimensional H- and H_i-invariant subspaces of V, respectively. Then $X_1^{(d)} \supseteq X_2^{(d)} \supseteq \cdots$, and since the union of the groups H_i is dense in H,

$$X^{(d)} = \bigcap_{i \geq 0} X_i^{(d)}.$$

By the Noetherian property of $\mathrm{Gr}(d, V)$, we have $X^{(d)} = X_{m_d}^{(d)}$ for some $m_d \geq 0$.

Since V does not have any H-invariant d-dimensional l-subspaces, we know that $X^{(d)}(l) = \varnothing$. Thus, $X_{m_d}^{(d)}(l) = \varnothing$, i.e., V does not have any H_{m_d}-invariant d-dimensional l-subspaces. Setting $m := \max\{m_1, \dots, m_{\dim V - 1}\}$, we see that $\rho|_{H_i}$ is irreducible for any $i \geq m$. □

We now proceed with the proof of Proposition 4.2. By Lemmas 5.1 and 5.2, it suffices to construct a sequence of finite p-subgroups

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq G$$

defined over k whose union $\bigcup_{n \geq 1} F_n$ is Zariski dense in G. In fact, it suffices to construct one p-subgroup $F' \subseteq G$ defined over k such that F' surjects onto F. Once F' is constructed, we can define $F_i \subseteq G$ as the subgroup generated by F' and $T[p^i]$ for every $i \geq 0$. Here $T[m]$ denotes the m-torsion subgroup of T. Since $\bigcup_{n \geq 1} F_n$ contains both F' and $T[p^i]$ for every $i \geq 0$, it is Zariski dense in G, as desired.

The following lemma, which establishes the existence of F', is thus the final step in our proof of Proposition 4.2:

Lemma 5.3. *Let $1 \to T \to G \xrightarrow{\pi} F \to 1$ be an extension of a p-group F by a torus T over an arbitrary field k. Then G has a p-subgroup F' with $\pi(F') = F$.*

Here G and k are not subject to Conventions 1.1. In the case where F is split and k is algebraically closed, the above lemma is proved in [Chernousov et al. 2006, page 564]; cf. also the proof of [Borel and Serre 1964, Lemme 5.11].

Proof. Denote by $\widetilde{\mathrm{Ex}}^1(F, T)$ the group of equivalence classes of extensions of F by T. We claim that $\widetilde{\mathrm{Ex}}^1(F, T)$ is torsion. Let $\mathrm{Ex}^1(F, T) \subseteq \widetilde{\mathrm{Ex}}^1(F, T)$ be the classes of extensions that have a scheme-theoretic section (i.e., $G(K) \to F(K)$ is surjective for all K/k). There is a natural isomorphism $\mathrm{Ex}^1(F, T) \simeq H^2(F, T)$, where H^2 denotes Hochschild cohomology; see [Demazure and Gabriel 1970, III.6.2, Proposition]. By [Schneider 1981], the usual restriction-corestriction arguments can be applied in Hochschild cohomology, and in particular, $m \cdot H^2(F, T) = 0$, where m is the order of F. Now recall that $M \mapsto \widetilde{\mathrm{Ex}}^i(F, M)$ and $M \mapsto \mathrm{Ex}^i(F, M)$ are both derived functors of the crossed homomorphisms $M \mapsto \mathrm{Ex}^0(F, M)$, where in the first case M is in the category of F-module sheaves and in the second F-module functors, cf. [Demazure and Gabriel 1970, III.6.2]. Since F is finite and T an affine scheme, by [Schneider 1980b, Sätze 1.2 and 3.3] there is an exact sequence of F-module schemes $1 \to T \to M_1 \to M_2 \to 1$ and an exact sequence $\mathrm{Ex}^0(F, M_1) \to \mathrm{Ex}^0(F, M_2) \to \widetilde{\mathrm{Ex}}^1(F, T) \to H^2(F, M_1) \simeq \mathrm{Ex}^1(F, M_1)$. The F-module sequence also induces a long exact sequence on $\mathrm{Ex}(F, *)$, and we have

$$\begin{tikzcd}[column sep=small]
 & & \widetilde{\mathrm{Ex}}^1(F, T) \arrow[dr] & \\
\mathrm{Ex}^0(F, M_1) \arrow[r] & \mathrm{Ex}^0(F, M_2) \arrow[ur] \arrow[dr] & & \mathrm{Ex}^1(F, M_1). \\
 & & \mathrm{Ex}^1(F, T) \arrow[uu, hook] \arrow[ur] &
\end{tikzcd}$$

An element in $\widetilde{\mathrm{Ex}}^1(F, T)$ can thus be killed first in $\mathrm{Ex}^1(F, M_1)$, so it comes from $\mathrm{Ex}^0(F, M_2)$. Then kill its image in $\mathrm{Ex}^1(F, T) \simeq H^2(F, T)$, so it comes from $\mathrm{Ex}^0(F, M_1)$ and hence is zero in $\widetilde{\mathrm{Ex}}^1(F, T)$. In particular, multiplying twice by the order m of F, we see that $m^2 \cdot \widetilde{\mathrm{Ex}}^1(F, T) = 0$. This proves the claim.

Now let us consider the exact sequence $1 \to N \to T \xrightarrow{\times m^2} T \to 1$, where N is the kernel of multiplication by m^2. Clearly N is finite, and we have an induced exact sequence

$$\widetilde{\mathrm{Ex}}^1(F, N) \to \widetilde{\mathrm{Ex}}^1(F, T) \xrightarrow{\times m^2} \widetilde{\mathrm{Ex}}^1(F, T),$$

which shows that the given extension G comes from an extension F' of F by N. Then G is the pushout of F' by $N \to T$, and we can identify F' with a subgroup of G. □

6. Proof of Proposition 4.3

We will prove Proposition 6.1 below; Proposition 4.3 is an immediate consequence with $N = \ker \rho$. Once again, please note that Conventions 1.1 are in force.

Proposition 6.1. *Let N be a normal k-subgroup of G. The following conditions are equivalent*:

(a) *N is finite of order prime to p.*

(b) $N \cap C(G) = \{1\}$.

(c) $N \cap Z(G)[p] = \{1\}$.

In particular, taking $N = G$, we see that $C(G) \neq \{1\}$ if $G \neq \{1\}$.

Proof. (a) $\Longrightarrow$ (b) This is obvious since $C(G)$ is a p-group.

(b) $\Longrightarrow$ (c) Assume the contrary: $A := N \cap Z(G)[p] \neq \{1\}$. By Lemma 5.1, $Z(G)[p]$ becomes constant over a field extension k'/k of p-power degree. Since k contains ζ_p, the group $Z(G)[p]$ splits over k' as a group of multiplicative type. It is shown in [Lötscher et al. 2013, Section 2] that $C(A) \neq \{1\}$. Thus,

$$\{1\} \neq C(A) \subseteq N \cap C(G),$$

contradicting (b).

(c) $\Longrightarrow$ (a) Our proof of this implication will rely on the following assertion:

Claim. Let M be a nontrivial normal finite p-subgroup of G such that the commutator (T, M) is trivial. Then $M \cap Z(G)[p] \neq \{1\}$.

To prove the claim, note that $M(k_{\mathrm{sep}})$ is nontrivial and the conjugation action of $G(k_{\mathrm{sep}})$ on $M(k_{\mathrm{sep}})$ factors through an action of the p-group $F(k_{\mathrm{sep}})$. Thus, each orbit has p^n elements for some $n \geq 0$; consequently, the number of fixed points is divisible by p. The intersection $(M \cap Z(G))(k_{\mathrm{sep}})$ is precisely the fixed point set for this action; hence, $M \cap Z(G)[p] \neq \{1\}$. This proves the claim.

We now continue with the proof of the implication (c) $\Longrightarrow$ (a). Assume that $N \lhd G$ and $N \cap Z(G)[p] = \{1\}$. Applying the claim to the normal subgroup $M := (N \cap T)[p]$ of G, we see that $(N \cap T)[p] = \{1\}$, i.e., $N \cap T$ is a finite group of order prime to p. The exact sequence

$$1 \to N \cap T \to N \to \overline{N} \to 1, \tag{6-1}$$

where $\overline{N}$ is the image of N in $F := G/T$, shows that N is finite. Now observe that for every $r \geq 1$, the commutator $(N, T[p^r])$ is a p-subgroup of $N \cap T$. Thus, $(N, T[p^r]) = \{1\}$ for every $r \geq 1$. We claim that this implies $(N, T) = \{1\}$. If N is smooth, this is straightforward; see [Borel 1969, Proposition 2.4, page 59]. If N is not smooth, note that the map $c : N \times T \to G$ sending (n, t) to the commutator $ntn^{-1}t^{-1}$ descends to $\bar{c} : \overline{N} \times T \to G$ (indeed, $N \cap T$ clearly commutes with T). Since $|\overline{N}|$ is a power of p and $\mathrm{char}(k) \neq p$, $\overline{N}$ is smooth over k, and we can pass to the separable closure k_{sep} and apply the usual Zariski density argument to show that the image of $\bar{c}$ is trivial.

We thus conclude that $N \cap T$ is central in N. Since $\gcd(|N \cap T|, \overline{N}) = 1$, by [Schneider 1980a, Corollary 5.4] the extension (6-1) splits, i.e., $N \simeq (N \cap T) \times \overline{N}$. This turns $\overline{N}$ into a finite p-subgroup of G with $(T, \overline{N}) = \{1\}$. The claim implies that $\overline{N}$ is trivial. Hence, $N = N \cap T$ is a finite group of order prime to p, as claimed.

This completes the proof of Proposition 6.1 and thus of Theorem 1.2. □

7. Tame groups

As we have seen in Section 1, some groups G satisfying Conventions 1.1 have faithful linear representations that are not generically free. In this section, we take a closer look at this phenomenon.

If F' is a subgroup of F, then we will use the notation $G_{F'}$ to denote the subgroup $\pi^{-1}(F')$ of G. Here π is the natural projection $G \to G/T = F$ as in (1-1).

Lemma 7.1. *Suppose T is central in G. Then*

(a) *G has only finitely many k-subgroups S such that $S \cap T = \{1\}$, and*

(b) *every faithful action of G on a geometrically irreducible variety X is generically free.*

Proof. After replacing k by its algebraic closure k_{alg}, we may assume without loss of generality that k is algebraically closed.

(a) Since F has finitely many subgroups, it suffices to show that for every subgroup $F_0 \subseteq F$, there are only finitely many $S \subseteq G$ such that $\pi(S) = F_0$ and $S \cap T = \{1\}$.

After replacing G by G_{F_0}, we may assume that $F_0 = F$. In other words, we will show that π has at most finitely many sections $s : F \to G$. Fix one such section, $s_0 : F \to G$. Denote the exponent of F by e. Suppose $s : F \to G$ is another section. Then for every $f \in F(k)$, we can write $s(f) = s_0(f)t$ for some $t \in T(k)$. Since T is central in G, t and $s_0(f)$ commute. Since $s(f)^e = s_0(f)^e = 1$, we see that $t^e = 1$. In other words, $t \in T(k)$ is an e-torsion element, and there are only finitely many e-torsion elements in $T(k)$. We conclude that there are only finitely many choices of $s(f)$ for each $f \in F(k)$. Hence, there are only finitely many sections $F \to G$, as claimed.

(b) The restriction of the G-action on X to T is faithful and hence generically free; cf., e.g., [Lötscher 2010, Proposition 3.7(A)]. Hence, there exists a dense open T-invariant subset $U \subseteq X$ such that $\operatorname{Stab}_T(u) = \{1\}$ for all $u \in U$. In other words, if $S = \operatorname{Stab}_G(u)$, then $S \cap T = \{1\}$. By (a), G has finitely many nontrivial subgroups S with this property. Denote them by $S_1, \dots, S_n$. Since G acts faithfully, X^{S_i} is a proper closed subvariety of X for any $i = 1, \dots, n$. Since X is irreducible,

$$U' = U \setminus (X^{S_1} \cup \dots \cup X^{S_n})$$

is a dense open T-invariant subset of X, and the stabilizer $\operatorname{Stab}_G(u)$ is trivial for every $u \in U'$. Replacing U' by the intersection of its (finitely many) $G(k_{\text{alg}})$-translates, we may assume that U' is G-invariant. This shows that the G-action on X is generically free. □

Proposition 7.2. (a) *A faithful action of G on a geometrically irreducible variety X is generically free if and only if the action of the subgroup $G_{C(F)} \subseteq G$ on X is generically free.*

(b) *A p-faithful action of G on a geometrically irreducible variety X is p-generically free if and only if the action of the subgroup $G_{C(F)} \subseteq G$ on X is p-generically free.*

Proof. (a) The (faithful) T-action on X is necessarily generically free; cf. [Lötscher 2010, Proposition 3.7(A)]. Thus, by [Gabriel 2011, Exposé V, Théorème 10.3.1] or [Berhuy and Favi 2003, Theorem 4.7], X has a dense open T-invariant subvariety U defined over k, which is the total space of a T-torsor, $U \to Y := U/T$, where Y is also smooth and geometrically irreducible. Since G/T is finite, after replacing U by the intersection of its (finitely many) $G(k_{\text{alg}})$-translates, we may assume that U is G-invariant.

The G-action on U gives rise to an F-action on Y (by descent). Now it is easy to see (cf. [Lorenz and Reichstein 2000, Lemma 2.1]) that the following conditions are equivalent:

(i) The G-action on X is generically free.

(ii) The F-action on Y is generically free.

Since F is finite, (ii) is equivalent to

(iii) F acts faithfully on Y.

Proposition 6.1 tells us that the kernel of the F-action on Y is trivial if and only if the kernel of the $C(F)$-action on Y is trivial. In other words, (iii) is equivalent to

(iv) $C(F)$ acts faithfully (or equivalently, generically freely) on Y

and consequently to

(v) the $G_{C(F)}$-action on U (or, equivalently, on X) is generically free.

Note that (iv) and (v) are the same as (ii) and (i), respectively, except that F is replaced by $C(F)$ and G by $G_{C(F)}$. Thus, the equivalence of (iv) and (v) follows like the equivalence of (i) and (ii). We conclude that (i) and (v) are equivalent, as desired.

(b) Let K be the kernel of the G-action on X, which is contained in T by assumption. Note that $(G/K)/(T/K) = G/T = F$, so (a) says the G/K-action on X is generically free if and only if the $G_{C(F)}/K$-action on X is generically free, and (b) follows. □

The following definition is natural in view of Proposition 7.2:

Definition 7.3. Consider the action of F on T induced by conjugation in G. We say that G is *tame* if $C(F)$ lies in the kernel of this action. Equivalently, G is tame if T is central in $G_{C(F)}$.

Recall in Section 1 we defined $\text{gap}(G; p)$ as the difference between the minimal dimension of a p-generically free representation and the minimal dimension of a p-faithful representation of G (all representations are assumed to be defined over k).

Corollary 7.4. *Let G be a tame k-group and X be a geometrically irreducible k-variety X.*

(a) *Every faithful G-action on X is generically free.*

(b) *Every p-faithful G-action on X is p-generically free.*

(c) *We have* $\operatorname{gap}(G;\, p) = 0$. *In other words,*

$$\operatorname{ed}(G;\, p) = \min \dim \rho - \dim G,$$

where the minimum is taken over all p-faithful k-representations of G.

Proof. (a) Since G is tame, T is central in $G_{C(F)}$. Hence, the $G_{C(F)}$-action on X is generically free by Lemma 7.1(b). By Proposition 7.2(a), the G-action on X is generically free.

(b) Let K be the kernel of the action. Note that G/K is also tame. Now apply (a) to G/K.

(c) This follows immediately from (b) and Theorem 1.2. □

8. Proof of Theorem 1.3

In this section, we will prove the following proposition, which implies Theorem 1.3:

Proposition 8.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G.*

(a) *If ρ is faithful, then G has a generically free representation of dimension at most* $\dim \rho + \dim T - \dim T^{C(F)}$.

(b) *If ρ is p-faithful, then G has a p-generically free representation of dimension at most* $\dim \rho + \dim T - \dim T^{C(F)}$.

Proof. (a) The subgroup $T^{C(F)}$ is preserved by the conjugation action of G, so the adjoint representation of G decomposes as $\operatorname{Lie}(T) = \operatorname{Lie}(T^{C(F)}) \oplus W$ for some G-representation W. Since the G-action on $\operatorname{Lie}(T)$ factors through F, the existence of W follows from Maschke's theorem. Let μ be the G-representation on $V \oplus W$. Since $\dim \operatorname{Lie}(T^{C(F)}) \geq \dim T^{C(F)}$, we have $\dim \mu \leq \dim \rho + \dim T - \dim T^{C(F)}$. It thus remains to show that μ is a generically free representation of G.

Let K be the kernel of the $G_{C(F)}$-action on $\operatorname{Lie}(T)$. We claim T is central in K. The finite p-group K/T acts on T (by conjugation), and it fixes the identity. By construction, K/T acts trivially on the tangent space at the identity, which implies K/T acts trivially on T since the characteristic is not equal to p; cf. [Gille and Reichstein 2009, Proof of Lemma 4.1]. This proves the claim.

By Lemma 7.1, the K-action on V is generically free. Now $G_{C(F)}$ acts trivially on $\operatorname{Lie}(T^{C(F)})$, so $G_{C(F)}/K$ acts faithfully on W. Since $G_{C(F)}/K$ is finite, this action is also generically free. Therefore, $G_{C(F)}$ acts generically freely on $V \oplus W$ [Meyer and Reichstein 2009, Lemma 3.2]. Finally, by Proposition 7.2(a), G acts generically freely on $V \oplus W$, as desired.

(b) By our assumption, $\ker \rho \subseteq T$. Set $\overline{T} := T/\ker \rho$. It is easy to see that $\dim T^{C(F)} \leq \dim \overline{T}^{C(F)}$. Hence, by (a) there exists a generically free representation of $G/\ker \rho$ of dimension at most

$$\dim \overline{T} - \dim \overline{T}^{C(F)} \leq \dim T - \dim T^{C(F)}.$$

We may now view this representation as a p-generically free representation of G. This completes the proof of Theorem 1.3. $\square$

Remark 8.2. A similar argument shows that for any tame normal subgroup $H \subseteq G$ over k, $\operatorname{gap}(G; p) \leq \operatorname{ed}(G/H; p)$.

9. Additivity

Our proof of the additivity Theorem 1.4 relies on the following lemma. Let G be an algebraic group defined over a field k and C be a k-subgroup of G. Denote the minimal dimension of a representation ρ of G such that $\rho|_C$ is faithful by $f(G, C)$.

Lemma 9.1. *Let k be an arbitrary field. For $i = 1, 2$, let G_i be an arbitrary* (*linear*) *algebraic group defined over k, and let C_i be a central k-subgroup of G_i. Assume that C_i is isomorphic to $\mu_p^{r_i}$ over k for some $r_1, r_2 \geq 0$. Then*

$$f(G_1 \times G_2; C_1 \times C_2) = f(G_1; C_1) + f(G_2; C_2).$$

Our argument below is a variant of the proof of [Karpenko and Merkurjev 2008, Theorem 5.1], where G is assumed to be a (constant) finite p-group and $C = C(G)$ (recall that $C(G)$ is defined at the beginning of Section 4).

Proof. For $i = 1, 2$, let $\pi_i : G_1 \times G_2 \to G_i$ be the natural projection, and let $\epsilon_i : G_i \to G_1 \times G_2$ be the natural inclusion.

If ρ_i is a d_i-dimensional representation of G_i whose restriction to C_i is faithful, then clearly $\rho_1 \circ \pi_1 \oplus \rho_2 \circ \pi_2$ is a $(d_1 + d_2)$-dimensional representation of $G_1 \times G_2$ whose restriction to $C_1 \times C_2$ is faithful. This shows that

$$f(G_1 \times G_2; C_1 \times C_2) \leq f(G_1; C_1) + f(G_2; C_2).$$

To prove the opposite inequality, let $\rho : G_1 \times G_2 \to \mathrm{GL}(V)$ be a representation such that $\rho|_{C_1 \times C_2}$ is faithful and of minimal dimension

$$d = f(G_1 \times G_2; C_1 \times C_2)$$

with this property. Let $\rho_1, \rho_2, \ldots, \rho_n$ denote the irreducible decomposition factors in a Jordan–Hölder series for ρ. (Note that since G_1 and G_2 are arbitrary linear algebraic groups, ρ may not be completely reducible.) Since $C_1 \times C_2$ is central in $G_1 \times G_2$, each ρ_i restricts to a multiplicative character of $C_1 \times C_2$, which we will denote by χ_i. Moreover, since $C_1 \times C_2 \simeq \mu_p^{r_1+r_2}$ is linearly reductive, $\rho|_{C_1 \times C_2}$

is a direct sum $\chi_1^{\oplus d_1} \oplus \cdots \oplus \chi_n^{\oplus d_n}$, where $d_i = \dim \rho_i$. It is easy to see that the following conditions are equivalent:

(i) $\rho|_{C_1 \times C_2}$ is faithful.

(ii) $\chi_1, \ldots, \chi_n$ generate $(C_1 \times C_2)^*$ as an abelian group.

In particular, we may replace ρ by the direct sum $\rho_1 \oplus \cdots \oplus \rho_n$. Since C_i is isomorphic to $\mu_p^{r_i}$, we will think of $(C_1 \times C_2)^*$ as an $\mathbb{F}_p$-vector space of dimension $r_1 + r_2$. Since (i) $\iff$ (ii) above, we know that $\chi_1, \ldots, \chi_n$ span $(C_1 \times C_2)^*$. In fact, they form a basis of $(C_1 \times C_2)^*$, i.e., $n = r_1 + r_2$. Indeed, if they were not linearly independent, we would be able to drop some of the terms in the irreducible decomposition $\rho_1 \oplus \cdots \oplus \rho_n$ so that the restriction of the resulting representation to $C_1 \times C_2$ would still be faithful, contradicting the minimality of $\dim \rho$.

We claim that it is always possible to replace each ρ_j by ρ'_j, where ρ'_j is either $\rho_j \circ \epsilon_1 \circ \pi_1$ or $\rho_j \circ \epsilon_2 \circ \pi_2$ such that the restriction of the resulting representation $\rho' = \rho'_1 \oplus \cdots \oplus \rho'_n$ to $C_1 \times C_2$ remains faithful. Since $\dim \rho_i = \dim \rho'_i$, we see that $\dim \rho' = \dim \rho$. Moreover, ρ' will then be of the form $\alpha_1 \circ \pi_1 \oplus \alpha_2 \circ \pi_2$, where α_i is a representation of G_i whose restriction to C_i is faithful. Thus, if we can prove the above claim, we will have

$$\begin{aligned} f(G_1 \times G_2; C_1 \times C_2) = \dim \rho = \dim \rho' &= \dim \alpha_1 + \dim \alpha_2 \\ &\geq f(G_1, C_1) + f(G_2, C_2), \end{aligned}$$

as desired.

To prove the claim, we will define ρ'_j recursively for $j = 1, \ldots, n$. Suppose $\rho'_1, \ldots, \rho'_{j-1}$ have already be defined so that the restriction of

$$\rho'_1 \oplus \cdots \oplus \rho'_{j-1} \oplus \rho_j \oplus \cdots \oplus \rho_n$$

to $C_1 \times C_2$ is faithful. For notational simplicity, we assume $\rho_1 = \rho'_1, \ldots, \rho_{j-1} = \rho'_{j-1}$. Note that

$$\chi_j = (\chi_j \circ \epsilon_1 \circ \pi_1) \oplus (\chi_j \circ \epsilon_2 \circ \pi_2).$$

Since $\chi_1, \ldots, \chi_n$ form a basis of $(C_1 \times C_2)^*$ as an $\mathbb{F}_p$-vector space, we see that (a) $\chi_j \circ \epsilon_1 \circ \pi_1$ or (b) $\chi_j \circ \epsilon_2 \circ \pi_2$ does not lie in $\mathrm{Span}_{\mathbb{F}_p}(\chi_1, \ldots, \chi_{j-1}, \chi_{j+1}, \ldots, \chi_n)$. Set

$$\rho'_j := \begin{cases} \rho_j \circ \epsilon_1 \circ \pi_1 & \text{in case (a),} \\ \rho_j \circ \epsilon_2 \circ \pi_2 & \text{otherwise.} \end{cases}$$

Using the equivalence of (i) and (ii) above, we see that the restriction of

$$\rho_1 \oplus \cdots \oplus \rho_{j-1} \oplus \rho'_j \oplus \rho_{j+1} \oplus \cdots \oplus \rho_n$$

to C is faithful. This completes the proof of the claim and thus of Lemma 9.1. □

Proof of Theorem 1.4. The groups G_1 and G_2 in the statement of Theorem 1.4 are assumed to satisfy Conventions 1.1 and hence so does $G := G_1 \times G_2$.

Recall also that $C(G)$ is defined as the maximal split p-torsion subgroup of the center of G; see Section 4. It follows from this definition that

$$C(G) = C(G_1) \times C(G_2).$$

By Lemma 9.1 and Proposition 4.3, the minimal dimension of a p-faithful representation is

$$f(G, C(G)) = f(G_1, C(G_1)) + f(G_2, C(G_2)),$$

which is the sum of the minimal dimensions of p-faithful representations of G_1 and G_2. For $i \in \{1, 2\}$ since $\mathrm{gap}(G_i; p) = 0$, there exists a p-generically free representation ρ_i of G_i of dimension $f(G_i, C(G_i))$. The direct sum $\rho_1 \oplus \rho_2$ is a p-generically free representation of G, and its dimension is $f(G, C(G))$. It follows that $\mathrm{gap}(G; p) = 0$. By Theorem 1.2,

$$\mathrm{ed}(G; p) = f(G, C(G)) - \dim G$$

and similarly for G_1 and G_2; cf. Proposition 4.3. Hence, as desired, we have $\mathrm{ed}(G; p) = \mathrm{ed}(G_1; p) + \mathrm{ed}(G_2; p)$. □

Example 9.2. Let T be a torus over a field k of characteristic not equal to 2. Suppose there exists an element τ in the absolute Galois group $\mathrm{Gal}(k_{\mathrm{sep}}/k)$ that acts on the character lattice $X(T)$ via multiplication by -1. Then $\mathrm{ed}(T; 2) \geq \dim T$.

Proof. Let $n := \dim T$. Over the fixed field $K := (k_{\mathrm{sep}})^{\tau}$, the torus T becomes isomorphic to a direct product of n copies of a nonsplit one-dimensional torus T_1. Using [Lötscher et al. 2013, Theorem 1.1], it is easy to see that $\mathrm{ed}(T_1; 2) = 1$. By Theorem 1.4, we conclude that

$$\mathrm{ed}(T; 2) \geq \mathrm{ed}(T_K; 2) = \mathrm{ed}((T_1)^n; 2) = n\,\mathrm{ed}(T_1; 2) = \dim T. \qquad \square$$

We end this section with an example that shows that the property $\mathrm{gap}(G; p) = 0$ is not preserved under base field extensions.

Example 9.3. Let k be as in Conventions 1.1, T be an algebraic k-torus that splits over a field extension of k of p-power degree and F be a nontrivial p-subgroup of the constant group S_n. Form the wreath product

$$T \wr F := T^n \rtimes F,$$

where F acts on T^n by permutations.

Then $\mathrm{gap}(T \wr F; p) = 0$ if and only if $\mathrm{ed}(T; p) > 0$. Moreover,

$$\mathrm{ed}(T \wr F; p) = \begin{cases} \mathrm{ed}(T^n; p) = n\,\mathrm{ed}(T; p) & \text{if } \mathrm{ed}(T; p) > 0, \\ \mathrm{ed}(F; p) & \text{otherwise.} \end{cases}$$

Proof. Let W be a p-faithful T-representation of minimal dimension. By [Lötscher et al. 2013, Theorem 1.1], $\mathrm{ed}(T;\, p) = \dim W - \dim T$.

Then $W^{\oplus n}$ is naturally a p-faithful $(T \wr F)$-representation. Lemma 9.1 and Proposition 4.3 applied to T^n tell us that $W^{\oplus n}$ has minimal dimension among all p-faithful representations of $T \wr F$.

Suppose $\mathrm{ed}(T;\, p) > 0$, i.e., $\dim W > \dim T$. The group F acts faithfully on the rational quotient $W^{\oplus n}/T^n = (W/T)^n$ since $\dim W/T = \dim W - \dim T > 0$. It is easy to see that the $(T \wr F)$-action on $W^{\oplus n}$ is p-generically free; cf., e.g, [Meyer and Reichstein 2009, Lemma 3.3]. In particular, $\mathrm{gap}(T \wr F;\, p) = 0$ and

$$\mathrm{ed}(T\wr F;\, p) = \dim W^{\oplus n} - \dim(T\wr F) = n(\dim W - \dim T) = n\,\mathrm{ed}(T;\, p) = \mathrm{ed}(T^n;\, p),$$

where the last equality follows from the additivity Theorem 1.4.

Now assume that $\mathrm{ed}(T;\, p) = 0$, i.e., $\dim W = \dim T$. The group $T \wr F$ cannot have a p-generically free representation V of dimension $\dim W^{\oplus n} = \dim T \wr F$ since T^n would then have a dense orbit in V. It follows that $\mathrm{gap}(T \wr F;\, p) > 0$. In order to compute its essential p-dimension of $T \wr F$, we use the fact that the natural projection $T \wr F \to F$ has a section. Hence, the map $H^1(*, T \wr F) \to H^1(*, F)$ also has a section and is consequently a surjection. This implies $\mathrm{ed}(T \wr F;\, p) \geq \mathrm{ed}(F;\, p)$. Let W' be a faithful F-representation of dimension $\mathrm{ed}(F;\, p)$. The direct sum $W^{\oplus n} \oplus W'$ considered as a $T \wr F$ representation is p-generically free, so $\mathrm{ed}(T \wr F;\, p) = \mathrm{ed}(F;\, p)$. $\square$

10. Groups of low essential p-dimension

In [Lötscher et al. 2013], we have identified tori of essential dimension 0 as those tori whose character lattice is invertible, i.e., a direct summand of a permutation module; see [Lötscher et al. 2013, Example 5.4]. The following lemma (with $H = G$) shows that among the algebraic groups G studied in this paper, i.e., extensions of p-groups by tori, there are no other examples of groups of $\mathrm{ed}(G;\, p) = 0$:

Lemma 10.1. *Let H be an algebraic group over a field l such that H/H^0 is a p-group. If $\mathrm{ed}(H;\, p) = 0$, then H is connected.*

Proof. Assume the contrary: $F := H/H^0 \neq \{1\}$. Let X be an irreducible H-torsor over some field K/l. For example, we can construct X as follows. Start with a faithful linear representation $H \hookrightarrow \mathrm{GL}_n$ for some $n \geq 0$. The natural projection $\mathrm{GL}_n \to \mathrm{GL}_n / H$ is an H-torsor. Pulling back to the generic point $\mathrm{Spec}(K) \to \mathrm{GL}_n / H$, we obtain an irreducible H-torsor over K.

Now $X/H^0 \to \mathrm{Spec}(K)$ is an irreducible F-torsor. Since $F \neq \{1\}$ is not connected, this torsor is nonsplit. As F is a p-group, X/H^0 remains nonsplit over every prime-to-p extension L/K. It follows that the degree of every closed point of X is divisible by p; hence, p is a torsion prime of H. Therefore, $\mathrm{ed}(H;\, p) > 0$ by

[Merkurjev 2009, Proposition 4.4]. This contradicts the assumption $\mathrm{ed}(H;\, p) = 0$, so F must be trivial. □

Proposition 10.2. *Let G be a central extension of a p-group F by a torus T over a field k of characteristic not p. If $\mathrm{ed}(G;\, p) \le p-2$, then G is of multiplicative type.*

Proof. Without loss of generality, assume $k = k_{\mathrm{alg}}$. By Theorem 1.2, there is a p-faithful representation V of G with $\dim V \le \dim T + p - 2$.

First consider the case where V is faithful. By the theorem of Nagata [1961], G is linearly reductive; hence, we can write $V = \bigoplus_{i=1}^{r} V_i$ for some nontrivial irreducible G-representations V_i. Since T is central and diagonalizable, it acts by a fixed character on V_i for every i. Hence, $r \ge \dim T$ by faithfulness of V. It follows that $1 \le \dim V_i \le p-1$ for each i. But every irreducible G-representation has dimension a power of p (Proposition 4.2), so each V_i is one-dimensional. In other words, G is of multiplicative type.

Now consider the general case, where V is only p-faithful, and let $K \subseteq G$ be the kernel of that representation. Then G/K is of multiplicative type, so it embeds into a torus T_1. Since T is central in G, a subgroup F' as in Lemma 5.3 is normal, so let $T_2 = G/F'$, which is also a torus. The kernel of the natural map $G \to T_1 \times T_2$ is contained in $K \cap F'$. On the other hand, $K \cap F' = \{1\}$ because p does not divide the order of K. This shows that G embeds into the torus $T_1 \times T_2$ and hence is of multiplicative type. □

Example 10.3. Proposition 10.2 does not generalize to tame groups. For a counterexample, assume that the field k contains a primitive root of unity of order p^2, and consider the group $G = \mathbb{G}_m^p \rtimes \mathbb{Z}/p^2\mathbb{Z}$, where a generator in $\mathbb{Z}/p^2\mathbb{Z}$ acts by cyclically permuting the p copies of $\mathbb{G}_m$. The group G is tame since $C(\mathbb{Z}/p^2\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} = \mu_p$ acts trivially on $\mathbb{G}_m^p$. On the other hand, G is not abelian and hence is not of multiplicative type.

We claim that $\mathrm{ed}(G;\, p) = 1$ and hence $\mathrm{ed}(G;\, p) \le p-2$ for every odd prime p. There is a natural p-dimensional faithful representation ρ of G; ρ embeds $\mathbb{G}_m^p$ into GL_p diagonally in the standard basis $e_1, \dots, e_p$, and $\mathbb{Z}/p^2\mathbb{Z}$ cyclically permutes $e_1, \dots, e_p$. Taking the direct sum of ρ with the one-dimensional representation $\chi : G \to \mathbb{Z}/p^2\mathbb{Z} = \mu_{p^2} \hookrightarrow \mathbb{G}_m = \mathrm{GL}_1$, we obtain a faithful $(p+1)$-dimensional representation $\rho \oplus \chi$, which is therefore generically free by Corollary 7.4 (this can also be verified directly). Hence, $\mathrm{ed}(G;\, p) \le (p+1) - \dim(G) = 1$. On the other hand, by Lemma 10.1, we see that $\mathrm{ed}(G;\, p) \ge 1$ and thus $\mathrm{ed}(G;\, p) = 1$, as claimed. □

Let Γ_p be a finite p-group, and let $\phi : P \to X$ be a map of $\mathbb{Z}[\Gamma_p]$-modules. As in [Lötscher et al. 2013], we will call ϕ a *p-presentation* if P is permutation and the cokernel is finite of order prime to p. We will denote by I the augmentation ideal of $\mathbb{Z}[\Gamma_p]$ and by $\overline{X} := X/(pX + IX)$ the largest p-torsion quotient with trivial Γ_p-action. The induced map on quotient modules will be denoted by $\overline{\phi} : \overline{P} \to \overline{X}$.

Lemma 10.4. *Let $\phi : P \to X$ be a map of $\mathbb{Z}[\Gamma_p]$-modules. Then the cokernel of ϕ is finite of order prime to p if and only if $\overline{\phi}$ is surjective.*

Proof. This is shown in [Merkurjev 2010, Proof of Theorem 4.3] and from a different perspective in [Lötscher et al. 2013, Lemma 2.2]. □

In the sequel, for G a group of multiplicative type over k, the group Γ_p in the definition of "p-presentation" is understood to be a Sylow p-subgroup of $\Gamma = \mathrm{Gal}(\ell/k)$, where ℓ/k is a Galois splitting field of G.

Proposition 10.5. *Let G be a central extension of a p-group F by a torus T, and let $0 \le r \le p-2$. The following statements are equivalent*:

(a) $\mathrm{ed}(G; p) \le r$.

(b) *G is of multiplicative type, and there is a p-presentation $P \to X(G)$ whose kernel is isomorphic to the trivial $\mathbb{Z}[\Gamma_p]$-module $\mathbb{Z}^r$.*

Proof. (a) $\Longrightarrow$ (b) Assuming (a) by Proposition 10.2, G is of multiplicative type. By [Lötscher et al. 2013, Corollary 5.1], we know there is a p-presentation $P \to X(G)$ whose kernel L is free of rank $\mathrm{ed}(G; p) \le p-2$. By [Abold and Plesken 1978, Satz], Γ_p must act trivially on L.

(b) $\Longrightarrow$ (a) This direction follows from [Lötscher et al. 2013, Corollary 5.1]. □

Proposition 10.6. *Assume that G is of multiplicative type with a p-presentation $\phi : P \to X(G)$ whose kernel is isomorphic to the trivial $\mathbb{Z}[\Gamma_p]$-module $\mathbb{Z}^r$ for some $r \ge 0$. Then $\mathrm{ed}(G; p) \le r$, and the following conditions are equivalent*:

(a) $\mathrm{ed}(G; p) = r$.

(b) $\ker\phi$ *is contained in* $pP + IP$.

(c) $\ker\phi$ *is contained in*

$$\left\{ \sum_{\lambda\in\Lambda} a_\lambda \lambda \in P \;\middle|\; a_\lambda \equiv 0 \pmod{p},\ \forall \lambda \in \Lambda^{\Gamma_p} \right\}.$$

Here I denotes the augmentation ideal in $\mathbb{Z}[\Gamma_p]$, and Λ is a Γ_p-invariant basis of P.

Proof. (a) $\Longleftrightarrow$ (b) We have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}^r \arrow[r] \arrow[d] & P \arrow[r, "\phi"] \arrow[d] \arrow[dr] & X(G) \arrow[d] \\
& (\mathbb{Z}/p\mathbb{Z})^r \arrow[r] & \overline{P} \arrow[r, "\overline{\phi}"] & \overline{X(G)}
\end{tikzcd}$$

with exact rows. By Lemma 10.4, $\overline{\phi}$ is a surjection. Therefore, $\ker\phi \subseteq pP + IP$ if and only if $\overline{\phi}$ is an isomorphism.

Write P as a direct sum $P \simeq \bigoplus_{j=1}^{m} P_j$ of transitive permutation $\mathbb{Z}[\Gamma_p]$-modules $P_1, \dots, P_m$. Then $P/(pP+IP) \simeq \bigoplus_{j=1}^{m} P_j/(pP_j+IP_j) \simeq (\mathbb{Z}/p\mathbb{Z})^m$. If $\overline{\phi}$ is not an isomorphism, we can replace P by the direct sum $\hat{P}$ of only $m-1$ P_js without losing surjectivity of $\overline{\phi}$. The composition $\hat{P} \hookrightarrow P \to X(G)$ is then still a p-presentation of $X(G)$ by Lemma 10.4, so $\operatorname{ed}(G; p) \le \operatorname{rank} \hat{P} - \dim G < \operatorname{rank} P - \dim G = r$.

Conversely, assume that $\overline{\phi}$ is an isomorphism. Let $\psi : P' \to X(G)$ be a p-presentation such that $\operatorname{ed}(G; p) = \operatorname{rank} \ker \psi$. Let d be the index $[X(G) : \phi(P)]$, which is finite and prime to p. Since the map $X(G) \to d \cdot X(G)$, $x \mapsto dx$ is an isomorphism, we may assume that the image of ψ is contained in $\phi(P)$. We have an exact sequence $\operatorname{Hom}_{\mathbb{Z}[\Gamma_p]}(P', P) \to \operatorname{Hom}_{\mathbb{Z}[\Gamma_p]}(P', \phi(P)) \to \operatorname{Ext}^1_{\mathbb{Z}[\Gamma_p]}(P', \mathbb{Z}^r)$, and the last group is zero by [Lorenz 2005, Lemma 2.5.1]. Therefore, $\psi = \phi \circ \psi'$ for some map $\psi' : P' \to P$ of $\mathbb{Z}[\Gamma_p]$-modules. Since $\overline{\phi}$ is an isomorphism and ψ is a p-presentation, it follows from Lemma 10.4 that ψ' is a p-presentation as well and in particular that $\operatorname{rank} P' \ge \operatorname{rank} P$. Thus, $\operatorname{ed}(G; p) = \operatorname{rank} \ker \psi \ge \operatorname{rank} \ker \phi = r$.

(b) $\iff$ (c). It suffices to show that $P^{\Gamma_p} \cap (pP + IP)$ consists precisely of the elements of P^{Γ_p} of the form $\sum_{\lambda \in \Lambda} a_\lambda \lambda$ with $a_\lambda \equiv 0 \pmod p$ for all $\lambda \in \Lambda^{\Gamma_p}$ for any permutation $\mathbb{Z}[\Gamma_p]$-module P. One easily reduces to the case where P is a transitive permutation module. Then P^{Γ_p} consists precisely of the $\mathbb{Z}$-multiples of $\sum_{\lambda \in \Lambda} \lambda$, and $pP + IP$ are the elements $\sum_{\lambda \in \Lambda} a_\lambda \lambda$ with $\sum_{\lambda \in \Lambda} a_\lambda \equiv 0 \pmod p$. Thus, for $n \in \mathbb{Z}$, the element $n \sum_{\lambda \in \Lambda} \lambda$ lies in $pP + IP$ if and only if $n \cdot |\Lambda| \equiv 0 \pmod p$ if and only if $n \equiv 0 \pmod p$ or $|\Lambda| \equiv 0 \pmod p$. Since $|\Lambda|$ is a power of p, the claim follows. □

Example 10.7. Let E be an étale algebra over k. We can write $E = \ell_1 \times \cdots \times \ell_m$ with some separable field extensions ℓ_i/k. The kernel of the norm map

$$n : \mathbb{R}_{E/k}(\mathbb{G}_m) \to \mathbb{G}_m$$

is denoted by $\mathbb{R}^{(1)}_{E/k}(\mathbb{G}_m)$. Let $G = n^{-1}(\mu_{p^r})$ for some $r \ge 0$. It is a group of multiplicative type fitting into an exact sequence

$$1 \to \mathbb{R}^{(1)}_{E/k}(\mathbb{G}_m) \to G \to \mu_{p^r} \to 1.$$

Let ℓ be a finite Galois extension of k containing $\ell_1, \dots, \ell_m$ (so ℓ splits G), let $\Gamma = \operatorname{Gal}(\ell/k)$ and $\Gamma_{\ell_i} = \operatorname{Gal}(\ell/\ell_i)$, and let Γ_p be a p-Sylow subgroup of Γ. The character module of G has a p-presentation

$$P := \bigoplus_{i=1}^{m} \mathbb{Z}[\Gamma/\Gamma_{\ell_i}] \to X(G)$$

with kernel generated by the element $(p^r, \dots, p^r) \in P$. This element is fixed by Γ_p, so $\operatorname{ed}(G; p) \le 1$. It satisfies condition (c) of Proposition 10.6 if and only if $r > 0$ or every Γ_p-set Γ/Γ_{ℓ_i} is fixed-point free. Note that Γ/Γ_{ℓ_i} has Γ_p-fixed points if and only if $[\ell_i : k] = |\Gamma / \Gamma_{\ell_i}|$ is prime to p. We thus have

$$\operatorname{ed}(G; p) = \begin{cases} 0 & \text{if } r = 0 \text{ and } [\ell_i : k] \text{ is prime to } p \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

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roland.loetscher@mathematik.uni-muenchen.de
Mathematisches Institut, Ludwig-Maximilians-Universität München, D-80333 München, Germany
http://www.mathematik.uni-muenchen.de/~lotscher/

m.macdonald@lancaster.ac.uk
Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF, United Kingdom
http://www.maths.lancs.ac.uk/~macdonam/

aurel.meyer@gmail.com
Départment de Mathématiques, Université Paris-Sud, Bâtiment 425, 91405 Orsay, France
http://www.math.u-psud.fr/~ameyer/

reichst@math.ubc.ca
Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC V6T1Z2, Canada
http://www.math.ubc.ca/~reichst

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Differential characterization of Wilson primes for $\mathbb{F}_q[t]$

Dinesh S. Thakur

Dedicated to Barry Mazur on his 75th birthday

We consider an analog, when $\mathbb{Z}$ is replaced by $\mathbb{F}_q[t]$, of Wilson primes, namely the primes satisfying Wilson's congruence $(p-1)! \equiv -1$ to modulus p^2 rather than the usual prime modulus p. We fully characterize these primes by connecting these or higher power congruences to other fundamental quantities such as higher derivatives and higher difference quotients as well as higher Fermat quotients. For example, in characteristic $p > 2$, we show that a prime $\wp$ of $\mathbb{F}_q[t]$ is a Wilson prime if and only if its second derivative with respect to t is 0 and in this case, further, that the congruence holds automatically modulo $\wp^{p-1}$. For $p = 2$, the power $p-1$ is replaced by $4-1=3$. For every q, we show that there are infinitely many such primes.

1. Introduction

For a prime p, the well-known Wilson congruence says that $(p-1)! \equiv -1 \bmod p$. A prime p is called a Wilson prime if the congruence above holds modulo p^2. Only three such primes are known, and we refer to [Ribenboim 1996, pp. 346 and 350] for history and [Sauerberg et al. 2013] for more references.

Many strong analogies [Goss 1996; Rosen 2002; Thakur 2004] between number fields and function fields over finite fields have been used to benefit the study of both. These analogies are even stronger in the base case $\mathbb{Q}, \mathbb{Z} \leftrightarrow F(t), F[t]$, where F is a finite field. We will study the concept of Wilson prime in this function field context and find interesting differential characterizations for them with the usual and arithmetic derivatives. In [Sauerberg et al. 2013], we exhibited infinitely many of them, at least for many F. Our characterization gives easier alternate proof generalizing to all F.

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2. Wilson primes

Let us fix some basic notation. We use the standard conventions that empty sums are zero and empty products are one. Further,

q	is a power of a prime p,
A	$=\mathbb{F}_q[t]$,
A_d	$=\{$elements of A of degree $d\}$,
$[n]$	$=t^{q^n}-t$,
D_n	$=\prod_{i=0}^{n-1}(t^{q^n}-t^{q^i})=\prod[n-i]^{q^i}$,
L_n	$=\prod_{i=1}^{n}(t^{q^i}-t)=\prod[i]$,
F_i	is the product of all (nonzero) elements of A of degree less than i,
$\mathcal{N}a$	$=q^d$ for $a\in A_d$, i.e., the norm of a and
$\wp$	is a monic irreducible polynomial in A of degree d.

If we interpret the factorial of $n-1$ as the product of nonzero “remainders” when we divide by n, we get F_i as a naïve analog of factorial of $a\in A_i$. Note that it just depends on the degree of a. By the usual group theory argument with pairing of elements with their inverses, we get an analog of Wilson's theorem that $F_d\equiv -1 \bmod \wp$ for $\wp$ a prime of degree d. Though not strictly necessary for this paper, we now introduce a more refined notion of factorial due to Carlitz. For $n\in\mathbb{Z}$ and $n\geq 0$, we define its factorial by

$$n! := \prod D_i^{n_i}\in A \quad \text{for } n=\sum n_i q^i,\ 0\leq n_i<q.$$

See [Thakur 2004, 4.5–4.8, 4.12 and 4.13; 2012] for its properties such as prime factorization, divisibilities, functional equations, interpolations and arithmetic of special values and congruences, which are analogous to those of the classical factorial. See also [Bhargava 2000], which gives many interesting divisibility properties in great generality.

Carlitz proved D_n is the product of monics of degree n. This gives the connection between the two notions above, that for $a\in A_i$, $(\mathcal{N}a-1)!=(-1)^iF_i$. (See [Thakur 2012, Theorem 4.1, Section 6] for more on these analogies and some refinements of analogs of Wilson's theorem.) This also implies

$$F_d=(-1)^d\prod_{j=1}^{d-1}[d-j]^{q^j-1}=(-1)^dD_d/L_d. \tag{1}$$

So let us restate the above well-known analog of Wilson's theorem.

Theorem 2.1. *If $\wp$ is a prime of A of degree d, then*

$$(-1)^d(\mathcal{N}\wp-1)!=F_d\equiv -1 \bmod \wp.$$

This naturally leads to:

Definition 2.2. A prime $\wp \in A_d$ is a Wilson prime if $F_d \equiv -1 \bmod \wp^2$.

Remarks 2.3. If $d = 1$, then $F_d = -1$. So the primes of degree 1 are Wilson primes. If $\wp(t)$ is Wilson prime, then so are $\wp(t+\theta)$ and $\wp(\mu t)$ for $\theta \in \mathbb{F}_q$ and $\mu \in \mathbb{F}_q^*$ as follows immediately from the formula for F_d.

We introduce some differential, difference and arithmetic differential operators.

Definition 2.4. (1) For $\wp$ as above and $a \in A$, let $Q_\wp(a) := (a^{q^d} - a)/\wp$ be the Fermat quotient. We denote its i-th iteration by $Q_\wp^{(i)}$.

(2) For $a = a(t) \in A$, we denote by $D^{(i)}(a) = a^{(i)}$ its i-th derivative with respect to t. We also use the usual short forms $a' = a^{(1)}$ and $a'' = a^{(2)}$.

(3) We define the higher difference quotients $\Delta^{(i)}(a) = a^{[i]}$ of $a \in A$ (with respect to t and θ to be fixed later) by

$$a^{[0]}(t) = a(t) \quad \text{and} \quad a^{[i+1]}(t) = (a^{[i]}(t) - a^{[i]}(\theta))/(t-\theta).$$

Theorem 2.5. *Let $d := \deg \wp$. If $d = 1$, then $F_d = -1$ and the valuation of $Q_\wp(t)$ at $\wp$ is $q-2$.*

Let $d > 1$ and $k \le q$. Then $F_d \equiv -1 \bmod \wp^k$ if and only if $Q_\wp^{(2)}(t) \equiv 0 \bmod \wp^{k-1}$ if and only if $Q_\wp^{(r)}(t) \equiv 0 \bmod \wp$ for $2 \le r \le k$.

Proof. The $d = 1$ case follows immediately from the definitions. Let $d > 1$. We recall (see, e.g., [Thakur 2004, pp. 7 and 103; 2012, proof of Theorem 7.2]) some facts, which we use below.

(i) The product of elements of $(A/\wp^k)^*$ is -1 unless $q = 2$, $d = 1$, and $k = 2$ or 3, as seen by pairing elements with their inverses and counting order-2 elements.

(ii) The product of all monic elements prime to $\wp$ and of degree i is $D_i/(\wp^r D_{i-d})$, where r is uniquely determined by the condition that the quantity is prime to $\wp$.

(iii) Since the valuation of $[m]$ at $\wp$ is 1 or 0 according to whether d divides m, we have $[i+kd] \equiv [i] \bmod \wp^{q^i}$ and thus $[kd]/\wp \equiv [d]/\wp \bmod \wp^{q^d-1}$ for k a positive integer. In particular, these congruences hold modulo $\wp^q$.

Hence, by (1), we have modulo $\wp^k$ (with s appropriate to make the second quantity below a unit at $\wp$)

$$\begin{aligned}
-1 &\equiv (-1)^d \wp^s \frac{D_{kd} L_{(k-1)d}}{L_{kd} D_{(k-1)d}} \\
&\equiv \big((-1)^d [kd-1]^{q-1} \cdots [(k-1)d+1]^{q^{d-1}-1}\big) \wp^s [(k-1)d]^{q^d-1} \\
&\qquad \times \big([(k-1)d-1]^{q^{d+1}-q} \cdots\big)[(k-2)d]^{q^{2d}-q^d} (\cdots) \cdots \\
&\equiv F_d([d]/\wp)^{(q^d-1)+(q^{2d}-q^d)+\cdots} (D_{d-1}^q)^{q^d-1} \big(D_{d-1}^{q^{d+1}}\big)^{q^d-1} \cdots \\
&\equiv F_d([d]/\wp)^{q^{(k-1)d}-1},
\end{aligned}$$

where we used that, for a prime to $\wp$, we have $a^{q^{id}-1} \equiv 1 \bmod \wp$ and thus $a^{q(q^{id}-1)} \equiv 1 \bmod \wp^q$.

Hence, if $Q_\wp^{(2)}(t)$ is 0 modulo $\wp^{k-1}$, then $([d]/\wp)^{q^d-1} \equiv 1 \bmod \wp^k$, and thus, $F_d \equiv -1 \bmod \wp^k$. Conversely, writing $([d]/\wp)^{q^d-1} = 1 + a\wp$ for some $a \in A$, we see that if $F_d \equiv -1 \bmod \wp^k$, then modulo $\wp^k$, we have

$$1 \equiv (1 + a\wp)^{1+q^d+\cdots+q^{(k-2)d}} \equiv 1 + a\wp$$

so that $a\wp \equiv 0$ as desired. The other implications are immediate. $\square$

This generalizes the $k = 2$ case [Sauerberg et al. 2013, Theorem 2.6] with a different manipulation of the quantities even in that case.

Next, we use this to give another criterion for Wilson prime now using the derivative of the Fermat quotient instead of iterated Fermat quotient! For a general study of differential operators in the arithmetic context, their classification and applications, we refer to [Buium 2005] and references there. See also [Ihara 1992].

Theorem 2.6. *Assume $q > 2$ or $d > 1$. The prime $\wp$ is a Wilson prime if and only if $\wp$ divides the derivative of $[d]/\wp$ with respect to t.*

Proof. Let $a = [d]/\wp = \sum a_i t^i$. Then by the binomial theorem, modulo $[d]^2$, we have

$$a^{q^d} - a \equiv \sum a_i t^i \big((t^{q^d-1} - 1 + 1)^i - 1\big) \equiv \sum a_i t^i \binom{i}{1} ([d]/t)^1 \equiv a'[d].$$

(In words, the Frobenius difference quotient $(a^{q^d} - a)/(t^{q^d} - t)$ of $a = Q_\wp(t)$ with respect to t is congruent to the derivative of a with respect to t modulo any prime of degree dividing the degree of $\wp$.) Now since a is square-free and, in particular, not a p-th power, a' is nonzero, and since the valuation of $[d]$ at $\wp$ is 1, the claim follows from Theorem 2.5. $\square$

This reduces computations from dq^d-degree polynomials occurring in F_d to just q^d-degree or from iterates of Fermat quotients to the first one. Also, the derivative kills $1/p$ of the coefficients on average. In fact, we will improve further.

Now we consider $\wp$-adic expansion of t using Teichmüller representatives. Let $A_\wp$ be the completion of A at $\wp$, and let $\mathbb{F}_\wp$ be its residue field. Let $\theta \in \mathbb{F}_\wp$ be the Teichmüller representative of t modulo $\wp$.

Lemma 2.7. *Let $t = \theta + \sum \mu_i \wp^i$ be the $\wp$-adic expansion of t with Teichmüller representatives $\mu_i \in \mathbb{F}_\wp$. Then*

$$\mu_1 = \frac{1}{\wp^{[1]}(\theta)} = \frac{1}{\wp^{(1)}(\theta)} \quad \text{and} \quad \mu_2 = -\frac{\wp^{[2]}(\theta)}{\wp^{[1]}(\theta)^3}.$$

More generally, if $(t-\theta)^r$ divides $\wp^{[2]}$, then $\mu_i = \wp^{[i]}(\theta) = 0$ for $2 \le i < r$, and for $2 \le i \le r$, we have

$$\mu_i = -\frac{\wp^{[i]}(\theta)}{\wp^{[1]}(\theta)^{i+1}}.$$

Proof. For $d=1$, we have $t=\theta+\wp$, whereas for $d>1$ the expansion is an infinite sum. Noting that $\wp=\prod(t-\theta^{q^i})$, where i runs from 0 to $d-1$, the claim follows inductively on i by starting with the unknown $\wp$-adic expansion and by dividing by $t-\theta$ and then putting $t=\theta$ in each step.

In more detail, in the first step, we have $1=\mu_1\prod_{d>i>0}(t-\theta^{q^i})$ plus terms divisible by $t-\theta$ so that $\mu_1=1/\prod(\theta-\theta^{q^i})=1/\wp^{(1)}(\theta)$. In the next step, we have $-\wp^{[2]}/(\wp^{[1]}(\theta)(\wp^{[1]})^2)=\mu_2+\mu_3(t-\theta)\wp^{[1]}+\cdots$, proving the claim for μ_2. Under the hypothesis of divisibility, the claims are clear inductively on i. □

Remarks 2.8. We record in passing that without any hypothesis as in the second part of Lemma 2.7, a similar manipulation leads to

$$\mu_3=-\frac{\wp^{[3]}(\theta)}{\wp^{[1]}(\theta)^4}+2\frac{\wp^{[2]}(\theta)^2}{\wp^{[1]}(\theta)^5}.$$

Note that the second term vanishes if $\wp^{[2]}(\theta)=0$ (or if $p=2$).

We now use Theorem 2.5 and Lemma 2.7 to get our main theorem, a criterion for Wilson prime in terms of vanishing at θ of the second difference quotient value as well as in terms of the total vanishing of the second derivative of $\wp$ with respect to t:

Theorem 2.9. (i) *A prime $\wp$ is a Wilson prime if and only if $\wp^{[2]}(\theta)=0$.*

(ii) *When $p>2$, $\wp$ is a Wilson prime if and only if $\wp''=d^2\wp/dt^2$ is identically zero. In other words, the Wilson primes are exactly the primes of the form $\sum p_it^i$ with p_i nonzero implying $i\equiv 0, 1 \bmod p$.*

(iii) *When $p>2$, if $\wp$ is a Wilson prime, then the Wilson congruence holds modulo $\wp^{p-1}$. Also, $\wp^{[i]}(\theta)=0$ for $1<i<p$.*

(iv) *When $p=2$, the Wilson primes are exactly the primes of the form $\sum p_it^i$ with p_i nonzero implying $i\equiv 0, 1 \bmod 4$. For such $\wp$, the Wilson congruence holds modulo $\wp^3$, and $\wp^{[i]}(\theta)=0$ for $1<i<4$.*

Proof. We have $Q_\wp(t)=-\mu_1-\mu_2\wp-\cdots-\mu_{q^d-1}\wp^{q^d-2} \bmod \wp^{q^d-1}$ and

$$Q_\wp(Q_\wp(t))=\mu_2+\mu_3\wp+\cdots+\mu_{q^d-1}\wp^{q^d-3} \bmod \wp^{q^d-2}.$$

Hence, (i) follows by Lemma 2.7.

Let $\alpha:=\wp^{(1)}(\theta)$ and $f(t)=\wp(t)-\alpha(t-\theta)$. Then $\wp^{[2]}(\theta)=0$ is equivalent to $(t-\theta)^3$ dividing $f(t)$. This condition implies $f''(\theta)=\wp''(\theta)=0$, but $\wp$ being an irreducible polynomial with θ as a root, this implies that the lower degree second derivative is identically zero. Conversely, $f(\theta)=f'(\theta)=0$ implies, if $d>1$, $f(t)=(t-\theta)^2h(t)$, and $f''(\theta)=0$ then implies that $2h(\theta)=0$ so that if $p>2$, h is divisible by $t-\theta$, implying (ii).

Once the second derivative is identically zero, the higher derivatives are also zero. (Note the $(d+1)$-th derivative or p-th derivative is identically zero anyway for any $\wp$.) The vanishing of first i derivatives implies at least $i+1$ multiplicity for $i < p$, which implies vanishing of higher difference quotients (which decrease in degree by 1 in each step). This implies (iii) by Lemma 2.7 and Theorem 2.5.

Here is an another way to see the last part. If we write $\wp(t) = \sum p_i t^i$, then $\wp(t+\theta) = \sum \alpha_i t^i$ with $\alpha_i = \sum p_k \binom{i}{k}\theta^{k-i}$. Our condition translates to t^3 dividing $f(t+\theta)$ so that $\alpha_2 = 0$. By Lucas' theorem or directly, if $p = 2$, $\binom{i}{2} = 0$ implies $\binom{i}{3} = 0$ so that $\alpha_3 = 0$. Similarly, for general p, $\binom{i}{2} = 0$ implies $\binom{i}{r} = 0$ for $2 \le r \le p-1$, implying $\alpha_r = 0$ for those r. This also proves (iv). □

Theorem 2.10. *There are infinitely many Wilson primes for* $\mathbb{F}_q[t]$.

Proof. First let q be odd. It is enough to produce infinitely many irreducible elements in A that have powers of t occurring only with exponents that are 0 or 1 modulo p. Let n be a positive integer. Then by consideration of factorization of the cyclotomic polynomial, we see that there are $\phi(q^n-1)/n$ primitive monic polynomials of degree n, where (as usual) we mean by a primitive polynomial of degree n a minimal polynomial over $\mathbb{F}_q$ of a generator of $\mathbb{F}_{q^n}$. For each such irreducible polynomial $P(t) = \sum p_i t^i$, the polynomial $\sum p_i t^{(q^i-1)/(q-1)}$ is of the form we want and is irreducible by a theorem of Ore [1934, Chapter 3, Theorem 1].

The same method works for $q = 2^s$ with $s > 1$ since the exponents are then 0, 1 mod 4 as we require. The remaining case $q = 2$ can not be handled by this method. In this case, applying Serret's theorem [Lidl and Niederreiter 1996, Theorem 3.3.5] (or the special case recalled in [Sauerberg et al. 2013, Theorem 2.8]) to the (Wilson) prime $f(t) = t^4 + t + 1$ and $s = 5^n$, we get infinitely many primes $f(t^{5^n})$, which are Wilson primes by Theorem 2.9(iv). □

Remarks 2.11 (Heuristic counts and exact multiplicity). In the $\mathbb{Z}$ case, the number of Wilson primes less than x grows like $\sum_{p<x} 1/p \sim \log\log(x)$ under the naïve heuristics of $((p-1)!+1)/p$ being randomly distributed modulo p, and we expect at most finitely many primes giving the congruence to power p^3. In [Sauerberg et al. 2013] for some q, we produced families of Wilson primes for A with $\log\log(x)$ growth of the size, but now with Theorem 2.9(ii), we can show that there are many more. In fact, if we let π_d and w_d denote the number of primes and Wilson primes, respectively, of A of degree d, then under the naïve heuristics of randomness of p_i in Theorem 2.9(ii) for primes, we see that as d tends to infinity and $(\log w_d)/(\log \pi_d)$ approaches $2/p$ if p is odd and $1/2$ if $p = 2$. It should be possible to prove these asymptotics using Theorem 2.9(ii). In our case, the congruence holds to power $\wp^{p-1}$ for the Wilson primes (to power $\wp^3$ if $p = 2$). It is unclear whether this power can be increased for some primes. Though the correspondence of Theorem 2.5 goes up to power $\wp^{q-1}$, the small amount of numerical data calculated by the author's masters student George Todd (for which the author thanks him) showed exactness of the power $\wp^{p-1}$ even for q not prime.

Remarks 2.12. We finish by giving quick sketches of alternate and simplified proofs of earlier results.

(1) We know that for $a \in \mathbb{F}_q$, $\wp = t^p - t - a$ is a prime of A if and only if trace of a to $\mathbb{F}_p$ is nonzero. Assume $\wp$ is a prime and $q = p^m$. Then

$$t^{q^p} = \wp^{p^{mp-1}} + t^{p^{mp-1}} + a^{p^{mp-1}} = \cdots = \wp^{p^{mp-1}} + \wp^{p^{mp-2}} + \cdots + \wp + t$$

so that $Q_\wp(t) = (t^{q^p} - t)/\wp = \wp^{p^{mp-1}-1} + \cdots + \wp^{p-1} + 1$. If $Q_\wp^{(r)}(t)$ denotes the r-th iteration of $Q_\wp$, we see immediately by induction that for $p \geq r > 1$, the valuation at $\wp$ of $Q_\wp^{(r)}(t)$ is $p - r$. Similarly, it is easy to check that $q = 2$ and $\wp = t^4 + t + 1$ satisfies the Wilson congruence modulo $\wp^3$ but not $\wp^4$, and similarly, a calculation as above shows that in this case $Q_\wp^{(3)}(t)$ vanishes modulo $\wp$ but not $Q_\wp^{(4)}(t)$.

This gives another proof of [Thakur 2012, Theorem 7.1], which says that such $\wp$'s are Wilson primes (even to the exact $(p-1)$-th power congruence) if $p > 2$.

(2) Theorem 2.6 allows us to give a simple alternate proof of [Sauerberg et al. 2013, Theorem 2.9]. By the theorem above, $\wp(t)^2$ divides $1 + (t^{q^d} - t)\wp'(t)/\wp(t)$ so that modulo $\wp(t^s)^2$,

$$0 \equiv 1 + \bigl(t^{s(q^d-1)} - 1\bigr) t^s \wp'(t^s)/\wp(t^s) \equiv 1 + \bigl(t^{q^{ds}-1} - 1\bigr) t \wp'(t^s) s t^{s-1}/\wp(t^s),$$

exactly as in the middle part of the proof of [Sauerberg et al. 2013, Theorem 2.9]. This implies by Theorem 2.5 that $\wp(t^s)$ is Wilson prime as desired.

(3) Theorem 2.6 also provides another proof for the reciprocal prime theorem [Sauerberg et al. 2013, Theorem 3.3] when p is odd. If $f(t) = t^d \wp(1/t)$ and $\wp$ is a Wilson prime, then $\wp'' = 0$ and $d(d-1) = 0 \bmod p$ so that taking derivatives with the product and chain rules simplifies to $f'' = -2(d-1)t^{d-3}\wp'(1/t)$, which is 0 if and only if $d \equiv 1 \bmod p$.

Using Theorem 2.9(ii) and (iv), instead of Theorem 2.6, gives even simpler proofs of results in (2) and (3) (and also (1) except for the exactness of the exponent $p-1$ in the modulus). We leave it as a straightforward exercise.

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dinesh.thakur@rochester.edu *1013 Hylan Building, Department of Mathematics, University of Rochester, RC Box 270138, Rochester, NY 14627, United States*

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Principal W-algebras for $\mathrm{GL}(m|n)$

Jonathan Brown, Jonathan Brundan and Simon M. Goodwin

We consider the (finite) W-algebra $W_{m|n}$ attached to the principal nilpotent orbit in the general linear Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{C})$. Our main result gives an explicit description of $W_{m|n}$ as a certain truncation of a shifted version of the Yangian $Y(\mathfrak{gl}_{1|1})$. We also show that $W_{m|n}$ admits a triangular decomposition and construct its irreducible representations.

1. Introduction

A (*finite*) *W-algebra* is a certain filtered deformation of the Slodowy slice to a nilpotent orbit in a complex semisimple Lie algebra $\mathfrak{g}$. Although the terminology is more recent, the construction has its origins in the classic work of Kostant [1978]. In particular, Kostant showed that the principal W-algebra—the one associated to the principal nilpotent orbit in $\mathfrak{g}$—is isomorphic to the center of the universal enveloping algebra $U(\mathfrak{g})$. In the last few years, there has been some substantial progress in understanding W-algebras for other nilpotent orbits thanks to works of Premet, Losev and others; see [Losev 2011] for a survey. The story is most complete (also easiest) for $\mathfrak{sl}_n(\mathbb{C})$. In this case, the W-algebras are closely related to *shifted Yangians*; see [Brundan and Kleshchev 2006].

Analogues of W-algebras have also been defined for Lie superalgebras; see, for example, the work of De Sole and Kac [2006, §5.2] (where they are defined in terms of BRST cohomology) or the more recent paper of Zhao [2012] (which focuses mainly on the queer Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$). In this article, we consider the easiest of all the "super" situations: the *principal W-algebra $W_{m|n}$ for the general linear Lie superalgebra* $\mathfrak{gl}_{m|n}(\mathbb{C})$. Our main result gives an explicit isomorphism between $W_{m|n}$ and a certain truncation of a shifted subalgebra of the Yangian $Y(\mathfrak{gl}_{1|1})$; see Theorem 4.5. Its proof is very similar to the proof of the analogous result for nilpotent matrices of Jordan type (m, n) in $\mathfrak{gl}_{m+n}(\mathbb{C})$ from [Brundan and Kleshchev 2006].

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The (super)algebra $W_{m|n}$ turns out to be quite close to being supercommutative. More precisely, we show that it admits a triangular decomposition

$$W_{m|n} = W_{m|n}^- W_{m|n}^0 W_{m|n}^+$$

in which $W_{m|n}^-$ and $W_{m|n}^+$ are exterior algebras of dimension $2^{\min(m,n)}$ and $W_{m|n}^0$ is a symmetric algebra of rank $m+n$; see Theorem 6.1. This implies that all the irreducible $W_{m|n}$-modules are finite-dimensional; see Theorem 7.2. We show further that they all arise as certain tensor products of irreducible $\mathfrak{gl}_{1|1}(\mathbb{C})$- and $\mathfrak{gl}_1(\mathbb{C})$-modules; see Theorem 8.4. In particular, all irreducible $W_{m|n}$-modules are of dimension dividing $2^{\min(m,n)}$. A closely related assertion is that all irreducible highest-weight representations of $Y(\mathfrak{gl}_{1|1})$ are tensor products of evaluation modules; this is similar to a well-known phenomenon for $Y(\mathfrak{gl}_2)$ going back to [Tarasov 1985].

Some related results about $W_{m|n}$ have been obtained independently by Poletaeva and Serganova [2013]. In fact, the connection between $W_{m|n}$ and the Yangian $Y(\mathfrak{gl}_{1|1})$ was foreseen long ago by Briot and Ragoucy [2003], who also looked at certain nonprincipal nilpotent orbits, which they assert are connected to higher-rank super Yangians although we do not understand their approach. It should be possible to combine the methods of this article with those of [Brundan and Kleshchev 2006] to establish such a connection for *all* nilpotent orbits in $\mathfrak{gl}_{m|n}(\mathbb{C})$. However, this is not trivial and will require some new presentations for the higher-rank super Yangians adapted to arbitrary parity sequences; the ones in [Gow 2007; Peng 2011] are not sufficient as they only apply to the standard parity sequence.

By analogy with the results of Kostant [1978], our expectation is that $W_{m|n}$ will play a distinguished role in the representation theory of $\mathfrak{gl}_{m|n}(\mathbb{C})$. In a forthcoming article [Brown et al.], we will investigate the *Whittaker coinvariants functor* H_0, a certain exact functor from the analogue of category $\mathbb{O}$ for $\mathfrak{gl}_{m|n}(\mathbb{C})$ to the category of finite-dimensional $W_{m|n}$-modules. We view this as a replacement for the functor $\mathbb{V}$ of Soergel [1990]; see also [Backelin 1997]. We will show that H_0 sends irreducible modules in $\mathbb{O}$ to irreducible $W_{m|n}$-modules or 0 and that all irreducible $W_{m|n}$-modules occur in this way; this should be compared with the analogous result for parabolic category $\mathbb{O}$ for $\mathfrak{gl}_{m+n}(\mathbb{C})$ obtained in [Brundan and Kleshchev 2008, Theorem E]. We will also use properties of H_0 to prove that the center of $W_{m|n}$ is isomorphic to the center of the universal enveloping superalgebra of $\mathfrak{gl}_{m|n}(\mathbb{C})$.

Notation. We denote the parity of a homogeneous vector x in a $\mathbb{Z}/2$-graded vector space by $|x| \in \{\bar{0}, \bar{1}\}$. A *superalgebra* means a $\mathbb{Z}/2$-graded algebra over $\mathbb{C}$. For homogeneous x and y in an associative superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, their *supercommutator* is $[x, y] := xy - (-1)^{|x||y|}yx$. We say that A is *supercommutative* if $[x, y] = 0$ for all homogeneous $x, y \in A$. Also for homogeneous $x_1, \dots, x_n \in A$, an *ordered supermonomial* in $x_1, \dots, x_n$ means a monomial of the form $x_1^{i_1} \cdots x_n^{i_n}$ for $i_1, \dots, i_n \geq 0$ such that $i_j \leq 1$ if x_j is odd.

2. Shifted Yangians

Recall that $\mathfrak{gl}_{m|n}(\mathbb{C})$ is the Lie superalgebra of all $(m+n)\times(m+n)$ complex matrices under the supercommutator with $\mathbb{Z}/2$-grading defined so that the matrix unit $e_{i,j}$ is even if $1 \le i, j \le m$ or $m+1 \le i, j \le m+n$ and $e_{i,j}$ is odd otherwise. We denote its universal enveloping superalgebra $U(\mathfrak{gl}_{m|n})$; it has basis given by all ordered supermonomials in the matrix units.

The Yangian $Y(\mathfrak{gl}_{m|n})$ was introduced originally by Nazarov [1991]; see also [Gow 2007]. We only need here the special case of $Y = Y(\mathfrak{gl}_{1|1})$. For its definition, we fix a choice of *parity sequence*

$$(|1|, |2|) \in \mathbb{Z}/2 \times \mathbb{Z}/2 \tag{2-1}$$

with $|1| \neq |2|$. All subsequent notation in the remainder of the article depends implicitly on this choice. Then we define Y to be the associative superalgebra on generators $\{t_{i,j}^{(r)} \mid 1 \le i, j \le 2,\ r > 0\}$, with $t_{i,j}^{(r)}$ of parity $|i|+|j|$, subject to the relations

$$[t_{i,j}^{(r)}, t_{p,q}^{(s)}] = (-1)^{|i||j|+|i||p|+|j||p|} \sum_{a=0}^{\min(r,s)-1} (t_{p,j}^{(a)} t_{i,q}^{(r+s-1-a)} - t_{p,j}^{(r+s-1-a)} t_{i,q}^{(a)}),$$

adopting the convention that $t_{i,j}^{(0)} = \delta_{i,j}$ (Kronecker delta).

Remark 2.1. In the literature, one typically only finds results about $Y(\mathfrak{gl}_{1|1})$ proved for the definition coming from the parity sequence $(|1|, |2|) = (\bar{0}, \bar{1})$. To aid in translating between this and the other possibility, we note that the map $t_{i,j}^{(r)} \mapsto (-1)^r t_{i,j}^{(r)}$ defines an isomorphism between the realizations of $Y(\mathfrak{gl}_{1|1})$ arising from the two choices of parity sequence.

As in [Nazarov 1991], we introduce the generating function

$$t_{i,j}(u) := \sum_{r \ge 0} t_{i,j}^{(r)} u^{-r} \in Y[\![u^{-1}]\!].$$

Then Y is a Hopf superalgebra with comultiplication Δ and counit ε given in terms of generating functions by

$$\Delta(t_{i,j}(u)) = \sum_{h=1}^{2} t_{i,h}(u) \otimes t_{h,j}(u), \tag{2-2}$$

$$\varepsilon(t_{i,j}(u)) = \delta_{i,j}. \tag{2-3}$$

There are also algebra homomorphisms

$$\mathrm{in} : U(\mathfrak{gl}_{1|1}) \to Y, \quad e_{i,j} \mapsto (-1)^{|i|} t_{i,j}^{(1)}, \tag{2-4}$$

$$\mathrm{ev} : Y \to U(\mathfrak{gl}_{1|1}), \quad t_{i,j}^{(r)} \mapsto \delta_{r,0}\delta_{i,j} + (-1)^{|i|}\delta_{r,1} e_{i,j}. \tag{2-5}$$

The composite $\mathrm{ev} \circ \mathrm{in}$ is the identity; hence, in is injective and ev is surjective. We call ev the *evaluation homomorphism*.

We need another set of generators for Y called *Drinfeld generators*. To define these, we consider the Gauss factorization $T(u) = F(u)D(u)E(u)$ of the matrix

$$T(u) := \begin{pmatrix} t_{1,1}(u) & t_{1,2}(u) \\ t_{2,1}(u) & t_{2,2}(u) \end{pmatrix}.$$

This defines power series $d_i(u), e(u), f(u) \in Y[\![u^{-1}]\!]$ such that

$$D(u) = \begin{pmatrix} d_1(u) & 0 \\ 0 & d_2(u) \end{pmatrix}, \qquad E(u) = \begin{pmatrix} 1 & e(u) \\ 0 & 1 \end{pmatrix}, \qquad F(u) = \begin{pmatrix} 1 & 0 \\ f(u) & 1 \end{pmatrix}.$$

Thus, we have that

$$d_1(u) = t_{1,1}(u), \qquad d_2(u) = t_{2,2}(u) - t_{2,1}(u)t_{1,1}(u)^{-1}t_{1,2}(u), \tag{2-6}$$

$$e(u) = t_{1,1}(u)^{-1}t_{1,2}(u), \qquad f(u) = t_{2,1}(u)t_{1,1}(u)^{-1}. \tag{2-7}$$

Equivalently,

$$t_{1,1}(u) = d_1(u), \qquad t_{2,2}(u) = d_2(u) + f(u)d_1(u)e(u), \tag{2-8}$$

$$t_{1,2}(u) = d_1(u)e(u), \qquad t_{2,1}(u) = f(u)d_1(u). \tag{2-9}$$

The Drinfeld generators are the elements $d_i^{(r)}$, $e^{(r)}$ and $f^{(r)}$ of Y defined from the expansions $d_i(u) = \sum_{r\geq 0} d_i^{(r)}u^{-r}$, $e(u) = \sum_{r\geq 1} e^{(r)}u^{-r}$ and $f(u) = \sum_{r\geq 1} f^{(r)}u^{-r}$. Also define $\tilde{d}_i^{(r)} \in Y$ from the identity $\tilde{d}_i(u) = \sum_{r\geq 0} \tilde{d}_i^{(r)}u^{-r} := d_i(u)^{-1}$.

Theorem 2.2 [Gow 2007, Theorem 3]. *The superalgebra Y is generated by the even elements $\{d_i^{(r)} \mid i = 1, 2,\ r > 0\}$ and odd elements $\{e^{(r)}, f^{(r)} \mid r > 0\}$ subject only to the following relations*:

$$[d_i^{(r)}, d_j^{(s)}] = 0, \qquad [e^{(r)}, f^{(s)}] = (-1)^{|1|} \sum_{a=0}^{r+s-1} \tilde{d}_1^{(a)} d_2^{(r+s-1-a)},$$

$$[e^{(r)}, e^{(s)}] = 0, \qquad [d_i^{(r)}, e^{(s)}] = (-1)^{|1|} \sum_{a=0}^{r-1} d_i^{(a)} e^{(r+s-1-a)},$$

$$[f^{(r)}, f^{(s)}] = 0, \qquad [d_i^{(r)}, f^{(s)}] = -(-1)^{|1|} \sum_{a=0}^{r-1} f^{(r+s-1-a)} d_i^{(a)}.$$

Here $d_i^{(0)} = 1$ and $\tilde{d}_i^{(r)}$ is defined recursively from $\sum_{a=0}^{r} \tilde{d}_i^{(a)} d_i^{(r-a)} = \delta_{r,0}$.

Remark 2.3. By [Gow 2007, Theorem 4], the coefficients $\{c^{(r)} \mid r > 0\}$ of the power series

$$c(u) = \sum_{r\geq 0} c^{(r)}u^{-r} := \tilde{d}_1(u)d_2(u) \tag{2-10}$$

generate the center of Y. Moreover, $[e^{(r)}, f^{(s)}] = (-1)^{|1|}c^{(r+s-1)}$, so these supercommutators are central.

Remark 2.4. Using the relations in Theorem 2.2, one can check that Y admits an algebra automorphism

$$\zeta : Y \to Y, \quad d_1^{(r)} \mapsto \tilde{d}_2^{(r)},\ d_2^{(r)} \mapsto \tilde{d}_1^{(r)},\ e^{(r)} \mapsto -f^{(r)},\ f^{(r)} \mapsto -e^{(r)}. \tag{2-11}$$

By [Gow 2007, Proposition 4.3], this satisfies

$$\Delta \circ \zeta = P \circ (\zeta \otimes \zeta) \circ \Delta, \tag{2-12}$$

where $P(x \otimes y) = (-1)^{|x||y|} y \otimes x$.

Proposition 2.5. *The comultiplication Δ is given on Drinfeld generators by the following*:

$$\begin{aligned}
\Delta(d_1(u)) &= d_1(u) \otimes d_1(u) + d_1(u)e(u) \otimes f(u)d_1(u),\\
\Delta(\tilde{d}_1(u)) &= \sum_{n\geq 0}(-1)^{\lceil n/2\rceil} e(u)^n \tilde{d}_1(u) \otimes \tilde{d}_1(u) f(u)^n,\\
\Delta(d_2(u)) &= \sum_{n\geq 0}(-1)^{\lfloor n/2\rfloor} d_2(u)e(u)^n \otimes f(u)^n d_2(u),\\
\Delta(\tilde{d}_2(u)) &= \tilde{d}_2(u) \otimes \tilde{d}_2(u) - e(u)\tilde{d}_2(u) \otimes \tilde{d}_2(u) f(u),\\
\Delta(e(u)) &= 1 \otimes e(u) - \sum_{n\geq 1}(-1)^{\lceil n/2\rceil} e(u)^n \otimes \tilde{d}_1(u) f(u)^{n-1} d_2(u),\\
\Delta(f(u)) &= f(u) \otimes 1 - \sum_{n\geq 1}(-1)^{\lceil n/2\rceil} d_2(u)e(u)^{n-1}\tilde{d}_1(u) \otimes f(u)^n.
\end{aligned}$$

Proof. Check the formulae for $d_1(u)$, $\tilde{d}_1(u)$ and $e(u)$ directly using (2-2), (2-6) and (2-7). The other formulae then follow using (2-12). □

Here is the *PBW theorem* for Y.

Theorem 2.6 [Gow 2007, Theorem 1]. *Order the set $\{t_{i,j}^{(r)} \mid 1 \leq i, j \leq 2,\ r > 0\}$ in some way. The ordered supermonomials in these generators give a basis for Y.*

There are two important filtrations on Y. First we have the *Kazhdan filtration*, which is defined by declaring that the generator $t_{i,j}^{(r)}$ is in degree r, i.e., the filtered degree-r part $F_r Y$ of Y with respect to the Kazhdan filtration is the span of all monomials of the form $t_{i_1,j_1}^{(r_1)} \cdots t_{i_n,j_n}^{(r_n)}$ such that $r_1 + \cdots + r_n \leq r$. The defining relations imply that the associated graded superalgebra $\operatorname{gr} Y$ is supercommutative. Let $\mathfrak{gl}_{1|1}[x]$ denote the current Lie superalgebra $\mathfrak{gl}_{1|1}(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[x]$ with basis $\{e_{i,j}x^r \mid 1 \leq i, j \leq 2,\ r \geq 0\}$. Then Theorem 2.6 implies that $\operatorname{gr} Y$ can be identified with the symmetric superalgebra $S(\mathfrak{gl}_{1|1}[x])$ of the vector superspace $\mathfrak{gl}_{1|1}[x]$ so that $\operatorname{gr}_r t_{i,j}^{(r)} = (-1)^{|i|} e_{i,j} x^{r-1}$.

The other filtration on Y, which we call the *Lie filtration*, is defined similarly by declaring that $t_{i,j}^{(r)}$ is in degree $r-1$. In this case, we denote the filtered degree-r part of Y by $F'_r Y$ and the associated graded superalgebra by $\mathrm{gr}' Y$. By Theorem 2.6 and the defining relations once again, $\mathrm{gr}' Y$ can be identified with the universal enveloping superalgebra $U(\mathfrak{gl}_{1|1}[x])$ so that $\mathrm{gr}'_{r-1} t_{i,j}^{(r)} = (-1)^{|i|} e_{i,j} x^{r-1}$. The Drinfeld generators $d_i^{(r)}$, $e^{(r)}$ and $f^{(r)}$ all lie in $F'_{r-1} Y$, and we have that

$$\mathrm{gr}'_{r-1} d_i^{(r)} = \mathrm{gr}'_{r-1} t_{i,i}^{(r)}, \qquad \mathrm{gr}'_{r-1} e^{(r)} = \mathrm{gr}'_{r-1} t_{1,2}^{(r)}, \qquad \mathrm{gr}'_{r-1} f^{(r)} = \mathrm{gr}'_{r-1} t_{2,1}^{(r)}.$$

(The situation for the Kazhdan filtration is more complicated: although $d_i^{(r)}$, $e^{(r)}$ and $f^{(r)}$ do all lie in $F_r Y$, their images in $\mathrm{gr}_r Y$ are not in general equal to the images of $t_{i,i}^{(r)}$, $t_{1,2}^{(r)}$ or $t_{2,1}^{(r)}$, but they can expressed in terms of them via (2-6) and (2-7).)

Combining the preceding discussion of the Lie filtration with Theorem 2.6, we obtain the following basis for Y in terms of Drinfeld generators. (One can also deduce this by working with the Kazhdan filtration and using (2-6)–(2-9).)

Corollary 2.7. *Order the set* $\{d_i^{(r)} \mid i = 1, 2,\ r > 0\} \cup \{e^{(r)}, f^{(r)} \mid r > 0\}$ *in some way. The ordered supermonomials in these generators give a basis for* Y.

Now we are ready to introduce the *shifted Yangians* for $\mathfrak{gl}_{1|1}(\mathbb{C})$. This parallels the definition of shifted Yangians in the purely even case from [Brundan and Kleshchev 2006, §2]. Let $\sigma = (s_{i,j})_{1\le i,j\le 2}$ be a 2×2 matrix of nonnegative integers with $s_{1,1} = s_{2,2} = 0$. We refer to such a matrix as a *shift matrix*. Let Y_σ be the superalgebra with even generators $\{d_i^{(r)} \mid i = 1, 2,\ r > 0\}$ and odd generators $\{e^{(r)} \mid r > s_{1,2}\} \cup \{f^{(r)} \mid r > s_{2,1}\}$ subject to all of the relations from Theorem 2.2 that make sense, bearing in mind that we no longer have available the generators $e^{(r)}$ for $0 < r \le s_{1,2}$ or $f^{(r)}$ for $0 < r \le s_{2,1}$. Clearly there is a homomorphism $Y_\sigma \to Y$ that sends the generators of Y_σ to the generators with the same name in Y.

Theorem 2.8. *Order the set*

$$\{d_i^{(r)} \mid i = 1, 2,\ r > 0\} \cup \{e^{(r)} \mid r > s_{1,2}\} \cup \{f^{(r)} \mid r > s_{2,1}\}$$

in some way. The ordered supermonomials in these generators give a basis for Y_σ. *In particular, the homomorphism* $Y_\sigma \to Y$ *is injective.*

Proof. It is easy to see from the defining relations that the monomials span, and their images in Y are linearly independent by Corollary 2.7. □

From now on, we will identify Y_σ with a subalgebra of Y via the injective homomorphism $Y_\sigma \hookrightarrow Y$. The Kazhdan and Lie filtrations on Y induce filtrations on Y_σ such that $\mathrm{gr}\, Y_\sigma \subseteq \mathrm{gr}\, Y$ and $\mathrm{gr}' Y_\sigma \subseteq \mathrm{gr}' Y$. Let $\mathfrak{gl}_{1|1}^\sigma[x]$ be the Lie subalgebra of $\mathfrak{gl}_{1|1}[x]$ spanned by the vectors $e_{i,j} x^r$ for $1 \le i, j \le 2$ and $r \ge s_{i,j}$. Then we have that $\mathrm{gr}\, Y_\sigma = S(\mathfrak{gl}_{1|1}^\sigma[x])$ and $\mathrm{gr}' Y_\sigma = U(\mathfrak{gl}_{1|1}^\sigma[x])$.

Remark 2.9. For another shift matrix $\sigma' = (s'_{i,j})_{1 \le i,j \le 2}$ with $s'_{2,1} + s'_{1,2} = s_{2,1} + s_{1,2}$, there is an isomorphism

$$\iota : Y_\sigma \xrightarrow{\sim} Y_{\sigma'}, \quad d_i^{(r)} \mapsto d_i^{(r)},\ e^{(r)} \mapsto e^{(s'_{1,2} - s_{1,2} + r)},\ f^{(r)} \mapsto f^{(s'_{2,1} - s_{2,1} + r)}. \tag{2-13}$$

This follows from the defining relations. Thus, up to isomorphism, Y_σ depends only on the integer $s_{2,1} + s_{1,2} \ge 0$, not on σ itself. Beware though that the isomorphism ι does not respect the Kazhdan or Lie filtrations.

For $\sigma \neq 0$, Y_σ is not a Hopf subalgebra of Y. However, there are some useful comultiplication-like homomorphisms between different shifted Yangians. To start with, let σ^{up} and σ^{lo} be the upper and lower triangular shift matrices obtained from σ by setting $s_{2,1}$ and $s_{1,2}$, respectively, equal to 0. Then, by Proposition 2.5, the restriction of the comultiplication Δ on Y gives a homomorphism

$$\Delta : Y_\sigma \to Y_{\sigma^{\mathrm{lo}}} \otimes Y_{\sigma^{\mathrm{up}}}. \tag{2-14}$$

The remaining comultiplication-like homomorphisms involve the universal enveloping algebra $U(\mathfrak{gl}_1) = \mathbb{C}[e_{1,1}]$. Assuming that $s_{1,2} > 0$, let σ_+ be the shift matrix obtained from σ by subtracting 1 from the entry $s_{1,2}$. Then the relations imply that there is a well-defined algebra homomorphism

$$\begin{aligned} &\Delta_+ : Y_\sigma \to Y_{\sigma_+} \otimes U(\mathfrak{gl}_1), \\ &d_1^{(r)} \mapsto d_1^{(r)} \otimes 1, \qquad d_2^{(r)} \mapsto d_2^{(r)} \otimes 1 + (-1)^{|2|} d_2^{(r-1)} \otimes e_{1,1}, \\ &e^{(r)} \mapsto e^{(r)} \otimes 1 + (-1)^{|2|} e^{(r-1)} \otimes e_{1,1}, \qquad f^{(r)} \mapsto f^{(r)} \otimes 1. \end{aligned} \tag{2-15}$$

Finally, assuming that $s_{2,1} > 0$, let σ_- be the shift matrix obtained from σ by subtracting 1 from $s_{2,1}$. Then there is an algebra homomorphism

$$\begin{aligned} &\Delta_- : Y_\sigma \to U(\mathfrak{gl}_1) \otimes Y_{\sigma_-}, \\ &d_1^{(r)} \mapsto 1 \otimes d_1^{(r)}, \qquad d_2^{(r)} \mapsto 1 \otimes d_2^{(r)} + (-1)^{|2|} e_{1,1} \otimes d_2^{(r-1)}, \\ &f^{(r)} \mapsto 1 \otimes f^{(r)} + (-1)^{|2|} e_{1,1} \otimes f^{(r-1)}, \qquad e^{(r)} \mapsto 1 \otimes e^{(r)}. \end{aligned} \tag{2-16}$$

If $s_{1,2} > 0$, we denote $(\sigma^{\mathrm{up}})_+ = (\sigma_+)^{\mathrm{up}}$ by σ_+^{up}. If $s_{2,1} > 0$, we denote $(\sigma^{\mathrm{lo}})_- = (\sigma_-)^{\mathrm{lo}}$ by σ_-^{lo}. If both $s_{1,2} > 0$ and $s_{2,1} > 0$, we denote $(\sigma_+)_- = (\sigma_-)_+$ by $\sigma_\pm$.

Lemma 2.10. *Assuming that $s_{1,2} > 0$ in the first diagram, $s_{2,1} > 0$ in the second diagram and both $s_{1,2} > 0$ and $s_{2,1} > 0$ in the final diagram, the following commute*:

$$\begin{tikzcd} Y_\sigma \arrow[r, "\Delta_+"] \arrow[d, "\Delta"'] & Y_{\sigma_+} \otimes U(\mathfrak{gl}_1) \arrow[d, "\Delta \otimes \mathrm{id}"] \\ Y_{\sigma^{\mathrm{lo}}} \otimes Y_{\sigma^{\mathrm{up}}} \arrow[r, "\mathrm{id} \otimes \Delta_+"] & Y_{\sigma^{\mathrm{lo}}} \otimes Y_{\sigma_+^{\mathrm{up}}} \otimes U(\mathfrak{gl}_1) \end{tikzcd} \tag{2-17}$$

$$\begin{tikzcd} Y_\sigma \arrow[r, "\Delta"] \arrow[d, "\Delta_-"'] & Y_{\sigma^{\mathrm{lo}}} \otimes Y_{\sigma^{\mathrm{up}}} \arrow[d, "\Delta_-\otimes\mathrm{id}"] \\ U(\mathfrak{gl}_1)\otimes Y_{\sigma_-} \arrow[r, "\mathrm{id}\otimes\Delta"] & U(\mathfrak{gl}_1)\otimes Y_{\sigma_-^{\mathrm{lo}}}\otimes Y_{\sigma^{\mathrm{up}}} \end{tikzcd} \tag{2-18}$$

$$\begin{tikzcd} Y_\sigma \arrow[r, "\Delta_+"] \arrow[d, "\Delta_-"'] & Y_{\sigma_+} \otimes U(\mathfrak{gl}_1) \arrow[d, "\Delta_-\otimes\mathrm{id}"] \\ U(\mathfrak{gl}_1)\otimes Y_{\sigma_-} \arrow[r, "\mathrm{id}\otimes\Delta_+"] & U(\mathfrak{gl}_1)\otimes Y_{\sigma_\pm}\otimes U(\mathfrak{gl}_1) \end{tikzcd} \tag{2-19}$$

Proof. Check on Drinfeld generators using (2-15) and (2-16) and Proposition 2.5. □

Remark 2.11. Writing $\varepsilon : U(\mathfrak{gl}_1) \to \mathbb{C}$ for the counit, the maps $(\mathrm{id}\,\bar{\otimes}\varepsilon)\circ\Delta_+$ and $(\varepsilon\,\bar{\otimes}\,\mathrm{id})\circ\Delta_-$ are the natural inclusions $Y_\sigma \to Y_{\sigma_+}$ and $Y_\sigma \to Y_{\sigma_-}$, respectively. Hence, the maps Δ_+ and Δ_- are injective.

3. Truncation

Let $\sigma = (s_{i,j})_{1\le i,j\le 2}$ be a shift matrix. Suppose also that we are given an integer $l \ge s_{2,1}+s_{1,2}$, and set

$$k := l - s_{2,1} - s_{1,2} \ge 0.$$

In view of Lemma 2.10, we can iterate Δ_+ a total of $s_{1,2}$ times, Δ_- a total of $s_{2,1}$ times and Δ a total of $k-1$ times in any order that makes sense (when $k=0$, this means we apply the counit ε once at the very end) to obtain a well-defined homomorphism

$$\Delta^l_\sigma : Y_\sigma \to U(\mathfrak{gl}_1)^{\otimes s_{2,1}} \otimes Y^{\otimes k} \otimes U(\mathfrak{gl}_1)^{\otimes s_{1,2}}.$$

For example, if

$$\sigma = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

then

$$\begin{aligned} \Delta^3_\sigma &= (\mathrm{id}\otimes\varepsilon\,\bar{\otimes}\,\mathrm{id}\otimes\mathrm{id})\circ(\Delta_-\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta_+\otimes\mathrm{id})\circ\Delta_+, \\ \Delta^4_\sigma &= (\mathrm{id}\otimes\Delta_+\otimes\mathrm{id})\circ(\Delta_-\otimes\mathrm{id})\circ\Delta_+ = (\mathrm{id}\otimes\Delta_+\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta_+)\circ\Delta_-, \\ \Delta^5_\sigma &= (\Delta_-\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta_+\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta_+)\circ\Delta \\ &= (\mathrm{id}\otimes\Delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(\Delta_-\otimes\mathrm{id}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta_+)\circ\Delta_+. \end{aligned}$$

Let

$$U^l_\sigma := U(\mathfrak{gl}_1)^{\otimes s_{2,1}} \otimes U(\mathfrak{gl}_{1|1})^{\otimes k} \otimes U(\mathfrak{gl}_1)^{\otimes s_{1,2}}, \tag{3-1}$$

viewed as a superalgebra using the usual sign convention. Recalling (2-5), we obtain a homomorphism

$$\mathrm{ev}_\sigma^l := (\mathrm{id}^{\otimes s_{2,1}} \otimes \mathrm{ev}^{\otimes k} \otimes \mathrm{id}^{\otimes s_{1,2}}) \circ \Delta_\sigma^l : Y_\sigma \to U_\sigma^l. \tag{3-2}$$

Let

$$Y_\sigma^l := \mathrm{ev}_\sigma^l(Y_\sigma) \subseteq U_\sigma^l. \tag{3-3}$$

This is the *shifted Yangian of level l*.

In the special case that $\sigma = 0$, we denote ev_σ^l, Y_σ^l and U_σ^l simply by ev^l, Y^l and U^l, respectively, so that $Y^l = \mathrm{ev}^l(Y) \subseteq U^l$. We call Y^l the *Yangian of level l*. Writing $\bar{e}_{i,j}^{[c]} := (-1)^{|i|} 1^{\otimes(c-1)} \otimes e_{i,j} \otimes 1^{\otimes(l-c)}$, we have simply that

$$\mathrm{ev}^l(t_{i,j}^{(r)}) = \sum_{1<c_1<\cdots<c_r\leq l} \; \sum_{1\leq h_1,\ldots,h_{r-1}\leq 2} \bar{e}_{i,h_1}^{[c_1]} \bar{e}_{h_1,h_2}^{[c_2]} \cdots \bar{e}_{h_{r-1},j}^{[c_r]} \tag{3-4}$$

for any $1 \leq i, j \leq 2$ and $r \geq 0$. In particular, $\mathrm{ev}^l(t_{i,j}^{(r)}) = 0$ for $r > l$. Gow [2007, proof of Theorem 1] shows that the kernel of $\mathrm{ev}^l : Y \twoheadrightarrow Y^l$ is generated by $\{t_{i,j}^{(r)} \mid 1 \leq i, j \leq 2,\ r > l\}$ and, moreover, the images of the ordered supermonomials in the remaining elements $\{t_{i,j}^{(r)} \mid 1 \leq i, j \leq 2,\ 0 < r \leq l\}$ give a basis for Y^l. (Actually, she proves this for all $Y(\mathfrak{gl}_{m|n})$ and not just $Y(\mathfrak{gl}_{1|1})$.) The goal in this section is to prove analogues of these statements for Y_σ with $\sigma \neq 0$.

Let I_σ^l be the two-sided ideal of Y_σ generated by the elements $d_1^{(r)}$ for $r > k$.

Lemma 3.1. $I_\sigma^l \subseteq \ker \mathrm{ev}_\sigma^l$.

Proof. We need to show that $\mathrm{ev}_\sigma^l(d_1^{(r)}) = 0$ for all $r > k$. We calculate this by first applying all the maps Δ_+ and Δ_- to deduce that

$$\mathrm{ev}_\sigma^l(d_1^{(r)}) = 1^{\otimes s_{2,1}} \otimes \mathrm{ev}^k(d_1^{(r)}) \otimes 1^{\otimes s_{1,2}}.$$

Since $d_1^{(r)} = t_{1,1}^{(r)}$, it is then clear from (3-4) that $\mathrm{ev}^k(d_1^{(r)}) = 0$ for $r > k$. □

Proposition 3.2. *The ideal I_σ^l contains all of the following elements*:

$$\sum_{s_{1,2}<a\leq r} d_1^{(r-a)} e^{(a)} \qquad \textit{for } r > s_{1,2}+k, \tag{3-5}$$

$$\sum_{s_{2,1}<b\leq r} f^{(b)} d_1^{(r-b)} \qquad \textit{for } r > s_{2,1}+k, \tag{3-6}$$

$$d_2^{(r)} + \sum_{\substack{s_{1,2}<a \\ s_{2,1}<b \\ a+b\leq r}} f^{(b)} d_1^{(r-a-b)} e^{(a)} \qquad \textit{for } r > l. \tag{3-7}$$

Proof. Consider the algebra $Y_\sigma[\![u^{-1}]\!][u]$ of formal Laurent series in the variable u^{-1} with coefficients in Y_σ. For any such formal Laurent series $p = \sum_{r\leq N} p_r u^r$, we write $[p]_{\geq 0}$ for its polynomial part $\sum_{r=0}^{N} p_r u^r$. Also write $\equiv$ for congruence modulo $Y_\sigma[u] + u^{-1} I_\sigma^l [\![u^{-1}]\!]$, so $p \equiv 0$ means that the u^r-coefficients of p lie in I_σ^l for all $r < 0$. Note that if $p \equiv 0$, $q \in Y_\sigma[u]$, then $pq \equiv 0$. In this notation, we have by definition of I_σ^l that $u^k d_1(u) \equiv 0$. Introduce the power series

$$e_\sigma(u) := \sum_{r > s_{1,2}} e^{(r)} u^{-r}, \qquad f_\sigma(u) := \sum_{r > s_{2,1}} f^{(r)} u^{-r}.$$

The proposition is equivalent to the following assertions:

$$u^{s_{1,2}+k} d_1(u) e_\sigma(u) \equiv 0, \tag{3-8}$$

$$u^{s_{2,1}+k} f_\sigma(u) d_1(u) \equiv 0, \tag{3-9}$$

$$u^l (d_2(u) + f_\sigma(u) d_1(u) e_\sigma(u)) \equiv 0. \tag{3-10}$$

For the first two, we use the identities

$$(-1)^{|1|}[d_1(u), e^{(s_{1,2}+1)}] = u^{s_{1,2}} d_1(u) e_\sigma(u), \tag{3-11}$$

$$(-1)^{|1|}[f^{(s_{2,1}+1)}, d_1(u)] = u^{s_{2,1}} f_\sigma(u) d_1(u). \tag{3-12}$$

These are easily checked by considering the u^{-r}-coefficients on each side and using the relations in Theorem 2.2. Assertions (3-8) and (3-9) follow from (3-11) and (3-12) on multiplying by u^k as $u^k d_1(u) \equiv 0$. For the final assertion (3-10), recall the elements $c^{(r)}$ from (2-10). Let $c_\sigma(u) := \sum_{r > s_{2,1}+s_{1,2}} c^{(r)} u^{-r}$. Another routine check using the relations shows that

$$(-1)^{|1|}[f^{(s_{2,1}+1)}, e_\sigma(u)] = u^{s_{2,1}} c_\sigma(u). \tag{3-13}$$

Using (3-8), (3-12) and (3-13), we deduce that

$$\begin{aligned} 0 &\equiv (-1)^{|1|} u^{s_{1,2}+k} [f^{(s_{2,1}+1)}, d_1(u) e_\sigma(u)] \\ &= u^{s_{1,2}+k} d_1(u) (-1)^{|1|} [f^{(s_{2,1}+1)}, e_\sigma(u)] + u^{s_{1,2}+k} (-1)^{|1|} [f^{(s_{2,1}+1)}, d_1(u)] e_\sigma(u) \\ &= u^l d_1(u) c_\sigma(u) + u^l f_\sigma(u) d_1(u) e_\sigma(u). \end{aligned}$$

To complete the proof of (3-10), it remains to observe that

$$u^{s_{2,1}+s_{1,2}} c_\sigma(u) = u^{s_{2,1}+s_{1,2}} \tilde{d}_1(u) d_2(u) - [u^{s_{2,1}+s_{1,2}} \tilde{d}_1(u) d_2(u)]_{\geq 0};$$

hence, $u^l d_1(u) c_\sigma(u) \equiv u^l d_2(u)$. □

For the rest of the section, we fix some total ordering on the set

$$\begin{aligned} \Omega := \{d_1^{(r)} \mid 0 < r \leq k\} \cup \{d_2^{(r)} \mid 0 < r \leq l\} \\ \cup \{e^{(r)} \mid s_{1,2} < r \leq s_{1,2} + k\} \cup \{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + k\}. \end{aligned} \tag{3-14}$$

Lemma 3.3. *The quotient algebra Y_σ / I_σ^l is spanned by the images of the ordered supermonomials in the elements of Ω.*

Proof. The Kazhdan filtration on Y_σ induces a filtration on Y_σ/I_σ^l with respect to which $\mathrm{gr}(Y_\sigma/I_\sigma^l)$ is a graded quotient of $\mathrm{gr}\, Y_\sigma$. We already know that $\mathrm{gr}\, Y_\sigma$ is supercommutative, so $\mathrm{gr}(Y_\sigma/I_\sigma^l)$ is too. Let $\underline{d}_i^{(r)} := \mathrm{gr}_r(d_i^{(r)} + I_\sigma^l)$, $\underline{e}^{(r)} := \mathrm{gr}_r(e^{(r)} + I_\sigma^l)$ and $\underline{f}^{(r)} := \mathrm{gr}_r(f^{(r)} + I_\sigma^l)$.

To prove the lemma, it is enough to show that $\mathrm{gr}(Y_\sigma/I_\sigma^l)$ is generated by

$$\{\underline{d}_1^{(r)} \mid 0 < r \le k\} \cup \{\underline{d}_2^{(r)} \mid 0 < r \le l\} \\ \cup \{\underline{e}^{(r)} \mid s_{1,2} < r \le s_{1,2} + k\} \cup \{\underline{f}^{(r)} \mid s_{2,1} < r \le s_{2,1} + k\}.$$

This follows because $\underline{d}_1^{(r)} = 0$ for $r > k$, and each of the elements $\underline{d}_2^{(r)}$ for $r > l$, $\underline{e}^{(r)}$ for $r > s_{1,2} + k$ and $\underline{f}^{(r)}$ for $r > s_{2,1} + k$ can be expressed as polynomials in generators of strictly smaller degrees by Proposition 3.2. $\square$

Lemma 3.4. *The image under* ev_σ^l *of the ordered supermonomials in the elements of* Ω *are linearly independent in* Y_σ^l.

Proof. Consider the standard filtration on U_σ^l generated by declaring that all the elements of the form $1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ for $x \in \mathfrak{gl}_1$ or $\mathfrak{gl}_{1|1}$ are in degree 1. It induces a filtration on Y_σ^l so that $\mathrm{gr}\, Y_\sigma^l$ is a graded subalgebra of $\mathrm{gr}\, U_\sigma^l$. Note that $\mathrm{gr}\, U_\sigma^l$ is supercommutative, so the subalgebra $\mathrm{gr}\, Y_\sigma^l$ is too. Each of the elements $\mathrm{ev}_\sigma^l(d_i^{(r)})$, $\mathrm{ev}_\sigma^l(e^{(r)})$ and $\mathrm{ev}_\sigma^l(f^{(r)})$ are in filtered degree r by the definition of ev_σ^l. Let $\underline{d}_i^{(r)} := \mathrm{gr}_r(\mathrm{ev}_\sigma^l(d_i^{(r)}))$, $\underline{e}^{(r)} := \mathrm{gr}_r(\mathrm{ev}_\sigma^l(e^{(r)}))$ and $\underline{f}^{(r)} := \mathrm{gr}_r(\mathrm{ev}_\sigma^l(f^{(r)}))$.

Let M be the set of ordered supermonomials in

$$\{\underline{d}_1^{(r)} \mid 0 < r \le k\} \cup \{\underline{d}_2^{(r)} \mid 0 < r \le l\} \\ \cup \{\underline{e}^{(r)} \mid s_{1,2} < r \le s_{1,2} + k\} \cup \{\underline{f}^{(r)} \mid s_{2,1} < r \le s_{2,1} + k\}.$$

To prove the lemma, it suffices to show that M is linearly independent in $\mathrm{gr}\, Y_\sigma^l$. For this, we proceed by induction on $s_{2,1} + s_{1,2}$.

To establish the base case $s_{2,1} + s_{1,2} = 0$, i.e., $\sigma = 0$, $Y_\sigma = Y$ and $Y_\sigma^l = Y^l$, let $\underline{t}_{i,j}^{(r)}$ denote $\mathrm{gr}_r(\mathrm{ev}_\sigma^l(t_{i,j}^{(r)}))$. Fix a total order on $\{\underline{t}_{i,j}^{(r)} \mid 1 \le i,\ j \le 2,\ 0 < r \le l\}$, and let M' be the resulting set of ordered supermonomials. Exploiting the explicit formula (3-4), Gow [2007, proof of Theorem 1] shows that M' is linearly independent. By (2-6)–(2-9), any element of M is a linear combination of elements of M' of the same degree and vice versa. So we deduce that M is linearly independent too.

For the induction step, suppose that $s_{2,1} + s_{1,2} > 0$. Then we either have $s_{2,1} > 0$ or $s_{1,2} > 0$. We just explain the argument for the latter case; the proof in the former case is entirely similar replacing Δ_+ with Δ_-. Recall that σ_+ denotes the shift matrix obtained from σ by subtracting 1 from $s_{1,2}$. So $U_\sigma^l = U_{\sigma_+}^{l-1} \otimes U(\mathfrak{gl}_1)$. By its definition, we have that $\mathrm{ev}_\sigma^l = (\mathrm{ev}_{\sigma_+}^{l-1} \otimes \mathrm{id}) \circ \Delta_+$; hence, $Y_\sigma^l \subseteq Y_{\sigma_+}^{l-1} \otimes U(\mathfrak{gl}_1)$. Let

$x := \mathrm{gr}_1\, e_{1,1} \in \mathrm{gr}\, U(\mathfrak{gl}_1)$. Then

$$\underline{d}_1^{(r)} = \dot{\underline{d}}_1^{(r)} \otimes 1, \qquad \underline{d}_2^{(r)} = \dot{\underline{d}}_2^{(r)} \otimes 1 + (-1)^{|2|} \dot{\underline{d}}_2^{(r-1)} \otimes x,$$
$$\underline{f}^{(r)} = \dot{\underline{f}}^{(r)} \otimes 1, \qquad \underline{e}^{(r)} = \dot{\underline{e}}^{(r)} \otimes 1 + (-1)^{|2|} \dot{\underline{e}}^{(r-1)} \otimes x.$$

The notation is potentially confusing here, so we have decorated elements of $\mathrm{gr}\, Y_{\sigma_+}^{l-1} \subseteq \mathrm{gr}\, U_{\sigma_+}^{l-1}$ with a dot. It remains to observe from the induction hypothesis applied to $\mathrm{gr}\, Y_{\sigma_+}^{l-1}$ that ordered supermonomials in

$$\{\dot{\underline{d}}_1^{(r)} \otimes 1 \mid 0 < r \leq k\} \cup \{\dot{\underline{d}}_2^{(r-1)} \otimes x \mid 0 < r \leq l\}$$
$$\cup \{\dot{\underline{e}}^{(r-1)} \otimes x \mid s_{1,2} < r \leq s_{1,2} + k\} \cup \{\dot{\underline{f}}^{(r)} \otimes 1 \mid 0 < r < s_{1,2} + k\}$$

are linearly independent. □

Theorem 3.5. *The kernel of* $\mathrm{ev}_\sigma^l : Y_\sigma \to Y_\sigma^l$ *is equal to the two-sided ideal* I_σ^l *generated by the elements* $\{d_1^{(r)} \mid r > k\}$. *Hence,* ev_σ^l *induces an algebra isomorphism between* Y_σ / I_σ^l *and* Y_σ^l.

Proof. By Lemma 3.1, ev_σ^l induces a surjection $Y_\sigma / I_\sigma^l \twoheadrightarrow Y_\sigma^l$. It maps the spanning set from Lemma 3.3 onto the linearly independent set from Lemma 3.4. Hence, it is an isomorphism and both sets are actually bases. □

Henceforth, we will *identify* Y_σ^l with the quotient Y_σ / I_σ^l, and we will abuse notation by denoting the canonical images in Y_σ^l of the elements $d_i^{(r)}, e^{(r)}, \dots$ of Y_σ by the same symbols $d_i^{(r)}, e^{(r)}, \dots$. This will not cause any confusion as we will not work with Y_σ again.

Here is the PBW theorem for Y_σ^l, which was noted already in the proof of Theorem 3.5.

Corollary 3.6. *Order the set*

$$\{d_1^{(r)} \mid 0 < r \leq k\} \cup \{d_2^{(r)} \mid 0 < r \leq l\}$$
$$\cup \{e^{(r)} \mid s_{1,2} < r \leq s_{1,2} + k\} \cup \{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + k\}$$

in some way. The ordered supermonomials in these elements give a basis for Y_σ^l.

Remark 3.7. In the arguments in this section, we have defined *two* filtrations on Y_σ^l: one in the proof of Lemma 3.3 induced by the Kazhdan filtration on Y_σ and the other in the proof of Lemma 3.4 induced by the standard filtration on U_σ^l. Using Corollary 3.6, one can check that these two filtrations coincide.

Remark 3.8. Theorem 3.5 shows that Y_σ^l has generators

$$\{d_i^{(r)} \mid i = 1, 2,\ r > 0\} \cup \{e^{(r)} \mid r > s_{1,2}\} \cup \{f^{(r)} \mid r > s_{2,1}\}$$

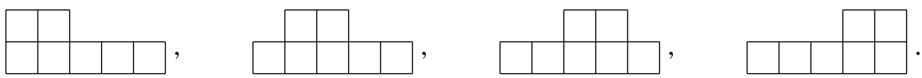
subject only to the relations from Theorem 2.2 and the additional truncation relations $d_1^{(r)} = 0$ for $r > k$. Corollary 3.6 shows that all but finitely many of the generators are redundant. In special cases, it is possible to optimize the relations too. For example, if $l = s_{2,1} + s_{1,2} + 1$ and we set $d := d_1^{(1)}$, $e := e^{(s_{1,2}+1)}$ and $f := f^{(s_{2,1}+1)}$, then Y_σ^l is generated by its even central elements $c^{(1)}, \dots, c^{(l)}$ from (2-10), the even element d and the odd elements e and f subject only to the relations

$$[d, e] = (-1)^{|1|} e, \qquad [d, f] = -(-1)^{|1|} f, \qquad [e, f] = (-1)^{|1|} c^{(l)},$$
$$[c^{(r)}, c^{(s)}] = [c^{(r)}, d] = [c^{(r)}, e] = [c^{(r)}, f] = [e, e] = [f, f] = 0,$$

for $r, s = 1, \dots, l$. To see this, observe that these elements generate Y_σ^l and they satisfy the given relations; then apply Corollary 3.6.

4. Principal W-algebras

We turn to the W-algebra side of the story. Let π be a (two-rowed) *pyramid*, that is, a collection of boxes in the plane arranged in two connected rows such that each box in the first (top) row lies directly above a box in the second (bottom) row. For example, here are all the pyramids with two boxes in the first row and five in the second:



Let k and l denote the number of boxes in the first and second rows of π, respectively, so that $k \le l$. The parity sequence fixed in (2-1) allows us to talk about the parities of the rows of π: the i-th row is of parity $|i|$. Let m be the number of boxes in the even row, i.e., the row with parity $\bar{0}$, and n be the number of boxes in the odd row, i.e., the row with parity $\bar{1}$. Then label the boxes in the even and odd rows from left to right by the numbers $1, \dots, m$ and $m+1, \dots, m+n$, respectively. For example, here is one of the above pyramids with boxes labeled in this way assuming that $(|1|, |2|) = (\bar{1}, \bar{0})$, i.e., the bottom row is even and the top row is odd:

$$\begin{array}{ccccc} & \boxed{6} & \boxed{7} & & \\ \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} \end{array}\,. \tag{4-1}$$

Numbering the columns of π $1, \dots, l$ in order from left to right, we write $\mathrm{row}(i)$ and $\mathrm{col}(i)$ for the row and column numbers of the i-th box in this labeling.

Now let $\mathfrak{g} := \mathfrak{gl}_{m|n}(\mathbb{C})$ for m and n coming from the pyramid π and the fixed parity sequence as in the previous paragraph. Let $\mathfrak{t}$ be the Cartan subalgebra consisting of all diagonal matrices and $\varepsilon_1, \dots, \varepsilon_{m+n} \in \mathfrak{t}^*$ the basis such that $\varepsilon_i(e_{j,j}) = \delta_{i,j}$ for each $j = 1, \dots, m+n$. The *supertrace form* $(\cdot|\cdot)$ on $\mathfrak{g}$ is the nondegenerate invariant supersymmetric bilinear form defined by $(x|y) = \mathrm{str}(xy)$, where the supertrace $\mathrm{str}\, A$ of matrix $A = (a_{i,j})_{1 \le i,j \le m+n}$ means $a_{1,1} + \dots + a_{m,m} - a_{m+1,m+1} - \dots - a_{m+n,m+n}$. It induces a bilinear form $(\cdot|\cdot)$ on $\mathfrak{t}^*$ such that $(\varepsilon_i|\varepsilon_j) = (-1)^{|\mathrm{row}(i)|}\delta_{i,j}$.

We have the explicit principal nilpotent element

$$e := \sum_{i,j} e_{i,j} \in \mathfrak{g}_{\bar{0}} \tag{4-2}$$

summing over all adjacent pairs $\boxed{i \mid j}$ of boxes in the pyramid π. In the example above, we have that $e = e_{1,2} + e_{2,3} + e_{3,4} + e_{4,5} + e_{6,7}$. Let $\chi \in \mathfrak{g}^*$ be defined by $\chi(x) := (x|e)$. If we set

$$\bar{e}_{i,j} := (-1)^{|\mathrm{row}(i)|} e_{i,j}, \tag{4-3}$$

then we have that

$$\chi(\bar{e}_{i,j}) = \begin{cases} 1 & \text{if } \boxed{j \mid i} \text{ is an adjacent pair of boxes in } \pi, \\ 0 & \text{otherwise.} \end{cases} \tag{4-4}$$

Introduce a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}(r)$ by declaring that $e_{i,j}$ is of degree

$$\deg(e_{i,j}) := \mathrm{col}(j) - \mathrm{col}(i). \tag{4-5}$$

This is a *good grading* for e, which means that $e \in \mathfrak{g}(1)$ and the centralizer $\mathfrak{g}^e$ of e in $\mathfrak{g}$ is contained in $\bigoplus_{r \geq 0} \mathfrak{g}(r)$; see [Hoyt 2012] for more about good gradings on Lie superalgebras (one should double the degrees of our grading to agree with the terminology there). Set

$$\mathfrak{p} := \bigoplus_{r \geq 0} \mathfrak{g}(r), \qquad \mathfrak{h} := \mathfrak{g}(0), \qquad \mathfrak{m} := \bigoplus_{r < 0} \mathfrak{g}(r).$$

Note that χ restricts to a character of $\mathfrak{m}$. Let $\mathfrak{m}_\chi := \{x - \chi(x) \mid x \in \mathfrak{m}\}$, which is a shifted copy of $\mathfrak{m}$ inside $U(\mathfrak{m})$. Then the *principal W-algebra* associated to the pyramid π is

$$W_\pi := \{u \in U(\mathfrak{p}) \mid u\mathfrak{m}_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g})\}. \tag{4-6}$$

It is straightforward to check that W_π is a subalgebra of $U(\mathfrak{p})$.

The first important result about W_π is its *PBW theorem*. This is noted already in [Zhao 2012, Remark 3.10], where it is described for arbitrary basic classical Lie superalgebras modulo a mild assumption on e (which is trivially satisfied here). To formulate the result precisely, embed e into an $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}_{\bar{0}}$ such that $h \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-1)$. It follows from $\mathfrak{sl}_2$ representation theory that

$$\mathfrak{p} = \mathfrak{g}^e \oplus [\mathfrak{p}^\perp, f], \tag{4-7}$$

where $\mathfrak{p}^\perp = \bigoplus_{r>0} \mathfrak{g}(r)$ denotes the nilradical of $\mathfrak{p}$. Also introduce the *Kazhdan filtration* on $U(\mathfrak{p})$, which is generated by declaring for each $r \geq 0$ that $x \in \mathfrak{g}(r)$ is of Kazhdan degree $r+1$. The associated graded superalgebra $\mathrm{gr}\, U(\mathfrak{p})$ is supercommutative and is naturally identified with the symmetric superalgebra $S(\mathfrak{p})$ viewed as a positively graded algebra via the analogously defined *Kazhdan grading*. The

Kazhdan filtration on $U(\mathfrak{p})$ induces a Kazhdan filtration on $W_\pi \subseteq U(\mathfrak{p})$ so that $\operatorname{gr} W_\pi \subseteq \operatorname{gr} U(\mathfrak{p}) = S(\mathfrak{p})$.

Theorem 4.1. *Let $p : S(\mathfrak{p}) \to S(\mathfrak{g}^e)$ be the homomorphism induced by the projection of $\mathfrak{p}$ onto $\mathfrak{g}^e$ along (4-7). The restriction of p defines an isomorphism of Kazhdan-graded superalgebras $\operatorname{gr} W_\pi \xrightarrow{\sim} S(\mathfrak{g}^e)$.*

Proof. Superize the arguments in [Gan and Ginzburg 2002] as suggested in [Zhao 2012, Remark 3.10]. □

In order to apply Theorem 4.1, it is helpful to have available an explicit basis for the centralizer $\mathfrak{g}^e$. We say that a shift matrix $\sigma = (s_{i,j})_{1\le i,j\le 2}$ is *compatible with* π if either $k > 0$ and π has $s_{2,1}$ columns of height 1 on its left side and $s_{1,2}$ columns of height 1 on its right side or if $k = 0$ and $l = s_{2,1} + s_{1,2}$. These conditions determine a unique shift matrix σ when $k > 0$, but there is some minor ambiguity if $k = 0$ (which should never cause any concern). For example, if π is as in (4-1), then

$$\sigma = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

is the only compatible shift matrix.

Lemma 4.2. *Let $\sigma = (s_{i,j})_{1\le i,j\le 2}$ be a shift matrix compatible with π. For $r \ge 0$, let*

$$x_{i,j}^{(r)} := \sum_{\substack{1\le p,q\le m+n \\ \operatorname{row}(p)=i,\ \operatorname{row}(q)=j \\ \deg(e_{p,q})=r-1}} \bar{e}_{p,q} \in \mathfrak{g}(r-1).$$

Then the elements

$$\{x_{1,1}^{(r)} \mid 0 < r \le k\} \cup \{x_{2,2}^{(r)} \mid 0 < r \le l\} \\ \cup \{x_{1,2}^{(r)} \mid s_{1,2} < r \le s_{1,2} + k\} \cup \{x_{2,1}^{(r)} \mid s_{2,1} < r \le s_{2,1} + k\}$$

give a homogeneous basis for $\mathfrak{g}^e$.

Proof. As e is even, the centralizer of e in $\mathfrak{g}$ is just the same as a vector space as the centralizer of e viewed as an element of $\mathfrak{gl}_{m+n}(\mathbb{C})$, so this follows as a special case of [Brundan and Kleshchev 2006, Lemma 7.3] (which is [Springer and Steinberg 1970, IV.1.6]). □

We come to the key ingredient in our approach: the explicit definition of special elements of $U(\mathfrak{p})$, some of which turn out to generate W_π. Define another ordering $\prec$ on the set $\{1, \dots, m+n\}$ by declaring that $i \prec j$ if $\operatorname{col}(i) < \operatorname{col}(j)$ or if $\operatorname{col}(i) = \operatorname{col}(j)$ and $\operatorname{row}(i) < \operatorname{row}(j)$. Let $\tilde{\rho} \in \mathfrak{t}^*$ be the weight with

$$(\tilde{\rho}|\varepsilon_j) = \#\{i \mid i \preceq j \text{ and } |\operatorname{row}(i)| = \bar{1}\} - \#\{i \mid i \prec j \text{ and } |\operatorname{row}(i)| = \bar{0}\}. \tag{4-8}$$

For example, if π is as in (4-1), then $\tilde{\rho} = -\varepsilon_4 - 2\varepsilon_5$. The weight $\tilde{\rho}$ extends to a character of $\mathfrak{p}$, so there are automorphisms

$$S_{\pm\tilde{\rho}} : U(\mathfrak{p}) \to U(\mathfrak{p}), \qquad e_{i,j} \mapsto e_{i,j} \pm \delta_{i,j}\tilde{\rho}(e_{i,i}). \tag{4-9}$$

Finally, given $1 \le i, j \le 2$, $0 \le \varsigma \le 2$ and $r \ge 1$, we define

$$t^{(r)}_{i,j;\varsigma} := S_{\tilde{\rho}}\Bigg(\sum_{s=1}^{r} (-1)^{r-s} \sum_{\substack{i_1,\dots,i_s \\ j_1,\dots,j_s}} (-1)^{\#\{a=1,\dots,s-1 \mid \mathrm{row}(j_a)\le\varsigma\}} \bar{e}_{i_1,j_1} \cdots \bar{e}_{i_s,j_s}\Bigg), \tag{4-10}$$

where the sum is over all $1 \le i_1, \dots, i_s, j_1, \dots, j_s \le m+n$ such that

- $\mathrm{row}(i_1) = i$ and $\mathrm{row}(j_s) = j$,
- $\mathrm{col}(i_a) \le \mathrm{col}(j_a)$ $(a = 1, \dots, s)$,
- $\mathrm{row}(i_{a+1}) = \mathrm{row}(j_a)$ $(a = 1, \dots, s-1)$,
- if $\mathrm{row}(j_a) > \varsigma$, then $\mathrm{col}(i_{a+1}) > \mathrm{col}(j_a)$ $(a = 1, \dots, s-1)$,
- if $\mathrm{row}(j_a) \le \varsigma$, then $\mathrm{col}(i_{a+1}) \le \mathrm{col}(j_a)$ $(a = 1, \dots, s-1)$ and
- $\deg(e_{i_1,j_1}) + \cdots + \deg(e_{i_s,j_s}) = r - s$.

It is convenient to collect these elements together into the generating function

$$t_{i,j;\varsigma}(u) := \sum_{r\ge 0} t^{(r)}_{i,j;\varsigma} u^{-r} \in U(\mathfrak{p})[\![u^{-1}]\!] \tag{4-11}$$

setting $t^{(0)}_{i,j;\varsigma} := \delta_{i,j}$. The following two propositions should already convince the reader of the remarkable nature of these elements:

Proposition 4.3. *The following identities hold in $U(\mathfrak{p})[\![u^{-1}]\!]$:*

$$t_{1,1;1}(u) = t_{1,1;0}(u)^{-1}, \tag{4-12}$$

$$t_{2,2;2}(u) = t_{2,2;1}(u)^{-1}, \tag{4-13}$$

$$t_{1,2;0}(u) = t_{1,1;0}(u)t_{1,2;1}(u), \tag{4-14}$$

$$t_{2,1;0}(u) = t_{2,1;1}(u)t_{1,1;0}(u), \tag{4-15}$$

$$t_{2,2;0}(u) = t_{2,2;1}(u) + t_{2,1;1}(u)t_{1,1;0}(u)t_{1,2;1}(u). \tag{4-16}$$

Proof. This is proved in [Brundan and Kleshchev 2006, Lemma 9.2]; the argument there is entirely formal and does not depend on the underlying associative algebra in which the calculations are performed. □

Proposition 4.4. *Let σ be a shift matrix compatible with π. The following elements of $U(\mathfrak{p})$ belong to W_π: all $t^{(r)}_{1,1;0}$, $t^{(r)}_{1,1;1}$, $t^{(r)}_{2,2;1}$ and $t^{(r)}_{2,2;2}$ for $r > 0$, all $t^{(r)}_{1,2;1}$ for $r > s_{1,2}$ and all $t^{(r)}_{2,1;1}$ for $r > s_{2,1}$.*

Proof. This is postponed to Section 5. □

Now we can deduce our main result. For any shift matrix σ compatible with π, we identify $U(\mathfrak{h})$ with the algebra U_σ^l from (3-1) so that

$$e_{i,j} \equiv \begin{cases} 1^{\otimes(c-1)} \otimes e_{\mathrm{row}(i),\mathrm{row}(j)} \otimes 1^{\otimes(l-c)} & \text{if } q_c = 2, \\ 1^{\otimes(c-1)} \otimes e_{1,1} \otimes 1^{\otimes(l-c)} & \text{if } q_c = 1 \end{cases}$$

for any $1 \le i, j \le m+n$ with $c := \mathrm{col}(i) = \mathrm{col}(j)$, where q_c denotes the number of boxes in this column of π. Define the *Miura transform*

$$\mu : W_\pi \to U(\mathfrak{h}) = U_\sigma^l \tag{4-17}$$

to be the restriction to W_π of the shift automorphism $S_{-\tilde{\rho}}$ composed with the natural homomorphism $\mathrm{pr} : U(\mathfrak{p}) \to U(\mathfrak{h})$ induced by the projection $\mathfrak{p} \twoheadrightarrow \mathfrak{h}$.

Theorem 4.5. *Let σ be a shift matrix compatible with π. The Miura transform is injective, and its image is the algebra $Y_\sigma^l \subseteq U_\sigma^l$ from (3-3). Hence, it defines a superalgebra isomorphism*

$$\mu : W_\pi \xrightarrow{\sim} Y_\sigma^l \tag{4-18}$$

between W_π and the shifted Yangian of level l. Moreover, μ maps the invariants from Proposition 4.4 to the Drinfeld generators of Y_σ^l as follows:

$$\mu(t_{1,1;0}^{(r)}) = d_1^{(r)} \quad (r > 0), \qquad \mu(t_{1,1;1}^{(r)}) = \tilde{d}_1^{(r)} \quad (r > 0), \tag{4-19}$$

$$\mu(t_{2,2;1}^{(r)}) = d_2^{(r)} \quad (r > 0), \qquad \mu(t_{2,2;2}^{(r)}) = \tilde{d}_2^{(r)} \quad (r > 0), \tag{4-20}$$

$$\mu(t_{1,2;1}^{(r)}) = e^{(r)} \quad (r > s_{1,2}), \qquad \mu(t_{2,1;1}^{(r)}) = f^{(r)} \quad (r > s_{2,1}). \tag{4-21}$$

Proof. We first establish the identities (4-19)–(4-21). Note that the identities involving $\tilde{d}_i^{(r)}$ are consequences of the ones involving $d_i^{(r)}$ thanks to (4-12) and (4-13) recalling also that $\tilde{d}_i(u) = d_i(u)^{-1}$. To prove all the other identities, we proceed by induction on $s_{2,1} + s_{1,2} = l - k$.

First consider the base case $l = k$. For $1 \le i, j \le 2$ and $r > 0$, we know in this situation that $t_{i,j;0}^{(r)} \in W_\pi$ since, using (4-14)–(4-16), it can be expanded in terms of elements all of which are known to lie in W_π by Proposition 4.4; see also Lemma 5.1. Moreover, we have directly from (4-10) and (3-4) that $\mu(t_{i,j;0}^{(r)}) = t_{i,j}^{(r)} \in Y_\sigma^l$. Hence, $\mu(t_{i,j;0}(u)) = t_{i,j}(u)$. The result follows from this, (2-6), (2-7) and the analogous expressions for $t_{1,1;0}(u)$, $t_{2,2;1}(u)$, $t_{1,2;1}(u)$ and $t_{2,1;1}(u)$ derived from (4-14)–(4-16).

Now consider the induction step, so $s_{2,1}+s_{1,2} > 0$. There are two cases according to whether $s_{2,1} > 0$ or $s_{1,2} > 0$. We just explain the argument for the latter situation since the former is entirely similar. Let $\dot{\pi}$ be the pyramid obtained from π by removing the rightmost column, and let $W_{\dot{\pi}}$ be the corresponding finite W-algebra. We denote its Miura transform by $\dot{\mu} : W_{\dot{\pi}} \to U_{\sigma_+}^{l-1}$ and similarly decorate all other notation related to $\dot{\pi}$ with a dot to avoid confusion. Now we proceed to show that $\mu(t_{1,2;1}^{(r)}) = e^{(r)}$ for each $r > s_{1,2}$. By induction, we know that $\dot{\mu}(\dot{t}_{1,2;1}^{(r)}) = \dot{e}^{(r)}$ for each $r \geq s_{1,2}$. But then it follows from the explicit form of (4-10), together with (2-15) and the definition of the evaluation homomorphism (3-2), that

$$\begin{aligned}\mu(t_{1,2;1}^{(r)}) &= \dot{\mu}(\dot{t}_{1,2;1}^{(r)}) \otimes 1 + (-1)^{|2|}\dot{\mu}(\dot{t}_{1,2;1}^{(r-1)}) \otimes e_{1,1} \\ &= \dot{e}^{(r)} \otimes 1 + (-1)^{|2|}\dot{e}^{(r-1)} \otimes e_{1,1} = e^{(r)}\end{aligned}$$

providing $r > s_{1,2}$. The other cases are similar.

Now we deduce the rest of the theorem from (4-19)–(4-21). Order the elements of

$$\Omega := \{t_{1,1;0}^{(r)} \mid 0 < r \leq k\} \cup \{t_{2,2;1}^{(r)} \mid 0 < r \leq l\} \\ \cup \{t_{1,2;1}^{(r)} \mid s_{1,2} < r \leq s_{1,2}+k\} \cup \{t_{2,1;1}^{(r)} \mid s_{2,1} < r \leq s_{2,1}+k\}$$

in some way. By Proposition 4.4, each $t_{i,j;\varsigma}^{(r)} \in \Omega$ belongs to W_π. Moreover, from the definition (4-10), it is in filtered degree r and $\operatorname{gr}_r t_{i,j;\varsigma}^{(r)}$ is equal up to a sign to the element $x_{i,j}^{(r)}$ from Lemma 4.2 plus a linear combination of monomials in elements of strictly smaller Kazhdan degree. Using Theorem 4.1, we deduce that the set of all ordered supermonomials in the set Ω gives a linear basis for W_π. By (4-19)–(4-21) and Corollary 3.6, μ maps this basis onto a basis for $Y_\sigma^l \subseteq U_\sigma^l$. Hence, μ is an isomorphism. □

Remark 4.6. The grading $\mathfrak{p} = \bigoplus_{r \geq 0} \mathfrak{g}(r)$ induces a grading on the superalgebra $U(\mathfrak{p})$. However, W_π is not a graded subalgebra. Instead, we get induced another filtration on W_π, with respect to which the associated graded superalgebra $\operatorname{gr}' W_\pi$ is identified with a graded subalgebra of $U(\mathfrak{p})$. From Proposition 4.4, each of the invariants $t_{i,j;\varsigma}^{(r)}$ belongs to filtered degree $r-1$ and has image $(-1)^{r-1}x_{i,j}^{(r)}$ in the associated graded algebra. Combined with Lemma 4.2 and the usual PBW theorem for $\mathfrak{g}^e$, it follows that $\operatorname{gr}' W_\pi = U(\mathfrak{g}^e)$. Moreover, this filtration on W_π corresponds under the isomorphism μ to the filtration on Y_σ^l induced by the Lie filtration on Y_σ.

Remark 4.7. In this section, we have worked with the "right-handed" definition (4-6) of the finite W-algebra. One can also consider the "left-handed" version

$$W_\pi^\dagger := \{u \in U(\mathfrak{p}) \mid \mathfrak{m}_\chi u \subseteq U(\mathfrak{g})\mathfrak{m}_\chi\}.$$

There is an analogue of Theorem 4.5 for $W_\pi^\dagger$, via which one sees that $W_\pi \cong W_\pi^\dagger$. More precisely, we define the "left-handed" Miura transform $\mu^\dagger : W_\pi^\dagger \to U(\mathfrak{h})$ as above but twisting with the shift automorphism $S_{-\tilde{\rho}^\dagger}$ rather than $S_{-\tilde{\rho}}$, where

$$(\tilde{\rho}^\dagger|\varepsilon_j) = \#\{i \mid i \preceq^\dagger j \text{ and } |\mathrm{row}(i)| = \bar{1}\} - \#\{i \mid i \prec^\dagger j \text{ and } |\mathrm{row}(i)| = \bar{0}\} \tag{4-22}$$

and $i \prec^\dagger j$ means either $\mathrm{col}(i) > \mathrm{col}(j)$, or $\mathrm{col}(i) = \mathrm{col}(j)$ and $\mathrm{row}(i) < \mathrm{row}(j)$. The analogue of Theorem 4.5 asserts that $\mu^\dagger$ is injective with the same image as μ. Hence, $\mu^{-1} \circ \mu^\dagger$, i.e., the restriction of the shift $S_{\tilde{\rho}-\tilde{\rho}^\dagger} : U(\mathfrak{p}) \to U(\mathfrak{p})$, gives an isomorphism between $W_\pi^\dagger$ and W_π. Noting that

$$\tilde{\rho} - \tilde{\rho}^\dagger = \sum_{\substack{1 \leq i,j \leq m+n \\ \mathrm{col}(i) < \mathrm{col}(j)}} (-1)^{|\mathrm{row}(i)|+|\mathrm{row}(j)|}(\varepsilon_i - \varepsilon_j), \tag{4-23}$$

there is a more conceptual explanation for this isomorphism along the lines of the proof given in the nonsuper case in [Brundan et al. 2008, Corollary 2.9].

Remark 4.8. Another consequence of Theorem 4.5 together with Remarks 2.9 and 2.1 is that up to isomorphism the algebra W_π depends only on the set $\{m, n\}$, i.e., on the isomorphism type of $\mathfrak{g}$ and not on the particular choice of the pyramid π or the parity sequence. As observed in [Zhao 2012, Remark 3.10], this can also be proved by mimicking [Brundan and Goodwin 2007, Theorem 2].

5. Proof of invariance

In this section, we prove Proposition 4.4. We keep all notation as in the statement of the proposition. Showing that $u \in U(\mathfrak{p})$ lies in the algebra W_π is equivalent to showing that $[x, u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for all $x \in \mathfrak{m}$ or even just for all x in a set of generators for $\mathfrak{m}$. Let

$$\Omega := \{t_{1,1;0}^{(r)} \mid r > 0\} \cup \{t_{1,2;1}^{(r)} \mid r > s_{1,2}\} \cup \{t_{2,1;1}^{(r)} \mid r > s_{2,1}\} \cup \{t_{2,2;1}^{(r)} \mid r > 0\}. \tag{5-1}$$

Our goal is to show that $[x, u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for x running over a set of generators of $\mathfrak{m}$ and $u \in \Omega$. Proposition 4.4 follows from this since all the other elements listed in the statement of the proposition can be expressed in terms of elements of Ω thanks to Proposition 4.3. Also observe for the present purposes that there is some freedom in the choice of the weight $\tilde{\rho}$: it can be adjusted by adding on any multiple of "supertrace" $\varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{m+1} - \cdots - \varepsilon_{m+n}$. This just twists the elements $t_{i,j;\varsigma}^{(r)}$ by an automorphism of $U(\mathfrak{g})$ so does not have any effect on whether they belong to W_π. So sometimes in this section we will allow ourselves to change the choice of $\tilde{\rho}$.

Lemma 5.1. *Assuming $k = l$, we have that $[x, t_{i,j;0}^{(r)}] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for all $x \in \mathfrak{m}$ and $r > 0$.*

Proof. Note when $k = l$ that $\tilde{\rho} = \varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{m+1} - \cdots - \varepsilon_{m+n}$ if $(|1|, |2|) = (\bar{1}, \bar{0})$ and $\tilde{\rho} = 0$ if $(|1|, |2|) = (\bar{0}, \bar{1})$. As noted above, it does no harm to change the choice of $\tilde{\rho}$ to assume in fact that $\tilde{\rho} = 0$ in both cases. Now we proceed to mimic the argument in [Brundan and Kleshchev 2006, §12].

Consider the tensor algebra $T(M_l)$ in the (purely even) vector space M_l of $l \times l$ matrices over $\mathbb{C}$. For $1 \le i, j \le 2$, define a linear map $t_{i,j} : T(M_l) \to U(\mathfrak{g})$ by setting

$$t_{i,j}(1) := \delta_{i,j}, \qquad t_{i,j}(e_{a,b}) := (-1)^{|i|} e_{i*a, j*b},$$
$$t_{i,j}(x_1 \otimes \cdots \otimes x_r) := \sum_{1 \le h_1, \ldots, h_{r-1} \le 2} t_{i,h_1}(x_1) t_{h_1,h_2}(x_2) \cdots t_{h_{r-1},j}(x_r)$$

for $1 \leq a, b \leq l$, $r \geq 1$ and $x_1, \dots, x_r \in M_l$, where $i * a$ denotes a if $|i| = \bar{0}$ and $l + a$ if $|i| = \bar{1}$. It is straightforward to check for $x, y_1, \dots, y_r \in M_l$ that

$$\begin{aligned}
&[t_{i,j}(x), t_{p,q}(y_1 \otimes \cdots \otimes y_r)] \\
&\quad = (-1)^{|i||j|+|i||p|+|j||p|} \sum_{s=1}^{r} \big(t_{p,j}(y_1 \otimes \cdots \otimes y_{s-1}) t_{i,q}(x y_s \otimes \cdots \otimes y_r) \\
&\qquad\qquad\qquad - t_{p,j}(y_1 \otimes \cdots \otimes y_s x) t_{i,q}(y_{s+1} \otimes \cdots \otimes y_r) \big), \qquad (5\text{-}2)
\end{aligned}$$

where the products xy_s and $y_s x$ on the right are ordinary matrix products in M_l. We extend $t_{i,j}$ to a $\mathbb{C}[u]$-module homomorphism $T(M_l)[u] \to U(\mathfrak{g})[u]$ in the obvious way. Introduce the following matrix with entries in the algebra $T(M_l)[u]$:

$$A(u) := \begin{pmatrix} u + e_{1,1} & e_{1,2} & e_{1,3} & \cdots & e_{1,l} \\ 1 & u + e_{2,2} & & & \vdots \\ 0 & & \ddots & & e_{l-2,l} \\ \vdots & & 1 & u + e_{l-1,l-1} & e_{l-1,l} \\ 0 & \cdots & 0 & 1 & u + e_{l,l} \end{pmatrix}.$$

The point is that $t_{i,j;0}(u) = u^{-l} t_{i,j}(\operatorname{cdet} A(u))$, where the *column determinant* of an $l \times l$ matrix $A = (a_{i,j})$ with entries in a noncommutative ring means the Laplace expansion keeping all the monomials in column order, i.e.,

$$\operatorname{cdet} A := \sum_{w \in S_l} \operatorname{sgn}(w) a_{w(1),1} \cdots a_{w(l),l}.$$

We also write $A_{c,d}(u)$ for the submatrix of $A(u)$ consisting only of rows and columns numbered $c, \dots, d$.

Since $\mathfrak{m}$ is generated by elements of the form $t_{i,j}(e_{c+1,c})$, it suffices now to show that $[t_{i,j}(e_{c+1,c}), t_{p,q}(\operatorname{cdet} A(u))] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for every $1 \leq i, j, p, q \leq 2$ and $c = 1, \dots, l-1$. To do this, we compute using the identity (5-2):

$$\begin{aligned}
&[t_{i,j}(e_{c+1,c}), t_{p,q}(\operatorname{cdet} A(u))] \\
&\quad = t_{p,j}(\operatorname{cdet} A_{1,c-1}(u)) t_{i,q}\left(\operatorname{cdet} \begin{pmatrix} e_{c+1,c} & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u + e_{c+1,c+1} & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u + e_{l,l} \end{pmatrix} \right) \\
&\qquad - t_{p,j}\left(\operatorname{cdet} \begin{pmatrix} u + e_{1,1} & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u + e_{c,c} & e_{c,c} \\ 0 & \cdots & 1 & e_{c+1,c} \end{pmatrix} \right) t_{i,q}(\operatorname{cdet} A_{c+2,l}(u)).
\end{aligned}$$

In order to simplify the second term on the right-hand side, we observe crucially for $h = 1, 2$ that $t_{h,j}((u + e_{c,c})e_{c+1,c}) \equiv t_{h,j}(u + e_{c,c}) \pmod{\mathfrak{m}_\chi U(\mathfrak{g})}$. Hence, we get that

$$
\begin{aligned}
&[t_{i,j}(e_{c+1,c}), t_{p,q}(\operatorname{cdet} A(u))] \\
&\qquad \equiv t_{p,j}(\operatorname{cdet} A_{1,c-1}(u)) t_{i,q}\left(\operatorname{cdet}\begin{pmatrix} 1 & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u + e_{c+1,c+1} & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u + e_{l,l} \end{pmatrix}\right) \\
&\qquad\qquad - t_{p,j}\left(\operatorname{cdet}\begin{pmatrix} u + e_{1,1} & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u + e_{c,c} & e_{c,c} \\ 0 & \cdots & 1 & 1 \end{pmatrix}\right) t_{i,q}(\operatorname{cdet} A_{c+2,l}(u))
\end{aligned}
$$

modulo $\mathfrak{m}_\chi U(\mathfrak{g})$. Making the obvious row and column operations gives that

$$
\operatorname{cdet}\begin{pmatrix} 1 & e_{c+1,c+1} & \cdots & e_{c+1,l} \\ 1 & u + e_{c+1,c+1} & \cdots & e_{c+1,l} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 1 & u + e_{l,l} \end{pmatrix} = u \operatorname{cdet} A_{c+2,l}(u),
$$

$$
\operatorname{cdet}\begin{pmatrix} u + e_{1,1} & \cdots & e_{1,c} & e_{1,c} \\ 1 & \ddots & & \vdots \\ \vdots & & u + e_{c,c} & e_{c,c} \\ 0 & \cdots & 1 & 1 \end{pmatrix} = u \operatorname{cdet} A_{1,c-1}(u).
$$

It remains to substitute these into the preceding formula. □

Proof of Proposition 4.4. Our argument goes by induction on $s_{2,1} + s_{1,2} = l - k$. For the base case $k = l$, we use Proposition 4.3 to rewrite the elements of Ω in terms of the elements $t_{i,j;0}^{(r)}$. The latter lie in W_π by Lemma 5.1. Hence, so do the former.

Now assume that $s_{2,1} + s_{1,2} > 0$. There are two cases according to whether $s_{1,2} \geq s_{2,1}$ or $s_{2,1} > s_{1,2}$. Suppose first that $s_{1,2} \geq s_{2,1}$ and hence that $s_{1,2} > 0$. We may as well assume in addition that $l \geq 2$: the result is trivial for $l \leq 1$ as $\mathfrak{m} = \{0\}$. Let $\dot{\pi}$ be the pyramid obtained from π by removing the rightmost column. We will decorate all notation related to $\dot{\pi}$ with a dot to avoid any confusion. In particular, $W_{\dot{\pi}}$ is a subalgebra of $U(\dot{\mathfrak{p}}) \subseteq U(\dot{\mathfrak{g}})$. Let

$$\theta : U(\dot{\mathfrak{g}}) \hookrightarrow U(\mathfrak{g})$$

be the embedding sending $e_{i,j} \in \dot{\mathfrak{g}}$ to $e_{i',j'} \in \mathfrak{g}$ if the i-th and j-th boxes of $\dot{\pi}$ correspond to the i'-th and j'-th boxes of π, respectively. Let b be the label of the box at the end of the second row of π, i.e., the box that gets removed when passing from π to $\dot{\pi}$. Also in the case that $s_{1,2} = 1$, let c be the label of the box at the end of the first row of π.

Lemma 5.2. *In the above notation, the following hold*:

(i) $t^{(r)}_{1,1;0} = \theta(\dot{t}^{(r)}_{1,1;0})$ *for all* $r > 0$,

(ii) $t^{(r)}_{2,1;1} = \theta(\dot{t}^{(r)}_{2,1;1})$ *for all* $r > s_{2,1}$,

(iii) $t^{(r)}_{1,2;1} = \theta(\dot{t}^{(r)}_{1,2;1}) + \theta(\dot{t}^{(r-1)}_{1,2;1}) S_{\tilde{\rho}}(\bar{e}_{b,b}) - [\theta(\dot{t}^{(r-1)}_{1,2;1}), e_{b-1,b}]$ *for all* $r > s_{1,2}$ *and*

(iv) $t^{(r)}_{2,2;1} = \theta(\dot{t}^{(r)}_{2,2;1}) + \theta(\dot{t}^{(r-1)}_{2,2;1}) S_{\tilde{\rho}}(\bar{e}_{b,b}) - [\theta(\dot{t}^{(r-1)}_{2,2;1}), e_{b-1,b}]$ *for all* $r > 0$.

Proof. This follows directly from the definition of these elements using also that $\theta \circ S_{\dot{\tilde{\rho}}} = S_{\tilde{\rho}} \circ \theta$ on elements of $U(\dot{\mathfrak{p}})$. □

Observe next that $\mathfrak{m}$ is generated by $\theta(\dot{\mathfrak{m}}) \cup J$, where

$$J := \begin{cases} \{e_{b,c}, e_{b,b-1}\} & \text{if } s_{1,2} = 1, \\ \{e_{b,b-1}\} & \text{if } s_{1,2} > 1. \end{cases} \tag{5-3}$$

We know by induction that the following elements of $U(\dot{\mathfrak{p}})$ belong to $W_{\dot{\pi}}$: all $\dot{t}^{(r)}_{1,1;0}$ and $\dot{t}^{(r)}_{2,2;1}$ for $r \geq 0$, all $\dot{t}^{(r)}_{1,2;1}$ for $r \geq s_{1,2}$ and all $\dot{t}^{(r)}_{2,1;1}$ for $r > s_{2,1}$. Also note that the elements of $\theta(\dot{\mathfrak{m}})$ commute with $e_{b-1,b}$ and $S_{\tilde{\rho}}(\bar{e}_{b,b})$. Combined with Lemma 5.2, we deduce that $[\theta(x), u] \in \theta(\dot{\mathfrak{m}}_\chi) U(\mathfrak{g}) \subseteq \mathfrak{m}_\chi U(\mathfrak{g})$ for any $x \in \dot{\mathfrak{m}}$ and $u \in \Omega$. It remains to show that $[x, u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for each $x \in J$ and $u \in \Omega$. This is done in Lemmas 5.3, 5.4 and 5.6 below.

Lemma 5.3. *For* $x \in J$ *and* $u \in \{t^{(r)}_{1,1;0} \mid r > 0\} \cup \{t^{(r)}_{2,1;1} \mid r > s_{2,1}\}$, *we have that* $[x, u] \in \mathfrak{m}_\chi U(\mathfrak{g})$.

Proof. Take $e_{b,d} \in J$. Consider a monomial $S_{\tilde{\rho}}(\bar{e}_{i_1,j_1} \cdots \bar{e}_{i_s,j_s})$ in the expansion of u from (4-10). The only way it could fail to supercommute with $e_{b,d}$ is if it involves some $\bar{e}_{i_h,j_h}$ with $j_h = b$ or $i_h = d$. Since $\mathrm{row}(j_s) = 1$ and $\mathrm{col}(i_{h+1}) > \mathrm{col}(j_h)$ when $\mathrm{row}(j_h) = 2$, this situation arises only if $s_{1,2} = 1$, $i_h = d$ and $j_h = c$. Then the supercommutator $[e_{b,d}, \bar{e}_{i_h,j_h}]$ equals $\pm e_{b,c}$. It remains to repeat this argument to see that we can move the resulting $e_{b,c} \in \mathfrak{m}_\chi$ to the beginning. □

It is harder to deal with the remaining elements $t^{(r)}_{1,2;1}$ and $t^{(r)}_{2,2;1}$ of Ω. We follow different approaches according to whether $s_{1,2} > 1$ or $s_{1,2} = 1$.

Lemma 5.4. *Assume that* $s_{1,2} > 1$. *We have that* $[e_{b,b-1}, u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ *for all* $u \in \{t^{(r)}_{1,2;1} \mid r > s_{1,2}\} \cup \{t^{(r)}_{2,2;1} \mid r > 0\}$.

Proof. We just explain in detail for $u = t^{(r)}_{1,2;1}$; the other case follows the same pattern. Let $\ddot{\pi}$ be the pyramid obtained from π by removing its rightmost two columns. We decorate all notation associated to $W_{\ddot\pi}$ with a double dot, so $W_{\ddot\pi} \subseteq U(\ddot{\mathfrak{p}}) \subseteq U(\ddot{\mathfrak{g}})$ and so on. Let

$$\phi : U(\ddot{\mathfrak{g}}) \hookrightarrow U(\mathfrak{g})$$

be the embedding sending $e_{i,j} \in \ddot{\mathfrak{g}}$ to $e_{i',j'} \in \mathfrak{g}$, where the i-th and j-th boxes of $\ddot\pi$ are labeled by i and j in π, respectively. For $r \geq s_{1,2}$, we have by analogy with Lemma 5.2(iii) that

$$\theta(\dot t^{(r)}_{1,2;1}) = \phi(\ddot t^{(r)}_{1,2;1}) + \phi(\ddot t^{(r-1)}_{1,2;1}) S_{\tilde\rho}(\bar e_{b-1,b-1}) - [\phi(\ddot t^{(r-1)}_{1,2;1}), e_{b-2,b-1}].$$

We combine this with Lemma 5.2(iii) to deduce for $r > s_{1,2}$ that

$$\begin{aligned} t^{(r)}_{1,2;1} = \phi(\ddot t^{(r)}_{1,2;1}) &+ \phi(\ddot t^{(r-1)}_{1,2;1}) S_{\tilde\rho}(\bar e_{b-1,b-1}) - [\phi(\ddot t^{(r-1)}_{1,2;1}), e_{b-2,b-1}] \\ &+ \phi(\ddot t^{(r-1)}_{1,2;1}) S_{\tilde\rho}(\bar e_{b,b}) + \phi(\ddot t^{(r-2)}_{1,2;1}) S_{\tilde\rho}(\bar e_{b-1,b-1}) S_{\tilde\rho}(\bar e_{b,b}) \\ &- [\phi(\ddot t^{(r-2)}_{1,2;1}), e_{b-2,b-1}] S_{\tilde\rho}(\bar e_{b,b}) - \phi(\ddot t^{(r-2)}_{1,2;1}) \bar e_{b-1,b} + [\phi(\ddot t^{(r-2)}_{1,2;1}), e_{b-2,b}]. \end{aligned}$$

We deduce that

$$\begin{aligned} [e_{b,b-1}, t^{(r)}_{1,2;1}] &= \phi(\ddot t^{(r-2)}_{1,2;1})(\bar e_{b,b-1} S_{\tilde\rho}(\bar e_{b,b}) - \bar e_{b,b-1} S_{\tilde\rho}(\bar e_{b-1,b-1}) + (-1)^{|2|} \bar e_{b,b-1}) \\ &+ [\phi(\ddot t^{(r-2)}_{1,2;1}), e_{b-2,b-1}] \bar e_{b,b-1} - \phi(\ddot t^{(r-2)}_{1,2;1})(\bar e_{b,b} - \bar e_{b-1,b-1}) - [\phi(\ddot t^{(r-2)}_{1,2;1}), e_{b-2,b-1}]. \end{aligned}$$

Working modulo $\mathfrak{m}_\chi U(\mathfrak{g})$, we can replace all $\bar e_{b,b-1}$ by 1. Then we are reduced just to checking that

$$S_{\tilde\rho}(\bar e_{b,b}) - S_{\tilde\rho}(\bar e_{b-1,b-1}) + (-1)^{|2|} = \bar e_{b,b} - \bar e_{b-1,b-1}.$$

This follows because $(\tilde\rho|\varepsilon_b) - (\tilde\rho|\varepsilon_{b-1}) + (-1)^{|2|} = 0$ by the definition (4-8). $\square$

Lemma 5.5. *Assume that* $s_{1,2} = 1$. *For* $r > 2$, *we have that*

$$t^{(r)}_{1,2;1} = (-1)^{|1|}[t^{(2)}_{1,1;0}, t^{(r-1)}_{1,2;1}] - t^{(1)}_{1,1;0} t^{(r-1)}_{1,2;1}, \tag{5-4}$$

$$t^{(r)}_{2,2;1} = (-1)^{|1|}[t^{(2)}_{1,2;1}, t^{(r-1)}_{2,1;1}] - \sum_{a=0}^{r} t^{(a)}_{1,1;1} t^{(r-a)}_{2,2;1}. \tag{5-5}$$

Proof. We prove (5-4). The induction hypothesis means that we can appeal to Theorem 4.5 for the algebra $W_{\dot\pi}$. Hence, using the relations from Theorem 2.2, we know that the following holds in the algebra $W_{\dot\pi}$ for all $r \geq 2$:

$$\dot t^{(r)}_{1,2;1} = (-1)^{|1|}[\dot t^{(2)}_{1,1;0}, \dot t^{(r-1)}_{1,2;1}] - \dot t^{(1)}_{1,1;0} \dot t^{(r-1)}_{1,2;1}.$$

Using Lemma 5.2, we deduce for $r > 2$ that

$$\begin{aligned}
t_{1,2;1}^{(r)} &= \theta(\dot{t}_{1,2;1}^{(r)}) + \theta(\dot{t}_{1,2;1}^{(r-1)})S_{\tilde{\rho}}(\bar{e}_{b,b}) - [\theta(\dot{t}_{1,2;1}^{(r-1)}), e_{b-1,b}] \\
&= (-1)^{|1|}[t_{1,1;0}^{(2)}, \theta(\dot{t}_{1,2;1}^{(r-1)})] - t_{1,1;0}^{(1)}\theta(\dot{t}_{1,2;1}^{(r-1)}) \\
&\quad + (-1)^{|1|}[t_{1,1;0}^{(2)}, \theta(\dot{t}_{1,2;1}^{(r-2)})]S_{\tilde{\rho}}(\bar{e}_{b,b}) - t_{1,1;0}^{(1)}\theta(\dot{t}_{1,2;1}^{(r-2)})S_{\tilde{\rho}}(\bar{e}_{b,b}) \\
&\quad - (-1)^{|1|}\big[[t_{1,1;0}^{(2)}, \theta(\dot{t}_{1,2;1}^{(r-2)})], e_{b-1,b}\big] + [t_{1,1;0}^{(1)}\theta(\dot{t}_{1,2;1}^{(r-2)}), e_{b-1,b}] \\
&= (-1)^{|1|}\big[t_{1,1;0}^{(2)}, \theta(\dot{t}_{1,2;1}^{(r-1)}) + \theta(\dot{t}_{1,2;1}^{(r-2)})S_{\tilde{\rho}}(\bar{e}_{b,b}) - [\theta(\dot{t}_{1,2;1}^{(r-2)}), e_{b-1,b}]\big] \\
&\quad - t_{1,1;0}^{(1)}\big(\theta(\dot{t}_{1,2;1}^{(r-1)}) + \theta(\dot{t}_{1,2;1}^{(r-2)})S_{\tilde{\rho}}(\bar{e}_{b,b}) - [\theta(\dot{t}_{1,2;1}^{(r-2)}), e_{b-1,b}]\big) \\
&= (-1)^{|1|}[t_{1,1;0}^{(2)}, t_{1,2;1}^{(r-1)}] - t_{1,1;0}^{(1)}t_{1,2;1}^{(r-1)}.
\end{aligned}$$

The other equation (5-5) follows by a similar trick. □

Lemma 5.6. *Assume that $s_{1,2} = 1$. We have that $[x, u] \in \mathfrak{m}_\chi U(\mathfrak{g})$ for all $x \in J$ and $u \in \{t_{1,2;1}^{(r)} \mid r > s_{1,2}\} \cup \{t_{2,2;1}^{(r)} \mid r > 0\}$.*

Proof. Proceed by induction on r. The base cases when $r \leq 2$ are small enough that they can be checked directly from the definitions. Then for $r > 2$, use Lemma 5.5, noting by the induction hypothesis and Lemma 5.3 that all the terms on the right-hand side of (5-4) and (5-5) are already known to lie in $\mathfrak{m}_\chi U(\mathfrak{g})$. □

We have now verified the induction step in the case that $s_{1,2} \geq s_{2,1}$. It remains to establish the induction step when $s_{2,1} > s_{1,2}$. The strategy for this is sufficiently similar to the case just done (based on removing columns from the left of the pyramid π) that we leave the details to the reader. We just note one minor difference: in the proof of the analogue of Lemma 5.2, it is no longer the case that $\theta \circ S_{\dot{\tilde{\rho}}} = S_{\tilde{\rho}} \circ \theta$, but this can be fixed by allowing the choice of $\tilde{\rho}$ to change by a multiple of $\varepsilon_1 + \cdots + \varepsilon_m - \varepsilon_{m+1} - \cdots - \varepsilon_{m+n}$.

This completes the proof of Proposition 4.4. □

6. Triangular decomposition

Let W_π be the principal W-algebra in $\mathfrak{g} = \mathfrak{gl}_{m|n}(\mathbb{C})$ associated to pyramid π. We adopt all the notation from §4. So

- $(|1|, |2|)$ is a parity sequence chosen so that $(|1|, |2|) = (\bar{0}, \bar{1})$ if $m < n$ and $(|1|, |2|) = (\bar{1}, \bar{0})$ if $m > n$,
- π has $k = \min(m, n)$ boxes in its first row and $l = \max(m, n)$ boxes in its second row and
- $\sigma = (s_{i,j})_{1 \leq i,j \leq 2}$ is a shift matrix compatible with π.

We identify W_π with Y_σ^l, the shifted Yangian of level l, via the isomorphism μ from (4-18). Thus, we have available a set of Drinfeld generators for W_π satisfying the relations from Theorem 2.2 plus the additional truncation relations $d_1^{(r)} = 0$ for $r > k$. In view of (4-19)–(4-21) and (4-10), we even have available explicit formulae for these generators as elements of $U(\mathfrak{p})$ although we seldom need to use these (but see the proof of Lemma 8.3 below).

By the relations, W_π admits a $\mathbb{Z}$-grading

$$W_\pi = \bigoplus_{g \in \mathbb{Z}} W_{\pi;g}$$

such that the generators $d_i^{(r)}$ are of degree 0, the generators $e^{(r)}$ are of degree 1 and the generators $f^{(r)}$ are of degree -1. Moreover, the PBW theorem (Corollary 3.6) implies that $W_{\pi;g} = 0$ for $|g| > k$.

More surprisingly, the algebra W_π admits a triangular decomposition. To introduce this, let W_π^0, W_π^+ and W_π^- be the subalgebras of W_π generated by the elements $\Omega_0 := \{d_1^{(r)}, d_2^{(s)} \mid 0 < r \leq k,\ 0 < s \leq l\}$, $\Omega_+ := \{e^{(r)} \mid s_{1,2} < r \leq s_{1,2} + k\}$ and $\Omega_- := \{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + k\}$, respectively. Let $W_\pi^\sharp$ and $W_\pi^\flat$ be the subalgebras of W_π generated by $\Omega_0 \cup \Omega_+$ and $\Omega_- \cup \Omega_0$, respectively. We warn the reader that the elements $e^{(r)}$ $(r > s_{1,2} + k)$ do not necessarily lie in W_π^+ (but they do lie in $W_\pi^\sharp$ by (3-5)). Similarly, the elements $f^{(r)}$ for $r > s_{2,1} + k$ do not necessarily lie in W_π^- (but they do lie in $W_\pi^\flat$), and the elements $d_2^{(r)}$ for $r > l$ do not necessarily lie in any of W_π^0, $W_\pi^\sharp$ or $W_\pi^\flat$.

Theorem 6.1. *The algebras W_π^0, W_π^+ and W_π^- are free supercommutative superalgebras on generators Ω_0, Ω_+ and Ω_-, respectively. Multiplication defines vector space isomorphisms*

$$W_\pi^- \otimes W_\pi^0 \otimes W_\pi^+ \xrightarrow{\sim} W_\pi, \qquad W_\pi^0 \otimes W_\pi^+ \xrightarrow{\sim} W_\pi^\sharp, \qquad W_\pi^- \otimes W_\pi^0 \xrightarrow{\sim} W_\pi^\flat.$$

Moreover, there are unique surjective homomorphisms

$$W_\pi^\sharp \twoheadrightarrow W_\pi^0, \qquad W_\pi^\flat \twoheadrightarrow W_\pi^0$$

sending $e^{(r)} \mapsto 0$ for all $r > s_{1,2}$ or $f^{(r)} \mapsto 0$ for all $r > s_{2,1}$, respectively, such that the restriction of these maps to the subalgebra W_π^0 is the identity.

Proof. Throughout the proof, we repeatedly apply the PBW theorem (Corollary 3.6), choosing the order of generators so that $\Omega_- < \Omega_0 < \Omega_+$.

To start with, note by the left-hand relations in Theorem 2.2 that each of W_π^0, W_π^+ and W_π^- is supercommutative. Combined with the PBW theorem, we deduce that they are free supercommutative on the given generators. Moreover, the PBW theorem implies that the multiplication map $W_\pi^- \otimes W_\pi^0 \otimes W_\pi^+ \to W_\pi$ is a vector space isomorphism.

Next we observe that $W_\pi^\sharp$ contains all the elements $e^{(r)}$ for $r > s_{1,2}$. This follows from (3-5) by induction on r. Moreover, it is spanned as a vector space by the ordered supermonomials in the generators $\Omega_0 \cup \Omega_+$. This follows from (3-5), the relation for $[d_i^{(r)}, e^{(s)}]$ in Theorem 2.2 and induction on Kazhdan degree. Hence, the multiplication map $W_\pi^0 \otimes W_\pi^+ \to W_\pi^\sharp$ is surjective. It is injective by the PBW theorem, so it is an isomorphism. Similarly, $W_\pi^- \otimes W_\pi^0 \to W_\pi^\flat$ is an isomorphism.

Finally, let $J^\sharp$ be the two-sided ideal of $W_\pi^\sharp$ that is the sum of all of the graded components $W_{\pi;g}^\sharp := W_\pi^\sharp \cap W_{\pi;g}$ for $g > 0$. By the PBW theorem, The natural quotient map $W_\pi^0 \to W_\pi^\sharp / J^\sharp$ is an isomorphism. Hence, there is a surjection $W_\pi^\sharp \twoheadrightarrow W_\pi^0$ as in the statement of the theorem. A similar argument yields the desired surjection $W_\pi^\flat \twoheadrightarrow W_\pi^0$. □

7. Irreducible representations

Continue with the notation of Section 6. Using the triangular decomposition, we can classify irreducible W_π-modules by highest weight theory. Define a *π-tableau* to be a filling of the boxes of the pyramid π by arbitrary complex numbers. Let Tab_π denote the set of all such π-tableaux. We represent the π-tableau with entries $a_1, \dots, a_k$ along its first row and $b_1, \dots, b_l$ along its second row simply by the array ${}^{a_1 \cdots a_k}_{b_1 \cdots b_l}$. We say that $A, B \in \mathrm{Tab}_\pi$ are *row equivalent*, denoted $A \sim B$, if B can be obtained from A by permuting entries within each row.

Recall from Theorem 6.1 that W_π^0 is the polynomial algebra on

$$\{d_1^{(r)}, d_2^{(s)} \mid 0 < r \le k,\ 0 < s \le l\}.$$

For $A = {}^{a_1 \cdots a_k}_{b_1 \cdots b_l} \in \mathrm{Tab}_\pi$, let $\mathbb{C}_A$ be the one-dimensional W_π^0-module on basis 1_A such that

$$u^k d_1(u) 1_A = (u + a_1) \cdots (u + a_k) 1_A, \tag{7-1}$$

$$u^l d_2(u) 1_A = (u + b_1) \cdots (u + b_l) 1_A. \tag{7-2}$$

Thus, $d_1^{(r)} 1_A = e_r(a_1, \dots, a_k) 1_A$ and $d_2^{(r)} 1_A = e_r(b_1, \dots, b_l) 1_A$, where e_r denotes the r-th elementary symmetric polynomial. Every irreducible W_π^0-module is isomorphic to $\mathbb{C}_A$ for some $A \in \mathrm{Tab}_\pi$, and $\mathbb{C}_A \cong \mathbb{C}_B$ if and only if $A \sim B$.

Given $A \in \mathrm{Tab}_\pi$, we view $\mathbb{C}_A$ as a $W_\pi^\sharp$-module via the surjection $W_\pi^\sharp \twoheadrightarrow W_\pi^0$ from Theorem 6.1, i.e., $e^{(r)} 1_A = 0$ for all $r > s_{1,2}$. Then we induce to form the *Verma module*

$$\bar{M}(A) := W_\pi \otimes_{W_\pi^\sharp} \mathbb{C}_A. \tag{7-3}$$

Sometimes we need to view this as a supermodule, which we do by declaring that its cyclic generator $1 \otimes 1_A$ is even. By Theorem 6.1, W_π is a free right $W_\pi^\sharp$-module with basis given by the ordered supermonomials in the odd elements $\{f^{(r)} \mid s_{2,1} < r \le s_{2,1} + k\}$. Hence, $\bar{M}(A)$ has basis given by the vectors $x \otimes 1_A$ as x runs over this set of supermonomials. In particular, $\dim \bar{M}(A) = 2^k$.

The following lemma shows that $\bar{M}(A)$ has a unique irreducible quotient, which we denote by $\bar{L}(A)$; we write v_+ for the image of $1 \otimes 1_A \in \bar{M}(A)$ in $\bar{L}(A)$.

Lemma 7.1. *For $A = {}^{a_1\cdots a_k}_{b_1\cdots b_l} \in \mathrm{Tab}_\pi$, the Verma module $\overline{M}(A)$ has a unique irreducible quotient $\overline{L}(A)$. The image v_+ of $1 \otimes 1_A$ is the unique (up to scalars) nonzero vector in $\overline{L}(A)$ such that $e^{(r)}v_+ = 0$ for all $r > s_{1,2}$. Moreover, we have that $d_1^{(r)}v_+ = e_r(a_1, \dots, a_k)v_+$ and $d_2^{(r)}v_+ = e_r(b_1, \dots, b_l)v_+$ for all $r \geq 0$.*

Proof. Let $\lambda := (-1)^{|1|}(a_1 + \cdots + a_k)$. For any $\mu \in \mathbb{C}$, let $\overline{M}(A)_\mu$ be the μ-eigenspace of the endomorphism of $\overline{M}(A)$ defined by $d := (-1)^{|1|}d_1^{(1)} \in W_\pi$. Note by (7-1) and the relations that $d1_A = \lambda 1_A$ and $[d, f^{(r)}] = -f^{(r)}$ for each $r > s_{2,1}$. Using the PBW basis for $\overline{M}(A)$, it follows that

$$\overline{M}(A) = \bigoplus_{i=0}^{k} \overline{M}(A)_{\lambda-i} \tag{7-4}$$

and $\dim \overline{M}(A)_{\lambda-i} = \binom{k}{i}$ for each $0 \leq i \leq k$. In particular, $\overline{M}(A)_\lambda$ is one-dimensional, and it generates $\overline{M}(A)$ as a $W_\pi^\flat$-module. This is all that is needed to deduce that $\overline{M}(A)$ has a unique irreducible quotient $\overline{L}(A)$ following the standard argument of highest weight theory.

The vector v_+ is a nonzero vector annihilated by $e^{(r)}$ for $r > s_{1,2}$, and $d_1^{(r)}v_+$ and $d_2^{(r)}v_+$ are as stated thanks to (7-1) and (7-2). It just remains to show that any vector $v \in \overline{L}(A)$ annihilated by all $e^{(r)}$ is a multiple of v_+. The decomposition (7-4) induces an analogous decomposition

$$\overline{L}(A) = \bigoplus_{i=0}^{k} \overline{L}(A)_{\lambda-i} \tag{7-5}$$

although for $0 < i \leq k$ the eigenspace $\overline{L}(A)_{\lambda-i}$ may now be 0. Write $v = \sum_{i=0}^k v_i$ with $v_i \in \overline{L}(A)_{\lambda-i}$. Then we need to show that $v_i = 0$ for $i > 0$. We have that $e^{(r)}v = \sum_{i=1}^k e^{(r)}v_i = 0$; hence, $e^{(r)}v_i = 0$ for each i. But this means for $i > 0$ that the submodule $W_\pi v_i = W_\pi^\flat v_i$ has trivial intersection with $\overline{L}(A)_\lambda$, so it must be 0. □

Here is the classification of irreducible W_π-modules.

Theorem 7.2. *Every irreducible W_π-module is finite-dimensional and is isomorphic to one of the modules $\overline{L}(A)$ from Lemma 7.1 for some $A \in \mathrm{Tab}_\pi$. Moreover, $\overline{L}(A) \cong \overline{L}(B)$ if and only if $A \sim B$. Hence, fixing a set $\mathrm{Tab}_\pi /_\sim$ of representatives for the $\sim$-equivalence classes in Tab_π, the modules*

$$\{\overline{L}(A) \mid A \in \mathrm{Tab}_\pi /_\sim\}$$

give a complete set of pairwise inequivalent irreducible W_π-modules.

Proof. We note, to start with, for $A, B \in \mathrm{Tab}_\pi$ that $\overline{L}(A) \cong \overline{L}(B)$ if and only if $A \sim B$. This is clear from Lemma 7.1.

Now take an arbitrary (conceivably infinite-dimensional) irreducible W_π-module L. We want to show that $L \cong \bar{L}(A)$ for some $A \in \mathrm{Tab}_\pi$. For $i \geq 0$, let

$$L[i] := \{v \in L \mid W_{\pi;g} v = \{0\} \text{ if } g > 0 \text{ or } g \leq -i\}.$$

We claim initially that $L[k+1] \neq \{0\}$. To see this, recall that $W_{\pi;g} = \{0\}$ for $g \leq -k-1$, so by the PBW theorem, $L[k+1]$ is simply the set of all vectors $v \in L$ such that $e^{(r)} v = 0$ for all $s_{1,2} < r \leq s_{1,2}+k$. Now take any nonzero vector $v \in L$ such that $\#\{r = s_{1,2}+1, \dots, s_{1,2}+k \mid e^{(r)}v = 0\}$ is maximal. If $e^{(r)}v \neq 0$ for some $s_{1,2} < r \leq s_{1,2}+k$, we can replace v by $e^{(r)}v$ to get a nonzero vector annihilated by more $e^{(r)}$'s. Hence, $v \in L[k+1]$ by the maximality of the choice of v, and we have shown that $L[k+1] \neq \{0\}$.

Since $L[k+1] \neq \{0\}$, it makes sense to define $i \geq 0$ to be minimal such that $L[i] \neq \{0\}$. Since $L[0] = \{0\}$, we actually have that $i > 0$. Pick $0 \neq v \in L[i]$, and let $L' := W_\pi^\sharp v$. Actually, by the PBW theorem, we have that $L' = W_\pi^0 v$ and $L' \subseteq L[i]$. Suppose first that L' is irreducible as a W_π^0-module. Then $L' \cong \mathbb{C}_A$ for some $A \in \mathrm{Tab}_\pi$. The inclusion $L' \hookrightarrow L$ induces a nonzero W_π-module homomorphism

$$\bar{M}(A) \cong W_\pi \otimes_{W_\pi^\sharp} L' \to L,$$

which is surjective as L is irreducible. Hence, $L \cong \bar{L}(A)$.

It remains to rule out the possibility that L' is reducible. Suppose for a contradiction that L' possesses a nonzero proper W_π^0-submodule L''. As $L = W_\pi L''$ and $W_\pi^\sharp L'' = L''$, the PBW theorem implies that we can write

$$v = w + \sum_{h=1}^{k} \sum_{s_{2,1} < r_1 < \dots < r_h \leq s_{2,1}+k} f^{(r_1)} \cdots f^{(r_h)} v_{r_1,\dots,r_h}$$

for some vectors $v_{r_1,\dots,r_h}, w \in L''$. Then we have that

$$0 \neq v - w \in L[i] \cap \left(\sum_{g \leq -1} W_{\pi;g} L[i] \right) \subseteq L[i-1].$$

This shows $L[i-1] \neq \{0\}$, contradicting the minimality of the choice of i. $\square$

The final theorem of the section gives an explicit monomial basis for $\bar{L}(A)$. We only prove linear independence here; the spanning part of the argument will be given in Section 8.

Theorem 7.3. *Suppose $A = {}^{a_1 \cdots a_k}_{b_1 \cdots b_l} \in \mathrm{Tab}_\pi$. Let $h \geq 0$ be maximal such that there exist distinct $1 \leq i_1, \dots, i_h \leq k$ and distinct $1 \leq j_1, \dots, j_h \leq l$ with $a_{i_1} = b_{j_1}, \dots, a_{i_h} = b_{j_h}$. Then the irreducible module $\bar{L}(A)$ has basis given by the vectors $x v_+$ as x runs over all ordered supermonomials in the odd elements $\{f^{(r)} \mid s_{2,1} < r \leq s_{2,1}+k-h\}$.*

Proof. Let $\bar{k} := k - h$ and $\bar{l} := l - h$. Since $\bar{L}(A)$ only depends on the $\sim$-equivalence class of A, we can reindex to assume that $a_{\bar{k}+1} = b_{\bar{l}+1}, a_{\bar{k}+2} = b_{\bar{l}+2}, \ldots, a_k = b_l$. We proceed to show that the vectors xv_+ for all ordered supermonomials x in $\{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + \bar{k}\}$ are linearly independent in $\bar{L}(A)$. In fact, it is enough for this to show just that

$$f^{(s_{2,1}+1)} f^{(s_{2,1}+2)} \cdots f^{(s_{2,1}+\bar{k})} v_+ \neq 0. \tag{7-6}$$

Indeed, assuming (7-6), we can prove the linear independence in general by taking any nontrivial linear relation of the form

$$\sum_{a=0}^{\bar{k}} \sum_{s_{2,1} < r_1 < \cdots < r_a \leq s_{2,1} + \bar{k}} \lambda_{r_1,\ldots,r_a} f^{(r_1)} \cdots f^{(r_a)} v_+ = 0.$$

Let a be minimal such that $\lambda_{r_1,\ldots,r_a} \neq 0$ for some $r_1, \ldots, r_a$. Apply $f^{(s_1)} \cdots f^{(s_{\bar{k}-a})}$, where $s_{2,1} < s_1 < \cdots < s_{\bar{k}-a} \leq s_{2,1} + \bar{k}$ are different from $r_1 < \cdots < r_a$. All but one term of the summation becomes 0, and using (7-6), we can deduce that $\lambda_{r_1,\ldots,r_a} = 0$, a contradiction.

In this paragraph, we prove (7-6) by showing that

$$e^{(s_{1,2}+1)} e^{(s_{1,2}+2)} \cdots e^{(s_{1,2}+\bar{k})} f^{(s_{2,1}+1)} f^{(s_{2,1}+2)} \cdots f^{(s_{2,1}+\bar{k})} v_+ \neq 0. \tag{7-7}$$

The left-hand side of (7-7) equals

$$\sum_{w \in S_{\bar{k}}} \operatorname{sgn}(w) [e^{(\bar{k}+1+s_{1,2}-1)}, f^{(s_{2,1}+w(1))}] \cdots [e^{(\bar{k}+1+s_{1,2}-\bar{k})}, f^{(s_{2,1}+w(\bar{k}))}] v_+.$$

By Remark 2.3, up to a sign, this is $\det(c^{(\bar{l}-i+j)})_{1 \leq i,j \leq \bar{k}} v_+$. It is easy to see from Lemma 7.1 that $c^{(r)} v_+ = e_r(b_1, \ldots, b_{\bar{l}} / a_1, \ldots, a_{\bar{k}}) v_+$, where

$$e_r(b_1, \ldots, b_{\bar{l}} / a_1, \ldots, a_{\bar{k}}) := \sum_{s+t=r} (-1)^t e_s(b_1, \ldots, b_{\bar{l}}) h_t(a_1, \ldots, a_{\bar{k}})$$

is the r-th elementary supersymmetric function from [Macdonald 1995, Exercise I.3.23]. Thus, we need to show that $\det(e_{\bar{l}-i+j}(b_1, \ldots, b_{\bar{l}} / a_1, \ldots, a_{\bar{k}}))_{1 \leq i,j \leq \bar{k}} \neq 0$. But this determinant is the supersymmetric Schur function $s_\lambda(b_1, \ldots, b_{\bar{l}} / a_1, \ldots, a_{\bar{k}})$ for the partition $\lambda = (\bar{k}^{\bar{l}})$ defined in [Macdonald 1995, Exercise I.3.23]. Hence, by the factorization property described there, it is equal to $\prod_{1 \leq i \leq \bar{l}} \prod_{1 \leq j \leq \bar{k}} (b_i - a_j)$, which is indeed nonzero.

We have now proved the linear independence of the vectors xv_+ as x runs over all ordered supermonomials in $\{f^{(r)} \mid s_{2,1} < r \leq s_{2,1} + \bar{k}\}$. It remains to show that these vectors also span $\bar{L}(A)$. For this, it is enough to show that $\dim \bar{L}(A) \leq 2^{\bar{k}}$. This will be established in the next section by means of an explicit construction of a module of dimension $2^{\bar{k}}$ containing $\bar{L}(A)$ as a subquotient. $\square$

8. Tensor products

In this section, we define some more general comultiplications between the algebras W_π, allowing certain tensor products to be defined. We apply this to construct so-called *standard modules* $\overline{V}(A)$ for each $A \in \mathrm{Tab}_\pi$. Then we complete the proof of Theorem 7.3 by showing that every irreducible W_π-module is isomorphic to one of the modules $\overline{V}(A)$ for suitable A.

Recall that the pyramid π has l boxes on its second row. Suppose we are given $l_1, \dots, l_d \geq 0$ such that $l_1 + \cdots + l_d = l$. For each $c = 1, \dots, d$, let π_c be the pyramid consisting of columns $l_1 + \cdots + l_{c-1} + 1, \dots, l_1 + \cdots + l_c$ of π. Thus, π is the "concatenation" of the pyramids $\pi_1, \dots, \pi_d$. Let W_{π_c} be the principal W-algebra defined from π_c. Let $\sigma_1, \dots, \sigma_d$ be the unique shift matrices such that each σ_c is compatible with π_c and σ_c is lower or upper triangular if $s_{2,1} \geq l_1 + \cdots + l_c$ or $s_{1,2} \geq l_c + \cdots + l_d$, respectively. We denote the Miura transform for W_{π_c} by $\mu_c : W_{\pi_c} \hookrightarrow U_{\sigma_c}^{l_c}$.

Lemma 8.1. *With the above notation, there is a unique injective algebra homomorphism*

$$\Delta_{l_1,\dots,l_d} : W_\pi \hookrightarrow W_{\pi_1} \otimes \cdots \otimes W_{\pi_d} \tag{8-1}$$

such that $(\mu_1 \otimes \cdots \otimes \mu_d) \circ \Delta_{l_1,\dots,l_d} = \mu$.

Proof. Let us add the suffix c to all notation arising from the definition of W_{π_c} so that W_{π_c} is a subalgebra of $U(\mathfrak{p}_c)$, we have that $\mathfrak{g}_c = \mathfrak{m}_c \oplus \mathfrak{h}_c \oplus \mathfrak{p}_c^\perp$ and so on. We identify $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_d$ with a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ so that $e_{i,j} \in \mathfrak{g}_c$ is identified with $e_{i',j'} \in \mathfrak{g}$, where i' and j' are the labels of the boxes of π corresponding to the i-th and j-th boxes of π_c, respectively. Similarly, we identify $\mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_d$ with $\mathfrak{m}' \subseteq \mathfrak{m}$, $\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_d$ with $\mathfrak{p}' \subseteq \mathfrak{p}$ and $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_d$ with $\mathfrak{h}' = \mathfrak{h}$. Also let $\tilde{\rho}' := \tilde{\rho}_1 + \cdots + \tilde{\rho}_d$, a character of $\mathfrak{p}'$. In this way, $W_{\pi_1} \otimes \cdots \otimes W_{\pi_d}$ is identified with $W'_\pi := \{u \in U(\mathfrak{p}') \mid u\mathfrak{m}'_\chi \subseteq \mathfrak{m}'_\chi U(\mathfrak{g}')\}$, where $\mathfrak{m}'_\chi = \{x - \chi(x) \mid x \in \mathfrak{m}'\}$.

Let $\mathfrak{q}$ be the unique parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}'$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Let $\psi : U(\mathfrak{q}) \twoheadrightarrow U(\mathfrak{g}')$ be the homomorphism induced by the natural projection of $\mathfrak{q} \twoheadrightarrow \mathfrak{g}'$. The following diagram commutes:

$$\begin{tikzcd} U(\mathfrak{p}) \arrow[r, "S_{-\tilde{\rho}'} \circ \psi \circ S_{\tilde{\rho}}"] \arrow[d, "\mathrm{pr} \circ S_{\tilde{\rho}}"'] & U(\mathfrak{p}') \arrow[d, "\mathrm{pr}' \circ S_{\tilde{\rho}'}"] \\ U(\mathfrak{h}) \arrow[r, equal] & U(\mathfrak{h}') \end{tikzcd}$$

We claim that $S_{-\tilde{\rho}'} \circ \psi \circ S_{\tilde{\rho}}$ maps W_π into W'_π. The claim implies the lemma, for then it makes sense to *define* $\Delta_{l_1,\dots,l_d}$ to be the restriction of this map to W_π, and we are done by the commutativity of the above diagram and injectivity of the Miura transform.

To prove the claim, observe that $\tilde{\rho} - \tilde{\rho}'$ extends to a character of $\mathfrak{q}$; hence, there is a corresponding shift automorphism $S_{\tilde{\rho}-\tilde{\rho}'} : U(\mathfrak{q}) \to U(\mathfrak{q})$ that preserves W'_π. Moreover, $S_{-\tilde{\rho}'} \circ \psi \circ S_{\tilde{\rho}} = S_{\tilde{\rho}-\tilde{\rho}'} \circ \psi$. Therefore, it enough to check just that $\psi(W_\pi) \subseteq W'_\pi$. To see this, take $u \in W_\pi$ so that $u\mathfrak{m}_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g})$. This implies that $u\mathfrak{m}'_\chi \subseteq \mathfrak{m}_\chi U(\mathfrak{g}) \cap U(\mathfrak{q})$; hence, applying ψ we get that $\psi(u)\mathfrak{m}'_\chi \subseteq \mathfrak{m}'_\chi U(\mathfrak{g}')$. This shows that $\psi(u) \in W'_\pi$ as required. $\square$

Remark 8.2. Special cases of the maps (8-1) with $d = 2$ are related to the comultiplications Δ, Δ_+ and Δ_- from (2-14)–(2-16). Indeed, if $l = l_1 + l_2$ for $l_1 \geq s_{2,1}$ and $l_2 \geq s_{1,2}$, the shift matrices σ_1 and σ_2 above are equal to σ^{lo} and σ^{up}, respectively. Both squares in the following diagram commute:

$$\begin{tikzcd}
Y_\sigma \arrow[r, "\Delta"] \arrow[d, "\mathrm{ev}^l_\sigma"'] & Y_{\sigma_1} \otimes Y_{\sigma_2} \arrow[d, "\mathrm{ev}^{l_1}_{\sigma_1} \otimes \mathrm{ev}^{l_2}_{\sigma_2}"] \\
U^l_\sigma \arrow[r, equal] & U^{l_1}_{\sigma_1} \otimes U^{l_2}_{\sigma_2} \\
W_\pi \arrow[r, "\Delta_{l_1,l_2}"] \arrow[u, "\mu"] & W_{\pi_1} \otimes W_{\pi_2} \arrow[u, "\mu_1 \otimes \mu_2"']
\end{tikzcd}$$

Indeed, the top square commutes by the definition of the evaluation homomorphisms from (3-2) while the bottom square commutes by Lemma 8.1. Hence, under our isomorphism between principal W-algebras and truncated shifted Yangians, $\Delta_{l_1,l_2} : W_\pi \to W_{\pi_1} \otimes W_{\pi_2}$ corresponds exactly to the map $Y^l_\sigma \to Y^{l_1}_{\sigma_1} \otimes Y^{l_2}_{\sigma_2}$ induced by the comultiplication $\Delta : Y_\sigma \to Y_{\sigma_1} \otimes Y_{\sigma_2}$.

Instead, if $l_1 = l - 1$, $l_2 = 1$ and the rightmost column of π consists of a single box, the map $\Delta_{l-1,1} : W_\pi \to W_{\pi_1} \otimes U(\mathfrak{gl}_1)$ corresponds exactly to the map $Y^l_\sigma \to Y^{l-1}_{\sigma_+} \otimes U(\mathfrak{gl}_1)$ induced by $\Delta_+ : Y_\sigma \to Y_{\sigma_+} \otimes U(\mathfrak{gl}_1)$. Similarly, if $l_1 = 1$, $l_2 = l - 1$ and the leftmost column of π consists of a single box, $\Delta_{1,l-1} : W_\pi \to U(\mathfrak{gl}_1) \otimes W_{\pi_2}$ corresponds exactly to the map $Y^l_\sigma \to U(\mathfrak{gl}_1) \otimes Y^{l-1}_{\sigma_-}$ induced by $\Delta_- : Y_\sigma \to U(\mathfrak{gl}_1) \otimes Y_{\sigma_-}$.

Using (8-1), we can make sense of tensor products: if we are given W_{π_c}-modules V_c for each $c = 1, \dots, d$, then we obtain a well-defined W_π-module

$$V_1 \otimes \cdots \otimes V_d := \Delta^*_{l_1,\dots,l_d}(V_1 \boxtimes \cdots \boxtimes V_d), \tag{8-2}$$

i.e., we take the pull-back of their outer tensor product (viewed as a module via the usual sign convention).

Now specialize to the situation that $d = l$ and $l_1 = \cdots = l_d = 1$. Then each pyramid π_c is a single column of height 1 or 2. In the former case, $W_{\pi_c} = U(\mathfrak{gl}_1)$, and in the latter, $W_{\pi_c} = U(\mathfrak{gl}_{1|1})$. So we have that $W_{\pi_1} \otimes \cdots \otimes W_{\pi_l} = U^l_\sigma$, and the map $\Delta_{1,\dots,1}$ coincides with the Miura transform μ.

Given $A \in \mathrm{Tab}_\pi$, let $A_c \in \mathrm{Tab}_{\pi_c}$ be its c-th column and $\bar{L}(A_c)$ be the corresponding irreducible W_{π_c}-module. Let us decode this notation a little. If $W_{\pi_c} = U(\mathfrak{gl}_1)$, then A_c has just a single entry b and $\bar{L}(A_c)$ is the one-dimensional module with an even basis vector v_+ such that $e_{1,1}v_+ = (-1)^{|2|}bv_+$. If $W_{\pi_c} = U(\mathfrak{gl}_{1|1})$, then A_c has two entries, a in the first row and b in the second row, and $\bar{L}(A_c)$ is one- or two-dimensional according to whether $a = b$; in both cases $\bar{L}(A_c)$ is generated by an even vector v_+ such that $e_{1,1}v_+ = (-1)^{|1|}av_+$, $e_{2,2}v_+ = (-1)^{|2|}bv_+$ and $e_{1,2}v_+ = 0$. Let

$$\bar{V}(A) := \bar{L}(A_1) \otimes \cdots \otimes \bar{L}(A_l). \tag{8-3}$$

Note that $\dim \bar{V}(A) = 2^{k-h}$, where h is the number of $c = 1, \dots, l$ such that A_c has two equal entries.

Lemma 8.3. *For any $A \in \mathrm{Tab}_\pi$, there is a nonzero homomorphism*

$$\bar{M}(A) \to \bar{V}(A)$$

sending the cyclic vector $1 \otimes 1_A \in \bar{M}(A)$ to $v_+ \otimes \cdots \otimes v_+ \in \bar{V}(A)$. In particular, $\bar{V}(A)$ contains a subquotient isomorphic to $\bar{L}(A)$.

Proof. Suppose that $A = {}^{a_1 \cdots a_k}_{b_1 \cdots b_l}$. By the definition of $\bar{M}(A)$ as an induced module, it suffices to show that $v := v_+ \otimes \cdots \otimes v_+ \in \bar{V}(A)$ is annihilated by all $e^{(r)}$ for $r > s_{1,2}$ and that $d_1^{(r)}v = e_r(a_1, \dots, a_k)v$ and $d_2^{(r)}v = e_r(b_1, \dots, b_l)v$ for all $r > 0$. For this, we calculate from the explicit formulae for the invariants $d_1^{(r)}$, $d_2^{(r)}$ and $e^{(r)}$ given by (4-10) and (4-19)–(4-21), remembering that their action on v is defined via the Miura transform $\mu = \Delta_{1,\dots,1}$. It is convenient in this proof to set

$$\bar{e}_{i,j}^{[c]} := \begin{cases} (-1)^{|i|} 1^{\otimes(c-1)} \otimes e_{i,j} \otimes 1^{\otimes(l-c)} & \text{if } q_c = 2, \\ (-1)^{|2|} 1^{\otimes(c-1)} \otimes e_{1,1} \otimes 1^{\otimes(l-c)} & \text{if } q_c = 1 \text{ and } i = j = 2, \\ 0 & \text{otherwise} \end{cases}$$

for any $1 \le i, j \le 2$ and $1 \le c \le l$, where q_c is the number of boxes in the c-th column of π. First we have that

$$d_1^{(r)}v = \sum_{1 \le c_1, \dots, c_r \le l} \; \sum_{1 \le h_1, \dots, h_{r-1} \le 2} \bar{e}_{1,h_1}^{[c_1]} \bar{e}_{h_1,h_2}^{[c_2]} \cdots \bar{e}_{h_{r-1},1}^{[c_r]} v$$

summing only over terms with $c_1 < \cdots < c_r$. The elements on the right commute (up to sign) because the c_i are all distinct, so any $\bar{e}_{1,2}^{[c_i]}$ produces 0 as $e_{1,2}v_+ = 0$. Thus, the summation reduces just to

$$\sum_{1 \le c_1 < \cdots < c_r \le l} \bar{e}_{1,1}^{[c_1]} \cdots \bar{e}_{1,1}^{[c_r]} v = e_r(a_1, \dots, a_k)v$$

as required. Next we have that

$$d_2^{(r)}v = \sum_{1 \le c_1, \dots, c_r \le l} \; \sum_{1 \le h_1, \dots, h_{r-1} \le 2} (-1)^{\#\{i=1,\dots,r-1 \mid \mathrm{row}(h_i)=1\}} \bar{e}_{2,h_1}^{[c_1]} \bar{e}_{h_1,h_2}^{[c_2]} \cdots \bar{e}_{h_{r-1},2}^{[c_r]} v$$

summing only over terms with $c_i \geq c_{i+1}$ if $\mathrm{row}(h_i) = 1$ and $c_i < c_{i+1}$ if $\mathrm{row}(h_i) = 2$. Here, if any monomial $\bar{e}_{1,2}^{[c_i]}$ appears, the rightmost such can be commuted to the end when it acts as 0. Thus, the summation reduces just to the terms with $h_1 = \cdots = h_{r-1} = 2$, and again we get the required elementary symmetric function $e_r(b_1, \ldots, b_l)$. Finally, we have that

$$e^{(r)}v = \sum_{1 \leq c_1, \ldots, c_r \leq l} \sum_{1 \leq h_1, \ldots, h_{r-1} \leq 2} (-1)^{\#\{i=1,\ldots,r-1 \mid \mathrm{row}(h_i)=1\}} \bar{e}_{1,h_1}^{[c_1]} \bar{e}_{h_1,h_2}^{[c_2]} \cdots \bar{e}_{h_{r-1},2}^{[c_r]} v$$

summing only over terms with $c_i \geq c_{i+1}$ if $\mathrm{row}(h_i) = 1$ and $c_i < c_{i+1}$ if $\mathrm{row}(h_i) = 2$. As before, this is 0 because the rightmost $\bar{e}_{1,2}^{[c_i]}$ can be commuted to the end. □

Theorem 8.4. *Take any $A = {}^{a_1 \cdots a_k}_{b_1 \cdots b_l} \in \mathrm{Tab}_\pi$, and let $h \geq 0$ be maximal such that distinct $1 \leq i_1, \ldots, i_h \leq k$ and $1 \leq j_1, \ldots, j_h \leq l$ with $a_{i_1} = b_{j_1}, \ldots, a_{i_h} = b_{j_h}$ exist. Choose $B \sim A$ so that B has h columns of height 2 containing equal entries. Then*

$$\bar{L}(A) \cong \bar{V}(B). \tag{8-4}$$

In particular, $\dim \bar{L}(A) = 2^{k-h}$.

Proof. By Lemma 8.3, $\bar{V}(B)$ has a subquotient isomorphic to $\bar{L}(B) \cong \bar{L}(A)$, which implies that $\dim \bar{L}(A) \leq \dim \bar{V}(B) = 2^{k-h}$. Also by the linear independence established in the partial proof of Theorem 7.3 given in Section 7, we know that $\dim \bar{L}(A) \geq 2^{k-h}$. □

Theorem 8.4 also establishes the fact about dimension needed to complete the proof of Theorem 7.3 in Section 7.

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brownj3@gonzaga.edu — *Department of Mathematics, Computer Science, and Statistics, State University of New York College at Oneonta, Oneonta, NY 13820, United States*

brundan@uoregon.edu — *Department of Mathematics, University of Oregon, Eugene, OR 97403, United States*

s.m.goodwin@bham.ac.uk — *School of Mathematics, University of Birmingham, Birmingham B15 2TT, United Kingdom*

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Kernels for products of L-functions

Nikolaos Diamantis and Cormac O'Sullivan

The Rankin–Cohen bracket of two Eisenstein series provides a kernel yielding products of the periods of Hecke eigenforms at critical values. Extending this idea leads to a new type of Eisenstein series built with a double sum. We develop the properties of these series and their nonholomorphic analogs and show their connection to values of L-functions outside the critical strip.

1. Introduction

Rankin [1952] introduced the fruitful idea of expressing the product of two critical values of the L-function of a weight-k Hecke eigenform f for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ in terms of the Petersson scalar product of f and a product of Eisenstein series:

$$\langle E_{k_1} E_{k_2}, f \rangle = (-1)^{k_1/2} 2^{3-k} \frac{k_1 k_2}{B_{k_1} B_{k_2}} L^*(f, 1) L^*(f, k_2) \tag{1-1}$$

for $k = k_1 + k_2$, the Bernoulli numbers B_j and the completed, entire L-function of f,

$$L^*(f, s) := \frac{\Gamma(s)}{(2\pi)^s} \sum_{m=1}^{\infty} \frac{a_f(m)}{m^s} = \int_0^\infty f(iy) y^{s-1}\, dy.$$

Zagier [1977, p. 149] extended (1-1) to get

$$\langle [E_{k_1}, E_{k_2}]_n, f \rangle = (-1)^{k_1/2} (2\pi i)^n 2^{3-k} \binom{k-2}{n} \frac{k_1 k_2}{B_{k_1} B_{k_2}} L^*(f, n+1) L^*(f, n+k_2), \tag{1-2}$$

where $k = k_1 + k_2 + 2n$ and $[g_1, g_2]_n$ stands for the *Rankin–Cohen bracket of index n* given by

$$[g_1, g_2]_n := \sum_{r=0}^{n} (-1)^r \binom{k_1+n-1}{n-r} \binom{k_2+n-1}{r} g_1^{(r)} g_2^{(n-r)}. \tag{1-3}$$

The *periods* of f in the critical strip are the numbers

$$L^*(f, 1), L^*(f, 2), \dots, L^*(f, k-1). \tag{1-4}$$

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Zagier [1977, §5] and Kohnen and Zagier [1984] proved important results of the Eichler–Shimura–Manin theory on the algebraicity of these critical values using (1-2). We describe this in more depth in Sections 2C and 8A.

On the face of it, the techniques of [Zagier 1977], employing (1-2), apply only to critical values; an extension to noncritical values, $L^*(f, j)$ for integers $j \leqslant 0$ or $j \geq k$, would seem to require Rankin–Cohen brackets of negative index n or holomorphic Eisenstein series of negative weight, neither of which are defined. Analyzing the structure of the Rankin–Cohen bracket of two Eisenstein series in Section 6 reveals a natural construction, which we call a *double Eisenstein series*:[1]

$$\sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1}\neq\Gamma_\infty}} (c_{\gamma\delta^{-1}})^l j(\gamma, z)^{-k_1} j(\delta, z)^{-k_2}, \tag{1-5}$$

where, for $\gamma \in \Gamma$, we write

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \quad \text{and} \quad j(\gamma, z) := c_\gamma z + d_\gamma.$$

By comparison, the usual holomorphic Eisenstein series is

$$E_k(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k}. \tag{1-6}$$

The double Eisenstein series (1-5) converges to a weight-$(k_1 + k_2)$ cusp form when $l < k_1 - 2, k_2 - 2$. For negative integers l, it behaves as a Rankin–Cohen bracket of negative index; see Proposition 2.4. This allows us to further generalize (1-1) and (1-2), and in Section 8, we characterize the field containing an arbitrary value of an L-function in terms of double Eisenstein series and their Fourier coefficients. In the interesting paper [Cohen et al. 1997], Rankin–Cohen brackets are linked to operations on automorphic pseudodifferential operators and may also be reinterpreted in this framework allowing for more general indices.

An extension of Zagier's kernel formula (1-2) in the nonholomorphic direction is given in Section 9C. There we show that the holomorphic double Eisenstein series have nonholomorphic counterparts:

$$\sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1}\neq\Gamma_\infty}} |c_{\gamma\delta^{-1}}|^{-s-s'} \operatorname{Im}(\gamma z)^s \operatorname{Im}(\delta z)^{s'}. \tag{1-7}$$

These weight-0 functions possess analytic continuations and functional equations resembling those for the classical nonholomorphic Eisenstein series. As kernels, they produce products of L-functions for *Maass cusp forms*; see Theorem 2.9. The main motivation for this construction was its potential use in the rapidly developing

[1] In the context of multiple zeta functions, the authors in [Gangl et al. 2006] give a different definition of "double Eisenstein series". See also [Deninger 1995], for example, for distinct "double Eisenstein–Kronecker series".

study of periods of Maass forms [Bruggeman et al. 2013; Lewis and Zagier 2001; Manin 2010; Mühlenbruch 2006]. In developing the properties of (1-7), we require a certain kernel $\mathcal{K}(z; s, s')$ as defined in (9-1). It is interesting to note that Diaconu and Goldfeld [2007] needed exactly the same series for their results on second moments of $L^*(f, s)$; see Section 9A.

2. Statement of main results

2A. *Preliminaries.* Our notation is as in [Diamantis and O'Sullivan 2010]. In all sections but two, Γ is the modular group $\mathrm{SL}(2, \mathbb{Z})$ acting on the upper half-plane $\mathbb{H}$. The definitions we give for double Eisenstein series extend easily to more general groups, so in Section 4, we prove their basic properties for Γ an arbitrary Fuchsian group of the first kind, and in Section 10, we see how some of our main results are valid in this general context.

Let $S_k(\Gamma)$ be the $\mathbb{C}$-vector space of holomorphic, weight-k cusp forms for Γ and $M_k(\Gamma)$ the space of modular forms. These spaces are acted on by the Hecke operators T_m; see (3-6). Let $\mathcal{B}_k$ be the unique basis of S_k consisting of Hecke eigenforms normalized to have first Fourier coefficient 1. We assume throughout this paper that $f \in \mathcal{B}_k$. Since $\langle T_m f, f\rangle = \langle f, T_m f\rangle$, it follows that all the Fourier coefficients of f are real, and hence, $\overline{L^*(f, s)} = L^*(f, \bar{s})$. Also, recall the functional equation

$$L^*(f, k-s) = (-1)^{k/2} L^*(f, s). \tag{2-1}$$

We summarize some standard properties of the nonholomorphic Eisenstein series; see for example [Iwaniec 2002, Chapters 3 and 6]. Throughout this paper, we use the variables $z = x + iy \in \mathbb{H}$ and $s = \sigma + it \in \mathbb{C}$.

Definition 2.1. For $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the weight-0, *nonholomorphic Eisenstein series* is

$$E(z, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s = \frac{y^s}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} |cz+d|^{-2s}. \tag{2-2}$$

Let $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$. Then $E(z, s)$ has a Fourier expansion [Iwaniec 2002, Theorem 3.4], which we may write in the form

$$E(z, s) = y^s + \frac{\theta(1-s)}{\theta(s)} y^{1-s} + \sum_{m \neq 0} \phi(m, s)|m|^{-1/2} W_s(mz), \tag{2-3}$$

where $W_s(mz) = 2(|m|y)^{1/2} K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}$ is the Whittaker function for $z \in \mathbb{H}$ and also $\theta(s)\phi(m, s) = \sigma_{2s-1}(|m|)|m|^{1/2-s}$. As usual, $\sigma_s(m) := \sum_{d|m} d^s$ is the divisor function.

For the weight-k ($k \in 2\mathbb{Z}$) nonholomorphic Eisenstein series, generalizing (2-2), set

$$E_k(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^s \left(\frac{j(\gamma,z)}{|j(\gamma,z)|}\right)^{-k}. \tag{2-4}$$

Then (2-4) converges to an analytic function of $s \in \mathbb{C}$ and a smooth function of $z \in \mathbb{H}$ for $\operatorname{Re}(s) > 1$. Also $y^{-k/2}E_k(z,s)$ has weight k in z. Define the *completed nonholomorphic Eisenstein series* as

$$E_k^*(z,s) := \theta_k(s)E_k(z,s) \quad \text{for } \theta_k(s) := \pi^{-s}\Gamma(s+|k|/2)\zeta(2s). \tag{2-5}$$

With (2-3), we see that $E(z,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$. The same is true of $E_k(z,s)$; see [Diamantis and O'Sullivan 2010, §2.1] for example. We have the functional equations

$$\theta(s/2) = \theta((1-s)/2), \tag{2-6}$$

$$E_k^*(z,s) = E_k^*(z,1-s). \tag{2-7}$$

2B. ***Holomorphic double Eisenstein series.*** Define the subgroup

$$B := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\} \subset \mathrm{SL}(2,\mathbb{Z}). \tag{2-8}$$

Then Γ_∞, the subgroup of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ fixing ∞, is $B \cup -B$. For $\gamma \in \Gamma_\infty \backslash \Gamma$, the quantities c_γ, d_γ and $j(\gamma,z)$ are only defined up to sign (though even powers are well-defined). For $\gamma \in B\backslash\Gamma$, there is no ambiguity in the signs of c_γ, d_γ and $j(\gamma,z)$.

Definition 2.2. Let $z \in \mathbb{H}$ and $w \in \mathbb{C}$. For integers $k_1, k_2 \geq 3$, we define the *double Eisenstein series*

$$\boldsymbol{E}_{k_1,k_2}(z,w) := \sum_{\substack{\gamma,\delta\in B\backslash\Gamma \\ c_{\gamma\delta^{-1}}>0}} (c_{\gamma\delta^{-1}})^{w-1} j(\gamma,z)^{-k_1} j(\delta,z)^{-k_2}. \tag{2-9}$$

This series is well-defined and converges to a holomorphic function of z that is a weight-$(k = k_1 + k_2)$ cusp form for $\operatorname{Re}(w) < k_1 - 1, k_2 - 1$, as we see in Proposition 4.2. It vanishes identically when k_1 and k_2 have different parity.

Let k be even. To get the most general kernel, with $s \in \mathbb{C}$ set

$$\boldsymbol{E}_{s,k-s}(z,w) := \sum_{\substack{\gamma,\delta\in B\backslash\Gamma \\ c_{\gamma\delta^{-1}}>0}} (c_{\gamma\delta^{-1}})^{w-1} \left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k}. \tag{2-10}$$

In the usual convention, for $\rho \in \mathbb{C}$ with $\rho \neq 0$, write

$$\rho = |\rho|e^{i\arg(\rho)} \quad \text{for } -\pi < \arg(\rho) \leqslant \pi$$

and

$$\rho^s = |\rho|^s e^{i\arg(\rho)s} \quad \text{for } s \in \mathbb{C}. \tag{2-11}$$

Note that

$$c_{\gamma\delta^{-1}} = \begin{vmatrix} c_\gamma & d_\gamma \\ c_\delta & d_\delta \end{vmatrix} > 0 \implies \frac{j(\gamma, z)}{j(\delta, z)} \in \mathbb{H} \quad \text{for } z \in \mathbb{H},$$

and so $(j(\gamma, z)/j(\delta, z))^{-s}$ in (2-10) is well-defined and a holomorphic function of $s \in \mathbb{C}$ and $z \in \mathbb{H}$. Proposition 4.2 shows that $\boldsymbol{E}_{s,k-s}(z, w)$ converges absolutely and uniformly on compact sets for which $2 < \sigma < k-2$ and $\mathrm{Re}(w) < \sigma - 1, k-1-\sigma$.

Define the *completed double Eisenstein series* as

$$\boldsymbol{E}^*_{s,k-s}(z, w) \tag{2-12}$$
$$:= \left[\frac{e^{s i\pi/2}\Gamma(s)\Gamma(k-s)\Gamma(k-w)\zeta(1-w+s)\zeta(1-w+k-s)}{2^{3-w}\pi^{k+1-w}\Gamma(k-1)}\right]\boldsymbol{E}_{s,k-s}(z, w).$$

Theorem 2.3. *Let $k \geq 6$ be even. The series $\boldsymbol{E}^*_{s,k-s}(z, w)$ has an analytic continuation to all $s, w \in \mathbb{C}$ and as a function of z is always in $S_k(\Gamma)$. For any f in $\mathcal{B}_k$, we have*

$$\langle \boldsymbol{E}^*_{s,k-s}(\,\cdot\,, w), f\rangle = L^*(f, s)L^*(f, w). \tag{2-13}$$

It follows directly from (2-13) and (2-1) that $\boldsymbol{E}^*_{s,k-s}(z, w)$ satisfies eight functional equations generated by

$$\boldsymbol{E}^*_{s,k-s}(z, w) = \boldsymbol{E}^*_{w,k-w}(z, s), \tag{2-14}$$

$$\boldsymbol{E}^*_{s,k-s}(z, w) = (-1)^{k/2}\boldsymbol{E}^*_{k-s,s}(z, w). \tag{2-15}$$

The next result shows how $\boldsymbol{E}^*_{s,k-s}$ is a generalization of the Rankin–Cohen bracket $[E_{k_1}, E_{k_2}]_n$.

Proposition 2.4. *For $n \in \mathbb{Z}_{\geq 1}$ and even $k_1, k_2 \geq 4$,*

$$n!\,[E_{k_1}, E_{k_2}]_n = \frac{2(-1)^{k_1/2}\pi^k\Gamma(k-1)}{(2\pi i)^n\zeta(k_1)\zeta(k_2)\Gamma(k_1)\Gamma(k_2)\Gamma(k-n-1)}\boldsymbol{E}^*_{k_1+n,k_2+n}(z, n+1).$$

Another way to understand these double Eisenstein series is through their connections to nonholomorphic Eisenstein series. Any smooth function transforming with weight k and with polynomial growth as $y \to \infty$ may be projected into S_k with respect to the Petersson scalar product. See [Diamantis and O'Sullivan 2010, §3.2] and the contained references. Denote this holomorphic projection by π_{hol}.

Proposition 2.5. *Let $k = k_1 + k_2 \geq 6$ for even $k_1, k_2 \geq 0$. Then for all $s, w \in \mathbb{C}$*

$$\boldsymbol{E}^*_{s,k-s}(z, w) = \pi_{\mathrm{hol}}\big[(-1)^{k_2/2}y^{-k/2}E^*_{k_1}(z, u)E^*_{k_2}(z, v)/(2\pi^{k/2})\big],$$

where

$$u = (s+w-k+1)/2 \quad \textit{and} \quad v = (-s+w+1)/2. \tag{2-16}$$

2C. *Values of L-functions.* For $f \in \mathcal{B}_k$, let K_f be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of f. We will recall Zagier's proof of the next result in Section 8A.

Theorem 2.6 (Manin's periods theorem). *For each $f \in \mathcal{B}_k$ there exist real numbers $\omega_+(f), \omega_-(f)$ such that*

$$L^*(f,s)/\omega_+(f),\, L^*(f,w)/\omega_-(f) \in K_f$$

for all s and w with $1 \leqslant s, w \leqslant k-1$ and s even and w odd.

Let $m \in \mathbb{Z}$ satisfy $m \leqslant 0$ or $m \geq k$. Then for these values outside the critical strip we have, according to [Kontsevich and Zagier 2001, §3.4] and the references therein,

$$L^*(f,m) \in \mathcal{P}[1/\pi],$$

where $\mathcal{P}$ is the ring of periods: complex numbers that may be expressed as an integral of an algebraic function over an algebraic domain. In contrast to the periods (1-4), we do not have much more precise information about the algebraic properties of the values $L^*(f,m)$. A special case of a theorem by Koblic [1975] shows, for example, that

$$L^*(f,m) \notin \mathbb{Z}\cdot L^*(f,1) + \mathbb{Z}\cdot L^*(f,2) + \cdots + \mathbb{Z}\cdot L^*(f,k-1).$$

Let $K(\boldsymbol{E}^*_{s,k-s}(\cdot\,,w))$ be the field obtained by adjoining to $\mathbb{Q}$ the Fourier coefficients of $\boldsymbol{E}^*_{s,k-s}(\cdot\,,w)$, and let $\omega_+(f)$ and $\omega_-(f)$ be as given in Theorem 2.6. Then we have:

Theorem 2.7. *For all $f \in \mathcal{B}_k$ and $s \in \mathbb{C}$,*

$$L^*(f,s)/\omega_+(f) \in K(\boldsymbol{E}^*_{s,k-s}(\cdot\,,k-1))K_f,$$
$$L^*(f,s)/\omega_-(f) \in K(\boldsymbol{E}^*_{k-2,2}(\cdot\,,s))K_f.$$

The above theorem gives the link between Fourier coefficients of double Eisenstein series and arbitrary values of L-functions. We hope that this interesting connection will help shed light on $L^*(f,s)$ for s outside the set $\{1,2,\ldots,k-1\}$. See the further discussion in Section 8B for the case when $s \in \mathbb{Z}$.

In Section 8C, we also prove results analogous to Theorem 2.7 for the completed L-function of f twisted by $e^{2\pi imp/q}$ for $p/q \in \mathbb{Q}$:

$$L^*(f,s;p/q) := \frac{\Gamma(s)}{(2\pi)^s}\sum_{m=1}^{\infty}\frac{a_f(m)e^{2\pi imp/q}}{m^s} = \int_0^\infty f(iy+p/q)y^{s-1}\,dy. \quad (2\text{-}17)$$

2D. *Nonholomorphic double Eisenstein series.*

Definition 2.8. For $z \in \mathbb{H}$ and $w, s, s' \in \mathbb{C}$, we define the *nonholomorphic double Eisenstein series* as

$$\mathscr{E}(z, w; s, s') := \sum_{\substack{\gamma,\delta \in \Gamma_\infty\backslash\Gamma \\ \gamma\delta^{-1} \neq \Gamma_\infty}} \frac{\operatorname{Im}(\gamma z)^s \operatorname{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{2-18}$$

A simple comparison with (2-2) shows it is absolutely and uniformly convergent for $\operatorname{Re}(s), \operatorname{Re}(s') > 1$ and $\operatorname{Re}(w) > 0$. (This domain of convergence is improved in Proposition 4.3.) The most symmetric form of (2-18) is when $w = s + s'$. Define

$$\mathscr{E}^*(z; s, s') := 4\pi^{-s-s'}\Gamma(s)\Gamma(s')\zeta(3s+s')\zeta(s+3s')\mathscr{E}(z, s+s'; s, s') \\ + 2\theta(s)\theta(s')E(z, s+s'). \tag{2-19}$$

Theorem 2.9. *The completed double Eisenstein series* $\mathscr{E}^*(z; s, s')$ *has a meromorphic continuation to all* $s, s' \in \mathbb{C}$ *and satisfies the functional equations*

$$\mathscr{E}^*(z; s, s') = \mathscr{E}^*(z; s', s),$$
$$\mathscr{E}^*(z; s, s') = \mathscr{E}^*(z; 1-s, 1-s').$$

For any even Maass Hecke eigenform u_j,

$$\langle \mathscr{E}^*(z; s, s'), u_j \rangle = L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2).$$

3. Further background results and notation

We need to introduce two more families of modular forms.

Definition 3.1. For $z \in \mathbb{H}$, $k \geq 4$ in $2\mathbb{Z}$ and $m \in \mathbb{Z}_{\geq 0}$, the holomorphic *Poincaré series* is

$$P_k(z; m) := \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \frac{e^{2\pi i m \gamma z}}{j(\gamma, z)^k} = \frac{1}{2} \sum_{\gamma \in B\backslash\Gamma} \frac{e^{2\pi i m \gamma z}}{j(\gamma, z)^k}. \tag{3-1}$$

For $m \geq 1$, the series $P_k(z; m)$ span $S_k(\Gamma)$. The Eisenstein series $E_k(z) = P_k(z; 0)$ is not a cusp form but is in the space $M_k(\Gamma)$. The second family of modular forms is based on a series due to Cohen [1981].

Definition 3.2. The *generalized Cohen kernel* is given by

$$\mathscr{C}_k(z, s; p/q) := \frac{1}{2} \sum_{\gamma \in \Gamma} (\gamma z + p/q)^{-s} j(\gamma, z)^{-k} \tag{3-2}$$

for $p/q \in \mathbb{Q}$ and $s \in \mathbb{C}$ with $1 < \operatorname{Re}(s) < k - 1$.

In [Diamantis and O'Sullivan 2010, §5], we studied $\mathscr{C}_k(z,s;\,p/q)$ (the factor $1/2$ is included to keep the notation consistent with that article, where $\Gamma=\mathrm{PSL}(2,\mathbb{Z})$). We showed that, for each $s\in\mathbb{C}$ with $1<\mathrm{Re}(s)<k-1$, $\mathscr{C}_k(z,s;\,p/q)$ converges to an element of $S_k(\Gamma)$ with a meromorphic continuation to all $s\in\mathbb{C}$. From Proposition 5.4 of the same work, we have

$$\langle \mathscr{C}_k(\,\cdot\,,s;\,p/q),\,f\rangle = 2^{2-k}\pi e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)}L^*(f,k-s;\,p/q), \tag{3-3}$$

which is a generalization of Cohen's lemma in [Kohnen and Zagier 1984, §1.2]. For simplicity, we write $\mathscr{C}_k(z,s)$ for $\mathscr{C}_k(z,s;\,0)$. The twisted L-functions satisfy

$$\overline{L^*(f,s;\,p/q)} = L^*(f,\bar{s};\,-p/q), \tag{3-4}$$

$$q^s L^*(f,s;\,p/q) = (-1)^{k/2}q^{k-s}L^*(f,k-s;\,-p'/q) \tag{3-5}$$

for $pp'\equiv 1 \bmod q$ as in [Kowalski et al. 2002, Appendix A.3].

Define $\mathcal{M}_n := \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \,\middle|\, a,b,c,d\in\mathbb{Z},\ ad-bc=n\right\}$. Thus, $\mathcal{M}_1=\Gamma$. For $k\in\mathbb{Z}$ and $g:\mathbb{H}\to\mathbb{C}$, set

$$(g|_k\gamma)(z) := \det(\gamma)^{k/2}g(\gamma z)j(\gamma,z)^{-k}$$

for all $\gamma\in\mathcal{M}_n$. The weight-k Hecke operator T_n acts on $g\in M_k$ by

$$(T_n g)(z) := n^{k/2-1}\sum_{\gamma\in\Gamma\backslash\mathcal{M}_n}(g|_k\gamma)(z) = n^{k-1}\sum_{\substack{ad=n\\ a,d>0}} d^{-k}\sum_{0\leqslant b<d} g\left(\frac{az+b}{d}\right). \tag{3-6}$$

4. Basic properties of double Eisenstein series

We work more generally in this section with Γ a Fuchsian group of the first kind containing at least one cusp. Set

$$\varepsilon_\Gamma := \#\{\Gamma\cap\{-I\}\}. \tag{4-1}$$

Label the finite number of inequivalent cusps $\mathfrak{a}$, $\mathfrak{b}$, etc., and let $\Gamma_\mathfrak{a}$ be the subgroup of Γ fixing $\mathfrak{a}$. There exists a corresponding scaling matrix $\sigma_\mathfrak{a}\in\mathrm{SL}(2,\mathbb{R})$ such that $\sigma_\mathfrak{a}\infty=\mathfrak{a}$ and

$$\sigma_\mathfrak{a}{}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \begin{cases} B\cup -B & \text{if } -I\in\Gamma\ (\varepsilon_\Gamma=1),\\ B & \text{if } -I\notin\Gamma\ (\varepsilon_\Gamma=0).\end{cases}$$

Also set $\Gamma_\mathfrak{a}^* := \sigma_\mathfrak{a}B\sigma_\mathfrak{a}{}^{-1}$.

We recall some facts about $E_{k,\mathfrak{a}}(z,s)$, the nonholomorphic Eisenstein series associated to the cusp $\mathfrak{a}$; see for example [Iwaniec 2002, Chapter 3; Diamantis and O'Sullivan 2010, §2.1]. It is defined as

$$E_{k,\mathfrak{a}}(z,s) := \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma}\mathrm{Im}(\sigma_\mathfrak{a}{}^{-1}\gamma z)^s\left(\frac{j(\sigma_\mathfrak{a}{}^{-1}\gamma,z)}{|j(\sigma_\mathfrak{a}{}^{-1}\gamma,z)|}\right)^{-k}$$

and absolutely convergent for $\mathrm{Re}(s) > 1$. Put $E^*_{k,\mathfrak{a}}(z,s) := \theta_k(s)E_{k,\mathfrak{a}}(z,s)$ as in (2-5). Then we have the expansion

$$E^*_{0,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{a}\mathfrak{b}}\theta(s)y^s + \theta(1-s)Y_{\mathfrak{a}\mathfrak{b}}(s)y^{1-s} + \sum_{l\neq 0} Y_{\mathfrak{a}\mathfrak{b}}(l,s)W_s(lz), \tag{4-2}$$

and

$$E^*_{k,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{a}\mathfrak{b}}\theta_k(s)y^s + \theta_k(1-s)Y_{\mathfrak{a}\mathfrak{b}}(s)y^{1-s} + O(e^{-2\pi y}) \tag{4-3}$$

as $y \to \infty$ for all $k \in 2\mathbb{Z}$. Also, its functional equation is

$$E^*_{k,\mathfrak{a}}(z,1-s) = \sum_{\mathfrak{b}} Y_{\mathfrak{a}\mathfrak{b}}(1-s)E^*_{k,\mathfrak{b}}(z,s). \tag{4-4}$$

We gave the coefficients $Y_{\mathfrak{a}\mathfrak{b}}(s)$ and $Y_{\mathfrak{a}\mathfrak{b}}(l,s)$ explicitly in the case of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ following (2-3), and in general, they involve series containing Kloosterman sums; see [Iwaniec 2002, (3.21) and (3.22)].

For the natural generalization of (2-10), we define the *double Eisenstein series associated to the cusp* $\mathfrak{a}$ as

$$\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w) := \sum_{\substack{\gamma,\delta\in\Gamma^*_{\mathfrak{a}}\backslash\Gamma \\ c_{\sigma_{\mathfrak{a}}^{-1}\gamma\delta^{-1}\sigma_{\mathfrak{a}}}>0}} (c_{\sigma_{\mathfrak{a}}^{-1}\gamma\delta^{-1}\sigma_{\mathfrak{a}}})^{w-1}\left(\frac{j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)}{j(\sigma_{\mathfrak{a}}^{-1}\delta,z)}\right)^{-s} j(\sigma_{\mathfrak{a}}^{-1}\delta,z)^{-k} \tag{4-5}$$

so that

$$\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_{\mathfrak{a}}z,w) = j(\sigma_{\mathfrak{a}},z)^k \sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} (c_{\gamma\delta^{-1}})^{w-1}\left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k} \tag{4-6}$$

for $\Gamma' = \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$, which is also a Fuchsian group of the first kind. To establish an initial domain of absolute convergence for (4-6), we consider

$$\sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} \left|(c_{\gamma\delta^{-1}})^{w-1}\left(\frac{j(\gamma,z)}{j(\delta,z)}\right)^{-s} j(\delta,z)^{-k}\right|. \tag{4-7}$$

Recalling (2-11), we see that

$$|\rho^s| = |\rho|^\sigma e^{-t\arg(\rho)} \ll_t |\rho|^\sigma \quad \text{for } s = \sigma + it \in \mathbb{C}.$$

Therefore, with $r = \mathrm{Re}(w)$ and $\mathrm{Im}(\gamma z) = y|j(\gamma,z)|^{-2}$, we deduce that (4-7) is bounded by a constant depending on s times

$$y^{-k/2} \sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\Gamma' \\ \gamma\delta^{-1}\notin\Gamma_\infty}} |c_{\gamma\delta^{-1}}|^{r-1}\,\mathrm{Im}(\gamma z)^{\sigma/2}\,\mathrm{Im}(\delta z)^{(k-\sigma)/2}. \tag{4-8}$$

Lemma 4.1. *There exists a constant $\kappa_\Gamma > 0$ so that for all $\gamma,\delta \in \Gamma$ with $c_{\gamma\delta^{-1}} > 0$*

$$\kappa_\Gamma \leqslant c_{\gamma\delta^{-1}} \leqslant \mathrm{Im}(\gamma z)^{-1/2}\,\mathrm{Im}(\delta z)^{-1/2}.$$

Proof. The existence of κ_Γ is described in [Iwaniec 2002, §2.5 and §2.6; Shimura 1971, Lemma 1.25]. Set $\varepsilon(\gamma, z) := j(\gamma, z)/|j(\gamma, z)| = e^{i\arg(j(\gamma,z))}$. It is easy to verify that, for all $\gamma, \delta \in \Gamma$ and $z \in \mathbb{H}$,

$$\begin{aligned} c_{\gamma\delta^{-1}} &= c_\gamma j(\delta, z) - c_\delta j(\gamma, z) \\ &= \left(\frac{j(\gamma, z) - \overline{j(\gamma, z)}}{2iy}\right) j(\delta, z) - \left(\frac{j(\delta, z) - \overline{j(\delta, z)}}{2iy}\right) j(\gamma, z) \\ &= (\varepsilon(\delta, z)^{-2} - \varepsilon(\gamma, z)^{-2}) j(\gamma, z) j(\delta, z)/(2iy). \end{aligned}$$

Therefore,

$$\begin{aligned} |c_{\gamma\delta^{-1}}| &= \left|\frac{\varepsilon(\gamma, z)}{\varepsilon(\delta, z)} - \frac{\varepsilon(\delta, z)}{\varepsilon(\gamma, z)}\right| \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2}/2 \\ &= \left|\operatorname{Im}\left(\frac{\varepsilon(\gamma, z)}{\varepsilon(\delta, z)}\right)\right| \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2} \\ &\leqslant \operatorname{Im}(\gamma z)^{-1/2} \operatorname{Im}(\delta z)^{-1/2}. \end{aligned}$$ □

It follows that for $r' = \max(r, 1)$ and $\gamma\delta^{-1} \notin \Gamma_\infty$

$$|c_{\gamma\delta^{-1}}|^{r-1} \ll \operatorname{Im}(\gamma z)^{(1-r')/2} \operatorname{Im}(\delta z)^{(1-r')/2} \tag{4-9}$$

for an implied constant depending on Γ and r. Combining (4-8) and (4-9) shows

$$\frac{\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a} z, w)}{j(\sigma_\mathfrak{a}, z)^k} \ll y^{-k/2} \sum_{\substack{\gamma,\delta \in \Gamma_\infty\backslash\Gamma' \\ \gamma\delta^{-1} \neq \Gamma_\infty}} \operatorname{Im}(\gamma z)^{(1-r'+\sigma)/2} \operatorname{Im}(\delta z)^{(1-r'+k-\sigma)/2} \tag{4-10}$$

$$= y^{-k/2}\left[E_\mathfrak{a}\left(\sigma_\mathfrak{a} z, \frac{1-r'+\sigma}{2}\right) E_\mathfrak{a}\left(\sigma_\mathfrak{a} z, \frac{1-r'+k-\sigma}{2}\right) - E_\mathfrak{a}\left(\sigma_\mathfrak{a} z, 1-r'+\frac{k}{2}\right)\right]$$

on noting that $\operatorname{Im}(\gamma z) = \operatorname{Im}(\delta z)$ for $\gamma\delta^{-1} \in \Gamma_\infty$. Since $E_\mathfrak{a}(z, s)$ is absolutely convergent for $\sigma = \operatorname{Re}(s) > 1$, we have proved that the series $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_\mathfrak{a} z, w)$, defined in (4-6), is absolutely convergent for $2 < \sigma < k-2$ and $\operatorname{Re}(w) < \sigma - 1, k-1-\sigma$. This convergence is uniform for z in compact sets of $\mathbb{H}$ and for s and w in compact sets in $\mathbb{C}$ satisfying the above constraints.

We next verify that $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z, w)$ has weight k in the z variable. We have

$$f(z) \in M_k(\Gamma) \iff f(\sigma_\mathfrak{a} z) j(\sigma_\mathfrak{a}, z)^{-k} \in M_k(\sigma_\mathfrak{a}{}^{-1}\Gamma\sigma_\mathfrak{a}),$$

so with (4-6), we must prove that

$$g(z) := \sum_{\substack{\gamma,\delta \in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \left(\frac{j(\gamma, z)}{j(\delta, z)}\right)^{-s} j(\delta, z)^{-k}$$

is in $M_k(\Gamma')$. For all $\tau \in \Gamma'$,

$$\frac{g(\tau z)}{j(\tau,z)^k} = \sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}}>0}} (c_{\gamma\delta^{-1}})^{w-1}\left(\frac{j(\gamma,\tau z)}{j(\delta,\tau z)}\right)^{-s} j(\delta,\tau z)^{-k} j(\tau,z)^{-k}$$
$$= \sum_{\substack{\gamma,\delta\in B\backslash\Gamma' \\ c_{(\gamma\tau)(\delta\tau)^{-1}}>0}} (c_{(\gamma\tau)(\delta\tau)^{-1}})^{w-1}\left(\frac{j(\gamma\tau,z)}{j(\delta\tau,z)}\right)^{-s} j(\delta\tau,z)^{-k} = g(z)$$

as required.

We finally show that $\boldsymbol{E}_{s,k-s}$ is a cusp form. By (4-10), replacing z by $\sigma_{\mathfrak{a}}{}^{-1}\sigma_{\mathfrak{b}}z$ and using (4-3), for any cusp $\mathfrak{b}$ we obtain

$$\frac{\boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_{\mathfrak{b}}z,w)}{j(\sigma_{\mathfrak{b}},z)^k}$$
$$\ll y^{-k/2}\left[E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}}z,\frac{1-r'+\sigma}{2}\right)E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}}z,\frac{1-r'+k-\sigma}{2}\right)-E_{\mathfrak{a}}\left(\sigma_{\mathfrak{b}}z,1-r'+\frac{k}{2}\right)\right]$$
$$\ll y^{1+\sigma-k}+y^{1-\sigma}+y^{1+r'-k}+y^{r'-k}$$

and approaches 0 as $y\to\infty$. Thus, by a standard argument (see for example [Diamantis and O'Sullivan 2010, Proposition 5.3]), $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$ is a cusp form. Assembling these results, we have shown the following:

Proposition 4.2. *Let $z\in\mathbb{H}$ and $k\in\mathbb{Z}$, and let $s,w\in\mathbb{C}$ satisfy $2<\sigma<k-2$ and* $\mathrm{Re}(w)<\sigma-1, k-1-\sigma$. *For Γ a Fuchsian group of the first kind with cusp $\mathfrak{a}$, the series $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$ is absolutely and uniformly convergent for s, w and z in compact sets satisfying the above constraints. For each such s and w, we have $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)\in S_k(\Gamma)$ as a function of z.*

The same techniques prove the next result for the nonholomorphic double Eisenstein series. Generalizing (2-18), we set

$$\mathscr{E}_{\mathfrak{a}}(\sigma_{\mathfrak{a}}z,w;s,s') := \sum_{\substack{\gamma,\delta\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}{}^{-1}\Gamma\sigma_{\mathfrak{a}} \\ \gamma\delta^{-1}\neq\Gamma_\infty}} \frac{\mathrm{Im}(\gamma z)^s\,\mathrm{Im}(\delta z)^{s'}}{|c_{\gamma\delta^{-1}}|^w}. \tag{4-11}$$

Proposition 4.3. *Let $z\in\mathbb{H}$ and $s,s',w\in\mathbb{C}$ with $\sigma=\mathrm{Re}(s)$ and $\sigma'=\mathrm{Re}(s')$. The series $\mathscr{E}_{\mathfrak{a}}(z,w;s,s')$ defined in (4-11) is absolutely and uniformly convergent for z, w, s and s' in compact sets satisfying*

$$\sigma,\sigma'>1 \quad \textit{and} \quad \mathrm{Re}(w)>2-2\sigma, 2-2\sigma'.$$

Unlike $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z,w)$, the series $\mathscr{E}_{\mathfrak{a}}(z,w;s,s')$ may have polynomial growth at cusps.

5. Further results on double Eisenstein series

5A. ***Analytic continuation: proof of Theorem 2.3.*** Our next task is to prove the meromorphic continuation of $\boldsymbol{E}_{s,k-s}(z,w)$ in s and w. For s and w in the initial domain of convergence, we begin with

$$\begin{aligned}
&\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w)\\
&\qquad=\sum_{u,v=1}^{\infty} u^{w-1-s}v^{w-1-k+s}\sum_{\substack{a,b,c,d\in\mathbb{Z}\\(a,b)=(c,d)=1\\ad-bc>0}}(ad-bc)^{w-1}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k}\\
&\qquad=\sum_{u,v=1}^{\infty}\sum_{\substack{a,b,c,d\in\mathbb{Z}\\(a,b)=(c,d)=1\\ad-bc>0}}(au\cdot dv-bu\cdot cv)^{w-1}\left(\frac{au\cdot z+bu}{cv\cdot z+dv}\right)^{-s}(cv\cdot z+dv)^{-k}\\
&\qquad=\sum_{\substack{a,b,c,d\in\mathbb{Z}\\ad-bc>0}}(ad-bc)^{w-1}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k} &&(5\text{-}1)\\
&\qquad=\sum_{n=1}^{\infty}\frac{1}{n^{1-w}}\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathcal{M}_n}\left(\frac{az+b}{cz+d}\right)^{-s}(cz+d)^{-k}\\
&\qquad=2\sum_{n=1}^{\infty}\frac{T_n\mathscr{C}_k(z,s)}{n^{k-w}}, &&(5\text{-}2)
\end{aligned}$$

recalling (3-2). With Proposition 4.2, we know $\boldsymbol{E}_{s,k-s}(z,w)\in S_k(\Gamma)$ so that

$$\boldsymbol{E}_{s,k-s}(z,w)=\sum_{f\in\mathscr{B}_k}\frac{\langle\boldsymbol{E}_{s,k-s}(\,\cdot\,,w),f\rangle}{\langle f,f\rangle}f(z)\implies$$
$$\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w)=2\sum_{n=1}^{\infty}\frac{1}{n^{k-w}}\sum_{f\in\mathscr{B}_k}\frac{\langle T_n\mathscr{C}_k(\,\cdot\,,s),f\rangle}{\langle f,f\rangle}f(z).$$

Then

$$\langle T_n\mathscr{C}_k(z,s),f\rangle=\langle\mathscr{C}_k(z,s),T_nf\rangle=a_f(n)\langle\mathscr{C}_k(z,s),f\rangle,$$

and with (3-3), we obtain

$$\begin{aligned}
&\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w)\\
&\qquad=2^{3-w}\pi^{k+1-w}e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\\
&\qquad\qquad\times\sum_{f\in\mathscr{B}_k}L^*(f,k-s)L^*(f,k-w)\frac{f(z)}{\langle f,f\rangle}. &&(5\text{-}3)
\end{aligned}$$

Define the completed double Eisenstein series $\boldsymbol{E}^*$ with (2-12). Then (5-3) becomes

$$\boldsymbol{E}^*_{s,k-s}(z,w) = \sum_{f\in\mathcal{B}_k} L^*(f,s)L^*(f,w)\frac{f(z)}{\langle f,f\rangle}. \tag{5-4}$$

We also now see from (5-4) that $\boldsymbol{E}^*_{s,k-s}(z,w)$ has an analytic continuation to all s and w in $\mathbb{C}$ and satisfies (2-13) and the two functional equations (2-14) and (2-15). The dihedral group D_8 generated by (2-14) and (2-15) is described in [Diamantis and O'Sullivan 2010, §4.4]. □

5B. *Twisted double Eisenstein series.* In this section, we define the *twisted double Eisenstein series* by

$$\begin{aligned}\zeta(1-w+s)\zeta(1-w+k-s)&\boldsymbol{E}_{s,k-s}(z,w;p/q)\\ &:= \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ ad-bc>0}} (ad-bc)^{w-1}\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s}(cz+d)^{-k}\end{aligned} \tag{5-5}$$

for $p/q\in\mathbb{Q}$ with $q>0$ and establish its basic required properties. We remark that the above definition of $\boldsymbol{E}_{s,k-s}(z,w;p/q)$ comes from generalizing (5-1), but it is not clear how it can be extended to general Fuchsian groups.

Writing

$$\begin{aligned}(ad-bc)^{w-1}&\left(\frac{az+b}{cz+d}+\frac{p}{q}\right)^{-s}\\ &= q^{1-w+s}((aq+cp)d-(bq+dp)c)^{w-1}\left(\frac{(aq+cp)z+(bq+dp)}{cz+d}\right)^{-s},\end{aligned}$$

we see that (5-5) equals

$$q^{1-w+s}\sum_{\substack{a',b',c,d\in\mathbb{Z}\\ a'd-b'c>0}} (a'd-b'c)^{w-1}\left(\frac{a'z+b'}{cz+d}\right)^{-s}(cz+d)^{-k}$$

with $a'\equiv cp \bmod q$ and $b'\equiv dp \bmod q$. Hence, $\boldsymbol{E}_{s,k-s}(z,w;p/q)$ is a subseries of $\boldsymbol{E}_{s,k-s}(z,w)$ and, in the same domain of initial convergence, is an element of S_k.

The analog of (5-2) is

$$\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w;p/q) = 2\sum_{n=1}^{\infty}\frac{T_n\mathscr{C}_k(z,s;p/q)}{n^{k-w}}. \tag{5-6}$$

Hence, with (3-3),

$$\begin{aligned}\zeta(1-w+s)\zeta(1-w+k-s)&\boldsymbol{E}_{s,k-s}(z,w;\,p/q)\\ &=2^{3-w}\pi^{k+1-w}e^{-si\pi/2}\frac{\Gamma(k-1)}{\Gamma(s)\Gamma(k-s)\Gamma(k-w)}\\ &\qquad\times\sum_{f\in\mathcal{B}_k}L^*(f,k-s;\,p/q)L^*(f,k-w)\frac{f(z)}{\langle f,f\rangle}.\end{aligned}\tag{5-7}$$

Define the completed double Eisenstein series $\boldsymbol{E}^*_{s,k-s}(z,w;\,p/q)$ with the same factor as (2-12), and we obtain

$$\langle\boldsymbol{E}^*_{s,k-s}(\,\cdot\,,w;\,p/q),f\rangle=L^*(f,k-s;\,p/q)L^*(f,k-w)\tag{5-8}$$

for any f in $\mathcal{B}_k$. Then (5-7) implies $\boldsymbol{E}^*_{s,k-s}(z,w;\,p/q)$ has an analytic continuation to all s and w in $\mathbb{C}$. It satisfies the two functional equations

$$\begin{aligned}\boldsymbol{E}^*_{s,k-s}(z,k-w;\,p/q)&=(-1)^{k/2}\boldsymbol{E}^*_{s,k-s}(z,w;\,p/q),\\ q^s\boldsymbol{E}^*_{k-s,s}(z,w;\,p/q)&=(-1)^{k/2}q^{k-s}\boldsymbol{E}^*_{s,k-s}(z,w;\,-p'/q)\end{aligned}$$

for $pp'\equiv 1 \bmod q$ using (2-1) and (3-5), respectively.

6. Applying the Rankin–Cohen bracket to Poincaré series

The main objective of this section is to show how double Eisenstein series arise naturally when the Rankin–Cohen bracket is applied to the usual Eisenstein series E_k. Proposition 2.4 will be a consequence of this. In fact, since there is no difficulty in extending these methods, we compute the Rankin–Cohen bracket of two arbitrary Poincaré series

$$[P_{k_1}(z;m_1),\,P_{k_2}(z;m_2)]_n$$

for $m_1,m_2\geq 0$. The result may be expressed in terms of the *double Poincaré series* defined below. In this way, the action of the Rankin–Cohen brackets on spaces of modular forms can be completely described. See also Corollary 6.5 at the end of this section.

Definition 6.1. Let $z\in\mathbb{H}$, $k_1,k_2\geq 3$ in $\mathbb{Z}$ and $m_1,m_2\in\mathbb{Z}_{\geq 0}$. For $w\in\mathbb{C}$ with $\mathrm{Re}(w)<k_1-1,k_2-1$, we define the *double Poincaré series*

$$\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2):=\sum_{\substack{\gamma,\delta\in B\backslash\Gamma\\ c_{\gamma\delta^{-1}}>0}}(c_{\gamma\delta^{-1}})^{w-1}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1}j(\delta,z)^{k_2}}.\tag{6-1}$$

The series (6-1) will vanish identically unless k_1 and k_2 have the same parity. Clearly, we have $\boldsymbol{E}_{k_1,k_2}(z,w)=\boldsymbol{P}_{k_1,k_2}(z,w;0,0)$. Since $|e^{2\pi i(m_1\gamma z+m_2\delta z)}|\leqslant 1$, it is a simple matter to verify that the work in Section 4 proves that $\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2)$ converges absolutely and uniformly on compacta to a cusp form in $S_{k_1+k_2}(\Gamma)$.

For $l \in \mathbb{Z}_{\geq 0}$, it is convenient to set

$$Q_k(z,l;m) := \begin{cases} P_k(z;m) & \text{if } l = 0, \\ \frac{1}{2}\sum_{\gamma \in B\backslash\Gamma} \frac{e^{2\pi i m\gamma z}(c_\gamma)^l}{j(\gamma,z)^{k+l}} & \text{if } l \geq 1. \end{cases} \tag{6-2}$$

As in the proof of Proposition 4.2, Q_k is an absolutely convergent series for k even and at least 4. The next result may be verified by induction.

Lemma 6.2. *For every $j \in \mathbb{Z}_{\geq 0}$, we have the formulas*

$$\frac{d^j}{dz^j} E_k(z) = (-1)^j \frac{(k+j-1)!}{(k-1)!} Q_k(z,j;0),$$

$$\frac{d^j}{dz^j} P_k(z;m) = \sum_{l=0}^{j} (-1)^{l+j} (2\pi i m)^l \frac{j!}{l!} \binom{k+j-1}{k+l-1} Q_{k+2l}(z, j-l; m) \quad \textit{for } m > 0.$$

Set

$$A_{k_1,k_2}(l,u)_n := \frac{(k_1+n-1)!\,(k_2+n-1)!}{l!\,u!\,(n-l-u)!\,(k_1+l-1)!\,(k_2+u-1)!}.$$

Proposition 6.3. *For $m_1, m_2 \in \mathbb{Z}_{\geq 1}$,*

$$\begin{aligned}[P_{k_1}(z;m_1), P_{k_2}(z;m_2)]_n = &\sum_{\substack{l,u \geq 0 \\ l+u \leqslant n}} A_{k_1,k_2}(l,u)_n (-2\pi i m_1)^l (2\pi i m_2)^u \\ &\times \boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z, n+1-l-u; m_1, m_2)/2 \\ &+ P_{k_1+k_2+2n}(z; m_1+m_2) \sum_{\substack{l,u \geq 0 \\ l+u = n}} A_{k_1,k_2}(l,u)_n (-2\pi i m_1)^l (2\pi i m_2)^u.\end{aligned}$$

Proof. With Lemma 6.2,

$$\begin{aligned}&[P_{k_1}(z;m_1), P_{k_2}(z;m_2)]_n \\ &\quad = \sum_{l=0}^{n} \sum_{u=0}^{n} (2\pi i m_1)^l (2\pi i m_2)^u \frac{(k_1+n-1)!\,(k_2+n-1)!}{l!\,u!\,(k_1+l-1)!\,(k_2+u-1)!} \\ &\quad\quad \times \sum_{r=l}^{n-u} (-1)^{n+l+u+r} \frac{Q_{k_1+2l}(z, r-l; m_1) Q_{k_2+2u}(z, n-r-u; m_2)}{(r-l)!\,(n-r-u)!}.\end{aligned} \tag{6-3}$$

The inner sum over r is

$$\begin{aligned}&\frac{(-1)^l}{4(n-l-u)!} \sum_{\gamma,\delta \in B\backslash\Gamma} \frac{e^{2\pi i(m_1\gamma z + m_2 \delta z)}}{j(\gamma,z)^{k_1+2l} j(\delta,z)^{k_2+2u}} \\ &\quad\quad \times \sum_{r=l}^{n-u} \binom{n-l-u}{r-l} \left(\frac{c_\gamma}{j(\gamma,z)}\right)^{r-l} \left(\frac{-c_\delta}{j(\delta,z)}\right)^{n-r-u},\end{aligned} \tag{6-4}$$

and, employing the binomial theorem, (6-4) reduces to

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in B\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+n+l-u}j(\delta,z)^{k_2+n-l+u}}(c_\gamma j(\delta,z)-c_\delta j(\gamma,z))^{n-l-u} \tag{6-5}$$

for $l+u<n$ and

$$\frac{(-1)^l}{4(n-l-u)!}\sum_{\gamma,\delta\in B\backslash\Gamma}\frac{e^{2\pi i(m_1\gamma z+m_2\delta z)}}{j(\gamma,z)^{k_1+n+l-u}j(\delta,z)^{k_2+n-l+u}} \tag{6-6}$$

for $l+u=n$. Noting that

$$c_\gamma j(\delta,z)-c_\delta j(\gamma,z)=\begin{vmatrix}c_\gamma & d_\gamma\\ c_\delta & d_\delta\end{vmatrix}=c_{\gamma\delta^{-1}}$$

means that (6-5) becomes

$$\frac{(-1)^l}{2(n-l-u)!}\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,n+1-l-u;m_1,m_2) \tag{6-7}$$

and (6-6) equals

$$\frac{(-1)^l}{(n-l-u)!}\biggl(\frac{\boldsymbol{P}_{k_1+n+l-u,k_2+n-l+u}(z,n+1-l-u;m_1,m_2)}{2} + P_{k_1+k_2+2n}(z;m_1+m_2)\biggr). \tag{6-8}$$

Putting (6-7) and (6-8) into (6-3) finishes the proof. □

In fact, Proposition 6.3 is also valid for m_1 or m_2 equaling 0 provided we agree that $(-2\pi i m_1)^l=1$ in the ambiguous case where $m_1=l=0$ and similarly that $(2\pi i m_2)^u=1$ when $m_2=u=0$. With this notational convention, the proof of the last proposition gives:

Corollary 6.4. *For $m>0$, we have*

$$[E_{k_1}(z),P_{k_2}(z;m)]_n=\sum_{u=0}^{n}A_{k_1,k_2}(0,u)_n(2\pi i m)^u \times\frac{\boldsymbol{P}_{k_1+n-u,k_2+n+u}(z,n+1-u;0,m)}{2}+P_{k_1+k_2+2n}(z;m)\cdot A_{k_1,k_2}(0,n)_n(2\pi i m)^n,$$
$$[E_{k_1}(z),E_{k_2}(z)]_n=A_{k_1,k_2}(0,0)_n\boldsymbol{E}_{k_1+n,k_2+n}(z,n+1)/2+E_{k_1+k_2}(z)\cdot\delta_{n,0}. \tag{6-9}$$

Proposition 2.4 follows directly from (6-9). Combining Proposition 2.4 with Theorem 2.3 gives a new proof of Zagier's formula (1-2). His original proof in [1977, Proposition 6] employed Poincaré series.

Proof of Proposition 2.5. Let $F_{s,w}(z)=(-1)^{k_2/2}y^{-k/2}E^*_{k_1}(z,u)E^*_{k_2}(z,v)/(2\pi^{k/2})$ with $u=(s+w-k+1)/2$ and $v=(-s+w+1)/2$ as before in (2-16). Then

$F_{s,w}(z)$ has weight k and polynomial growth as $y \to \infty$. It is proved in [Diamantis and O'Sullivan 2010, Proposition 2.1] that

$$\langle F_{s,w}, f\rangle = L^*(f,s)L^*(f,w) \tag{6-10}$$

for all $f \in B_k$. Comparing (6-10) with (2-13) shows that

$$\boldsymbol{E}^*_{s,k-s}(\,\cdot\,,w) = \pi_{\mathrm{hol}}(F_{s,w}),$$

as required. □

A basic property of Rankin–Cohen brackets naturally emerges from Proposition 6.3 and Corollary 6.4.

Corollary 6.5. *For $g_1 \in M_{k_1}(\Gamma)$ and $g_2 \in M_{k_2}(\Gamma)$, we have $[g_1, g_2]_n \in S_{k_1+k_2+2n}(\Gamma)$ for $n > 0$.*

Proof. The space $M_{k_1}(\Gamma)$ is spanned by E_{k_1} and the Poincaré series $P_{k_1}(z;m)$ for $m \in \mathbb{Z}_{\geq 1}$. So we may write g_1, and similarly g_2, as a linear combination of Eisenstein and Poincaré series. Hence, $[g_1, g_2]_n$ is a linear combination of the Rankin–Cohen brackets appearing in Proposition 6.3 and Corollary 6.4. By these results, $[g_1, g_2]_n$ is a linear combination of double Poincaré and double Eisenstein series, which are in $S_{k_1+k_2+2n}(\Gamma)$ as we have already shown. □

It would be interesting to know if $\boldsymbol{P}_{k_1,k_2}(z,w;m_1,m_2)$ has a meromorphic continuation in w. As a corollary of work in the next section, we establish the continuation of $\boldsymbol{P}_{k_1,k_2}(z,w;0,0)$ to all $w \in \mathbb{C}$.

7. The Hecke action

The expression (5-2), giving $\boldsymbol{E}_{s,k-s}$ in terms of $\mathscr{C}_k$ acted upon by the Hecke operators, can be studied further and yields an interesting relation between $\boldsymbol{E}_{s,k-s}(z,w)$ and the generalized Cohen kernel $\mathscr{C}_k(z,s;p/q)$.

We have

$$\begin{aligned} T_n\mathscr{C}_k(z,s;p/q) &= n^{k-1}\sum_{\rho\in\Gamma\backslash\mathcal{M}_n}\mathscr{C}_k(\rho z,s;p/q)\cdot j(\rho,z)^{-k} \\ &= \tfrac{1}{2}n^{k-1}\sum_{\gamma\in\mathcal{M}_n}\left(\gamma z+\frac{p}{q}\right)^{-s} j(\gamma,z)^{-k}. \end{aligned}$$

To decompose $\mathcal{M}_n$ into left Γ-cosets, set

$$\mathscr{H} := \left\{ \begin{pmatrix} a & b \\ 0 & d\end{pmatrix} \,\middle|\, a,b,d \in \mathbb{Z}_{\geq 0},\, ad = n,\, 0 \leqslant b < a \right\}$$

so that $\mathcal{M}_n = \bigcup_{\rho\in\mathcal{H}} \rho\Gamma$, a disjoint union. Hence,

$$\begin{aligned}
T_n\mathscr{C}_k(z,s;p/q) &= \tfrac{1}{2}n^{k-1}\sum_{\rho\in\mathcal{H}}\sum_{\gamma\in\Gamma}\left(\rho\gamma z+\frac{p}{q}\right)^{-s} j(\rho,\gamma z)^{-k} j(\gamma,z)^{-k}\\
&= \tfrac{1}{2}n^{k-1}\sum_{a|n}\left(\frac{n}{a}\right)^{-k}\left(\frac{a^2}{n}\right)^{-s}\sum_{0\leqslant b<a}\sum_{\gamma\in\Gamma}\left(\gamma z+\frac{b}{a}+\frac{n}{a^2}\frac{p}{q}\right)^{-s} j(\gamma,z)^{-k}\\
&= n^{s-1}\sum_{a|n}a^{k-2s}\sum_{0\leqslant b<a}\mathscr{C}_k\left(z,s;\frac{b}{a}+\frac{n}{a^2}\frac{p}{q}\right). \qquad (7\text{-}1)
\end{aligned}$$

Combining (7-1) in the case $p/q=0$, with (5-2) we find

$$\begin{aligned}
&\frac{\zeta(1-w+s)\zeta(1-w+k-s)\boldsymbol{E}_{s,k-s}(z,w)}{2}\\
&\qquad= \sum_{n=1}^{\infty}\frac{T_n\mathscr{C}_k(z,s)}{n^{k-w}}\\
&\qquad= \sum_{n=1}^{\infty}n^{s+w-k-1}\sum_{a|n}a^{k-2s}\sum_{0\leqslant b<a}\mathscr{C}_k\left(z,s;\frac{b}{a}\right)\\
&\qquad= \sum_{a=1}^{\infty}a^{k-2s}\sum_{v=1}^{\infty}(av)^{s+w-k-1}\sum_{0\leqslant b<a}\mathscr{C}_k\left(z,s;\frac{b}{a}\right)\\
&\qquad= \zeta(k+1-s-w)\sum_{a=1}^{\infty}a^{w-s-1}\sum_{0\leqslant b<a}\mathscr{C}_k\left(z,s;\frac{b}{a}\right).
\end{aligned}$$

Consequently, for $2<\sigma<k-2$ and $\mathrm{Re}(w)<\sigma-1,k-1-\sigma$,

$$\zeta(1-w+s)\boldsymbol{E}_{s,k-s}(z,w) = 2\sum_{a=1}^{\infty}a^{w-s-1}\sum_{b=0}^{a-1}\mathscr{C}_k\left(z,s;\frac{b}{a}\right). \qquad (7\text{-}2)$$

Upon taking the inner product of both sides with $f\in\mathscr{B}_k$, by using (2-13) and (3-3) and then simplifying, we obtain

$$\begin{aligned}
&\frac{(2\pi)^{k-w}}{\Gamma(k-w)}L^*(f,s)L^*(f,w)\\
&\qquad= \zeta(k+1-s-w)\sum_{a=1}^{\infty}a^{w-s-1}\sum_{b=0}^{a-1}L^*\left(f,k-s;\frac{b}{a}\right). \qquad (7\text{-}3)
\end{aligned}$$

Since the eigenforms f in $\mathscr{B}_k$ span S_k, we may verify (7-2) by giving another proof of (7-3). Note that the right side of (7-3) equals

$$\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}}\sum_{a=1}^{\infty}a^{w-s-1}\sum_{b=0}^{a-1}\sum_{m=1}^{\infty}\frac{a_f(m)e^{2\pi imb/a}}{m^{k-s}}$$
$$=\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}}\sum_{m=1}^{\infty}\sum_{a|m}a^{w-s}\frac{a_f(m)}{m^{k-s}}$$
$$=\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}}\sum_{m=1}^{\infty}\frac{a_f(m)\sigma_{w-s}(m)}{m^{k-s}}.$$

The series

$$L(f\otimes E(\cdot\,,v),k-s):=\sum_{m=1}^{\infty}\frac{a_f(m)\sigma_{w-s}(m)}{m^{k-s}}$$

is a convolution L-series involving the Fourier coefficients of $f(z)$ and $E(z,v)$ for $2v=-s+w+1$ (as in (2-16)) and, recalling [Zagier 1977, (72)] or [Diamantis and O'Sullivan 2010, (2.11)],

$$\zeta(k+1-s-w)\frac{\Gamma(k-s)}{(2\pi)^{k-s}}L(f\otimes E(\cdot\,,v),k-s)=\frac{(2\pi)^{k-w}}{\Gamma(k-w)}L^*(f,k-s)L^*(f,k-w). \tag{7-4}$$

Applying the functional equation (2-1) confirms that the right side of (7-4) equals the left side of (7-3).

Looking to simplify (7-2) leads to the natural question, what are the relations between the $\mathscr{C}_k(z,s;p/q)$ for rational p/q in the interval $[0,1)$? For example, it is a simple exercise with (3-3) and (3-5) to show that

$$q^{-s}\mathscr{C}_k(z,s;p/q)=e^{-si\pi}q^{-k+s}\mathscr{C}_k(z,k-s;-p'/q)$$

for $pp'\equiv 1 \bmod q$. With $s=k/2$ at the center of the critical strip, we get an even simpler relation:

$$\mathscr{C}_k(z,k/2;p/q)=(-1)^{k/2}\mathscr{C}_k(z,k/2;-p'/q). \tag{7-5}$$

A more interesting, but speculative, possibility would be to argue in the reverse direction in order to derive information about L-functions twisted by exponentials with *nonrational* exponents. Specifically, if we established, by other means, relations between the $\mathscr{C}_k(z,s;x)$ for $x\notin\mathbb{Q}$, then (7-2) and other results proven here might lead to relations for L-functions twisted by exponentials with nonrational exponents. That would be important because such L-functions play a prominent role in Kaczorowski and Perelli's program of classifying the Selberg class (see, e.g., [Kaczorowski and Perelli 1999]). Relations between these L-functions seem to be necessary for the extension of Kaczorowski and Perelli's classification to degree 2, to which L-functions of GL(2) cusp forms belong.

8. Periods of cusp forms

8A. ***Values of L-functions inside the critical strip.*** We first review Zagier's proof in [1977, §5] of Manin's periods theorem. This exhibits a general principle of proving algebraicity we will be using in the next sections.

For all $s, w \in \mathbb{C}$, it is convenient to define $H_{s,w} \in S_k$ by the conditions

$$\langle H_{s,w}, f\rangle = L^*(f, s)L^*(f, w) \quad \text{for all } f \in \mathcal{B}_k.$$

We need the following result:

Lemma 8.1. *For $g \in S_k$ with Fourier coefficients in the field K_g and $f \in \mathcal{B}_k$ with coefficients in K_f,*

$$\langle g, f\rangle/\langle f, f\rangle \in K_g K_f.$$

Proof. See the general result of Shimura [1976, Lemma 4]. It is also a simple extension of [Diamantis and O'Sullivan 2010, Lemma 4.3]. □

Let K_{critical} be the field obtained by adjoining to $\mathbb{Q}$ all the Fourier coefficients of

$$\{H_{s,k-1}, H_{k-2,w} \mid 1 \leqslant s, w \leqslant k-1,\ s \text{ even},\ w \text{ odd}\}.$$

Thus, with $f \in \mathcal{B}_k$ and employing Lemma 8.1,

$$L^*(f, k-1)L^*(f, k-2) = \langle H_{k-1,k-2}, f\rangle = c_f\langle f, f\rangle \tag{8-1}$$

for $c_f \in K_{\text{critical}}K_f$, and the left side of (8-1) is nonzero because the Euler product for $L^*(f, s)$ converges for $\operatorname{Re}(s) > k/2 + 1/2$. Set

$$\omega_+(f) := \frac{c_f\langle f, f\rangle}{L^*(f, k-1)} \quad \text{and} \quad \omega_-(f) := \frac{\langle f, f\rangle}{L^*(f, k-2)}. \tag{8-2}$$

Then $\omega_+(f)\omega_-(f) = \langle f, f\rangle$, and we have:

Lemma 8.2. *For each $f \in \mathcal{B}_k$,*

$$L^*(f, s)/\omega_+(f) \quad \textit{and} \quad L^*(f, w)/\omega_-(f) \in K_{\text{critical}}K_f$$

for all s and w with $1 \leqslant s, w \leqslant k-1$, s even and w odd.

Proof. For such s and w,

$$\frac{L^*(f, s)}{\omega_+(f)} = \frac{L^*(f, s)L^*(f, k-1)}{c_f\langle f, f\rangle} = \frac{\langle H_{s,k-1}, f\rangle}{c_f\langle f, f\rangle} = \frac{c'_f\langle f, f\rangle}{c_f\langle f, f\rangle} \in K_{\text{critical}}K_f,$$

$$\frac{L^*(f, w)}{\omega_-(f)} = \frac{L^*(f, w)L^*(f, k-2)}{c_f\langle f, f\rangle} = \frac{\langle H_{k-2,w}, f\rangle}{c_f\langle f, f\rangle} = \frac{c''_f\langle f, f\rangle}{c_f\langle f, f\rangle} \in K_{\text{critical}}K_f. \quad □$$

To deduce Manin's theorem from Lemma 8.2, we use Zagier's explicit expression for $H_{s,w}$. For $n \geq 0$, even $k_1, k_2 \geq 4$ and $k = k_1 + k_2 + 2n$, (1-2) implies

$$(-1)^{k_1/2} 2^{3-k} \frac{k_1 k_2}{B_{k_1} B_{k_2}} \binom{k-2}{n} H_{n+1,n+k_2} = \frac{[E_{k_1}, E_{k_2}]_n}{(2\pi i)^n}. \tag{8-3}$$

The Fourier coefficients of E_{k_1} and E_{k_2} are rational, and hence, the right side of (8-3) has rational coefficients. Then $H_{n+1,n+k_2}$ has Fourier coefficients in $\mathbb{Q}$ (and also for $k_1, k_2 = 2$ [Kohnen and Zagier 1984, p. 214]). It follows that $K_{\text{critical}} = \mathbb{Q}$ and Lemma 8.2 becomes Theorem 2.6, Manin's periods theorem.

8B. *Arbitrary L-values.* With the results of the last section, we may now give the proof of Theorem 2.7, restated here:

Theorem 8.3. *For all $f \in \mathcal{B}_k$ and $s \in \mathbb{C}$, with $\omega_+(f)$ and $\omega_-(f)$ as in Manin's theorem,*

$$L^*(f,s)/\omega_+(f) \in K(\boldsymbol{E}^*_{s,k-s}(\,\cdot\,, k-1))K_f,$$
$$L^*(f,s)/\omega_-(f) \in K(\boldsymbol{E}^*_{k-2,2}(\,\cdot\,, s))K_f.$$

Proof. By Theorem 2.3, we have $H_{s,w}(z) = \boldsymbol{E}^*_{s,k-s}(z,w)$ for all $s, w \in \mathbb{C}$. Thus, arguing as in Lemma 8.2 with $\boldsymbol{E}^*_{s,k-s}(\,\cdot\,, k-1) = H_{s,k-1}$ and $\boldsymbol{E}^*_{k-2,2}(\,\cdot\,, s) = H_{k-2,s}$ yields the theorem. □

We indicate briefly how the double Eisenstein series Fourier coefficients required to define $K(\boldsymbol{E}^*_{s,k-s}(\,\cdot\,, k-1))$ and $K(\boldsymbol{E}^*_{k-2,2}(\,\cdot\,, s))$ in Theorem 2.7 may be calculated when $s \in \mathbb{Z}$, using a slight extension of the methods in [Diamantis and O'Sullivan 2010, §3]. We wish to find the l-th Fourier coefficient, $a_{s,w}(l)$, of $H_{s,w}(z) = \boldsymbol{E}^*_{s,k-s}(z,w)$ for s even and w odd (and we assume $s, w \geq k/2 > 1$). With Proposition 2.5, this is $(-1)^{k_2/2}/(2\pi^{k/2})$ times the l-th Fourier coefficient of

$$\pi_{\text{hol}}\big[y^{-k/2} E^*_{k_1}(z,u) E^*_{k_2}(z,v)\big]$$

for $u = (s+w-k+1)/2$ and $v = (-s+w+1)/2$ both in $\mathbb{Z}$. Let

$$F(z) := y^{-k/2} E^*_{k_1}(z,u) E^*_{k_2}(z,v) - \frac{\theta_{k_1}(u)\theta_{k_2}(1-v)}{\theta_k(s+1-k/2)} y^{-k/2} E^*_k(z, s+1-k/2) - \frac{\theta_{k_1}(u)\theta_{k_2}(v)}{\theta_k(w+1-k/2)} y^{-k/2} E^*_k(z, w+1-k/2).$$

Then $\pi_{\text{hol}}(y^{-k/2} E^*_{k_1}(z,u) E^*_{k_2}(z,v)) = \pi_{\text{hol}}(F(z))$ because $\pi_{\text{hol}}(y^{-k/2} E^*_k(z,s)) = 0$ for every s. We have constructed F so that $F(z) \ll y^{-\varepsilon}$ as $y \to \infty$, and we may use [Diamantis and O'Sullivan 2010, Lemma 3.3] to obtain

$$a_{s,w}(l) = \frac{(-1)^{k_2/2}(4\pi l)^{k-1}}{(2\pi^{k/2})(k-2)!} \int_0^\infty F_l(y) e^{-2\pi l y} y^{k-2}\, dy$$

on writing $F(z) = \sum_{l \in \mathbb{Z}} e^{2\pi i l x} y^{-k/2} F_l(y)$. The functions $F_l(y)$ are sums involving the Fourier coefficients of $E^*_{k_1}(z, u)$ and $E^*_{k_2}(z, v)$ with $u, v \in \mathbb{Z}$. As shown in [Diamantis and O'Sullivan 2010, Theorem 3.1], these coefficients are simply expressed in terms of divisor functions, Bernoulli numbers and a combinatorial part. For s and w in the critical strip, this calculation yields an explicit finite formula for $a_{s,w}(l)$ in [Diamantis and O'Sullivan 2010, Theorem 1.3] (and another proof that $H_{s,w}$ in (8-3) has rational Fourier coefficients and that $K_{\text{critical}} = \mathbb{Q}$). For s and w outside the critical strip, we obtain infinite series representations for $a_{s,w}(l)$ but again involving nothing more complicated than divisor functions and Bernoulli numbers. Further details of this computation will appear in [O'Sullivan 2013].

8C. *Twisted periods.* There is an analog of Manin's periods theorem for twisted L-functions. Let $p/q \in \mathbb{Q}$, and let u be an integer with $1 \leqslant u \leqslant k-1$. Manin shows in [1973, (13)] (see also [Lang 1976, Chapter 5]) that $i^u \int_0^{p/q} f(iy) y^{u-1}\, dy$ is an integral linear combination of periods $i^v \int_0^\infty f(iy) y^{v-1}\, dy$ for $v = 1, \dots, k-1$. With (2-17), this proves

$$i^u q^{k-2} L^*(f, u; p/q) \in \mathbb{Z} \cdot i L^*(f, 1) + \mathbb{Z} \cdot i^2 L^*(f, 2) + \cdots + \mathbb{Z} \cdot i^{k-1} L^*(f, k-1).$$

Therefore, Theorem 2.6 implies the next result.

Proposition 8.4. *For all $f \in \mathcal{B}_k$, $p/q \in \mathbb{Q}$ and integers u with $1 \leqslant u \leqslant k-1$,*

$$L^*(f, u; p/q) \in K_f(i)\omega_+(f) + K_f(i)\omega_-(f).$$

Employing (5-8), a similar proof to that of Theorem 2.7 in the last section shows the following:

Proposition 8.5. *For all $f \in \mathcal{B}_k$, $p/q \in \mathbb{Q}$ and $s \in \mathbb{C}$ with $\omega_+(f)$ and $\omega_-(f)$ as in Manin's theorem,*

$$L^*(f, s; p/q)/\omega_+(f) \in K(\boldsymbol{E}^*_{k-s,s}(\,\cdot\,, 1; p/q))K_f,$$
$$L^*(f, s; p/q)/\omega_-(f) \in K(\boldsymbol{E}^*_{k-s,s}(\,\cdot\,, 2; p/q))K_f.$$

9. The nonholomorphic case

9A. *Background results and notation.* We will need a nonholomorphic analog of the Cohen kernel $\mathscr{C}_k(z, s)$.

Definition 9.1. With $z \in \mathbb{H}$ and $s, s' \in \mathbb{C}$, define the *nonholomorphic kernel* $\mathscr{K}$ as

$$\mathscr{K}(z; s, s') := \frac{1}{2} \sum_{\gamma \in \Gamma} \frac{\operatorname{Im}(\gamma z)^{s+s'}}{|\gamma z|^{2s}}. \tag{9-1}$$

Following directly from the results in [Diamantis and O'Sullivan 2010, §5.2], it is absolutely convergent, uniformly on compacta, for $z \in \mathbb{H}$ and $\operatorname{Re}(s), \operatorname{Re}(s') > 1/2$.

The kernel $\mathcal{K}(z; s, s')$ was introduced by Diaconu and Goldfeld [2007, (2.1)] (though they describe it there as a Poincaré series and their kernel is a product of Γ factors). Starting with the identity [Diaconu and Goldfeld 2007, Proposition 3.5]

$$\langle f \cdot \mathcal{K}(\,\cdot\,; s, s'), g\rangle \\ = \frac{\Gamma(s+s'+k-1)}{2^{s+s'+k-1}} \int_{-\infty}^{\infty} \frac{L^*(f, \alpha+i\beta)L^*(g, -s+s'+k-\alpha-i\beta)}{\Gamma(s+\alpha+i\beta)\Gamma(-s+s'+k-\alpha-i\beta)}\, d\beta$$

for f and g in $\mathcal{B}_k$, they provide a new method to establish estimates for the second moment of $L^*(f, s)$ along the critical line $\mathrm{Re}(s) = k/2$. They give similar results for $L^*(u_j, s)$, the L-function associated to a Maass form u_j as defined below.

The spectral decomposition of $\mathcal{K}(z; s, s')$ and its meromorphic continuation in the s and s' variables is shown in [Diaconu and Goldfeld 2007, §5]. We do the same; our treatment is slightly different, and we include it in Section 9B for completeness.

For $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, the discrete spectrum of the Laplace operator $\Delta = -4y^2\partial_z\partial_{\bar{z}}$ is given by u_0, the constant eigenfunction, and u_j for $j \in \mathbb{Z}_{\geq 1}$ an orthogonal system of Maass cusp forms (see, e.g., [Iwaniec 2002, Chapters 4 and 7]) with Fourier expansions

$$u_j(z) = \sum_{n \neq 0} |n|^{-1/2} \nu_j(n) W_{s_j}(nz),$$

where u_j has eigenvalue $s_j(1-s_j)$ and by Weyl's law [Iwaniec 2002, (11.5)]

$$\#\{j \mid |\mathrm{Im}(s_j)| \leqslant T\} = T^2/12 + O(T \log T). \tag{9-2}$$

We may assume the u_j are Hecke eigenforms normalized to have $\nu_j(1) = 1$. Necessarily we have $\nu_j(n) \in \mathbb{R}$. Let ι be the antiholomorphic involution $(\iota u_j)(z) := u_j(-\bar{z})$. We may also assume each u_j is an eigenfunction of this operator, necessarily with eigenvalues ± 1. If $\iota u_j = u_j$, then $\nu_j(n) = \nu_j(-n)$ and u_j is called *even*. If $\iota u_j = -u_j$, then $\nu_j(n) = -\nu_j(-n)$ and u_j is *odd*.

The L-function associated to the Maass cusp form u_j is

$$L(u_j, s) = \sum_{n=1}^{\infty} \nu_j(n)/n^s,$$

convergent for $\mathrm{Re}(s) > 3/2$ since $\nu_j(n) \ll n^{1/2}$ by [Iwaniec 2002, (8.8)]. The completed L-function for an even form u_j is

$$L^*(u_j, s) := \pi^{-s}\Gamma\left(\frac{s+s_j-1/2}{2}\right)\Gamma\left(\frac{s-s_j+1/2}{2}\right)L(u_j, s), \tag{9-3}$$

and it satisfies

$$L^*(u_j, 1-s) = L^*(u_j, s) = \overline{L^*(u_j, \bar{s})}. \tag{9-4}$$

See [Bump 1997, p. 107] for (9-3), (9-4) and the analogous odd case.

To $E(z,s)$ (recall (2-3)) we associate the L-function

$$L(E(\,\cdot\,,s),w) := \sum_{m=1}^{\infty} \frac{\phi(m,s)}{m^w}.$$

The well-known identity $\sum_{m=1}^{\infty} \sigma_x(m)/m^w = \zeta(w)\zeta(w-x)$ implies

$$L(E(\,\cdot\,,s),w) = \frac{2\pi^s}{\Gamma(s)} \frac{\zeta(w+s-1/2)\zeta(w-s+1/2)}{\zeta(2s)}. \tag{9-5}$$

9B. ***The nonholomorphic kernel*** $\mathcal{K}$**.** Throughout this section, we use $s=\sigma+it$ and $s'=\sigma'+it'$. Recall $\mathcal{K}(z;s,s')$ defined in (9-1) for $\operatorname{Re}(s), \operatorname{Re}(s') > 1/2$. Our goal is to find the spectral decomposition of $\mathcal{K}(z;s,s')$ and prove its meromorphic continuation in s and s'. See [Diaconu and Goldfeld 2007, §5] and also [Iwaniec 2002, §7.4] for a similar decomposition and continuation of the automorphic Green function.

A routine verification (using [Jorgenson and O'Sullivan 2005, Lemma 9.2], for example) yields

$$\Delta\mathcal{K}(z;s,s') = (s+s')(1-s-s')\mathcal{K}(z;s,s') + 4ss'\mathcal{K}(z;s+1,s'+1). \tag{9-6}$$

Put

$$\xi_{\mathbb{Z}}(z,s) := \sum_{m\in\mathbb{Z}} \frac{1}{|z+m|^{2s}}.$$

Then

$$\mathcal{K}(z;s,s') = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma z)^{s+s'} \xi_{\mathbb{Z}}(\gamma z,s). \tag{9-7}$$

Use the Poisson summation formula as in [Iwaniec 2002, §3.4] or [Goldfeld 2006, Theorem 3.1.8] to see that

$$\xi_{\mathbb{Z}}(z,s) = \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} y^{1-2s} + \frac{2\pi^s}{\Gamma(s)} y^{1/2-s} \sum_{m\neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x} \tag{9-8}$$

for $\operatorname{Re}(s) > 1/2$. Set

$$\xi^{\sharp}_{\mathbb{Z}}(z,s) := \sum_{m\neq 0} |m|^{s-1/2} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}. \tag{9-9}$$

Let $B_\rho := \{z\in\mathbb{C} \mid |z|\leqslant\rho\}$. Then with [Jorgenson and O'Sullivan 2008, Lemma 6.4],

$$\sqrt{y}K_{s-1/2}(2\pi y) \ll e^{-2\pi y}(y^{\rho+3} + y^{-\rho-3})$$

for all $s\in B_\rho$ and $\rho, y>0$ with the implied constant depending only on ρ. Hence,

$$\xi^{\sharp}_{\mathbb{Z}}(z,s) \ll \sum_{m=1}^{\infty} e^{-2\pi m y}(m^{\rho+\sigma+2}y^{\rho+5/2} + m^{-\rho+\sigma-4}y^{-\rho-7/2}).$$

We also have [Jorgenson and O'Sullivan 2008, Lemma 6.2]

$$\sum_{m=1}^{\infty} m^\rho e^{-2m\pi y} \ll e^{-2\pi y}(1+y^{-\rho-1})$$

for all $y > 0$ with the implied constant depending only on $\rho \geq 0$. Therefore,

$$\xi^\sharp_{\mathbb{Z}}(z,s) \ll e^{-2\pi y}(y^{\rho+5/2}+y^{-\rho-9/2}). \tag{9-10}$$

Consider the weight-0 series

$$\mathcal{K}^\sharp(z;s,s') := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^{s'+1/2}\xi^\sharp_{\mathbb{Z}}(\gamma z,s). \tag{9-11}$$

With (9-10), we have

$$\mathcal{K}^\sharp(z;s,s') \ll \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\mathrm{Im}(\gamma z)^{\sigma'+\rho+3}+\mathrm{Im}(\gamma z)^{\sigma'-\rho-4})e^{-2\pi\,\mathrm{Im}(\gamma z)} \tag{9-12}$$

so that $\mathcal{K}^\sharp(z;s,s')$ is absolutely convergent for $\mathrm{Re}(s') > \rho+5$.

Proposition 9.2. *Let $\rho > 0$ and $s, s' \in \mathbb{C}$ satisfy $\mathrm{Re}(s) > 1/2$, $\mathrm{Re}(s') > \rho+5$ and $s \in B_\rho$. Then*

$$\mathcal{K}(z;s,s') = \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)}E(z,s'-s+1)+\frac{2\pi^s}{\Gamma(s)}\mathcal{K}^\sharp(z;s,s'), \tag{9-13}$$

and, for an implied constant depending only on s and s',

$$\mathcal{K}^\sharp(z;s,s') \ll y^{5+\rho-\sigma'} \quad \text{as } y\to\infty. \tag{9-14}$$

Proof. It is clear that (9-13) follows from (9-7), (9-8), (9-9) and (9-11) when s and s' are in the stated range. With (9-12) and employing (4-3), we deduce that as $y\to\infty$,

$$\begin{aligned}\mathcal{K}^\sharp(z;s,s') &\ll (y^{\sigma'+\rho+3}+y^{\sigma'-\rho-4})e^{-2\pi y}\\ &\qquad + \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma\\ \gamma\neq\Gamma_\infty}} (\mathrm{Im}(\gamma z)^{\sigma'+\rho+3}+\mathrm{Im}(\gamma z)^{\sigma'-\rho-4})\\ &\ll e^{-\pi y}+y^{1-(\sigma'+\rho+3)}+y^{1-(\sigma'-\rho-4)}\\ &\ll y^{5+\rho-\sigma'}.\end{aligned}$$

□

Clearly, for $\mathrm{Re}(s') > \rho+5$, (9-13) gives the meromorphic continuation of $\mathcal{K}(z;s,s')$ to all $s \in B_\rho$. For these s and s', it follows from (9-14) that $\mathcal{K}^\sharp$, as a function of z, is bounded. Also use (9-6) and (9-13) to show that

$$\Delta\mathcal{K}^\sharp(z;s,s') = (s+s')(1-s-s')\mathcal{K}^\sharp(z;s,s')+4\pi s'\mathcal{K}^\sharp(z;s+1,s'+1),$$

and hence, $\Delta\mathcal{K}^\sharp$ is also bounded. Therefore, with [Iwaniec 2002, Theorems 4.7 and 7.3], $\mathcal{K}^\sharp$ has the spectral decomposition

$$\mathcal{K}^\sharp(z;s,s') = \sum_{j=0}^{\infty} \frac{\langle \mathcal{K}^\sharp(\,\cdot\,;s,s'),u_j\rangle}{\langle u_j,u_j\rangle} u_j(z) \\ + \frac{1}{4\pi i}\int_{(1/2)} \langle \mathcal{K}^\sharp(\,\cdot\,;s,s'),E(\,\cdot\,,r)\rangle E(z,r)\,dr, \quad (9\text{-}15)$$

where the integral is from $1/2-i\infty$ to $1/2+i\infty$ and the convergence of (9-15) is pointwise absolute in z and uniform on compacta.

Lemma 9.3. *For $s \in B_\rho$ and $\mathrm{Re}(s') > \rho+5$, we have*

$$\langle \mathcal{K}^\sharp(\,\cdot\,;s,s'),u_j\rangle = \frac{\pi^{1/2-s}}{4\Gamma(s')} L^*(u_j,s'-s+1/2)\Gamma\left(\frac{s'+s+s_j-1}{2}\right)\Gamma\left(\frac{s'+s-s_j}{2}\right)$$

when u_j is an even Maass cusp form. If u_j is odd or constant, then the inner product is zero.

Proof. Unfolding,

$$\begin{aligned}\langle \mathcal{K}^\sharp(\,\cdot\,;s,s'),u_j\rangle &= \int_{\Gamma\backslash\mathbb{H}} \mathcal{K}^\sharp(z;s,s')\overline{u_j(z)}\,d\mu(z) \\ &= \int_0^\infty\int_0^1 \left(\sum_{m\neq 0} y^{s'+1/2}|m|^{s-1/2}K_{s-1/2}(2\pi|m|y)e^{2\pi i m x}\right)\overline{u_j(z)}\,\frac{dx\,dy}{y^2} \\ &= 2\sum_{m\neq 0}\nu_j(m)|m|^{s-1/2}\int_0^\infty y^{s'}K_{s-1/2}(2\pi|m|y)K_{\overline{s_j}-1/2}(2\pi|m|y)\,\frac{dy}{y}.\end{aligned}$$

Evaluating the integral [Iwaniec 2002, p. 205] yields

$$\langle \mathcal{K}^\sharp(\,\cdot\,;s,s'),u_j\rangle = \frac{L(u_j,s'-s+1/2)}{4\pi^{s'}\Gamma(s')}\prod\Gamma\left(\frac{s'\pm(s-1/2)\pm(\overline{s_j}-1/2)}{2}\right).$$

Using (9-3) and that $\overline{s_j} = 1-s_j$ finishes the proof. □

In the same way, when $\mathrm{Re}(r) = 1/2$,

$$\langle \mathcal{K}^\sharp(\,\cdot\,;s,s'),E(\,\cdot\,,r)\rangle \\ = \frac{L(\overline{E(\,\cdot\,,r)},s'-s+1/2)}{4\pi^{s'}\Gamma(s')}\prod\Gamma\left(\frac{s'\pm(s-1/2)\pm(\bar r-1/2)}{2}\right).$$

Further, $\overline{E(z,r)} = E(z,\bar r) = E(z,1-r)$, and with (9-5) we have shown the following:

Lemma 9.4. *For $s \in B_\rho$ and* $\operatorname{Re}(s') > \rho + 5$,

$$\langle \mathcal{K}^\sharp(\,\cdot\,; s, s'), E(\,\cdot\,, r)\rangle = \frac{\pi^{1/2-s}}{2\Gamma(s')\theta(1-r)} \Gamma\Big(\frac{s'+s-r}{2}\Big) \times \Gamma\Big(\frac{s'+s-1+r}{2}\Big)\theta\Big(\frac{s'-s+r}{2}\Big)\theta\Big(\frac{s'-s+1-r}{2}\Big).$$

Recall that $\theta(s) := \pi^{-s}\Gamma(s)\zeta(2s)$ as in (2-5). Let

$$\mathcal{K}_1(z; s, s') := \frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} E(z, s'-s+1),$$

$$\mathcal{K}_2(z; s, s') := \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s'-s+1/2)\Gamma\Big(\frac{s'+s+s_j-1}{2}\Big) \times \Gamma\Big(\frac{s'+s-s_j}{2}\Big)\frac{u_j(z)}{\langle u_j, u_j\rangle},$$

$$\mathcal{K}_3(z; s, s') := \frac{\pi^{1/2}}{\Gamma(s)\Gamma(s')}\frac{1}{4\pi i}\int_{(1/2)} \Gamma\Big(\frac{s'+s-r}{2}\Big)\Gamma\Big(\frac{s'+s-1+r}{2}\Big) \times \theta\Big(\frac{s'-s+r}{2}\Big)\theta\Big(\frac{s'-s+1-r}{2}\Big)\frac{E(z,r)}{\theta(1-r)}\,dr.$$

Assembling Proposition 9.2, (9-15) and Lemmas 9.3 and 9.4, we have proven the decomposition

$$\mathcal{K}(z; s, s') = \mathcal{K}_1(z; s, s') + \mathcal{K}_2(z; s, s') + \mathcal{K}_3(z; s, s') \tag{9-16}$$

for $s \in B_\rho$ and $\operatorname{Re}(s') > \rho + 5$. This agrees exactly with [Diaconu and Goldfeld 2007, (5.8)].

Clearly $\mathcal{K}_1(z; s, s')$ is a meromorphic function of s and s' in all of $\mathbb{C}$. The same is true for $\mathcal{K}_2(z; s, s')$ since the factors $L(u_j, s'-s+1/2)u_j(z)/\langle u_j, u_j\rangle$ have at most polynomial growth as $\operatorname{Im}(s_j) \to \infty$ while the Γ factors have exponential decay by Stirling's formula. See (9-2) and [Iwaniec 2002, §7 and §8] for the necessary bounds. The next result was first established in [Diaconu and Goldfeld 2007, §5].

Theorem 9.5. *The nonholomorphic kernel $\mathcal{K}(z; s, s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$.*

Proof. As we have discussed, $\mathcal{K}_1(z; s, s')$ and $\mathcal{K}_2(z; s, s')$ are meromorphic functions of $s, s' \in \mathbb{C}$. The poles of $\Gamma(w)$ are at $w = 0, -1, -2, \dots$, and $\theta(w)$ has poles exactly at $w = 0, 1/2$ (with residues $-1/2$ and $1/2$, respectively). Therefore, the integral in $\mathcal{K}_3(z; s, s')$ is certainly an analytic function of s and s' for $\sigma' > \sigma + 1/2$ and $\sigma > 1/2$ since the Γ and θ factors have exponential decay as $|r| \to \infty$. Next, consider s fixed (with $\sigma > 1/2$) and s' varying. Consider a point r_0 with $\operatorname{Re}(r_0) = 1/2$.

Let $B(r_0)$ be a small disc centered at r_0 and $B(1-r_0)$ an identical disc at $1-r_0$. By deforming the path of integration to a new path C to the left of $B(r_0)$ and to the right of $B(1-r_0)$, we may, by Cauchy's theorem, analytically continue $\mathcal{K}_3(z;\, s, s')$ to s' with $s'-s \in B(r_0)$. Let C_1 be a clockwise contour around the left side of $B(r_0)$ and C_2 be a counterclockwise contour around the right side of $B(1-r_0)$ so that $C=(1/2)+C_1+C_2$. For $s'-s$ inside C_1 (and $1-(s'-s)$ inside C_2), we have

$$\pi^{-1/2}\Gamma(s)\Gamma(s')\cdot\mathcal{K}_3(z;\, s, s') = \frac{1}{4\pi i}\int_C * = \frac{1}{4\pi i}\int_{(1/2)} * + \frac{1}{4\pi i}\int_{C_1} * + \frac{1}{4\pi i}\int_{C_2} *,$$

where $*$ denotes the integrand in the definition of $\mathcal{K}_3$. Then

$$\begin{aligned}\frac{1}{4\pi i}\int_{C_1} * &= \frac{-2\pi i}{4\pi i}\left(\operatorname*{Res}_{r=s'-s}\theta\left(\frac{s'-s+1-r}{2}\right)\right)\\ &\qquad\times\Gamma(s)\Gamma(s'-1/2)\frac{\theta(s'-s)}{\theta(1-s'+s)}E(z, s'-s)\\ &= \tfrac{1}{2}\Gamma(s)\Gamma(s'-1/2)\frac{\theta(s'-s)}{\theta(1-s'+s)}E(z, s'-s)\\ &= \tfrac{1}{2}\Gamma(s)\Gamma(s'-1/2)E(z, s-s'+1).\end{aligned}$$

We get the same result for $(1/4\pi i)\int_{C_2}$, and for all s' with $\sigma-1/2<\operatorname{Re}(s')<\sigma+1/2$, it follows that the continuation of $\mathcal{K}_3(z;\, s, s')$ is given by

$$\begin{aligned}&\pi^{-1/2}\Gamma(s)\Gamma(s')\cdot\mathcal{K}_3(z;\, s, s')\\ &\qquad\qquad= \Gamma(s)\Gamma(s'-1/2)E(z, s-s'+1) + \frac{1}{4\pi i}\int_{(1/2)} *. \qquad (9\text{-}17)\end{aligned}$$

Similarly, as s' crosses the line with real part $\sigma-1/2$, the term

$$-\Gamma(s-1/2)\Gamma(s')E(z, s'-s+1)$$

must be added to the right side of (9-17). Thus, for all s' with $1/2<\operatorname{Re}(s')<\sigma-1/2$, the continuation of $\mathcal{K}(z;\, s, s')$ is

$$\mathcal{K}(z;\, s, s') = \frac{\pi^{1/2}\Gamma(s'-1/2)}{\Gamma(s')}E(z, s-s'+1) + \mathcal{K}_2(z;\, s, s') + \mathcal{K}_3(z;\, s, s'). \quad (9\text{-}18)$$

Clearly, with (9-17) and (9-18) we have demonstrated the meromorphic continuation of $\mathcal{K}(z;\, s, s')$ to all $s, s' \in \mathbb{C}$ with $\operatorname{Re}(s), \operatorname{Re}(s') > 1/2$. The continuation to all $s, s' \in \mathbb{C}$ follows in the same way with further terms in the expression for $\mathcal{K}(z;\, s, s')$ appearing from the residues of the poles of $\Gamma((s'+s-r)/2)\Gamma((s'+s-1+r)/2)$ as $\operatorname{Re}(s'+s) \to -\infty$. □

Proposition 9.6. *We have the functional equation*

$$\mathcal{K}(z;\, s, s') = \mathcal{K}(z;\, s', s). \qquad (9\text{-}19)$$

Proof. We may verify (9-19) by comparing (9-16) with (9-18) and using that $\mathcal{K}_2(z; s, s') = \mathcal{K}_2(z; s', s)$ by (9-4) and $\mathcal{K}_3(z; s, s') = \mathcal{K}_3(z; s', s)$ by (2-6). There is a second, easier proof: with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, replace γ in (9-1) by $S\gamma$. □

Proposition 9.7. *For all $s, s' \in \mathbb{C}$ and any even Maass Hecke eigenform u_j,*

$$\langle \mathcal{K}(\,\cdot\,; s, s'), u_j \rangle = \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \Gamma\left(\frac{s'+s+s_j-1}{2}\right) \Gamma\left(\frac{s'+s-s_j}{2}\right) L^*\big(u_j, s'-s+\tfrac{1}{2}\big).$$

Proof. Since each u_j is orthogonal to Eisenstein series, we have by (9-16) (for $s \in B_\rho$ and $\mathrm{Re}(s') > \rho + 5$) that

$$\langle \mathcal{K}(\,\cdot\,; s, s'), u_j \rangle = \langle \mathcal{K}_2(\,\cdot\,; s, s'), u_j \rangle.$$

The result follows, extending to all $s, s' \in \mathbb{C}$ by analytic continuation. □

9C. *Nonholomorphic double Eisenstein series.* A similar argument to the proof of (5-2) shows that, for $\mathrm{Re}(s), \mathrm{Re}(s') > 1$ and $\mathrm{Re}(w) \geq 0$,

$$\zeta(w+2s)\zeta(w+2s')\mathcal{E}(z, w; s, s') = \frac{1}{2} \sum_{n=1}^{\infty} \frac{T_n \mathcal{K}(z; s, s')}{n^{w-1/2}}, \tag{9-20}$$

where, in this context [Goldfeld 2006, (3.12.3)], the appropriately normalized Hecke operator acts as

$$T_n \mathcal{K}(z) = \frac{1}{n^{1/2}} \sum_{\gamma \in \Gamma \backslash \mathcal{M}_n} \mathcal{K}(\gamma z).$$

For each Maass form, we have $T_n u_j = \nu_j(n) u_j$, and for the Eisenstein series, [Goldfeld 2006, Proposition 3.14.2] implies $T_n E(z, s) = n^{s-1/2} \sigma_{1-2s}(n) E(z, s)$. Therefore, as in (9-5),

$$\sum_{n=1}^{\infty} \frac{T_n E(z, s)}{n^{w-1/2}} = E(z, s) \sum_{n=1}^{\infty} \frac{\sigma_{1-2s}(n)}{n^{w-s}} = E(z, s)\zeta(w-s)\zeta(w+s-1).$$

Now choose any $\rho > 0$. For $s \in B_\rho$, $\mathrm{Re}(s) > 1$, $\mathrm{Re}(s') > \rho + 5$ and $\mathrm{Re}(w) \geq 0$, we may apply T_n to both sides of (9-16) and obtain

$$\begin{aligned}
&\zeta(w+2s)\zeta(w+2s')\mathcal{E}(z, w; s, s') \\
&= \frac{\pi^{1/2}\Gamma(s-1/2)}{2\Gamma(s)} \zeta(s'-s+w)\zeta(s-s'+w-1) E(z, s'-s+1) \\
&+ \frac{\pi^{1/2}}{4\Gamma(s)\Gamma(s')} \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s'-s+1/2) \Gamma\left(\frac{s'+s+s_j-1}{2}\right) \Gamma\left(\frac{s'+s-s_j}{2}\right) \\
&\times L(u_j, w-1/2) \frac{u_j(z)}{\langle u_j, u_j \rangle} + \frac{\pi^{1/2}}{2\Gamma(s)\Gamma(s')} \frac{1}{4\pi i} \int_{(1/2)} \theta\left(\frac{s'-s+r}{2}\right) \theta\left(\frac{s'-s+1-r}{2}\right) \\
&\qquad \times \Gamma\left(\frac{s'+s-r}{2}\right) \Gamma\left(\frac{s'+s-1+r}{2}\right) \zeta(w-r)\zeta(w-1+r) \frac{E(z, r)}{\theta(1-r)}\, dr.
\end{aligned} \tag{9-21}$$

Put

$$\Omega(s,s';r) := \theta\Big(\frac{s'+s-r}{2}\Big)\theta\Big(\frac{s'+s-1+r}{2}\Big) \times \theta\Big(\frac{s'-s+r}{2}\Big)\theta\Big(\frac{s'-s+1-r}{2}\Big) \Big/ \theta(1-r).$$

Define the completed double Eisenstein series as in (2-19) and write

$$U(z;s,s') := \sum_{\substack{j=1 \\ u_j \text{ even}}}^{\infty} L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2)\frac{u_j(z)}{\langle u_j, u_j\rangle}.$$

As in the last section, Ω and U have exponential decay as $|r|, |\mathrm{Im}(s_j)| \to \infty$. Specializing (9-21) to $w = s+s'$, we have proved the next result.

Lemma 9.8. *For $s \in B_\rho$, $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s') > \rho+5$,*

$$\mathscr{E}^*(z;s,s') = 2\theta(s)\theta(s')E(z;s+s') + 2\theta(1-s)\theta(s')E(z,s'-s+1) + U(z;s,s') + \frac{1}{2\pi i}\int_{(1/2)} \Omega(s,s';r)E(z,r)\,dr. \quad (9\text{-}22)$$

From this, we show the following:

Theorem 9.9. *The completed double Eisenstein series $\mathscr{E}^*(z;s,s')$ has a meromorphic continuation to all $s, s' \in \mathbb{C}$, and we have the functional equations*

$$\mathscr{E}^*(z;s,s') = \mathscr{E}^*(z;s',s), \quad (9\text{-}23)$$

$$\mathscr{E}^*(z;s,s') = \mathscr{E}^*(z;1-s,1-s'). \quad (9\text{-}24)$$

Proof. First note that (9-22) gives the meromorphic continuation of $\mathscr{E}^*(z;s,s')$ to all s and s' with $s \in B_\rho$ and $\mathrm{Re}(s') > \rho+5$. As in the proof of Theorem 9.5, we see that the further continuation in s' is given by (9-22) along with residues that are picked up as the line of integration is crossed; for $s \in B_\rho$ fixed and $\mathrm{Re}(s') \to -\infty$, the continuation of $\mathscr{E}^*(z;s,s')$ is given by (9-22) plus each of the following:

$$\begin{aligned} 2\theta(s)\theta(1-s')E(z,s-s'+1) &\quad \text{when } \mathrm{Re}(s') < \sigma+1/2, \\ -2\theta(1-s)\theta(s')E(z,s'-s+1) &\quad \text{when } \mathrm{Re}(s') < \sigma-1/2, \\ 2\theta(1-s)\theta(1-s')E(z,2-s-s') &\quad \text{when } \mathrm{Re}(s') < -\sigma+1/2, \\ -2\theta(s)\theta(s')E(z,s+s') &\quad \text{when } \mathrm{Re}(s') < -\sigma-1/2. \end{aligned}$$

We have therefore shown the meromorphic continuation of $\mathscr{E}^*(z;s,s')$ to all $s \in B_\rho$ and $s' \in \mathbb{C}$. Hence, for all s' with $\mathrm{Re}(s') < -\rho-4$, say, we have

$$\mathscr{E}^*(z;s,s') = 2\theta(1-s)\theta(1-s')E(z,2-s-s') + 2\theta(s)\theta(1-s')E(z,s-s'+1) + U(z;s,s') + \frac{1}{2\pi i}\int_{(1/2)} \Omega(s,s';r)E(z,r)\,dr. \quad (9\text{-}25)$$

The functional Equation (9-24) is a consequence of the easily checked symmetries $U(z; 1-s, 1-s') = U(z; s, s')$ and $\Omega(1-s, 1-s'; r) = \Omega(s, s'; r)$ and a comparison of (9-22) and (9-25). The Equation (9-23) has a similar proof or more simply follows from the definition (2-19). □

Proposition 9.10. *For any even Maass Hecke eigenform u_j (as in Section 9A) and all $s, s' \in \mathbb{C}$,*

$$\langle \mathscr{E}^*(\,\cdot\,; s, s'), u_j\rangle = L^*(u_j, s+s'-1/2)L^*(u_j, s'-s+1/2).$$

Proof. As in Proposition 9.7, only $U(z; s, s')$ in (9-22) will contribute to the inner product. □

With Theorem 9.9 and Proposition 9.10, we have proved Theorem 2.9.

10. Double Eisenstein series for general groups

We proved in Section 5A that for $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ the holomorphic double Eisenstein series $\boldsymbol{E}_{s,k-s}(z, w)$ may be continued to all s and w in $\mathbb{C}$ and satisfies a family of functional equations. That proof does not extend to groups where Hecke operators are not available. To show the continuation of $\boldsymbol{E}_{s,k-s,\mathfrak{a}}(z, w)$ for Γ an arbitrary Fuchsian group of the first kind, we first demonstrate a generalization of Proposition 2.5. Recall the definitions of u and v in (2-16) and ε_Γ in (4-1).

Theorem 10.1. *For s and w in the initial domain of convergence and even $k_1, k_2 \geq 0$ with $k = k_1 + k_2$, we have*

$$\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w) = 2^{\varepsilon_\Gamma - 1}\pi_{\mathrm{hol}}\big[(-1)^{k_2/2} y^{-k/2} E^*_{k_1,\mathfrak{a}}(\,\cdot\,, 1-u) E^*_{k_2,\mathfrak{a}}(\,\cdot\,, 1-v)/(2\pi^{k/2})\big]. \tag{10-1}$$

Proof. Let $g \in S_k(\Gamma)$, and set $\Gamma' = \sigma_{\mathfrak{a}}{}^{-1}\Gamma\sigma_{\mathfrak{a}}$. Then

$$\langle \boldsymbol{E}_{s,k-s,\mathfrak{a}}(\,\cdot\,, w), g\rangle = \int_{\Gamma'\backslash\mathbb{H}} \mathrm{Im}(\sigma_{\mathfrak{a}} z)^k \bar{g}(\sigma_{\mathfrak{a}} z) \boldsymbol{E}_{s,k-s,\mathfrak{a}}(\sigma_{\mathfrak{a}} z, w)\, d\mu z \tag{10-2}$$

$$= \int_{\Gamma'\backslash\mathbb{H}} y^k \frac{\bar{g}(\sigma_{\mathfrak{a}} z)}{\bar{j}(\sigma_{\mathfrak{a}}, z)^k} \sum_{\delta \in B\backslash\Gamma'} j(\delta, z)^{-k} \Bigg[\sum_{\substack{\gamma \in B\backslash\Gamma' \\ c_{\gamma\delta^{-1}} > 0}} (c_{\gamma\delta^{-1}})^{w-1} \bigg(\frac{j(\gamma, z)}{j(\delta, z)}\bigg)^{-s}\Bigg] d\mu z.$$

Since $g(\sigma_{\mathfrak{a}} z) j(\sigma_{\mathfrak{a}}, z)^{-k} \in S_k(\Gamma')$, we have

$$y^k \frac{\bar{g}(\sigma_{\mathfrak{a}} z)}{\bar{j}(\sigma_{\mathfrak{a}}, z)^k j(\delta, z)^k} = \mathrm{Im}(\delta z)^k \frac{\bar{g}(\sigma_{\mathfrak{a}} \delta z)}{\bar{j}(\sigma_{\mathfrak{a}}, \delta z)^k}.$$

Note also that $j(\gamma, z)/j(\delta, z) = j(\gamma\delta^{-1}, \delta z)$. Hence, (10-2) equals

$$2^{\varepsilon_\Gamma} \int_{\Gamma_\infty\backslash\mathbb{H}} y^k \frac{\bar{g}(\sigma_{\mathfrak{a}} z)}{\bar{j}(\sigma_{\mathfrak{a}}, z)^k} \Bigg[\sum_{\substack{\gamma \in B\backslash\Gamma' \\ c_\gamma > 0}} (c_\gamma)^{w-1} j(\gamma, z)^{-s}\Bigg] d\mu z. \tag{10-3}$$

Writing

$$\sum_{\substack{\gamma\in B\backslash\Gamma' \\ c_\gamma>0}} (c_\gamma)^{w-1} j(\gamma,z)^{-s} = \sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} (c_\gamma)^{w-1} \sum_{m\in\mathbb{Z}} j(\gamma,z+m)^{-s}$$

and using the Fourier expansion of g at $\mathfrak{a}$, $j(\sigma_\mathfrak{a},z)^{-k} g(\sigma_\mathfrak{a} z) = \sum_{n=1}^\infty a_{g,\mathfrak{a}}(n) e^{2\pi i n z}$, we get that (10-3) equals

$$2^{\varepsilon_\Gamma} \sum_{n=1}^\infty \overline{a_{g,\mathfrak{a}}}(n) \sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} \frac{1}{(c_\gamma)^{s+1-w}} \int_0^\infty \int_{-\infty}^\infty y^{k-2} \frac{e^{-2\pi i n x - 2\pi n y}}{(x+d_\gamma/c_\gamma+iy)^s}\,dx\,dy$$
$$= 2^{\varepsilon_\Gamma} I_k(s) \sum_{n=1}^\infty \frac{\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s}} \sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} \frac{e^{2\pi i n d_\gamma/c_\gamma}}{(c_\gamma)^{s+1-w}}$$

for

$$I_k(s) := \int_0^\infty \int_{-\infty}^\infty y^{k-2} \frac{e^{-2\pi i x - 2\pi y}}{(x+iy)^s}\,dx\,dy.$$

The inner integral over x may be evaluated with a formula of Laplace [Whittaker and Watson 1927, p. 246]:

$$\int_{-\infty}^\infty \frac{e^{-2\pi i x}}{(x+iy)^s}\,dx = e^{-2\pi y} \frac{(2\pi)^s}{\Gamma(s) e^{s i\pi/2}}$$

so that

$$I_k(s) = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \frac{(2\pi)^s}{\Gamma(s) e^{s i\pi/2}}.$$

With (4-2) and, for example, [Iwaniec 2002, Chapter 3], we recognize

$$\sum_{\substack{\gamma\in B\backslash\Gamma'/B \\ c_\gamma>0}} \frac{e^{2\pi i n d_\gamma/c_\gamma}}{(c_\gamma)^{2s}} = \sum_{\substack{\gamma\in \Gamma_\infty\backslash\Gamma'/\Gamma_\infty \\ c_\gamma>0}} \frac{e^{2\pi i n d_\gamma/c_\gamma}}{(c_\gamma)^{2s}} = \frac{Y_{\mathfrak{a}\mathfrak{a}}(n,s)}{\zeta(2s) n^{s-1}}.$$

It follows that we have shown

$$\langle \boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(\cdot,w), g\rangle = 2^{\varepsilon_\Gamma - 1} \frac{\zeta(2-2u)\Gamma(k-s)\Gamma(k-w)}{(2\pi)^{2k-s-w}} \sum_{n=1}^\infty \frac{Y_{\mathfrak{a}\mathfrak{a}}(n,1-v)\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s-v}}.$$

Reasoning as in the proof of [Diamantis and O'Sullivan 2010, (2.10)], we also find, for all even $k_1,k_2\geq 0$ with $k_1+k_2=k$,

$$\langle (-1)^{k_2/2} y^{-k/2} E^*_{k_1,\mathfrak{a}}(\cdot,1-u) E^*_{k_2,\mathfrak{b}}(\cdot,1-v)/(2\pi^{k/2}), g\rangle$$
$$= \frac{\zeta(2-2u)\Gamma(k-s)\Gamma(k-w)}{(2\pi)^{2k-s-w}} \sum_{n=1}^\infty \frac{Y_{\mathfrak{b}\mathfrak{a}}(n,1-v)\overline{a_{g,\mathfrak{a}}}(n)}{n^{k-s-v}}.$$

Since $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z,w)\in S_k(\Gamma)$ and $g\in S_k(\Gamma)$ is arbitrary, (10-1) follows. $\square$

Corollary 10.2. *The double Eisenstein series $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w)$ has a meromorphic continuation to all $s, w \in \mathbb{C}$ and as a function of z is always in $S_k(\Gamma)$. It satisfies the functional equation*

$$\boldsymbol{E}^*_{k-s,s,\mathfrak{a}}(z, w) = (-1)^{k/2}\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w). \tag{10-4}$$

Proof. Since $E^*_{k,\mathfrak{a}}(z, s)$ has a well-known continuation to all $s \in \mathbb{C}$, due to Selberg, the continuation of $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w)$ follows from (10-1). The change of variables $(s, w) \to (k-s, w)$ corresponds to $(u, v) \to (v, u)$, and so (10-4) is also a consequence of (10-1). □

If Γ has more than one cusp, then $\boldsymbol{E}^*_{s,k-s,\mathfrak{a}}(z, w)$ does not appear to possess a functional equation of the type (2-14) as $(s, w) \to (w, s)$. This corresponds on the right of (10-1) to $(u, v) \to (u, 1-v)$, and the functional equation for $E^*_{k_2,\mathfrak{a}}(\cdot, 1-v)$ involves a sum over cusps as in (4-4).

We remark that the functional Equation (10-4) also follows directly from (4-6) if $-I \in \Gamma$: replace γ and δ in the sum by $-\delta$ and γ, respectively.

Finally, it would be interesting to find the continuation in s and s' of the non-holomorphic double Eisenstein series $\mathscr{E}^*_{\mathfrak{a}}(z; s, s')$ for general groups. We expect that a similar decomposition to (9-22) should be true.

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diamant@mpim-bonn.mpg.de *School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom*

cosullivan@gc.cuny.edu *Department of Mathematics, The City University of New York Graduate Center, 365 Fifth Avenue, New York, NY 10016-4309, United States*

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Division algebras and quadratic forms over fraction fields of two-dimensional henselian domains

Yong Hu

Let K be the fraction field of a two-dimensional, henselian, excellent local domain with finite residue field k. When the characteristic of k is not 2, we prove that every quadratic form of rank ≥ 9 is isotropic over K using methods of Parimala and Suresh, and we obtain the local-global principle for isotropy of quadratic forms of rank 5 with respect to discrete valuations of K. The latter result is proved by making a careful study of ramification and cyclicity of division algebras over the field K, following Saltman's methods. A key step is the proof of the following result, which answers a question of Colliot-Thélène, Ojanguren and Parimala: for a Brauer class over K of prime order q different from the characteristic of k, if it is cyclic of degree q over the completed field K_v for every discrete valuation v of K, then the same holds over K. This local-global principle for cyclicity is also established over function fields of p-adic curves with the same method.

1. Introduction

Division algebras and quadratic forms over a field have been objects of interest in classical and modern theories of algebra and number theory. They may also be naturally and closely related to the study of semisimple algebraic groups of classical types. In recent years, there has been much interest in problems on division algebras and quadratic forms over function fields of two-dimensional integral schemes (which we call *surfaces*).

Mostly, surfaces that have been studied are those equipped with a dominant quasiprojective morphism to the spectrum of a normal, henselian, excellent local domain A. If A is of (Krull) dimension 0, these are algebraic surfaces over a field. Over function fields of these surfaces, de Jong [2004] and Lieblich [2011b] have proven remarkable theorems concerning the period-index problem. If A is of dimension 1, the surfaces of interest are called *arithmetic surfaces* by some authors. Over function fields of arithmetic surfaces, several methods have been developed to study division algebras and/or quadratic forms, for example in [Saltman 1997;

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2007; 2008; Lieblich 2011a; Harbater et al. 2009]. The methods pioneered in the series of papers by Saltman have been important ingredients in several works by others, including the proof of Parimala and Suresh [2010; 2012] of the fact that over a nondyadic p-adic function field every quadratic form of dimension ≥ 9 has a nontrivial zero. In contrast with the arithmetic case, it seems that in the case where A is two-dimensional, fewer results have been established in earlier work.

In this paper, we concentrate on the study of division algebras and quadratic forms over the function field K of a surface that admits a proper birational morphism to the spectrum of a two-dimensional, henselian, excellent local domain R. The spectrum $\operatorname{Spec} R$ will sometimes be called a *local henselian surface*, and a regular surface X equipped with a proper birational morphism $X \to \operatorname{Spec} R$ will be referred to as a *regular proper model* of $\operatorname{Spec} R$. As typical examples, one may take R to be the henselization at a closed point of an algebraic or an arithmetic surface or the integral closure of the ring $A[\![t]\!]$ of formal power series in a finite extension of its fraction field $\operatorname{Frac}(A[\![t]\!])$, where A is a complete discrete valuation ring. Note that the ring R need not be regular in our context.

Let k denote the residue field of R. When k is separably closed, many problems over the function field K (e.g., period-index, cyclicity of division algebras, u-invariant and local-global principle for quadratic forms of lower dimension) have been solved by Colliot-Thélène, Ojanguren and Parimala [Colliot-Thélène et al. 2002]. In the case with k finite, only the local-global principle for quadratic forms of rank 3 or 4 is proved in that paper. Harbater, Hartmann, and Krashen [Harbater et al. 2011] obtained some results with less restrictive assumptions on the residue field but more restrictions on the shape of the ring R.

While the proofs pass through many analyses on ramification of division algebras, our primary goals are the following two theorems on quadratic forms:

Theorem 1.1. *Let R be a two-dimensional, henselian, excellent local domain with finite residue field k and fraction field K. Assume that 2 is invertible in k. Let Ω_R be the set of discrete valuations of K that correspond to codimension-1 points of regular proper models of* $\operatorname{Spec} R$.

Then quadratic forms of rank 5 over K satisfy the local-global principle with respect to discrete valuations in Ω_R; namely, if a quadratic form ϕ of rank 5 over K has a nontrivial zero over the completed field K_v for every $v \in \Omega_R$, then ϕ has nontrivial zero over K.

The next theorem amounts to saying that the field K has u-invariant (page 1945) equal to 8:

Theorem 1.2. *Let R be a two-dimensional, henselian, excellent local domain with finite residue field k and fraction field K. Assume that 2 is invertible in k.*

Then every quadratic form of rank ≥ 9 has a nontrivial zero over K.

Over the function field of an arithmetic surface over a complete discrete valuation ring, the same local-global principle as in Theorem 1.1 is proved for all quadratic forms of rank ≥ 3 in [Colliot-Thélène et al. 2012, Theorem 3.1] by using the patching method of [Harbater et al. 2009]. In the case that $R = A[\![t]\!]$ is a ring of formal power series in one variable over a complete discrete valuation ring A, the same type of local-global principle has been proven for quadratic forms of rank ≥ 5 in [Hu 2012b] using the arithmetic case established by Colliot-Thélène, Parimala and Suresh [Colliot-Thélène et al. 2012]. (See Remark 4.3 for more information.) However, in the general local henselian case, the lack of an appropriate patching method has been an obstacle to proving the parallel local-global result. So for a field K as in Theorem 1.1, the local-global principle for quadratic forms of rank 6, 7 or 8 remains open.

In the case of a p-adic function field, it is known that at least three methods can be used to determine the u-invariant: the cohomological method of [Parimala and Suresh 2010], the patching method of [Harbater et al. 2009] and the method of [Leep 2013], which is built on results from [Heath-Brown 2010]. But in the case of the function field of a local henselian surface considered here, not all of them seem to still work. For the fraction field of a power series ring $R = A[\![t]\!]$ over a complete discrete valuation ring with finite residue field, it is known that the u-invariant is at most 8 [Harbater et al. 2009, Corollary 4.19]. Our proof of this result for general R (with finite residue field) follows the method of Parimala and Suresh [2010; 2012].

Theorem 1.2 implies that the u-invariants $u(K)$ of the fraction field K and $u(k)$ of the residue field k satisfy the relation $u(K) = 4u(k)$ when the residue field k is finite. A question of Suresh asks if this relation still holds when k is an arbitrary field of characteristic $\neq 2$. The answer is known to be affirmative in some other special cases, but the general case seems to remain open. (See Question 4.8 for more information.)

As a byproduct, we also obtain (under the assumption of Theorem 1.2) a local-global principle for torsors of the special orthogonal group $\mathrm{SO}(\phi)$ of a quadratic form ϕ of rank ≥ 2 over K (Theorem 4.9). In fact, Theorem 1.2 will also be useful in the study of local-global principle for torsors under some simply connected groups of classical types over K [Hu 2012a].

The main tools we will need to prove Theorem 1.1 come from technical analyses of ramification behaviors of division algebras using methods developed by Saltman [1997; 2007; 2008]. A key ingredient is the following result:

Theorem 1.3. *Let R be a two-dimensional, henselian, excellent local domain with finite residue field k, q a prime number unequal to the characteristic of k, K the fraction field of R and $\alpha \in \mathrm{Br}(K)$ a Brauer class of order q. Let Ω_R be the set of discrete valuations of K that correspond to codimension-*1 *points of regular proper models of* Spec *R.*

If for every $v \in \Omega_R$, the Brauer class

$$\alpha \otimes_K K_v \in \mathrm{Br}(K_v)$$

is represented by a cyclic algebra of degree q over the completed field K_v, then α is represented by a cyclic algebra of degree q over K.

Actually, as the same proof applies to the function field of a p-adic curve, a similar result over p-adic function fields, which seems not to have been treated in the literature, holds as well (Theorem 3.21). Note that a special case of Theorem 1.3 answers a question raised in [Colliot-Thélène et al. 2002, Remark 3.7].

Here is a brief description of the organization of the paper, together with some auxiliary results obtained in the process of proving the above-mentioned theorems.

Section 2 is concerned with preliminary reviews on Brauer groups and Galois symbols. The goal is to introduce some basic notions and recall standard results that we will frequently use later. In Section 3, we recall some of the most useful techniques and results from Saltman's papers and we prove Theorem 1.3. We also prove over the field K considered in Theorem 1.1 two local analogs of more global results Saltman had shown: that the index of a Brauer class of period prime to the residue characteristic divides the square of its period and that a class of prime index q that is different from the residue characteristic is represented by a cyclic algebra of degree q. This last statement is proved by generalizing a result of Saltman on modified Picard groups. Finally, we will concentrate on results about quadratic forms in Section 4. The proofs of Theorems 1.1 and 1.2 build upon the work of Parimala and Suresh and on a result from Saito's class field theory for two-dimensional local rings [Saito 1987].

To ease the discussions, we fix some notations and terminological conventions for all the rest of the paper.

- All schemes are assumed to be noetherian and separated. All rings under consideration will be noetherian (commutative with 1).
- A *curve* or *surface* means an integral scheme of dimension 1 or 2, respectively.
- Given a scheme X, we denote by $\mathrm{Br}(X)$ its cohomological Brauer group, i.e., $\mathrm{Br}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)$. If $X = \operatorname{Spec} A$ is affine, we write $\mathrm{Br}(A)$ instead of $\mathrm{Br}(\operatorname{Spec} A)$.
- If X is a scheme and $x \in X$, we write $\kappa(x)$ for the residue field of x, and if $Z \subseteq X$ is an irreducible closed subset with generic point η, then we write $\kappa(Z) := \kappa(\eta)$.
- The reduced closed subscheme of a scheme X will be written as X_{red}.
- A discrete valuation will always be assumed normalized (nontrivial) and of rank 1.

- Given a field F and a scheme X together with a morphism $\operatorname{Spec} F \to X$, $\Omega(F/X)$ will denote the set of discrete valuations of F that have a center on X. If $X = \operatorname{Spec} A$ is affine, we write $\Omega(F/A)$ instead of $\Omega(F/\operatorname{Spec} A)$.
- Given a scheme X and $i \in \mathbb{N}$, we denote by $X^{(i)}$ the set of codimension-i points of X, i.e., $X^{(i)} := \{x \in X \mid \dim \mathcal{O}_{X,x} = i\}$. If X is a normal integral scheme with function field F, we will sometimes identify $X^{(1)}$ with the set of discrete valuations of F corresponding to points in $X^{(1)}$.
- For an abelian group A and a positive integer n, let $A[n]$ denote the subgroup consisting of n-torsion elements of A and let $A/n = A/nA$ so that there is a natural exact sequence
$$0 \to A[n] \to A \xrightarrow{n} A \to A/n \to 0.$$
- Given a field F, let F_s be a fixed separable closure of F and $G_F := \operatorname{Gal}(F_s/F)$ the absolute Galois group. Galois cohomology $H^i(G_F, \cdot)$ of the group G_F will be written $H^i(F, \cdot)$ instead.
- R will always denote a two-dimensional, henselian, excellent local domain with fraction field K and residue field k.
- By a *regular proper model* of $\operatorname{Spec} R$, we mean a regular integral scheme $\mathscr{X}$ equipped with a proper birational morphism $\mathscr{X} \to \operatorname{Spec} R$. A discrete valuation of K that corresponds to a codimension-1 point of a regular proper model of $\operatorname{Spec} R$ will be referred to as a *divisorial valuation* of K. We denote by Ω_R the set of divisorial valuations of K.

2. Some preliminaries

Brauer groups of low-dimensional schemes. Since we will often use arguments related to Brauer groups of curves or surfaces, let us briefly review some basic facts in this respect.

Theorem 2.1 [Grothendieck 1968a; Colliot-Thélène et al. 2002]. *Let X be a (noetherian) scheme of dimension d.*

(i) *If $d \le 1$, then the natural map* $\operatorname{Br}(X) \to \operatorname{Br}(X_{\mathrm{red}})$ *is an isomorphism.*

(ii) *If X is regular and integral with function field F, then the natural map* $\operatorname{Br}(X) \to \operatorname{Br}(F)$ *is injective.*

(iii) *If X is regular, integral with function field F and of dimension $d \le 2$, then* $\operatorname{Br}(X) = \bigcap_{x \in X^{(1)}} \operatorname{Br}(\mathcal{O}_{X,x})$ *inside* $\operatorname{Br}(F)$.

(iv) *Let A be a henselian local ring, and let $X \to \operatorname{Spec} A$ be a proper morphism whose closed fiber X_0 has dimension ≤ 1. If X is regular and of dimension 2, then the natural map* $\operatorname{Br}(X) \to \operatorname{Br}(X_0)$ *is an isomorphism.*

Proof. See [Colliot-Thélène et al. 2002, Lemma 1.6] for (i), [Grothendieck 1968a, Corollary 1.8] for (ii), [Grothendieck 1968a, Corollary 2.2, Proposition 2.3] for (iii) and [Colliot-Thélène et al. 2002, Theorem 1.8(c)] for (iv). □

The following property for fields, already considered in [Saltman 1997], will be of interest to us:

Definition 2.2. We say a field k has *property* B_1 or k is a B_1 *field* if, for every proper regular integral (not necessarily geometrically integral) curve C over the field k, one has $\mathrm{Br}(C) = 0$.

Example 2.3. Here are some examples of B_1 fields:

(1) A separably closed field k has property B_1 [Grothendieck 1968b, Corollary 5.8].

(2) A finite field k has property B_1. This is classical by class field theory; see also [Grothendieck 1968b, p. 97].

(3) If k has property B_1, then so does any algebraic field extension k' of k.

Proposition 2.4. *Let k be a B_1 field.*

(i) *For any proper k-scheme X of dimension ≤ 1, one has* $\mathrm{Br}(X) = 0$.

(ii) *The cohomological dimension* $\mathrm{cd}(k)$ *of k is ≤ 1; i.e., for every torsion G_k-module A, $H^i(k, A) = 0$ for all $i \geq 2$.*

(iii) *If the characteristic of k is not 2, then every quadratic form of rank ≥ 3 has a nontrivial zero over k.*

Proof. (i) By Theorem 2.1(i), we may assume X is reduced.

For the zero-dimensional case, it suffices to prove that $\mathrm{Br}(L) = 0$ for a finite extension field L of k. Indeed, the B_1 property implies that $\mathrm{Br}(\mathbb{P}^1_L) = 0$. The existence of L-rational points on $\mathbb{P}^1_L$ shows that the natural map $\mathrm{Br}(L) \to \mathrm{Br}(\mathbb{P}^1_L)$ induced by the structural morphism $\mathbb{P}^1_L \to \mathrm{Spec}\, L$ is injective. Hence, $\mathrm{Br}(L) = 0$.

Now assume that X is reduced of dimension 1. Let $X' \to X$ be the normalization of X. By [Colliot-Thélène et al. 2002, Proposition 1.14], there is a zero-dimensional closed subscheme D of X such that the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X') \times \mathrm{Br}(D)$ is injective. Now X' is a disjoint union of finitely many proper regular k-curves, so $\mathrm{Br}(X') = 0$ by the B_1 property. We have $\mathrm{Br}(D) = 0$ by the zero-dimensional case, whence $\mathrm{Br}(X) = 0$ as desired.

(ii) As a special case of (i), we have $\mathrm{Br}(k') = 0$ for every finite separable extension field k' of k. This implies $\mathrm{cd}(k) \leq 1$ by [Serre 1994, p. 88, Proposition 5].

(iii) By (ii), we have in particular $\mathrm{Br}(k)[2] = H^2(k, \mu_2) = 0$. Thus, every quaternion algebra over k is split and the associated quadric has a k-rational point. Up to a scalar multiple, every nonsingular three-dimensional quadratic form is associated to a quaternion algebra and hence isotropic. □

The following corollary is essentially proven in [Colliot-Thélène et al. 2002, Corollaries 1.10 and 1.11]:

Corollary 2.5. *Let A be a (noetherian) henselian local ring, and let $X \to \operatorname{Spec} A$ be a proper morphism whose closed fiber X_0 is of dimension ≤ 1. Assume that the residue field of A has property B_1.*

If X is regular and of dimension 2, then $\operatorname{Br}(X) = 0$.

Proof. Combine Theorem 2.1(iv) and Proposition 2.4(i). □

Symbols and unramified cohomology. This subsection is devoted to a quick review of a few standard facts about Galois symbols and residue maps. For more information, we refer the reader to [Colliot-Thélène 1995].

Let F be a field and v a discrete valuation of F with valuation ring $\mathcal{O}_v$ and residue field $\kappa(v)$. Let $n > 0$ be a positive integer unequal to the characteristic of $\kappa(v)$. Let μ_n be the Galois module on the group of n-th roots of unity. For an integer $j \ge 1$, let $\mu_n^{\otimes j}$ denote the Galois module given by the tensor product of j copies of μ_n, and define

$$\mu_n^{\otimes 0} := \mathbb{Z}/n \quad \text{and} \quad \mu_n^{\otimes(-j)} := \operatorname{Hom}(\mu_n^{\otimes j}, \mathbb{Z}/n),$$

where as usual $\mathbb{Z}/n$ is regarded as a trivial Galois module. Kummer theory gives a canonical isomorphism $H^1(F, \mu_n) \cong F^*/F^{*n}$. For an element $a \in F^*$, we denote by (a) its canonical image in $H^1(F, \mu_n) = F^*/F^{*n}$. For $\alpha \in H^i(F, \mu_n^{\otimes j})$, the cup product $\alpha \cup (a) \in H^{i+1}(F, \mu_n^{\otimes(j+1)})$ will be simply written as (α, a). In particular, if $a_1, \dots, a_i \in F^*$, $(a_1, \dots, a_i) \in H^i(F, \mu_n^{\otimes i})$ will denote the cup product $(a_1) \cup \dots \cup (a_i) \in H^i(F, \mu_n^{\otimes i})$. Such a cohomology class is called a *symbol class*.

By standard theories from Galois or étale cohomology, there are *residue homomorphisms* for all $i \ge 1$ and all $j \in \mathbb{Z}$

$$\partial_v^{i,j} : H^i(F, \mu_n^{\otimes j}) \to H^{i-1}(\kappa(v), \mu_n^{\otimes(j-1)})$$

that fit into a long exact sequence

$$\cdots \to H^i_{\text{ét}}(\mathcal{O}_v, \mu_n^{\otimes j}) \to H^i(F, \mu_n^{\otimes j}) \xrightarrow{\partial_v^{i,j}} H^{i-1}(\kappa(v), \mu_n^{\otimes(j-1)}) \to H^{i+1}_{\text{ét}}(\mathcal{O}_v, \mu_n^{\otimes j}) \to \cdots.$$

An element $\alpha \in H^i(F, \mu_n^{\otimes j})$ is called *unramified* at v if $\partial_v^{i,j}(\alpha) = 0$.

Now consider the case of Brauer groups. By Theorem 2.1(ii), $\operatorname{Br}(\mathcal{O}_v)$ gets identified with a subgroup of $\operatorname{Br}(F)$. An element $\alpha \in \operatorname{Br}(F)$ is called *unramified* at v if it lies in the subgroup $\operatorname{Br}(\mathcal{O}_v) \subseteq \operatorname{Br}(F)$. If $n > 0$ is a positive integer that is invertible in $\kappa(v)$, then an element $\alpha \in \operatorname{Br}(F)[n]$ is unramified at v if and only if $\partial_v(\alpha) = 0$, where ∂_v denotes the residue map

$$\partial_v = \partial_v^{2,1} : \operatorname{Br}(F)[n] = H^2(F, \mu_n) \to H^1(\kappa(v), \mathbb{Z}/n).$$

As we will frequently speak of ramification of division algebras, the above residue map $\partial_v = \partial_v^{2,1}$ will often be called the *ramification map* and denoted by ram_v.

Let X be a scheme equipped with a morphism $\mathrm{Spec}\, F \to X$. The subgroup

$$\mathrm{Br}_{\mathrm{nr}}(F/X) := \bigcap_{v\in\Omega(F/X)} \mathrm{Br}(\mathbb{O}_v) \subseteq \mathrm{Br}(F),$$

where $\Omega(F/X)$ denotes the set of discrete valuations of K that have a center on X, is referred to as the (relative) *unramified Brauer group* of F over X. A Brauer class $\alpha \in \mathrm{Br}(F)$ is called *unramified over* X if it lies in the subgroup $\mathrm{Br}_{\mathrm{nr}}(F/X)$. We say a field extension M/F *splits all ramification of* $\alpha \in \mathrm{Br}(F)$ *over* X if $\alpha_M \in \mathrm{Br}(M)$ is unramified over X. When $X = \mathrm{Spec}\, A$ is affine, we write $\mathrm{Br}_{\mathrm{nr}}(F/A)$ instead of $\mathrm{Br}_{\mathrm{nr}}(F/\mathrm{Spec}\, A)$.

If X is an integral scheme with function field F and if $X \to Y$ is a proper morphism, then $\Omega(F/X) = \Omega(F/Y)$ and hence $\mathrm{Br}_{\mathrm{nr}}(F/X) = \mathrm{Br}_{\mathrm{nr}}(F/Y)$. If X is a regular curve or surface with function field F, then Theorem 2.1 implies that $\mathrm{Br}_{\mathrm{nr}}(F/X) \subseteq \mathrm{Br}(X)$.

Note that for any field κ, the Galois cohomology group $H^1(\kappa, \mathbb{Q}/\mathbb{Z})$ is identified with the group of characters of the absolute Galois group G_κ, i.e., the group $\mathrm{Hom}_{\mathrm{cts}}(G_\kappa, \mathbb{Q}/\mathbb{Z})$ of continuous homomorphisms $f : G_\kappa \to \mathbb{Q}/\mathbb{Z}$. Any character $f \in \mathrm{Hom}_{\mathrm{cts}}(G_\kappa, \mathbb{Q}/\mathbb{Z})$ must have image of the form $\mathbb{Z}/m \subseteq \mathbb{Q}/\mathbb{Z}$ for some positive integer m, and its kernel is equal to $G_{\kappa'}$ for some cyclic Galois extension κ'/κ of degree m. There is a generator $\sigma \in \mathrm{Gal}(\kappa'/\kappa)$ such that $f(\sigma) = 1 + m\mathbb{Z} \in \mathbb{Z}/m$. The function $f \in \mathrm{Hom}_{\mathrm{cts}}(G_\kappa, \mathbb{Q}/\mathbb{Z})$ is uniquely determined by the pair $(\kappa'/\kappa, \sigma)$. In this paper, we will often write an element of $H^1(\kappa, \mathbb{Q}/\mathbb{Z})$ in this way. In particular, the ramification $\mathrm{ram}_v(\alpha) \in H^1(\kappa(v), \mathbb{Z}/n)$ of a Brauer class $\alpha \in \mathrm{Br}(F)[n]$ at a discrete valuation $v \in \Omega(F/X)$ will be represented in this way.

Let $\chi \in H^1(F, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cts}}(G_F, \mathbb{Q}/\mathbb{Z})$ be a character of G_F with image $\mathbb{Z}/n \subseteq \mathbb{Q}/\mathbb{Z}$, represented by a pair $(L/F, \sigma)$; i.e., L/F is a finite cyclic Galois extension of degree n such that

$$G_L = \mathrm{Ker}(\chi : G_F \to \mathbb{Q}/\mathbb{Z})$$

and $\sigma \in \mathrm{Gal}(L/F)$ is a generator such that $\chi(\sigma) = 1 + n\mathbb{Z} \in \mathbb{Z}/n$. Recall that [Gille and Szamuely 2006, §2.5] the *cyclic algebra* (χ, b) associated with χ and an element $b \in F^*$ is the F-algebra generated by L and a word y subject to the following multiplication relations:

$$y^n = b \quad \text{and} \quad \lambda y = y\sigma(\lambda) \quad \text{for all } \lambda \in L.$$

It is a standard fact that (χ, b) is a central simple algebra of degree n over F. The class of the cyclic algebra (χ, b) in $\mathrm{Br}(F)[n] = H^2(F, \mu_n)$ coincides with the cup product of $\chi \in H^1(F, \mathbb{Z}/n)$ and $(b) \in H^1(F, \mu_n)$.

If $\mu_n \subseteq F$, then by Kummer theory L is of the form $L = F(\sqrt[n]{a})$ for some $a \in F^*$. There is a primitive n-th root of unity $\xi_n \in F$ such that $\sigma(\sqrt[n]{a}) = \xi_n \sqrt[n]{a}$. The cyclic algebra (χ, b) is isomorphic to the F-algebra $(a, b)_{\xi_n}$, which by definition is the F-algebra generated by two words x, y subject to the relations

$$x^n = a, \quad y^n = b \quad \text{and} \quad xy = \xi_n yx.$$

Conversely, when F contains a primitive n-th root of unity ξ_n, the algebra $(a, b)_{\xi_n}$ associated to elements $a, b \in F^*$ is isomorphic to (χ, b), where $\chi \in H^1(F, \mathbb{Q}/\mathbb{Z})$ is the character represented by the cyclic extension $L/F = F(\sqrt[n]{a})/F$ and the F-automorphism $\sigma \in \mathrm{Gal}(L/F)$ that sends $\sqrt[n]{a}$ to $\xi_n \sqrt[n]{a}$. The class of the algebra $(a, b)_{\xi_n}$ in $\mathrm{Br}(F)$ will be denoted by (a, b) when the degree n and the choice of $\xi_n \in F$ are clear from the context. This notation is compatible with the notion of symbol classes via the isomorphism $\mathrm{Br}(F)[n] = H^2(F, \mu_n) \cong H^2(F, \mu_n^{\otimes 2})$ corresponding to the choice of $\xi_n \in F$.

3. Division algebras over local henselian surfaces

In this section, we first recall a number of techniques in Saltman's method of detecting ramification of division algebras [Saltman 1997; 2007] and then we will prove Theorem 1.3.

Ramification of division algebras over surfaces. In this subsection, let X be a regular excellent surface and let F be the function field of X. By resolution of embedded singularities [Shafarevich 1966, Theorem on p. 38 and Remark on p. 43; Lipman 1975, p. 193], for any effective divisor D on X, there exists a regular surface X' together with a proper birational morphism $X' \to X$, obtained by a sequence of blow-ups, such that the total transform D' of D in X' is a *simple normal crossing* (snc) divisor (i.e., the reduced subschemes on the irreducible components of D' are regular curves and they meet transversally everywhere). We will use this result without further reference.

Let n be a positive integer that is invertible on X, and let $\alpha \in \mathrm{Br}(F)[n]$ be a Brauer class of order dividing n. For any discrete valuation $v \in \Omega(F/X)$, let ram_v denote the ramification map (or the residue map)

$$\mathrm{ram}_v = \partial_v^{2,1} : \mathrm{Br}(F)[n] = H^2(F, \mu_n) \to H^1(\kappa(v), \mathbb{Z}/n).$$

If $v = v_C$ is the discrete valuation centered at the generic point of a curve $C \subseteq X$, we write $\mathrm{ram}_C = \mathrm{ram}_{v_C}$. The *ramification locus* of $\alpha \in \mathrm{Br}(F)[n]$ on X, denoted $\mathrm{Ram}_X(\alpha)$, is by definition the (finite) union of curves $C \subseteq X$ such that $\mathrm{ram}_C(\alpha) \neq 0 \in H^1(\kappa(C), \mathbb{Z}/n)$. The *ramification divisor* of α on X, denoted again by $\mathrm{Ram}_X(\alpha)$ by abuse of notation, is the reduced divisor supported on the ramification locus. After several blow-ups, we may assume $\mathrm{Ram}_X(\alpha)$ is an snc divisor on X.

Definition 3.1 [Saltman 2007, §2]. Let X, F and α be as above. Assume that $\mathrm{Ram}_X(\alpha)$ is an snc divisor on X. A closed point $P \in X$ is called

(1) a *distant point* for α if $P \notin \mathrm{Ram}_X(\alpha)$,

(2) a *curve point* for α if P lies on one and only one irreducible component of $\mathrm{Ram}_X(\alpha)$ and

(3) a *nodal point* for α if P lies on two different irreducible components of $\mathrm{Ram}_X(\alpha)$.

Saltman essentially derived the following theorem from a local study of a Brauer class at closed points in its ramification locus [Saltman 1997, Proposition 1.2]:

Theorem 3.2 [Saltman 1997, Theorem 2.1]. *Let X be a regular excellent surface that is quasiprojective over a ring, F the function field of X, $n > 0$ a positive integer that is invertible on X and $\alpha \in \mathrm{Br}(F)[n]$. Assume $\mu_n \subseteq F$.*

Then there exist $f, g \in F^*$ such that the field extension $M/F := F(\sqrt[n]{f}, \sqrt[n]{g})/F$ splits all ramification of α over X, i.e., $\alpha_M \in \mathrm{Br}_{\mathrm{nr}}(M/X)$.

(Although our setup here differs from that of Saltman's Theorem 2.1, a careful verification shows that his proof — with Gabber's corrections given in [Saltman 1998] — still works. One can also find a proof of Theorem 3.2 in [Brussel 2010, Lemma 7.8]. When n is prime, a stronger statement holds; see [Saltman 2008, Theorem 7.13] and Proposition 3.11.)

Remark 3.3. Let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. By resolution of singularities for surfaces (see [Lipman 1975; 1978]), there exists a regular proper model $\mathscr{X} \to \mathrm{Spec}\, R$. The structural morphism $\mathscr{X} \to \mathrm{Spec}\, R$ is actually projective by [Grothendieck 1967, IV.21.9.13]. So Theorem 3.2 applies to such a regular proper model $\mathscr{X} \to \mathrm{Spec}\, R$.

If the residue field k of R is finite, Theorem 3.2 has the following refined form over the fraction field K:

Let $n > 0$ be a positive integer that is invertible in the finite residue field k. Assume that $\mu_n \subseteq R$. Then for any finite collection of Brauer classes $\alpha_i \in \mathrm{Br}(K)[n]$, $1 \le i \le m$, there exist $f, g \in K^*$ such that the field extension $M/K := K(\sqrt[n]{f}, \sqrt[n]{g})/K$ splits all the α_i, $i = 1, \dots, m$.

In the literature, this result has been established in the case where K is a function field of a p-adic curve and where n is a prime number, and the proof is essentially an observation of Gabber and Colliot-Thélène [Colliot-Thélène 1998; Hoffmann and Van Geel 1998, Theorem 2.5]. One may verify that essentially the same arguments work in the local henselian case considered here.

We will need the following analog of [Saltman 1997, Theorem 3.4] in the sequel:

Theorem 3.4. *Let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. Let $n > 0$ be a positive integer that is invertible in k. Assume that k is a B_1 field.*

Then any Brauer class $\alpha \in \mathrm{Br}(K)$ of order n has index dividing n^2.

Proof. This follows on parallel lines along the proof of [Saltman 1997, Theorem 3.4] with suitable substitutions of the ingredients used in the case of p-adic function fields. For the sake of the reader's convenience, we recall the argument.

We may assume $n = q^r$ is a power of a prime number q. Let $\mathscr{X} \to \operatorname{Spec} R$ be a regular proper model. For any finite field extension K'/K, the integral closure R' of R in K' satisfies the same assumptions as R and K' is the function field of a regular proper model $\mathscr{X}'$ of $\operatorname{Spec} R'$. One has $\Omega(K'/\mathscr{X}') = \Omega(K'/\mathscr{X})$ and $\mathrm{Br}_{\mathrm{nr}}(K'/\mathscr{X}) = \mathrm{Br}_{\mathrm{nr}}(K'/\mathscr{X}') = 0$ by Theorem 2.1(iii) and Corollary 2.5. So it suffices to find a finite separable field extension K'/K of degree $q^{2r}m$ with $q \nmid m$ such that K'/K splits all ramification of α over $\mathscr{X}$.

Now we proceed by induction on r. First assume $r = 1$. Then the result is immediate from Theorem 3.2 if $\mu_q \subseteq F$. The general case follows by passing to the extension $F(\mu_q)/F$, which has degree prime to q.

For general r, the inductive hypothesis applied to the Brauer class $q\alpha$ implies that there is a separable field extension K'/K splitting all ramification of $q\alpha$ over $\mathscr{X}$, which has degree $q^{2r-2}m'$, where $q \nmid m'$. But $q\alpha_{K'} = 0 \in \mathrm{Br}(K')$ by Corollary 2.5. By the case with $r = 1$, we can find a separable extension K''/K' of degree q^2m'' with $q \nmid m''$ that splits all ramification of $\alpha_{K'}$ over $\mathscr{X}$. Now K''/K is a separable extension of degree $[K'' : K] = q^{2r}m$ with $m = m'm''$ coprime to q and K''/K splits all ramification of α over $\mathscr{X}$, as desired. □

We will give an example of a Brauer class $\alpha \in \mathrm{Br}(K)$ of order n that is of index n^2 in Example 3.18.

Classification of nodal points. To prove further results, we need more analysis on ramification at nodal points, for which we briefly recall in this subsection some basic notions and results due to Saltman. The reader is referred to [Saltman 2007, §§2–3] or [Brussel 2010, §§7–8] for more details.

Let X be a regular excellent surface with function field F, and let q be a prime number that is invertible on X. Let $\alpha \in \mathrm{Br}(F)[q]$. Assume that $\mathrm{Ram}_X(\alpha)$ is an snc divisor on X. Let $P \in X$ be a nodal point for α (Definition 3.1), lying on two distinct irreducible components C_1 and C_2 of $\mathrm{Ram}_X(\alpha)$. Let $\chi_1 = \mathrm{ram}_{C_1}(\alpha)$ and $\chi_2 = \mathrm{ram}_{C_2}(\alpha)$ be respectively the ramifications of α at C_1 and C_2. Since the natural sequence induced by residue maps

$$H^2(F, \mu_q) \to \bigoplus_{v \in (\operatorname{Spec} \mathcal{O}_{X,P})^{(1)}} H^1(\kappa(v), \mathbb{Z}/q) \to H^0(\kappa(P), \mu_q^{\otimes(-1)})$$

is a complex (see [Kato 1986] or [Colliot-Thélène 2006, Proposition 2.3]), $\chi_1 = \mathrm{ram}_{C_1}(\alpha) \in H^1(\kappa(C_1), \mathbb{Z}/q)$ is unramified at P if and only if $\chi_2 = \mathrm{ram}_{C_2} \in H^1(\kappa(C_2), \mathbb{Z}/q)$ is unramified at P.

Definition 3.5 [Saltman 2007, §§2–3]. Let X, F, q, α and so on be as above. Assume that $\mathrm{Ram}_X(\alpha)$ is an snc divisor on X. Let $P \in X$ be a nodal point for α, lying on two distinct irreducible components C_1 and C_2 of $\mathrm{Ram}_X(\alpha)$.

(1) P is called a *cold point* for α if $\chi_1 = \mathrm{ram}_{C_1}(\alpha) \in H^1(\kappa(C_1), \mathbb{Z}/q)$ (and hence also $\chi_2 = \mathrm{ram}_{C_2}(\alpha) \in H^1(\kappa(C_2), \mathbb{Z}/q)$) is ramified at P.

(2) Assume now χ_1 and χ_2 are unramified at P so that they lie respectively in $H^1(\mathcal{O}_{C_1,P}, \mathbb{Z}/q)$ and $H^1(\mathcal{O}_{C_2,P}, \mathbb{Z}/q)$. Let $\chi_i(P) \in H^1(\kappa(P), \mathbb{Z}/q)$, $i = 1, 2$, be their specializations and $\langle \chi_i(P) \rangle$, $i=1, 2$, be the subgroups of $H^1(\kappa(P), \mathbb{Z}/q)$ generated by $\chi_i(P)$, respectively. Then P is called
   (a) a *cool point* for α if $\langle \chi_1(P) \rangle = \langle \chi_2(P) \rangle = 0$,
   (b) a *chilly point* for α if $\langle \chi_1(P) \rangle = \langle \chi_2(P) \rangle \neq 0$ and
   (c) a *hot point* for α if $\langle \chi_1(P) \rangle \neq \langle \chi_2(P) \rangle$.

When P is a chilly point, there is a unique $s = s(C_2/C_1) \in (\mathbb{Z}/q)^*$ such that

$$\chi_2(P) = s.\chi_1(P) \in H^1(\kappa(P), \mathbb{Z}/q).$$

One says that $s = s(C_2/C_1)$ is the *coefficient* of the chilly point P with respect to C_1.

Remark 3.6. One may verify without much pain that our classification of nodal points, following [Brussel 2010, Definition 8.5], is equivalent to Saltman's original formulation, which goes as follows. First consider the case $\mu_q \subseteq F$. Then

$$\alpha \equiv (u, \pi) + (v, \delta) + r.(\pi, \delta) \pmod{\mathrm{Br}(\mathcal{O}_{X,P})}$$

by [Saltman 1997, Proposition 1.2]. Here $u, v \in \mathcal{O}_{X,P}^*$, $r \in \mathbb{Z}/q$ and $\pi, \delta \in \mathcal{O}_{X,P}$ are local equations of the two components of $\mathrm{Ram}_X(\alpha)$ passing through P. The point P is a *cold point* if $r \neq 0 \in \mathbb{Z}/q$. Assume next $r = 0 \in \mathbb{Z}/q$. Then P is a *cool point* if $u(P), v(P) \in \kappa(P)^{*q}$, a *chilly point* if $u(P), v(P) \notin \kappa(P)^{*q}$ and they generate the same subgroup of $\kappa(P)^*/\kappa(P)^{*q}$ or a *hot point* otherwise. In the general case, let $X' \to X$ be the connected finite étale cover obtained by adjoining all q-th roots of unity and let α' be the canonical image of α in $\mathrm{Br}(F')$, where F' denotes the function field of X'. Then for any two points $P_1', P_2' \in X'$, both lying over $P \in X$, P_1' is a cold, cool, chilly or hot point for α' if and only if P_2' is, and in that case, one says that P is a cold, cool, chilly or hot point for α, respectively. When P is chilly, the coefficient of P with respect to a component through it is also well-defined, as the coefficient of any preimage P' of P.

To get some compatibility for coefficients of chilly points, one has to eliminate the so-called *chilly loops*, i.e., loops in the following graph. The set of vertices is the set of irreducible components of $\mathrm{Ram}_X(\alpha)$, and the number $r \geq 0$ of edges linking two vertices is equal to the number of chilly points in the intersection of the two curves corresponding to the two vertices. (Two vertices may be joined by two or more edges and thus contribute to some loops.)

Proposition 3.7 [Saltman 2007, Proposition 3.8]. *Let X, F, q and $\alpha \in \mathrm{Br}(F)[q]$ be as above. Assume that $\mathrm{Ram}_X(\alpha)$ is an snc divisor on X. Then there exists a proper birational morphism $X' \to X$, obtained by a finite number of blow-ups, such that α has no cool points and no chilly loops on X'.*

We also need the notion of *residual class* at a ramified place. Let C be an irreducible component of $\mathrm{Ram}_X(\alpha)$, and let $v = v_C$ be the associated discrete valuation of F. Choose any $x \in F^*$ with $q \nmid v(x)$ so that the extension $M/F := F(\sqrt[q]{x})/F$ is totally ramified at $v = v_C$ and $\alpha_M = \alpha \otimes_F M \in \mathrm{Br}(M)$ is unramified at the unique discrete valuation w of M that lies over v. One has $\kappa(w) = \kappa(v) = \kappa(C)$ and hence a well-defined Brauer class $\beta_{C,x} \in \mathrm{Br}(\kappa(C))$ given by the specialization of $\alpha_M \in \mathrm{Br}(M)$ in $\mathrm{Br}(\kappa(w)) = \mathrm{Br}(\kappa(C))$. Let $(L/\kappa(C), \sigma) = \mathrm{ram}_C(\alpha)$ be the ramification of α at C. Whether $\beta_{C,x} \in \mathrm{Br}(\kappa(C))$ is split by the field extension $L/\kappa(C)$ does not depend on the choice of $M = F(\sqrt[q]{x})$ [Saltman 2007, Corollary 0.7]. We say that the *residual classes of α at C are split by the ramification* if, for one (and hence for all) choice of $M = F(\sqrt[q]{x})$, the residual class $\beta_{C,x} \in \mathrm{Br}(\kappa(C))$ is split by $L/\kappa(C)$ [Saltman 2007, p. 821 Remark]. When we are only interested in this property, we will simply write β_C for $\beta_{C,x} \in \mathrm{Br}(\kappa(C))$ with respect to any choice of x.

It is proved in [Saltman 2007, Propositions 0.5 and 3.10(d)] that if α has index q, then all the residual classes β_C of α at all components C of $\mathrm{Ram}_X(\alpha)$ are split by the ramification and there are no hot points for α on X.

Splitting over a Kummer extension. Let X be a reduced scheme that is projective over a ring. Let $\mathcal{P} \subseteq X$ be a finite set of closed points of X. Denote by $\mathcal{K}_X$ the sheaf of meromorphic functions on X, and set $\mathcal{P}^* := \bigoplus_{P \in \mathcal{P}} \kappa(P)^*$. Let $\mathcal{O}_{X,\mathcal{P}}^*$ denote the kernel of the natural surjection of sheaves $\mathcal{O}_X^* \to \mathcal{P}^*$ so that there is a natural exact sequence

$$1 \to \mathcal{O}_{X,\mathcal{P}}^* \to \mathcal{O}_X^* \to \mathcal{P}^* \to 1.$$

Define subgroups $K_{\mathcal{P}}^* \subseteq H^0(X, \mathcal{K}_X^*)$ and $H_{\mathcal{P}}^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \subseteq H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ by

$$K_{\mathcal{P}}^* := \{f \in H^0(X, \mathcal{K}_X^*) \mid f \in \mathcal{O}_{X,P}^* \quad \text{for all } \ P \in \mathcal{P}\},$$
$$H_{\mathcal{P}}^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) := \{D \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \mid \mathrm{Supp}(D) \cap \mathcal{P} = \varnothing\}.$$

Consider the natural map

$$\phi : K_{\mathcal{P}}^* \to H_{\mathcal{P}}^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \oplus \Big(\bigoplus_{P \in \mathcal{P}} \kappa(P)^*\Big), \quad f \mapsto \Big(\mathrm{div}_X(f), \bigoplus f(P)\Big).$$

Proposition 3.8 [Saltman 2007, Proposition 1.6]. *With notation as above, there is a natural isomorphism*

$$H^1(X, \mathcal{O}^*_{X,\mathcal{P}}) \cong \frac{H^0_{\mathcal{P}}(X, \mathcal{K}^*_X/\mathcal{O}^*_X) \oplus (\bigoplus_{P\in\mathcal{P}} \kappa(P)^*)}{\phi(K^*_{\mathcal{P}})}.$$

The analog in the arithmetic case of the following proposition is [Saltman 2007, Proposition 1.7]. The following generalization to the case where A is two-dimensional will be indispensable in the proofs of our results:

Proposition 3.9. *Let A be a (noetherian) normal, henselian local domain with residue field κ, X an integral scheme and $X \to \operatorname{Spec} A$ a proper morphism whose closed fiber X_0 has dimension ≤ 1 and whose generic fiber is geometrically integral. Let m be a positive integer invertible in A. Let $\overline{X} = (X_0)_{\mathrm{red}}$ be the reduced closed subscheme on the closed fiber X_0. Suppose that $\overline{X}$ is geometrically reduced over κ (e.g., κ is perfect). Then for any finite set $\mathcal{P}$ of closed points of X, the natural map*

$$H^1(X, \mathcal{O}^*_{X,\mathcal{P}}) \to H^1(\overline{X}, \mathcal{O}^*_{\overline{X},\mathcal{P}})$$

is surjective and induces a canonical isomorphism

$$H^1(X, \mathcal{O}^*_{X,\mathcal{P}})/m \cong H^1(\overline{X}, \mathcal{O}^*_{\overline{X},\mathcal{P}})/m.$$

To prove the proposition, we need a well-known lemma.

Lemma 3.10. *Let A be a (noetherian) henselian local ring, $X \to \operatorname{Spec} A$ a proper morphism with closed fiber X_0 of dimension ≤ 1, $m > 0$ a positive integer that is invertible in A and $\overline{X} = (X_0)_{\mathrm{red}}$ the reduced closed subscheme on the closed fiber X_0.*

Then the natural map $\operatorname{Pic}(X) \to \operatorname{Pic}(\overline{X})$ is surjective and induces an isomorphism

$$\operatorname{Pic}(X)/m \xrightarrow{\sim} \operatorname{Pic}(\overline{X})/m.$$

Proof. The surjectivity of $\operatorname{Pic}(X) \to \operatorname{Pic}(\overline{X})$ follows from [Grothendieck 1967, IV.21.9.12]. Then the commutative diagram with exact rows, which comes from the Kummer sequence,

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Pic}(X)/m \arrow[r] \arrow[d] & H^2_{\text{ét}}(X, \mu_m) \arrow[r] \arrow[d] & \operatorname{Br}(X)[m] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Pic}(\overline{X})/m \arrow[r] & H^2_{\text{ét}}(\overline{X}, \mu_m) \arrow[r] & \operatorname{Br}(\overline{X})[m] \arrow[r] & 0
\end{tikzcd}$$

yields the desired isomorphism $\operatorname{Pic}(X)/m \xrightarrow{\sim} \operatorname{Pic}(\overline{X})/m$ since the vertical map in the middle is an isomorphism by proper base change [Milne 1980, p. 224, Corollary 2.7], noticing also that any scheme Y has the same étale cohomology with value in a commutative étale group scheme as its reduced closed subscheme Y_{red} [SGA 4.2 1972, Exposé VIII, Corollary 1.2]. □

Proof of Proposition 3.9. Consider the commutative diagram with exact rows

$$\begin{tikzcd} H^0(X,\mathbb{O}^*) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H^0(X,\mathcal{P}^*) \arrow[r] \arrow[d, "\mathrm{id}"] & H^1(X,\mathbb{O}^*_{X,\mathcal{P}}) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\ H^0(\overline{X},\mathbb{O}^*) \arrow[r, "\theta"] & H^0(\overline{X},\mathcal{P}^*) \arrow[r] & H^1(\overline{X},\mathbb{O}^*_{\overline{X},\mathcal{P}}) \arrow[r] & \mathrm{Pic}(\overline{X}) \arrow[r] & 0 \end{tikzcd}$$

from which the surjectivity of $H^1(X,\mathbb{O}^*_{X,\mathcal{P}}) \to H^1(\overline{X},\mathbb{O}^*_{\overline{X},\mathcal{P}})$ is immediate since $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})$ is surjective by Lemma 3.10. Put $M := \Im(\varphi) \subseteq N := \Im(\theta)$.

We claim that π is surjective. Indeed, by Zariski's connectedness theorem [Grothendieck 1961, III.4.3.12], the hypotheses that A is normal and the generic fiber of $X \to \operatorname{Spec} A$ is geometrically integral imply that the closed fiber X_0 is geometrically connected. The reduced closed fiber $\overline{X} = (X_0)_{\mathrm{red}}$ is geometrically connected as well. Since $\overline{X}$ is assumed to be geometrically reduced, we have $H^0(\overline{X},\mathbb{O}^*) = \kappa^*$. Thus, the map $\pi : H^0(X,\mathbb{O}^*) \to H^0(\overline{X},\mathbb{O}^*)$ is clearly surjective since $A^* \subseteq H^0(X,\mathbb{O}^*)$.

Now our claim shows that $M = N$, and then it follows that

$$\mathrm{Ker}(H^1(X,\mathbb{O}^*_{X,\mathcal{P}}) \to H^1(\overline{X},\mathbb{O}^*_{\overline{X},\mathcal{P}})) \cong B := \mathrm{Ker}(\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})).$$

It's sufficient to show $B/m = 0$. From the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathbb{O}^*)/m \arrow[r] \arrow[d, "\pi"] & H^1_{\text{ét}}(X,\mu_m) \arrow[r] \arrow[d] & \mathrm{Pic}(X)[m] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^0(\overline{X},\mathbb{O}^*)/m \arrow[r] & H^1_{\text{ét}}(\overline{X},\mu_m) \arrow[r] & \mathrm{Pic}(\overline{X})[m] \arrow[r] & 0 \end{tikzcd}$$

it follows that $\mathrm{Pic}(X)[m] \cong \mathrm{Pic}(\overline{X})[m]$ since the vertical map in the middle is an isomorphism by proper base change and the left vertical map is already shown to be surjective. Now applying the snake lemma to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "m"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "m"] & \mathrm{Pic}(\overline{X}) \arrow[r] \arrow[d, "m"] & 0 \\ 0 \arrow[r] & B \arrow[r] & \mathrm{Pic}(X) \arrow[r] & \mathrm{Pic}(\overline{X}) \arrow[r] & 0 \end{tikzcd}$$

and using Lemma 3.10, we easily find $B/m = 0$, which completes the proof. □

The following result is proved in [Saltman 2008, Theorem 7.13] in the case where $\mu_q \subseteq F$ without assuming the residue field κ perfect. It says essentially that the conclusion of Theorem 3.2 can be strengthened for Brauer classes of prime order.

Proposition 3.11. *Let A be a (noetherian) henselian local domain with residue field κ, q a prime number unequal to the characteristic of κ and X a regular excellent surface equipped with a proper dominant morphism $X \to \operatorname{Spec} A$ whose* *closed fiber is of dimension* ≤ 1. *Let* F *be the function field of* X *and* $\alpha \in \mathrm{Br}(F)[q]$. *Assume that* κ *is perfect and that* α *has index* q.

Then there is some $g \in F^*$ *such that the field extension* $M/F := F(\sqrt[q]{g})/F$ *splits all ramification of* α *over* X, *i.e.*, $\alpha_M \in \mathrm{Br}_{\mathrm{nr}}(M/X)$.

Proof. Replacing A by its normalization if necessary, we may assume that A is normal. Let $\mathrm{Ram}_X(\alpha) = \sum C_i$ be the ramification divisor of α on X, and let $\overline{X} = (X_0)_{\mathrm{red}}$ be the reduced closed subscheme on the closed fiber X_0. After a finite number of blow-ups, we may assume that X_0 is purely of dimension 1, that $B := (\mathrm{Ram}_X(\alpha) \cup \overline{X})_{\mathrm{red}}$ is an snc divisor and that there are no cool points or chilly loops for α on X (Proposition 3.7). Write

$$(L_i/\kappa(C_i), \sigma_i) = \mathrm{ram}_{C_i}(\alpha) \in H^1(\kappa(C_i), \mathbb{Z}/q)$$

for the ramification of α at C_i. By the assumption on the index, there are no hot points for α on X and the residual classes of α at C_i are split by the ramification $L_i/\kappa(C_i)$ for every i [Saltman 2007, Propositions 0.5 and 3.10(d)]. Using [Saltman 2007, Theorem 4.6], we can find $\pi \in F^*$ having the following properties:

(P1) The valuation $v_{C_i}(\pi) = s_i$ is not divisible by q.

(P2) If P is a chilly point in the intersection of C_i and C_j, then the coefficient $s(C_j/C_i)$ of P with respect to C_i (Definition 3.5) satisfies $s(C_j/C_i)s_i = s_j \in \mathbb{Z}/q\mathbb{Z}$.

(P3) The divisor $E := \mathrm{div}_X(\pi) - \sum s_i C_i$ does not contain any singular points of $B = (\mathrm{Ram}_X(\alpha) \cup X_0)_{\mathrm{red}}$ or any irreducible component of B in its support.

(P4) With respect to $F' := F(\pi^{1/q})$, the residue Brauer classes $\beta_{C_i,F'} = \beta_{C_i,\pi} \in \mathrm{Br}(\kappa(C_i))$ of α at all the C_i are trivial.

(P5) For any closed point P in the intersection of E and some C_i, the intersection multiplicity $(C_i \cdot E)_P$ is a multiple of q if the corresponding field extension $L_i/\kappa(C_i)$ is nonsplit at P.

Let $\gamma \in \mathrm{Pic}(X)$ be the class of $\mathcal{O}_X(-E)$, and let $\bar{\gamma} \in \mathrm{Pic}(\overline{X})$ be its canonical image. By property (P3), E and $\overline{X}$ only intersect in nonsingular points of $\overline{X}$. So we can represent $\bar{\gamma}$ as a Cartier divisor on $\overline{X}$ using the intersection of $-E$ and $\overline{X}$. This divisor can be chosen in the form

$$\sum q n_j Q_j + \sum n_l Q_l', \tag{3-1}$$

where Q_j and Q_l' are nonsingular points on $\overline{X}$, and for each Q_l', one has either $Q_l' \notin \mathrm{Ram}_X(\alpha)$ or $Q_l' \in C_i$ for exactly one C_i and the corresponding field extension $L_i/\kappa(C_i)$ is split at Q_l' (by property (P5)).

By [Grothendieck 1967, IV.21.9.11 and IV.21.9.12], there exists a prime divisor E_l' on X such that $E_l'|_{\overline{X}} = Q_l'$ as Cartier divisors on $\overline{X}$. Note that $E_l' \nsubseteq \mathrm{Ram}_X(\alpha)$

because otherwise $Q_l' \in E_l' \cap \overline{X}$ would be a singular point of $E_l' \cup \overline{X} \subseteq B = (\mathrm{Ram}_X(\alpha) \cup \overline{X})_{\mathrm{red}}$. Set $E' = -E - \sum n_l E_l'$. Let $\mathcal{P}$ be the set of all singular points of B (in particular, $\mathcal{P}$ contains all nodal points for α).

Let $\gamma' \in H^1(X, \mathcal{O}^*_{X,\mathcal{P}})$ be the element represented by the pair

$$\Bigl(E', \bigoplus 1\Bigr) \in H^0_{\mathcal{P}}(X, \mathcal{K}^*/\mathcal{O}^*) \oplus \Bigl(\bigoplus_{P \in \mathcal{P}} \kappa(P)^*\Bigr)$$

via the isomorphism

$$H^1(X, \mathcal{O}^*_{X,\mathcal{P}}) \cong \frac{H^0_{\mathcal{P}}(X, \mathcal{K}^*/\mathcal{O}^*) \oplus (\bigoplus_{P \in \mathcal{P}} \kappa(P)^*)}{K^*_{\mathcal{P}}}$$

in Proposition 3.8. (Here $E' \in H^0_{\mathcal{P}}(X, \mathcal{K}^*/\mathcal{O}^*)$ since by the choice of π, E does not contain any singular points of $B = (\mathrm{Ram}_X(\alpha) \cup \overline{X})_{\mathrm{red}}$.) The image $\overline{\gamma}'$ of γ' in $H^1(\overline{X}, \mathcal{O}^*_{\overline{X},\mathcal{P}})$ lies in $q.H^1(\overline{X}, \mathcal{O}^*_{\overline{X},\mathcal{P}})$ by the expression (3-1).

From Proposition 3.9, it follows that $\gamma' \in q.H^1(X, \mathcal{O}^*_{X,\mathcal{P}})$. Thus, by Proposition 3.8, there is a divisor $E'' \in H^0_{\mathcal{P}}(X, \mathcal{K}^*/\mathcal{O}^*)$, elements $a(P) \in \kappa(P)^*$ for each $P \in \mathcal{P}$ and $f \in F^*$ such that f is a unit at every $P \in \mathcal{P}$, $\mathrm{div}_X(f) = E' + qE''$ and $f(P) = a(P)^q$ for all $P \in \mathcal{P}$. We now compute

$$\begin{aligned}
\mathrm{div}_X(f\pi) = \mathrm{div}_X(f) + \mathrm{div}_X(\pi) &= (E' + qE'') + \Bigl(\sum s_i C_i + E\Bigr) \\
&= -E - \sum n_l E_l' + qE'' + E + \sum s_i C_i \\
&= \sum s_i C_i + \Bigl(qE'' - \sum n_l E_l'\Bigr) \\
&=: \sum s_i C_i + \sum \tilde{n}_j D_j.
\end{aligned} \tag{3-2}$$

For any D_j, the following properties hold:

(P6) D_j can only intersect B in nonsingular points of B.

(P7) If $q \nmid \tilde{n}_j$, then $D_j \in \{E_l'\}$ so that either $D_j \cap \mathrm{Ram}_X(\alpha) = \varnothing$ or $D_j \cap \mathrm{Ram}_X(\alpha)$ consists of a single point P that lies on one C_i and the corresponding field extension $L_i/\kappa(C_i)$ splits at P.

Now we claim that $g = f\pi$ satisfies the required property. That is, putting $M = F((f\pi)^{1/q})$, $\alpha_M \in \mathrm{Br}(M)$ is unramified at every discrete valuation of M that lies over a point or a curve on X.

Consider a discrete valuation of M lying over some $v \in \Omega(F/X)$.

If v is centered at some C_i, then M/F is totally ramified at v since the coefficient s_i of $f\pi$ at C_i is prime to q (3-2); hence, in particular, M/F splits the ramification of α at v. As α is unramified at all other curves on X, we may restrict to the case where v is centered at a closed point P of X. By [Saltman 2007, Theorem 3.4], we can also ignore distant points and curve points $P \in C_i$ where $L_i/\kappa(C_i)$ splits at P.

Now assume that P is a curve point lying on some $C_1 \in \{C_i\}$ where the corresponding field extension $L_1/\kappa(C_1)$ is nonsplit at P. By property (P7), the only curves other than C_1 in the support of $\mathrm{div}_X(f\pi)$ that can pass through P have coefficients a multiple of q. Therefore, in $R_P = \mathcal{O}_{X,P}$, we have $f\pi = u\pi_1^{s_1}\delta^q$ with $u \in R_P^*$, $\pi_1 \in R_P$ a uniformizer of C_1 at P and $\delta \in R_P$ prime to π_1. Using [Saltman 2007, Proposition 3.5], we then conclude that M/F splits all ramification of α at v.

Recall that we have assumed there are no cool points or hot points for α. So in the only remaining cases, P is either a cold point or a chilly point.

Assume first that P is a cold point for α. By property (P4) and [Saltman 2007, Corollary 0.7], the residual class $\beta_{C_i,M}$ of α at any C_i with respect to $M = F((f\pi)^{1/q})$ is given by the class of a cyclic algebra $(\chi_i, \bar{f}^{-t})$, where

$$\chi_i = (L_i/\kappa(C_i), \sigma_i) = \mathrm{ram}_{C_i}(\alpha) \in H^1(\kappa(C_i), \mathbb{Z}/q),$$

t is an integer prime to q and $\bar{f}$ denotes the canonical image of f in $\kappa(C_i)$. Since f is a q-th power in $\kappa(P)$ by the choice, it follows easily that $\beta_{C_i,M}$ is unramified at P. In the local ring $R_P = \mathcal{O}_{X,P}$, we have $f\pi = u_P\pi_1^{s_1}\pi_2^{s_2}$ for some $u_P \in R_P^*$ by (3-2) and property (P6). Hence, by [Saltman 2007, Proposition 3.10(c)], M splits all ramification of α at v.

Finally, consider the case where P is a chilly point. Let $C_1, C_2 \in \{C_i\}$ be the two different irreducible components of $\mathrm{Ram}_X(\alpha)$ through P, and let $\pi_1, \pi_2 \in R_P = \mathcal{O}_{X,P}$ be uniformizers of C_1 and C_2 at P. Again by (3-2) and property (P6), we have $f\pi = u_P\pi_1^{s_1}\pi_2^{s_2}$ for some $u_P \in R_P^*$. Let $s = s(C_2/C_1)$ be the coefficient of P with respect to C_1. Using property (P2), we find that $M = F((f\pi)^{1/q})$ may be written in the form $M = F((\pi_1'\pi_2^s)^{1/q})$, where $\pi_1' \in R_P$ is a uniformizer of C_1 at P. Thus, by [Saltman 2007, Proposition 3.9(a)], M/F splits all ramification of α at v, which completes the proof. □

Corollary 3.12. *Let A be a (noetherian) henselian local domain with residue field κ, q a prime number unequal to the characteristic of κ, and X a regular excellent surface equipped with a proper dominant morphism $X \to \mathrm{Spec}\, A$ whose closed fiber is of dimension ≤ 1. Let F be the function field of X and $\alpha \in \mathrm{Br}(F)[q]$. Assume that κ is a B_1 field and that α has index q.*

If either $\mu_q \subseteq F$ or κ is perfect, then α is represented by a cyclic algebra of degree q.

Proof. If $\mu_q \subseteq F$, we may use [Saltman 2008, Theorem 7.13] to find a degree-q Kummer extension $M/F = F(\sqrt[q]{g})/F$ that splits all ramification of α over X. If κ is perfect, such an extension exists by Proposition 3.11. As in the proof of Theorem 3.4, we have $\mathrm{Br}_{\mathrm{nr}}(M/X) = 0$ by Corollary 2.5. Hence, $\alpha_M = 0 \in \mathrm{Br}(M)$. Then by a theorem of Albert [Saltman 2007, Proposition 0.1], which is rather immediate when assuming the existence of a primitive q-th root of unity, α is represented by a cyclic algebra of degree q. □

Recall that R always denotes a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. Applying Corollary 3.12 to a regular proper model $\mathscr{X} \to \operatorname{Spec} R$ yields the following:

Theorem 3.13. *Assume that the residue field k of R has property B_1. Let q be a prime number unequal to the characteristic of k.*

If either $\mu_q \subseteq R$ or k is perfect, then any Brauer class $\alpha \in \operatorname{Br}(K)[q]$ of index q is represented by a cyclic algebra of degree q.

Remark 3.14. (1) In Proposition 3.11 or Corollary 3.12, according to the above proof, if we assume the morphism $X \to \operatorname{Spec} A$ is chosen such that $\operatorname{Ram}_X(\alpha)$ is an snc divisor and that α has no cool points or chilly loops on X, then the hypothesis that α has index q may be replaced by the weaker condition that all the residual classes β_C of α at all components C of $\operatorname{Ram}_X(\alpha)$ are split by the ramification.

(2) Similarly, let $\mathscr{X} \to \operatorname{Spec} R$ be a regular proper model such that $\operatorname{Ram}_{\mathscr{X}}(\alpha)$ is an snc divisor and that $\alpha \in \operatorname{Br}(K)[q]$ has no cool points or chilly loops on $\mathscr{X}$. Then the conclusion in Theorem 3.13 remains valid if instead of assuming α has index q we only require that all the residual classes of α at all components of $\operatorname{Ram}_{\mathscr{X}}(\alpha)$ are split by the ramification.

(3) In the context of Theorem 3.13, if k is a separably closed field, [Colliot-Thélène et al. 2002, Theorem 2.1] proved a stronger result: any Brauer class $\alpha \in \operatorname{Br}(K)$ of order n that is invertible in R (but not necessarily a prime) is represented by a cyclic algebra of index n.

Some corollaries. As applications of results obtained previously, we give a criterion for $\alpha \in \operatorname{Br}(K)[q]$ to have index q. Also, we will prove Theorem 1.3.

We begin with the following easy and standard fact:

Lemma 3.15. *Let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. Let $\mathscr{X} \to \operatorname{Spec} R$ be a regular proper model. Then for any curve $C \subseteq \mathscr{X}$, one has either*

(i) *C is a proper curve over k or*

(ii) *$C = \operatorname{Spec} B$, where B is a domain whose normalization B' is a henselian discrete valuation ring with residue field finite over k.*

Proof. After replacing R by its normalization, we may assume R is normal.

Consider the scheme-theoretic image D of $C \subseteq \mathscr{X}$ by the structural morphism $\mathscr{X} \to \operatorname{Spec} R$. If D is the closed point of $\operatorname{Spec} R$, then C is a proper curve over the residue field k. Otherwise, D is the closed subscheme of $\operatorname{Spec} R$ defined by a height-1 prime ideal $\mathfrak{p} \subseteq R$. Since R is two-dimensional and normal, the proper birational morphism $\mathscr{X} \to \operatorname{Spec} R$ is an isomorphism over codimension-1 points [Liu 2002, p. 150, Corollary 4.4.3]. Thus, the induced morphism $C \to D$ is proper, birational and quasifinite, and hence, by Chevalley's theorem, it is finite. Write $A = R/\mathfrak{p}$ so that $D = \operatorname{Spec} A$. Then $C = \operatorname{Spec} B$ for some domain $B \subseteq \kappa(C) = \kappa(D)$ that is finite over A. Since A is a henselian excellent local domain, the same is true for B. The normalization B' of B is finite over B and hence a henselian local domain as well, and it coincides with the normalization of A in its fraction field $\operatorname{Frac}(A) = \kappa(C) = \kappa(D)$. Then it is clear that B' is a henselian discrete valuation ring with residue field finite over k. This finishes the proof. $\square$

Recall that Ω_R denotes the set of discrete valuations of K that are centered at codimension-1 points of regular proper models.

Corollary 3.16. *For any $v \in \Omega_R$, the residue field $\kappa(v)$ is either the function field of a curve over k or the fraction field of a henselian discrete valuation ring whose residue field is finite over k.*

Now we can prove the following variant of [Saltman 2007, Corollary 5.2]:

Corollary 3.17. *Let q be a prime number unequal to the characteristic of the residue field k and $\alpha \in \operatorname{Br}(K)[q]$ a Brauer class of order q. Let $\mathscr{X} \to \operatorname{Spec} R$ be a regular proper model such that the ramification divisor $\operatorname{Ram}_{\mathscr{X}}(\alpha)$ of α on $\mathscr{X}$ has only simple normal crossings and that α has no cool points or chilly loops on $\mathscr{X}$. Write $\operatorname{ram}_{C_i}(\alpha) = (L_i/\kappa(C_i), \sigma_i)$ for the ramification data and $\beta_i \in \operatorname{Br}(\kappa(C_i))$ for the residual classes.*

Suppose that k is a finite field. Then the following conditions are equivalent:

(i) *α has index q,*

(ii) *$\beta_i \in \operatorname{Br}(\kappa(C_i))$ is split by $L_i/\kappa(C_i)$ for every i and*

(iii) *there are no hot points for α on $\mathscr{X}$.*

Proof. Propositions 0.5 and 3.10(d) of [Saltman 2007] give (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

To see (ii) $\Longrightarrow$ (i), note that by Proposition 3.11 and Remark 3.14 there is a degree-q Kummer extension $M/K = K(\sqrt[q]{g})/K$ that splits all the ramification of α over R. As the residue field is finite, we have $\operatorname{Br}_{\mathrm{nr}}(M/X) = 0$ and in particular $\alpha_M = 0$. Hence, the index of α divides q, the degree of the extension M/K. Since α has order q, it follows that the index of α is q.

To show (iii) $\Longrightarrow$ (ii), let C be a fixed irreducible component of the ramification divisor $\operatorname{Ram}_{\mathscr{X}}(\alpha)$ with associated ramification data $\operatorname{ram}_C(\alpha) = (L/\kappa(C), \sigma)$. By [Saltman 2007, Lemma 4.1], there exists $\pi \in K^*$ having the following properties: (1) the valuation $s_i := v_{C_i}(\pi)$ with respect to every component C_i of $\operatorname{Ram}_{\mathscr{X}}(\alpha)$ is prime to q and (2) whenever there is a chilly point P in the intersection of two components C_i and C_j, the coefficient of P with respect to C_i is equal to $s_j/s_i = v_{C_j}(\pi)/v_{C_i}(\pi) \bmod q$.

Put $M := K(\sqrt[q]{\pi})$. Let β denote the residual class of α with respect to M, i.e., the specialization of $\alpha_M \in \mathrm{Br}(M)$ in $\mathrm{Br}(\kappa(C))$. We want to show that β is split by $L/\kappa(C)$.

By Corollary 3.16, $\kappa(C)$ is either a function field in one variable over the finite field k or the fraction field of a henselian discrete valuation ring with finite residue field. The same is true for L. So in either case, $\beta \in \mathrm{Br}(\kappa(C))[q]$ is split by $L/\kappa(C)$ if and only if $L/\kappa(C)$ splits all ramification of β at every closed point P of C.

Assume first that $L/\kappa(C)$ is split at P. Then P is either a chilly point or a curve point (P is not cold because $L/\kappa(C)$ is unramified at P; see Definition 3.5). If P is chilly, β is unramified at P by [Saltman 2007, Proposition 3.10(b)]. If P is a curve point, then we conclude by [Saltman 2007, Proposition 3.11].

Next consider the case where $L/\kappa(C)$ is nonsplit at P. Then the P-adic valuation v_P of $\kappa(C)$ extends uniquely to a discrete valuation w_P of L. If $L/\kappa(C)$ is ramified at P, it is obvious that $L/\kappa(C)$ splits the ramification of β at P. If $L/\kappa(C)$ is unramified at P, then $\kappa(w_P)$ is the unique degree-q extension of the finite field $\kappa(v_P) = \kappa(P)$. Thus, the restriction map

$$\mathrm{Res} : H^1(\kappa(P), \mathbb{Z}/q\mathbb{Z}) \to H^1(\kappa(w_P), \mathbb{Z}/q\mathbb{Z})$$

is the zero map, which implies that $L/\kappa(C)$ splits the ramification of β at P. The corollary is thus proved. $\square$

Example 3.18. Here is a concrete example that shows the bound on the period-index exponent in Theorem 3.4 is sharp. The criterion in the above corollary will be used in the argument.

Let p be a prime number such that $p \equiv 3 \pmod 4$. Let $k = \mathbb{F}_p$ be the finite field of cardinality p, and let $R = k[\![x, y]\!]$ be the ring of formal power series in two variables x and y over k. Let $\mathscr{X} \to \operatorname{Spec} R$ be the blow-up of $\operatorname{Spec} R$ at the closed point, and let $E \subseteq \mathscr{X}$ be the exceptional divisor. We have

$$\operatorname{Proj}(k[T, S]) \cong E = \operatorname{Proj}(R[T, S]/(x, y)) \subseteq \mathscr{X} = \operatorname{Proj}(R[T, S]/(xS - yT)).$$

Let $f_1 = y$, $f_2 = x$ and $f_3 = y + x$, and let $C_i \subseteq \mathscr{X}$ be the strict transform of the curve defined by $f_i = 0$ in $\operatorname{Spec} R$ for each $i = 1, 2, 3$. Each intersection $C_i \cap E$ consists of a single point P_i.

Let α be the Brauer class of the biquaternion algebra $(-1, y) \otimes (y + x, x)$ over $K = k((x, y))$. The ramification divisor $\mathrm{Ram}_{\mathscr{X}}(\alpha)$ is $C_1 + C_2 + C_3 + E$. The set of nodal points for α on $\mathscr{X}$ is $\{P_1, P_2, P_3\}$. Locally at P_1, we may choose s and x as local equations for C_1 and E, respectively, where $s = S/T = y/x \in K$. Thus, in the Brauer group $\mathrm{Br}(K)$, we have

$$\alpha = (-1, y) + (y + x, x) = (-1, xs) + (xs + x, x) = (-1, s) + (s + 1, x).$$

The function s vanishes at P_1, and $-1 \neq 1 \in \kappa(P_1)^*/\kappa(P_1)^{*2} = k^*/k^{*2}$, so P_1 is a hot point by definition (Remark 3.6). One may verify that P_2 and P_3 are cold points.

As for the residual classes, one may check that for each i the residual classes of α at C_i are split by the ramification. Let us now show that at E the residual classes are not split by the ramification. Indeed, if v_E denotes the discrete valuation of K defined by E, we have $v_E(x) = v_E(y) = v_E(y+x) = 1$. Then it is easy to see that the ramification $\mathrm{ram}_E(\alpha)$ of α at E is represented by the quadratic extension $k(s)(\sqrt{s+1})$ of $\kappa(E) = k(s)$. Putting $M = K(\sqrt{x})$, $\alpha_M = (-1, y) = (-1, s) \in \mathrm{Br}(M)$, and hence, the residue class of α at E with respect to M/K is $\beta_E = (-1, s) \in \mathrm{Br}(\kappa(E)) = \mathrm{Br}(k(s))$. Putting $u = \sqrt{s+1}$, it is easy to see that the quaternion algebra $(-1, s) = (-1, u^2 - 1)$ is not split over $k(u) = k(s)(\sqrt{s+1})$ (in fact, it ramifies at $u = 1$ since -1 is not a square in $k = \mathbb{F}_p$).

By Theorem 3.4 and Corollary 3.17, we conclude that $\alpha \in \mathrm{Br}(K)[2]$ is of index 4.

We shall now prove Theorem 1.3 in a slightly generalized form.

Theorem 3.19. *Let R be a two-dimensional, henselian, excellent local domain whose residue field k has property B_1, K the fraction field of R, q a prime number unequal to the characteristic of k and $\alpha \in \mathrm{Br}(K)[q]$. Assume either $\mu_q \subseteq R$ or k is perfect.*

If for every v in the set Ω_R of discrete valuations of K that correspond to codimension-1 points of regular proper models, the Brauer class $\alpha_v = \alpha \otimes_K K_v \in \mathrm{Br}(K_v)$ is represented by a cyclic algebra of degree q over K_v, then α is represented by a cyclic algebra of degree q over K.

Proof. Let $\mathscr{X} \to \mathrm{Spec}\, R$ be a regular proper model such that $\mathrm{Ram}_{\mathscr{X}}(\alpha)$ is an snc divisor and that α has no cool points or chilly loops on $\mathscr{X}$. By Theorem 3.13 and Remark 3.14(2), it suffices to prove that all the residual classes of α at all components of $\mathrm{Ram}_{\mathscr{X}}(\alpha)$ are split by the ramification.

Assume the contrary. Then there is an irreducible component C of $\mathrm{Ram}_{\mathscr{X}}(\alpha)$ such that the residual classes of α at C are not split by the ramification. Now consider the discrete valuation $v = v_C$ of K defined by C. By assumption, $\alpha_v \in \mathrm{Br}(K_v)[q]$ is cyclic of degree q so that $\alpha_v = (\chi_v, b_v)$ for some $\chi_v \in H^1(K_v, \mathbb{Z}/q)$ and $b_v \in K_v^*$. Without loss of generality, we may assume $b_v = w\pi^t$, where $\pi \in K_v^*$ is a uniformizer for v, $w \in K_v^*$ is a unit for v and t is an integer such that $0 \leq t \leq q-1$.

Let $(L/K_v, \sigma_v)$ be the pair representing the character $\chi_v \in \mathrm{Hom}_{\mathrm{cts}}(G_{K_v}, \mathbb{Z}/q)$. If L/K_v is unramified, then $t \neq 0$ because α is ramified at $v = v_C$. Then there exist integers $r, s \in \mathbb{Z}$ such that $1 = rq + st$. Putting $\pi' = w^s\pi$, we have $b_v = w\pi^t = (w^r)^q(\pi')^t$. Then

$$\alpha_v = (\chi_v, b_v) = (\chi_v, (\pi')^t) \in \mathrm{Br}(K_v)[q]$$

is clearly split by the totally ramified extension $K_v'/K_v := K_v(\sqrt[q]{\pi'})/K_v$. In particular, the residual class of α with respect to K_v'/K_v is 0 and a fortiori the residual classes of α at C are split by the ramification. But this contradicts our choice of C.

If L/K_v is ramified, then it is totally and tamely ramified. So $L = K_v(\sqrt[q]{\theta})$ for some $\theta \in K_v$. The extension being Galois, it follows that $\mu_q \subseteq L$. Since the residue fields of L and K_v are the same, Hensel's lemma implies that $\mu_q \subseteq K_v$. We may thus assume $\alpha_v = (u\pi^s, b_v) = (u\pi^s, w\pi^t)$ for some unit $u \in K_v^*$ and some integer s such that $0 \le s \le q-1$. Since α is ramified at $v = v_C$, s and t cannot both be 0. Assume for instance $s > 0$. A similar argument as before shows that α_v is split by a totally ramified extension $K_v(\sqrt[q]{\pi})/K_v$, which leads to a contradiction again. This proves the theorem. □

The following special case, which answers a question in [Colliot-Thélène et al. 2002, Remark 3.7], will be used in Section 4:

Corollary 3.20. *Let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. Assume that k is a finite field of characteristic $\neq 2$. Let D be a central simple algebra over K of period 2.*

If for every $v \in \Omega_R$, $D \otimes_K K_v$ is Brauer equivalent to a quaternion algebra over K_v, then D is Brauer equivalent to a quaternion algebra over K.

The analog of Theorem 3.19 in the case of a p-adic function field does not seem to have been noticed. Let us prove it in the following general form:

Theorem 3.21. *Let A be a henselian, excellent discrete valuation ring whose residue field κ has property B_1. Let q be a prime number unequal to the characteristic of κ, F the function field of an algebraic curve over the fraction field of A and $\alpha \in \mathrm{Br}(F)[q]$. For every regular surface $\mathcal{Y}$ with function field F equipped with a proper flat morphism $\mathcal{Y} \to \operatorname{Spec} A$, let $\Omega(F/\mathcal{Y}^{(1)})$ denote the set of discrete valuations of F corresponding to codimension-1 points of $\mathcal{Y}$. Let $\Omega_{A,F}$ be the union of all $\Omega(F/\mathcal{Y}^{(1)})$, where $\mathcal{Y}$ runs over regular surfaces as above.*

Assume either $\mu_q \subseteq F$ or κ is perfect. If, for every $v \in \Omega_{A,F}$,

$$\alpha_v = \alpha \otimes_F F_v \in \mathrm{Br}(F_v)$$

is represented by a cyclic algebra of degree q over F_v, then α is represented by a cyclic algebra of degree q over F.

Proof. By resolution of singularities, there exists a regular surface X with function field F, together with a proper flat morphism $X \to \operatorname{Spec} A$, such that the ramification divisor $\mathrm{Ram}_X(\alpha)$ has only simple normal crossings and α has no cool points or chilly loops on X. By Corollary 3.12 and Remark 3.14, it suffices to show that all the residual classes of α are split by the ramification. This may be done as in the proof of Theorem 3.19. □

4. Quadratic forms and the u-invariant

A local-global principle from class field theory. To prove our results on quadratic forms, a result coming from Saito's work on class field theory for two-dimensional local rings will be used at a few crucial points.

As before, let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. The following proposition grows out of a conversation with S. Saito:

Proposition 4.1. *Assume R is normal and the residue field k is finite. Suppose $\mathscr{X} \to \operatorname{Spec} R$ is a regular proper model such that the reduced divisor on the closed fiber has only simple normal crossings. Let $n > 0$ be an integer that is invertible in k.*

Then the natural map $H^3(K, \mu_n^{\otimes 2}) \to \prod_{v \in \mathscr{X}^{(1)}} H^3(K_v, \mu_n^{\otimes 2})$. is injective.

Proof. Let Y be the reduced subscheme on the closed fiber of $\mathscr{X} \to \operatorname{Spec} R$ and $U = \mathscr{X} \setminus Y$. Let $i : Y \to \mathscr{X}$ and $j : U \to \mathscr{X}$ denote the natural inclusions, and put $\mathscr{F} := i^* Rj_* \mu_n^{\otimes 2}$. Let $P = (\operatorname{Spec} R)^{(1)}$ be the set of codimension-1 points of $\operatorname{Spec} R$. We may identify P with the set of closed points of U via the structural map $\mathscr{X} \to \operatorname{Spec} R$. From localization theories, we obtain exact sequences [Saito 1987, pp. 358–360]

$$H^3(U, \mu_n^{\otimes 2}) \xrightarrow{\phi} H^3(K, \mu_n^{\otimes 2}) \xrightarrow{\iota} \bigoplus_{\mathfrak{p} \in P} H^3(K_{\mathfrak{p}}, \mu_n^{\otimes 2}), \tag{4-1}$$

$$H^3(Y, \mathscr{F}) \to \bigoplus_{\eta \in Y^{(0)}} H^3(\kappa(\eta), \mathscr{F}) \xrightarrow{\theta'} \bigoplus_{x \in Y^{(1)}} \mathbb{Z}/n, \tag{4-2}$$

where the map ι in (4-1) induced by the natural maps $H^3(K, \mu_n^{\otimes 2}) \to H^3(K_{\mathfrak{p}}, \mu_n^{\otimes 2})$. For each $\eta \in Y^{(0)} \subseteq \mathscr{X}^{(1)}$, let A_η be the completion of the discrete valuation ring $\mathbb{O}_{\mathscr{X},\eta}$. By the functoriality of the functor Rj_*, we have the commutative diagram

$$\begin{tikzcd}
H^3(\mathscr{X}, Rj_*\mu_n^{\otimes 2}) \arrow[r] \arrow[d, "\cong"'] & H^3(\mathbb{O}_{\mathscr{X},\eta}, Rj_*\mu_n^{\otimes 2}) \arrow[r] \arrow[d, "\cong"] & H^3(A_\eta, Rj_*\mu_n^{\otimes 2}) \arrow[d, "\cong"] \\
H^3(U, \mu_n^{\otimes 2}) \arrow[r] & H^3(K, \mu_n^{\otimes 2}) \arrow[r] & H^3(K_\eta, \mu_n^{\otimes 2})
\end{tikzcd}$$

where the vertical maps are canonical isomorphisms. On the other hand, we have a commutative diagram

$$\begin{tikzcd}
H^3(\mathscr{X}, Rj_*\mu_n^{\otimes 2}) \arrow[r] \arrow[d, "\cong"'] & H^3(\mathbb{O}_{\mathscr{X},\eta}, Rj_*\mu_n^{\otimes 2}) \arrow[r] \arrow[d] & H^3(A_\eta, Rj_*\mu_n^{\otimes 2}) \arrow[d, "\cong"] \\
H^3(Y, \mathscr{F}) \arrow[r] & H^3(\kappa(\eta), \mathscr{F}) \arrow[r, "\mathrm{id}"] & H^3(\kappa(\eta), \mathscr{F})
\end{tikzcd}$$

where the two vertical isomorphisms come from proper base change. Let

$$\theta : \bigoplus_{\eta\in Y^{(0)}} H^3(K_\eta, \mu_n^{\otimes 2}) \to \bigoplus_{x\in Y^{(1)}} \mathbb{Z}/n$$

be the composition of the map θ' in (4-2) with the canonical isomorphism

$$\bigoplus_{\eta\in Y^{(0)}} H^3(K_\eta, \mu_n^{\otimes 2}) \cong \bigoplus_{\eta\in Y^{(0)}} H^3(\kappa(\eta), \mathscr{F}).$$

Putting all things together, we get a commutative diagram with exact rows

$$\begin{tikzcd}
H^3(U, \mu_n^{\otimes 2}) \arrow[r, "\phi"] \arrow[d, "\cong"'] & H^3(K, \mu_n^{\otimes 2}) \arrow[r, "\iota"] \arrow[d, "\varphi"'] & \bigoplus_{\mathfrak{p}\in P} H^3(K_{\mathfrak{p}}, \mu_n^{\otimes 2}) \\
H^3(Y, \mathscr{F}) \arrow[r, "\tau"] & \bigoplus_{\eta\in Y^{(0)}} H^3(K_\eta, \mu_n^{\otimes 2}) \arrow[r, "\theta"] & \bigoplus_{x\in Y^{(1)}} \mathbb{Z}/n
\end{tikzcd}$$

where the map φ is induced by the restriction maps $H^3(K, \mu_n^{\otimes 2}) \to H^3(K_\eta, \mu_n^{\otimes 2})$. By [Saito 1987, p. 361, Lemma 5.13], the induced map $\operatorname{Ker}\phi \to \operatorname{Ker}\tau$ is an isomorphism. Hence, φ induces an isomorphism $\operatorname{Ker}\iota \cong \operatorname{Ker}\theta$. In particular, an element $\zeta \in H^3(K, \mu_n^{\otimes 2})$ vanishes if and only if $\iota(\zeta) = 0 = \varphi(\zeta)$. The result then follows immediately since $\mathscr{X}^{(1)} = P \cup Y^{(0)}$. □

An application of Proposition 4.1 that we will need is the following variant of [Parimala and Suresh 2010, Theorem 3.5] (see also [Parimala and Suresh 2012, Corollary 4.3]):

Theorem 4.2. *Let R be a two-dimensional, henselian, excellent local domain with finite residue field k and fraction field K. Let q be a prime number that is different from the characteristic of k. If $\mu_q \subseteq K$, then every element in $H^3(K, \mu_q^{\otimes 3})$ is a symbol.*

Proof. First note that as in [Parimala and Suresh 2010, Corollary 1.2], one can show that for any $v \in \Omega_R$, the kernel $H^3_{\mathrm{nr}}(K_v, \mu_q^{\otimes 3})$ of the residue map

$$H^3(K_v, \mu_q^{\otimes 3}) \xrightarrow{\partial_v} H^2(\kappa(v), \mu_q^{\otimes 2})$$

is trivial. Indeed, by Corollary 3.16, the residue field $\kappa(v)$ is either a function field in one variable over k or the fraction field of a henselian discrete valuation ring B whose residue field k' is finite over k. In the former case, we have $\mathrm{cd}_q(\kappa(v)) \le 2$ by [Serre 1994, p. 93, Proposition 11] and hence $H^3(\kappa(v), \mu_q^{\otimes 3}) = 0$. In the latter case, the exact sequence

$$H^3(k', \mu_q^{\otimes 3}) \cong H^3_{\text{ét}}(B, \mu_q^{\otimes 3}) \to H^3(\kappa(v), \mu_q^{\otimes 3}) \to H^2(k', \mu_q^{\otimes 2})$$

implies $H^3(\kappa(v), \mu_q^{\otimes 3}) = 0$ since $\mathrm{cd}(k') \le 1$.

Let $\mathscr{X} \to \operatorname{Spec} R$ be a regular proper model. Since $H^3_{\mathrm{nr}}(K_v, \mu_q^{\otimes 3}) = 0$ for all $v \in \Omega_R$, it follows that the kernel of the natural map

$$H^3(K, \mu_q^{\otimes 3}) \to \prod_{v \in \mathscr{X}^{(1)}} H^3(K_v, \mu_q^{\otimes 3})$$

coincides with

$$H^3_{\mathrm{nr}}(K/\mathscr{X}^{(1)}, \mu_q^{\otimes 3}) := \bigcap_{v \in \mathscr{X}^{(1)}} \operatorname{Ker}\bigl(\partial_v : H^3(K, \mu_q^{\otimes 3}) \to H^2(\kappa(v), \mu_q^{\otimes 2})\bigr).$$

As $\mu_q \subseteq K$, it follows from Proposition 4.1 that $H^3_{\mathrm{nr}}(K/\mathscr{X}^{(1)}, \mu_q^{\otimes 3}) = 0$. On the other hand, Lemma 3.15 implies that $\mathscr{X}$ is a q-special surface in the sense of [Parimala and Suresh 2012, §3]. So the result follows from [Parimala and Suresh 2012, Theorem 4.2]. □

Local-global principle for quadratic forms. Thanks to Corollary 3.20, we are now in a position to prove the local-global principle for quadratic forms of rank 5 with respect to the set Ω_R of divisorial valuations as asserted in Theorem 1.1. Standard notations from the algebraic theory of quadratic forms (as in [Lam 2005] or [Scharlau 1985]) will be used as of now.

Proof of Theorem 1.1. We follow the ideas in the proof of [Colliot-Thélène et al. 2002, Theorem 3.6]. Let ϕ be a five-dimensional nonsingular quadratic form over K. Assume that ϕ_v is isotropic over K_v for every $v \in \Omega_R$.

The six-dimensional form $\psi := \phi \perp \langle -\det(\phi)\rangle$ is similar to a so-called Albert form $\langle a, b, -ab, -c, -d, cd\rangle$. By the general theory of Albert forms [Gille and Szamuely 2006, p. 14, Theorem 1.5.5], the form $\langle a, b, -ab, -c, -d, cd\rangle$ is isotropic if and only if the biquaternion algebra $D := (a, b) \otimes (c, d)$ is not a division algebra. By assumption, for every $v \in \Omega_R$, ψ_v is isotropic over K_v, so the biquaternion algebra $D_v = (a, b)_{K_v} \otimes (c, d)_{K_v}$ is not a division algebra. The index of D_v must be smaller than 4, which is the degree. Therefore, D_v is Brauer equivalent to a quaternion algebra over K_v. By Corollary 3.20, $D = (a, b) \otimes (c, d)$ is Brauer equivalent to a quaternion algebra over K. In particular, D is not a division algebra over K. Hence, ψ is isotropic over K. This implies that ϕ may be written in the form

$$\phi = \det(\phi).\langle 1, a, b, c, abc\rangle$$

over K. In particular, ϕ is similar to a subform of the threefold Pfister form

$$\rho = \langle\langle -a, -b, -c\rangle\rangle := \langle 1, a\rangle \otimes \langle 1, b\rangle \otimes \langle 1, c\rangle.$$

By Merkurjev's theorem [Arason 1984, p. 129, Proposition 2], the form ρ is isotropic if and only if the symbol class $(-a, -b, -c)$ vanishes. For each $v \in \Omega_R$, as the subform ϕ_v of ρ_v is isotropic over K_v, we have $(-a, -b, -c) = 0$ in $H^3(K_v, \mathbb{Z}/2)$.

Then it follows from Proposition 4.1 that $(-a, -b, -c) = 0$ in $H^3(K, \mathbb{Z}/2)$ (noticing that we may assume R is normal). Thus, the Pfister form ρ is isotropic over K and hence hyperbolic [Lam 2005, p. 319, Theorem 1.7]. The form ρ then contains a four-dimensional totally isotropic subspace, which must intersect the underlying space of the five-dimensional subform ϕ in a nontrivial subspace. Hence, ϕ is isotropic over K. □

Remark 4.3. Let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. Assume the characteristic of k is not 2. We record results and open questions on local-global principle for isotropy of quadratic forms over K as far as we know.

(1) When the residue field k is finite, Theorem 1.1 establishes the local-global principle with respect to discrete valuations in Ω_R only for rank-5 forms. If R is of the form $R = A[[t]]$, where A is a complete discrete valuation ring, then the same local-global principle is proved for quadratic forms of any rank ≥ 5 in [Hu 2012b, Theorem 1.2]. There the residue field may be any C_1-field.

(2) For general R with finite residue field, it remains an open question whether the local-global principle holds for quadratic forms of rank 6, 7 or 8.

(3) Generalizing an earlier result of [Colliot-Thélène et al. 2002], [Hu 2012b, Theorem 1.1] proves the local-global principle for forms of rank 3 or 4 when the residue field k is arbitrary (not necessarily finite, C_1 or B_1).

(4) The above results do not extend to binary forms even if k is finite. For example, Jaworski [2001, Theorem 1.5] shows that if K is the fraction field of the ring

$$R = k[[x, y, z]]/(z^2 - (y^2 - x^3)(x^2 - y^3)),$$

then $y^2 - x^3$ is a square in K_v for every discrete valuation v of K, but it is not a square in K.

The u-invariant. Let F be a field of characteristic $\neq 2$. Let $W(F)$ denote the Witt ring of quadratic forms over F, and $I(F)$ the fundamental ideal. For $a_1, \dots, a_n \in F^*$, let $\langle\langle a_1, \dots, a_n \rangle\rangle$ denote the n-fold Pfister form $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$. The n-th power $I^n(F)$ of the fundamental ideal $I(F)$ is generated by the n-fold Pfister forms. Recall that the *u-invariant* of F, denoted $u(F)$, is defined as the supremum of dimensions of anisotropic quadratic forms over F (so $u(F) = \infty$ if such dimensions can be arbitrarily large).

Let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. Assume that the residue field k of R is finite of characteristic $\neq 2$. Then the inequality $u(K) \geq 8$ may be easily seen as follows. Take any discrete valuation v corresponding to a height-1 prime ideal of the normalization of R. It follows from Corollary 3.16 and a theorem of Springer [Scharlau 1985,

p. 209, Corollary 2.6] that $u(\kappa(v))=4$. Take a four-dimensional diagonal form φ over K whose coefficients are units for v such that its residue form $\bar{\varphi}$ over $\kappa(v)$ is anisotropic, and let $\pi\in K$ be a uniformizer for v. Then $\varphi\perp\pi.\varphi$ is an eight-dimensional form over K that is anisotropic over K_v.

The rest of this subsection is devoted to the proof of Theorem 1.2, which asserts that if k is a finite field of characteristic $\neq 2$, then $u(K)\leq 8$ (or equivalently $u(K)=8$ according to the proceeding paragraph). Basically, we will follow the method of [Parimala and Suresh 2010] (see also [2012, Appendix]).

Proposition 4.4 [Parimala and Suresh 2010, Proposition 4.3]. *Let F be a field of characteristic $\neq 2$. Assume the following properties hold*:

(i) $I^4(F)=0$.

(ii) *Every element of $I^3(F)$ is represented by a threefold Pfister form.*

(iii) *Every element of $H^3(F,\mathbb{Z}/2)$ is the sum of two symbols.*

(iv) *If ϕ is a threefold Pfister form and φ_2 is two-dimensional quadratic form over F, then there exist $f,a,b\in F^*$ such that f is a value of φ_2 and $\phi=\langle 1,f\rangle\otimes\langle 1,a\rangle\otimes\langle 1,b\rangle$ in the Witt group $W(F)$.*

(v) *If $\phi=\langle 1,f\rangle\otimes\langle 1,a\rangle\otimes\langle 1,b\rangle$ is a threefold Pfister form and φ_3 is three-dimensional quadratic form over F, then there exist $g,h\in F^*$ such that g is a value of φ_3 and $\phi=\langle 1,f\rangle\otimes\langle 1,g\rangle\otimes\langle 1,h\rangle$ in the Witt group $W(F)$.*

Then $u(F)\leq 8$.

Property (i) in the above proposition is verified for the field K by using the following deep, well-known theorem:

Theorem 4.5 (Artin–Gabber). *Let R be a two-dimensional, henselian, excellent local domain with fraction field K and finite residue field k.*

Then for every prime number p different from the characteristic of k, the p-cohomological dimension $\mathrm{cd}_p(K)$ *of K is* 3.

When K and k have the same characteristic, this follows from a theorem of Artin [SGA 4.3 1973, Exposé XIX, Corollary 6.3]. When the characteristic of K is different from that of k, Gabber proved the analog of Artin's result. A different proof due to Kato may be given along the lines of the case treated in [Saito 1986, §5].

Corollary 4.6. *Let R be a two-dimensional, henselian, excellent local domain with finite residue field of characteristic $\neq 2$. Let K be the fraction field of R.*

Then $I^4(K)=0$.

Proof. This follows by combining Theorem 4.5 and a result of Arason, Elman and Jacob [Arason et al. 1986, Corollary 4]. □

Lemma 4.7. *Let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. Assume that k is a B_1 field of characteristic $\neq 2$.*

Then every element in $H^2(K, \mathbb{Z}/2)$ is the sum of two symbols.

Proof. By Theorem 3.4, every element in $\mathrm{Br}(K)[2] \cong H^2(K, \mathbb{Z}/2)$ has index dividing 4. A well-known theorem of Albert [1939, Chapter XI, §6, Theorem 9] then implies that it is the class of a tensor product of two quaternion algebras. □

Proof of Theorem 1.2. We have $I^3(K) \cong H^3(K, \mathbb{Z}/2)$ in view of Corollary 4.6 [Arason et al. 1986]. Thus, every element of $I^3(K)$ is represented by a threefold Pfister form by Theorem 4.2. That the field K has property (iii) in Proposition 4.4 is proved in Lemma 4.7. Finally, the same argument as in the proof of [Parimala and Suresh 2012, Appendix, Proposition 3] proves that the field K has properties (iv) and (v) in Proposition 4.4. The theorem is thus proved. □

Question 4.8 (Suresh). Let R, K and k be as usual, and assume the residue field k is an arbitrary (not necessarily finite) field of characteristic $\neq 2$. It is known that $u(K) = 4u(k)$ in each of the following special cases:

(1) k is finite (Theorem 1.2).

(2) k is hereditarily quadratically closed (i.e., every finite extension field of k is quadratically closed). This basically follows from the proof of [Colliot-Thélène et al. 2002, Theorem 3.6].

(3) Assume that $u(L) \leq 2^d u(k)$ for every finitely generated field extension L/k of transcendence degree $d \leq 1$. Then Harbater, Hartmann and Krashen have proved that $u(K) = 4u(k)$ holds in the following cases:

   (a) $R = A[\![t]\!]$, where A is a complete discrete valuation ring [Harbater et al. 2009, Corollary 4.19] and

   (b) K is a finite separable extension of $k((x, y))$ [Harbater et al. 2011, Corollary 4.2].

Question: Is the relation $u(K) = 4u(k)$ always true under the hypothesis of (3)? Is it true when assuming moreover R is complete?

Torsors under special orthogonal groups.

Theorem 4.9. *Let R be a two-dimensional, henselian, excellent local domain with fraction field K and residue field k. Assume k is a finite field of characteristic $\neq 2$.*

Then for any nonsingular quadratic form ϕ of rank ≥ 2 over K, the natural map

$$H^1(K, \mathrm{SO}(\phi)) \to \prod_{v \in \Omega_R} H^1(K_v, \mathrm{SO}(\phi))$$

is injective.

Proof. Let ψ and ψ' be nonsingular quadratic forms representing classes in $H^1(K, \mathrm{SO}(\phi))$. As they have the same dimension, the forms ψ and ψ' are isometric if and only if they represent the same class in the Witt group. Since ψ and ψ' also have the same discriminant, it follows from [Scharlau 1985, p. 82, Chapter 2, Lemma 12.10] that $\psi - \psi' \in I^2(K)$. Now it suffices to apply Lemma 4.10 below. □

Lemma 4.10. *Let R, K and k be as in Theorem 4.9. The natural map*

$$I^2(K) \to \prod_{v \in \Omega_R} I^2(K_v).$$

is injective.

Proof. In the case where R is the henselization of an algebraic surface over a finite field at a closed point, this is already established in [Colliot-Thélène et al. 2002, Theorem 3.10]. Here the argument is essentially the same with Proposition 4.1 and Corollary 4.6 substituting appropriate ingredients in that case. □

Remark 4.11. In Theorem 4.9, if ϕ is of dimension 2 or 3, one need not assume the residue field k finite.

Indeed, let ψ and ψ' be nonsingular forms representing classes in $H^1(K, \mathrm{SO}(\phi))$. In the two-dimensional case, assume $\psi' \cong \langle a, b\rangle$ and $\psi \cong \langle \alpha, \beta\rangle$. Then $\psi' \cong \psi$ if and only if the quaternion algebras (a, b) and (α, β) are isomorphic since the two forms have the same discriminant [Scharlau 1985, Chapter 2, Corollary 11.11]. In the three-dimensional case, assume $\psi' \cong \langle a, b, c\rangle$ and $\psi \cong \langle \alpha, \beta, \gamma\rangle$. Then $\psi' \cong \psi$ if and only if the quaternion algebras $(-ac, -bc)$ and $(-\alpha\gamma, -\beta\gamma)$ are isomorphic [Scharlau 1985, Chapter 2, Theorem 11.12]. Since two quaternion algebras are isomorphic if and only if their classes in the Brauer group coincide, the result then follows from the injectivity of the natural map

$$\mathrm{Br}(K) \to \prod_{v \in \Omega_R} \mathrm{Br}(K_v),$$

this last local-global statement being essentially established in [Colliot-Thélène et al. 2002, §1] (see also the proof of [Hu 2012b, Theorem 1.1]).

Remark 4.12. Let F be a field of characteristic $\neq 2$ and Ω a set of discrete valuations of F. For each integer $r \geq 2$, consider the following statements:

(LG_r) For any two nonsingular quadratic forms of rank r that have the same discriminant over F, if they are isometric over F_v for every $v \in \Omega$, then they must already be isometric over F.

(LG'_r) If a nonsingular quadratic form of rank r over F is isotropic over F_v for every $v \in \Omega$, then it is isotropic over F.

Theorem 4.9 amounts to saying that for every $r \geq 2$, (LG_r) is true for the field K with respect to its divisorial valuations. Our proof of this theorem does not rely on the local-global principle for the *isotropy* of quadratic forms. Note however that over an *arbitrary* field F (of characteristic $\neq 2$) one has $(\mathrm{LG}_r)+(\mathrm{LG}'_{r+2}) \Longrightarrow (\mathrm{LG}_{r+1})$.

Indeed, let ψ, ψ' be nonsingular quadratic forms of rank $r+1$ over F that have the same discriminant. Assume $\psi \cong \langle a_1\rangle \perp \psi_1$ with ψ_1 of rank r. If $(\psi)_{F_v} \cong (\psi')_{F_v}$ for every $v \in \Omega$, then $(\psi' \perp \langle -a_1\rangle)_{F_v}$ is isotropic for every $v \in \Omega$. By the local-global principle (LG'_{r+2}), ψ' represents a_1 over F whence a decomposition $\psi' \cong \langle a_1\rangle \perp \psi'_1$. It then suffices to apply (LG_r) to the forms ψ_1 and ψ'_1, thanks to Witt's cancellation theorem.

Together with the argument in Remark 4.11, this shows that if the natural map $\mathrm{Br}(F) \to \prod_{v\in\Omega} \mathrm{Br}(F_v)$ is injective and if the local-global principle with respect to Ω holds for quadratic forms of rank ≥ 5 over F, then (LG_r) is true for all $r \geq 2$. In particular, if F is the function field of an algebraic curve over the fraction field of a complete discrete valuation ring with *arbitrary* residue field of characteristic $\neq 2$, then the analog of Theorem 4.9 over F is true by [Colliot-Thélène et al. 2012, Theorems 3.1 and 4.3]. Note also that in this situation (LG'_2) is false in general.

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hu1983yong@gmail.com *Université Paris-Sud 11, 15 rue Georges Clemenceau, Mathématiques, Bâtiment 425, 91405 Orsay Cedex, France*

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The operad structure of admissible G-covers

Dan Petersen

We describe the modular operad structure on the moduli spaces of pointed stable curves equipped with an admissible G-cover. To do this we are forced to introduce the notion of an operad colored not by a set but by the objects of a category. This construction interpolates in a sense between "framed" and "colored" versions of operads; we hope that it will be of independent interest. An algebra over the cohomology of this operad is the same thing as a G-equivariant CohFT, as defined by Jarvis, Kaufmann and Kimura. We prove that the (orbifold) Gromov–Witten invariants of global quotients $[X/G]$ give examples of G-CohFTs.

1. Introduction

The notion of a cohomological field theory (CohFT) was introduced by Kontsevich and Manin [1994] as a simpler algebro-geometric relative of the notion of a (1+1)-dimensional topological conformal field theory, where holomorphic holes have been replaced with marked points (so one gets a theory modeled on gluing of compact Riemann surfaces along markings) and singular chains on moduli space have been replaced by (co)homology. One can give a succinct definition of a CohFT in the language of modular operads [Getzler and Kapranov 1998]: a CohFT is nothing but a coalgebra over the modular co-operad $H^\bullet(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. The main examples of CohFTs are the Gromov–Witten invariants of smooth projective varieties [Behrend and Manin 1996; Behrend 1997; Behrend and Fantechi 1997].

Jarvis, Kaufmann and Kimura [Jarvis et al. 2005] defined a generalization called a G-CohFT, where G is a finite group. Here one glues instead marked Riemann surfaces C equipped with a branched covering $P \to C$ which forms a G-torsor away from the markings. The gluing rules need to be slightly modified: firstly because one needs a marked point on P over each marked point on C in order that

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the gluing is independent of choices, secondly because one needs to impose the condition that the monodromies around the respective markings should be inverse to each other. In algebraic language, going from CohFTs to G-CohFTs corresponds to going from $\overline{\mathcal{M}}_{g,n}$ to spaces $\overline{\mathcal{M}}^G_{g,n}$ of *admissible G-covers*. One expects the main source of G-CohFTs to be the Gromov–Witten invariants of a global quotient $[X/G]$ (in the sense of orbifolds or stacks) of a smooth projective variety by a finite group [Chen and Ruan 2002; Abramovich et al. 2008]. Similar ideas can be found in a letter from Kontsevich to Borisov from 1996, published in [Abramovich 2008].

Analogous constructions have existed for a longer time in the physics literature, arising from Chern–Simons theory with a finite gauge group, see for example [Dijkgraaf and Witten 1990; Freed 1994]. Also closely related is Turaev's notion of a homotopy quantum field theory [Turaev 2010], which is a TQFT where all spaces and cobordisms are equipped with a map up to homotopy to a fixed target space X. Taking X a $K(G, 1)$ shows the similarity with G-CohFTs.

The definition of a G-CohFT in [Jarvis et al. 2005] is unsatisfactory in one minor respect. A G-CohFT is defined by a list of axioms, but just as for ordinary CohFTs one would expect it to be possible to bundle together these axioms by stating that a G-CohFT is an algebra over a certain operad. And it is clear from the definition that a G-CohFT is an algebra over *something*, it is just not clear in what sense the spaces $\overline{\mathcal{M}}^G_{g,n}$ form an operad.

We claim that the correct definition is that $\{\overline{\mathcal{M}}^G_{g,n}\}$ forms a modular operad colored by a *category*. The category in question is the action groupoid of G acting on itself by conjugation, the so-called loop groupoid of the group G. Moreover, this groupoid carries an involution given by "changing orientation of the loop", which corresponds to inversion in the group, and the gluing rules need to be modified in order to accommodate this involution.

Let us give a brief outline of the article. Section 2 contains background. We recall the notions of an admissible G-cover, of a category with duality, and of the loop groupoid of a finite group. As we will see in this paper, the structure of a category with duality is the "correct" structure to put on a category in order that it can serve as the collection of colors of a modular operad. The loop groupoid of a finite group is a category with duality, with the duality operation given by the inversion described in the preceding paragraph.

Section 3 contains a formal definition of a colored modular operad where the colors form a category with duality. We have not seen this defined in the literature. Although it is quite easy to define what this should mean for an ordinary operad, it is a bit subtle to come up with the "right" definition when one considers structures defined by more general graphs than trees (that is, cyclic, wheeled, modular, etc. versions of operads).

After this we explain in Section 4 how the work of Jarvis, Kaufmann and Kimura fits into this framework. We prove a result left open in their article, that the Gromov–Witten invariants of a global quotient $[X/G]$ endow the ring

$$H^{\bullet}(X, G)$$

of Fantechi and Göttsche with the structure of a G-CohFT.

In a sequel to this paper, we will extend the formalism of symmetric functions to this setting, and prove an analogue of Getzler and Kapranov's formula [Getzler and Kapranov 1998] for the effect of the "free modular operad" functor on the level of symmetric functions.

2. Background

In this section we begin by explaining the definition of an admissible G-cover, and the stratification of the moduli space of such covers, in an operad-like way. After that we recall the notion of a category with duality, that is, a category $\mathcal{C}$ equipped with a coherent equivalence $\mathcal{C} \cong \mathcal{C}^{\mathrm{op}}$. It turns out that whereas any category can serve as the collection of colors of an ordinary operad, only a category with duality can be the collection of colors of a cyclic or modular operad (or a similar operad-like structure modeled on undirected graphs). This is analogous to how any vector space can be an algebra over an operad, but only a vector space with an inner product can be an algebra over a cyclic operad.

Finally we recall the notion of the loop groupoid $\mathscr{L}G$ associated to the finite group G, and define the way in which we shall consider $\mathscr{L}G$ a category with duality. The relevance of this groupoid is that the spaces of admissible G-covers turn out to be an operad colored by $\mathscr{L}G$ whose algebras are exactly G-CohFTs. Let us remark that the appearance of the groupoid $\mathscr{L}G$ is not a coincidence. For one thing, it turns out that an algebra over a $\mathcal{C}$-colored operad needs in particular to be a representation of $\mathcal{C}$. Moreover, a representation of $\mathscr{L}G$ is exactly the same as a module over the Drinfel'd (quantum) double of the group G. This module structure is well known in Dijkgraaf–Witten theory, see for example [Dijkgraaf et al. 1991; Freed 1994], and the more recent references [Kaufmann and Pham 2009; Willerton 2008] on the mathematical side.

Moduli of admissible G-covers. Consider first the topological version of the story: let G be a (finite) group, and consider a variant of 2-dimensional TQFT modeled on sewing of compact oriented surfaces with boundary, equipped with a G-bundle. Then there is a basic compatibility condition needed in the definition of the sewing: for each boundary component, we get a G-bundle on S^1, and to glue surfaces we need an isomorphism between these G-bundles.

In the algebraic version, there is no analogue of gluing surfaces with boundary, and one is forced to work with punctured or marked surfaces. Since the G-cover will not in general extend across the punctures, one is moreover forced to work with ramified covers instead.

Definition 2.1. Let G be a finite group, and C an n-pointed nodal curve. An *admissible G-cover* is a covering $\pi : P \to C$ and a G-action on P, such that:

(1) the quotient P/G is identified with C via π;

(2) the map π is a G-torsor away from the nodes and markings;

(3) if $x \in P$ is a node, then the stabilizer G_x acts on the tangent spaces of the two branches at x by characters which are inverses of each other.

Condition (3) is the algebraic analogue of the sewing condition in the topological setting. Suppose we are given two Riemann surfaces C and C' with marked points y and y'. Let $\overline{C}$ be the nodal surface obtained by gluing y and y'. Let $P \to C \setminus \{y\}$ and $P' \to C' \setminus \{y'\}$ be G-torsors. These extend uniquely to ramified covers of C and C', and by choosing points x, x' in the fibers over y and y' they can be glued together to a covering $\overline{P} \to \overline{C}$ whenever the isotropy groups G_x and $G_{x'}$ coincide. But in general the resulting covering will not be smoothable, in the sense that there is no family of G-covers $P_t \to C_t$ of *smooth* curves, such that the limit as $t \to 0$ of this family is $\overline{P} \to \overline{C}$. Clearly, the topological obstruction to such a smoothing is that the monodromies of $P \to C \setminus \{y\}$ and $P' \to C' \setminus \{y'\}$, computed with respect to x and x', are inverse to each other in G. This final condition is equivalent to condition (3), which however makes sense over an arbitrary base field. Nevertheless, we shall stick to the language of Riemann surfaces in this article.

Though the notion of an admissible cover predates their work (admissible covers traditionally arise when one tries to compactify moduli spaces of unramified covers: see [Beauville 1977; Harris and Mumford 1982]), Definition 2.1 was first written down in this form in [Abramovich et al. 2003]. (They call coverings satisfying (3) *balanced*. We omit this adjective, as there will be no need for unbalanced coverings.) They also construct a moduli space for such covers. This theory arises from Abramovich, Vistoli and their coauthors' work on defining Gromov–Witten invariants of stacks: it is the special case of stable maps where the target space is the stack BG.

Definition 2.2. We denote by $\overline{\mathcal{M}}^G_{g,n}$ the moduli stack parametrizing admissible G-covers $P \to C$, where C is a stable n-pointed curve of genus g, together with a choice of a point $x_i \in P$ over every marked point $y_i \in C$.

That we include liftings x_i of the points y_i is crucial in order for there to be a natural operad structure.

The operadic structure. The spaces $\overline{\mathcal{M}}_{g,n}^G$ admit a kind of stratification by topological type, analogous to that of $\overline{\mathcal{M}}_{g,n}$. To an admissible cover $P \to C$ we associate a stable graph, namely the dual graph of C. The choice of a point in the fiber over each marking on C produces extra structure on this graph: by considering the monodromy of the covering over each marked point, we find that the legs of the graph are decorated by elements of G. Condition (3) above implies that the spaces $\overline{\mathcal{M}}_{g,n}^G$ have partially defined analogues of the gluing maps for $\overline{\mathcal{M}}_{g,n}$: one can glue together two legs precisely when they have mutually inverse decorations. So it would seem that they form a kind of colored operad where there is an involution on the collection of colors.

However, there is further structure present: the wreath product $G \wr \mathbb{S}_n$ acts on $\overline{\mathcal{M}}_{g,n}^G$, where $\mathbb{S}_n$ acts by permuting the markings and each copy of G acts by changing the choice of the lifted point $x_i \in P$. Changing the point x_i to $g \cdot x_i$ has the effect of changing the monodromy by conjugation with g. Hence G acts both on the spaces involved and on the set of colors (by conjugation), and the gluing maps are equivariant for this G-action.

Moreover, since there are no distinguished points in P in the fibers over the nodes of C, we see that gluing two points together also involves simultaneously forgetting the choices of liftings over the two markings, that is, quotienting by a diagonal action of G acting on both markings that are being glued together. It is instructive to compare this to the framed little disks operad, which parametrizes little disks equipped with a marked point on their boundaries, and gluing involves forgetting about this marked point.

We claim that the correct formalism for describing all this data — the presence of a coloring, the fact that gluing means simultaneously quotienting by the action of a group acting "on the legs", and compatibility with the action of the group on the set of colors — is the following: the spaces $\overline{\mathcal{M}}_{g,n}^G$ form a colored operad where the colors are the objects of the action groupoid $[G/G]$ in which G acts on its underlying set by conjugation. Finally there is the condition of inverse monodromy, which is now most easily described as an involution of this groupoid.

Categories with duality. The notion of a category with duality appears to have first arisen in K-theory, see for example [Knus 1991].

Definition 2.3. A *category with duality* is a category $\mathcal{C}$ equipped with a contravariant functor $\vee\colon \mathcal{C} \to \mathcal{C}$, and a natural isomorphism

$$\eta\colon\ \mathrm{id}_{\mathcal{C}} \to \vee \circ \vee,$$

such that the composition

$$\vee \xrightarrow{\eta\vee} \vee \circ \vee \circ \vee \xrightarrow{\vee\eta} \vee$$

is the identity.

Remark 2.4. To make sense of the last equation in the preceding definition, recall that if $\epsilon\colon F \to G$ is a natural transformation, and H is a contravariant functor, then the horizontal composition has reversed direction: one has $H\epsilon\colon HG \to HF$.

We write $x^\vee$ rather than $\vee(x)$, where x is either an object or a morphism in $\mathcal{C}$. An equivalent, more symmetric, definition is the following:

Definition 2.5. A *category with duality* is a category $\mathcal{C}$ equipped with a functor $\vee\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$, such that $\vee$ and $\vee^{\mathrm{op}}$ are quasi-inverses, and the resulting counit and unit $\vee^{\mathrm{op}}\vee \to \mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \to \vee\vee^{\mathrm{op}}$ are opposites of each other.

Example 2.6. The category of finitely generated projective modules over a ring A becomes a category with duality if we define $M^\vee = \mathrm{Hom}(M, A)$. More generally, any compact closed category is a category with duality.

Example 2.7. Any groupoid is a category with duality, with $\vee$ the identity on objects and $g^\vee = g^{-1}$ on morphisms.

Example 2.8. A discrete category with duality is a set with an involution.

Definition 2.9. A *pairing* between two objects x and y of a category with duality is a morphism $\phi\colon x \to y^\vee$. (Equivalently, it is a morphism $y \to x^\vee$.)

Definition 2.10. A pairing between x and itself is said to be *symmetric* if $\phi^\vee \circ \eta_x = \phi$.

Example 2.11. In the category of finitely generated projective A-modules, a pairing between M and N is a map $M \otimes N \to A$, and a symmetric pairing is a symmetric bilinear form.

If $\mathcal{C}$ and $\mathcal{D}$ are categories with duality, then so is the functor category $[\mathcal{C}, \mathcal{D}]$: if $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, its dual is defined as $\vee_{\mathcal{D}} \circ F \circ \vee_{\mathcal{C}}$.

Definition 2.12. A *weak symmetric functor* $\mathcal{C} \to \mathcal{D}$ is a functor F in $[\mathcal{C}, \mathcal{D}]$ with a symmetric pairing.

Explicitly, this means we have a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a natural transformation

$$\rho\colon F \circ \vee_{\mathcal{C}} \to \vee_{\mathcal{D}} \circ F$$

such that the diagram

$$\begin{tikzcd}
F \circ \vee_{\mathcal{C}} \arrow[rr, "\rho"] \arrow[d, "\eta_{\mathcal{D}}"] & & \vee_{\mathcal{D}} \circ F \\
\vee_{\mathcal{D}} \circ \vee_{\mathcal{D}} \circ F \circ \vee_{\mathcal{C}} \arrow[rr, "\rho"] & & \vee_{\mathcal{D}} \circ F \circ \vee_{\mathcal{C}} \circ \vee_{\mathcal{C}} \arrow[u, "\eta_{\mathcal{C}}"']
\end{tikzcd}$$

commutes. If ρ is an isomorphism, then F is *strong symmetric*.

Example 2.13. A weak symmetric functor from the one-object one-morphism category into $\mathcal{C}$ is an object of $\mathcal{C}$ with a symmetric pairing.

Example 2.14. The category **fdHilb** is naturally a category with duality, with $\vee$ the identity on objects and $T^\vee$ the adjoint of T. Let G be a group, considered as a category with duality as in Example 2.7. A (weak or strong) symmetric functor $G \to$ **fdHilb** is a unitary representation of G.

Example 2.15. If F is weak symmetric, then a pairing between x and y induces a pairing between $F(x)$ and $F(y)$.

The loop groupoid.

Definition 2.16. Let G be a group. We denote by $\mathscr{L}G$ the action groupoid of G acting on its underlying set by conjugation, and call this the *loop groupoid* of G.

Remark 2.17. The groupoid $\mathscr{L}G$ can equivalently (and more generally) be described as the functor category $\mathrm{Fun}(\mathbb{Z}, G)$, where $\mathbb{Z}$ and G are considered as one-object categories. Since $|\mathbb{Z}| \simeq S^1$, where $|*|$ denotes geometric realization, this explains the terminology.

Remark 2.18. One can show that for any two groupoids $\mathcal{G}$ and $\mathcal{H}$, there is a homotopy equivalence

$$|\mathrm{Fun}(\mathcal{H}, \mathcal{G})| \simeq \mathrm{map}(|\mathcal{H}|, |\mathcal{G}|),$$

see for instance [Strickland 2000]. In particular, $|\mathscr{L}G|$ is the space LBG of free loops on the classifying space BG. Another way to think about this is that $\mathscr{L}G$ is isomorphic to the groupoid of $\mathbb{C}$-points of the inertia stack of BG (see [Abramovich 2008, Section 5], for instance). The relationship between these viewpoints is that the inertia stack $I(\mathcal{X})$ is in general defined as the fiber product $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$. On the other hand, LX is given by the homotopy pullback $X \times^h_{X\times X} X$, for any space X.

In any case, this leads to a geometrically appealing situation. We are trying to combinatorially model gluing of surfaces equipped with G-torsors. In the topological setting, we needed for any two boundary circles an isomorphism between the respective G-bundles, which are (up to homotopy) points of LBG. Now we replace surfaces with their dual graphs, and find that we must decorate legs by $\mathscr{L}G$, which is a combinatorial model of LBG.

Definition 2.19. Let $\mathcal{C}$ be a groupoid and k a field. We define the *groupoid algebra* $k[\mathcal{C}]$ to be the k-algebra which is spanned as a vector space by the morphisms in $\mathcal{C}$, and whose product is defined on generators by

$$f * g = \begin{cases} f \circ g & \text{if this composition makes sense,} \\ 0 & \text{otherwise.} \end{cases}$$

This is extended bilinearly.

Just as for finite groups, $k[\mathcal{C}]$ is naturally a Hopf algebra, and a representation of $\mathcal{C}$ is a $k[\mathcal{C}]$-module. If G is a finite group, then $k[\mathscr{L}G]$ is exactly the Drinfel'd double of the usual group algebra $k[G]$, as mentioned in the introduction.

We shall always consider $\mathscr{L}G$ as a category with duality in the following way: first observe that $\mathbb{Z}$ and G, both being groups, carry a natural structure of category with duality. As remarked earlier, the category of functors between two categories with duality is again a category with duality, which gives a canonical such structure on $\mathscr{L}G = \mathrm{Fun}(\mathbb{Z}, G)$. More explicitly, the equivalence $\mathscr{L}G \to (\mathscr{L}G)^{\mathrm{op}}$ is defined on objects by $g \mapsto g^{-1}$, on morphisms by

$$(g \xrightarrow{h} hgh^{-1}) \mapsto (g^{-1} \xrightarrow{h^{-1}} hg^{-1}h^{-1}).$$

3. Operads colored by categories

In this section we give the general definition of an operad-like structure colored by a category. By an operad-like structure we mean, for example, a cyclic or modular operad, a (wheeled) PROP, a properad, a dioperad, etc. As we have remarked earlier, there is a distinction between directed and undirected graphs. As we shall explain in this section, the directed case is really a special case of the undirected one, so that it suffices to give a definition of an undirected operad-like structure colored by a category with duality.

However, we begin by giving a direct definition that works for ordinary operads, and which is very similar to the usual one. The general case requires some more combinatorics with graphs: in order to give a suitably general definition we define a category of graphs colored, in an appropriate sense, by some fixed category with duality, and construct the "free operad" functor combinatorially in terms of sums over such graphs. This functor is naturally a monad and one can then define an operad as an algebra over it. A pedagogical introduction to this point of view on operads and related structures can be found in [Markl 2008].

The case of ordinary operads.

Definition 3.1. Suppose a finite group G acts on a category $\mathcal{C}$. We define the *semidirect product* $\mathcal{C} \rtimes G$ to be the category with the same objects as $\mathcal{C}$, and whose morphisms $x \to y$ are pairs (ϕ, g), where $g \in G$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(x, yg)$. The composition is defined by

$$(\phi, g) \circ (\psi, h) = \big((\phi h) \circ \psi, g \cdot h\big).$$

Definition 3.2. The *wreath product* $\mathcal{C} \wr \mathbb{S}_n$ of a category with the symmetric group on n letters is the semidirect product $\mathcal{C}^n \rtimes \mathbb{S}_n$ with the obvious $\mathbb{S}_n$-action.

For the remainder of this section, we fix a cocomplete symmetric monoidal category $\mathcal{E}$, and a small category $\mathcal{C}$. We shall consider operads colored by $\mathcal{C}$ taking values in $\mathcal{E}$.

Definition 3.3. A $\mathcal{C} \wr \mathbb{S}$*-module* is a sequence $V(n)$, $n \geq 0$, of functors

$$V(n)\colon \mathcal{C}^{\mathrm{op}} \times (\mathcal{C} \wr \mathbb{S}_n) \to \mathcal{E}.$$

Definition 3.4. The *tensor product* of two $\mathcal{C} \wr \mathbb{S}$-modules is defined by

$$(V \otimes W)(n) = \coprod_{k+l=n} \mathrm{Ind}_{\mathcal{C}\wr\mathbb{S}_k \times \mathcal{C}\wr\mathbb{S}_l}^{\mathcal{C}\wr\mathbb{S}_n} V(k) \otimes W(l).$$

By induction we mean here the left Kan extension along $\mathcal{C} \wr \mathbb{S}_k \times \mathcal{C} \wr \mathbb{S}_l \hookrightarrow \mathcal{C} \wr \mathbb{S}_n$, which is the usual induction functor when $\mathcal{C}$ is a group.

Definition 3.5. The *plethysm* of two $\mathcal{C} \wr \mathbb{S}$-modules is defined by the coend

$$(V \circ W)(n) = \coprod_{k \geq 0} V(k) \otimes_{\mathcal{C}\wr\mathbb{S}_k} W^{\bigotimes k}(n) \stackrel{\text{def}}{=} \coprod_{k \geq 0} \int^{\mathcal{C}\wr\mathbb{S}_k} V(k) \otimes W^{\bigotimes k}(n),$$

where $W^{\bigotimes k}(n)$ is considered as a $\mathcal{C}^{\mathrm{op}} \wr \mathbb{S}_k$-module by virtue of the fact that a k-fold tensor product of a representation of $\mathcal{C}^{\mathrm{op}}$ is a representation of $\mathcal{C}^{\mathrm{op}} \wr \mathbb{S}_k$, using the symmetric monoidal structure on $\mathcal{E}$.

Proposition 3.6. *The category of $\mathcal{C} \wr \mathbb{S}$-modules is monoidal with plethysm as product.*

Proof. Let e be the $\mathcal{C} \wr \mathbb{S}$-module concentrated in degree one, where it is given by the composition

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}(-,-)} \mathbf{Set} \xrightarrow{\phi} \mathcal{E},$$

where $\phi(X) = \coprod_{x \in X} \mathbf{1}$, with $\mathbf{1}$ the monoidal identity in $\mathcal{E}$. In other words, we are forming the copower $\mathrm{Hom}(-,-) \odot \mathbf{1}$. Then e is both a left and right unit for plethysm, as one verifies using the canonical isomorphism (the "co-Yoneda lemma")

$$F(x) = \int^{\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(-,x) \odot F(-)$$

for any functor F defined on a category $\mathcal{C}$. Associativity is immediate from the fact that coproducts and coends can be freely commuted past each other, both being colimits. □

Example 3.7. If $\mathcal{C} = G$ is a group and $\mathcal{E} = R$-Mod, then $e(1)$ is given by the group ring $R[G]$, considered as a left and right G-module.

Definition 3.8. A $\mathcal{C}$*-operad* is a monoid in the monoidal category of $\mathcal{C} \wr \mathbb{S}$-modules.

Remark 3.9. In the usual theory of operads one often visualizes $V(n)$, the part of the operad in arity n, as a vertex with n incoming legs (inputs) and one outgoing leg (output). Then the $\mathbb{S}_n$-action on $V(n)$ arises by permuting the input legs, and the gluing maps of the operad correspond to attaching inputs to outputs.

In the $\mathcal{C}$-colored case we imagine that there is a representation of $\mathcal{C}$ attached to each input, and a representation of $\mathcal{C}^{\mathrm{op}}$ attached to each output, which explains why each $V(n)$ is now a representation of $\mathcal{C}^{\mathrm{op}} \times (\mathcal{C} \wr \mathbb{S}_n)$ — there is one output and n inputs. The gluing maps of the operad are defined by gluing input to output as before, except we must in addition form the coend of the representation of $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ obtained from the input and output legs which are identified.

Example 3.10. Let $\mathcal{C} = X$ be a set, thought of as a discrete category. An X-operad is the same thing as an operad colored by the set X.

Example 3.11. Let $\mathcal{C} = G$ be a group. A natural example here is the *framed little disks operad* of [Getzler 1994], for $G = \mathrm{SO}(N)$, which we claim can be thought of as a colored operad which has only one color, but where this color has a nontrivial automorphism group.

Let D_N be the closed unit disk in $\mathbb{R}^N$. Let $f\mathcal{D}_N(n)$ be the topological space parametrizing maps

$$\coprod_{i=1}^{n} D_N \hookrightarrow D_N,$$

where each factor is a composition of rotations, translations and positive dilations, and the images are disjoint. Then $\{f\mathcal{D}_N(n)\}$ is an $\mathrm{SO}(N)$-operad in **Spaces**, with edge contractions defined by composing embeddings with each other. In particular the space $f\mathcal{D}_N(n)$ has an action of

$$\mathrm{SO}(N)^{\mathrm{op}} \times (\mathrm{SO}(N)^n \rtimes \mathbb{S}_n).$$

We define this action by letting the first factor act by rotating the entire disk, and the second factor act by rotations and permutations of the individual embedded disks. The gluing maps are $\mathrm{SO}(N)$-equivariant as required, in the sense that any gluing map is invariant under the simultaneous action of $\mathrm{SO}(N)$ on the input and output legs that are being glued together.

More generally, any semidirect product operad $\mathcal{P} \rtimes G$ in the sense of [Salvatore and Wahl 2003] is an example of a G-operad in our sense. The notion of a G-operad is, however, more general. (Note that there is an unfortunate clash of notation: Salvatore and Wahl use the word G-operad to mean an operad in the category of spaces with a G-action.)

Remark 3.12. The preceding example also demonstrates that one should really be working throughout in an enriched setting, although we have not done so for readability's sake. Indeed, we do not want to think of $\mathrm{SO}(N)$ as just a group, but a topological group, and we want its actions on spaces to be continuous. One should therefore consider categories enriched over some closed symmetric monoidal category $\mathcal{V}$ (in the preceding example, $\mathcal{V} = \mathbf{Spaces}$): $\mathcal{E}$ is a $\mathcal{V}$-cocomplete symmetric monoidal $\mathcal{V}$-category, $\mathcal{C}$ is a small $\mathcal{V}$-category, and we are given a $\mathcal{V}$-functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \wr \mathbb{S}_n$ to $\mathcal{E}$. All coends, copowers, Kan extensions, etc. need to be replaced with their $\mathcal{V}$-analogues. We leave the details to the reader.

Remark 3.13. The author does not know a natural example of an operad colored by a category where that category is not in fact a groupoid. Such an example would perhaps be interesting.

Undirected graphs. We now wish to generalize to cyclic or modular operads, where there is no distinction between input and output. In light of Remark 3.9, it will thus be necessary to be able to identify $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$. So from now on we demand in addition that our category of colors $\mathcal{C}$ is a category with duality.

We shall follow the definitions and conventions of [Getzler and Kapranov 1998] regarding graphs, which we recall for the reader's convenience. A *graph* Γ is a finite set F of *flags*, a finite set V of *vertices*, a function $h\colon F \to V$, and an involution τ on F. The fixed points of τ are called *legs* and the orbits of length two are called *edges*.

A *morphism of graphs* $f : \Gamma \to \Gamma'$ consists of two functions $f_*\colon V \to V'$ and $f^*\colon F' \to F$ such that f^* is bijective on legs, injective on edges, and for which

$$F \setminus f^*(F') \underset{h\tau}{\overset{h}{\rightrightarrows}} V \xrightarrow{f_*} V'$$

is a coequalizer. Informally, f is a composition of automorphisms and edge contractions.

A graph with one vertex and no edges is called a *corolla*. For every $v \in V$ we denote by $\gamma(v)$ the corolla with flag set $h^{-1}(v)$.

A *dual graph* is a graph with a *genus function* $g\colon V \to \{0, 1, 2, \dots\}$. We denote by $n(\Gamma)$ the number of legs of a graph Γ. For a vertex v, we use the shorthand $n(v) = n(\gamma(v))$. A *morphism of dual graphs* is a morphism $f : \Gamma \to \Gamma'$ of the underlying graphs such that for all $v' \in V'$ we have

$$2g(v') - 2 + n(v') = \sum_{f_*(v)=v'} \big(2g(v) - 2 + n(v)\big).$$

If Γ is a dual graph, then we declare its *genus* $g(\Gamma)$ to be the unique integer satisfying

$$2g(\Gamma) - 2 + n(\Gamma) = \sum_{v \in V} \big(2g(v) - 2 + n(v)\big).$$

A simple lemma shows that if $f\colon \Gamma \to \Gamma'$ is a morphism of dual graphs, then $g(\Gamma) = g(\Gamma')$. A dual graph is called *stable* if for each vertex v the inequality

$$2g(v) - 2 + n(v) > 0$$

is satisfied.

Remark 3.14. The idea of a dual graph is best thought of topologically as follows. We imagine that a vertex of genus g with n adjacent legs describes a compact oriented surface of genus g with n boundary circles. Then the number $2g-2+n$ is just the negative of the Euler characteristic of the surface. If we think of an edge contraction as an operation which glues together the corresponding boundary components, then the formulas in the definition of a dual graph express that Euler characteristic should be additive over gluing of circles.

Definition 3.15. A *$\mathcal{C}$-graph* is a graph Γ with the following extra data: for every flag x we are given an object A_x of $\mathcal{C}$, and for an edge connecting the flags x and y we are given a pairing between A_x and A_y.

Definition 3.16. A *morphism of $\mathcal{C}$-graphs* is a morphism $\Gamma \to \Gamma'$ of underlying graphs, together with a morphism $q_x\colon A_{f^*(x)} \to A_x$ for every flag x of Γ', such that for an edge between x and y in Γ', the following diagram commutes:

$$\begin{tikzcd} A_{f^*(x)} \arrow[r] \arrow[d, "q_x"'] & A^\vee_{f^*(y)} \\ A_x \arrow[r] & A^\vee_y \arrow[u, "q^\vee_y"'] \end{tikzcd}$$

Remark 3.17. One can describe a $\mathcal{C}$-graph as a graph Γ together with a symmetric functor $\mathcal{F} \to \mathcal{C}$, where $\mathcal{F}$ is an appropriate category with duality defined in terms of the flags and edges of Γ. Then a morphism of $\mathcal{C}$-graphs can be defined more simply in terms of a natural transformation. We leave the details to the reader.

Operads as algebras.

Notation 3.18. Let $\boldsymbol{S}$ be the category of stable $\mathcal{C}$-graphs. Let $\boldsymbol{S}^0$ be the full subcategory of corollas in $\boldsymbol{S}$. Let $[\boldsymbol{S}^0, \mathcal{E}]$ denote the category of functors $\boldsymbol{S}^0 \to \mathcal{E}$.

Definition 3.19. We call the objects of $[\boldsymbol{S}^0, \mathcal{E}]$ *stable $\mathcal{C} \wr \mathbb{S}$-modules*.

Remark 3.20. Suppose $\mathcal{C}$ is trivial. Then a functor $\boldsymbol{S}^0 \to \mathcal{E}$ is the same thing as a *stable $\mathbb{S}$-module* in the terminology of [Getzler and Kapranov 1998], as $\boldsymbol{S}^0$ has the obvious skeleton

$$\boldsymbol{S}^0 \cong \coprod_{\substack{g,n\geq 0\\ 2g-2+n>0}} \mathbb{S}_n.$$

Hence a functor from $\boldsymbol{S}^0$ to $\mathcal{E}$ is just a family of $\mathbb{S}_n$-representations indexed by g and n, which recovers the definition of Getzler and Kapranov and justifies our terminology. More generally one has for any $\mathcal{C}$ that

$$\boldsymbol{S}^0 \cong \coprod_{\substack{g,n\geq 0 \\ 2g-2+n>0}} \mathcal{C} \wr \mathbb{S}_n.$$

Notation 3.21. Let $\mathrm{Bij}(\boldsymbol{S})$ denote the full subcategory of $\boldsymbol{S}$ consisting of graph morphisms which do not contract any edge.

Remark 3.22. Any functor $V\colon \boldsymbol{S}^0 \to \mathcal{E}$ can be extended to a functor $\mathrm{Bij}(\boldsymbol{S}) \to \mathcal{E}$ via

$$V(\Gamma) = \bigotimes_{v \in V(\Gamma)} V(\gamma(v)).$$

Note that if Γ is stable then so are all the $\gamma(v)$.

Definition 3.23. Let $\mathbb{M}$ be the endofunctor on $[\boldsymbol{S}^0, \mathcal{E}]$ defined by

$$\mathbb{M}V(\gamma) = \operatorname*{colim}_{\Gamma \in \mathrm{Bij}(\boldsymbol{S})\downarrow\gamma} V(\Gamma)$$

for any corolla γ. Here $\mathrm{Bij}(\boldsymbol{S}) \downarrow \gamma$ denotes the slice category over γ; its objects are graphs in $\boldsymbol{S}$ with a map to γ, and its morphisms are morphisms over γ which do not contract any edges.

For any corolla $\gamma \in \boldsymbol{S}^0$ there is a natural map $V(\gamma) \to \mathbb{M}V(\gamma)$ induced by sending id_γ to the corresponding morphism in $\mathrm{Bij}(\boldsymbol{S}) \downarrow \gamma$. This defines a natural transformation $\eta\colon \mathrm{id}_{[\boldsymbol{S}^0,\mathcal{E}]} \to \mathbb{M}$. There is also a natural transformation $\mu\colon \mathbb{M}^2 \to \mathbb{M}$, defined as usual by "erasing braces" (see [Markl 2008]).

Proposition 3.24. *The functor $\mathbb{M}$ is a monad with unit η and multiplication μ.*

Proof. A rather conceptual proof can be found in [Getzler and Kapranov 1998], which carries through with only minor changes to the $\mathcal{C}$-colored setting. The necessary commutative diagrams can also be checked somewhat tediously by hand. □

Definition 3.25. A *modular $\mathcal{C}$-operad* is an $\mathbb{M}$-algebra.

Remark 3.26. A posteriori, the fact that $\mathbb{M}$ turns out to be a monad can be explained by saying that $\mathbb{M}$ maps a stable $\mathcal{C}\mathbb{S}$-module V to the underlying stable $\mathcal{C} \wr \mathbb{S}$-module of the free modular $\mathcal{C}$-operad generated by V. Hence the fact that $\mathbb{M}$ is a monad expresses the fact that the free modular operad functor is left adjoint to the forgetful functor sending a modular operad to its underlying stable $\mathcal{C} \wr \mathbb{S}$-module.

Remark 3.27. One can describe modular $\mathcal{C}$-operads more explicitly in the following way. A modular $\mathcal{C}$-operad $\mathcal{A}$ consists of:

(1) for any $g, n \geq 0$ such that $2g - 2 + n > 0$, and any n-tuple $(x_1, \dots, x_n)$ of objects of $\mathcal{C}$, an object

$$\mathcal{A}(g, x_1, \dots, x_n)$$

of $\mathcal{E}$;

(2) for any $\sigma \in \mathbb{S}_n$ a map

$$\mathcal{A}(g, x_1, \dots, x_n) \to \mathcal{A}(g, x_{\sigma(1)}, \dots, x_{\sigma(n)});$$

(3) for any morphism $x_i \mapsto x_i'$ in $\mathcal{C}$ a map

$$\mathcal{A}(g, x_1, \dots, x_i, \dots, x_n) \to \mathcal{A}(g, x_1, \dots, x_i', \dots, x_n);$$

(4) for any i and j and for every pairing between x_i and y_j, a gluing map

$$\mathcal{A}(g_1, x_1, \dots, x_n) \otimes \mathcal{A}(h, y_1, \dots, y_m) \to \mathcal{A}(g + h, x_1, \dots, \widehat{x_i}, \dots, \widehat{y_j}, \dots, y_m);$$

(5) for any $i \neq j$ and for every pairing between x_i and x_j, a gluing map

$$\mathcal{A}(g, x_1, \dots, x_n) \to \mathcal{A}(g + 1, x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n).$$

One thinks of $\mathcal{A}(g, x_1, \dots, x_n)$ as the value of $\mathcal{A}$ on a corolla of genus g with n legs decorated by $x_1, \dots, x_n$. We will not list the functoriality conditions and commutative diagrams that these maps must satisfy.

Algebras over operads. The notion of an algebra over an operad can be defined in various levels of generality. We assume in this section that the target category $\mathcal{E}$ is compact closed, that is, every object is dualizable, which will be sufficient for this article. In particular, this implies that $\mathcal{E}$ is a category with duality.

Definition 3.28. Suppose given a weak symmetric functor $\rho \colon \mathcal{C} \to \mathcal{E}$. We associate to ρ its *endomorphism operad* End_ρ. In the notation of Remark 3.27, it is defined on objects by

$$\mathrm{End}_\rho(g, x_1, \dots, x_n) = \bigotimes_{i=1}^{n} \rho(x_i).$$

Every pairing between x and y in $\mathcal{C}$ gives a pairing between $\rho(x)$ and $\rho(y)$ in $\mathcal{E}$ in the usual sense, that is, a map

$$\rho(x) \otimes \rho(y) \to \mathbf{1},$$

where $\mathbf{1}$ is the monoidal unit in $\mathcal{E}$. This pairing defines the gluing maps for the modular $\mathcal{C}$-operad End_ρ.

Definition 3.29. An *algebra* over a modular $\mathcal{C}$-operad $\mathcal{A}$ is a weak symmetric functor $\rho \colon \mathcal{C} \to \mathcal{E}$ and a morphism $\mathcal{A} \to \mathrm{End}_\rho$.

Other operad-like structures. By considering some other category of graphs $\boldsymbol{G}$ instead of $\boldsymbol{S}$ one can define in a similar way $\mathcal{C}$-colored versions of other operad-like constructions. One lets $\boldsymbol{G}^0$ be the subcategory of corollas. In order for the definition of $\mathbb{M}$ to make sense, one needs to assume that for any $\Gamma \in \mathrm{ob}(\boldsymbol{G})$ and $v \in V(\Gamma)$, we also have $\gamma(v) \in \mathrm{ob}(\boldsymbol{G})$. To define the multiplication map μ one needs to assume that $\boldsymbol{G}$ is closed under "erasing braces". With these assumptions, it will remain true that $\mathbb{M}$ is a monad.

For example, take $\boldsymbol{G}$ to be the full subcategory of trees in $\boldsymbol{S}$. The algebras over the corresponding monad are exactly the *cyclic $\mathcal{C}$-operads*.

We would also like to be able to define $\mathcal{C}$-colored versions of more ordinary things like operads and PROPs, which are modeled on directed graphs. One could repeat appropriate modification of all our definitions for digraphs, but there is a quicker way. This is based on the observation that an ordinary operad is the same thing as a two-colored cyclic operad whose colors are {input, output}, and where the gluing rules have been twisted by an involution: one is only allowed to glue an input leg to an output, and vice versa.

Observe that for any category $\mathcal{C}$, there is an obvious structure of category with duality on the disjoint union $\mathcal{C} \amalg \mathcal{C}^{\mathrm{op}}$.

Definition 3.30. We define a *$\mathcal{C}$-digraph* to be a $(\mathcal{C} \amalg \mathcal{C}^{\mathrm{op}})$-graph. Flags decorated by objects in $\mathcal{C}$ are called *incoming* and flags decorated by objects in $\mathcal{C}^{\mathrm{op}}$ are *outgoing*.

Remark 3.31. Note that every edge in a $\mathcal{C}$-digraph consists of exactly one incoming and one outgoing flag, by our definition of a pairing.

Let then for instance $\boldsymbol{G}$ be the category of $\mathcal{C}$-digraphs which are trees, and where each vertex is adjacent exactly one outgoing flag. Algebras over the resulting monad are called $\mathcal{C}$-operads. If $\boldsymbol{G}$ consists of arbitrary $\mathcal{C}$-digraphs which are trees, then we have defined the notion of a $\mathcal{C}$-PROP. This also gives the correct notions of algebras over $\mathcal{C}$-operads and $\mathcal{C}$-PROPs.

Proposition 3.32. *This definition of a $\mathcal{C}$-operad coincides with Definition 3.8.*

Proof. We allow ourselves to be brief, as the proof is similar to the uncolored case [Markl 2008, Theorem 40]. The only new subtlety in the $\mathcal{C}$-colored situation is that we must compare the coend appearing in Definition 3.5 with the colimit in Definition 3.23.

Consider the full subcategory $\mathcal{G}$ of $\mathrm{Bij}(\boldsymbol{G}) \downarrow \gamma$ where the underlying graph is given by some fixed graph Γ with a single edge. An object of $\mathcal{G}$ consists of a decoration of this edge, that is, two objects x and y of $\mathcal{C} \amalg \mathcal{C}^{\mathrm{op}}$, and a pairing between x and y. It follows that an object of $\mathcal{G}$ is an arrow in $\mathcal{C}$. By comparing with Definition 3.16, we see that a morphism between $x \to y$ and $x' \to y'$ is a commutative square

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ x' \arrow[r] & y' \arrow[u] \end{tikzcd}.$$

In other words, $\mathcal{G}$ coincides with the so-called *twisted arrow category* of $\mathcal{C}$, with its natural map to $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. If F is any functor on $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, then

$$\operatorname*{colim}_{\mathcal{G}} F = \int^{\mathcal{C}} F,$$

see [MacLane 1971, Example IX.6.3]. For a graph Γ with n edges, we find instead the category $\mathcal{C} \wr \mathbb{S}_n$, and the coend over $\mathcal{C} \wr \mathbb{S}_n$. It is now not hard to show that the two definitions of a $\mathcal{C}$-operad coincide. □

4. Equivariant CohFTs

The definition of a G-CohFT. Recall that $\overline{\mathcal{M}}^G_{g,n}$ is the moduli stack parametrizing stable n-pointed curves C of genus g equipped with an admissible G-torsor $P \to C$ and liftings of the n markings to P. Let $\boldsymbol{S}$ be the category of stable $\mathscr{L}G$-graphs, and again $\boldsymbol{S}^0$ the full subcategory of corollas. Let **Stack** be the category of DM-stacks over some fixed base k where $|G|$ is invertible. The analytically inclined reader can also take **Stack** to be the category of complex orbispaces.

Remark 4.1. There are two minor issues at this point. We wish to consider operads in **Stack**. Unfortunately, we formulated the earlier theory in a cocomplete symmetric monoidal category, but **Stack** is not cocomplete, and it is a 2-category! However, neither of these are serious problems. First of all, even though **Stack** is not cocomplete, all colimits that occur in the definition of a modular $\mathscr{L}G$-operad will exist: indeed, whenever the category of colors is a finite groupoid, it is easy to see that one only needs to assume the existence of coproducts and quotients by actions of finite groups. Secondly, there are no 2-categorical surprises, either. As mentioned in Remark 3.12 the definitions carry over to the enriched case, in particular when the target category is **Cat**-enriched, that is, a strict 2-category. Then $\mathbb{M}$ becomes a **Cat**-enriched monad, that is, a strict 2-monad. However, one should not define an operad in this case as a strict algebra over it but as a pseudoalgebra: for instance, the diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1} \times \overline{\mathcal{M}}_{g',n'+2} \times \overline{\mathcal{M}}_{g'',n''+1} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n+1} \times \overline{\mathcal{M}}_{g'+g'',n'+n''+1} \arrow[d] \\ \overline{\mathcal{M}}_{g+g',n+n'+1} \times \overline{\mathcal{M}}_{g'',n''+1} \arrow[r] & \overline{\mathcal{M}}_{g+g'+g'',n+n'+n''} \end{tikzcd}$$

does not commute strictly but only up to a canonical natural transformation.

Definition 4.2. For a corolla $\gamma \in \mathrm{ob}(\boldsymbol{S}^0)$ with genus g, and legs decorated by $\gamma_1, \ldots, \gamma_n$, let $\mathcal{M}(\gamma)$ be the open and closed substack of $\overline{\mathcal{M}}^G_{g,n}$ where the monodromy around the i-th marking is given by γ_i, for $i = 1, \ldots, n$. Then $\mathcal{M}$ naturally becomes a stable $\mathscr{L}G \wr \mathbb{S}$-module in **Stack**.

Theorem 4.3. *The functor $\mathcal{M}$ extends naturally to a modular $\mathscr{L}G$-operad in* **Stack**.

Proof. The structure maps in the operad $\mathcal{M}$ are given by gluing together admissible covers along markings. The monodromy condition ensures that this is well defined. For the necessary associativity conditions, apply the 2-Yoneda lemma: on the level of moduli functors, associativity is clear. □

Since homology is a symmetric monoidal functor, one immediately obtains a modular $\mathscr{L}G$-operad $H_\bullet(\mathcal{M})$ in the category of graded $\mathbb{Q}$-vector spaces (assuming that we are working over the complex numbers). Algebraically, it is more natural to consider the co-operad $H^\bullet(\mathcal{M})$ associated to some Weil cohomology theory. In any case one can consider (co)algebras over the resulting (co)operads. The main examples of such algebras are the G-equivariant cohomological field theories of [Jarvis et al. 2005]. They assume the existence of a flat identity, which is not always natural from the operadic perspective. If we agree that a nonunital CohFT is defined by omitting axioms (iii) and (iv) from [loc. cit., Definition 4.1], then we can state the following result.

Proposition 4.4. *A coalgebra $\mathcal{H}$ over $H^\bullet(\mathcal{M}, \mathbb{Q})$ (in the category of finite-dimensional vector spaces) is the same thing as a nonunital G-CohFT.*

Proof. The usual proof that a coalgebra over $H^\bullet(\overline{\mathcal{M}}_{g,n})$ is the same thing as a CohFT carries through with only minor changes. □

Remark 4.5. Axiom (i), that $\mathcal{H}$ is a G-graded G-module, just says that $\mathcal{H}$ is a representation of $\mathscr{L}G$. Write $\mathcal{H} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma$. We remark that any algebra over $H^\bullet(\mathcal{M}, \mathbb{Q})$ has a natural structure of a nonunital braided commutative G-Frobenius algebra obtained by imitating the construction in [Jarvis et al. 2005]. The multiplication is defined by noting that $\overline{\mathcal{M}}^G_{0,3}$ is a finite union of points (generally with nontrivial automorphism group), each of which defines a partial multiplication on $\mathcal{H}$:

$$\mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} \to \mathcal{H}_{\gamma_3},$$

where γ_i is the monodromy around the i-th marked point. A total multiplication can then be defined by summing over the distinguished points $\xi(\gamma_1, \gamma_2, \gamma_2^{-1}\gamma_1^{-1})$; see Section 2.5 of [Jarvis et al. 2005]. The arguments there extend to show associativity (that is, the WDVV equation, via $\overline{\mathcal{M}}^G_{0,4}$) and the trace axiom (via $\overline{\mathcal{M}}^G_{1,1}$).

Gromov–Witten invariants of global quotients. Just as the main example of a CohFT is the cohomology of a smooth projective variety, it is expected that the main example of a G-CohFT comes from a smooth projective variety with a G-action. So let for the remainder of this section X be a smooth projective variety acted upon by G. For simplicity, we work over the complex numbers, so that classes of curves lie in the second integral homology group; it is well known also how to describe this algebraically.

Definition 4.6. Let $\beta \in H_2(X/G, \mathbb{Z})$. Define $\overline{\mathcal{M}}^G_{g,n}(X, \beta)$ to be the moduli stack parametrizing the following data:

- an admissible G-cover $P \to C$, where C is a *prestable* n-pointed curve of genus g
- a G-equivariant map $f : P \to X$, such that the induced map $\overline{f} : C \to X/G$ is stable in the sense of Kontsevich and $\overline{f}_*[C] = \beta$;
- a section of $P \to C$ over each marked point of C.

Equivalently, we have

$$\overline{\mathcal{M}}^G_{g,n}(X, \beta) = \overline{\mathcal{M}}_{g,n}([X/G], \beta) \times_{\overline{\mathcal{M}}_{g,n}(BG)} \overline{\mathcal{M}}^G_{g,n},$$

where $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ denotes the usual space of stable maps to a stack.

It follows from [Behrend and Fantechi 1997; Abramovich et al. 2008] that $\overline{\mathcal{M}}^G_{g,n}(X, \beta)$ has a virtual fundamental class $[\overline{\mathcal{M}}^G_{g,n}(X, \beta)]^{\mathrm{vir}}$ defined by the relative obstruction theory given by the G-invariants of $\boldsymbol{R}\pi_* f^* T_X$, where $\pi : P \to \overline{\mathcal{M}}^G_{g,n}(X, \beta)$ is the natural projection.

Definition 4.7. Denote by $\mathcal{M}(X, \beta)$ the stable $\mathcal{L}G \wr \mathbb{S}$-module in **Stack** given by the spaces $\overline{\mathcal{M}}^G_{g,n}(X, \beta)$. We extend $\mathcal{M}(X, \beta)$ to a functor from stable $\mathcal{L}G$-graphs to stacks, but in a slightly different way than in Remark 3.22: for an $\mathcal{L}G$-graph Γ with n vertices, we define

$$\mathcal{M}(X, \beta)(\Gamma) = \coprod_{\beta_1+\cdots+\beta_n=\beta} \prod_{v\in V(\Gamma)} \mathcal{M}(X, \beta_i)(\gamma(v)).$$

Definition 4.8. The *inertia variety* of X is defined by

$$\mathrm{I}X = \coprod_{g\in G} X^g.$$

Note that $\mathrm{I}X$ is naturally a representation of $\mathcal{L}G$ in the category of algebraic varieties, since the element $h \in G$ carries X^g to $X^{hgh^{-1}}$.

Since X is smooth, its inertia variety is smooth too, see [Iversen 1972].

Definition 4.9. Let **Corr** be the $\mathbb{Q}$-linear category, whose objects are smooth and proper DM-stacks, and whose morphisms are given by

$$\mathrm{Hom}_{\mathbf{Corr}}(\mathcal{X}, \mathcal{Y}) = A^\bullet(\mathcal{Y} \times \mathcal{X}),$$

where the latter denotes the Chow ring with rational coefficients. Composition is defined via the formula

$$f \circ g = p_{13,*}(p_{12}^* f \cup p_{23}^* g).$$

Remark 4.10. The category of spans of smooth proper DM-stacks, with morphisms defined via pullbacks, sits naturally inside **Corr**: a span

$$\mathcal{X} \xleftarrow{f} \mathcal{Z} \xrightarrow{g} \mathcal{Y}$$

defines a morphism $\mathcal{X} \to \mathcal{Y}$ in **Corr** via $(g \times f)_*[\mathcal{Z}]$.

Remark 4.11. Let **Corr**$'$ be the category defined in the same way, except with varieties instead of stacks. The natural inclusion **Corr**$' \hookrightarrow$ **Corr** induces an equivalence of categories once one takes the pseudoabelian completion of both categories, see [Toen 2000].

The category **Corr** is compact closed with every object equal to its own dual. The counit is given by the span

$$\mathcal{X} \times \mathcal{X} \xleftarrow{\Delta} \mathcal{X} \to \operatorname{Spec} k,$$

and vice versa for the unit. This is a kind of motivic Poincaré duality; it gives the usual Poincaré duality on any realization functor $H^\bullet$. Moreover, $\mathrm{I}X$ is a symmetric functor $\mathscr{L}G \to$ **Corr** since $X^g = X^{g^{-1}}$. It follows that we can talk about the endomorphism operad $\mathrm{End}(\mathrm{I}X)$, which is a modular $\mathscr{L}G$-operad in **Corr**. Its value on an n-tuple $(g_1, \ldots, g_n)$ of elements of G is the product $\prod_{i=1}^n X^{g_i}$.

There are natural evaluation maps $\overline{\mathcal{M}}^G_{g,n}(X, \beta) \to \mathrm{I}X$, giving a diagram

$$\overline{\mathcal{M}}^G_{g,n} \leftarrow \overline{\mathcal{M}}^G_{g,n}(X, \beta) \to (\mathrm{I}X)^n,$$

equivariant for the $\mathscr{L}G \wr \mathbb{S}_n$-action on all three spaces. We can write this as a diagram of stable $\mathscr{L}G \wr \mathbb{S}$-modules in **Stack**:

$$\mathcal{M} \xleftarrow{\pi} \mathcal{M}(X, \beta) \xrightarrow{\mathrm{ev}} \mathrm{End}(\mathrm{I}X).$$

Pushing forward the virtual fundamental class defines a morphism $\mathcal{M} \to \mathrm{End}(\mathrm{I}X)$ of $\mathscr{L}G \wr \mathbb{S}$-modules in **Corr**,

$$(\mathrm{ev} \times \pi)_*[\mathcal{M}(X, \beta)]^{\mathrm{vir}} \in A^\bullet(\mathrm{End}(\mathrm{I}X) \times \mathcal{M}).$$

Theorem 4.12. *For any fixed $\beta \in H_2(X/G, \mathbb{Z})$, the morphism just defined gives the inertia variety $\mathrm{I}X$ the structure of an algebra over $\mathcal{M}$ in* **Corr**.

Proof. We need to show that for any morphism $\Gamma \to \Gamma'$ in $\boldsymbol{S}$, the diagram

$$\begin{tikzcd}
\mathcal{M}(\Gamma') \arrow[r] & \mathrm{End}(\mathrm{I}X)(\Gamma') \\
\mathcal{M}(\Gamma) \arrow[r] \arrow[u] & \mathrm{End}(\mathrm{I}X)(\Gamma) \arrow[u]
\end{tikzcd}$$

in **Corr** commutes. We may assume that $\Gamma \to \Gamma'$ is given by contracting a single edge, which is decorated by $g, g^{-1} \in G$. In this case we have

$$\mathrm{End}(\mathrm{I}X)(\Gamma) = \mathrm{End}(\mathrm{I}X)(\Gamma') \times X^g \times X^{g^{-1}}.$$

Unwinding the definition of composition in **Corr**, we see that we must study the following diagram in **Stack**:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}(X,\beta)(\Gamma') \arrow[r] \arrow[d] & \mathrm{End}(\mathrm{I}X)(\Gamma') \\
\mathcal{M}(\Gamma) \arrow[r, "\mathrm{gl}"'] \arrow[dd, equal] & \mathcal{M}(\Gamma') & \\
 & \mathrm{End}(\mathrm{I}X)(\Gamma) \arrow[dr, phantom, "\square"] & \mathrm{End}(\mathrm{I}X)(\Gamma') \times X^g \arrow[l, "\mathrm{id}\times\Delta"'] \arrow[uu] \\
\mathcal{M}(\Gamma) & \mathcal{M}(X,\beta)(\Gamma) \arrow[l] \arrow[u] & B. \arrow[l] \arrow[u]
\end{tikzcd}$$

Here Δ is the diagonal map $X^g \to X^g \times X^{g^{-1}} = X^g \times X^g$, and gl is the gluing map of the operad $\mathcal{M}$ in **Stack**. The spaces A and B are defined by the requirement that the smaller squares are cartesian. What we need to show is that the pushforwards of $\mathrm{gl}^![\mathcal{M}(X,\beta)(\Gamma')]^{\mathrm{vir}}$ and $\Delta^![\mathcal{M}(X,\beta)(\Gamma)]^{\mathrm{vir}}$ to $A^\bullet(\mathrm{End}(\mathrm{I}X)(\Gamma') \times \mathcal{M}(\Gamma))$ coincide.

There is a natural morphism $h\colon B \to A$, which is not an isomorphism. Indeed, after unwinding the fiber products one finds that B parametrizes all the same data as $\mathcal{M}(X,\beta)(\Gamma')$, together with a decomposition of the admissible cover $P \to C$ into two components whose genera and markings are determined by Γ. The stack A parametrizes the same thing, except one only has a decomposition of the *stabilization* of $P \to C$ into two components. However, one can show that h is an isomorphism on an open set, and then prove that $h_*\Delta^![\mathcal{M}(X,\beta)(\Gamma)]^{\mathrm{vir}} = \mathrm{gl}^![\mathcal{M}(X,\beta)(\Gamma')]^{\mathrm{vir}}$, which proves the claim. What we need are exactly the properties (III) and (IV) in [Behrend and Manin 1996], which they refer to as "cutting edges" and "isogenies". These are not proven exactly in this form in [Abramovich et al. 2008], but they follow by combining Propositions 5.3.1 and 5.3.2 there, the arguments of [Behrend 1997, Proposition 8], and the calculation immediately following Lemma 10 in this last reference, which generalize from prestable pointed curves to prestable pointed curves with an admissible cover. □

Definition 4.13. We define Θ_X to be the usual Novikov ring of X/G, that is, the ring of formal power series in the variables q^β, where $\beta \in H_2(X/G, \mathbb{Z})$ is the class of a curve, and $q^\beta q^{\beta'} = q^{\beta+\beta'}$.

Definition 4.14. Let $\mathbf{Corr} \otimes \Theta_X$ be the category obtained by tensoring all hom-spaces in $\mathbf{Corr}$ with Θ_X.

We define a morphism $\phi\colon \mathcal{M} \to \mathrm{End}(\mathrm{I}X)$ in $\mathbf{Corr} \otimes \Theta_X$ by

$$\sum_\beta (\mathrm{ev} \times \pi)_*[\mathcal{M}(X, \beta)]^{\mathrm{vir}} q^\beta \in A^\bullet(\mathrm{End}(\mathrm{I}X) \times \mathcal{M}) \otimes \Theta_X.$$

Theorem 4.15. *With these maps*, $\mathrm{I}X$ *is an algebra over* $\mathcal{M}$ *in* $\mathbf{Corr} \otimes \Theta_X$.

Proof. This is clear from the preceding theorem. □

The category $\mathbf{Corr}$ is equipped with realization functors associated to (Weil) cohomology theories; similarly, the category $\mathbf{Corr} \otimes \Theta_X$ has functors $Y \mapsto H^\bullet(Y, \Theta_X)$ by the universal coefficients theorem. The cohomology of $\mathrm{I}X$ is exactly Fantechi and Göttsche's ring $H^\bullet(X, G)$. Applying $H^\bullet$ to the morphism $\mathcal{M} \to \mathrm{End}\,\mathrm{I}X$, one finds the following result:

Theorem 4.16. *Let X be a smooth projective variety with an action of the finite group G. Then the stringy cohomology ring $H^\bullet(X, G)$, taken with coefficients in the Novikov ring of X, is in a canonical way a G-CohFT.*

Remark 4.17. In the above statement, we consider $H^\bullet(X, G)$ just as a super vector space, but one can with some care introduce a grading compatible with the algebra. To do this, one needs to introduce a grading on Θ via $\deg(q^\beta) = -2c_1[X/G] \cap \beta$, and equip $H^\bullet(X, G)$ with the so-called age grading. We omit the details as this is well known.

The above theorem was announced in [Jarvis et al. 2005], but a proof has not appeared. Although it is certainly possible to prove this without the language of operads, the author believes that the operadic framework has simplified the proof.

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danpete@math.kth.se *Department of Mathematics, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden*

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The p-adic monodromy theorem in the imperfect residue field case

Shun Ohkubo

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ and G_K the absolute Galois group of K. In this paper, we will prove the p-adic monodromy theorem for p-adic representations of G_K without any assumption on the residue field of K, for example the finiteness of a p-basis of the residue field of K. The main point of the proof is a construction of (φ, G_K)-module $\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V)$ for a de Rham representation V, which is a generalization of Pierre Colmez's $\widetilde{\mathbb{N}}_{\mathrm{rig}}^{+}(V)$. In particular, our proof is essentially different from Kazuma Morita's proof in the case when the residue field admits a finite p-basis.

We also give a few applications of the p-adic monodromy theorem, which are not mentioned in the literature. First, we prove a horizontal analogue of the p-adic monodromy theorem. Secondly, we prove an equivalence of categories between the category of horizontal de Rham representations of G_K and the category of de Rham representations of an absolute Galois group of the canonical subfield of K. Finally, we compute H^1 of some p-adic representations of G_K, which is a generalization of Osamu Hyodo's results.

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Keywords: p-adic Hodge theory, p-adic representations.

Introduction

Let p be a prime and K a complete discrete valuation field of mixed characteristic $(0, p)$ with residue field k_K. Let G_K be the absolute Galois group of K. When k_K is perfect, Jean-Marc Fontaine defined the notions of crystalline, semistable, de Rham, Hodge–Tate representations for p-adic representations of G_K (see [Fontaine 1994a; 1994b] for example). The p-adic monodromy conjecture, which asserts that de Rham representations are potentially semistable, was first proved by Laurent Berger [2002, Théorème 0.7] by using the theory of p-adic differential equations. Precisely speaking, Berger used the p-adic local monodromy theorem for p-adic differential equations with Frobenius structure due to Yves André, Zoghman Mebkhout, and Kiran Kedlaya.

The notions of the above categories of representations were defined by Olivier Brinon [2006] when k_K admits a finite p-basis. In this case, the p-adic monodromy theorem was proved by Kazuma Morita [2011, Corollary 1.2]. Roughly speaking, he proved the p-adic monodromy theorem by studying some differential equations, which are defined by Sen's theory of $\mathbb{B}_{\mathrm{dR}}$ due to Fabrizio Andreatta and Olivier Brinon [2010]. In that reference, Tate–Sen formalism for a quotient Γ_K of G_K is applied to establish Sen's theory of $\mathbb{B}_{\mathrm{dR}}$, where Γ_K is isomorphic to an open subgroup of $\mathbb{Z}_p^\times \ltimes \mathbb{Z}_p(1)^{J_K}$ with $J_K := \dim_{k_K} \Omega^1_{k_K/\mathbb{Z}} < \infty$. To prove Tate–Sen formalism, we iteratively use analogues of the normalized trace map due to John Tate. Hence, we can not use Morita's approach in the case $J_K = \infty$.

Our main theorem in this paper is the p-adic monodromy theorem without any assumption on the residue field k_K. We also give the following applications of the p-adic monodromy theorem, which are not mentioned in the literature: First, we will prove a horizontal analogue of the p-adic monodromy theorem (Theorem 7.4). Secondly, we will prove that the category of horizontal de Rham representations of G_K is canonically equivalent to the category of de Rham representations of $G_{K_{\mathrm{can}}}$ (Theorem 7.6), where K_{can} is the canonical subfield of K. Finally, we will calculate H^1 of horizontal de Rham representations under a certain condition on Hodge–Tate weights (Theorem 7.8). This calculation is a generalization of calculations done by Hyodo for $\mathbb{Z}_p(n)$ with $n \in \mathbb{Z}$ (Theorem 1.16).

Statement of Main Theorem. Let K and G_K be as above. We do not put any assumption on the residue field k_K of K, in particular, we may consider the case that k_K is imperfect with $[k_K : k_K^p] = \infty$. In this setup, the notions of crystalline, semistable, de Rham, Hodge–Tate representations are also defined (see Section 3). Then, our main theorem is the following:

Main Theorem (p-adic monodromy theorem). *Let V be a de Rham representation of G_K. Then, there exists a finite extension L/K such that the restriction $V|_L$ is semistable.*

Note that the converse can be easily proved by using Hilbert 90.

Strategy of proof. As is mentioned above, Kazuma Morita's proof can not be generalized directly. When the residue field k_K is perfect, an alternative proof of the p-adic monodromy theorem due to Pierre Colmez is available, which does not need the theory of p-adic differential equations. We will prove the Main Theorem by generalizing Colmez's method. In the following, we will explain our strategy after recalling Colmez's proof in the case that V is a 2-dimensional de Rham representation. (We can prove the higher-dimensional case in a similar way.) After replacing K by the maximal unramified extension of K and taking a Tate twist of V, we may also assume that we have $\mathbb{D}_{\mathrm{dR}}(V) = (\mathbb{B}_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and k_K is separably closed.

In this paragraph, assume that the residue field of K is perfect, that is, k_K is algebraically closed. We first fix notation: Let $\tilde{\mathbb{B}}_{\mathrm{rig}}^{+} := \bigcap_{n\in\mathbb{N}} \varphi^n(\mathbb{B}_{\mathrm{cris}}^{+})$. For $h \in \mathbb{N}_{>0}$ and $a \in \mathbb{N}$, denote $\mathbb{U}_{h,a} := (\mathbb{B}_{\mathrm{cris}}^{+})^{\varphi^h=p^a}$ and $\mathbb{U}'_{h,a} := (\mathbb{B}_{\mathrm{st}}^{+})^{\varphi^h=p^a}$. Note that we have $\mathbb{U}_{h,0} = \mathbb{U}'_{h,0} = \mathbb{Q}_{p^h}$, where $\mathbb{Q}_{p^h}$ denotes the unramified extension of $\mathbb{Q}_p$ with $[\mathbb{Q}_{p^h} : \mathbb{Q}_p] = h$. We will recall Colmez's proof: His proof has the following two key ingredients. One is Dieudonné–Manin classification theorem over $\tilde{\mathbb{B}}_{\mathrm{rig}}^{+}$. Then, he applies this theorem to construct a rank 2 free $\tilde{\mathbb{B}}_{\mathrm{rig}}^{+}$-submodule $\tilde{\mathbb{N}}_{\mathrm{rig}}^{+}(V)$ of $\tilde{\mathbb{B}}_{\mathrm{rig}}^{+} \otimes_{\mathbb{Q}_p} V$ with basis e_1, e_2. Moreover, $\tilde{\mathbb{N}}_{\mathrm{rig}}^{+}(V)$ is stable by φ and G_K-actions and the following properties are satisfied:

(i) We have an isomorphism of $\mathbb{B}_{\mathrm{dR}}^{+}[G_K]$-modules

$$\mathbb{B}_{\mathrm{dR}}^{+} \otimes_{\tilde{\mathbb{B}}_{\mathrm{rig}}^{+}} \tilde{\mathbb{N}}_{\mathrm{rig}}^{+}(V) \cong (\mathbb{B}_{\mathrm{dR}}^{+})^2.$$

(ii) There exist $h \in \mathbb{N}_{>0}$ and a 1-cocycle

$$C : G_K \to \begin{pmatrix} \mathbb{Q}_{p^h}^{\times} & \mathbb{U}_{h,a} \\ 0 & \mathbb{Q}_{p^h}^{\times} \end{pmatrix}; \quad g \mapsto C_g := \begin{pmatrix} \chi_1(g) & c_g \\ 0 & \chi_2(g) \end{pmatrix}$$

such that we have $g(e_1, e_2) = (e_1, e_2)C_g$ for all $g \in G_K$.

The second key ingredient is the $H_g^1 = H_{\mathrm{st}}^1$-theorem for $\mathbb{U}'_{h,a}$ with $h, a \in \mathbb{N}_{>0}$: Let L/K be a finite extension. If a 1-cocycle $G_L \to \mathbb{U}'_{h,a}$ is a 1-coboundary in $\mathbb{B}_{\mathrm{dR}}^{+}$, then it is a 1-coboundary in $\mathbb{U}'_{h,a}$. By using these facts, Colmez proved the Main Theorem as follows. When $h = 0$, we may regard C as a p-adic representation of G_K, which is Hodge–Tate of weights 0 by (i). By Sen's theorem on $\mathbb{C}_p$-representations, C has a finite image, which implies the assertion. Therefore, we may assume $h > 0$. By the cocycle condition of C, χ_i for $i = 1, 2$ is a character. By (i), χ_i for $i = 1, 2$ is Hodge–Tate with weights 0 as a p-adic representation. By Sen's theorem again, there exists a finite extension L/K such that $\chi_i(G_L) = 1$ for $i = 1, 2$. By the cocycle condition of C again, $c : G_L \to \mathbb{U}_{h,a}$ is a 1-cocycle, which is a 1-coboundary in $\mathbb{B}_{\mathrm{dR}}^{+}$

by (i). By the $H^1_g = H^1_{\mathrm{st}}$-theorem, there exists $x \in \mathbb{U}'_{h,a}$ such that $c_g = (g-1)(x)$ for all $g \in G_L$. Therefore,

$$e_1, -xe_1 + e_2 \in \mathbb{B}^+_{\mathrm{st}} \otimes_{\tilde{\mathbb{B}}^+_{\mathrm{rig}}} \tilde{\mathbb{N}}^+_{\mathrm{rig}}(V) \subset \mathbb{B}^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$$

form a basis of $\mathbb{D}_{\mathrm{st}}(V|_L)$, which implies that $V|_L$ is semistable.

We will outline our proof of the Main Theorem in the following: For simplicity, we omit some details. We first fix notation: In the imperfect residue field case, we can construct rings of p-adic periods $\mathbb{B}^+_{\mathrm{cris}}$, $\mathbb{B}^+_{\mathrm{st}}$ and $\mathbb{B}^+_{\mathrm{dR}}$, on which connections ∇ act. Let $\mathbb{B}^{\nabla+}_{\mathrm{cris}}$ and $\mathbb{B}^{\nabla+}_{\mathrm{st}}$ be the rings of the horizontal sections of $\mathbb{B}^+_{\mathrm{cris}}$ and $\mathbb{B}^+_{\mathrm{st}}$ respectively. Let $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} := \bigcap_{n\in\mathbb{N}} \varphi^n(\mathbb{B}^{\nabla+}_{\mathrm{cris}})$. For $h \in \mathbb{N}_{>0}$ and $a \in \mathbb{N}$, let $\mathbb{U}_{h,a} := (\mathbb{B}^{\nabla+}_{\mathrm{cris}})^{\varphi^h = p^a}$ and $\mathbb{U}'_{h,a} := (\mathbb{B}^{\nabla+}_{\mathrm{st}})^{\varphi^h = p^a}$. Even when k_K may not be perfect, we can easily prove a generalization of Sen's theorem (Theorem 2.1) and an analogue of Colmez's Dieudonné–Manin classification theorem in an appropriate setting (see Section 5). By using Dieudonné–Manin theorem, we can also give a functorial construction $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ for a de Rham representation V. Our object $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ is a rank 2 free $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}$-submodule of $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V$ with basis e_1, e_2. Moreover, $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ is stable by φ and G_K-actions and the following properties are satisfied:

(i) We have an isomorphism of $\mathbb{B}^+_{\mathrm{dR}}[G_K]$-modules

$$\mathbb{B}^+_{\mathrm{dR}} \otimes_{\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}} \tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V) \cong (\mathbb{B}^+_{\mathrm{dR}})^2.$$

(ii) There exist $h \in \mathbb{N}_{>0}$ and a 1-cocycle

$$C : G_K \to \begin{pmatrix} \mathbb{Q}^\times_{p^h} & \mathbb{U}_{h,a} \\ 0 & \mathbb{Q}^\times_{p^h} \end{pmatrix}; \quad g \mapsto C_g := \begin{pmatrix} \chi_1(g) & c_g \\ 0 & \chi_2(g) \end{pmatrix}$$

such that we have $g(e_1, e_2) = (e_1, e_2)C_g$ for all $g \in G_K$.

By using $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$, we prove the Main Theorem as follows. In the case $h = 0$, the same proof as above is valid, hence we assume $h > 0$. By the cocycle condition of C, χ_i for $i = 1, 2$ is a character, which is Hodge–Tate with weights 0 by (i). By a generalization of Sen's theorem, we may assume that $\chi_i(G_K) = 1$ for $i = 1, 2$ after replacing K by some finite extension. Then, by the cocycle condition of C, $c : G_K \to \mathbb{U}_{h,a}$ is a 1-cocycle, which is a 1-coboundary in $\mathbb{B}^+_{\mathrm{dR}}$. Unfortunately, an analogue of the above $H^1_{\mathrm{st}} = H^1_g$-theorem does not hold in the imperfect residue field case. Instead, we will prove that there exists $x \in (\mathbb{B}^+_{\mathrm{cris}})^{G_{K^{\mathrm{pf}}}}$ and $y \in \mathbb{B}^{\nabla+}_{\mathrm{dR}}$ such that $c_g = (g-1)(x+y)$ for $g \in G_K$ (a special case of Lemma 6.6). Here K^{pf} denotes a "perfection" of K, which is a complete discrete valuation field of mixed characteristic $(0, p)$ with residue field k_K^{pf} and we can regard an absolute Galois group $G_{K^{\mathrm{pf}}}$ of K^{pf} as a closed subgroup of G_K. Since we have a canonical isomorphism $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)|_{G_{K^{\mathrm{pf}}}} \cong \tilde{\mathbb{N}}^+_{\mathrm{rig}}(V|_{G_{K^{\mathrm{pf}}}})$ by functoriality, we can apply Colmez's $H^1_g = H^1_{\mathrm{st}}$-theorem to the 1-cocycle $c|_{G_{K^{\mathrm{pf}}}}$. As a consequence, there exists $z \in \mathbb{U}'_{h,a}$ such that

$c_g = (g-1)(z)$ for all $g \in G_{K^{\mathrm{pf}}}$. Since we have $c_g = (g-1)(y)$ for all $g \in G_{K^{\mathrm{pf}}}$, we have $z - y \in (\mathbb{B}_{\mathrm{dR}}^{\nabla+})^{G_{K^{\mathrm{pf}}}}$, which is included in $\mathbb{B}_{\mathrm{cris}}^{\nabla+}$ by a calculation. Hence, $e_1, -\{x + (y-z) + z\}e_1 + e_2 \in \mathbb{B}_{\mathrm{st}}^{+} \otimes_{\tilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+}} \widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V) \subset \mathbb{B}_{\mathrm{st}}^{+} \otimes V$ forms a basis of $\mathbb{D}_{\mathrm{st}}(V|_K)$, which implies that $V|_K$ is semistable.

Structure of paper. In Section 1, we will recall the preliminary facts used in the paper. In Section 2, we will generalize Sen's theorem on $\mathbb{C}_p$-admissible representations, which is a special case of the Main Theorem and will be used in the following. The next two sections are devoted to review rings of p-adic periods in the imperfect residue field case. Although most of the results seem to be well-known, we will give proofs for the convenience of the reader. In Section 3, we will recall basic constructions and algebraic properties of rings of p-adic periods used in p-adic Hodge theory in the imperfect residue field case. In Section 4, we will recall Galois-theoretic properties of rings of p-adic periods constructed in the previous section. In Section 5, we will construct the (φ, G_K)-modules $\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V)$ for de Rham representations V after Tate twist. In Section 6, we will prove the Main Theorem combining the results proved in the previous sections. In Section 7, we will give applications of the Main Theorem.

Conventions

Throughout this paper, let p be a prime and K a complete discrete valuation field of mixed characteristic $(0, p)$. Denote the integer ring of K by $\mathcal{O}_K$ and a uniformizer of $\mathcal{O}_K$ by π_K. Put $U_K^{(n)} := 1 + \pi_K^n \mathcal{O}_K$ for $n \in \mathbb{N}_{>0}$. Denote by k_K the residue field of K. We denote by K^{ur} the p-adic completion of the maximal unramified extension of K. Denote by e_K the absolute ramification index of K. For an extension L/K of complete discrete valuation fields, we define the relative ramification index $e_{L/K}$ of L/K by $e_{L/K} := e_L/e_K$.

For a field F, fix an algebraic closure (resp. a separable closure) of F, denote it by F^{alg} or $\overline{F}$ (resp. F^{sep}) and let G_F be the absolute Galois group of F. For a field k of characteristic p, let $k^{\mathrm{pf}} := k^{p^{-\infty}}$ be the perfect closure in a fixed algebraic closure of k. Let $k^{p^{\infty}} := \bigcap_{n\in\mathbb{N}} k^{p^n}$ be the maximal perfect subfield of k. Denote by $\mathbb{C}_p$ and $\mathcal{O}_{\mathbb{C}_p}$ the p-adic completion of $\overline{K}$ and its integer ring. Let v_p be the p-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p) = 1$.

We fix a system of p-power roots of unity $\{\zeta_{p^n}\}_{n\in\mathbb{N}_{>0}}$ in $\overline{K}$, that is, ζ_p is a primitive p-th root of unity and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all $n \in \mathbb{N}_{>0}$. Let $\chi : G_K \to \mathbb{Z}_p^{\times}$ be the cyclotomic character defined by $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ for $n \in \mathbb{N}_{>0}$.

For a set S, denote by $|S|$ the cardinality of S. Let J_K be an index set such that we have an isomorphism $\Omega^1_{k_K/\mathbb{Z}} \cong k_K^{\bigoplus J_K}$ as k_K-vector spaces. In this paper, we do not assume $|J_K| < \infty$. Unless a particular mention is stated, we always fix a lift $\{t_j\}_{j\in J_K}$ of a p-basis of k_K and sequences of p-power roots $\{t_j^{p^{-n}}\}_{n\in\mathbb{N},j\in J_K}$ in $\overline{K}$, that is, we have $(t_j^{p^{-n-1}})^p = t_j^{p^{-n}}$ for $n\in\mathbb{N}_{>0}$.

For a ring R, denote the Witt ring with coefficients in R by $W(R)$. If R has characteristic p, then we denote the absolute Frobenius on R by $\varphi : R\to R$ and also denote the ring homomorphism $W(\varphi) : W(R)\to W(R)$ by φ. Denote by $[x]\in W(R)$ the Teichmüller lift of $x\in R$.

For a p-adically Hausdorff abelian group M in which p is not a nonzero divisor, we define the p-adic semivaluation of M as the map $v : M\to\mathbb{Z}\cup\{\infty\}$ such that $v(0)=\infty$ and $v(x)=n$ if $x\in p^nM\setminus p^{n+1}M$. We have the properties

$$v(px)=1+v(x),\quad v(x+y)\geq\inf(v(x),v(y)),\quad v(x)=\infty \iff x=0,$$

for $x,y\in M$. We can extend v to $v : M[p^{-1}]\to\mathbb{Z}\cup\{\infty\}$, which we call the p-adic semivaluation defined by the lattice M. We also call the topology induced by v the p-adic topology defined by the lattice M.

Let F be a nontrivial nonarchimedean complete valuation field with valuation v_F. Assume that an F-vector space V is endowed with a countable decreasing sequence of valuations $\{v^{(n)} : V\to\mathbb{R}\cup\{\infty\}\}_{n\in\mathbb{N}}$ over F, that is, we have

$$v^{(0)}(x)\geq v^{(1)}(x)\geq\cdots,\quad v^{(n)}(\lambda x)=v_F(\lambda)+v^{(n)}(x),$$
$$v^{(n)}(x+y)\geq\inf(v^{(n)}(x),v^{(n)}(y))$$

for $\lambda\in F$ and $x,y\in V$. We regard V as a topological F-vector space whose topology is generated by $V_r^{(n)} := \{x\in V \mid v^{(n)}(x)\geq r\}$ for $n,r\in\mathbb{N}$. Then, we call V a Fréchet space (over F) if V is complete with respect to this topology (see [Schneider 2002, Section 8]). For Fréchet spaces V and W, we define the completed tensor product $V\widehat{\otimes}_F W$ as the inverse limit $\varprojlim_{n,r\in\mathbb{N}} V/V_r^{(n)}\otimes_F W/W_r^{(n)}$ (see [Schneider 2002, Section 17]).

For a multiset $\{a_i\}_{i\in I}$ of elements in $\mathbb{R}\cup\{\infty\}$, we denote $\{a_i\}_{i\in I}\to\infty$ if the set $\{i\in I, a_i<n\}$ is finite for all $n\in\mathbb{N}_{>0}$. Note that if $|I|<\infty$, then the condition $\{a_i\}_{i\in I}\to\infty$ is always satisfied.

In this paper, we refer to the continuous cohomology group as the group cohomology. For a profinite group G and a topological G-module M, denote by $H^n(G,M)$ the n-th continuous group cohomology with coefficients in M. We also denote $H^0(G,M)$ by M^G. We also consider $H^q(G,M)$ for $q=0,1$ if M is a (noncommutative) topological G-group M.

We denote by $\boldsymbol{e}_i\in\mathbb{N}^{\oplus I}$ the element whose i-th component is equal to 1 and zero otherwise. We will use the following multi-index notation: Let M be a monoid. For a subset $\{x_i\}_{i\in I}$ of M and $\boldsymbol{n}=(n_i)_{i\in I}\in\mathbb{N}^{\oplus I}$, we define $\boldsymbol{x}^{\boldsymbol{n}} := \prod_{i\in I} x_i^{n_i}$ and $\boldsymbol{x}^{[\boldsymbol{n}]} := \prod_{i\in I} u_i^{n_i}/n_i!$ when it has a meaning. We denote by $|\boldsymbol{n}|$ the sum $\sum_{i\in I} n_i$ for

$(n_i)_{i\in I} \in \mathbb{N}^{\oplus I}$. If no particular mention is stated, for an index set I, we denote by $\mathbf{u}_I$ or $\mathbf{v}_I$ the formal variables $\{\mathrm{u}_i\}_{i\in I}$ or $\{\mathrm{v}_i\}_{i\in I}$ respectively.

For group homomorphisms $f, g : M \to N$ of abelian groups, we denote by $M^{f=g}$ the kernel of the map $f - g : M \to N$.

1. Preliminaries

This preliminary section is a miscellany of basic definitions, facts, conventions, and remarks used in the paper. Although we will give some proofs for convenience, the reader may skip the proofs by admitting the facts.

1A. ***Cohen ring.*** Let k be a field of characteristic p. Let $C(k)$ be a Cohen ring of k, that is, a complete discrete valuation ring with maximal ideal generated by p and residue field k. This is unique up to a canonical isomorphism if k is perfect (in fact, $C(k) \cong W(k)$) and unique up to noncanonical isomorphisms in general. Denote $J_{C(k)[p^{-1}]}$ by J for a while. For a lift $\{t_j\}_{j\in J} \subset C(k)$ of a p-basis of k, we regard $C(k)$ as a $\mathbb{Z}[T_j]_{j\in J}$-algebra by $T_j \mapsto t_j$. This morphism is formally étale for the p-adic topologies. In fact, we may replace $\mathbb{Z}[T_j]_{j\in J}$ by $R := (\mathbb{Z}[T_j]_{j\in J})_{(p)}$. Since $C(k)/R$ is flat and $k/\mathbb{F}_p(T_j)_{j\in J}$ is formally étale for the discrete topologies, $C(k)/R$ is formally étale by [Grothendieck 1964, 0.19.7.1 and 0.20.7.5].

By the lifting property, we have $C(k_K) \to \mathcal{O}_K$, an injective algebra homomorphism which is totally ramified of degree e_K. We will denote by K_0 the fraction field of the image of $C(k)$ in K. We also note that $\mathcal{O}_{K_0}$ is unique if k_K is perfect and nonunique otherwise. By the lifting property again, we have a lift $\varphi : \mathcal{O}_{K_0} \to \mathcal{O}_{K_0}$ of the absolute Frobenius of k_K: It is unique if k_K is perfect and nonunique otherwise. An example of such a φ is $\varphi(t_j) = t_j^p$ for all $j \in J_{K_0}$. Moreover, when k_K is imperfect, the construction of K_0 cannot be functorial in the following sense: For a finite extension L/K, we cannot always choose $K_0 \subset K$ and $L_0 \subset L$ such that $K_0 \subset L_0$.

Finally, note that for a given lift $\{t_j\}_{j\in J_K} \subset \mathcal{O}_K$ of a p-basis of k_K, we can choose $\mathcal{O}_{K_0}$ such that $\{t_j\}_{j\in J_K} \subset \mathcal{O}_{K_0}$. In fact, we regard $\mathcal{O}_K$ as a $\mathbb{Z}[T_j]_{j\in J_K}$-algebra by sending T_j to t_j. We choose a lift $\{t'_j\}_{j\in J_K} \subset C(k_K)$ of the p-basis $\{\bar{t}_j\}_{j\in J_K} \subset k_K$ and we regard $C(k_K)$ as a $\mathbb{Z}[T_j]_{j\in J_K}$-algebra by $T_j \mapsto t'_j$. Then, we lift the projection $C(k_K) \to k_K$ to a $\mathbb{Z}[T_j]_{j\in J_K}$-algebra homomorphism $C(k_K) \to \mathcal{O}_K$ by the lifting property, whose image satisfies the condition. Thus, if we choose a lift $\{t_j\}_{j\in J_K}$ of a p-basis of k_K, we may always assume that we have $\{t_j\}_{j\in J_K} \subset K_0$.

1B. ***Canonical subfield.*** We first recall the following two lemmas, which are proved in [Epp 1973, 0.4]. We give proofs for the reader.

Lemma 1.1. *Let k be a field of characteristic p.*

(i) *The field k^{p^∞} is algebraically closed in k. In particular, the fields $(k^{p^\infty})^{\mathrm{sep}}$ and k are linearly disjoint over k^{p^∞}.*

(ii) *For a finite extension k'/k^{p^∞}, we have $k' = (kk')^{p^\infty}$.*

Proof.

(i) The assertion follows from the fact that any algebraic extension over a perfect field is perfect.

(ii) As is mentioned in the above proof, k' is perfect. We have $kk' = k \otimes_{k^{p^\infty}} k'$ by (i). Hence, we have $(kk')^{p^n} = k^{p^n} \otimes_{k^{p^\infty}} k'$ and

$$(kk')^{p^\infty} = \bigcap_n (k^{p^n} \otimes_{k^{p^\infty}} k') = k^{p^\infty} \otimes_{k^{p^\infty}} k' = k'. \qquad \square$$

Lemma 1.2. *Let l/k be an algebraic extension of fields of characteristic p.*

(i) *If l/k is a (possibly infinite) Galois extension, then $l^{p^\infty}/k^{p^\infty}$ is also a (possible infinite) Galois extension. Moreover, the canonical map*

$$G_{l/k} \to G_{l^{p^\infty}/k^{p^\infty}}$$

is surjective.

(ii) *If l/k is finite, then $l^{p^\infty}/k^{p^\infty}$ is also a finite extension. Moreover, we have $[l^{p^\infty} : k^{p^\infty}] \le [l : k]$.*

Proof. (i) We may easily reduce to the case that l/k is finite Galois. Obviously any k-algebra endomorphism on l induces a k^{p^n}-algebra endomorphism on l^{p^n}. In particular, l^{p^n} and l^{p^∞} are $G_{l/k}$-stable. Since the Frobenius commutes with the action of $G_{l/k}$, we have $(l^{p^n})^{G_{l/k}} = (l^{G_{l/k}})^{p^n} = k^{p^n}$. By taking the intersection, we have $(l^{p^\infty})^{G_{l/k}} = k^{p^\infty}$. For $x \in l^{p^\infty}$, let $f(X) \in k[X]$ be the monic irreducible separable polynomial such that $f(x) = 0$. Then all the solutions of f belong to l^{p^∞} and we have $f(X) \in (l^{p^\infty})^{G_{l/k}}[X] = k^{p^\infty}[X]$. This implies that $l^{p^\infty}/k^{p^\infty}$ is a Galois extension. The latter assertion follows from the equality $(l^{p^\infty})^{G_{l/k}} = k^{p^\infty}$.

(ii) We may assume that l/k is purely inseparable or separable. If l/k is purely inseparable, then l is generated by finitely many elements of the form $x^{p^{-n}}$ with $n \in \mathbb{N}$ and $x \in k$ as a k-algebra. Hence we have $l^{p^n} \subset k$ for some n, that is, $k^{p^\infty} = l^{p^\infty}$. Assume that l/k is separable. The first assertion is reduced to the case that l/k is a Galois extension, which follows from (i). Since the canonical k-algebra homomorphism $l^{p^\infty} \otimes_{k^{p^\infty}} k \to l$ is injective by Lemma 1.1(i), we have $[l^{p^\infty} : k^{p^\infty}] \le [l : k]$. $\square$

Defintion 1.3. (i) (Compare [Hyodo 1986, Theorem 2].) We define the canonical subfield K_{can} of K as the algebraic closure of $W(k_K^{p^\infty})[p^{-1}]$ in K.

(ii) (Compare [Hyodo 1986, (0-5)].) We define condition (H) as follows:

$$K \text{ contains a primitive } p^2\text{-th root of unity and we have } e_{K/K_{\mathrm{can}}} = 1.$$

Note that K_{can} is a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field $k_K^{p^\infty}$. If k_K is perfect, then we have $K_{\mathrm{can}} = K$. We also note that the restriction $G_K \to G_{K_{\mathrm{can}}}$ is surjective since K_{can} is algebraically closed in K. We will regard $G_{K_{\mathrm{can}}}$ as a quotient of G_K in the rest of the paper.

Remark 1.4. (i) In [Brinon 2006, Notation 2.29], K_{can} is denoted by K^∇ since K_{can} coincides with the kernel of the canonical derivation $d : K \to \widehat{\Omega}^1_K$ (Proposition 1.13 below).

(ii) The canonical morphism

$$K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} K_0 \to K$$

is injective since we have $e_{K_0/K_{\mathrm{can},0}} = 1$ and $K_{\mathrm{can}}/K_{\mathrm{can},0}$ is totally ramified. Note that we have $e_{K/K_{\mathrm{can}}} = 1$ if and only if the above morphism is surjective.

The following are the basic properties of the canonical subfields used in this paper.

Lemma 1.5. *Let L/K be a finite extension.*

(i) *The fields $(K_{\mathrm{can}})^{\mathrm{alg}}$ and K are linearly disjoint over K_{can}.*

(ii) *If L/K is Galois, then $L_{\mathrm{can}}/K_{\mathrm{can}}$ is also a finite Galois extension. Moreover, the canonical map $G_{L/K} \to G_{L_{\mathrm{can}}/K_{\mathrm{can}}}$ is surjective.*

(iii) *The field extension $L_{\mathrm{can}}/K_{\mathrm{can}}$ is finite with $[L_{\mathrm{can}} : K_{\mathrm{can}}] \leq [L : K]$.*

(iv) *If K'/K_{can} is a finite extension, then we have $(KK')_{\mathrm{can}} = K'$.*

Proof. (i) Since K_{can} is algebraically closed in K, we have $(K_{\mathrm{can}})^{\mathrm{alg}} \cap K = K_{\mathrm{can}}$, which implies the assertion.

(ii) Since $k_L^{p^\infty}/k_K^{p^\infty}$ is finite by Lemma 1.2(ii), we have $L_{\mathrm{can}} = L \cap (K_{\mathrm{can}})^{\mathrm{alg}}$. Hence we have $L_{\mathrm{can}} \cap K = K_{\mathrm{can}}$. Since $L_{\mathrm{can}}/K_{\mathrm{can}}$ is algebraic, L_{can} and K are linearly disjoint over K_{can} by (i). Let $x \in L_{\mathrm{can}}$ and $f(X) \in K_{\mathrm{can}}[X]$ be the monic irreducible polynomial such that $f(x) = 0$. By the linearly disjointness, $f(X)$ is irreducible in $K[X]$. Since L/K is Galois, all the solutions of $f(X) = 0$ belong to $L \cap (K_{\mathrm{can}})^{\mathrm{alg}} = L_{\mathrm{can}}$. This implies that $L_{\mathrm{can}}/K_{\mathrm{can}}$ is Galois. Since we have $(L_{\mathrm{can}})^{G_{L/K}} = L_{\mathrm{can}} \cap K = K_{\mathrm{can}}$, we have the rest of the assertion.

(iii) The finiteness of $L_{\mathrm{can}}/K_{\mathrm{can}}$ is reduced to the case that L/K is Galois, which follows from (ii). Since the canonical K-algebra homomorphism $L_{\mathrm{can}} \otimes_{K_{\mathrm{can}}} K \to L$ is injective by (i), we have $[L_{\mathrm{can}} : K_{\mathrm{can}}] \leq [L : K]$.

(iv) The assertion follows from the inequalities

$$[K' : K_{\mathrm{can}}] \leq [(KK')_{\mathrm{can}} : K_{\mathrm{can}}] \leq [KK' : K] = [K' : K_{\mathrm{can}}],$$

where the second inequality follows from (iii) and the last equality follows from the linear disjointness of K and K' over K_{can} by (i). $\square$

Theorem 1.6 (the complete case of Epp's theorem [1973]). *There exists a finite Galois extension of K'/K_{can} such that KK' satisfies condition* (H).

Proof. By the original Epp's theorem, we have a finite extension K'/K_{can} such that we have $e_{KK'/K'} = 1$. We have only to prove that we have $e_{KK''/K''} = 1$ for any finite extension K''/K'. In fact, if we choose K'' as the Galois closure of $K'(\mu_{p^2})$ over K_{can}, then K'' satisfies the condition by Lemma 1.5(iv). Since we have $KK'' = (KK') \otimes_{K'} K''$ by Lemma 1.5(i) and (iv), we have $e_{KK''/KK'} \le e_{K''/K'}$. By multiplying with $e_{KK'} = e_{K'}$, we have $e_{KK''} \le e_{K''}$, implying the assertion. $\square$

Example 1.7 (the higher-dimensional local fields case). We say that K has a structure of a higher-dimensional local field if K is isomorphic to a finite extension over the fractional field of a Cohen ring of the field

$$\mathbb{F}_q((X_1))((X_2))\dots((X_d))$$

with $q = p^f$ (see [Zhukov 2000] about higher-dimensional local fields). In this case, K_{can} coincides with the algebraic closure of $\mathbb{Q}_p$ in K. In fact, we have only to prove that $k_K^{p^\infty}$ is a finite field. By Lemma 1.2(ii), we may reduce to the case $k_K = \mathbb{F}_q((X_1))\dots((X_d))$. Then, the assertion follows from an iterative use of the following fact: If k is a field of characteristic p, then we have $k((X))^{p^\infty} = k^{p^\infty}$. Obviously, the RHS is contained in the LHS. Let $f = \sum_{n\gg-\infty} a_n X^n \in k((X))^{p^\infty}$ with $a_n \in k$. Since $f \in k((X))^p$, we have $a_n = 0$ if $p \nmid n$ and $a_n \in k^p$ otherwise. By repeating this argument, we have $a_n = 0$ for $n \neq 0$ and $f = a_0 \in k^{p^\infty}$.

1C. *Canonical derivation.*

Defintion 1.8 (Compare [Hyodo 1986, Section 4].). Let $q \in \mathbb{N}$. For a complete discrete valuation ring R with mixed characteristic $(0, p)$, let

$$\widehat{\Omega}^q_R := \varprojlim_n \Omega^q_{R/\mathbb{Z}}/p^n\Omega^q_{R/\mathbb{Z}}$$

and let $d : R \to \widehat{\Omega}^1_R$ be the canonical derivation. Let $\widehat{\Omega}^q_{R[p^{-1}]} := \widehat{\Omega}^q_R[p^{-1}]$ for $q \in \mathbb{Z}$ and let $d : R[p^{-1}] \to \widehat{\Omega}^1_{R[p^{-1}]}$ be the canonical derivation and $d_q : \widehat{\Omega}^q_{R[p^{-1}]} \to \widehat{\Omega}^{q+1}_{R[p^{-1}]}$ the morphism induced by the exterior derivation, which satisfies the usual formula $d_q(\lambda\omega) = \lambda d_q\omega + (-1)^q\omega \wedge d\lambda$ for $\lambda \in K$ and $\omega \in \widehat{\Omega}^q_K$. We endow $\widehat{\Omega}^q_{R[p^{-1}]}$ with the p-adic topology defined by the lattice $\mathrm{Im}(\widehat{\Omega}^q_R \xrightarrow{\mathrm{can}} \widehat{\Omega}^q_{R[p^{-1}]})$. Obviously, the derivation d_q is continuous.

For $q \in \mathbb{Z}_{<0}$, we put $\widehat{\Omega}^q_{R[p^{-1}]} := 0$ as a matter of convention.

The following are the basic properties of the canonical derivations used in the sequel.

Lemma 1.9. *Let R be a discrete valuation ring with uniformizer π_R and $\alpha : M \to M'$ a morphism of R-modules whose kernel and cokernel are killed by π_R^c for $c \in \mathbb{N}$. Then, for any R-module M'', the kernel and cokernel of the morphism* $\mathrm{id} \otimes \alpha : M'' \otimes_R M \to M'' \otimes_R M'$ *are killed by π_R^{2c}. In particular, the kernel and cokernel of $\alpha^{\otimes q} : M^{\otimes q} \to M'^{\otimes q}$ are killed by π_R^{2qc}.*

Proof. We prove the first assertion. If α is injective or surjective, then the cokernel and kernel are killed by π_R^c by the calculation of $\mathrm{Tor}_R^{\bullet}$. The general case follows easily from these cases by writing α as a composition of an injection and a surjection.

The last assertion follows from the following decomposition and induction on q:

$$M^{\otimes(q+1)} = M \otimes_R M^{\otimes q} \xrightarrow{\mathrm{id}\otimes\alpha^{\otimes q}} M \otimes_R M'^{\otimes q} \xrightarrow{\alpha\otimes\mathrm{id}} M' \otimes_R M'^{\otimes q} = M'^{\otimes(q+1)}$$

□

Lemma 1.10 [Hyodo 1986]. *Let $q \in \mathbb{N}$.*

(i) *We have the $\mathbb{O}_{K_0}$-linear isomorphism*

$$\hat{\Omega}^q_{\mathbb{O}_{K_0}} \cong \varprojlim_n \big((\mathbb{O}_{K_0}/p^n\mathbb{O}_{K_0}) \otimes_{\mathbb{Z}} \textstyle\bigwedge^q_{\mathbb{Z}} \mathbb{Z}^{\bigoplus J_K}\big); \;\; dt_{j_1} \wedge \cdots \wedge dt_{j_q} \mapsto 1 \otimes \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_q}.$$

In particular, $\hat{\Omega}^q_{\mathbb{O}_{K_0}}/(p^n)$ is a free $\mathbb{O}_{K_0}/(p^n)$-module.

(ii) *We have a canonical isomorphism*

$$\big(\textstyle\bigwedge^q_K \hat{\Omega}^1_K\big)\widehat{\;} \to \hat{\Omega}^q_K.$$

(iii) *Let L be a finite extension over the completion of an unramified extension of K. Then, we have a canonical isomorphism*

$$L \otimes_K \hat{\Omega}^q_K \to \hat{\Omega}^q_L.$$

Proof. The assertions (i) and (ii) follow from [Hyodo 1986, Lemma (4.4), Remark 3] respectively. The canonical exact sequence

$$0 \to \mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}} \to \Omega^1_{\mathbb{O}_L/\mathbb{Z}} \to \Omega^1_{\mathbb{O}_L/\mathbb{O}_K} \to 0$$

(from [Scholl 1998, Section 3.4, footnote]) induces the exact sequence

$$\Omega^1_{\mathbb{O}_L/\mathbb{O}_K}[p^n] \to \mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}/(p^n) \xrightarrow{\alpha_n} \Omega^1_{\mathbb{O}_L/\mathbb{Z}}/(p^n) \to \Omega^1_{\mathbb{O}_L/\mathbb{O}_K}/(p^n) \to 0,$$

where $\Omega^1_{\mathbb{O}_L/\mathbb{O}_K}[p^n]$ denotes the kernel of the multiplication by p^n on $\Omega^1_{\mathbb{O}_L/\mathbb{O}_K}$. Fix $c \in \mathbb{N}$ such that $p^c\Omega^1_{\mathbb{O}_L/\mathbb{O}_K} = 0$. Then, the kernel and cokernel of α_n are killed by p^c. Denote by $\mathcal{Q}_n$ and Q_n the kernel of the canonical maps

$$\textstyle\bigotimes^q_{\mathbb{O}_L} \big(\mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}/(p^n)\big) \to \textstyle\bigwedge^q_{\mathbb{O}_L} \big(\mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}/(p^n)\big),$$
$$\textstyle\bigotimes^q_{\mathbb{O}_L} \Omega^1_{\mathbb{O}_L/\mathbb{Z}}/(p^n) \to \Omega^q_{\mathbb{O}_L/\mathbb{Z}}/(p^n).$$

We consider the commutative diagram

$$\begin{tikzcd}
\mathscr{O}_L \otimes_{\mathscr{O}_K} \dfrac{\bigotimes^q_{\mathscr{O}_K} \Omega^1_{\mathscr{O}_K/\mathbb{Z}}}{(p^n)} \arrow[r, "\text{can.}", "\cong"'] \arrow[d, "\text{can.}"] & \bigotimes^q_{\mathscr{O}_L} \dfrac{\mathscr{O}_L \otimes_{\mathscr{O}_K} \Omega^1_{\mathscr{O}_K/\mathbb{Z}}}{(p^n)} \arrow[r, "\alpha_n^{\otimes q}"] \arrow[d, "\text{can.}"] & \bigotimes^q_{\mathscr{O}_L} \dfrac{\Omega^1_{\mathscr{O}_L/\mathbb{Z}}}{(p^n)} \arrow[d, "\text{can.}"] \\
\mathscr{O}_L \otimes_{\mathscr{O}_K} \dfrac{\Omega^q_{\mathscr{O}_K/\mathbb{Z}}}{(p^n)} \arrow[r, "\text{can.}", "\cong"'] & \bigwedge^q_{\mathscr{O}_L} \dfrac{\mathscr{O}_L \otimes_{\mathscr{O}_K} \Omega^1_{\mathscr{O}_K/\mathbb{Z}}}{(p^n)} \arrow[r, "\bigwedge^q \alpha_n"] & \dfrac{\Omega^q_{\mathscr{O}_L/\mathbb{Z}}}{(p^n)}.
\end{tikzcd}$$

We have only to prove that the kernel and cokernel of $\bigwedge^q \alpha_n$ are killed by p^{3qc}. Indeed, if this is true, then we decompose the canonical map

$$\alpha_n^q : \mathscr{O}_L \otimes_{\mathscr{O}_K} \Omega^q_{\mathscr{O}_K/\mathbb{Z}}/(p^n) \to \Omega^q_{\mathscr{O}_L/\mathbb{Z}}/(p^n)$$

into the following exact sequences:

$$0 \longrightarrow \ker \alpha_n^q \xrightarrow{\text{inc.}} \mathscr{O}_L \otimes_{\mathscr{O}_K} \Omega^q_{\mathscr{O}_K/\mathbb{Z}}/(p^n) \xrightarrow{\alpha_n^q} \operatorname{Im} \alpha_n^q \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Im} \alpha_n^q \xrightarrow{\text{inc.}} \Omega^q_{\mathscr{O}_L/\mathbb{Z}}/(p^n) \xrightarrow{\text{pr.}} \operatorname{cok} \alpha_n^q \longrightarrow 0.$$

By passing to limits, we obtain the following exact sequences:

$$0 \longrightarrow \varprojlim_n \ker \alpha_n^q \xrightarrow{\text{inc.}} \mathscr{O}_L \otimes_{\mathscr{O}_K} \widehat{\Omega}^q_{\mathscr{O}_K} \xrightarrow{\text{can.}} \varprojlim_n \operatorname{Im} \alpha_n^q \xrightarrow{\delta} \varprojlim_n{}^1 \ker \alpha_n^q,$$

$$0 \longrightarrow \varprojlim_n \operatorname{Im} \alpha_n^q \xrightarrow{\text{inc.}} \widehat{\Omega}^q_{\mathscr{O}_L} \xrightarrow{\text{pr.}} \varprojlim_n \operatorname{cok} \alpha_n^q.$$

Since $\ker \alpha_n^q$ and $\operatorname{cok} \alpha_n^q$ are killed by p^{3qc}, $\varprojlim_n \ker \alpha_n^q$ and $\varprojlim_n^1 \ker \alpha_n^q$, $\varprojlim \operatorname{cok} \alpha_n^q$ are also killed by p^{3qc} [Neukirch et al. 2008, Proposition 2.7.4]. Hence, the kernel and cokernel of the canonical map $\mathscr{O}_L \otimes_{\mathscr{O}_K} \widehat{\Omega}^q_{\mathscr{O}_K} \to \widehat{\Omega}^q_{\mathscr{O}_L}$ are killed by p^{3qc} and p^{6qc} respectively. By inverting p, we obtain the assertion.

Note that the kernel and cokernel of $\alpha_n^{\otimes q}$ are killed by p^{2qc} by Lemma 1.9. By the snake lemma, it suffices to prove that the cokernel of the map $\alpha_n^{\otimes q} : \mathscr{Q}_n \to Q_n$ is killed by p^{qc}. The $\mathscr{O}_L$-module Q_n is generated by the elements of the form $x := x_1 \otimes \cdots \otimes x_q$ with $x_i \in \Omega^1_{\mathscr{O}_L/\mathbb{Z}}/(p^n)$ such that $x_i = x_j$ for some $i \neq j$. Since the cokernel of α_n is killed by p^c, there exist $y_1, \dots, y_q \in \mathscr{O}_L \otimes_{\mathscr{O}_K} \Omega^1_{\mathscr{O}_L/\mathbb{Z}}/(p^n)$ such that $p^c x_i = \alpha_n(y_i)$ and $y_i = y_j$. Hence we have $p^{qc} x = (p^c x_1) \otimes \cdots \otimes (p^c x_q) = \alpha_n^{\otimes q}(y_1 \otimes \cdots \otimes y_q)$ and $y_1 \otimes \cdots \otimes y_q \in \mathscr{Q}_n$, which implies the assertion. □

Remark 1.11. If $[k_K : k_K^p] = p^d < \infty$, then $\dim_K \widehat{\Omega}^q_K = \binom{d}{q} < \infty$ for $q \in \mathbb{N}$ by Lemma 1.10. In particular, the canonical derivation d is K_{can}-linear since the restriction $d|_{K_{\text{can}}}$ factors through $\widehat{\Omega}^1_{K_{\text{can}}} = 0$ by functoriality.

Defintion 1.12. Fix a lift $\{t_j\}_{j\in J_K}\subset \mathcal{O}_{K_0}$ of a p-basis of k_K. By Lemma 1.10(i), dx for $x\in\mathcal{O}_{K_0}$ can be uniquely written in the form $\sum_{j\in J_K} dt_j\otimes\partial_j(x)$, where $\{\partial_j(x)\}_{j\in J_K}\subset\mathcal{O}_{K_0}$ is such that $\{v_p(\partial_j(x))\}_{j\in J_K}\to\infty$. Note that $\{\partial_j\}_{j\in J_K}$ are mutually commutative derivations of $\mathcal{O}_{K_0}$ by the formula $d_1\circ d=0$. We also note that ∂_j is continuous since we have the inequality $v_p(\partial_j(x))\geq v_p(x)$ for $x\in\mathcal{O}_{K_0}$, which we can check by taking modulo p.

The following is another characterization of the canonical subfields.

Proposition 1.13 [Brinon 2006, Proposition 2.28]. *We have the exact sequence*

$$0\longrightarrow K_{\mathrm{can}}\xrightarrow{\text{inc.}} K\xrightarrow{\ d\ }\widehat{\Omega}^1_K.$$

Proof. We first reduce to the case $K=K_0$. In the case that K satisfies condition (H), we obtain the exact sequence by applying $K_{\mathrm{can}}\otimes_{K_{\mathrm{can},0}}$ to the exact sequence for $K=K_0$ by Remark 1.4(ii) and Lemma 1.10(iii). In the general case, we choose a finite Galois extension K'/K_{can} such that KK' satisfies condition (H) by Epp's Theorem 1.6. Since we have $(KK')_{\mathrm{can}}=K'$ by Lemma 1.5(iv), $K'\otimes_{K_{\mathrm{can}}}K=KK'$ by Lemma 1.5(i) and $(\widehat{\Omega}^1_{KK'})^{G_{K'/K_{\mathrm{can}}}}=\widehat{\Omega}^1_K$ by Lemma 1.10(iii), the assertion follows from Galois descent.

We will prove the assertion in the case $K=K_0$. We may replace K_{can}, K and $\widehat{\Omega}^1_K$ by $\mathcal{O}_{K_{\mathrm{can}}},\mathcal{O}_K$ and $\widehat{\Omega}^1_{\mathcal{O}_K}$ respectively. Notation is as above. Let φ be the Frobenius on $\mathcal{O}_K$ given by $\varphi(t_j)=t_j^p$ for $j\in J_K$. Let $\varphi_*:\widehat{\Omega}^1_K\to\widehat{\Omega}^1_K$ be the Frobenius induced by φ. Since we have $d\circ\varphi=\varphi_*\circ d$, by a simple calculation, we have $\partial_j\circ\varphi=pt_j^{p-1}\varphi\circ\partial_j$, that is, $(t_j\partial_j)\circ\varphi=p\varphi\circ(t_j\partial_j)$ for $j\in J_K$.

The ring $\varphi(\mathcal{O}_K)$ is a complete discrete valuation ring of mixed characteristic $(0,p)$ and we may regard its residue field as k_K^p. Let $\Lambda:=\{0,\dots,p-1\}^{\oplus J_K}$. Since the image of $\{\boldsymbol{t}^{\boldsymbol{n}}\}_{\boldsymbol{n}\in\Lambda}$ in k_K forms a k_K^p-basis of k_K, by approximation, every element $x\in\mathcal{O}_K$ can be uniquely written in the form $x=\sum_{\boldsymbol{n}\in\Lambda}\varphi(a_{\boldsymbol{n}})\boldsymbol{t}^{\boldsymbol{n}}$, where $a_{\boldsymbol{n}}\in\mathcal{O}_K$ is such that $\{v_p(a_{\boldsymbol{n}})\}_{\boldsymbol{n}\in\Lambda}\to\infty$. We claim that if $\varphi^n(x)\in\ker d$ with $n\in\mathbb{N}$ and $x\in\mathcal{O}_K$, we have $x\in\varphi(\mathcal{O}_K)$. Since the Frobenius φ_* on $\widehat{\Omega}^1_{\mathcal{O}_K}$ is injective by Lemma 1.10(i) and the commutativity $d\circ\varphi=\varphi_*\circ d$, we may assume $n=0$. By definition, we have $\partial_j(x)=0$ for all $j\in J_K$. We have

$$t_j\partial_j(x)=\sum_{\boldsymbol{n}\in\Lambda}(t_j\partial_j\circ\varphi)(a_{\boldsymbol{n}})\boldsymbol{t}^{\boldsymbol{n}}+\sum_{\boldsymbol{n}\in\Lambda}\varphi(a_{\boldsymbol{n}})t_j\partial_j(\boldsymbol{t}^{\boldsymbol{n}})=\sum_{\boldsymbol{n}\in\Lambda}\varphi(pt_j\partial_j(a_{\boldsymbol{n}})+n_ja_{\boldsymbol{n}})\boldsymbol{t}^{\boldsymbol{n}}.$$

Hence, we have $a_{\boldsymbol{n}}=-n_j^{-1}pt_j\partial_j(a_{\boldsymbol{n}})$ if $n_j\neq 0$. Therefore, for $\boldsymbol{n}\in\Lambda\setminus\{\boldsymbol{0}\}$, we have $v_p(a_{\boldsymbol{n}})\geq v_p(a_{\boldsymbol{n}})+1$, that is, $a_{\boldsymbol{n}}=0$, which implies the claim. By using the claim, if we have $x\in\ker d$, then we have $x\in\bigcap_{n\in\mathbb{N}}\varphi^n(\mathcal{O}_K)$. Since the complete discrete valuation ring $\bigcap_{n\in\mathbb{N}}\varphi^n(\mathcal{O}_K)$ is absolutely unramified with residue field $k_K^{p^\infty}$, the inclusion $\mathcal{O}_{K_{\mathrm{can}}}\subset\bigcap_{n\in\mathbb{N}}\varphi^n(\mathcal{O}_K)$ is an equality by approximation, which implies the assertion. $\square$

1D. ***A spectral sequence of continuous group cohomology.*** The following lemma is a basic fact when we calculate continuous Galois cohomology whose coefficient is an inverse limit of p-adic Banach spaces with surjective transition maps. For example, we need it later when we calculate cohomology of $\mathbb{B}^+_{\mathrm{dR}}$-modules.

Lemma 1.14 (Compare [Neukirch et al. 2008, Theorem 2.7.5].). *Let G be a profinite group and $\{M_n\}_{n\in\mathbb{N}}$ be an inverse system of continuous G-modules (each M_n may not be discrete) such that the transition map $M_{n+1}\to M_n$ admits a continuous section (as topological spaces) for all $n\in\mathbb{N}$. Let M_∞ be the continuous G-module $\varprojlim M_n$ with the inverse limit topology. Then, we have a canonical exact sequence*

$$0 \longrightarrow \varprojlim_n{}^1 H^{q-1}(G,M_n) \longrightarrow H^q(G,M_\infty) \longrightarrow \varprojlim_n H^q(G,M_n) \longrightarrow 0$$

for all $q\in\mathbb{N}$, where $\varprojlim^\bullet$ is the derived functor of $\varprojlim$ in the category of inverse systems of abelian groups indexed by $\mathbb{N}$.

Proof. Let $\mathscr{C}^\bullet_\infty := \mathscr{C}^\bullet_{\mathrm{cont.}}(G,M_\infty)$ (resp. $\mathscr{C}^\bullet_n := \mathscr{C}^\bullet_{\mathrm{cont.}}(G,M_n)$) be the continuous cochain complex of G with coefficients in M_∞ (resp. M_n). Then, $\{\mathscr{C}^\bullet_n\}_{n\in\mathbb{N}}$ forms an inverse system of cochain complexes and we have $\mathscr{C}^\bullet_\infty=\varprojlim_n \mathscr{C}^\bullet_n$. Moreover, the transition maps of the inverse system $\{\mathscr{C}^\bullet_n\}_{n\in\mathbb{N}}$ are surjective by the existence of continuous sections, in particular, $\{\mathscr{C}^\bullet_n\}_{n\in\mathbb{N}}$ satisfies the Mittag–Leffler condition. Then, the assertion follows from [Weibel 1994, Variant in pp.84]. □

1E. ***Hyodo's calculations of Galois cohomology.*** We will recall Hyodo's calculations of Galois cohomology. For $n\in\mathbb{Z}$, denote by $\mathbb{Z}_p(n)$ the n-th Tate twist of $\mathbb{Z}_p$. For a $\mathbb{Z}_p[G_K]$-module V, let $V(n):=V\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(n)$.

Theorem 1.15 [Hyodo 1986, Theorem 1]. *For $n\in\mathbb{N}$ and $q\in\mathbb{Z}$, we have canonical isomorphisms*

$$H^n(G_K,\mathbb{C}_p(q))\cong\begin{cases}0 & q\neq n,n-1,\\ \hat{\Omega}^q_K & \textit{otherwise}.\end{cases}$$

We will generalize the following theorem as an application of the Main Theorem in Section 7.

Theorem 1.16. (i) [Hyodo 1986, Theorem 2] *We have the exact sequence*

$$0 \longrightarrow H^1(G_{K_{\mathrm{can}}},\mathbb{Z}_p(1)) \xrightarrow{\mathrm{Inf}} H^1(G_K,\mathbb{Z}_p(1)) \xrightarrow{\mathrm{can.}} H^1(G_K,\mathbb{C}_p(1)).$$

(ii) [Hyodo 1987, Theorem (0-2)] *If k_K is separably closed, then*

$$\mathrm{Inf}: H^1(G_{K_{\mathrm{can}}},\mathbb{Z}_p(n))\to H^1(G_K,\mathbb{Z}_p(n))$$

is an isomorphism for $n\neq 1$.

1F. *Closed subgroups of G_K.* Let L be an algebraic extension of K in $\mathbb{C}_p$. Let $\widehat{L}^{\mathrm{alg}}$ be the algebraic closure of $\widehat{L}$ in $\mathbb{C}_p$. Let M be a finite extension of $\widehat{L}$ and choose a polynomial $f(X) \in \widehat{L}[X]$ such that $M \cong \widehat{L}[X]/(f(X))$. Let $f_0(X) \in L[X]$ be a polynomial such that the p-adic valuations of the coefficients of $f - f_0$ are large enough. Then, we have $M \cong \widehat{L}[X]/(f_0(X))$ by Krasner's lemma. In particular, the algebraic extension $(M \cap L^{\mathrm{alg}})/L$ is dense in M. Hence, we have a canonical morphism of profinite groups $G_L \to G_{\widehat{L}}$, which is an isomorphism whose inverse $G_{\widehat{L}} \to G_L$ maps g to $g|_{L^{\mathrm{alg}}}$. In the sequel, we will identify G_L with $G_{\widehat{L}}$ and we also regard $G_{\widehat{L}}$ as a closed subgroup of G_K.

1G. *Perfection.* For a subset J of J_K, we denote the p-adic completion of the field $\bigcup_{n\in\mathbb{N}} K(\{t_j^{p^{-n}}\}_{j\in J})$ by K_J. Then, K_J is a complete discrete valuation field of mixed characteristic $(0, p)$ with $e_{K_J/K} = 1$ and its residue field is isomorphic to $\bigcup_{n\in\mathbb{N}} k_K(\{\bar{t}_j^{p^{-n}}\}_{j\in J})$. We also denote K_{J_K} by K^{pf}, which is referred as a perfection of K since the residue field $k_{K^{\mathrm{pf}}} \cong k_K^{\mathrm{pf}}$ of K^{pf} is perfect. Since we may assume that $\{t_j\}_{j\in J_K}$ is contained in K_0 (Section 1A), we may assume $(K_0)_J = (K_J)_0$, which is denoted by $K_{J,0}$ for simplicity.

Let $\mathcal{P}(J_K)$ be the subsets of J_K consisting of subsets $J \in J_K$ such that $J_K \setminus J$ is finite. Note that we have $[k_{K_J} : k_{K_J}^p] = p^{|J_K\setminus J|} < \infty$ for $J \in \mathcal{P}(J_K)$ since $\{\bar{t}_j\}_{j\in J_K\setminus J}$ forms of a p-basis of k_{K_J}. We regard $\mathcal{P}(J_K)$ as an inverse system with respect to the reverse inclusion. Then, we have

$$K \cong \varprojlim_{J\in\mathcal{P}(J_K)} K_J = \bigcap_{J\in\mathcal{P}(J_K)} K_J,$$

that is, K is represented by an inverse limit of complete discrete valuation fields, whose residue fields admit a finite p-basis. In fact, if we endow J_K with a well-order $\precsim$ by the axiom of choice, then for $J \in \mathcal{P}(J_K)$, the subset

$$\mathcal{E}_J := \{1\} \cup \left\{ t_{j_1}^{a_1 p^{-n_1}} \dots t_{j_m}^{a_m p^{-n_m}} \;\middle|\; \begin{array}{l} j_1 \precnsim \cdots \precnsim j_m \in J, 0 < a_{j_i} < p^{n_{j_i}} \in \mathbb{N}_{>0} \\ (p, a_{j_i}) = 1 \text{ for } 1 \le i \le m \in \mathbb{N}_{>0} \end{array} \right\}$$

of K_J forms a basis of K_J as a K-Banach space. If $J_1 \subset J_2$ are in $\mathcal{P}(J_K)$, then we have $\mathcal{E}_{J_1} \subset \mathcal{E}_{J_2}$ and the assertion follows from the fact $\{1\} = \bigcap_{J\in\mathcal{P}(J_K)} \mathcal{E}_J$.

1H. *G-regular ring.* We will recall basic facts about G-regular rings. For details, see [Fontaine 1994b, Section 1].

Let E be a topological field and G a topological group. A finite-dimensional E-vector space V is an E-representation of G if V has a continuous E-linear action of G. We denote the category of E-representations of G by $\mathrm{Rep}_E G$. We call B an (E, G)-ring if B is a commutative E-algebra and G acts on B by E-algebra automorphisms. Let B be an (E, G)-ring. For $V \in \mathrm{Rep}_E G$, let $D_B(V) := (B \otimes_E V)^G$

and we will call the following canonical homomorphism the comparison map:

$$\alpha_B(V) : B \otimes_{B^G} D_B(V) \to B \otimes_E V.$$

We say that an (E, G)-ring B is G-regular if the following is satisfied:

($G\cdot R_1$) The ring B is reduced.

($G\cdot R_2$) For all $V \in \mathrm{Rep}_E G$, $\alpha_B(V)$ is injective.

($G\cdot R_3$) Every G-stable E-line in B is generated by an invertible element of B.

Here, a G-stable E-line in B means one-dimensional G-stable E-vector space in B. The condition ($G\cdot R_3$) implies that B^G is a field. We say that $V \in \mathrm{Rep}_E G$ is B-admissible if $\alpha_B(V)$ is an isomorphism. We denote the category of B-admissible E-representations of G by $\mathrm{Rep}_{B/E} G$, which is a Tannakian full subcategory of $\mathrm{Rep}_E G$ [Fontaine 1994b, Proposition 1.5.2].

Notation. We will call an object of $\mathrm{Rep}_{\mathbb{Q}_p} G_K$ a p-adic representation of G_K. For a $(\mathbb{Q}_p, G_K)$-ring B, we denote $\mathrm{Rep}_{B/\mathbb{Q}_p} G_K$ by $\mathrm{Rep}^{\mathrm{adm}}_B G_K$ if no confusion arises.

We recall the basic facts about G-regular rings.

Lemma 1.17. *Let B be a field and G a group acting on B by ring automorphisms. Let M be a finite-dimensional B-vector space with semilinear G-action. Then, the canonical map*

$$B \otimes_{B^G} M^G \to M$$

is injective. In particular, we have $\dim_{B^G} M^G \leq \dim_B M$.

Proof. Suppose that the assertion does not hold. Let $n \in \mathbb{N}$ be the smallest integer such that there exist n elements $v_1, \ldots, v_n \in M^G$ which are linearly independent over B^G but not over B. Let $\sum_{1\leq i\leq n} \lambda_i v_i = 0$ be a nontrivial relation with $\lambda_i \in B$. Since B is a field, we may assume that $\lambda_1 = 1$. Then, we have

$$0 = (g-1)\Bigl(\sum_{1\leq i\leq n} \lambda_i v_i\Bigr) = \sum_{1<i\leq n} (g(\lambda_i) - \lambda_i) v_i.$$

Hence, we have $\lambda_i \in B^G$ by assumption, which is a contradiction. □

Example 1.18 [Fontaine 1994b, Proposition 1.6.1]. All (E, G)-rings which are fields are G-regular. In fact, we have only to verify ($G\cdot R_2$), which follows by applying the above lemma to $M := B \otimes_E V$.

Lemma 1.19 [Fontaine 1994b, Proposition 1.4.2]. *Let B be a G-regular (E, G)-ring and V an E-representation of G. Then, we have* $\dim_{B^G} D_B(V) \leq \dim_E V$. *Moreover, the equality holds if and only if V is B-admissible.*

Lemma 1.20 [Fontaine 1994b, Proposition 1.6.5]. *Let B be a G-regular (E,G)-ring and B' an E-subalgebra of B stable by G. Assume that B' satisfies $(G\text{-}R_3)$ and that the canonical map $B^G \otimes_{B'^G} B' \to B$ is injective. Then, B' is a G-regular (E,G)-ring. Moreover, if $V \in \mathrm{Rep}_E G$ is B'-admissible, then V is B-admissible and the canonical map*

$$B^G \otimes_{B'^G} D_{B'}(V) \to D_B(V)$$

is an isomorphism.

Lemma 1.21 [Fontaine 1994b, Corollaire 1.6.6]. *Let B' be an integral domain which is an (E,G)-ring, and B the fraction field of B'. If B' satisfies $(G\text{-}R_3)$ and $B'^G = B^G$, then B' is G_K-regular.*

Remark 1.22 (restriction). Let B be a G-regular (E,G)-ring and H a subgroup of G such that B is H-regular as an (E,H)-ring. If $V \in \mathrm{Rep}_E G$ is B-admissible, then $V|_H$ is also B-admissible in $\mathrm{Rep}_E H$. Moreover, we have a canonical isomorphism $B^H \otimes_{B^G} D_B(V) \cong D_B(V|_H)$. Indeed, the admissibility of V implies that we have the comparison isomorphism $B \otimes_{B^G} D_B(V) \cong B \otimes_E V$ as $B[G_K]$-modules. By taking H-invariants, we have $B^H \otimes_{B^G} D_B(V) \cong D_B(V|_H)$. In particular, we have $\dim_{B^H} D_B(V|_H) = \dim_{B^G} D_B(V) = \dim_E V$, which implies the B-admissibility of $V|_H$ by Lemma 1.19.

2. A generalization of Sen's theorem

The aim of this section is to prove the following generalization of Sen's theorem on $\mathbb{C}_p$-admissible representations [Sen 1980, Corollary in (3.2)].

Theorem 2.1. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$. The following are equivalent*:

(i) *There exists a finite extension L over the maximal unramified extension of K such that G_L acts trivially on V.*

(ii) *V is $\mathbb{C}_p$-admissible.*

(iii) *$V|_{K^{\mathrm{pf}}}$ is $\mathbb{C}_p$-admissible as an object of $\mathrm{Rep}_{\mathbb{Q}_p} G_{K^{\mathrm{pf}}}$.*

Lemma 2.2. *Let E be a field of characteristic 0 and $\rho : U^{(n)}_{\mathbb{Q}_p} \ltimes \prod_{i\in I} p^{n_i}\mathbb{Z}_p \to \mathrm{GL}_r(E)$ a group homomorphism with $n, r \in \mathbb{N}_{>0}$ and $(n_i)_{i\in I} \in \mathbb{N}^I$, where the action of $U^{(n)}_{\mathbb{Q}_p}$ on $\prod_{i\in I} p^{n_i}\mathbb{Z}_p$ is given by scalar multiplication. If $\ker\rho$ contains an open subgroup of $U^{(n)}_{\mathbb{Q}_p}$, then the image of ρ is finite.*

Proof. By shrinking $U^{(n)}_{\mathbb{Q}_p}$, we may assume that $\ker\rho$ contains $U^{(n)}_{\mathbb{Q}_p}$. Also, we may assume that E is algebraically closed. Let $x_0 := 1 + p^n \in U^{(n)}_{\mathbb{Q}_p}$, $\boldsymbol{x} \in \prod_{i\in I} p^{n_i}\mathbb{Z}_p$. By the fact that $\ker\rho$ is a normal subgroup of $U := U^{(n)}_{\mathbb{Q}_p} \ltimes \prod_{i\in I} p^{n_i}\mathbb{Z}_p$ and a simple calculation, we have

$$(1,\boldsymbol{x})^{-1}(x_0,\boldsymbol{0})(1,\boldsymbol{x})(x_0^{-1},\boldsymbol{0}) = (1,(x_0-1)\boldsymbol{x}) = (1,p^n\boldsymbol{x}) \in \ker\rho.$$

In particular, $\ker\rho$ contains $U_{\mathbb{Q}_p}^{(n)} \ltimes \prod_{i\in I} p^{n+n_i}\mathbb{Z}_p$ as a normal subgroup. By taking the quotient of U by this subgroup, ρ factors through a group homomorphism $\bar{\rho} : (\mathbb{Z}/p^n\mathbb{Z})^I \to \mathrm{GL}_r(E)$.

To prove the assertion, it suffices to prove that for any finite subset S of $\mathrm{Im}\,\bar{\rho}$, we have $|S| \le p^{nr}$. Any $g \in \mathrm{Im}\,\bar{\rho}$ is conjugate to a diagonal matrix whose diagonal entries are in $\mu_{p^n}(E)$ since the order of g divides p^n. Since the elements of S commute, S is simultaneously diagonalizable. Hence, up to conjugation, S is contained in the set $\{\mathrm{diag}(a_1,\dots,a_r) \mid a_i \in \mu_{p^n}(E)\}$, whose order is p^{nr}. □

Proof of Theorem 2.1. The implication (i) ⇒ (ii) follows from Hilbert 90 and (ii) ⇒ (iii) follows from Remark 1.22. We will prove (iii) ⇒ (i). Note that if k_K is perfect, then the assertion is a theorem of Sen ([1980, Corollary in (3.2)]).

By replacing K by a finite extension of K^{ur}, we may assume that k_K is separably closed and K satisfies condition (H). In this case, the assertion to prove is that G_K acts on V via a finite quotient. Since the residue field k_K^{pf} of K^{pf} is algebraically closed, $G_{K^{\mathrm{pf}}} = G_{K^{\mathrm{geo}}}$ acts on V via a finite quotient by Sen's theorem, where $K^{\mathrm{geo}} := \bigcup_{n\in\mathbb{N}} K(\{t_j^{p^{-n}}\}_{j\in J_K})$. Hence, there exists a finite extension L/K such that $G_{LK^{\mathrm{geo}}}$ acts trivially on V. In particular, if we put $K_\infty := K^{\mathrm{geo}}(\mu_{p^\infty})$, then G_{LK_∞} acts trivially on V. In the following, we regard V as a p-adic representation of $G_{LK_\infty/L}$. Take a basis of V and let $\rho' : G_{LK_\infty/L} \to \mathrm{GL}_r(\mathbb{Q}_p)$ be the corresponding matrix presentation of V with $r := \dim_{\mathbb{Q}_p} V$. We have only to prove that the image of ρ' is finite.

Since K satisfies condition (H), we have an isomorphism $G_{K_\infty/K} \cong U_{\mathbb{Q}_p}^{(n_0)} \ltimes \mathbb{Z}_p^{J_K}$, where $n_0 \in \mathbb{N}_{>1}$ satisfies $G_{K(\mu_{p^\infty})/K} \cong U_{\mathbb{Q}_p}^{(n_0)}$ via the cyclotomic character and $U_{\mathbb{Q}_p}^{(n_0)}$ acts on $\mathbb{Z}_p^{J_K}$ by scalar multiplication (see [Hyodo 1986, Section 1] for details). We have $G_{LK_\infty/LK^{\mathrm{geo}}} \le \ker\rho' \trianglelefteq_c G_{LK_\infty/L}$. By using the restriction map $\mathrm{Res}^{LK_\infty}_{K_\infty}$ and the above isomorphism, we may regard these groups as subgroups of $U_{\mathbb{Q}_p}^{(n_0)} \ltimes \mathbb{Z}_p^{J_K}$. Since $G_{LK_\infty/L}$ is an open subgroup of $G_{K_\infty/K}$, there exists $n \in \mathbb{N}$ and $(n_j)_{j\in J_K} \in \mathbb{N}^{J_K}$ such that $G_{LK_\infty/L}$ contains $U := U_{\mathbb{Q}_p}^{(n)} \ltimes \prod_{j\in J_K} p^{n_j}\mathbb{Z}_p$ as an open subgroup. Since $G_{LK_\infty/LK^{\mathrm{geo}}}$ is an open subgroup of $G_{K_\infty/K^{\mathrm{geo}}} \cong G_{K(\mu_{p^\infty})/K} \cong U_{\mathbb{Q}_p}^{(n_0)} \cong \mathbb{Z}_p$, $\ker\rho'$ contains an open subgroup of $U_{\mathbb{Q}_p}^{(n)}$. Therefore, the group homomorphism $\rho := \rho'|_U : U \to \mathrm{GL}_r(\mathbb{Q}_p)$ satisfies the assumption of Lemma 2.2, hence, the image of ρ is finite. Since U is open in $G_{LK_\infty/L}$, we obtain the assertion. □

3. Basic construction of rings of p-adic periods

Throughout this section, let $\mathcal{K}$ be a closed subfield of $\mathbb{C}_p$ whose value group $v_p(\mathcal{K}^\times)$ is discrete. We will recall the construction of rings of p-adic periods

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}},\quad \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}},\quad \mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}},\quad \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}},\quad \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$$

due to Fontaine [1994a], which is functorial with respect to $\mathbb{C}_p$ and $\mathcal{K}$. We also recall abstract algebraic properties of these rings as in [Brinon 2006]. Although we do not assume $\mathcal{K} = K$, standard techniques of proofs in the case $\mathcal{K} = K$, which are developed in [Fontaine 1994a; Brinon 2006], can be applied to our situation.

3A. *Universal pro-infinitesimal thickenings.*

Defintion 3.1 [Fontaine 1994a, Section 1]. A p-adically formal pro-infinitesimal $\mathcal{O}_{\mathcal{K}}$-thickening of $\mathcal{O}_{\mathbb{C}_p}$ is a pair (D, θ_D), where

- D is an $\mathcal{O}_{\mathcal{K}}$-algebra,
- $\theta_D : D \to \mathcal{O}_{\mathbb{C}_p}$ is a surjective $\mathcal{O}_{\mathcal{K}}$-algebra homomorphism such that D is $(p, \ker \theta_D)$-adic Hausdorff complete.

Obviously, p-adically formal $\mathcal{O}_{\mathcal{K}}$-thickenings of $\mathcal{O}_{\mathbb{C}_p}$ form a category.

Theorem 3.2 [Fontaine 1994a, Théorème 1.2.1]. *The category of p-adically formal pro-infinitesimal $\mathcal{O}_{\mathcal{K}}$-thickenings of $\mathcal{O}_{\mathbb{C}_p}$ admits a universal object, that is, an initial object.*

Such an object is unique up to a canonical isomorphism and we denote it by $(\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}, \theta_{\mathbb{C}_p/\mathcal{K}})$. Note that $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is functorial with respect to $\mathbb{C}_p$ and $\mathcal{K}$. We recall the construction. Let $R_{\mathbb{C}_p} := \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ be the perfection of the ring $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$. We have the canonical isomorphism

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \to R_{\mathbb{C}_p}; \quad (x^{(n)})_{n\in\mathbb{N}} \mapsto (x^{(n)} \bmod p\mathcal{O}_{\mathbb{C}_p})_{n\in\mathbb{N}},$$

where the addition and the multiplication of the LHS are given by

$$((x^{(n)}) + (y^{(n)}))_n = \lim_m (x^{(n+m)} + y^{(n+m)})^{p^m}, \quad (x^{(n)}) \cdot (y^{(n)}) = (x^{(n)} y^{(n)}).$$

Let $\theta_{\mathbb{C}_p/\mathbb{Q}_p} : W(R_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}$ be defined by $\sum_{n\in\mathbb{N}} p^n[x_n] \mapsto \sum_{n\in\mathbb{N}} p^n x_n^{(0)}$. This is a surjective $\mathbb{Z}_p$-algebra homomorphism. Let $\theta_{\mathbb{C}_p/\mathcal{K}} : \mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}$ be the linear extension of $\theta_{\mathbb{C}_p/\mathbb{Q}_p}$. Then, $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is the Hausdorff completion of $\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p})$ with respect to the $(p, \ker \theta_{\mathbb{C}_p/\mathcal{K}})$-adic topology. We will give an explicit description of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ later: Note that the description, together with the isomorphism $W(R_{\mathbb{C}_p}) \cong \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$ (Remark 3.5), immediately implies that $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is an integral domain (at least) when we have $\mathcal{K} = \mathcal{K}_0$.

We define $\tilde{t}_j := (t_j, t_j^{p^{-1}}, \dots) \in R_{\mathbb{C}_p}$ and $u_j := t_j - [\tilde{t}_j] \in \ker \theta_{\mathbb{C}_p/\mathcal{K}_0}$. Let $v_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ be the p-adic semivaluation of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. We put

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\} := \left\{ \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}} \mathbf{u}^{\boldsymbol{n}} \;\middle|\; a_{\boldsymbol{n}} \in \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}, \{v_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{|\boldsymbol{n}|=n} \to \infty \text{ for all } n \in \mathbb{N} \right\}.$$

If $J_{\mathcal{K}}$ is finite, $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\}$ is a ring of formal power series with coefficients in $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$. We extend $\theta_{\mathbb{C}_p/\mathbb{Q}_p}$ to a surjective $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism $\vartheta_{\mathbb{C}_p/\mathcal{K}}:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\}\to\mathcal{O}_{\mathbb{C}_p}$ by $\vartheta_{\mathbb{C}_p/\mathcal{K}}(\mathrm{u}_j)=0$. Then, $(\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\},\vartheta_{\mathbb{C}_p/\mathcal{K}})$ is a p-adically formal $\mathbb{Z}_p$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$. We have a canonical $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism

$$\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}};\quad \mathbf{u}^{\boldsymbol{n}}\mapsto\boldsymbol{u}^{\boldsymbol{n}}.$$

Lemma 3.3. *If we assume $\mathcal{K}=\mathcal{K}_0$, then $\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is an isomorphism. In particular, we have*

$$v_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}(x)=\inf_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}}v_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})$$

for $x=\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}}a_{\boldsymbol{n}}\boldsymbol{u}^{\boldsymbol{n}}$ with $a_{\boldsymbol{n}}\in\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$.

Proof. Denote $\mathcal{A}=\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\}$ and $\vartheta=\vartheta_{\mathbb{C}_p/\mathcal{K}}$. We regard $\mathcal{O}_{\mathcal{K}}$ as a $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$-algebra as in Section 1A. We recall that since $\mathcal{K}=\mathcal{K}_0$, the map $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}\to\mathcal{O}_{\mathcal{K}}$ is formally étale for the p-adic topology. We also regard $\mathcal{A}$ as a $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$-algebra by $T_j\mapsto[\tilde{t}_j]+\mathrm{u}_j$. Then, by the lifting property, we can lift the canonical $\mathcal{O}_{\mathcal{K}}$-algebra structure on $\mathcal{A}/(p,\ker\vartheta)\cong\mathcal{O}_{\mathbb{C}_p}/(p)$ to an $\mathcal{O}_{\mathcal{K}}$-algebra structure on $\mathcal{A}\cong\varprojlim_n\mathcal{A}/(p,\ker\vartheta)^n$:

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{K}} \arrow[r, "\text{can.}"] \arrow[dr, dashed, "\exists!"] & \mathcal{O}_{\mathbb{C}_p} \\
\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}} \arrow[u, "\text{str.}"] \arrow[r, "\text{str.}"] & \mathcal{A} \arrow[u, "\vartheta"']
\end{tikzcd}$$

By this structure map, we may regard $\mathcal{A}$ as a pro-infinitesimal $\mathcal{O}_{\mathcal{K}}$-thickening of $\mathcal{O}_{\mathbb{C}_p}$. By universality, we have only to prove that $\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is an $\mathcal{O}_{\mathcal{K}}$-algebra homomorphism. Let $\alpha:\mathcal{O}_{\mathcal{K}}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ be the composition of the structure map $\mathcal{O}_{\mathcal{K}}\to\mathcal{A}$ and $\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. Since $\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ commutes with the projections ϑ and $\theta_{\mathbb{C}_p/\mathcal{K}}$, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{K}} \arrow[r, "\text{can.}"] \arrow[dr, "\alpha"] & \mathcal{O}_{\mathbb{C}_p} \\
\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}} \arrow[u, "\text{str.}"] \arrow[r, "\text{str.}"] & \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}, \arrow[u, "\theta_{\mathbb{C}_p/\mathcal{K}}"']
\end{tikzcd}$$

where the horizontal structure map is given by $T_j\mapsto t_j$. By this diagram and the lifting property, α coincides with the structure map $\mathcal{O}_{\mathcal{K}}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ modulo $(p,\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$ for all $n\in\mathbb{N}$. Since $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is $(p,\ker\theta_{\mathbb{C}_p/\mathcal{K}})$-adically Hausdorff complete, α coincides with the structure map $\mathcal{O}_{\mathcal{K}}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$, which implies the assertion. $\square$

For general $\mathcal{K}$, we have:

Lemma 3.4. (i) *The canonical map*

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}^{\mathrm{ur}}}$$

is an isomorphism.

(ii) *If $\mathcal{L}/\mathcal{K}$ is a finite extension with $[k_{\mathcal{L}} : k_{\mathcal{K}}]_{\mathrm{sep}} = 1$, then the canonical map*

$$\mathcal{O}_{\mathcal{L}} \otimes_{\mathcal{O}_{\mathcal{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}$$

is an isomorphism.

(iii) *Let $\mathcal{L}$ be a finite extension of the p-adic completion of an unramified extension of $\mathcal{K}$. Then, the canonical map*

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{L}})^n$$

is an isomorphism for all $n \in \mathbb{N}$.

Proof. (i) The assertion is equivalent to saying that the category of p-adically formal $\mathcal{O}_{\mathcal{K}}$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$ is equivalent to the category of p-adically formal $\mathcal{O}_{\mathcal{K}^{\mathrm{ur}}}$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$. Let (D, θ_D) be a p-adically formal $\mathcal{O}_{\mathcal{K}}$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$. Then, we have only to prove that there exists a unique $\mathcal{O}_{\mathcal{K}}$-algebra homomorphism $\mathcal{O}_{\mathcal{K}^{\mathrm{ur}}} \to D$ such that θ_D is an $\mathcal{O}_{\mathcal{K}^{\mathrm{ur}}}$-algebra homomorphism. By dévissage, we may replace D by $D/(p, \ker\theta_D)^n$ with $n \in \mathbb{N}$. Since θ_D induces an isomorphism $D/(p, \ker\theta_D) \cong \mathcal{O}_{\mathbb{C}_p}/(p)$ and $\mathcal{O}_{\mathcal{K}^{\mathrm{ur}}}/\mathcal{O}_{\mathcal{K}}$ is p-adically formally étale, the assertion follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{K}^{\mathrm{ur}}} \arrow[r, "\text{can.}"] \arrow[dr, dashed, "\exists!"] & \mathcal{O}_{\mathbb{C}_p}/(p) \\
\mathcal{O}_{\mathcal{K}} \arrow[u, "\text{can.}"] \arrow[r, "\text{str.}"] & D/(p, \ker\theta_D)^n, \arrow[u, "(\theta_D)_*"']
\end{tikzcd}$$

where $(\theta_D)_*$ is the ring homomorphism induced by θ_D.

(ii) By assumption, the canonical map $\mathcal{O}_{\mathcal{L}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathcal{K}^{\mathrm{ur}}} \to \mathcal{O}_{\mathcal{L}^{\mathrm{ur}}}$ is an isomorphism. By using this fact and (i), we may assume that $\mathcal{K} = \mathcal{K}^{\mathrm{ur}}$ and $\mathcal{L} = \mathcal{L}^{\mathrm{ur}}$. In particular, we may consider the case that $k_{\mathcal{K}}$ is separably closed, where the condition $[k_{\mathcal{L}} : k_{\mathcal{K}}]_{\mathrm{sep}} = 1$ is always satisfied. By faithfully flat descent, the assertion is reduced to the case that $\mathcal{L}/\mathcal{K}$ is Galois. Since $\mathcal{L}/\mathcal{K}$ is a solvable extension [Fesenko and Vostokov 2002, Exercise 2, Section 2, Chapter II], we may assume that $\mathcal{L}/\mathcal{K}$ has prime degree.

By universality, we have only to prove that the LHS is a p-adically formal $\mathcal{O}_{\mathcal{L}}$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$. Hence, it suffices to verify that $\mathcal{O}_{\mathcal{L}} \otimes_{\mathcal{O}_{\mathcal{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is (p, I)-adically Hausdorff complete, where I denotes the kernel of the canonical map $1 \otimes \theta_{\mathbb{C}_p/\mathcal{K}} : \mathcal{O}_{\mathcal{L}} \otimes_{\mathcal{O}_{\mathcal{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \to \mathcal{O}_{\mathbb{C}_p}$. Since we have an isomorphism of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$-modules $\mathbb{O}_{\mathcal{L}} \otimes_{\mathbb{O}_{\mathcal{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \cong (\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}})^{[\mathcal{L}:\mathcal{K}]}$, we have only to prove that the topologies on $\mathbb{O}_{\mathcal{L}} \otimes_{\mathbb{O}_{\mathcal{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ defined by the ideals (p, I) and (p, I') are equivalent, where I' denotes the ideal of $\mathbb{O}_{\mathcal{L}} \otimes_{\mathbb{O}_{\mathcal{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ generated by $\ker(\theta_{\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \to \mathbb{O}_{\mathbb{C}_p})$. By definition, we have $(p, I') \subset (p, I)$. We have only to prove that we have $I^n \subset (\pi_{\mathcal{K}} \otimes 1, I')$ for some $n \in \mathbb{N}$ since p divides $\pi_{\mathcal{K}}^{e_{\mathcal{K}}}$.

In the following, for $x \in \mathbb{O}_{\mathbb{C}_p}$, we denote by $\tilde{x}$ any element $\tilde{x} \in R_{\mathbb{C}_p}$ such that $\tilde{x}^{(0)} = x$. Since we have $\pi_{\mathcal{K}} \otimes 1 - 1 \otimes [\tilde{\pi}_{\mathcal{K}}] \in I'$, we have $(\pi_{\mathcal{K}} \otimes 1, 1 \otimes [\tilde{\pi}_{\mathcal{K}}]) \subset (\pi_{\mathcal{K}} \otimes 1, I')$. Note that if $x \in \mathbb{O}_{\mathcal{L}}$ is primitive, that is, $1, x, \dots, x^{[\mathcal{L}:\mathcal{K}]-1}$ is an $\mathbb{O}_{\mathcal{K}}$-basis of $\mathbb{O}_{\mathcal{L}}$, then we have $I \subset (x \otimes 1 - 1 \otimes [\tilde{x}], I')$. Hence, we have only to prove the existence of a primitive element $x \in \mathbb{O}_{\mathcal{L}}$ satisfying $(x \otimes 1 - 1 \otimes [\tilde{x}])^n \in (\pi_{\mathcal{K}} \otimes 1, I')$ for some $n \in \mathbb{N}$. In the case $[\mathcal{L} : \mathcal{K}] = e_{\mathcal{L}/\mathcal{K}}$, $\pi_{\mathcal{L}}$ is a primitive element of $\mathbb{O}_{\mathcal{L}}$ and we have $(\pi_{\mathcal{L}} \otimes 1 - 1 \otimes [\tilde{\pi}_{\mathcal{L}}])^{2e_{\mathcal{L}/\mathcal{K}}} \in (\pi_{\mathcal{K}} \otimes 1, 1 \otimes [\tilde{\pi}_{\mathcal{K}}])$. Otherwise, we have $[\mathcal{L} : \mathcal{K}] = [k_{\mathcal{L}} : k_{\mathcal{K}}]_{\mathrm{insep}} = p$. If we choose $x \in \mathbb{O}_{\mathcal{L}}$ whose image in $\mathbb{O}_{\mathcal{L}}/\pi_{\mathcal{K}}\mathbb{O}_{\mathcal{L}}$ does not belong to $k_{\mathcal{K}}$, then x is primitive by Nakayama's lemma. Moreover, if we choose $a \in \mathbb{O}_{\mathcal{K}}$ such that $x^p \equiv a \mod \pi_{\mathcal{K}}\mathbb{O}_{\mathcal{L}}$, then we have

$$(x \otimes 1 - 1 \otimes [\tilde{x}])^p \equiv a \otimes 1 - 1 \otimes [\tilde{a}] \mod (\pi_{\mathcal{K}} \otimes 1, 1 \otimes [\tilde{\pi}_{\mathcal{K}}])$$

and $a \otimes 1 - 1 \otimes [\tilde{a}] \in I'$, which implies the assertion.

(iii) We denote the map by i and we will construct the inverse. By replacing $\mathcal{K}$ and $\mathcal{L}$ by $\mathcal{K}^{\mathrm{ur}}$ and $\mathcal{L}^{\mathrm{ur}}$, we may assume $[k_{\mathcal{L}} : k_{\mathcal{K}}]_{\mathrm{sep}} = 1$. By (ii), we identify $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}$ with $\mathbb{O}_{\mathcal{L}} \otimes_{\mathbb{O}_{\mathcal{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. Since $\mathcal{L}/\mathcal{K}$ is étale, by a similar argument as in the proof of (i), we have a unique $\mathcal{K}$-algebra homomorphism

$$j : \mathcal{L} \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$$

such that $\theta_{\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n \to \mathbb{C}_p$ is an $\mathcal{L}$-algebra homomorphism. Hence, we have the $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$-algebra homomorphism

$$j \otimes \mathrm{id} : \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{L}})^n \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n.$$

By construction, we have $(j \otimes \mathrm{id}) \circ i = \mathrm{id}$. To prove $i \circ (j \otimes \mathrm{id}) = \mathrm{id}$, we have only to prove that $i \circ (j \otimes \mathrm{id})$ is an $\mathcal{L}$-algebra homomorphism, which follows from the uniqueness of j. □

Remark 3.5. We may identify $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$ with $W(R_{\mathbb{C}_p})$ [Fontaine 1994a, 1.2.4(e)] and the kernel of $\theta_{\mathbb{C}_p/\mathbb{Q}_p}$ is principal by [Fontaine 1994a, 2.3.3]. Moreover, if $\mathcal{K} = \mathcal{K}_0$ and $k_{\mathcal{K}}$ is perfect, then the canonical map $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is an isomorphism [Fontaine 1994a, 1.2.4(e)]. Note that we have no canonical choice of an embedding $W(k_K^{\mathrm{alg}})[p^{-1}] \to \mathbb{C}_p$ when k_K is imperfect, since different perfections of K induce different embeddings. Thus, we can not endow $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$ with a canonical $W(k_K^{\mathrm{alg}})$-algebra structure induced by that of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/W(k_K^{\mathrm{alg}})[p^{-1}]}$ via the above isomorphism as in the perfect residue field case.

3B. $\mathbb{B}_{\mathbf{dR}}$ ***and*** $\mathbb{B}_{\mathbf{HT}}$**.** We define $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} := \varprojlim_n \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$ and

$$t := \log([\varepsilon]) = \sum_{n\in\mathbb{N}_{>0}} (-1)^{n-1}\frac{([\varepsilon]-1)^n}{n} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$$

with $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in R_{\mathbb{C}_p}$. We also define $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} := \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}[t^{-1}]$. We denote the projection $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \to \mathbb{C}_p$ by $\theta_{\mathbb{C}_p/\mathcal{K}}$ again. Then, $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ is a Hausdorff complete local ring with maximal ideal $\ker\theta_{\mathbb{C}_p/\mathcal{K}}$. Moreover, $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ is an integral domain. In fact, by the following explicit description of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$, it follows from the fact that $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ is a field (Remark 3.6(ii) below).

We define the canonical topology on $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ as follows. We regard

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$$

as a p-adic Banach space whose lattice is given by the image of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. Then, we endow $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with the inverse limit topology, which is a Fréchet complete $\mathcal{K}$-algebra. We also endow $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with a limit of Fréchet topology by regarding $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ as the direct limit of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with respect to the multiplication by t^{-1}. Let $v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ be the semivaluation of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ induced by the p-adic semivaluation of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$ defined by the lattice

$$\mathrm{Im}(\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \xrightarrow{\text{can.}} \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n).$$

Obviously, the semivaluations $\{v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}\}_{n\in\mathbb{N}}$ are decreasing.

We will give an explicit description of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$. Let

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\} := \Big\{ \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}} a_{\boldsymbol{n}}\mathbf{u}^{\boldsymbol{n}} \;\Big|\; a_{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}, \{v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{|\boldsymbol{n}|=n} \to \infty \text{ for all } n, r \in \mathbb{N} \Big\}.$$

This is a $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-algebra. Then, the canonical $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism

$$\iota_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} : \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}; \quad \mathbf{u}^{\boldsymbol{n}} \mapsto \boldsymbol{u}^{\boldsymbol{n}}$$

is an isomorphism. To prove this, by Remark 3.6(ii) below, we may reduce to the case $\mathcal{K} = \mathcal{K}_0$. In this case, the assertion follows from the explicit description of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$.

For $n \in \mathbb{N}$, let $\mathrm{Fil}^n\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ be the closed ideal of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ generated by the ideal $(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$. We endow $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with the decreasing filtration defined by $\mathrm{Fil}^n\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} := \sum_{i+j=n} t^i\,\mathrm{Fil}^j\,\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$. Denote the graded $\mathbb{C}_p$-algebra associated to the filtration by $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$. We also denote by v_j the image of u_j/t in $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K,0}$ for $j \in J_K$. Since the filtration is compatible with the multiplication by t, that is, $t^m\mathrm{Fil}^n\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} = \mathrm{Fil}^{n+m}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$, we have an isomorphism $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}} \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K},0}t^n$.

For $n \in \mathbb{N}$, let

$$\mathbb{C}_p\{\mathbf{v}_{J_{\mathcal{K}}}\}_n := \left\{ \sum_{\boldsymbol{n} \in \mathbb{N}^{\oplus J_{\mathcal{K}}} : |\boldsymbol{n}| = n} a_{\boldsymbol{n}} \mathbf{v}^{\boldsymbol{n}} \;\middle|\; a_{\boldsymbol{n}} \in \mathbb{C}_p, \{v_p(a_{\boldsymbol{n}})\}_{\boldsymbol{n}} \to \infty \right\}$$

and $\mathbb{C}_p\{\mathbf{v}_{J_{\mathcal{K}}}\} := \bigoplus_{n \in \mathbb{N}} \mathbb{C}_p\{\mathbf{v}_{J_{\mathcal{K}}}\}_n$. We have a $\mathbb{C}_p$-algebra homomorphism

$$\iota_{\mathrm{HT},\mathbb{C}_p/\mathcal{K},0} : \mathbb{C}_p\{\mathbf{v}_{J_{\mathcal{K}}}\} \to \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K},0}; \quad \mathbf{v}^{\boldsymbol{n}} \mapsto \boldsymbol{v}^{\boldsymbol{n}},$$

which is an isomorphism. One reduces to the case $\mathcal{K} = \mathcal{K}_0$ by Remark 3.6(ii) below. Then, the assertion follows from the above explicit description of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and the formula of the semivaluation $v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ (Remark 3.6(iii) below). By this description, $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ is an integral domain.

Remark 3.6. (i) (The perfect residue field case) Assume that $k_{\mathcal{K}}$ is perfect. Then, we have a canonical isomorphism $\mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathcal{K}}$ for $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$. Moreover, $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ is a complete discrete valuation field of equal characteristic 0 with valuation ring $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$, t is a uniformizer and the residue field is $\mathbb{C}_p$. We also have an isomorphism $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p} \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p t^n$. In fact, the first assertion follows from Remark 3.5 and the latter assertion reduces to the case where k_K is perfect by regarding $\mathbb{C}_p$ as the p-adic completion of $(K^{\mathrm{pf}})^{\mathrm{alg}}$ [Fontaine 1994a, 1.5.1].

(ii) (Invariance) The above structures on $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ (ring structure, filtration, topology) are invariant under finite or unramified extensions. As a consequence, we may regard $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ as a $\mathcal{K}^{\mathrm{alg}}$-algebra and a similar invariance for $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ as a graded $\mathbb{C}_p$-algebra also holds. As for a filtered ring, the invariance follows from Lemma 3.4(iii). To prove the rest of the assertion, we have only to prove that for an unramified extension or a finite extension $\mathcal{L}/\mathcal{K}$, the p-adic semivaluations $v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{L}}$ are equivalent for all $n \in \mathbb{N}$. The unramified case follows from Lemma 3.4(i). In the other case, let $\Lambda^{(n)}_{\mathcal{K}}$ (resp. $\Lambda^{(n)}_{\mathcal{L}}$) be the image of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ (resp. $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}$) in $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$. Replacing $\mathcal{K}$ by the maximal unramified extension of $\mathcal{K}$ in $\mathcal{L}$, we may assume that $\mathcal{L}/\mathcal{K}$ satisfies the assumption in Lemma 3.4(ii). Since $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}$ is a finite $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$-module by Lemma 3.4(ii), there exists $m \in \mathbb{N}$ such that $p^m\Lambda^{(n)}_{\mathcal{L}} \subset \Lambda^{(n)}_{\mathcal{K}}$ by Lemma 3.4(iii). Since we have $\Lambda^{(n)}_{\mathcal{K}} \subset \Lambda^{(n)}_{\mathcal{L}}$ by definition, the two p-adic topologies induced by the lattices $\Lambda^{(n)}_{\mathcal{K}}$ and $\Lambda^{(n)}_{\mathcal{L}}$ respectively are equivalent, which implies the assertion.

(iii) Assume $\mathcal{K} = \mathcal{K}_0$. Then, we have the formula

$$v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(x) = \inf_{|\boldsymbol{n}|<n} v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}),$$

where we have $x = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}}\boldsymbol{u}^{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with $a_{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$. This follows from the explicit description of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$.

3C. *Connections on* $\mathbb{B}_{\mathbf{dR}}$ *and* $\mathbb{B}_{\mathbf{HT}}$. We denote by $\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ the direct limit $\varinjlim \widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$, where the transition maps are the multiplication by $1\otimes t^{-1}$. Then, the canonical derivation $d:\mathcal{K}\to\widehat{\Omega}^1_{\mathcal{K}}$ uniquely extends to a $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-linear continuous derivation

$$\nabla:\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}\to\widehat{\Omega}^1_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}.$$

Indeed, the canonical derivation $d:\mathcal{O}_{\mathcal{K}}\to\widehat{\Omega}^1_{\mathcal{O}_{\mathcal{K}}}$ extends to an $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$-linear derivation $d:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}\to\widehat{\Omega}^1_{\mathcal{O}_{\mathcal{K}}}\widehat{\otimes}_{\mathcal{O}_{\mathcal{K}}}\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ by the construction of $\mathbb{A}_{\mathrm{inf}}$. After inverting p, then taking the $\ker\theta_{\mathbb{C}_p/\mathcal{K}}$-adic Hausdorff completion, we obtain a desired derivation. Since the image of $\mathcal{K}\otimes_{\mathbb{Z}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ is dense in $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ by construction, the uniqueness follows. More precisely, if we denote by $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ the derivations on $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ given by $\nabla(x)=\sum_{j\in J_{\mathcal{K}}}dt_j\otimes\partial_j(x)$, then $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ are mutually commutative continuous $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-derivations and we have $\partial_j=\partial/\partial u_j$. More generally, the exterior derivation $d_q:\widehat{\Omega}^q_{\mathcal{K}}\to\widehat{\Omega}^{q+1}_{\mathcal{K}}$ for $q\in\mathbb{N}_{>0}$ uniquely extends to a $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-linear continuous homomorphism

$$\nabla_q:\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}\to\widehat{\Omega}^{q+1}_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$$

such that we have $\nabla_q(\omega\otimes x)=\nabla_q(\omega)\otimes x+(-1)^q\omega\wedge\nabla(x)$ for $x\in\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $\omega\in\widehat{\Omega}^q_{\mathcal{K}}$. Obviously, the connection ∇ satisfies Griffith transversality

$$\nabla(\mathrm{Fil}^n\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}})\subset\widehat{\Omega}^1_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathrm{Fil}^{n-1}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$$

for $n\in\mathbb{Z}$. These connections are invariant under finite or unramified extensions by Lemma 1.10(iii) and Remark 3.6(ii).

Notation. We will use the following notation:

$$\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}:=(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}})^{\nabla=0},\ \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}:=(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}})^{\nabla=0}$$
$$\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}:=\mathrm{Im}(\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p}\xrightarrow{\text{can.}}\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}).$$

We endow the first two rings with induced filtrations and the last one with an induced graded structure. Note that these rings are invariant under finite or unramified extensions of $\mathcal{K}$ and that $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ (resp. $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$) have a canonical $(\mathcal{K}_{\mathrm{can}})^{\mathrm{alg}}$-algebra (resp. $\mathbb{C}_p$-algebra) structure. By the above description of the connection and the explicit descriptions of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$, we have:

Lemma 3.7. *The canonical maps*

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}},\quad \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}},\quad \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$$

are isomorphisms. These maps are compatible with filtrations and gradings.

Remark 3.8. Assume that $[k_{\mathcal{K}} : k_{\mathcal{K}}^p] < \infty$. Since $\widehat{\Omega}^1_{\mathcal{K}}$ is a finite-dimensional $\mathcal{K}$-vector space (Remark 1.11), the connection $\nabla : \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \to \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ induces a $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p}$-linear derivation

$$\nabla : \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}} \to \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}.$$

More precisely, if we denote by $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ the derivations on $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ defined as above, then, by the explicit description of $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$, $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ are commuting $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p}$-linear derivations and we have $\partial_j = t\partial/\partial v_j$. In particular, $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ coincides with $(\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}})^{\nabla=0}$. In the general case, we must handle complicated topologies to define such a connection. To avoid it, we define $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ in an ad-hoc way as above.

We also have an analogue of Poincaré lemma.

Lemma 3.9. *The complex*

$$0 \longrightarrow \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \xrightarrow{\text{inc.}} \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \xrightarrow{\nabla} \widehat{\Omega}^1_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \xrightarrow{\nabla_1} \widehat{\Omega}^2_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$$

is exact.

Proof. By the invariance of the above complex under a finite extension, we may assume $\mathcal{K} = \mathcal{K}_0$. Recall the explicit description of $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ in Section 3B. Since we have $v_p(\boldsymbol{n}!) \le |\boldsymbol{n}|$ for $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_{\mathcal{K}}}$, $x \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ is written uniquely in the form $x = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}} \boldsymbol{u}^{[\boldsymbol{n}]}$ with $a_{\boldsymbol{n}} \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ such that $\{v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{|\boldsymbol{n}|=n} \to \infty$ for all $r, n \in \mathbb{N}$. Moreover, we have the inequality

$$\inf_{|\boldsymbol{n}|<r} v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}) + r > \inf_{|\boldsymbol{n}|<r} v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(\boldsymbol{n}!\, a_{\boldsymbol{n}}) = v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(x) \tag{1}$$

by Remark 3.6(iii). We have only to prove that there exists $x \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ such that $\nabla(x) = \omega$ for $\omega \in \ker \nabla_1$. Write $\omega = \sum_{j\in J_{\mathcal{K}}} dt_j \otimes \lambda_j$ with $\lambda_j \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ such that $\{v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(\lambda_j)\}_{j\in J_{\mathcal{K}}} \to \infty$ for all $r \in \mathbb{N}$. The assumption $\omega \in \ker \nabla_1$ implies that we have $\partial_{j'}(\lambda_j) = \partial_j(\lambda_{j'})$ for $j, j' \in J_{\mathcal{K}}$. As above, we can write $\lambda_j = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} \lambda_{j,\boldsymbol{n}} \boldsymbol{u}^{[\boldsymbol{n}]}$, where $\lambda_{j,\boldsymbol{n}} \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ satisfies the convergence condition as above. We have the relation $\lambda_{j,\boldsymbol{n}+\mathbf{e}_{j'}} = \lambda_{j',\boldsymbol{n}+\mathbf{e}_j}$ for $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_{\mathcal{K}}}$ and $j, j' \in J_{\mathcal{K}}$. We will define a sequence $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}}$ in $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ as follows: Put $a_{\boldsymbol{0}}$ equal to 0. For $\boldsymbol{n} \neq \boldsymbol{0}$, choose any $j \in J_{\mathcal{K}}$ such that $n_j \neq 0$ and define $a_{\boldsymbol{n}} := \lambda_{j,\boldsymbol{n}-\mathbf{e}_j}$. By the above relation, this is independent of the choice of j. To prove the assertion, it suffices to prove that we have $\{v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{|\boldsymbol{n}|=n} \to \infty$ for all $r, n \in \mathbb{N}$. Indeed, if this is proved, we see that the element $x := \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}} \mathbf{u}^{[\boldsymbol{n}]}$ belongs to $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and we have $\nabla(x) = \omega$. We have only to prove that, for fixed $r, n, N \in \mathbb{N}$, we have $v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}) \ge N$ for all but finitely many $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_{\mathcal{K}}}$ such that $|\boldsymbol{n}| = n$. We may assume $r \ge n$. Choose a finite subset J of $J_{\mathcal{K}}$ such that $v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(\lambda_j) \ge r + N$

for $j \in J_{\mathcal{K}} \setminus J$. Let $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_K}$ such that $|\boldsymbol{n}| = n$. If there exists $j \in J_{\mathcal{K}} \setminus J$ such that $n_j \neq 0$, then we have

$$v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}) = v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}-\mathbf{e}_j}) > v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(\lambda_j) - r \geq r + N - r = N,$$

where the first inequality follows from inequality (1). This implies the assertion since our exceptional set $\{\boldsymbol{n} \in \mathbb{N}^J \mid |\boldsymbol{n}| = n\}$ is finite. □

3D. *Universal PD-thickenings.*

Defintion 3.10. A p-adically formal $\mathcal{O}_{\mathcal{K}}$-PD-thickening of $\mathcal{O}_{\mathbb{C}_p}$ is a triple

$$(D, \theta_D, \gamma_D),$$

where

- D is a p-adically Hausdorff complete $\mathcal{O}_{\mathcal{K}}$-algebra,
- $\theta_D : D \to \mathcal{O}_{\mathbb{C}_p}$ is a surjective $\mathcal{O}_{\mathcal{K}}$-algebra homomorphism,
- γ_D is a PD-structure on $\ker \theta_D$, compatible with the canonical PD-structure on the ideal (p).

Obviously, p-adically formal $\mathcal{O}_{\mathcal{K}}$-thickenings of $\mathcal{O}_{\mathbb{C}_p}$ form a category.

Theorem 3.11 [Fontaine 1994b, Théorème 2.2.1]. *The category of p-adically formal $\mathcal{O}_{\mathcal{K}}$-thickenings of $\mathcal{O}_{\mathbb{C}_p}$ admits a universal object, that is, an initial object.*

Such an object is unique up to a canonical isomorphism and we denote it by $(\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}, \theta_{\mathbb{C}_p/\mathcal{K}}, \gamma)$. Let's recall the construction. Let $(\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}))^{\mathrm{PD}}$ be the PD-envelope of $\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p})$ with respect to the ideal

$$\ker (\theta_{\mathbb{C}_p/\mathcal{K}} : \mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}),$$

compatible with the canonical PD-structure on the ideal (p). Then, $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is the p-adic Hausdorff completion of $(\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}))^{\mathrm{PD}}$.

Remark 3.12. (i) By [Fontaine 1994a, Remarques 2.2.3], if we have $\mathcal{K} = \mathcal{K}_0$ and $k_{\mathcal{K}}$ is perfect, then the canonical map $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is an isomorphism.

(ii) By a similar proof as Lemma 3.4(i), the canonical map

$$\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}^{\mathrm{ur}}}$$

is an isomorphism. In general, we have no invariance for $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ as in Remark 3.6(ii) even after inverting p.

If $\mathcal{K} = \mathcal{K}_0$ and $k_{\mathcal{K}}$ is perfect, then we have an explicit description of $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$:

$$\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} = \left\{ \sum_{n\in\mathbb{N}} a_n \frac{\omega^n}{n!} \;\middle|\; a_n \in \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}, \{v_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}(a_n)\}_{n\in\mathbb{N}} \to \infty \right\},$$

where ω denotes a generator of $\ker(\theta_{\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \to \mathbb{O}_{\mathbb{C}_p})$. Note that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is not uniquely determined. Moreover, we have $t \in \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ and $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is an integral domain of characteristic 0 whose PD-structure is given by $\gamma_n(x) = x^{[n]} = x^n/n!$ for $x \in \ker\theta_{\mathbb{C}_p/\mathcal{K}}$. In fact, the assertions follow from the case $\mathcal{K} = K_0^{\mathrm{pf}}$ by Remark 3.5 and Remark 3.12(i), and the assertion in this case follows from [Fontaine 1994a, 2.3.3].

We define $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} := \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}[p^{-1}]$ and $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} := \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}[t^{-1}]$. We also define $\mathbb{A}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} := \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}[\mathrm{x}]$, where x is a formal variable, and we set $\mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} := \mathbb{A}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}[p^{-1}]$ and $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} := \mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}[t^{-1}]$. We define a monodromy operator N on $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$ as the $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$-derivation $N := -d/d\mathrm{x}$. We denote by $v_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ the p-adic semivaluation on $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ (or $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$) defined by the lattice $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$.

In the following, we will give an explicit description of $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. Let

$$\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}\langle \mathbf{u}_{J_{\mathcal{K}}}\rangle$$

be the p-adic Hausdorff completion of the PD-polynomial $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-algebra on the indeterminates $\{\mathrm{u}_j\}_{j\in J_{\mathcal{K}}}$. Note that the PD-structure is given by $\gamma_n(\mathrm{u}_j) = \mathrm{u}_j^n/n! = \mathrm{u}_j^{[n]}$ for $n \in \mathbb{N}$ and $j \in J_{\mathcal{K}}$. We also have

$$\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}\langle \mathbf{u}_{J_{\mathcal{K}}}\rangle = \left\{ \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}}\mathbf{u}^{[\boldsymbol{n}]} \;\middle|\; a_{\boldsymbol{n}} \in \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}, \{v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} \to \infty \right\}.$$

We regard $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ as an $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-algebra by functoriality. Then, by the universal property of PD-polynomial algebras, we have the $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism

$$\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}\langle \mathbf{u}_{J_{\mathcal{K}}}\rangle \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}; \quad \mathbf{u}^{[\boldsymbol{n}]} \mapsto \boldsymbol{u}^{[\boldsymbol{n}]}.$$

Lemma 3.13. *If $\mathcal{K} = \mathcal{K}_0$, then $\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is an isomorphism. Moreover, we have*

$$v_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}(x) = \inf_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})$$

for $x = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}}\boldsymbol{u}^{[\boldsymbol{n}]} \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ with $a_{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$.

We use the following lemma in the proof:

Lemma 3.14. *We also assume that $\mathcal{K} = \mathcal{K}_0$ and we use the notation in Section 1A.*

(i) *If R is a p-adically Hausdorff complete $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$-algebra, then the canonical map*

$$\mathrm{Hom}_{\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}}(\mathbb{O}_{\mathcal{K}}, R) \to \mathrm{Hom}_{\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}, R/(p))$$

is bijective, where the $\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}$*-algebra structure on* $k_{\mathcal{K}}$ *(resp.* $R/(p)$*) is given by* $T_j \mapsto \bar{t}_j$ *(resp. is induced by* $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}} \to R$*). Moreover, the restriction map*

$$|_{k_{\mathcal{K}}^p} : \mathrm{Hom}_{\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}, R/(p)) \to \mathrm{Hom}_{\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}^p, R/(p))$$

is bijective, where the $\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}}$*-algebra structure on* $k_{\mathcal{K}}^p$ *(resp.* $R/(p)$*) is given by* $T_j^p \mapsto t_j^p$ *(resp. the composition of the inclusion* $\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}} \to \mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}$ *and the above structure map* $\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}} \to R/(p)$*).*

(ii) *Let* $\vartheta : S \to R$ *be a surjective homomorphism of p-adically Hausdorff complete* $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$*-algebras, whose kernel admits a PD-structure, compatible with the canonical PD-structure on the ideal* (p)*. Then, the canonical map*

$$\vartheta_* : \mathrm{Hom}_{\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}}(\mathbb{O}_{\mathcal{K}}, S) \to \mathrm{Hom}_{\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}}(\mathbb{O}_{\mathcal{K}}, R); \quad f \mapsto \vartheta \circ f$$

is bijective.

Proof. (i) The first claim follows from the p-adic formal étaleness of $\mathbb{O}_{\mathcal{K}}/\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$. The latter assertion follows by using the isomorphism of $k_{\mathcal{K}}^p$-algebras

$$k_{\mathcal{K}}^p[T_j]_{j\in J_{\mathcal{K}}}/(\{T_j^p - \bar{t}_j^p\}_{j\in J_{\mathcal{K}}}) \cong k_{\mathcal{K}}; \quad \overline{T}_j \mapsto \bar{t}_j.$$

(ii) We denote by $\vartheta_1 : S/(p) \to R/(p)$ the ring homomorphism induced by ϑ. By the first assertion of (i), we have only to prove that the canonical map

$$\mathrm{Hom}_{\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}, S/(p)) \to \mathrm{Hom}_{\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}, R/(p)); \quad f \mapsto \vartheta_1 \circ f,$$

which is denoted by ϑ_* again, is bijective.

We first note the following: We regard $R/(p)$ as a quotient of $S/(p)$ by ϑ_1. Let $x \in R/(p)$ and let $\hat{x}_1, \hat{x}_2 \in S/(p)$ be lifts of x. Then, we have $\hat{x}_1 - \hat{x}_2 \in \ker \vartheta_1$. Since $a^p = p!\gamma_p(a) \in pS$ for $a \in \ker \vartheta$, where γ denotes a PD-structure on $\ker \vartheta$, we have $\hat{x}_1^p = \hat{x}_2^p$. In particular, if we denote by $\hat{x} \in S/(p)$ a lift of $x \in R/(p)$, then $\hat{x}^p$ depends only on x.

We prove the injectivity. Let $\bar{f} : k_{\mathcal{K}} \to R/(p)$ be an $\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}$-algebra homomorphism and $f, f' : k_{\mathcal{K}} \to S/(p)$ lifts of $\bar{f}$, that is, $\vartheta_*(f) = \vartheta_*(f') = \bar{f}$. For $\bar{x} \in k_K$, $f(\bar{x})$ and $f'(\bar{x}) \in S/(p)$ are lifts of $\bar{f}(\bar{x}) \in R/(p)$, hence we have $f(\bar{x}^p) = f(\bar{x})^p = f'(\bar{x})^p = f'(\bar{x}^p)$ by the above remark. Hence, we have $f|_{k_{\mathcal{K}}^p} = f'|_{k_{\mathcal{K}}^p}$, that is, $f = f'$ by the latter assertion of (i).

We prove the surjectivity. Let $\bar{f} : k_{\mathcal{K}} \to R/(p)$ be an $\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}$-algebra homomorphism. We have only to construct an $\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}}$-algebra homomorphism $f : k_{\mathcal{K}}^p \to S/(p)$ such that $\vartheta_*(f)|_{k_{\mathcal{K}}^p}$ coincides with $\bar{f}|_{k_{\mathcal{K}}^p}$, where we endow $k_{\mathcal{K}}^p$ and $S/(p)$ with $\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}}$-algebra structures by a similar way as in the statement of (i). In fact, we can uniquely extend f to a $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$-algebra homomorphism $f : k_{\mathcal{K}} \to S/(p)$ by the latter assertion of (i). Moreover, $(\vartheta_*(f))|_{k_{\mathcal{K}}^p} = \vartheta_*(f|_{k_{\mathcal{K}}^p})$ coincides with $\bar{f}|_{k_{\mathcal{K}}^p}$, which implies $\vartheta_*(f) = \bar{f}$ by the latter assertion of (i) again. The set-theoretic map $f : k_{\mathcal{K}}^p \to S/(p)$ taking $\bar{y}$ to $\hat{x}^p$, where $\hat{x} \in S/(p)$ is any lift of $\bar{f}(\bar{y}^{p^{-1}}) \in R/(p)$, is well-defined by the above remark. Moreover, f is a $\mathbb{Z}[T_j]_{j \in J_{\mathcal{K}}}$-algebra homomorphism by a simple calculation and $\vartheta_*(f)|_{k_{\mathcal{K}}^p}$ coincides with $\bar{f}|_{k_{\mathcal{K}}^p}$ by construction, which implies the assertion. $\square$

Proof of Lemma 3.13. Obviously, we have only to prove the first assertion. Put $\mathcal{A} = \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}\langle \mathbf{u}_{J_{\mathcal{K}}} \rangle$. Extend $\theta_{\mathbb{C}_p/\mathbb{Q}_p} : \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{O}_{\mathbb{C}_p}$ to a surjective $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism $\vartheta : \mathcal{A} \to \mathbb{O}_{\mathbb{C}_p}$ by $\vartheta(\mathbf{u}^{[\boldsymbol{n}]}) = 0$. We first prove that $\mathcal{A}$ has an $\mathbb{O}_{\mathcal{K}}$-algebra structure such that ϑ is an $\mathbb{O}_{\mathcal{K}}$-algebra homomorphism.

Denote by ω a generator of the kernel of $\theta_{\mathbb{C}_p/\mathbb{Q}_p} : \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{O}_{\mathbb{C}_p}$. Then, the PD-structure on the ideal $\ker\theta_{\mathbb{C}_p/\mathbb{Q}_p}$ of $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ canonically extends to a PD-structure δ_1 on the ideal (ω) of $\mathcal{A}$, compatible with the canonical PD-structure on the ideal (p). By construction, the kernel of the map $\xi : \mathcal{A} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ taking $\mathbf{u}^{[\boldsymbol{n}]}$ to 0 is endowed with a PD-structure δ_2, compatible with the canonical PD-structure on the ideal (p). Since $\mathcal{A}$ is an integral domain of characteristic 0, δ_1 and δ_2 induce the same PD-structure on $(\omega) \cap \ker\xi$. Hence, by [Berthelot and Ogus 1978, Proposition 3.12], the ideal $\ker\vartheta = (\omega) + \ker\xi$ admits a PD-structure, compatible with the canonical PD-structure on the ideal (p). Then, the assertion follows by applying Lemma 3.14(ii) to ϑ:

$$\begin{tikzcd}
\mathbb{O}_{\mathcal{K}} \arrow[r, "\text{can.}"] \arrow[dr, dashed, "\exists!"] & \mathbb{O}_{\mathbb{C}_p} \\
\mathbb{Z}[T_j]_{j \in J_{\mathcal{K}}} \arrow[u, "\text{str.}"] \arrow[r, "\text{str.}"] & \mathcal{A}, \arrow[u, "\vartheta"']
\end{tikzcd}$$

where the horizontal structure map is given by $T_j \mapsto \mathrm{u}_j + [\tilde{t}_j] \in \mathcal{A}$.

By the above $\mathbb{O}_{\mathcal{K}}$-structure, we may regard $\mathcal{A}$ as a p-adically formal $\mathbb{O}_{\mathcal{K}}$-PD-thickening of $\mathbb{O}_{\mathbb{C}_p}$. By universality, we have only to prove that $\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is an $\mathbb{O}_{\mathcal{K}}$-algebra homomorphism. Let $\alpha : \mathbb{O}_{\mathcal{K}} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ be the composition of the structure map $\mathbb{O}_{\mathcal{K}} \to \mathcal{A}$ and $\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. Since $\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ commutes with the projections ϑ and $\theta_{\mathbb{C}_p/\mathcal{K}}$, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{O}_{\mathcal{K}} \arrow[r, "\text{can.}"] \arrow[dr, "\alpha"] & \mathbb{O}_{\mathbb{C}_p} \\
\mathbb{Z}[T_j]_{j \in J_{\mathcal{K}}} \arrow[u, "\text{str.}"] \arrow[r, "\text{str.}"] & \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}, \arrow[u, "\theta_{\mathbb{C}_p/\mathcal{K}}"']
\end{tikzcd}$$

where the horizontal structure map is given by $T_j \mapsto t_j$. By Lemma 3.14(ii), α coincides with the structure map $\mathbb{O}_{\mathcal{K}} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$, which implies the assertion. $\square$

Finally, we remark that if $\mathcal{K} = \mathcal{K}_0$, then $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ and $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$ are integral domains by the above explicit description of $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$.

3E. *Connections and Frobenius on $\mathbb{B}_{\mathbf{cris}}$ and $\mathbb{B}_{\mathbf{st}}$.* In this section, assume $\mathcal{K} = \mathcal{K}_0$. We endow $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ with the p-adic topology defined by the lattice $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. We regard $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ as the direct limit of $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ under the multiplication by t^{-1} and we set

$$\widehat{\Omega}^q_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} = \varinjlim \widehat{\Omega}^q_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}.$$

Then, the canonical derivation $d : \mathcal{K} \to \widehat{\Omega}^1_{\mathcal{K}}$ uniquely extends to a $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$-linear continuous derivation $\nabla : \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} \to \widehat{\Omega}^1_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ by the explicit description of $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. Note that $\nabla(x^{[n]}) = \nabla(x)\cdot x^{[n-1]}$ for $x \in \ker \theta_{\mathbb{C}_p/\mathcal{K}}$. As in Section 3C, if we denote by $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ the derivations on $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ given by $\nabla(x) = \sum_{j\in J_{\mathcal{K}}} dt_j \otimes \partial_j(x)$, then $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ are commuting continuous $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-derivations and we have $\partial_j = \partial/\partial u_j$. We also have a canonical extension ∇_q of exterior derivations d_q. Also, we can uniquely extend ∇_q to the map $\nabla_q : \widehat{\Omega}^q_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} \to \widehat{\Omega}^{q+1}_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$ by putting $\nabla(\mathrm{x}) = 0$, where we define $\widehat{\Omega}^q_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} := (\widehat{\Omega}^q_{\mathcal{K}} \widehat{\otimes}_{\mathcal{K}} \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}})[\mathrm{x}]$.

Let $\varphi : \mathbb{O}_{\mathcal{K}} \to \mathbb{O}_{\mathcal{K}}$ be a lift of the absolute Frobenius on $k_{\mathcal{K}}$. The ring homomorphism $\varphi \otimes \varphi : \mathbb{O}_{\mathcal{K}} \otimes W(R_{\mathbb{C}_p}) \to \mathbb{O}_{\mathcal{K}} \otimes W(R_{\mathbb{C}_p})$ induces a ring homomorphism on $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. Although the resulting map depends on the choice of a Frobenius lift of $\mathbb{O}_{\mathcal{K}}$ in general, we denote it by φ again. By defining $\varphi(\mathrm{x}) := p\mathrm{x}$, we also have a Frobenius on $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$. By construction, the connection and the Frobenius on $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ commute and we have the relation $N \circ \varphi = p\varphi \circ N$ by a simple calculation.

Notation. We define $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}} := (\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}})^{\nabla=0}$ for $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$.

By the commutativity of ∇ and φ, these rings are endowed with φ-actions. Obviously, $\mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$ is endowed with the monodromy operator N. By the explicit description of $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$, we have:

Lemma 3.15. *For $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$, the canonical map*

$$\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}}$$

is an isomorphism. Since this map is compatible with Frobenius, Frobenius on $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}}$ is independent of the choice of a Frobenius lift of $\mathbb{O}_{\mathcal{K}}$. In particular, the Frobenius on $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}}$ is injective.

3F. *Compatibility with limit.* When a p-basis of $k_{\mathcal{K}}$ is not finite, some technical difficulties occur. In this case, we will reduce to the finite p-basis case by using the results of Section 1G and the following inverse limits.

Let the notation be as in Section 1G. By functoriality, we have canonical maps

$$\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}_0} \to \varprojlim_{J\in\mathcal{P}(J_K)} \mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}_{J,0}}, \quad \mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathcal{K}} \to \varprojlim_{J\in\mathcal{P}(J_K)} \mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathcal{K}_J},$$

where $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$, $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$. Since these morphisms are compatible with the above explicit descriptions of these rings, it is easy to see that these maps are injective.

3G. ***Embeddings of $\mathbb{B}_{\mathrm{cris}}$ and $\mathbb{B}_{\mathrm{st}}$ into $\mathbb{B}_{\mathrm{dR}}$.*** Let

$$\mathbb{J}_{\mathbb{C}_p/\mathcal{K}} := \ker\left(\theta_{\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}] \to \mathbb{C}_p\right).$$

We endow the ideal $\mathbb{J}_{\mathbb{C}_p/\mathcal{K}}/\mathbb{J}^n_{\mathbb{C}_p/\mathcal{K}}$ of the $\mathbb{Q}$-algebra $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/\mathbb{J}^n_{\mathbb{C}_p/\mathcal{K}}$ with the unique PD-structure. This is compatible with the canonical PD-structure of $\mathcal{O}_{\mathcal{K}}$ on the ideal (p). Hence, the canonical map $\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}) \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/\mathbb{J}^n_{\mathbb{C}_p/\mathcal{K}}$ factors through $(\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}))^{\mathrm{PD}} \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/\mathbb{J}^n_{\mathbb{C}_p/\mathcal{K}}$. If we endow the LHS and the RHS with the p-adic topology and the p-adic Banach space topology respectively (see Section 3B), then the above morphism is continuous. In fact, the canonical map times $n!$ factors through the image of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. By passing to limit, the map extends to $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$. Thus, we have a canonical $\mathcal{K}$-algebra homomorphism $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$. Fixing $\tilde{p} \in R_{\mathbb{C}_p}$ such that $\tilde{p}^{(0)} = p$, we extend this map to $\mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ by sending x to $\log([\tilde{p}]/p) := \sum_{n\in\mathbb{N}_{>0}} (-1)^{n-1}([\tilde{p}]/p-1)^n/n$. Note that these morphisms are compatible with connections.

Proposition 3.16. *Assume that the algebraic closure of $\mathcal{K}$ in $\mathbb{C}_p$ is dense in $\mathbb{C}_p$. Then, the canonical maps*

$$\mathcal{K}_{\mathrm{can}} \otimes_{\mathcal{K}_{\mathrm{can},0}} \mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}_0} \to \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}, \quad \mathcal{K}_{\mathrm{can}} \otimes_{\mathcal{K}_{\mathrm{can},0}} \mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}_0} \to \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}},$$
$$\mathcal{K} \otimes_{\mathcal{K}_0} \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}_0} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}, \quad \mathcal{K} \otimes_{\mathcal{K}_0} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}_0} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$$

are injective.

Proof. By identifying $\mathbb{C}_p$ with the p-adic completion of $\mathcal{K}^{\mathrm{alg}}$, we may assume $\mathcal{K} = K$. Note that if k_K is perfect, then this is due to [Fontaine 1994a, 4.2.4]. We consider the general case. We first prove the first two cases. We have only to prove the semistable case. The canonical map $K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} K_0^{\mathrm{pf}} \to K^{\mathrm{pf}}$ is injective since $K_{\mathrm{can}}/K_{\mathrm{can},0}$ is totally ramified and K_0^{pf} is absolutely unramified. Hence, we have the commutative diagram

$$\begin{tikzcd}
K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/K_0} \arrow[rr, "\mathrm{can.}"] \arrow[d, "\cong"] & & \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K} \arrow[d, "\cong"] \\
K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/K_0^{\mathrm{pf}}} \arrow[r, hook, "\mathrm{can.}"] & K^{\mathrm{pf}} \otimes_{K_0^{\mathrm{pf}}} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/K_0^{\mathrm{pf}}} \arrow[r, hook, "\mathrm{can.}"] & \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}},
\end{tikzcd}$$

where the vertical arrows are induced by base changes and the injectivity of the bottom second arrow follows from the perfect residue field case. Then, the assertion follows from the above diagram. We consider the latter two cases. By passing to limit (Section 3F), we may assume $[k_K : k_K^p] < \infty$. Then, the crystalline case follows from [Brinon 2006, Proposition 2.47], where $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0}$ is denoted by $\mathrm{B}_{\mathrm{cris}}$. We will prove the semistable case. By regarding $K \otimes_{K_0} \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0}$ as a subring of $\mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})$, the assertion is equivalent to saying that x is transcendental over $\mathrm{Frac}(\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})$. Suppose that it is not the case. To deduce a contradiction, we have only to construct a nonzero polynomial in $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{pf}}}[X]$ which has x as a zero. By assumption, we have a nonzero polynomial $f(X) = \sum_i a_i X^i \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0}[X]$ such that $f(\mathrm{x}) = 0$. For $\boldsymbol{m} \in \mathbb{N}^{\bigoplus J_K}$, we denote by $\partial^{\boldsymbol{m}}$ the product $\prod_{j\in J_K} \partial_j^{m_j}$, where $\{\partial_j\}_{j\in J_K}$ are the derivations defined in Section 3C. Denote by $\tilde{f}^{(\boldsymbol{m})}(X) \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{pf}}}[X]$ the image of the polynomial $f^{(\boldsymbol{m})}(X) := \sum_i \partial^{\boldsymbol{m}}(a_i) X^i$ under the canonical homomorphism $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$. Then, $\tilde{f}^{(\boldsymbol{m})}(X)$ has x as a zero since we have $\mathrm{x} \in \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$. Write $a_i = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}} a_{i,\boldsymbol{n}} \boldsymbol{u}^{[\boldsymbol{n}]}$ with $a_{i,\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ by using the explicit description of $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0}$ given in Section 3D. We have $\partial^{\boldsymbol{m}}(a_i) = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus \mathcal{K}}} a_{i,\boldsymbol{n}+\boldsymbol{m}} \boldsymbol{u}^{[\boldsymbol{n}]}$ and $\tilde{f}^{(\boldsymbol{m})}(X) = \sum_i a_{i,\boldsymbol{m}} X^i$. Hence, we obtain the desired polynomial $\tilde{f}^{(\boldsymbol{m})}(X)$ by choosing $\boldsymbol{m} \in \mathbb{N}^{\bigoplus J_K}$ such that we have $a_{i,\boldsymbol{m}} \neq 0$ for some i. □

4. Basic properties of rings of p-adic periods

We will apply the preceding construction to the cases $\mathcal{K} = \mathbb{Q}_p, K, K^{\mathrm{pf}}$, among others. The resulting rings of p-adic periods will have an appropriate Galois action by the functoriality of the construction: For example, G_K acts on $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ and $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$, $G_{K^{\mathrm{pf}}}$ acts on $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$. In this section, we will review Galois theoretic properties of these rings. The proofs of the properties are somewhat technical and the reader may skip this section by admitting the results including the G_K-regularities just below. We keep the notation of the previous section.

4A. ***Calculations of H^0 and verification of G_K-regularity.*** In this subsection, we will prove the G_K-regularity of the $(\mathbb{Q}_p, G_K)$-rings

$$\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0},\quad \mathbb{B}_{\mathrm{st},\mathbb{C}_p/K_0},\quad \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K},\quad \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K},$$
$$\mathbb{B}^\nabla_{\mathrm{cris},\mathbb{C}_p/K_0},\quad \mathbb{B}^\nabla_{\mathrm{st},\mathbb{C}_p/K_0},\quad \mathbb{B}^\nabla_{\mathrm{dR},\mathbb{C}_p/K},\quad \mathbb{B}^\nabla_{\mathrm{HT},\mathbb{C}_p/K},$$

which are used later in the paper, and calculate their H^0. Note that these rings are integral domains by their explicit description.

Lemma 4.1. *Let* $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$.

(i) $H^0\big(G_K, \mathrm{Frac}(\mathbb{B}_{\heartsuit,\mathbb{C}_p/K})\big) = K$.

(ii) *The* $(\mathbb{Q}_p, G_K)$*-ring* $\mathbb{B}_{\heartsuit,\mathbb{C}_p/K}$ *satisfies condition* $(G\text{-}R_3)$ *of Section 1H.*

(iii) *The* $(\mathbb{Q}_p, G_K)$*-ring* $\mathbb{B}_{\heartsuit,\mathbb{C}_p/K}$ *is* G_K*-regular.*

Proof. Assertion (iii) follows from (i), (ii) and Lemma 1.21. We will prove (i) and (ii) separately in the Hodge–Tate case and the de Rham case.

(a) The Hodge–Tate case: We first verify (i). By Theorem 1.15, we have only to prove that if we have nonzero $x, y \in \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$ such that $g(x)y = xg(y)$ for all $g \in G_K$, then we have $x/y \in \mathbb{C}_p$. We first consider the case $|J_K| < \infty$. Note that $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K} \cong \mathbb{C}_p[t, t^{-1}, \{v_j\}_{j\in J_K}]$ is a uniquely factorization domain. Hence we may assume that x and y are relatively prime by dividing x and y by their greatest common divisor. Then we have $g(x) = c_g x$ and $g(y) = c_g y$ for $c_g \in (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^\times \cong \bigcup_{n\in\mathbb{Z}} \mathbb{C}_p^\times t^n$ by assumption. By the explicit description of $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$, we can choose $\boldsymbol{n} \in \mathbb{N}^{J_K}$ such that

$$\partial^{\boldsymbol{n}}(x) \in \mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K} \setminus \{0\} \cong \mathbb{C}_p[t, t^{-1}] \setminus \{0\},$$

where $\partial_j = t\partial/\partial v_j$ and $\partial^{\boldsymbol{n}} := \prod_j \partial_j^{n_j}$ (Remark 3.8). Write $\partial^{\boldsymbol{n}}(x) = \sum_{n\in\mathbb{Z}} a_n t^n$ with $a_n \in \mathbb{C}_p$. Then, we have $g(\partial^{\boldsymbol{n}}(x)) = c_g \partial^{\boldsymbol{n}}(x)$ by the commutativity of ∂_j and the G_K-action. Since c_g is homogeneous with respect to t, we have $c_g \in \mathbb{C}_p$ by comparing degrees. By comparing the leading terms, we have $c_g = g(a_n)/a_n \chi^n(g)$ for all $g \in G_K$, where n is the degree of $\partial^{\boldsymbol{n}}(x)$ with respect to t. Hence, we have $x/a_n t^n \in (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K}$. Note that we have $(\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K} = K$. This follows from the facts that we have $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K} = \bigcup_{r\in\mathbb{N}} t^{-r}\mathbb{C}_p[t, \{tv_j\}_{j\in J_K}]$ and

$$H^0(G_K, t^{-r}\mathbb{C}_p[t, \{tv_j\}_{j\in J_K}]) = K$$

by [Brinon 2006, Lemme 2.15], where $\mathbb{C}_p[t, \{tv_j\}_{j\in J_K}]$ is written $\bigoplus_{r\in\mathbb{N}} \mathrm{gr}^r(\mathbb{B}^+_{\mathrm{dR}})$ in the reference. Thus, we have $x \in \mathbb{C}_p^\times t^n$. By the same argument, we have $y \in \mathbb{C}_p^\times t^m$ for some $m \in \mathbb{Z}$. Write $x = at^n$, $y = bt^m$ with $a, b \in \mathbb{C}_p^\times$. Then, we have

$$g(a/b) = \chi^{m-n}(g)(a/b)$$

for $g \in G_K$. Since $H^0(G_K, \mathbb{C}_p(n-m))$ is nonzero if and only if $n = m$ by Theorem 1.15, we must have $n = m$. In particular, we have $x/y = a/b \in \mathbb{C}_p$.

We consider the general case. Recall the notation in Section 1G. Let $J \in \mathcal{P}(J_K)$ and denote by x_J, y_J the image of x, y in $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K_J}$. By applying the above result to $J_K = J$, if x_J and y_J are nonzero, then there exists $\lambda_J \in \mathbb{C}_p^\times$ such that $x_J = \lambda_J y_J$. Since this λ_J is uniquely determined, $\lambda = \lambda_J$ is independent of the choice of J. Since $S_{x,y} := \{J \in \mathcal{P}(J_K) \mid x_J \neq 0 \text{ and } y_J \neq 0\}$ is a cofinal subset of $\mathcal{P}(J_K)$ by the explicit description of $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$, we have $x = \lambda y$ by the injection in Section 1G.

We will verify (ii). Let $x \in \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$ be a generator of a G_K-stable $\mathbb{Q}_p$-line in $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$. Write $g(x) = c_g x$ with $c_g \in \mathbb{Q}_p^\times$. We use the same notation as above. By a similar argument as above, if $x_J \neq 0$, then we have $x_J = a_J t^{n_J}$ for $a_J \in \mathbb{C}_p^\times$ and $n_J \in \mathbb{N}$. Moreover, a_J and n_J are unique. In particular, $\{a_J\}$ and $\{n_J\}$ are constant on the cofinal subset $S_{x,x}$ of $\mathcal{P}(J_K)$ and we have $x \in \mathbb{C}_p^\times t^n \subset (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^\times$ by the injection in Section 1G.

(b) The de Rham case: To prove assertion (i), we have only to prove that if we have nonzero $x, y \in \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$ such that $g(x)y = xg(y)$ for all $g \in G_K$, then we have $x/y \in K$. Let $J \in \mathcal{P}(J_K)$ and denote by $x_J, y_J \in \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K_J}$ the image of x, y. If $x_J \neq 0$ and $y_J \neq 0$, then we have $x_J/y_J \in H^0(G_{K_J}, \mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K_J})) = K_J$ by [Brinon 2006, Proposition 2.18], where $\mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K_J})$ is denoted by C_{dR}. Since the set $\{J \in \mathcal{P}(J_K) \mid x_J \neq 0 \text{ and } y_J \neq 0\}$ is a cofinal subset of $\mathcal{P}(J_K)$ by the explicit description of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$, we have $x/y \in \bigcap_{J \in \mathcal{P}(J_K)} K_J = K$ by the injection in Section 1G. We will verify (ii). By Remark 3.5(i), we may assume $K = K^{\mathrm{ur}}$. Let V be a G_K-stable $\mathbb{Q}_p$-line in $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$ generated by x. By Lemma 4.2 below and Theorem 2.1, there exist $n \in \mathbb{Z}$ and a finite extension L/K such that $Vt^n \subset (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^{G_L} = (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/L})^{G_L} = L$; in particular, we have $x \in (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^\times$. □

Lemma 4.2. *Let V be a G_K-stable $\mathbb{Q}_p$-line in $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$. Then, up to a Tate twist, V is $\mathbb{C}_p$-admissible as a p-adic representation.*

Proof. We assume $K = K^{\mathrm{ur}}$ by Hilbert 90 and Remark 3.6(ii). Let $x \in \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$ be a generator of V. By multiplying by a power of t, we may assume $x \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$. Let $\rho : G_K \to \mathbb{Q}_p^\times$ be the character defined by $\rho(g) = g(x)/x$. By the explicit description of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$ (Section 3B), we have

$$x = \sum_{\boldsymbol{n} \in \mathbb{N}^{\oplus J_K}} a_{\boldsymbol{n}} \boldsymbol{u}^{\boldsymbol{n}}$$

with $a_{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$. Choose $\boldsymbol{n} \in \mathbb{N}^{\oplus J_K}$ such that $a_{\boldsymbol{n}} \neq 0$ and write $a_{\boldsymbol{n}} = t^n \lambda$ with $n \in \mathbb{N}$ and $\lambda \in (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^\times$. Since we have $g(a_{\boldsymbol{n}}) = \rho(g) a_{\boldsymbol{n}}$ for $g \in G_{K^{\mathrm{pf}}}$, we have $(\rho\chi^{-n})(g) = g(\lambda)/\lambda$ for $g \in G_{K^{\mathrm{pf}}}$. By taking the $\mathbb{Q}_p$-linear map $\theta_{\mathbb{C}_p/\mathbb{Q}_p}$, we have $(\rho\chi^{-n})(g) = g(\theta_{\mathbb{C}_p/\mathbb{Q}_p}(\lambda))/\theta_{\mathbb{C}_p/\mathbb{Q}_p}(\lambda)$ for $g \in G_{K^{\mathrm{pf}}}$, that is, $\rho\chi^{-n}|_{K^{\mathrm{pf}}}$ is $\mathbb{C}_p$-admissible. Hence, $\rho\chi^{-n}$ is $\mathbb{C}_p$-admissible by Theorem 2.1. □

Corollary 4.3. *We have*

$$\begin{aligned}
(\mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_K} &= (\mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/K_0})^{G_K} = K_{\mathrm{can},0},\\
(\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_K} &= (\mathbb{B}_{\mathrm{st},\mathbb{C}_p/K_0})^{G_K} = K_0,\\
(\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})^{G_K} &= (\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K})^{G_K} = K_{\mathrm{can}},\\
(\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K})^{G_K} &= (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^{G_K} = K,\\
(\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K} &= (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K} = K.
\end{aligned}$$

Proof. Since we have trivial inclusions (such as $K_0 \subset (\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_K}$), we have only to show the converse inclusions. By passing to limit (Section 1G and 3F), we may assume $[k_K : k_K^p] < \infty$. We prove the Hodge–Tate case first. Since we have $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K} \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)$ (Section 3B), the assertion for $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K}$ follows from Theorem 1.15. The assertion for $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$ follows from [Brinon 2006, Lemme 2.15].

We will prove the rest of the assertion. Since we have $K_{\mathrm{can},0} = (K_0)_{\mathrm{can}}$ by comparing the residue fields, the assertions in the horizontal case follow from those in the ∇-less case by taking horizontal sections. The de Rham case follows from Lemma 4.1(i) and the crystalline and semistable cases follow from de Rham case and Proposition 3.16. $\square$

Lemma 4.4. *The $(\mathbb{Q}_p, G_K)$-ring $\mathbb{B}_{\diamond,\mathbb{C}_p/K_0}$ satisfies $(G\cdot R_3)$ for $\diamond \in \{\mathrm{cris}, \mathrm{st}\}$. In particular, $\mathbb{B}_{\diamond,\mathbb{C}_p/K_0}$ is G_K-regular.*

Proof. Note that the last assertion is obtained by applying Lemma 1.20, whose assumptions are satisfied by Proposition 3.16, Lemma 4.1(iii) and Corollary 4.3. By Remark 3.12(ii), we may assume $K = K^{\mathrm{ur}}$. Let V be a G_K-stable $\mathbb{Q}_p$-line in $\mathbb{B}_{\diamond,\mathbb{C}_p/K_0}$ with generator x. By Lemma 4.2, there exists $n \in \mathbb{Z}$ such that Vt^n is $\mathbb{C}_p$-admissible as a p-adic representation of G_K. By Theorem 2.1, the image of the map $\rho: G_K \to \mathbb{Q}_p^\times$ that takes g to $g(xt^n)/(xt^n)$ is included in $(\mathbb{Q}_p^\times)_{\mathrm{tors}}$, which is killed by $2(p-1)$. Therefore, we have $(xt^n)^{2(p-1)} \in (\mathbb{B}_{\diamond,\mathbb{C}_p/K_0})^{G_K} = K_0$, which implies $x \in \mathbb{B}^{\times}_{\diamond,\mathbb{C}_p/K_0}$. $\square$

Lemma 4.5. *The $(\mathbb{Q}_p, G_K)$-rings*

$$\mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0},\quad \mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/K_0},\quad \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K},\quad \mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K}$$

are G_K-regular.

Proof. The G_K-regularity of the field $\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}$ follows from Example 1.18. Since we have a $G_{K^{\mathrm{pf}}}$-equivariant canonical isomorphism $\mathbb{B}^{\nabla}_{\diamond,\mathbb{C}_p/K_0} \cong \mathbb{B}_{\diamond,\mathbb{C}_p/K_0^{\mathrm{pf}}}$ for $\diamond \in \{\mathrm{cris}, \mathrm{st}\}$, the verification of $(G\cdot R_3)$ for $\mathbb{B}^{\nabla}_{\diamond,\mathbb{C}_p/K_0}$ is reduced to that for $\mathbb{B}_{\diamond,\mathbb{C}_p/K_0^{\mathrm{pf}}}$, which follows from [Fontaine 1994b, Proposition 5.1.2(ii)]. By a similar reason, $(G\cdot R_3)$ for $\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}$ is reduced to [Fontaine 1994b, Proposition 3.6]. The $(\mathbb{Q}_p, G_K)$-ring $\mathbb{C}_p((t))$ is a field containing the fractional field of $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K} \cong \mathbb{C}_p[t, t^{-1}]$. By Theorem 1.15 and dévissage, we have $\mathbb{C}_p((t))^{G_K} = K = (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K}$, where the last equality follows from Corollary 4.3. By applying Lemma 1.21, $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K}$ is G_K-regular. By Corollary 4.3, the G_K-regularity for $\mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0}$ and $\mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/K_0}$ follows from Lemma 1.20 and Proposition 3.16. $\square$

Remark 4.6. For $\bullet \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}, \mathrm{HT}\}$, the $(\mathbb{Q}_p, G_K)$-rings $\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p}$ and $\mathbb{B}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p}$ are G_K-regular We also have

$$(\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p})^{G_K} = (\mathbb{B}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p})^{G_K} \cong (\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K_0})^{G_K}.$$

In fact, the assertion follows from canonical isomorphisms $\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p} = \mathbb{B}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K_0}$ as $(\mathbb{Q}_p, G_K)$-rings.

Notation. (i) We define the category of crystalline (resp. horizontal crystalline) representations of G_K as $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0}} G_K$ (resp. $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0}} G_K$), and we denote it by $\mathrm{Rep}_{\mathrm{cris}} G_K$ (resp. $\mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K$). The corresponding functor $\mathbb{D}_B$ is denoted by $\mathbb{D}_{\mathrm{cris}}$ (resp. $\mathbb{D}^{\nabla}_{\mathrm{cris}}$) and the comparison map α_B by $\alpha_{\mathrm{cris},\mathbb{C}_p/K_0}$ (resp. $\alpha^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0}$). We define the category of semistable representations similarly, with "cris" in place of "st".

(ii) We define the category of de Rham (resp. horizontal de Rham) representations of G_K as $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}} G_K$ (resp. $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}} G_K$), and we denote it by $\mathrm{Rep}_{\mathrm{dR}} G_K$ (resp. $\mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$). The corresponding functor $\mathbb{D}_B$ is denoted by $\mathbb{D}_{\mathrm{dR}}$ (resp. $\mathbb{D}^{\nabla}_{\mathrm{dR}}$) and the comparison map α_B (loc. cit.) by $\alpha_{\mathrm{dR},\mathbb{C}_p/K}$ (resp. $\alpha^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}$). We define the category of Hodge–Tate representations similarly, with "dR" in place of "HT".

(iii) We define rings with G_K-actions and automorphisms φ by

$$\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} := \bigcap_{n\in\mathbb{N}} \varphi^n(\mathbb{B}^{\nabla+}_{\mathrm{cris},\mathbb{C}_p/K_0}), \quad \tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0} := \bigcap_{n\in\mathbb{N}} \varphi^n(\mathbb{B}^{\nabla+}_{\mathrm{st},\mathbb{C}_p/K_0}).$$

Note that we have $\tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0} \cong \tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0^{\mathrm{pf}}}$ for $\diamondsuit \in \{\mathrm{rig}, \log\}$.

(iv) In the rest of the paper, when k_K is perfect, we omit hyperscripts ∇ to be consistent with the usual notation; e.g., we write $\tilde{\mathbb{B}}^{+}_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$ instead of $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$.

Remark 4.7. As is explained in Section 1A, there is no canonical choice of a Cohen ring of k_K nor a Frobenius lift when k_K is not perfect. Since some definitions, such as the definition of crystalline representations, involve these choices, we make some remarks on the independence of definitions.

(i) Since we have a canonical isomorphism $\mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathbb{Q}_p} \cong \mathbb{B}^{\nabla}_{\heartsuit,\mathbb{C}_p/K}$ for $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$ (Lemma 3.7), $\mathbb{B}^{\nabla}_{\heartsuit,\mathbb{C}_p/K}$ depend only on $\mathbb{C}_p$ as an abstract ring.

(ii) Since we have a canonical isomorphism $\mathbb{B}^{+}_{\diamondsuit,\mathbb{C}_p/\mathbb{Q}_p} \cong \mathbb{B}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0}$ for $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$ (Lemma 3.15), the category $\mathrm{Rep}^{\nabla}_{\diamondsuit} G_K$ depends only on $\mathbb{C}_p$ but not on the choice of K_0. It also follows that $\tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0}$ for $\diamondsuit \in \{\mathrm{rig}, \log\}$ is independent of the choices of K_0 and φ as a $\mathbb{Q}_p$-algebra with φ-action. Moreover, for a finite extension L/K, $\tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0}$ coincides with $\tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/L_0}$ in $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/L}$.

(iii) By definition, the category $\mathrm{Rep}_{\diamondsuit} G_K$ for $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$ may depend on the choice of K_0. In the case $[k_K : k_K^p] < \infty$ with $\diamondsuit = \mathrm{cris}$, the independence is proved by Brinon [2006, Proposition 3.42]: He proves the assertion by introducing a ring $A_{\max,K}$, which is independent of the choice of K_0 and is slightly bigger than $\mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/K_0}$. Although a similar idea seems to work in the general case, we do not treat this problem in this paper. Instead, we will state a precise version of the Main Theorem later (see Section 6).

Remark 4.8 (Hilbert 90). Let $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$. Then, V is crystalline or semistable if and only if so is $V|_{K^{\mathrm{ur}}}$. In fact, we have $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0} \cong \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{ur}}}$ by Remark 3.12(i), whose $G_{K^{\mathrm{ur}}}$-invariant is K_0^{ur} by Corollary 4.3. Hence, the assertion in the crystalline case follows from Hilbert 90 and the same proof works also in the semistable case. We can also prove that V is de Rham or Hodge–Tate if and only if so is $V|_L$ for a finite extension L of the completion of an unramified extension of K. This follows from the cases when L/K is finite or unramified and in these cases the claim follows from Remark 3.6(ii) and Hilbert 90.

Algebraic structures of rings of p-adic period, which are compatible with the action of G_K, induce additional structures on the corresponding $\mathbb{D}$. We do not review these structures here since we do not need all of them to prove the Main Theorem. For the reader interested in these structures, see [Brinon 2006, 3.5] for example. We need only the connection on $\mathbb{D}_{\mathrm{dR}}$ for the proof of the Main Theorem: For $V \in \mathrm{Rep}_{\mathrm{dR}} G_K$, the finite-dimensional K-vector space $\mathbb{D}_{\mathrm{dR}}(V)$ has a connection $\nabla : \mathbb{D}_{\mathrm{dR}}(V) \to \widehat{\Omega}^1_K \otimes_K \mathbb{D}_{\mathrm{dR}}(V)$, which is compatible with the canonical derivation on K.

4B. ***Restriction to perfection.*** If we have $V \in \mathrm{Rep}_\bullet G_K$ with $\bullet \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}, \mathrm{HT}\}$, then we have $V|_{K^{\mathrm{pf}}} \in \mathrm{Rep}_\bullet G_{K^{\mathrm{pf}}}$. Moreover, we have canonical isomorphisms

$$K_0^{\mathrm{pf}} \otimes_{K_0} \mathbb{D}_\diamondsuit(V) \to \mathbb{D}_\diamondsuit(V|_{K_0^{\mathrm{pf}}}), \quad K^{\mathrm{pf}} \otimes_K \mathbb{D}_\heartsuit(V) \to \mathbb{D}_\heartsuit(V|_{K^{\mathrm{pf}}}),$$

induced by the canonical map $\mathbb{B}_{\diamondsuit,\mathbb{C}_p/K_0} \to \mathbb{B}_{\diamondsuit,\mathbb{C}_p/K_0^{\mathrm{pf}}}$ and $\mathbb{B}_{\heartsuit,\mathbb{C}_p/K} \to \mathbb{B}_{\heartsuit,\mathbb{C}_p/K^{\mathrm{pf}}}$ for $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$ and $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$. We first prove the de Rham case. By applying $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}} \otimes_{\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}}$ to the comparison isomorphism $\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V)$, we have a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism

$$\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V.$$

By taking $G_{K^{\mathrm{pf}}}$-invariant, we have an isomorphism $K^{\mathrm{pf}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{D}_{\mathrm{dR}}(V|_{K^{\mathrm{pf}}})$. The other cases follow similarly.

5. Construction of $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$

In this section, we construct a (φ, G_K)-module $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ for a de Rham representation V of G_K, possibly after a Tate twist. Our $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}$ coincides with Colmez's $\tilde{\mathbb{N}}^{+}_{\mathrm{rig}}$ when the residue field k_K is perfect.

We first recall Colmez's Dieudonné–Manin theorem, which is a key ingredient of the construction. Let M be a finite free $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-module of rank $r > 0$. We call N a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-lattice of M if N is a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-submodule of finite type of M such that $N[t^{-1}] = M[t^{-1}]$. Note that a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-lattice of M is finite free of rank r over $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$ since $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$ is a discrete valuation ring.

For $n \in \mathbb{Z}$, denote the composition

$$\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \overset{\varphi^n}{\hookrightarrow} \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \overset{\mathrm{inc.}}{\hookrightarrow} \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$$

by φ^n again. By the commutative diagram

$$\begin{tikzcd}
\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \arrow[r, hook, "\varphi^n"] \arrow[d, "\mathrm{can.}"', "\cong"] & \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \arrow[d, "\mathrm{can.}"', "\cong"] \\
\tilde{\mathbb{B}}^{+}_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}} \arrow[r, hook, "\varphi^n"] & \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}},
\end{tikzcd}$$

the proof of the following theorem is reduced to the perfect residue field case [Colmez 2008, Proposition 0.3] (see also the remark below).

Theorem 5.1 (Colmez's Dieudonné–Manin classification theorem). *Let $r \in \mathbb{N}_{>0}$ and M be a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-lattice of $(\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})^r$. Let*

$$M_{\mathrm{rig}} := \{x \in (\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})^r \mid \varphi^n(x) \in M \text{ for all } n \in \mathbb{Z}\}.$$

Then, M_{rig} is a finite free $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$-module of rank r with semilinear φ-action and there exists a basis $e_1, \dots, e_r$ of M_{rig} over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ such that:

(i) *There exist $h \in \mathbb{N}_{>0}$ and $a_1 \le \cdots \le a_r \in \mathbb{N}$ such that $\varphi^h(e_i) = p^{a_i} e_i$ for $1 \le i \le r$;*

(ii) *$e_1, \dots, e_r$ is a basis of M over $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$.*

Remark 5.2. Though our condition (ii) is weaker than that in [Colmez 2008], the conclusions of the theorem are the same for the following reason: By definition, φ acts on M_{rig}. Since φ^h is an automorphism on M_{rig} by (i), φ is also an automorphism on M_{rig}. Hence, (ii) implies that $\varphi^n(e_1), \dots, \varphi^n(e_r)$ is a $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$-basis of M_{rig} for all $n \in \mathbb{Z}$. In particular, $\varphi^n(e_1), \dots, \varphi^n(e_r)$ is a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-basis of M.

In the rest of this section, let V be a de Rham representation of G_K of dimension r such that $\mathbb{D}_{\mathrm{dR}}(V) = (\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V)^{G_K}$. Note that the last assumption is satisfied for any de Rham representation after some Tate twist. Let

$$\mathbb{N}^{+}_{\mathrm{dR}}(V) := \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K \mathbb{D}_{\mathrm{dR}}(V).$$

It is a finite free $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K}$-module of rank r with G_K-action and ∇-action which are commuting. By the comparison isomorphism $\alpha_{\mathrm{dR},\mathbb{C}_p/K}$, we have a canonical isomorphism $\mathbb{N}^{+}_{\mathrm{dR}}(V)[t^{-1}] \cong \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V$, in particular, we have

$$t^n \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V \subset \mathbb{N}^{+}_{\mathrm{dR}}(V) \subset \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V$$

for sufficiently large $n \in \mathbb{N}$. Taking horizontal sections, we see that $\mathbb{N}^{\nabla+}_{\mathrm{dR}}(V) := \mathbb{N}^{+}_{\mathrm{dR}}(V)^{\nabla=0}$ is a G_K-stable $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-lattice of $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V$. By applying Theorem 5.1 to $M = \mathbb{N}^{\nabla+}_{\mathrm{dR}}(V)$, we have the following proposition: (In the following, a (φ, G_K)-module over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ (of rank r) means a finite free module (of rank r)

over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ with a semilinear φ-action and a semilinear G_K-action, which are commuting.)

Proposition 5.3. *The $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$-module*

$$\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V) := \{x \in \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \otimes_{\mathbb{Q}_p} V \mid \varphi^n \otimes \mathrm{id}(x) \in \mathbb{N}^{\nabla+}_{\mathrm{dR}}(V) \text{ for all } n \in \mathbb{Z}\}$$

is a (φ, G_K)-module over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ of rank r. Moreover, we have a basis $e_1, \dots, e_r$ of $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ such that:

(i) *There exist $h \in \mathbb{N}_{>0}$ and $a_1 \leq \cdots \leq a_r \in \mathbb{N}$ such that $\varphi^h(e_i) = p^{a_i} e_i$ for $1 \leq i \leq r$;*

(ii) *$e_1, \dots, e_r$ is a basis of $\mathbb{N}^{\nabla+}_{\mathrm{dR}}(V)$ over $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$.*

Note that $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ is independent of the choice of K_0 by Remark 4.7(ii). We will use the following property of $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ in the proof of the Main Theorem.

Proposition 5.4. *The canonical map*

$$\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}} \mathbb{N}^{\nabla+}_{\mathrm{dR}}(V) \to \mathbb{N}^{+}_{\mathrm{dR}}(V)$$

is a G_K-equivariant isomorphism. In particular, $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}} \widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ is isomorphic to $(\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K})^r$ as a $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-module by Proposition 5.3(ii).

Proof. Since $V|_{K^{\mathrm{pf}}}$ is de Rham and we have the canonical isomorphism $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$, we have the comparison isomorphism

$$\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}} (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_{K^{\mathrm{pf}}}} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V.$$

By taking the base change of this isomorphism by $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$, we obtain a canonical isomorphism of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}[G_{K^{\mathrm{pf}}}]$-modules

$$\alpha : \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}} (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_{K^{\mathrm{pf}}}} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V. \quad (2)$$

We also have the comparison isomorphism

$$\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V) : \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K \mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V.$$

Note that we have $(\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}} = (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}$ since we have

$$(t^{-n}\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}/t^{-n+1}\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}} = (\mathbb{C}_p(-n))^{G_{K^{\mathrm{pf}}}} = 0$$

for $n \in \mathbb{N}_{>0}$. We have only to prove that there exists an isomorphism of $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K}$-modules

$$(\mathbb{N}^{+}_{\mathrm{dR}}(V) =)\ \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K \mathbb{D}_{\mathrm{dR}}(V) \cong \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{(\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}} (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_{K^{\mathrm{pf}}}}$$

which is compatible with the injections $\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V)$ and α. Indeed, by taking the horizontal sections of both sides, we have

$$\mathbb{N}_{\mathrm{dR}}^{\nabla+}(V)=\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{\nabla+}\otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}^{+})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}},$$

which implies the assertion.

We have

$$\mathbb{D}_{\mathrm{dR}}(V)\hookrightarrow(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}}=\\(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}\otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}},$$

where the equality follows by taking $G_{K^{\mathrm{pf}}}$-invariant of (2). Note that we have

$$(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{+})^{G_{K^{\mathrm{pf}}}}=(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}.$$

Indeed, if we write $x\in\mathrm{LHS}$ as $x=t^{-n}\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}}a_{\boldsymbol{n}}\boldsymbol{u}^{\boldsymbol{n}}$ with $a_{\boldsymbol{n}}\in\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}^{+}$, since $\{u_j\}_{j\in J_K}$ are invariant by the action of $G_{K^{\mathrm{pf}}}$, we have

$$b_{\boldsymbol{n}}:=a_{\boldsymbol{n}}/t^n\in(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}=(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}^{+})^{G_{K^{\mathrm{pf}}}}.$$

Therefore, we have $x=\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}}b_{\boldsymbol{n}}\boldsymbol{u}^{\boldsymbol{n}}\in(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{+})^{G_{K^{\mathrm{pf}}}}$. Hence we have a canonical map

$$\mathbb{D}_{\mathrm{dR}}(V)\hookrightarrow\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{+}\otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}^{+})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}}.$$

This induces a canonical homomorphism of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{+}$-modules

$$i:\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{+}\otimes_K\mathbb{D}_{\mathrm{dR}}(V)\to\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{+}\otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}^{+})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}},$$

which is compatible with the injections $\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V)$ and α by construction. We have only to prove the surjectivity of i. By Nakayama's lemma, we have only to prove the assertion after applying $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}^{+}\otimes_{\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{+}}$ (note that $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}^{+}\to\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}^{+}$ is a surjective homomorphism of local rings). We have the commutative diagram

$$\begin{tikzcd}
\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}^{+}\otimes_K\mathbb{D}_{\mathrm{dR}}(V) \arrow[r, "\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V)_*"] \arrow[d, "i_*"] & \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V \arrow[d, equal] \\
\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}^{+}\otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}^{+})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}} \arrow[r, "\alpha_*"] \arrow[d, "\mathrm{can.}"', "\cong"] & \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V \arrow[d, equal] \\
\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}^{+}\otimes_{K^{\mathrm{pf}}}\mathbb{D}_{\mathrm{dR}}(V|_{K^{\mathrm{pf}}}) \arrow[r, hook, "\alpha_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}(V|_{K^{\mathrm{pf}}})"] & \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V,
\end{tikzcd}$$

where the left lower arrow is induced by $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$, the $G_{K^{\mathrm{pf}}}$-equivariant isomorphism. Denote the composition of the left vertical arrows by i'. Since the canonical map $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$ is $G_{K^{\mathrm{pf}}}$-equivariant, by the diagram, the restriction of i' to $\mathbb{D}_{\mathrm{dR}}(V)$ coincides with the canonical map $\mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{D}_{\mathrm{dR}}(V|_{K^{\mathrm{pf}}})$, which is an isomorphism after tensoring K^{pf} (see Section 4B). Therefore, i' is an isomorphism and we obtain the assertion. □

6. Proof of the Main Theorem

We will restate our main theorem in the point of view of Remark 4.7(iii):

Main Theorem. *Let V be a de Rham representation of G_K. Then, there exists a finite extension L/K such that the restriction $V|_L$ is $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/L_0}$-admissible for any choice of L_0.*

In this section, we give a proof of the Main Theorem in this form. Before the proof, we prepare technical lemmas used in the proof. The reader may go to the proof of the Main Theorem and back to the lemmas if necessary.

We first recall a slightly modified version of [Colmez 2008, Proposition 0.6]. In the rest of this section, denote the unramified extension of $\mathbb{Q}_p$ of degree $h \in \mathbb{N}_{>0}$ by $\mathbb{Q}_{p^h}$.

Proposition 6.1. *Assume that k_K is perfect. Let $\mathbb{U}'_{h,a} := (\tilde{\mathbb{B}}^+_{\log,\mathbb{C}_p/K_0})^{\varphi^h = p^a}$ for h, $a \in \mathbb{N}$. Let M be a (φ, G_K)-module over $\tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K_0}$ of rank $r \in \mathbb{N}_{>0}$ with basis $e_1, \dots, e_r$. Assume that there exists an isomorphism of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K}} M \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^r$ and that $e_1, \dots, e_r$ satisfies the following conditions:*

- (i) *There exists $h \in \mathbb{N}_{>0}$ and $a_1 \leq \cdots \leq a_r \in \mathbb{N}$ such that $\varphi^h(e_i) = p^{a_i} e_i$ for $1 \leq i \leq r$.*
- (ii) *For all $g \in G_K$, there exists $c_g \in \mathrm{GL}_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$, a (unique) upper triangular matrix whose diagonal entries are 1, such that $g(e_1, \dots, e_r) = (e_1, \dots, e_r)c_g$.*

Then there exists a $\tilde{\mathbb{B}}^+_{\log,\mathbb{C}_p/K_0}$-basis $f_1, \dots, f_r$ of $\tilde{\mathbb{B}}^+_{\log,\mathbb{C}_p/K_0} \otimes_{\tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K_0}} M$ satisfying the following conditions:

- (a) *f_i is fixed by G_K;*
- (b) *$f_i = e_i + \sum_{1 \leq j \leq i-1} \alpha_{ji} e_j$ with $\alpha_{ji} \in \mathbb{U}'_{h,a_i - a_j}$ (hence $\varphi^h(f_i) = p^{a_i} f_i$).*

Proof. Note that we add the extra assumption (ii) and the slightly stronger conclusion (a) to the original proposition. Let U be the subgroup of $\mathrm{GL}_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ consisting of upper triangular matrices whose diagonal entries are 1 and whose (i, j)-component belongs to $\mathbb{U}'_{h,a_j - a_i}$ for $i < j$. We endow U with the subspace topology of $\mathrm{GL}_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$. Then, U is a topological G_K-group and the map $g \mapsto c_g; G_K \to U$ is a continuous 1-cocycle. By [Colmez 2008, Proposition 0.6], there exists a finite Galois extension L/K such that $[c]$ is mapped to the trivial class in $H^1(G_{L^{\mathrm{ur}}}, U)$ by the composite $\mathrm{Res}_L^{L^{\mathrm{ur}}} \circ \mathrm{Res}_K^L$, where $[c]$ denotes the class represented by c. Note that for all $a \in \mathbb{N}_{>0}$, we have

$$(\mathbb{U}'_{h,a})^{G_{L^{\mathrm{ur}}}} \subset ((\mathbb{B}_{\mathrm{st},\mathbb{C}_p/L_0})^{G_{L^{\mathrm{ur}}}})^{\varphi^h = p^a} = (L_0^{\mathrm{ur}})^{\varphi^h = p^a} = 0,$$

where the first equality follows from Remark 3.12(ii) and Corollary 4.3 and the last equality follows from [Colmez 2008, Lemme 10.9]. Hence $U^{G_{L^{\mathrm{ur}}}} = \{1\}$ and $[c]$ is mapped to the trivial class in $H^1(G_L, U)$ by the inflation-restriction exact sequence. Hence, we have only to prove that the inverse image of the trivial element by $\mathrm{Res}_K^L : H^1(G_K, U) \to H^1(G_L, U)$ consists of the trivial element.

We endow U with a G_K-stable decreasing filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ by $\mathcal{F}_n := \{(x_{ij}) \in U \mid x_{ij} = 0 \text{ for } 0 < j - i \le n\}$. Then, we have $\mathcal{F}_0 = U$, $\mathcal{F}_r = \{1\}$, $\mathcal{F}_{n+1} \trianglelefteq \mathcal{F}_n$ and $\mathcal{F}_n/\mathcal{F}_{n+1}$ is isomorphic to a direct sum of copies of $\mathbb{U}'_{h,a}$ with $a \in \mathbb{N}$. We have only to prove that the inverse image of the trivial element under the restriction map $\mathrm{Res}_K^L : H^1(G_K, \mathcal{F}_n) \to H^1(G_L, \mathcal{F}_n)$ for $n \in \mathbb{N}$ consists of the trivial element. Since there exists a G_K-equivariant set-theoretic section of the canonical projection $\mathcal{F}_n \to \mathcal{F}_n/\mathcal{F}_{n+1}$ (for example, we can identify

$$1 + \sum_i x_{i,i+n+1} E_{i,i+n+1} \in \mathcal{F}_n$$

with its image in $\mathcal{F}_n/\mathcal{F}_{n+1}$), the canonical maps $\mathcal{F}_n^{G_K} \to (\mathcal{F}_n/\mathcal{F}_{n+1})^{G_K}$ and $\mathcal{F}_n^{G_L} \to (\mathcal{F}_n/\mathcal{F}_{n+1})^{G_L}$ are surjective. By using long exact sequences, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(G_K, \mathcal{F}_{n+1}) \arrow[r, "\mathrm{can.}"] \arrow[d, "\mathrm{Res}_K^L"] & H^1(G_K, \mathcal{F}_n) \arrow[r, "\mathrm{can.}"] \arrow[d, "\mathrm{Res}_K^L"] & H^1(G_K, \mathcal{F}_n/\mathcal{F}_{n+1}) \arrow[d, "\mathrm{Res}_K^L"] \\
0 \arrow[r] & H^1(G_L, \mathcal{F}_{n+1}) \arrow[r, "\mathrm{can.}"] & H^1(G_L, \mathcal{F}_n) \arrow[r, "\mathrm{can.}"] & H^1(G_L, \mathcal{F}_n/\mathcal{F}_{n+1}),
\end{tikzcd}$$

whose rows are exact as pointed sets. To prove the assertion, it suffices to prove the injectivity of the restriction map $H^1(G_K, \mathbb{U}'_{h,a}) \to H^1(G_L, \mathbb{U}'_{h,a})$ for $h, a \in \mathbb{N}$. Indeed, it implies the injectivity of the right arrow in the diagram and we obtain the assertion by dévissage and diagram chasing. We first consider the case $a = 0$, that is, $\mathbb{U}'_{h,0} = \mathbb{Q}_{p^h}$ (Lemma 6.2 below). Since $H^1(G_{L/K}, \mathbb{Q}_{p^h}^{G_L})$ is killed by the multiplication by $[L : K]$ (using the corestriction) which induces an isomorphism on the coefficients, we have $H^1(G_{L/K}, \mathbb{Q}_{p^h}^{G_L}) = 0$. By the inflation-restriction sequence, we obtain the assertion. Consider the case $a > 0$. We denote by $\chi : G_K \to \mathbb{Z}_p^\times$ the cyclotomic character. Then, we obtain the assertion by the following commutative diagram:

$$\begin{tikzcd}
H^1(G_K,\mathbb{U}'_{h,a}) \arrow[r, hook, "\Pi(N^k\circ\varphi^{-n})_*"] \arrow[d, "\mathrm{Res}^L_K"] & \prod_{n,k\in\mathbb{N}} H^1(G_K,\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \cong \prod_{n,k\in\mathbb{N}} K\log\chi \arrow[d, hook, "\Pi\mathrm{Res}^L_K"] \\
H^1(G_L,\mathbb{U}'_{h,a}) \arrow[r, hook, "\Pi(N^k\circ\varphi^{-n})_*"] & \prod_{n,k\in\mathbb{N}} H^1(G_L,\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \cong \prod_{n,k\in\mathbb{N}} L\log\chi,
\end{tikzcd}$$

where two isomorphisms follow by dévissage and Lemma 1.14, Theorem 1.15 (a theorem of J. Tate) and the injectivity of the horizontal arrows follow from [Colmez 2008, Proposition 0.4(ii)]. □

Lemma 6.2. *We have*

$$(\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})^{\varphi^h=p^{-a}} = (\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0})^{\varphi^h=p^{-a}} = 0 \quad \textit{for } a\in\mathbb{N}_{>0},$$
$$(\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})^{\varphi^h=1} = (\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0})^{\varphi^h=1} = \mathbb{Q}_{p^h}.$$

Proof. We first prove the first assertion. Suppose that we have a nonzero element x in $(\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0})^{\varphi^h=p^{-a}}$. Since $\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0}$ is an integral domain, we may assume that we have $x\in\mathbb{A}_{\mathrm{st},\mathbb{C}_p/K_0}$ by multiplying by some power of p. By assumption and the φ-stability of $\mathbb{A}_{\mathrm{st},\mathbb{C}_p/K_0}$, $x=p^{na}\varphi^{nh}(x)\in p^n\mathbb{A}_{\mathrm{st},\mathbb{C}_p/K_0}$. Hence $x\in\bigcap_n p^n\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/K_0}[\mathrm{x}]=\{0\}$ since $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/K_0}$ is p-adically separated. Thus $x=0$, which is a contradiction.

We prove the latter assertion. By a simple calculation, we have

$$(\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0})^{\varphi^h=1} = (\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})^{\varphi^h=1}.$$

By the canonical isomorphism $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}\cong\tilde{\mathbb{B}}^{+}_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$, we may reduce to the perfect residue field case, which follows from [Colmez 2002, Proposition 9.2]. □

Lemma 6.3. *Let D be a finite free $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$-module with semilinear G_K-action. Then, the canonical map $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_K D^{G_K}\to D$ is injective. In particular, we have* $\dim_K D^{G_K}\le\mathrm{rank}_{\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}} D<\infty$.

Proof. Suppose that we have linearly independent elements $f_1,\dots,f_n\in D^{G_K}$ over K, which have a nontrivial relation $\sum_i\lambda_i f_i=0$ with $\lambda_i\in\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$. Choose the minimum n among such n's. Then for $1\le i\le n$, we have $g(\lambda_i/\lambda_1)=\lambda_i/\lambda_1$ in $\mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})$. Hence we have both $\lambda_i/\lambda_1\in H^0(G_K,\mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}))=K$ and $\sum_i(\lambda_i/\lambda_1)f_i=0$, a contradiction. □

Lemma 6.4. *Let W be an r-dimensional $\mathbb{Q}_{p^h}$-vector space with semilinear G_K-action. For $0\le i<h$, we define the $\mathbb{Q}_{p^h}$-vector space φ^i_*W with semilinear G_K-action by $\varphi^i_*W:=W$ as G_K-module with scalar multiplication*

$$\mathbb{Q}_{p^h}\times W\to W;\quad (\lambda,x)\mapsto\varphi^i(\lambda)x.$$

If we have an isomorphism of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$*-modules*

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{\mathbb{Q}_{p^h}}\varphi^i_*W\cong(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^r$$

for $0\le i<h$*, then* W *is* $\mathbb{C}_p$*-admissible as a* p*-adic representation of* G_K*.*

Proof. By assumption, we have isomorphisms

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{\mathbb{Q}_p}W\cong\bigoplus_{0\le i<h}\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{\mathbb{Q}_{p^h}}\varphi^i_*W\cong(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{hr}$$

of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules, which implies the assertion by tensoring with $\mathbb{C}_p$ over $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$. □

Lemma 6.5. *Assume that* $e_{K/K_{\mathrm{can}}}=1$*. Then, the complex*

$$K\otimes_{K_0}(\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}}\xrightarrow{\nabla}\Omega^1_K\hat{\otimes}_{K_0}(\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}}\xrightarrow{\nabla_1}\Omega^2_K\hat{\otimes}_{K_0}(\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}},$$

which is induced by the inclusion $K\otimes_{K_0}\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0}\to\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$ *(Proposition 3.16) and Lemma 3.9, is exact. Here, we endow* $(\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}}$ *with the* p*-adic topology induced by the* p*-adic semivaluation* $v_{\mathrm{cris},\mathbb{C}_p/K}$*.*

Proof. Note that the connections are K_{can}-linear by Proposition 1.13. We may reduce to the case $K=K_0$ by Remark 1.4(ii) and Lemma 1.10(iii). Let $\omega\in\ker\nabla_1$. We can write $\omega=\sum_{j\in J_K}dt_j\otimes\lambda_j$ with $\lambda_j\in\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K}$ such that

$$\{v_{\mathrm{cris},\mathbb{C}_p/K}(\lambda_j)\}_{j\in J_K}\to\infty.$$

We can also write $\lambda_j=\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}}\lambda_{j,\boldsymbol{n}}\boldsymbol{u}^{[\boldsymbol{n}]}$ with $\lambda_{j,\boldsymbol{n}}\in\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ such that $\{v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}})\}_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}}\to\infty$. Since u_j is invariant under the action of $G_{K^{\mathrm{pf}}}$, we have $\lambda_{j,\boldsymbol{n}}\in(\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}$. Recall the proof of Lemma 3.9: We define $a_{\boldsymbol{0}}=0$ and $a_{\boldsymbol{n}}=\lambda_{j,\boldsymbol{n}-\mathbf{e}_j}$ if $n_j\neq 0$. Then, we have

$$x=\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}}a_{\boldsymbol{n}}\boldsymbol{u}^{[\boldsymbol{n}]}\in\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$$

and $\nabla(x)=\omega$. Note that we have $x\in(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}$. Hence, we have only to prove $x\in\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K}$. Fix $N\in\mathbb{N}$: we have to show that $v_{\mathrm{cris},\mathbb{C}_p/K}(a_{\boldsymbol{n}})\ge N$ for all but finitely many $\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}$. Choose a finite subset J of J_K such that we have $v_{\mathrm{cris},\mathbb{C}_p/K}(\lambda_j)\ge N$ for $j\in J_K\setminus J$. We also choose $n\in\mathbb{N}$ such that we have $v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}})\ge N$ for $j\in J$ and $|\boldsymbol{n}|\ge n$. Let $\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}\setminus\mathbb{N}^J$. Then, we have $v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})=v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}-\boldsymbol{e}_j})\ge N$ for some $j\in J_K\setminus J$. Let $\boldsymbol{n}\in\mathbb{N}^J$ with $|\boldsymbol{n}|>n$. Then, we have $v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})=v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}-\boldsymbol{e}_j})\ge N$ for some $j\in J$. Since the set $\{\boldsymbol{n}\in\mathbb{N}^J\mid|\boldsymbol{n}|\le n\}$ is finite, these inequalities imply the assertion. □

Proof of Main Theorem. Obviously, we may assume $r := \dim_{\mathbb{Q}_p} V > 0$. By Hilbert 90, we may replace K by K^{ur}. Hence, we may assume that k_K is separably closed. After some Tate twist, we may also assume that V satisfies the assumption of Section 5, that is, we have $\mathbb{D}_{\mathrm{dR}}(V) = (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

We divide the rest of the proof into two steps: We will construct a finite extension L/K in Step 1 and after replacing K by L, we will prove the semistability of V in Step 2. Note that only Step 2 involves the choice of K_0.

Step 1: Set $\mathcal{M} := \widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ and let $e_1, \dots, e_r$ be as in Proposition 5.3. Also let $\{a'_1 < \cdots < a'_{r'}\}$ be the set of distinct elements in the multiset $\{a_1, \dots, a_r\}$ and m_i the multiplicity of a'_i in the multiset for $1 \le i \le r'$. Let $\{e_1^{(i)}, \dots, e_{m_i}^{(i)}\}$ be the subset of $e_l \in \{e_1, \dots, e_r\}$ satisfying $\varphi^h(e_l) = p^{a'_i} e_l$. We define an exhaustive and separated increasing filtration of $\mathcal{M}$ by

$$\mathcal{M}_n := \begin{cases} 0 & \text{if } n \le 0, \\ \bigoplus_{1 \le i \le n} (\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} e_1^{(i)} \oplus \cdots \oplus \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} e_{m_i}^{(i)}) & \text{if } 1 \le n < r', \\ \mathcal{M} & \text{otherwise.} \end{cases}$$

The filtration is stable under φ and G_K-actions. In fact, for $1 \le i \le n < r'$ and $g \in G_K$, we have

$$\varphi(e_1^{(i)}), \dots, \varphi(e_{m_i}^{(i)}), g(e_1^{(i)}), \dots, g(e_{m_i}^{(i)}) \in \mathcal{M}^{\varphi^h = p^{a'_i}} \subset \mathcal{M}_n,$$

where the last inclusion follows from Lemma 6.2. We also define

$$W_n := (\mathcal{M}_n/\mathcal{M}_{n-1})^{\varphi^h = p^{a'_n}}$$

for $1 \le n \le r'$. Since we have $W_n = \mathbb{Q}_{p^h} \bar{e}_1^{(n)} \oplus \cdots \oplus \mathbb{Q}_{p^h} \bar{e}_{m_n}^{(n)}$ by Lemma 6.2 (where $\bar{e}_i^{(n)}$ denotes the image of $e_i^{(n)}$ in $\mathcal{M}_n/\mathcal{M}_{n-1}$), W_n is an m_n-dimensional $\mathbb{Q}_{p^h}$-vector space with continuous semilinear G_K-action. Let

$$D_n := \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}} \mathcal{M}_n.$$

Then, we have the left exact sequence of finite K-vector spaces

$$0 \longrightarrow D_{n-1}^{G_K} \xrightarrow{\text{inc.}} D_n^{G_K} \xrightarrow{\text{pr.}} (D_n/D_{n-1})^{G_K}. \tag{3}$$

Hence, we have the inequalities

$$\dim_K D_n^{G_K} \le \dim_K D_{n-1}^{G_K} + \dim_K (D_n/D_{n-1})^{G_K} \le \dim_K D_{n-1}^{G_K} + m_n$$

for $n \in \mathbb{Z}$ by Lemma 6.3. By Proposition 5.4, we have an isomorphism of $\mathbb{B}^+_{\mathrm{dR}}[G_K]$-modules

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}} \mathcal{M} \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^r, \tag{4}$$

which implies $\dim_K D_n^{G_K} = r$ for $n \ge r'$. Hence, the summation of the above inequalities are equalities. Therefore, the above inequalities are equalities, in particular, the map $\mathrm{pr.}: D_n^{G_K} \to (D_n/D_{n-1})^{G_K}$ in (3) is surjective. Thus, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K D_{n-1}^{G_K} \arrow[r] \arrow[d, hook] & \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K D_n^{G_K} \arrow[r] \arrow[d, hook] & \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K (D_n/D_{n-1})^{G_K} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & D_{n-1} \arrow[r] & D_n \arrow[r] & D_n/D_{n-1} \arrow[r] & 0
\end{tikzcd}$$

with exact rows and injective vertical arrows by Lemma 6.3. Since the middle vertical arrow is an isomorphism for $n \geq r'$ by (4), all vertical arrows are isomorphisms. In particular, for $1 \leq n \leq r'$, we have isomorphisms of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_{p^h}} W_n \cong D_n/D_{n-1} \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{m_n}$. Since W_n is stable under the action of φ, the map $W_n \to \varphi^i_* W_n$ taking x to $\varphi^i(x)$ is an isomorphism of $\mathbb{Q}_{p^h}[G_K]$-modules. In particular, we have isomorphisms of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_{p^h}} \varphi^i_* W_n \cong \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_{p^h}} W_n \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{m_n}$$

for $1 \leq n \leq r'$ and $0 \leq i < h$, which implies the $\mathbb{C}_p$-admissibility of W_n by Lemma 6.4. Hence, G_K acts on W_n factoring through a finite quotient by Theorem 2.1. We choose a finite extension L/K such that G_L acts on W_n trivially for all $1 \leq n \leq r'$ and such that L satisfies condition (H).

Step 2: By replacing V by $V|_L$, we will prove that V is semistable by calculating Galois cohomology associated to $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$. In the following, we fix K_0 and a lift $\{t_j\}_{j \in J_K}$ of a p-basis of k_K in K_0. We regard $\{t_j\}_{j \in J_K}$ as a lift of a p-basis of k_K in K. We also fix notation: For a commutative ring R, let $U_r(R) \subset \mathrm{GL}_r(R)$ be the group of unipotent upper triangular matrices. Let $N_r(R) \subset M_r(R)$ be the Lie algebra of $U_r(R)$, that is, the group of nilpotent upper triangular matrices. We denote $U^+_{r,\mathrm{dR}} := U_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$, $U^{\nabla+}_{r,\mathrm{dR}} = U_r(\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})$ for simplicity.

By assumption, we have $g(e_1, \ldots, e_r) = (e_1, \ldots, e_r)c_g$ with 1-cocycle

$$c : G_K \to U_r(\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}).$$

Since we have $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V) \subset \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \otimes_{\mathbb{Q}_p} V$ and

$$(K \otimes_{K_0} \mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0} \otimes_{\mathbb{Q}_p} V)^{G_K} = K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0} \otimes_{\mathbb{Q}_p} V)^{G_K},$$

we have only to prove that c is a 1-coboundary in $U_r(K \otimes_{K_0} \mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0})$. We have the exact sequence of pointed sets

$$(U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K} \xrightarrow{\delta} H^1(G_K, U^{\nabla+}_{r,\mathrm{dR}}) \xrightarrow{\mathrm{inc.}_*} H^1(G_K, U^+_{r,\mathrm{dR}}), \tag{5}$$

where $U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}}$ denotes the left coset of $U^+_{r,\mathrm{dR}}$ by $U^{\nabla+}_{r,\mathrm{dR}}$, that is, $X \sim Y$ if $X^{-1}Y \in U^{\nabla+}_{r,\mathrm{dR}}$. The class $[c] \in H^1(G_K, U^{\nabla+}_{r,\mathrm{dR}})$ represented by c is mapped to the trivial class in $H^1(G_K, U^+_{r,\mathrm{dR}})$. In fact, since we have $\bar{e}^{(n)}_i \in (D_n/D_{n-1})^{G_K}$ for $1 \le n \le r'$ and $1 \le i \le m_n$ by assumption, there exists an element $\tilde{e}^{(n)}_i \in D_n^{G_K}$ such that $\tilde{e}^{(n)}_i - e^{(n)}_i \in D_{n-1}$ by the exactness of (3). Then,

$$(\tilde{e}^{(1)}_1, \dots, \tilde{e}^{(1)}_{m_1}, \dots, \tilde{e}^{(n)}_1, \dots, \tilde{e}^{(n)}_{m_n})$$

is a $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$-basis of D_n for $1 \le n \le r'$ and we have a unique matrix $U \in U^+_{r,\mathrm{dR}}$ such that

$$(e^{(1)}_1, \dots, e^{(1)}_{m_1}, \dots, e^{(r')}_1, \dots, e^{(r')}_{m_{r'}}) = (\tilde{e}^{(1)}_1, \dots, \tilde{e}^{(1)}_{m_1}, \dots, \tilde{e}^{(r')}_1, \dots, \tilde{e}^{(r')}_{m_{r'}})U.$$

By a simple calculation, we have $c_g = U^{-1}g(U)$ for all $g \in G_K$. Hence, the class $[c]$ is represented by an element of the image of $(U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K}$ under δ by the exact sequence (5). We regard $K \otimes_{K_0} \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0}$ as a subring of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$ by Proposition 3.16. Then, we have the following lemma:

Lemma 6.6. *Every element of $(U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K}$ is represented by an element in $U_r(K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}})$.*

We leave the proof of Lemma 6.6 to the end of the proof. Thanks to the lemma, there exist $X_1 \in U_r(K \otimes_{K_0} (\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}})$ and $X_2 \in U^{\nabla+}_{r,\mathrm{dR}}$ such that

$$c_g = X_2^{-1}X_1^{-1}g(X_1)g(X_2) \tag{6}$$

for all $g \in G_K$.

Since the canonical isomorphism $i : \tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \to \tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$ is compatible with the actions of φ and $G_{K^{\mathrm{pf}}}$, we may regard $M := i^*\mathcal{M}$ as a $(\varphi, G_{K^{\mathrm{pf}}})$-module over $\tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$. Then, the triple $(M, \{e_1, \dots, e_r\}, i^*c)$ satisfies the assumptions of Proposition 6.1. Indeed, assumption (i) follows from Proposition 5.3, Proposition 5.4 and the functoriality. The image of c is in $U_r(\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})$, which implies assumption (ii). Applying Proposition 6.1 to the above triple, we have $X_3' \in U_r(\mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0^{\mathrm{pf}}})$ such that $i(c_g) = (X_3')^{-1}g(X_3')$. Hence, $X_3 := i^{-1}(X_3') \in U_r(\mathbb{B}^{\nabla+}_{\mathrm{st},\mathbb{C}_p/K_0})$ satisfies $c_g = X_3^{-1}g(X_3)$ for $g \in G_{K^{\mathrm{pf}}}$. Since we have $c_g = X_2^{-1}g(X_2)$ for $g \in G_{K^{\mathrm{pf}}}$ by (6), we have

$$X_2X_3^{-1} \in (U^{\nabla+}_{r,\mathrm{dR}})^{G_{K^{\mathrm{pf}}}} = U_r((\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}).$$

Note that the canonical map

$$K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} (\mathbb{B}^{\nabla+}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}} \to (\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}$$

is an isomorphism. In fact, by using the canonical isomorphisms $\mathbb{B}^{\nabla+}_{\mathrm{cris},\mathbb{C}_p/K_0} \to \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{pf}}}$ and $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$, it follows from the isomorphisms

$$K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} K_0^{\mathrm{pf}} \cong K \otimes_{K_0} K_0^{\mathrm{pf}} \cong K^{\mathrm{pf}},$$

where the first isomorphism easily follows from Remark 1.4(ii) and the second one is trivial. Thus, we have

$$c_g = (X_1 \cdot X_2 X_3^{-1} \cdot X_3)^{-1} g(X_1 \cdot X_2 X_3^{-1} \cdot X_3)$$

for all $g \in G_K$ with X_1, $X_2 X_3^{-1}$, $X_3 \in U_r(K \otimes_{K_0} \mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0})$, which implies the assertion.

Now, we return to the proof of Lemma 6.6. We endow $M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{r^2}$ with the product topology. Let

$$\begin{aligned} d &: M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \to \widehat{\Omega}^1_K \widehat{\otimes}_K M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}); \quad (x_{ij}) \mapsto (\nabla(x_{ij})), \\ d_1 &: \widehat{\Omega}^1_K \widehat{\otimes}_K M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \to \widehat{\Omega}^2_K \widehat{\otimes}_K M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}); \quad (\omega_{ij}) \mapsto (\nabla_1(\omega_{ij}))\end{aligned}$$

be the derivations. For $i \in \{1, 2\}$, we endow $\widehat{\Omega}^i_K \widehat{\otimes}_K M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ with the left (resp. right) action of $M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ induced by the left (resp. right) multiplication on $M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$. We also have the wedge product

$$\begin{aligned} \wedge : \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(K) \times \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) &\to \widehat{\Omega}^2_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \\ (\omega_{ij}) \times (\omega'_{ij}) &\mapsto \Bigl(\sum_{1 \le k \le r} \omega_{ik} \wedge \omega'_{kj}\Bigr).\end{aligned}$$

Then, we have the formulas $d_1 \circ d = 0$, $d(XX') = dX \cdot X' + X \cdot dX'$, $d_1(\omega \cdot X) = d_1\omega \cdot X - \omega \wedge dX$, and $(\omega \wedge \omega') \cdot X = \omega \wedge (\omega' \cdot X)$, for $X, X' \in \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$, $\omega \in \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(K)$, and $\omega' \in \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$. We define a log differential

$$\mathrm{dlog} : U^+_{r,\mathrm{dR}} \to \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}); \quad X \mapsto dX \cdot X^{-1},$$

which is G_K-equivariant. (Note that it does not preserve the group laws in general.) Since we have $\mathrm{dlog}(XA) = \mathrm{dlog}(X)$ for $A \in U^{\nabla+}_{r,\mathrm{dR}}$ and $X \in U^+_{r,\mathrm{dR}}$ by the above formulas, dlog induces a morphism of G_K-sets

$$\mathrm{dlog}_* : U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}} \to \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}).$$

Moreover, dlog_* is injective. Indeed, let $X, Y \in U^+_{r,\mathrm{dR}}$ such that $\mathrm{dlog} X = \mathrm{dlog} Y$. By $d\boldsymbol{E} = d(Y^{-1}Y) = 0$ and the above formulas, we have $d(Y^{-1}) = -Y^{-1} dY \cdot Y^{-1}$. Hence, we have

$$\begin{aligned}\mathrm{dlog}(Y^{-1}X) &= (d(Y^{-1}) \cdot X + Y^{-1} dX) \cdot X^{-1} Y \\ &= -Y^{-1}(dY \cdot Y^{-1} - dX \cdot X^{-1}) \cdot Y = 0.\end{aligned}$$

Since the inverse image of $\{0\}$ by dlog is $U^{\nabla+}_{r,\mathrm{dR}}$, we have $X \sim Y$. By taking $H^0(G_K, -)$ of dlog_*, we have an injection of sets

$$\mathrm{dlog}_* : (U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K} \hookrightarrow \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(K).$$

We define a decreasing filtration on $N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ by

$$\mathrm{Fil}^n N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) := \{(a_{ij}) \in N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \mid a_{ij} = 0 \text{ if } j-i \le n\}.$$

Then, we have $\mathrm{Fil}^0 N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) = N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ and $\mathrm{Fil}^r N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) = 0$. Let $X \in U^+_{r,\mathrm{dR}}$ such that we have $[X] \in (U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K}$. Let $\omega := \mathrm{dlog}(X) \in \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(K)$. We will construct $X^{(n)} \in U_r(K \otimes_{K_0} (\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}})$ for $n \in \mathbb{N}$ satisfying

$$\omega \cdot X^{(n)} \equiv dX^{(n)} \mod \widehat{\Omega}^1_K \widehat{\otimes}_K \mathrm{Fil}^n N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}).$$

Set $X^{(0)} := 1$. Suppose that we have constructed $X^{(n)}$. Since we have $\omega \cdot X = dX$, we have $d_1\omega \cdot X = \omega \wedge dX$ by taking d_1. Hence, we have $d_1\omega = (\omega \wedge dX) \cdot X^{-1} = \omega \wedge \omega$. Let $\omega' = (\omega'_{ij}) := \omega \cdot X^{(n)} - dX^{(n)} \in \widehat{\Omega}^1_K \widehat{\otimes}_K \mathrm{Fil}^n N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$. Then, by a simple calculation using the above formulas, we have

$$d_1\omega' = \omega \wedge (\omega \cdot X^{(n)} - dX^{(n)}) = \omega \wedge \omega' \equiv 0 \mod \widehat{\Omega}^2_K \widehat{\otimes}_K \mathrm{Fil}^{n+1} N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}),$$

which implies $\nabla_1(\omega'_{i,i+n+1}) = 0$. Since we have

$$\omega'_{ij} \in \widehat{\Omega}^1_K \widehat{\otimes}_K (K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}}),$$

by Lemma 6.5, there exists $x'_{i,i+n+1} \in K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}}$ such that

$$\nabla(x'_{i,i+n+1}) = \omega'_{i,i+n+1}.$$

Let $X^{(n+1)} := X^{(n)} + \sum_i x'_{i,i+n+1} E_{i,i+n+1} \in U_r(K \otimes_{K_0} (\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}})$. Then, by a simple calculation, we have

$$\begin{aligned}
&\omega \cdot X^{(n+1)} - dX^{(n+1)} \\
&\qquad \equiv \omega \cdot X^{(n)} - dX^{(n)} - d\Big(\sum_i x'_{i,i+n+1} E_{i,i+n+1}\Big) \\
&\qquad \equiv \omega' - \sum_i \nabla(x'_{i,i+n+1}) E_{i,i+n+1} \equiv 0 \mod \widehat{\Omega}_K \widehat{\otimes}_K \mathrm{Fil}^{n+1} N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}).
\end{aligned}$$

Hence, we have $\mathrm{dlog}(X^{(r)}) = \omega$, which implies the assertion. □

7. Applications

We will give applications of the Main Theorem. In Section 7A, we will recall linear algebraic structures, which appear in the following. In Section 7B, we will prove a horizontal analogue of the p-adic monodromy theorem. The results of the next two subsections are applications of this theorem. In Section 7C, we will prove an equivalence between the category of horizontal de Rham representations of G_K and the category of de Rham representation of $G_{K_{\mathrm{can}}}$. In Section 7D, we will prove a generalization of Hyodo's Theorem 1.16.

In this section, unless particular mention is stated, we will denote $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/K_0}$ (resp. $\mathbb{B}^{\nabla}_{\heartsuit,\mathbb{C}_p/K}$) by $\mathbb{B}^{\nabla}_{\diamondsuit}$ (resp. $\mathbb{B}^{\nabla}_{\heartsuit}$) for $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$ (resp. $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$): This notation is justified by the facts that $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/K_0}$ and $\mathbb{B}^{\nabla}_{\heartsuit,\mathbb{C}_p/K}$ are isomorphic to $\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathbb{Q}_p}$ and $\mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathbb{Q}_p}$ as $(\mathbb{Q}_p, G_K)$-rings respectively.

7A. ***Additional structures.*** In the following, let $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$. The vector space $\mathbb{D}^{\nabla}_{\bullet}(V)$ has additional structures for $\bullet \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}, \mathrm{HT}\}$, which we will recall following [Fontaine 1994b].

- The Hodge–Tate case

We define a graded K-vector space as a finite-dimensional K-vector space D endowed with a decomposition $D = \bigoplus_{n\in\mathbb{Z}} D_n$. Denote by MG_K the category of graded K-vector spaces. The graded ring structure on $\mathbb{B}^{\nabla}_{\mathrm{HT}}$ induces a graded K-vector space structure on $\mathbb{D}^{\nabla}_{\mathrm{HT}}(V)$. Hence, we have a $\otimes$-functor

$$\mathbb{D}^{\nabla}_{\mathrm{HT}} : \mathrm{Rep}^{\nabla}_{\mathrm{HT}} G_K \to MG_K.$$

Assume that we have $V \in \mathrm{Rep}^{\nabla}_{\mathrm{HT}} G_K$. We define the Hodge–Tate weights of V as the multiset consisting of $n \in \mathbb{Z}$ with multiplicity $m_n := \dim_K (\mathbb{C}_p(-n) \otimes_{\mathbb{Q}_p} V)^{G_K}$. Since the comparison isomorphism $\alpha^{\nabla}_{\mathrm{HT}}$ is compatible with G_K-actions and gradings, by taking the degree zero part, we have an isomorphism of $\mathbb{C}_p[G_K]$-modules

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)^{m_n},$$

which is referred to as the Hodge–Tate decomposition of V. Note that if $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$ admits such a decomposition, then it is horizontal Hodge–Tate.

- The de Rham case

We define a filtered K_{can}-module as a finite-dimensional K_{can}-vector space endowed with a decreasing filtration $\{\mathrm{Fil}^n D\}_{n\in\mathbb{Z}}$ of K_{can}-subspaces such that $\mathrm{Fil}^n D = D$ for $n \ll 0$ and $\mathrm{Fil}^n D = 0$ for $n \gg 0$. Denote by $MF_{K_{\mathrm{can}}}$ the category of filtered K_{can}-modules. The filtration $\mathrm{Fil}^n \mathbb{B}^{\nabla}_{\mathrm{dR}} = t^n \mathbb{B}^{\nabla +}_{\mathrm{dR}}$ on $\mathbb{B}^{\nabla}_{\mathrm{dR}}$ induces a filtered K_{can}-module structure on $\mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$. Hence, we have a $\otimes$-functor

$$\mathbb{D}^{\nabla}_{\mathrm{dR}} : \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K \to MF_{K_{\mathrm{can}}}.$$

- The crystalline and semistable cases

We first define filtered $(\varphi, N, G_{L/K})$-modules for our later use.

Defintion 7.1. (i) Let L/K be a finite Galois extension. A filtered $(\varphi, N, G_{L/K})$-module is a finite-dimensional $L_{\mathrm{can},0}$-vector space D endowed with

- the Frobenius endomorphism: a bijective φ-semilinear map $\varphi : D \to D$;
- the monodromy operator: an $L_{\mathrm{can},0}$-linear map $N : D \to D$ such that $N\varphi = p\varphi N$;

- the Galois action: an $L_{\mathrm{can},0}$-semilinear action of $G_{L/K}$, which commutes with φ and N;
- the filtration: a decreasing filtration $\{\mathrm{Fil}^n D_{L_{\mathrm{can}}}\}_{n\in\mathbb{Z}}$ of $G_{L/K}$-stable L_{can}-subspaces of $D_{L_{\mathrm{can}}} := L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} D$ satisfying
$$\mathrm{Fil}^n D_{L_{\mathrm{can}}} = D_{L_{\mathrm{can}}} \quad \text{for } n \ll 0 \quad \text{and} \quad \mathrm{Fil}^n D_{L_{\mathrm{can}}} = 0 \quad \text{for } n \gg 0.$$

If $L = K$, then we call D a filtered (φ, N)-module relative to K_{can}. Moreover, if $N = 0$, then we call D a filtered φ-module relative to K_{can}.

A morphism $D_1 \to D_2$ of filtered $(\varphi, N, G_{L/K})$-modules is an $L_{\mathrm{can},0}$-linear map $f : D_1 \to D_2$ such that f commutes with φ and N, $G_{L/K}$-actions and we have $f(\mathrm{Fil}^n D_{1,L_{\mathrm{can}}}) \subset \mathrm{Fil}^n D_{2,L_{\mathrm{can}}}$ for all $n \in \mathbb{Z}$.

Denote by $MF(\varphi, N, G_{L/K})$ (resp. $MF_{K_{\mathrm{can}}}(\varphi, N)$, $MF_{K_{\mathrm{can}}}(\varphi)$) the category of filtered $(\varphi, N, G_{L/K})$-modules (resp. filtered (φ, N)-modules relative to K_{can}, filtered φ-modules relative to K_{can}).

(ii) Let $D \in MF_{K_{\mathrm{can}}}(\varphi, N)$ and $r := \dim_{K_{\mathrm{can},0}} D$. We define $t_N(D)$ and $t_H(D)$ in the following way: First, we consider the case $r = 1$. If we have $v \in D \setminus \{0\}$ and $\varphi(v) = \lambda v$, then $v_p(\lambda) \in \mathbb{Z}$ is independent of the choice of v. We denote it by $t_N(D)$. We denote by $t_H(D)$ the maximum number $n \in \mathbb{Z}$ such that $\mathrm{Fil}^n D_{K_{\mathrm{can}}} \neq 0$. In the general case, we define
$$t_N(D) := t_N\big(\textstyle\bigwedge^r D\big), \quad t_H(D) := t_H\big(\textstyle\bigwedge^r D\big).$$

We say that D is weakly admissible if we have $t_N(D) = t_H(D)$ and $t_N(D') \geq t_H(D')$ for any $K_{\mathrm{can},0}$-subspace D' of D which is stable by φ and N, with $D'_{K_{\mathrm{can}}}$ endowed with the induced filtration of $D_{K_{\mathrm{can}}}$.

Denote by $MF^{\mathrm{wa}}(\varphi, N, G_{L/K})$ the full subcategory of $MF(\varphi, N, G_{L/K})$ whose objects are weakly admissible as object of $MF_{L_{\mathrm{can}}}(\varphi, N)$. We define $MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N)$ and $MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi)$ similarly.

Let $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$. By Proposition 3.16, we have a K_{can}-linear injection
$$K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\diamondsuit}(V) \to \mathbb{D}^{\nabla}_{\mathrm{dR}}(V).$$

We endow $K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\diamondsuit}(V)$ with the induced filtration of $\mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$. Together with the Frobenius endomorphism φ and the monodromy operator N on $\mathbb{B}^{\nabla}_{\mathrm{st}}$, these data induce a structure of a filtered (φ, N)-module over K_{can} relative to $K_{\mathrm{can},0}$ on $\mathbb{D}^{\nabla}_{\diamondsuit}(V)$. Since we have $\mathbb{D}^{\nabla}_{\mathrm{cris}}(V) = (\mathbb{D}^{\nabla}_{\mathrm{st}}(V))^{N=0}$, $\mathbb{D}^{\nabla}_{\mathrm{cris}}(V)$ has a structure of a filtered φ-module over K_{can} relative to $K_{\mathrm{can},0}$. Therefore, we have $\otimes$-functors
$$\mathbb{D}^{\nabla}_{\mathrm{cris}} : \mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K \to MF_{K_{\mathrm{can}}}(\varphi), \quad \mathbb{D}^{\nabla}_{\mathrm{st}} : \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \to MF_{K_{\mathrm{can}}}(\varphi, N).$$

For $D \in MF_{K_{\mathrm{can}}}(\varphi, N)$, we define
$$\mathbb{V}_{\mathrm{st}}(D) := (\mathbb{B}^{\nabla}_{\mathrm{st}} \otimes_{K_{\mathrm{can},0}} D)^{N=0,\varphi=1} \cap \mathrm{Fil}^0(\mathbb{B}^{\nabla}_{\mathrm{dR}} \otimes_{K_{\mathrm{can}}} D_{K_{\mathrm{can}}}).$$

For $D \in MF_{K_{\mathrm{can}}}(\varphi)$, we define $\mathbb{V}_{\mathrm{cris}}(D) := \mathbb{V}_{\mathrm{st}}(D)$. These are (possibly infinite-dimensional) $\mathbb{Q}_p$-vector spaces with G_K-action.

Remark 7.2. Note that we have the hierarchy of full subcategories of $\mathrm{Rep}_{\mathbb{Q}_p} G_K$

$$\mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K \subset \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \subset \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K \subset \mathrm{Rep}^{\nabla}_{\mathrm{HT}} G_K.$$

In fact, if we have $V \in \mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K$, then we have $\dim_{\mathbb{Q}_p} V = \dim_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{cris}}(V) \leq \dim_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V)$, which implies that V is horizontal semistable by Lemma 1.19. In this case, the canonical injection $\mathbb{D}^{\nabla}_{\mathrm{cris}}(V) \hookrightarrow \mathbb{D}^{\nabla}_{\mathrm{st}}(V)$ is an isomorphism as filtered (φ, N)-modules relative to K_{can}. The inclusion $\mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \subset \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$ follows from Lemma 1.20, Proposition 3.16 and Corollary 4.3. Moreover, if we have $V \in \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K$, then the canonical map $K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V) \to \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$ is an isomorphism of filtered K_{can}-modules. Finally, let $V \in \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. We choose a lift of a K_{can}-basis of $\mathrm{gr}^n \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$ in $\mathrm{Fil}^n \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$ for all $n \in \mathbb{Z}$. We denote these lifts by $\{e_i\}$ and let $n_i \in \mathbb{Z}$ such that $e_i \in \mathrm{Fil}^{n_i} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \setminus \mathrm{Fil}^{n_i+1} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$. Then, $\{e_i\}$ forms a K_{can}-basis of $\mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$. Consider the comparison isomorphism

$$\mathbb{B}^{\nabla}_{\mathrm{dR}} \otimes_{K_{\mathrm{can}}} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \to \mathbb{B}^{\nabla}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V,$$

which is compatible with the filtrations. By taking Fil^n of both sides, we have

$$\sum_i t^{n-n_i} \mathbb{B}^{\nabla +}_{\mathrm{dR}} e_i = t^n \mathbb{B}^{\nabla +}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V.$$

By taking gr^n of both sides and taking $H^0(G_K, -)$, we have

$$K \otimes_{K_{\mathrm{can}}} \mathrm{gr}^n \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \cong \bigoplus_{i : n_i = n} K e_i \cong (\mathbb{C}_p(n) \otimes_{\mathbb{Q}_p} V)^{G_K}$$

by Theorem 1.15. Hence, we have an isomorphism $K \otimes_{K_{\mathrm{can}}} \mathrm{gr} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \cong \mathbb{D}^{\nabla}_{\mathrm{HT}}(V)$ of filtered K-vector spaces, which implies $V \in \mathrm{Rep}^{\nabla}_{\mathrm{HT}} G_K$ by Lemma 1.19. In particular, the multiset of Hodge–Tate weights of V coincides with the multiset consisting of $n \in \mathbb{Z}$ with multiplicity $\dim_{K_{\mathrm{can}}} \mathrm{Fil}^{-n} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) / \mathrm{Fil}^{-n+1} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$.

Proposition 7.3. *The functors $\mathbb{D}^{\nabla}_{\mathrm{cris}}$ and $\mathbb{D}^{\nabla}_{\mathrm{st}}$ induce the functors*

$$\mathbb{D}^{\nabla}_{\mathrm{cris}} : \mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K \to MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi), \quad \mathbb{D}^{\nabla}_{\mathrm{st}} : \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \to MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N).$$

Moreover, these functors are fully faithful.

Proof. We first prove the weak admissibilities of the images. As noted in Remark 7.2, if V is horizontal crystalline, then $\mathbb{D}^{\nabla}_{\mathrm{cris}}(V)$ coincides with $\mathbb{D}^{\nabla}_{\mathrm{st}}(V)$ as a filtered (φ, N)-module relative to K_{can}. Therefore, we may reduce to the case that V is horizontal semistable.

For a filtered (φ, N)-module D relative to K_{can}, we endow the finite K_0^{pf}-vector space $D_{K_0^{\mathrm{pf}}}$ with a structure of a filtered (φ, N)-module relative to K^{pf} as follows.

We extend the Frobenius φ on D to $D_{K_0^{\mathrm{pf}}}$ semilinearly and extend the monodromy operator N on D to $D_{K_0^{\mathrm{pf}}}$ linearly. We also define a filtration of $D_{K^{\mathrm{pf}}}$ as $\mathrm{Fil}^\bullet D_{K^{\mathrm{pf}}} := K^{\mathrm{pf}} \otimes_{K_{\mathrm{can}}} \mathrm{Fil}^\bullet D_{K_{\mathrm{can}}}$. Moreover, the scalar extension

$$K_0^{\mathrm{pf}} \otimes_{K_{\mathrm{can},0}} (-) : MF_{K_{\mathrm{can}}}(\varphi, N) \to MF_{K^{\mathrm{pf}}}(\varphi, N)$$

induces a functor. We have only to prove that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Rep}_{\mathrm{st}}^{\nabla} G_K \arrow[r, "\mathbb{D}_{\mathrm{st}}^{\nabla}"] \arrow[d, "\mathrm{Res}_K^{K^{\mathrm{pf}}}"] & MF_{K_{\mathrm{can}}}(\varphi, N) \arrow[d, "K_0^{\mathrm{pf}} \otimes_{K_{\mathrm{can},0}} (-)"] \\
\mathrm{Rep}_{\mathrm{st}} G_{K^{\mathrm{pf}}} \arrow[r, "\mathbb{D}_{\mathrm{st}}"] & MF_{K^{\mathrm{pf}}}(\varphi, N)
\end{tikzcd}$$

In fact, since $\mathbb{D}_{\mathrm{st}}(V|_{K^{\mathrm{pf}}}) = K_0^{\mathrm{pf}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}_{\mathrm{st}}^{\nabla}(V)$ is weakly admissible by [Fontaine 1994b, Proposition 5.4.2(i)], $\mathbb{D}_{\mathrm{st}}^{\nabla}(V)$ is weakly admissible by definition.

By functoriality, the canonical map $i : K_0^{\mathrm{pf}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}_{\mathrm{st}}^{\nabla}(V) \to \mathbb{D}_{\mathrm{st}}(V|_{K^{\mathrm{pf}}})$ is a morphism of filtered (φ, N)-modules relative to K_0^{pf}. Consider the associated graded homomorphism after applying $K^{\mathrm{pf}} \otimes_{K_0^{\mathrm{pf}}}$. The resulting homomorphism coincides with the canonical map $K^{\mathrm{pf}} \otimes_K \mathbb{D}_{\mathrm{HT}}^{\nabla}(V) \to \mathbb{D}_{\mathrm{HT}}(V|_{K^{\mathrm{pf}}})$. Since $V \in \mathrm{Rep}_{\mathrm{HT}}^{\nabla} G_K$ by Remark 7.2, a Hodge–Tate decomposition $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n \in \mathbb{N}} \mathbb{C}_p(n)^{m_n}$ of V induces a Hodge–Tate decomposition of $V|_{K^{\mathrm{pf}}}$. In particular, $V|_{K^{\mathrm{pf}}}$ is also Hodge–Tate and the above canonical map is an isomorphism. Since the filtrations of $\mathbb{D}_{\mathrm{st}}^{\nabla}(V)$ and $\mathbb{D}_{\mathrm{st}}(V|_{K^{\mathrm{pf}}})$ are separated and exhaustive, i is an isomorphism as filtered (φ, N)-modules relative to K_0^{pf}.

We prove the full faithfulness. We have the fundamental exact sequence

$$0 \longrightarrow \mathbb{Q}_p \xrightarrow{\text{inc.}} (\mathbb{B}_{\mathrm{cris}}^{\nabla})^{\varphi=1} \xrightarrow{\text{can.}} \mathbb{B}_{\mathrm{dR}}^{\nabla}/\mathbb{B}_{\mathrm{dR}}^{\nabla+} \longrightarrow 0.$$

Indeed, the exactness is reduced to [Colmez and Fontaine 2000, Proposition 9.25] by identifying $\mathbb{B}_{\mathrm{cris}}^{\nabla}$ (resp. $\mathbb{B}_{\mathrm{dR}}^{\nabla+}, \mathbb{B}_{\mathrm{dR}}^{\nabla}$) with $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{pf}}}$ (resp. $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}, \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$). By the fundamental exact sequence, we have $\mathbb{V}_{\mathrm{st}} \circ \mathbb{D}_{\mathrm{st}}^{\nabla}(V) = V$ (resp. $\mathbb{V}_{\mathrm{cris}} \circ \mathbb{D}_{\mathrm{cris}}^{\nabla}(V) = V$) for $V \in \mathrm{Rep}_{\mathrm{st}}^{\nabla} G_K$ (resp. $V \in \mathrm{Rep}_{\mathrm{cris}}^{\nabla} G_K$). This implies the full faithfulness. $\square$

In Proposition 7.5(ii), we will prove that the above functors induce equivalences of categories, that is, are essentially surjective.

7B. ***A horizontal analogue of the p-adic monodromy theorem.*** The following is a horizontal analogue of the p-adic monodromy theorem. Note that the converse is true by Hilbert 90 and Corollary 4.3.

Theorem 7.4. *Let $V \in \mathrm{Rep}_{\mathrm{dR}}^{\nabla} G_K$. Then, there exists a finite extension K'/K_{can} such that $V|_{KK'}$ is horizontal semistable.*

Proof. First, the comparison isomorphism $\alpha^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}$ induces an isomorphism of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules

$$\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{K_{\mathrm{can}}} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V.$$

By taking $H^0(G_K, -)$, we have $\dim_K \mathbb{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$ by Corollary 4.3, which implies $V \in \mathrm{Rep}_{\mathrm{dR}} G_K$ by Lemma 1.19. Hence, there exists a finite extension L/K such that $V|_L$ is semistable by the Main Theorem. We may assume that L/K is a finite Galois extension satisfying condition (H) by the proof of the Main Theorem (Step 1) and Epp's Theorem 1.6. The extension $L_{\mathrm{can}}/K_{\mathrm{can}}$ is finite Galois by Lemma 1.5(ii). We will prove the assertion for $K' = L_{\mathrm{can}}$.

We have canonical isomorphisms

$$L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}_{\mathrm{st}}(V|_L) \cong L \otimes_{L_0} \mathbb{D}_{\mathrm{st}}(V|_L) \cong \mathbb{D}_{\mathrm{dR}}(V|_L),$$

where the first one is induced by a canonical isomorphism $L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} L_0 \to L$ (Remark 1.4(ii)), the second one follows by using Lemma 1.20 and Proposition 3.16. Moreover, these maps are compatible with the residual $G_{L/K}$-actions and the ∇-actions. By taking the horizontal sections, we have

$$\begin{aligned}\mathbb{D}^{\nabla}_{\mathrm{dR}}(V|_L) &\cong \mathbb{D}_{\mathrm{dR}}(V|_L)^{\nabla=0} \cong (L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}_{\mathrm{st}}(V|_L))^{\nabla=0} \\ &\cong L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}_{\mathrm{st}}(V|_L)^{\nabla=0} \cong L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V|_L),\end{aligned}$$

where the third equality follows from the fact $\nabla|_{L_{\mathrm{can}}} = 0$. By taking $G_{L/K\cdot L_{\mathrm{can}}}$-invariants, we have $\mathbb{D}^{\nabla}_{\mathrm{dR}}(V|_{K\cdot L_{\mathrm{can}}}) = L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V|_{K\cdot L_{\mathrm{can}}})$. Since $V|_{K\cdot L_{\mathrm{can}}}$ is horizontal de Rham by Remark 1.22 and since $(K\cdot L_{\mathrm{can}})_{\mathrm{can}} = L_{\mathrm{can}}$ by Lemma 1.5(iv), we have

$$\dim_{L_{\mathrm{can}}} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V|_{K\cdot L_{\mathrm{can}}}) = \dim_{\mathbb{Q}_p} V = \dim_{L_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V|_{K\cdot L_{\mathrm{can}}}),$$

which implies that $V|_{K\cdot L_{\mathrm{can}}}$ is horizontal semistable. □

7C. *Equivalences of categories.* The surjection of profinite groups $\iota^* : G_K \to G_{K_{\mathrm{can}}}$ induces a $\otimes$-functor of Tannakian categories

$$\iota^* : \mathrm{Rep}_{\mathbb{Q}_p} G_{K_{\mathrm{can}}} \to \mathrm{Rep}_{\mathbb{Q}_p} G_K.$$

Obviously, the functor ι^* is fully faithful. Denote by $\boldsymbol{C_p}$ the p-adic completion of the algebraic closure of K_{can} in $\overline{K}$. For $\bullet \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$, we have a Galois equivariant canonical injection $\mathbb{B}_{\bullet,\boldsymbol{C_p}/K_{\mathrm{can}}} \to \mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K}$ by functoriality and we have $(\mathbb{B}_{\bullet,\boldsymbol{C_p}/K_{\mathrm{can}}})^{G_{K_{\mathrm{can}}}} \cong (\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K})^{G_K}$ ($= K_{\mathrm{can}}$ if $\bullet = \mathrm{dR}$, $K_{\mathrm{can},0}$ otherwise) by Proposition 3.16. Hence, if we have $V \in \mathrm{Rep}_{\bullet} G_{K_{\mathrm{can}}}$, then we have $\iota^* V \in \mathrm{Rep}^{\nabla}_{\bullet} G_K$. In fact, we have a canonical injection $\mathbb{D}_{\bullet}(V) \subset \mathbb{D}^{\nabla}_{\bullet}(\iota^* V)$ of $(\mathbb{B}_{\bullet,\boldsymbol{C_p}/K_{\mathrm{can}}})^{G_{K_{\mathrm{can}}}}$-vector spaces, which implies the $\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K}$-admissibility of $\iota^* V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$ by

Lemma 1.19. Hence, ι^* induces a fully faithful $\otimes$-functor

$$\iota^*_\bullet : \mathrm{Rep}_\bullet G_{K_{\mathrm{can}}} \to \mathrm{Rep}^\nabla_\bullet G_K.$$

The following proposition is a direct consequence of théorème 4.3 in [Colmez and Fontaine 2000].

Proposition 7.5 (horizontal analogue of Colmez–Fontaine). (i) *The functors ι^*_{cris} and ι^*_{st} are essentially surjective. In particular, ι^*_{cris} and ι^*_{st} induce equivalences of Tannakian categories.*

(ii) *The functors*

$$\mathbb{D}^\nabla_{\mathrm{cris}} : \mathrm{Rep}^\nabla_{\mathrm{cris}} G_K \to MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi), \quad \mathbb{D}^\nabla_{\mathrm{st}} : \mathrm{Rep}^\nabla_{\mathrm{st}} G_K \to MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N)$$

induce equivalences of categories with quasi-inverses $\mathbb{V}_{\mathrm{cris}}$, $\mathbb{V}_{\mathrm{st}}$.

Proof. We first prove the assertion in the semistable case. Together with the full faithfulness of $\mathbb{D}^\nabla_{\mathrm{st}}$, we have only to prove the commutativity of the diagram

$$\begin{tikzcd} \mathrm{Rep}_{\mathrm{st}} G_{K_{\mathrm{can}}} \arrow[r, "\iota^*_{\mathrm{st}}"] \arrow[d, "\mathbb{D}_{\mathrm{st}}", "\cong"'] & \mathrm{Rep}^\nabla_{\mathrm{st}} G_K \arrow[d, "\mathbb{D}^\nabla_{\mathrm{st}}"] \\ MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N) \arrow[r, "\mathrm{id}"] & MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N), \end{tikzcd}$$

where $\mathbb{D}_{\mathrm{st}}$ is an equivalence of categories by Colmez–Fontaine theorem [2000, Théorème 4.3]. As we mentioned above, the canonical map $\mathbb{D}_{\mathrm{st}}(V) \to \mathbb{D}^\nabla_{\mathrm{st}}(\iota^* V)$, which commutes with φ and N-actions, is an isomorphism of $K_{\mathrm{can},0}$-vector spaces. We have only to prove that the map also preserves the filtrations. Obviously, we have $\mathrm{Fil}^\bullet \mathbb{D}_{\mathrm{st}}(V) \subset \mathrm{Fil}^\bullet \mathbb{D}^\nabla_{\mathrm{st}}(\iota^* V)$. To prove the converse, it suffices to prove that the associated graded modules of both sides have the same dimension since the filtrations are exhaustive and separated. Let $\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n\in\mathbb{Z}} \boldsymbol{C}_p(n)^{m_n}$ be the Hodge–Tate decomposition of V. Then, it induces the Hodge–Tate decomposition of $\iota^* V$, that is, $\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)^{m_n}$, which implies the assertion.

In the horizontal crystalline case, a similar proof works by replacing $*_{\mathrm{st}}$ and $MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N)$ by $*_{\mathrm{cris}}$ and $MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi)$. ☐

Theorem 7.6. *The functor ι^*_{dR} is essentially surjective. In particular, ι^*_{dR} induces an equivalence of Tannakian categories.*

Proof. For a finite Galois extension L/K such that $K \cdot L_{\mathrm{can}} = L$, let $\mathscr{C}_{L/K}$ be the full subcategory of $\mathrm{Rep}^\nabla_{\mathrm{dR}} G_K$ whose objects consist of $V \in \mathrm{Rep}^\nabla_{\mathrm{dR}} G_K$ such that $V|_L$ is horizontal semistable. Recall the notation in Defintion 7.1. Then, we have an equivalence of categories

$$\mathbb{D}^\nabla_{\mathrm{st},L} : \mathscr{C}_{L/K} \to MF^{\mathrm{wa}}(\varphi, N, G_{L/K}); \quad V \mapsto \mathbb{D}^\nabla_{\mathrm{st}}(V|_L).$$

In fact, we have the following quasi-inverse $\mathbb{V}_{\mathrm{st},L}$: For $D \in MF^{\mathrm{wa}}(\varphi, N, G_{L/K})$, we regard D as an object of $MF^{\mathrm{wa}}_{L_{\mathrm{can}}}(\varphi, N)$ and let $\mathbb{V}_{\mathrm{st},L}(D) := \mathbb{V}_{\mathrm{st}}(D)$. We have $\mathbb{V}_{\mathrm{st},L}(D) \in \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_L$ by Proposition 7.5(ii) and $\mathbb{V}_{\mathrm{st},L}(D)$ has a canonical G_K-action, which is an extension of the action of G_L, induced by the $G_{L/K}$-action on D. We have $D \in \mathscr{C}_{L/K}$ by Remark 4.8 and Remark 7.2. We have $\mathbb{V}_{\mathrm{st},L} \circ \mathbb{D}^{\nabla}_{\mathrm{st},L} \cong \mathrm{id}_{\mathscr{C}_{L/K}}$ and $\mathbb{D}^{\nabla}_{\mathrm{st},L} \circ \mathbb{V}_{\mathrm{st},L} \cong \mathrm{id}_{MF^{\mathrm{wa}}(\varphi,N,G_{L/K})}$ by Proposition 7.5(ii).

The restriction map $\mathrm{Res}^{L}_{L_{\mathrm{can}}} : G_{L/K} \xrightarrow{\cong} G_{L_{\mathrm{can}}/K_{\mathrm{can}}}$ induces the equivalence of categories

$$(\mathrm{Res}^{L}_{L_{\mathrm{can}}})^* : MF^{\mathrm{wa}}(\varphi, N, G_{L_{\mathrm{can}}/K_{\mathrm{can}}}) \xrightarrow{\cong} MF^{\mathrm{wa}}(\varphi, N, G_{L/K}).$$

We will prove that the diagram

$$\begin{tikzcd}
MF^{\mathrm{wa}}(\varphi, N, G_{L_{\mathrm{can}}/K_{\mathrm{can}}}) \arrow[r, "(\mathrm{Res}^{L}_{L_{\mathrm{can}}})^*", "\cong"'] \arrow[d, "\mathbb{V}_{\mathrm{st},L_{\mathrm{can}}}", "\cong"'] & MF^{\mathrm{wa}}(\varphi, N, G_{L/K}) \arrow[d, "\mathbb{V}^{\nabla}_{\mathrm{st},L}", "\cong"'] \\
\mathscr{C}_{L_{\mathrm{can}}/K_{\mathrm{can}}} \arrow[r, "\iota^*_{\mathrm{dR}}"] & \mathscr{C}_{L/K}
\end{tikzcd}$$

is commutative, where the bottom horizontal arrow is induced by $\iota^*_{\mathrm{dR}} : \mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}} \to \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. Indeed, we have the G_K-equivariant inclusion

$$\iota^*_{\mathrm{dR}} \circ \mathbb{V}_{\mathrm{st},L_{\mathrm{can}}}(D) \subset \mathbb{V}^{\nabla}_{\mathrm{st},L} \circ (\mathrm{Res}^{L}_{L_{\mathrm{can}}})^*(D)$$

for $D \in MF^{\mathrm{wa}}(\varphi, N, G_{L_{\mathrm{can}}/K_{\mathrm{can}}})$ by construction. Since both sides have the same dimension over $\mathbb{Q}_p$, this inclusion is an equality. By the commutative diagram, the functor $\iota^*_{\mathrm{dR}} : \mathscr{C}_{L_{\mathrm{can}}/K_{\mathrm{can}}} \to \mathscr{C}_{L/K}$ is essentially surjective.

Let $V \in \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. By Theorem 7.4, we have a finite Galois extension K'/K_{can} such that $V|_{G_{KK'}}$ is horizontal semistable. Let $L := KK'$. By Lemma 1.5(iv), we have $L_{\mathrm{can}} = K'$, that is, L/K satisfies the above assumption. Since we have $V \in \mathscr{C}_{L/K}$, the assertion follows from the essential surjectivity of

$$\iota^*_{\mathrm{dR}} : \mathscr{C}_{L_{\mathrm{can}}/K_{\mathrm{can}}} \to \mathscr{C}_{L/K}. \qquad \square$$

The above equivalence induces a $\mathbb{Q}_p$-linear isomorphism of Ext^1 on $\mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}}$ and $\mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. Note that for $V \in \mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}}$, we may regard $\mathrm{Ext}^1_{\mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}}}(\mathbb{Q}_p, V)$ and $\mathrm{Ext}^1_{\mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K}(\mathbb{Q}_p, \iota^* V)$ as

$$H^1_g(G_{K_{\mathrm{can}}}, V) := \ker\left(H^1(G_K, V) \xrightarrow{(1\otimes\mathrm{id})_*} H^1(G_K, \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K_{\mathrm{can}}} \otimes_{\mathbb{Q}_p} V)\right),$$

$$H^{1,\nabla}_g(G_K, \iota^* V) := \ker\left(H^1(G_K, \iota^* V) \xrightarrow{(1\otimes\mathrm{id})_*} H^1(G_K, \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} \iota^* V)\right)$$

respectively. In particular:

Corollary 7.7. *For $V \in \mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}}$, the inflation map*

$$\mathrm{Inf} : H^1(G_{K_{\mathrm{can}}}, V) \to H^1(G_K, \iota^* V)$$

induces the isomorphism

$$\mathrm{Inf} : H^1_g(G_{K_{\mathrm{can}}}, V) \cong H^{1,\nabla}_g(G_K, \iota^* V).$$

7D. ***A comparison theorem on H^1.*** Notation is as in the previous subsection.

Theorem 7.8 (a generalization of Theorem 1.16). *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_{K_{\mathrm{can}}}$ be a de Rham representation whose Hodge–Tate weights are greater than or equal to 1. Then, we have the exact sequence*

$$0 \to H^1(G_{K_{\mathrm{can}}}, V) \xrightarrow{\mathrm{Inf}} H^1(G_K, \iota^* V) \xrightarrow{(1\otimes \mathrm{id})_*} H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V) \quad (7)$$

and a canonical isomorphism

$$(\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V(-1))^{G_{K_{\mathrm{can}}}} \otimes_{K_{\mathrm{can}}} \widehat{\Omega}^1_K \cong H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V). \quad (8)$$

Moreover, if the Hodge–Tate weights of V are greater than or equal to 2, then $H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V)$ vanishes, in particular, the inflation map

$$\mathrm{Inf} : H^1(G_{K_{\mathrm{can}}}, V) \to H^1(G_K, \iota^* V)$$

is an isomorphism.

Proof. We first prove the exactness of (7). Note that the injectivity of the inflation map follows by definition. We have the commutative diagram

$$\begin{tikzcd} H^1(G_{K_{\mathrm{can}}}, V) \arrow[d, "(1\otimes\mathrm{id})_*"] \arrow[dr, "(1\otimes\mathrm{id})_*\circ\mathrm{Inf}"] & \\ H^1(G_{K_{\mathrm{can}}}, \boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V) \arrow[r, "\mathrm{Inf}"] & H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V). \end{tikzcd}$$

Since we have a Hodge–Tate decomposition $\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n\in\mathbb{N}_{\geq 1}} \boldsymbol{C}_p(n)^{m_n}$, we have $H^1(G_{K_{\mathrm{can}}}, \boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V) = 0$ by Theorem 1.15, which implies $(1\otimes\mathrm{id})_* \circ \mathrm{Inf} = 0$.

Let $\mathscr{H} := \ker\{(1\otimes\mathrm{id})_* : H^1(G_K, \iota^* V) \to H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V)\}$. We have only to prove $\mathscr{H}$ is contained in the image of $\mathrm{Inf} : H^1(G_{K_{\mathrm{can}}}, V) \to H^1(G_K, \iota^* V)$. Consider the exact sequence

$$0 \longrightarrow t\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \xrightarrow{\mathrm{inc.}} \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \xrightarrow{\theta} \mathbb{C}_p \longrightarrow 0$$

with $\theta := \theta_{\mathbb{C}_p/K}$. By applying $\otimes_{\mathbb{Q}_p} \iota^* V$ and taking $H^\bullet(G_K, -)$, we have the commutative diagram with exact row, where S stands for $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$:

$$\begin{tikzcd} & H^1(G_K, \iota^* V) \arrow[d, "(1\otimes\mathrm{id})_*"'] \arrow[dr, "(1\otimes\mathrm{id})_*"] & \\ H^1(G_K, tS\otimes_{\mathbb{Q}_p}\iota^* V) \arrow[r, "(\mathrm{inc.}\otimes\mathrm{id})_*"] & H^1(G_K, S\otimes_{\mathbb{Q}_p}\iota^* V) \arrow[r, "(\theta\otimes\mathrm{id})_*"] & H^1(G_K, \mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota^* V). \end{tikzcd}$$

Since $V(1)$ is de Rham with Hodge–Tate weights ≥ 2, we have

$$H^1(G_K, t\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{\mathbb{Q}_p}\iota^* V) = 0$$

by Theorem 1.15, Lemma 1.14 and dévissage. Hence, the canonical map

$$(1\otimes\mathrm{id})_* : \mathscr{H} \to H^1(G_K, \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{\mathbb{Q}_p}\iota^* V)$$

vanishes by the above exact sequence. In particular, we have $\mathscr{H}\subset H^{1,\nabla}_g(G_K, \iota^* V)$. By Corollary 7.7, we have $\mathrm{Inf}: H^1_g(G_{K_{\mathrm{can}}}, V)\cong H^{1,\nabla}_g(G_K, \iota^* V)$, which implies (7).

Then, we will prove the existence of the canonical isomorphism (8). By the inclusion $(\boldsymbol{C}_p\otimes_{\mathbb{Q}_p} V(-1))^{G_{K_{\mathrm{can}}}}\subset(\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota^* V(-1))^{G_K}$ and the canonical isomorphism $\widehat{\Omega}^1_K\to H^1(G_K,\mathbb{C}_p(1))$ in Theorem 1.15, we can define a canonical map f as the composite

$$\begin{aligned}\bigl(\boldsymbol{C}_p\otimes_{\mathbb{Q}_p} V(-1)\bigr)^{G_{K_{\mathrm{can}}}}\otimes_{K_{\mathrm{can}}}\widehat{\Omega}^1_K &\xrightarrow{\mathrm{inc.}\otimes\mathrm{can.}} \\ &(\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota^* V(-1))^{G_K}\otimes_K H^1(G_K,\mathbb{C}_p(1)) \xrightarrow{\mathrm{cup.}} H^1(G_K,\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota^* V).\end{aligned}$$

We will prove that f is an isomorphism. A Hodge–Tate decomposition of V induces a Hodge–Tate decomposition $\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota^* V\cong\bigoplus_{n\in\mathbb{N}_{\geq 1}}\mathbb{C}_p(n)^{m_n}$ of $\iota^* V$. By replacing $\boldsymbol{C}_p\otimes_{\mathbb{Q}_p} V$ and $\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota^* V$ by their Hodge–Tate decompositions, we may reduce to the case $V=\mathbb{Q}_p(n)$ with $n\in\mathbb{N}_{\geq 1}$ since the cup product commutes with direct sums. Then, the assertion follows from Theorem 1.15.

We will prove the last assertion. The assumption implies that we have $m_1 = 0$ in the above notation, hence, we have $H^1(G_K,\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota^* V)=0$ by the Hodge–Tate decomposition of $\iota^* V$ and Theorem 1.15. □

Remark 7.9. (i) Originally, Theorem 1.16(i) and (ii) are proved separately by using ramification theory in some sense.

(ii) (Finiteness) Suppose that we have $[K_{\mathrm{can}}:\mathbb{Q}_p]<\infty$. For example, consider the case that K has a structure of a higher-dimensional local field (Example 1.7). Let $V\in\mathrm{Rep}_{\mathbb{Q}_p} G_K$ be horizontal de Rham of Hodge–Tate weights greater than or equal to 2. Then we have

$$\dim_{\mathbb{Q}_p} H^1(G_K, V) = [K_{\mathrm{can}}:\mathbb{Q}_p]\dim_{\mathbb{Q}_p} V<\infty.$$

Indeed, by Theorem 7.6 and 7.8, we may reduce to the case $K = K_{\mathrm{can}}$. By a Hodge–Tate decomposition $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n\in\mathbb{N}_{\geq 2}} \mathbb{C}_p(n)^{m_n}$ with $m_n \in \mathbb{N}$, we have $H^0(G_K, V) \subset H^0(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V) = 0$ and $H^2(G_K, V) \cong H^0(G_K, V^\vee(1)) \subset H^0(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V^\vee(1)) = 0$ by the local Tate duality [Herr 1998, Théorème in Introduction], where $^\vee$ denotes the dual. Then, the assertion follows from the Euler–Poincaré characteristic formula (loc. cit).

Note that $H^1(G_K, V)$ is not finite over $\mathbb{Q}_p$ without the condition on Hodge–Tate weights: For example, $H^1(G_K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n K^\times/(K^\times)^{p^n}$ contains $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_K$, which is infinite-dimensional over $\mathbb{Q}_p$ if k_K is imperfect, via the map $\mathcal{O}_K \hookrightarrow U_K^{(1)}$ that takes x to $\exp(2px)$.

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shuno@ms.u-tokyo.ac.jp *Department of Mathematical Sciences, University of Tokyo, Tokyo 153-8914, Japan*

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On the Manin–Mumford and Mordell–Lang conjectures in positive characteristic

Damian Rössler

We prove that in positive characteristic, the Manin–Mumford conjecture implies the Mordell–Lang conjecture in the situation where the ambient variety is an abelian variety defined over the function field of a smooth curve over a finite field and the relevant group is a finitely generated group. In particular, in the setting of the last sentence, we provide a proof of the Mordell–Lang conjecture that does not depend on tools coming from model theory.

1. Introduction

Let B be a semiabelian variety over an algebraically closed field F of characteristic $p > 0$. Let Y be an irreducible reduced closed subscheme of B. Let $\Lambda \subseteq B(F)$ be a subgroup. Suppose that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module (here, as is customary, we write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at the prime p).

Let $C := \mathrm{Stab}(Y)^{\mathrm{red}}$, where $\mathrm{Stab}(Y) = \mathrm{Stab}_B(Y)$ is the translation stabilizer of Y. This is the closed subgroup scheme of B that is characterized uniquely by the fact that for any scheme S and any morphism $b : S \to B$, translation by b on the product $B \times_F S$ maps the subscheme $Y \times S$ to itself if and only if b factors through $\mathrm{Stab}_B(Y)$. Its existence is proven in [Grothendieck et al. 1970b, Exposé VIII, Exemples 6.5(e)].

The Mordell–Lang conjecture for Y and B is now the following statement:

Theorem 1.1 (Mordell–Lang conjecture [Hrushovski 1996]). *If $Y \cap \Lambda$ is Zariski dense in Y, then there are*

- *a semiabelian variety B' over F,*
- *a homomorphism with finite kernel $h : B' \to B/C$,*
- *a model $\mathbf{B}'$ of B' over a finite subfield $\mathbb{F}_{p^r} \subset F$,*
- *an irreducible reduced closed subscheme $\mathbf{Y}' \hookrightarrow \mathbf{B}'$, and*
- *a point $b \in (B/C)(F)$ such that $Y/C = b + h_*(\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F)$.*

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Here $h_*(\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F)$ refers to the scheme-theoretic image of $\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F$ by h. Since h is finite and $\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F$ is reduced, this implies that $h_*(\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F)$ is simply the set-theoretic image of $\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F$ by h endowed with its reduced-induced scheme structure.

Theorem 1.1 in particular implies the following result, which will perhaps seem more striking on first reading. Suppose that there are no nontrivial homomorphisms from B to a semiabelian variety that has a model over a finite field. Then if $Y \cap \Lambda$ is Zariski dense in Y, then Y is the translate of an abelian subvariety of B.

Theorem 1.1 was first proven in 1996 by Hrushovski using deep results from model theory, in particular the Hrushovski–Zilber theory of Zariski geometries (see [Hrushovski and Zilber 1996]). An algebraic proof of Theorem 1.1 in the situation where B is an ordinary abelian variety was given by Abramovich and Voloch [1992]. In the situation where Y is a smooth curve embedded into B as its Jacobian, the theorem was known to be true much earlier. See for instance [Samuel 1966; Szpiro et al. 1981]. The earlier proofs for curves relied on the use of heights, which do not appear in the later approach of Voloch and Hrushovski, which is parallel and inspired by Buium's approach in characteristic 0 via differential equations (see below).

The *Manin–Mumford conjecture* has exactly the same form as the Mordell–Lang conjecture, but Λ is replaced by the group $\mathrm{Tor}(B(F))$ of points of finite order of $B(F)$. For the record, we state it in full.

Theorem 1.2 (Manin–Mumford conjecture [Pink and Rössler 2004]). *Suppose $Y \cap \mathrm{Tor}(B(F))$ is Zariski dense in Y. Then there are*

- *a semiabelian variety B' over F,*
- *a homomorphism with finite kernel $h : B' \to B/C$,*
- *a model $\mathbf{B}'$ of B' over a finite subfield $\mathbb{F}_{p^r} \subset F$,*
- *an irreducible reduced closed subscheme $\mathbf{Y}' \hookrightarrow \mathbf{B}'$, and*
- *a point $b \in (B/C)(F)$ such that $Y/C = b + h_*(\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F)$.*

See also [Scanlon 2005] for a model-theoretic proof of the Manin–Mumford conjecture.

Remark (Important). Notice that the Manin–Mumford conjecture is *not* a special case of the Mordell–Lang conjecture because $\mathrm{Tor}(A(F))$ is not in general a finitely generated $\mathbb{Z}_{(p)}$-module (because $\mathrm{Tor}(A(F))[p^\infty]$ is not finite in general). Nevertheless, it seems reasonable to conjecture that Theorem 1.1 should still be true when the hypothesis that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is finitely generated is replaced by the weaker hypothesis that $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is finitely generated. This last statement, which is still not proven in general, is often called the *full Mordell–Lang conjecture*, and it would have Theorems 1.1 and 1.2 as special cases. See [Ghioca and Moosa 2006] for more about this.

Now suppose that the group Λ is actually finitely generated and that B arises by base-change to F from an abelian variety B_0, which is defined over a function field of transcendence degree 1 over a finite field. The main result of this text is then the proof of the fact that the Manin–Mumford conjecture in general implies the Mordell–Lang conjecture in this situation. We follow here the lead of A. Pillay, who suggested in a talk he gave in Paris on December 17, 2010 that it should be possible to establish this logical link without proving the Mordell–Lang conjecture first. See Theorem 1.3 and its corollary below for a precise statement.

The interest of an algebraic-geometric (in contrast with model-theoretic) proof of the implication Manin–Mumford $\Longrightarrow$ Mordell–Lang is that it provides in particular an algebraic-geometric proof of the Mordell–Lang conjecture.

Let K_0 be the function field of a smooth curve over $\overline{\mathbb{F}}_p$. Let A be an abelian variety over K_0, and let $X \hookrightarrow A$ be a closed integral subscheme. We shall write $+$ for the group law on A.

Let $\Gamma \subseteq A(K_0)$ be a finitely generated subgroup.

Theorem 1.3. *Suppose that for any field extension $L_0|K_0$ and any $Q \in A(L_0)$, the set $X_{L_0}^{+Q} \cap \mathrm{Tor}(A(L_0))$ is not Zariski dense in $X_{L_0}^{+Q}$. Then $X \cap \Gamma$ is not Zariski dense in X.*

Here $X_{L_0}^{+Q}$ stands for the scheme-theoretic image of X_{L_0} under the morphism $+Q : A_{L_0} \to A_{L_0}$.

Corollary 1.4. *Suppose that $X \cap \Gamma$ is Zariski dense in X. Then the conclusion of the Mordell–Lang conjecture (Theorem 1.1) holds for $F = \overline{K}_0$, $B = A_{\overline{K}_0}$, and $Y = X_{\overline{K}_0}$.*

In an upcoming article, which builds on the present one, Corpet [2012] shows that Theorem 1.3 (and thus its corollary) can be generalized; more specifically, he shows that the hypothesis that K_0 is of transcendence degree 1 can be dropped, that the hypothesis that Γ is finitely generated can be weakened to the hypothesis that $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module, and finally that it can be assumed that A is only semiabelian. In particular, he gives a new proof of Theorem 1.1.

In the present article, we deliberately focus on the situation of an abelian variety and a finitely generated group (which is probably the most important situation) in order to avoid some technical issues, which we feel would obscure the structure of the proof.

The structure of the article is the following. Section 2 contains some general results on the geometry of relative jet schemes (or spaces), which are probably known to many specialists but for which there doesn't seem to be a coherent set of references in the literature. The jet spaces considered in [Moosa and Scanlon 2010] do not seem to suffice for our purposes because they are defined in an absolute situation, and the jet spaces considered in [Buium 1992] are only defined in characteristic 0 (although this is probably not an essential restriction); furthermore, the latter are defined in Buium's language of differential schemes whereas our definition has the philological advantage of being based on the older notion of Weil restriction. Section 2A contains the definition of jet schemes and a description of the various torsor structures on the latter. Section 2B contains a short discussion on the structure of the jet schemes of smooth commutative group schemes and various natural maps that are associated with them. In Section 3, we use jet schemes to construct some natural schemes in the geometrical context of the Mordell–Lang conjecture. These "critical schemes" are devised to "catch rational points"; we then proceed to show that these schemes must be of small dimension. This is deduced from a general result on the sparsity of points over finite fields that are liftable to highly p-divisible unramified points. This last result is proved in Section 4. Once we know that the critical schemes are small, it is but a small step to the proofs of Theorem 1.3 and Corollary 1.4. The terminology of the introduction is used in Sections 2 and 3, but Section 4 has its own terminology and is also technically independent of the rest of the text. A reader who would only be interested in its main result (i.e., Theorem 4.1) can skip to Section 4 directly.

The use that we make of jet schemes in this note is in many ways similar to the use that Buium [1992] makes of them in his article on the geometric Mordell–Lang conjecture in characteristic 0. In the article [Buium and Voloch 1996], where some of Buium's techniques are adapted to the context of positive characteristic, the authors give a proof of the Mordell conjecture for curves over function fields in positive characteristic, which has exactly the same structure as ours if one leaves out the proof of the result on the sparsity of liftable points mentioned above.

For more detailed explanations on this connection, see Remarks 4.8 and 4.9 at the end of the text.

2. Preliminaries

We first recall the definition and existence theorem for the Weil restriction functor. Let T be a scheme, and let $T' \to T$ be a morphism. Let Z be a scheme over T'. The Weil restriction $\mathfrak{R}_{T'/T}(Z)$ (if it exists) is a T-scheme that represents the functor $W/T \mapsto \mathrm{Hom}_{T'}(W \times_T T', Z)$. It is shown in [Bosch et al. 1990, Section 7.6] that $\mathfrak{R}_{T'/T}(Z)$ exists if T' is finite, flat, and locally of finite presentation over T. The Weil restriction is naturally functorial in Z and sends closed immersions to closed immersions. The same permanence property is satisfied for smooth and étale morphisms. Finally notice that the definition of the Weil restriction implies that there is a natural isomorphism $\mathfrak{R}_{T'/T}(Z)_{T_1} \simeq \mathfrak{R}_{T'_{T_1}/T_1}(Z_{T'_{T_1}})$ for any scheme T_1 over T (in words, Weil restriction is invariant under base-change on T). See [Bosch et al. 1990, Chapter 7.6] for all this.

2A. ***Jet schemes.*** Let k_0 be a field, let U be a smooth scheme over k_0, and let $\Delta : U \to U \times_{k_0} U$ be the diagonal immersion. Let $I_\Delta \subseteq \mathcal{O}_{U\times_{k_0} U}$ be the ideal sheaf of $\Delta_* U$. For all $n \in \mathbb{N}$, we let $U_n := \mathcal{O}_{U\times_{k_0} U}/I_\Delta^{n+1}$ be the n-th infinitesimal neighborhood of the diagonal in $U \times_{k_0} U$.

Write $\pi_1, \pi_2 : U \times_{k_0} U \to U$ for the first and second projection morphisms, respectively. Write $\pi_1^{U_n}, \pi_2^{U_n} : U_n \to U$ for the induced morphisms. We view U_n as a U-scheme via the *first* projection $\pi_1^{U_n}$.

We write $i_{m,n} : U_m \hookrightarrow U_n$ for the natural inclusion morphism.

Lemma 2.1. *The U-scheme U_n is flat and finite.*

Proof. As a U-scheme, U_n is finite because it is quasifinite and proper over U since $U_n^{\mathrm{red}} = \Delta_*(U)$. So we only have to prove that it is flat over U. For this purpose, we may view U_n as a coherent sheaf of $\mathcal{O}_U$-algebras (via the second projection).

Let $I := I_\Delta$. For any $n \geqslant 0$, there are exact sequences of $\mathcal{O}_{U_{n+1}}$-modules (and hence $\mathcal{O}_U$-modules)

$$0 \to I^{n+1}/I^{n+2} \to \mathcal{O}_{U_{n+1}} \to \mathcal{O}_{U_n} \to 0.$$

Furthermore, I^{n+1}/I^{n+2} is naturally a $\mathcal{O}_{U_0}$-module and isomorphic to $\mathrm{Sym}^{n+1}_{\mathcal{O}_{U_1}}(I/I^2)$ as a $\mathcal{O}_{U_0}$-module because I is locally generated by a regular sequence in $U \times_{k_0} U$ (U being smooth over k_0). See [Matsumura 1989, Chapters 6 and 16] for this. Hence, I^{n+1}/I^{n+2} is locally free as a $\mathcal{O}_U$-module. Since $U_0 = \Delta_*(U)$ is locally free as a $\mathcal{O}_U$-module, we may apply induction on n to prove that $\mathcal{O}_{U_n}$ is locally free, which is the claim. $\square$

Let W/U be a scheme over U.

Definition 2.2. The n-th jet scheme $J^n(W/U)$ of W over U is the U-scheme $\mathfrak{R}_{U_n/U}(\pi_2^{U_n,*} W)$.

By $\pi_2^{U_n,*} W$ we mean the base-change of W to U_n via the morphism $\pi_2^{U_n} : U_n \to U$ described above.

If W_1 is another scheme over U and $W \to W_1$ is a morphism of U-schemes, then the induced morphism $\pi_2^{U_n,*} W \to \pi_2^{U_n,*} W_1$ over U_n leads to a morphism of jet schemes $J^n(W/U) \to J^n(W_1/U)$ over U so that the construction of jet schemes is covariantly functorial in W.

Notice that the permanence properties of Weil restrictions show that if the morphism $W \to W_1$ is a closed immersion, then the morphism $J^n(W/U) \to J^n(W_1/U)$ is too. The same is true for smooth and étale morphisms.

To understand the nature of jet schemes better, let $u \in U$ be a closed point. Suppose until the end of this paragraph that k_0 is algebraically closed. View u as a closed reduced subscheme of U. Let u_n be the n-th infinitesimal neighborhood of u in U. From the definitions, we infer that there are canonical bijections

$$\begin{aligned} J^n(W/U)(u) &= J^n(W/U)_u(k_0) = \mathrm{Hom}_{U_n}(u \times_U U_n, \pi_2^{U_n,*}W) \\ &= \mathrm{Hom}_{U_n}(u_n, W_{u_n}) = \mathrm{Hom}_{u_n}(u_n, W_{u_n}) = W(u_n). \end{aligned} \tag{1}$$

In words, (1) says the set of geometric points of the fiber of $J^n(W/U)$ over u corresponds to the set of sections of W over the n-th infinitesimal neighborhood of u; the scheme $J^n(W/U)_u$ is often called the scheme of arcs of order n at u in the literature [Moosa and Scanlon 2010, Example 2.5].

The family of U-morphisms $i_{m,n} : U_m \to U_n$ induce U-morphisms

$$\Lambda^W_{n,m} : J^n(W/U) \to J^m(W/U)$$

for any $m \leqslant n$. These morphisms will be studied in detail in the proof of the next lemma.

Lemma 2.3. *Suppose that W is a smooth U-scheme. For all $n \geqslant 1$, the morphism*

$$\mathfrak{R}_{U_n/U}(\pi_2^{U_n,*}W) \to \mathfrak{R}_{U_{n-1}/U}(\pi_2^{U_{n-1},*}W)$$

makes $\mathfrak{R}_{U_n/U}(\pi_2^{U_n,*}W)$ into a $\mathfrak{R}_{U_{n-1}/U}(\pi_2^{U_{n-1},*}W)$-torsor under the vector bundle $\Lambda^{W,*}_{n,0}(\Omega^\vee_{W/U}) \otimes \mathrm{Sym}^n(\Omega_{U/k_0})$.

Proof. Let $T \to U$ be an affine U-scheme. By definition,

$$\begin{aligned} \mathfrak{R}_{U_n/U}(\pi_2^{U_n,*}W)(T) &\simeq \mathrm{Hom}_{U_n}(T \times_U U_n, \pi_2^{U_n,*}W), \\ \mathfrak{R}_{U_{n-1}/U}(\pi_2^{U_{n-1},*}W)(T) &\simeq \mathrm{Hom}_{U_{n-1}}(T \times_U U_{n-1}, \pi_2^{U_{n-1},*}W). \end{aligned}$$

Now the immersion $U_{n-1} \hookrightarrow U_n$ gives rise to a natural restriction map

$$\mathrm{Hom}_{U_n}(T \times_U U_n, \pi_2^{U_n,*}W) \to \mathrm{Hom}_{U_{n-1}}(T \times_U U_{n-1}, \pi_2^{U_{n-1},*}W). \tag{2}$$

This is the functorial description of the morphism

$$\mathfrak{R}_{U_n/U}(\pi_2^{U_n,*}W) \to \mathfrak{R}_{U_{n-1}/U}(\pi_2^{U_{n-1},*}W).$$

Now notice that the ideal of U_{n-1} in U_n is a square-0 ideal.

Let $f \in \mathrm{Hom}_{U_{n-1}}(T \times_U U_{n-1}, \pi_2^{U_{n-1},*}W)$. View f as a U_n-morphism

$$T \times_U U_{n-1} \to \pi_2^{U_n,*}W$$

via the canonical closed immersions $\pi_2^{U_{n-1},*}W \hookrightarrow \pi_2^{U_n,*}W$ and $U_{n-1} \hookrightarrow U_n$. The fiber over f of the map (2) then consists of the extensions of f to U_n-morphisms $T \times_U U_n \to \pi_2^{U_n,*}W$. The theory of infinitesimal extensions of morphisms to smooth schemes (see [Grothendieck 1963, Exposé III, Proposition 5.1]) implies that this fiber is an affine space under the group

$$H^0(T \times_U U_{n-1}, f^*\Omega^\vee_{\pi_2^{U_n,*}W/U_n} \otimes N),$$

where N is the conormal bundle of the closed immersion $T \times_U U_{n-1} \hookrightarrow T \times_U U_n$. Since U_n and U_{n-1} are flat over U, the coherent sheaf N is the pullback to $T \times_U U_{n-1}$ of the conormal bundle of the immersion $U_{n-1} \hookrightarrow U_n$. Now since the diagonal is regularly immersed in $U \times_{k_0} U$ (because U is smooth over k_0), the conormal bundle of the immersion $U_{n-1} \hookrightarrow U_n$ is $\mathrm{Sym}^n(\Omega_{U/k_0})$ (viewed as a sheaf in $\mathscr{O}_{U_{n-1}}$-modules via the closed immersion $U_0 \to U_{n-1}$). See [Matsumura 1989, Chapters 6 and 16]. Hence,

$$\begin{aligned} H^0(T \times_U U_{n-1}, f^*\Omega^\vee_{\pi_2^{U_n,*}W/U_n} \otimes N) & \\ &\simeq H^0(T \times_U U_{n-1}, f^*\Omega^\vee_{\pi_2^{U_n,*}W/U_n} \otimes \mathrm{Sym}^n(\Omega_{U/k_0})) \\ &\simeq H^0(T, f_0^*\Omega^\vee_{W/U} \otimes \mathrm{Sym}^n(\Omega_{U/k_0})), \end{aligned}$$

where f_0 is the U-morphism $T \to W$ arising from f by base-change to U. □

2B. ***The jet schemes of smooth commutative group schemes.*** We keep the terminology of Section 2A. Let $\mathscr{C}/U$ be a commutative group scheme over U with zero-section $\epsilon : U \to \mathscr{C}$. If $n \in \mathbb{N}$, we shall write $[n]_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ for the multiplication-by-n morphism. The schemes $J^n(\mathscr{C}/U)$ are then naturally group schemes over U. Furthermore, for each $n \geqslant m \geqslant 0$, the morphism $\Lambda^{\mathscr{C}}_{n,m} : J^n(\mathscr{C}/U) \to J^m(\mathscr{C}/U)$ is a morphism of group schemes. If $m = n - 1$, the kernel of $\Lambda^{\mathscr{C}}_{n,m}$ is the vector bundle $\epsilon^*(\Omega^\vee_{\mathscr{C}/U}) \otimes \mathrm{Sym}^n(\Omega_{U/k_0})$. The torsor structure is realized by the natural action of $\epsilon^*(\Omega^\vee_{\mathscr{C}/U}) \otimes \mathrm{Sym}^n(\Omega_{U/k_0})$ on $J^n(\mathscr{C}/U)$. The details of the verification of these facts are left to the reader.

Lemma 2.4. *Let $n \geqslant 1$. Suppose that* $\mathrm{char}(k_0) = p$. *There is a U-morphism $[p^n]^\circ : \mathscr{C} \to J^n(\mathscr{C}/U)$ such that $\Lambda^{\mathscr{C}}_{n,0} \circ [p^n]^\circ = [p^n]_{\mathscr{C}}$ and $[p^n]_{J^n(\mathscr{C}/U)} = [p^n]^\circ \circ \Lambda^{\mathscr{C}}_{n,0}$.*

Proof. Let $T \to U$ be an affine U-scheme. Define a map

$$\phi_{T,n} : \mathrm{Hom}_U(T, \mathscr{C}) \to \mathrm{Hom}_{U_n}(T \times_U U_n, \pi_2^*\mathscr{C})$$

by the following recipe. Let $f \in \mathrm{Hom}_U(T, \mathscr{C})$, and take any extension $\tilde{f}$ of f to a morphism $T \times_U U_n \to (\pi_2^*\mathscr{C})_{U_n}$; then define $\phi_{T,n}(f) = p^n \cdot \tilde{f}$. To see that this does not depend on the choice of the extension $\tilde{f}$, notice that the kernel K_n of the restriction map

$$\mathrm{Hom}_{U_n}(T \times_U U_n, \pi_2^*\mathscr{C}) \to \mathrm{Hom}_U(T, \mathscr{C})$$

is obtained by successive extensions by the groups $H^0(T, f^*\Omega^\vee_{\mathscr{C}/U} \otimes \mathrm{Sym}^i(\Omega_{U/k_0}))$ for $i = 1, \ldots, n$ (see [Grothendieck 1963, Exposé III, Corollaire 5.3] for all this). Hence, K_n is annihilated by multiplication by p^n because T is a scheme of characteristic p.

The definition of $\phi_{T,n}$ is functorial in T, and thus, by patching the morphisms $\phi_{T,n}$ as T runs over the elements of an affine cover of $\mathscr{C}$, we obtain the required morphism $[p^n]^\circ$. □

Now notice that there is a canonical map $\lambda_n^W : W(U) \to J^n(W/U)(U)$ that sends the U-morphism $f : U \to W$ to $J^n(f) : J^n(U/U) = U \to J^n(W/U)$ for any scheme W over U.

Lemma 2.5. *The maps λ_n^W have the following properties*:

(a) *For $n \geqslant m \geqslant 0$, the identity $\Lambda_{n,m}^W \circ \lambda_n^W = \lambda_m^W$ holds.*

(b) *If W/U is commutative group scheme over U, then λ_n^W is a homomorphism*; *furthermore, on $W(U)$ we then have the identity $[p^n]_{J^n(W/U)} \circ \lambda_n^W = [p^n]^\circ$.*

(c) *If $f : W \to W_1$ is a U-scheme morphism, then $J^n(f) \circ \lambda_n^W = \lambda_n^{W_1} \circ f$.*

Proof. This is left as an exercise for the reader. □

Remark 2.6. An interesting feature of the map λ_n^W is that it does *not* arise from a morphism of schemes $W \to J^n(W/U)$.

3. Proof of Theorem 1.3 and Corollary 1.4

We now turn to the proof of our main result. We shall use the terminology of the preliminaries. Let $k_0 := \overline{\mathbb{F}}_p$, and suppose now that U is a smooth curve over k_0 whose function field is K_0. We take U sufficiently small so that X extends to a flat scheme $\mathscr{X}$ over U and so that A extends to an abelian scheme $\mathscr{A}$ over U. We also suppose that the closed immersion $X \hookrightarrow A$ extends to a closed immersion $\mathscr{X} \to \mathscr{A}$.

Recall that the following hypothesis is supposed to hold: for any field extension $L_0|K_0$ and any $Q \in A(L_0)$, the set $X_{L_0}^{+Q} \cap \mathrm{Tor}(A(L_0))$ is not Zariski dense in $X_{L_0}^{+Q}$.

3A. ***The critical schemes.*** For all $n \geqslant 0$, we define

$$\mathrm{Crit}^n(\mathscr{X}, \mathscr{A}) := [p^n]_*(J^n(\mathscr{A}/U)) \cap J^n(\mathscr{X}/U).$$

Here $[p^n]_*(J^n(\mathscr{A}/U))$ is the scheme-theoretic image of $J^n(\mathscr{A}/U)$ by $[p^n]_{J^n(\mathscr{A}/U)}$. Notice that by Lemma 2.4, we have $[p^n](J^n(\mathscr{A}/U)) = [p^n]^\circ(\mathscr{A})$, and since $[p^n]$ is proper (because $\mathscr{A}$ is proper over U), we see that $[p^n](J^n(\mathscr{A}/U))$ is closed and that the natural morphism $[p^n]_*(J^n(\mathscr{A}/U)) \to \mathscr{A}$ is finite.

The morphisms $\Lambda_{n,n-1}^{\mathscr{A}} : J^n(\mathscr{A}/U) \to J^{n-1}(\mathscr{A}/U)$ lead to a projective system of U-schemes

$$\cdots \to \mathrm{Crit}^2(\mathscr{X}, \mathscr{A}) \to \mathrm{Crit}^1(\mathscr{X}, \mathscr{A}) \to \mathscr{X}$$

whose connecting morphisms are finite. We let $\mathrm{Exc}^n(\mathscr{A}, \mathscr{X}) \hookrightarrow \mathscr{X}$ be the scheme-theoretic image of $\mathrm{Crit}^n(\mathscr{A}, \mathscr{X})$ in $\mathscr{X}$.

For any $Q \in \mathscr{A}(U) = A(K_0)$, we shall write $\mathscr{X}^{+Q} = \mathscr{X} + Q$ for the translation of $\mathscr{X}$ by Q in $\mathscr{A}$.

Proposition 3.1. *There exists $\alpha = \alpha(\mathcal{A}, \mathcal{X}) \in \mathbb{N}$ such that for all $Q \in \Gamma$, the set* $\mathrm{Exc}^{\alpha}(\mathcal{A}, \mathcal{X}^{+Q})$ *is not dense in* $\mathcal{X}^{+Q}$.

Remark 3.2. Proposition 3.1 should be compared to [Buium 1992, Theorem 1].

The following theorem, proved by Galois-theoretic methods in Section 4, will play a crucial role in the proof of Proposition 3.1.

Let $S := \operatorname{Spec} k_0[[t]]$. Let $L := k_0((t))$ be the function field of S. For any $n \in \mathbb{N}$, let $S_n := \operatorname{Spec} k_0[t]/t^{n+1}$ be the n-th infinitesimal neighborhood of the closed point of S in S. Fix $\lambda_0 \in \mathbb{N}^*$, and let $R^{\mathrm{alg}} = R^{\mathrm{alg},\lambda_0} := \mathbb{F}_{p^{\lambda_0}}[[t]] \subseteq k_0[[t]]$. Let $S^{\mathrm{alg}} = S^{\mathrm{alg},\lambda_0} := \operatorname{Spec} R^{\mathrm{alg}}$. There is an obvious morphism $S \to S^{\mathrm{alg}}$.

Let $\mathfrak{D}$ be an abelian scheme over S, and let $\mathfrak{X} \hookrightarrow \mathfrak{D}$ be a closed integral subscheme. Suppose that the abelian scheme has a model $\mathfrak{D}^{\mathrm{alg}}$ over S^{alg} as an abelian scheme and that the immersion $\mathfrak{X} \hookrightarrow \mathfrak{D}$ has a model $\mathfrak{X}^{\mathrm{alg}} \hookrightarrow \mathfrak{D}^{\mathrm{alg}}$ over S^{alg}. If $c \in \mathfrak{D}(S)$, write as usual $\mathfrak{X}^{+c} := \mathfrak{X} + c$ for the translation of $\mathfrak{X}$ by c in $\mathfrak{D}$. Let D_0 and D be the fibers of $\mathfrak{D}$ over the closed and generic points of S, respectively. If $c \in \mathfrak{D}(S)$, let Z_0^{+c} and Z^{+c} be the fibers of $\mathfrak{X}^{+c}$ over the closed and generic points of S, respectively.

Notice that there is a natural inclusion $\mathfrak{D}^{\mathrm{alg}}(S^{\mathrm{alg}}) \subseteq \mathfrak{D}(S)$.

Theorem 3.3. *Suppose that* $\mathrm{Tor}(D(\bar{L})) \cap X_{\bar{L}}^{+c}$ *is not dense in* $X_{\bar{L}}^{+c}$ *for all* $c \in \mathfrak{D}^{\mathrm{alg}}(S^{\mathrm{alg}}) \subseteq \mathfrak{D}(S)$.

Then there exists a constant $n_0 = n_0(\mathfrak{D}, \mathfrak{X}) \in \mathbb{N}^*$ such that for all $c \in \mathfrak{D}^{\mathrm{alg}}(S^{\mathrm{alg}}) \subseteq \mathfrak{D}(S)$ the set

$$\{P \in Z_0^{+c}(k_0) \mid P \textit{ lifts to an element of } \mathfrak{X}^{+c}(S_{n_0}) \cap p^{n_0} \cdot \mathfrak{D}(S_{n_0})\}$$

is not Zariski dense in Z_0^{+c}.

Proof. This is a special case of Corollary 4.5. □

Proof Proposition 3.1. Since $\mathcal{X}$ is flat over U and X is integral, we see that $\mathcal{X}$ is also integral (see for instance [Liu 2002, 4.3.1, Proposition 3.8] for this). Hence, it is sufficient to show that $\mathrm{Exc}^n(\mathcal{A}, \mathcal{X}^{+Q})_u$ is not Zariski dense in $\mathcal{X}_u^{+Q}$ for some (any) closed point $u \in U$. Now using (1) in the previous section, we see that

$$\begin{aligned}\mathrm{Crit}^n(\mathcal{A}, \mathcal{X}^{+Q})_u(k_0) &= ([p^n]_*(J^n(\mathcal{A}/U)))_u(k_0) \cap J^n(\mathcal{X}^{+Q}/U)_u(k_0)\\ &= \{P \in J^n(\mathcal{X}^{+Q}/U)_u(k_0) \mid \exists \tilde{P} \in J^n(\mathcal{A}/U)_u(k_0),\ p^n \cdot \tilde{P} = P\}\\ &= \{P \in \mathcal{X}^{+Q}(u_n) \mid \exists \tilde{P} \in \mathcal{A}(u_n),\ p^n \cdot \tilde{P} = P\},\end{aligned}$$

and thus,

$$\begin{aligned}&\mathrm{Exc}^n(\mathcal{A}, \mathcal{X}^{+Q})_u(k_0)\\ &\qquad = \{P \in \mathcal{X}_u^{+Q}(k_0) \mid P \text{ lifts to an element of } \mathcal{X}^{+Q}(u_n) \cap p^n \cdot \mathcal{A}(u_n)\}.\end{aligned}$$

Now notice that $\mathcal{A}$ has a model $\tilde{\mathcal{A}}$ as an abelian scheme over a curve $\tilde{U}$, which is smooth over a finite field; also since the group Γ is finitely generated, we might assume that Γ is the image of a group $\tilde{\Gamma} \subseteq \tilde{\mathscr{A}}(\tilde{U})$. Finally, we might assume that the immersion $\mathscr{X} \hookrightarrow \mathscr{A}$ has a model $\tilde{\mathscr{X}} \hookrightarrow \tilde{\mathscr{A}}$ over $\tilde{U}$. We may thus apply Theorem 3.3 to the base-change of $\mathscr{X} \hookrightarrow \mathscr{A}$ to the completion of U at u. We obtain that there is an n_0 such that the set

$$\{P \in \mathscr{X}_u^{+Q}(k_0) \mid P \text{ lifts to an element of } \mathscr{X}^{+Q}(u_{n_0}) \cap p^{n_0} \cdot \mathscr{A}(u_{n_0})\}$$

is not Zariski dense in $\mathscr{X}_u$ for all $Q \in \Gamma$. So we may set $\alpha = n_0$. □

3B. *End of proof.* The proof of Theorem 1.3 is by contradiction. So suppose that $X \cap \Gamma$ is dense in X.

Let $P_1 \in \Gamma$ be such that $(X + P_1) \cap p \cdot \Gamma$ is dense, let $P_2 \in p \cdot \Gamma$ such that $(X + P_1 + P_2) \cap p^2 \cdot \Gamma$ is dense in $X + P_1 + P_2$, and so forth. The existence of the sequence of point $(P_i)_{i \in \mathbb{N}^*}$ is guaranteed by the assumption on Γ, which implies that $p^i\Gamma/p^{i+1}\Gamma$ is finite for all $i \geqslant 0$.

Now let $\alpha = \alpha(\mathscr{A}, \mathscr{X})$ be the natural number provided by Proposition 3.1. Let $Q = \sum_{i=1}^{\alpha} P_i$. By construction, the set $\mathscr{X}^{+Q} \cap p^\alpha \cdot \Gamma$ is dense in $\mathscr{X}^{+Q}$. On the other hand, by Lemma 2.5,

$$\begin{aligned}
&\mathscr{X}^{+Q}(U) \cap p^\alpha \cdot \Gamma \\
&\qquad = \Lambda^{\mathscr{A}}_{\alpha,0}(\lambda^{\mathscr{A}}_\alpha(\mathscr{X}^{+Q}(U) \cap p^\alpha \cdot \Gamma)) \subseteq \Lambda^{\mathscr{A}}_{\alpha,0}[\lambda^{\mathscr{X}}_\alpha(\mathscr{X}^{+Q}(U)) \cap \lambda^{\mathscr{A}}_\alpha(p^\alpha \cdot \Gamma)] \\
&\qquad \subseteq \Lambda^{\mathscr{A}}_{\alpha,0}[J^\alpha(\mathscr{X}^{+Q}/U) \cap p^\alpha \cdot J^\alpha(\mathscr{A}/U)(U)] \subseteq \Lambda^{\mathscr{X}}_{\alpha,0}[\mathrm{Crit}^\alpha(\mathscr{A}, \mathscr{X}^{+Q})] \\
&\qquad = \mathrm{Exc}^\alpha(\mathscr{A}, \mathscr{X}^{+Q}),
\end{aligned}$$

and thus, we deduce that $\mathrm{Exc}^\alpha(\mathscr{A}, \mathscr{X}^{+Q})$ is dense in $\mathscr{X}^{+Q}$. This contradicts Proposition 3.1 and concludes the proof of Theorem 1.3.

The proof of Corollary 1.4 now follows directly from Theorem 1.3 and from the following invariance lemma:

Lemma 3.4. *Suppose that the hypotheses of Theorem 1.1 hold. Let F' be an algebraically closed field, and let $F'|F$ be a field extension. Then Theorem 1.1 holds if and only if Theorem 1.1 holds with F' in place of F, $Y_{F'} \hookrightarrow B_{F'}$ in place of $Y \hookrightarrow B$, and the image $\Lambda_{F'} \subseteq B_{F'}(F')$ of Λ in place of Λ.*

Proof. The implication $\Longrightarrow$ follows from the fact that $Y_{F'} \cap \Lambda_{F'}$ is dense in $Y_{F'}$ if and only if $Y \cap \Lambda$ is dense in Y; indeed, the morphism $\operatorname{Spec} F' \to \operatorname{Spec} F$ is universally open (see [Grothendieck 1965, Exposé IV, 2.4.10] for this).

Now we prove the implication $\Longleftarrow$. Let $C_1 := \mathrm{Stab}(Y_{F'})^{\mathrm{red}}$, and suppose that there exist

- a semiabelian variety B_1' over F',
- a homomorphism with finite kernel $h_1 : B_1' \to B_{F'}/C_1$,
- a model $\mathbf{B}_1'$ of B_1' over a finite subfield $\mathbb{F}_{p^r} \subset F'$,

- an irreducible reduced closed subscheme $\mathbf{Y}'_1 \hookrightarrow \mathbf{B}'_1$, and
- a point $b_1 \in (B_{F'}/C_1)(F')$ such that $Y_{F'}/C_1 = b_1 + h_{1,*}(\mathbf{Y}'_1 \times_{\mathbb{F}_{p^r}} F')$.

Now, first notice that since $\mathrm{Stab}(\,\cdot\,)$ represents a functor, there is a natural isomorphism $\mathrm{Stab}(Y_{F'}) \simeq \mathrm{Stab}(Y)_{F'}$ and, since F is algebraically closed, also a natural isomorphism $\mathrm{Stab}(Y_{F'})^{\mathrm{red}} \simeq (\mathrm{Stab}(Y)^{\mathrm{red}})_{F'}$. Secondly, we have $\mathbb{F}_{p^r} \subset F$ since F is algebraically closed. Thirdly, if B_2 and B_3 are semiabelian varieties over F and $\phi : B_{2,F'} \to B_{3,F'}$ is a homomorphism of group schemes over F', then ϕ arises by base-change from an F-morphism $B_2 \to B_3$. This is a consequence of the facts that the graph of ϕ has a dense set of torsion points in $B_{2,F'} \times_{F'} B_{3,F'}$ and torsion points are defined in $B_2 \times_F B_3$. Putting these facts together, we deduce that there exist

- a semiabelian variety B' over F,
- a homomorphism with finite kernel $h : B' \to B/C$,
- a model $\mathbf{B}'$ of B' over a finite subfield $\mathbb{F}_{p^r} \subset F$,
- an irreducible reduced closed subscheme $\mathbf{Y}' \hookrightarrow \mathbf{B}'$, and
- a point $b_1 \in (B_{F'}/C_{F'})(F')$ such that $Y_{F'}/C_{F'} = b_1 + h_{F',*}(\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F')$,

where $C = \mathrm{Stab}(Y)^{\mathrm{red}}$. Now $\mathrm{Transp}(Y_{F'}/C_{F'}, h_{F',*}(\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F'))(F') \neq \varnothing$ by the last point in the list above. Here $\mathrm{Transp}(\,\cdot\,)$ is the transporter, which is a generalization of the stabilizer (see [Grothendieck et al. 1970b, Exposé VIII, 6] for the definition). Thus, $\mathrm{Transp}(Y/C, h_*(\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F))(F) \neq \varnothing$, which is to say that there also exists

- a point $b_1 \in (B/C)(F)$ such that $Y/C = b_1 + h_*(\mathbf{Y}' \times_{\mathbb{F}_{p^r}} F)$. □

4. Sparsity of highly p-divisible unramified liftings

This section can be read independently of the rest of the text, and its results do not rely on the previous ones. Also, unlike the previous sections, *the terminology of this section is independent of the terminology of Section 1*.

Let S be the spectrum of a complete discrete local ring. Let k be the residue field of its closed point. We suppose that k is a *finite field* of characteristic p. Let K be the fraction field of S. Let S^{sh} be the spectrum of the strict henselization of S, and let L be the fraction field of S^{sh}. We identify $\bar{k}$ with the residue field of the closed point of S^{sh}. For any $n \in \mathbb{N}$, we shall write S_n and S^{sh}_n for the n-th infinitesimal neighborhoods of the closed point of S in S and S^{sh} in S^{sh}, respectively.

Let $\mathscr{A}$ be an abelian scheme over S, and let $A := \mathscr{A}_K$. Write A_0 for the fiber of $\mathscr{A}$ over the closed point of S.

Theorem 4.1. *Let $\mathscr{X} \hookrightarrow \mathscr{A}$ be a closed integral subscheme. Let X_0 be the fiber of $\mathscr{X}$ over the closed point of S, and let $X := \mathscr{X}_K$.*

Suppose that $\mathrm{Tor}(A(\bar{K})) \cap X_{\bar{K}}$ *is not dense in* $X_{\bar{K}}$.

Then there exists a constant $m \in \mathbb{N}$ such that the set

$$\{P \in X_0(\bar{k}) \mid P \text{ lifts to an element of } \mathscr{X}(S_m^{\text{sh}}) \cap p^m \cdot \mathscr{A}(S_m^{\text{sh}})\}$$

is not Zariski dense in X_0.

Suppose for the next sentence that S is the spectrum of a complete discrete valuation ring that is absolutely unramified and is the completion of a number field along a nonarchimedean place. In this situation, Raynaud proves Theorem 4.1 and Corollary 4.5 below under the stronger hypothesis that $X_{\overline{K}}$ does not contain any translates of positive-dimensional abelian subvarieties of $A_{\overline{K}}$ [Raynaud 1983a, Proposition II.1.1]. See also [Raynaud 1983b, Theorem II, p. 207] for a more precise result in the situation where X is a smooth curve.

In the case where S is the spectrum of the ring of integers of a finite extension of $\mathbb{Q}_p$, Theorem 4.1 implies versions of the Tate–Voloch conjecture (see [Tate and Voloch 1996; Scanlon 1999]). We leave it to the reader to work out the details.

Preliminary to the proof of Theorem 4.1, we quote the following result. Let B be an abelian variety over an algebraically closed field F, and let $\psi : B \to B$ be an endomorphism. Let $R \in \mathbb{Z}[T]$ be a polynomial that has no roots of unity among its complex roots. Suppose that $R(\psi) = 0$ in the ring of endomorphisms of B.

Proposition 4.2 (Pink–Rössler). *Let $Z \subseteq B$ be a closed irreducible subset such that $\psi(Z) = Z$. Then* $\mathrm{Tor}(B(F)) \cap Z$ *is dense in Z.*

The proof of Proposition 4.2 is based on a spreading-out argument, which is used to reduce the problem to the case where F is the algebraic closure of a finite field. In this last case, the statement becomes obvious. See [Pink and Rössler 2004, Proposition 6.1] for the details.

We shall use the map $[p^\ell]^\circ : A_0(\bar{k}) \to \mathscr{A}(S_\ell^{\text{sh}})$, which is defined by the formula $[p^\ell]^\circ(x) = p^\ell \cdot \tilde{x}$, where $\tilde{x}$ is any lifting of x (this does not depend on the lifting; see [Katz 1981, after Theorem 2.1]).

Proof of Theorem 4.1. Let ϕ be a topological generator of $\mathrm{Gal}(\bar{k}|k)$. By the Weil conjectures for abelian varieties, there is a polynomial

$$Q(T) := T^{2g} - (a_{2g-1}T^{2g-1} + \cdots + a_0)$$

with $a_i \in \mathbb{Z}$ such that $Q(\phi)(x) = 0$ for all $x \in A_0(\bar{k})$ and such that $Q(T)$ has no roots of unity among its complex roots. Let M be the matrix

$$\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{2g-1} \end{bmatrix}.$$

We view M as an endomorphism of abelian S-schemes $\mathcal{A}^{2g} \to \mathcal{A}^{2g}$. Let $\tau \in \mathrm{Aut}_S(S^{\mathrm{sh}})$ be the canonical lifting of ϕ. By construction, τ induces an element of $\mathrm{Aut}_{S_n}(S_n^{\mathrm{sh}})$ for any $n \geqslant 0$, which we also call τ. The reduction map $\mathcal{A}(S^{\mathrm{sh}}) \to \mathcal{A}(S_n^{\mathrm{sh}})$ is compatible with the action of τ on both sides. Write

$$u(x) := (x, \tau(x), \tau^2(x), \dots, \tau^{2g-1}(x)) \in \left(\prod_{s=0}^{2g-1} \mathcal{A}\right)(S^{\mathrm{sh}})$$

for any element $x \in \mathcal{A}(S^{\mathrm{sh}})$. Abusing notation, we shall also write

$$u(x) := (x, \tau(x), \tau^2(x), \dots, \tau^{2g-1}(x)) \in \left(\prod_{s=0}^{2g-1} \mathcal{A}\right)(S_n^{\mathrm{sh}})$$

for any element $x \in \mathcal{A}(S_n^{\mathrm{sh}})$. By construction, for any $x \in \mathcal{A}(S^{\mathrm{sh}})$ and $x \in \mathcal{A}(S_n^{\mathrm{sh}})$, the equation $Q(\tau)(x) = 0$ implies the vector identity $M(u(x)) = u(\tau(x))$, respectively.

Now consider the closed S-subscheme of $\mathcal{A}^{2g}$

$$\mathfrak{X} := \bigcap_{t \geqslant 0} M_*^t\left(\bigcap_{r \geqslant 0} M^{r,*}\left(\prod_{s=0}^{2g-1} \mathcal{X}\right)\right),$$

where for any $r \geqslant 0$, M^r is the r-th power of M. The symbol $M_*^t(\cdot)$ refers to the scheme-theoretic image, and the intersections are the scheme-theoretic intersections. The intersections are finite by noetherianity.

Let $\lambda : J \to \mathcal{A}^{2g}$ be a morphism of schemes. The construction of $\mathfrak{X}$ implies that if

(i) $M^r \circ \lambda$ factors through $\prod_{s=0}^{2g-1} \mathcal{X}$ for all $r \geqslant 0$ and

(ii) for all $r \geqslant 0$, there is a morphism $\phi_r : J \to \bigcap_{r \geqslant 0} M^{r,*}\left(\prod_{s=0}^{2g-1} \mathcal{X}\right)$ such that $\lambda = M^r \circ \phi_r$,

then λ factors through $\mathfrak{X}$.

In particular, if (i) is verified and $M^{r_\lambda} \circ \lambda = \lambda$ for some $r_\lambda \geqslant 1$, then λ factors through $\mathfrak{X}$.

Remark 4.3. In particular, this implies that if $x \in \mathcal{X}(S^{\mathrm{sh}})$ and $x \in \mathcal{X}(S_n^{\mathrm{sh}})$ have the property that $Q(\tau)(x) = 0$, then $u(x) \in \mathfrak{X}(S^{\mathrm{sh}})$ and $u(x) \in \mathfrak{X}(S_n^{\mathrm{sh}})$, respectively.

Lemma 4.4. *There is a set-theoretic identity $M(\mathfrak{X}) = \mathfrak{X}$.*

Proof. Since M is proper, we have a set-theoretic identity

$$\mathfrak{X} = \bigcap_{t \geqslant 0} M^t\left(\bigcap_{r \geqslant 0} M^{r,-1}\left(\prod_{s=0}^{2g-1} \mathcal{X}\right)\right).$$

Now directly from the construction, we have

$$M\Big(\bigcap_{r\geqslant 0} M^{r,-1}\Big(\prod_{s=0}^{2g-1}\mathscr{X}\Big)\Big)\subseteq \bigcap_{r\geqslant 0} M^{r,-1}\Big(\prod_{s=0}^{2g-1}\mathscr{X}\Big),$$

and hence, we have inclusions

$$\bigcap_{r\geqslant 0} M^{r,-1}\Big(\prod_{s=0}^{2g-1}\mathscr{X}\Big)\supseteq M\Big(\bigcap_{r\geqslant 0} M^{r,-1}\Big(\prod_{s=0}^{2g-1}\mathscr{X}\Big)\Big)\supseteq M^2\Big(\bigcap_{r\geqslant 0} M^{r,-1}\Big(\prod_{s=0}^{2g-1}\mathscr{X}\Big)\Big)\supseteq\cdots,$$

and thus, by noetherianity

$$M^{\ell}\Big(\bigcap_{r\geqslant 0} M^{r,-1}\Big(\prod_{s=0}^{2g-1}\mathscr{X}\Big)\Big)= M^{\ell+1}\Big(\bigcap_{r\geqslant 0} M^{r,-1}\Big(\prod_{s=0}^{2g-1}\mathscr{X}\Big)\Big)$$

for some $\ell\geqslant 0$. This implies the result. □

Now we apply Proposition 4.2 and obtain that $\mathfrak{X}_{\bar{K},\mathrm{red}}\cap\mathrm{Tor}(\prod_{s=0}^{2g-1}A(\bar{K}))$ is dense in $\mathfrak{X}_{\bar{K},\mathrm{red}}$. Hence, the projection onto the first factor $\mathfrak{X}_K\to X$ is not surjective by hypothesis.

Let T be the scheme-theoretic image of the morphism $\mathfrak{X}\to\mathscr{X}$ given by the first projection. Notice that X_0 is a closed subscheme of T because every element P of $X_0(\bar{k})$ satisfies the equation $Q(\phi)(P)=0$. Let H be the closed subset of T that is the union of the irreducible components of T that surject onto S. A reduced irreducible component I of T that surjects onto S is flat over S; since $H\neq\mathscr{X}$, we have in particular $I\neq\mathscr{X}$, and so we see that the dimension of the fiber of I over the closed point of S is strictly smaller than the dimension of X_0. Hence, the intersection of H and X_0 is a proper closed subset of X_0. Let T_1 be the open subscheme $T\setminus H$ of T. From the previous discussion, we see that the underlying set of T_1 is a *nonempty open subset of* X_0.

We are now in a position to complete the proof of Theorem 4.1. The proof will be by contradiction. So suppose that for all $\ell\in\mathbb{N}$, the set

$$\{P\in X_0(\bar{k})\mid P \text{ lifts to an element of } \mathscr{X}(S_\ell^{\mathrm{sh}})\cap p^{\ell}\cdot\mathscr{A}(S_\ell^{\mathrm{sh}})\}$$

is Zariski dense in X_0.

Choose an arbitrary $\ell\in\mathbb{N}$, and let $P\in T_1(\bar{k})$ be a point that lifts to an element of $\mathscr{X}(S_\ell^{\mathrm{sh}})\cap p^{\ell}\cdot\mathscr{A}(S_\ell^{\mathrm{sh}})$. This exists because the set of points in $X_0(\bar{k})$ with this property is assumed to be dense in X_0. Let $\tilde{P}\in\mathscr{A}(S_\ell^{\mathrm{sh}})$ be such that $p^{\ell}\cdot\tilde{P}\in\mathscr{X}(S_\ell^{\mathrm{sh}})$ and such that $p^{\ell}\cdot\tilde{P}_0=P$. Here $\tilde{P}_0\in A_0(\bar{k})$ is the $\bar{k}$-point induced by $\tilde{P}$. Since the map $[p^{\ell}]^{\circ}:\mathscr{A}_0(\bar{k})\to\mathscr{A}(S_\ell^{\mathrm{sh}})$ intertwines ϕ and τ, we see that

$$Q(\tau)([p^{\ell}]^{\circ}(\tilde{P}_0))=0.$$

By Remark 4.3, we thus have

$$u([p^\ell]^\circ(\tilde{P}_0)) \in \mathfrak{X}(S_\ell^{\mathrm{sh}}).$$

Hence,

$$[p^\ell]^\circ(\tilde{P}_0) \in T_1(S_\ell^{\mathrm{sh}}) \subseteq T(S_\ell^{\mathrm{sh}}).$$

This shows that $T_1(S_\ell^{\mathrm{sh}}) \neq \varnothing$. Since ℓ was arbitrary, this shows that the generic fiber $T_{1,K}$ of T_1 is not empty, which is a contradiction. □

Corollary 4.5. *We keep the hypotheses of Theorem 4.1. We suppose furthermore that* $\mathrm{Tor}(A(\bar{K})) \cap X_{\bar{K}}^{+c}$ *is not dense in* $X_{\bar{K}}^{+c}$ *for all* $c \in \mathcal{A}(S)$*. Then there exists a constant* $m \in \mathbb{N}$ *such that for all* $c \in \mathcal{A}(S)$ *the set*

$$\{P \in X_0^{+c}(\bar{k}) \mid P \textit{ lifts to an element of } \mathcal{X}^{+c}(S_m^{\mathrm{sh}}) \cap p^m \cdot \mathcal{A}(S_m^{\mathrm{sh}})\}$$

is not Zariski dense in X_0^{+c}.

Here as usual $\mathcal{X}^{+c} = \mathcal{X} + c$ is the translate inside $\mathcal{A}$ of $\mathcal{X}$ by $c \in \mathcal{A}(S)$. Slightly abusing notation, we write X^{+c} for $(\mathcal{X}^{+c})_K$ and X_0^{+c} for $(\mathcal{X}^{+c})_k$.

Proof. We prove by contradiction. Write $m(\mathcal{X}^{+c})$ for the smallest integer m such that

$$\{P \in X_0^{+c}(\bar{k}) \mid P \text{ lifts to an element of } \mathcal{X}^{+c}(S_m^{\mathrm{sh}}) \cap p^m \cdot \mathcal{A}(S_m^{\mathrm{sh}})\}$$

is not Zariski dense in X_0. Suppose that there exists a sequence $(a_n \in \mathcal{A}(S))_{n\in\mathbb{N}}$ such that $m(\mathcal{X}^{+a_n})$ strictly increases. Replace $(a_n \in \mathcal{A}(S))_{n\in\mathbb{N}}$ by one of its subsequences so that $\lim_n a_n = a \in \mathcal{A}(S)$, where the convergence is for the topology given by the discrete valuation on the ring underlying S (notice that $\mathcal{A}(S)$ is compact for this topology because S is complete and has a finite residue field at its closed point). Replace $(a_n \in \mathcal{A}(S))_{n\in\mathbb{N}}$ by one of its subsequences again so that the image of a_n in $\mathcal{A}(S_n)$ equals the image of a in $\mathcal{A}(S_n)$. By construction, we have $m(\mathcal{X}^{+a_n}) \geqslant n$, and hence, by definition $m(\mathcal{X}^a) \geqslant n$. Since this is true for all $n \geqslant 0$, this contradicts Theorem 4.1. □

The following corollary should be viewed as a curiosity only since it is a special case of Theorem 1.3. The interest lies in its proof, which avoids the use of jet schemes, unlike the proof of Theorem 1.3.

Corollary 4.6. *We keep the notation and assumptions of Corollary 4.5. Suppose furthermore that* S *is a ring of characteristic* p *and that the fibers of* $\mathcal{A}$ *over* S *are ordinary abelian varieties. We also suppose that* $\mathcal{X}$ *is smooth over* S*. Let* $\Gamma \subseteq A(K)$ *be a finitely generated subgroup. Then the set* $X \cap \Gamma$ *is not Zariski dense in* X.

We shall call the topology on $A(K)$ induced by the discrete valuation *the v-adic topology*.

Before the proof of the corollary, recall a simple but crucial lemma of Voloch (see [Abramovich and Voloch 1992, Lemma 1]).

Lemma 4.7 (Voloch). *Let L_0 be a field, and let T be a reduced scheme of finite type over L_0. Then $T(L_0^{\mathrm{sep}})$ is dense in T if and only if T is geometrically reduced.*

Proof of Corollary 4.6. The proof is by contradiction. We shall exhibit a translate of X by an element of $A(K)$, which violates the conclusion of Theorem 4.1. Suppose that $X \cap \Gamma$ is Zariski dense in X. Let $P_1 \in \Gamma$ be such that $(X+P_1) \cap p \cdot \Gamma$ is dense, let $P_2 \in p \cdot \Gamma$ such that $(X+P_1+P_2) \cap p^2 \cdot \Gamma$ is dense in X, and so forth. The existence of the sequence of points (P_i) is guaranteed by the assumption on Γ, which implies that the group $p^\ell \Gamma / p^{\ell+1} \Gamma$ is finite for all $\ell \geqslant 0$. Since the v-adic topology on the set $A(K)$ is compact (because S is a discrete valuation ring with a finite residue field), the sequence $Q_i = \sum_{\ell \geqslant 1}^{i} P_\ell$ has a subsequence that converges in $A(K)$. Let Q be the limit point of such a subsequence. By construction, $(X+Q) \cap p^\ell \cdot A(K)$ is dense for all $\ell \geqslant 0$. Let $\mathscr{X}^{+Q} := \mathscr{X} + Q$.

Consider the morphism $([p^\ell]^* \mathscr{X}^{+Q})_{\mathrm{red}} \to \mathscr{X}^{+Q}$. There is a diagram

$$\begin{tikzcd}
([p^\ell]^*\mathscr{X}^{+Q})_{\mathrm{red}} \arrow[r, hook] & ([p^\ell]^*\mathscr{X}^{+Q}) \arrow[rr, "{[p^\ell]}"] \arrow[d, hook] & & \mathscr{X}^{+Q} \arrow[d, hook] \\
& \mathscr{A} \arrow[r, "\mathrm{Frob}^\ell_{\mathscr{A}/S}"] & \mathscr{A}^{(p^\ell)} \arrow[r, "\mathrm{Ver}^\ell"] & \mathscr{A}
\end{tikzcd}$$

where F_S is the absolute Frobenius morphism on S, $\mathscr{A}^{(p^\ell)} = F_S^{\ell,*}\mathscr{A}$ is the base-change of $\mathscr{A}$ by $F_S^{\ell,*}$, $\mathrm{Frob}_{\mathscr{A}/S}$ is the Frobenius morphism relatively to S, and Ver is the Verschiebung (see [Grothendieck et al. 1970a, Exposé $\mathrm{VII_A}$, 4.3] for the latter). The square is cartesian (by definition). By assumption, the morphism Ver is étale. Hence, $\mathrm{Ver}^{\ell,*}(\mathscr{X}^{+Q})$ is a disjoint union of schemes that are integral and smooth over S. Let $\mathscr{X}_1 \hookrightarrow \mathrm{Ver}^{\ell,*}(\mathscr{X}^{+Q})$ be an irreducible component such that $\mathscr{X}_{1,K} \cap \mathrm{Frob}^\ell_{A/K}(A(K))$ is dense. Let $\mathscr{X}_2 := (\mathrm{Frob}^{\ell,*}_{\mathscr{A}/S}(\mathscr{X}_1))_{\mathrm{red}}$ be the corresponding reduced irreducible component.

Now notice that $\mathscr{X}_{2,K}$ is geometrically reduced since $\mathscr{X}_2(K)$ is dense in $\mathscr{X}_{2,K}$ (Lemma 4.7). Furthermore, $\mathscr{X}_2$ is flat over S because it is reduced and dominates S. Hence, $(\mathscr{X}_2)^{(p^\ell)}$ is also flat over S. Furthermore, by its very construction $(\mathscr{X}_{2,K})^{(p^\ell)}$ is reduced since $\mathscr{X}_{2,K}$ is geometrically reduced. Hence, $(\mathscr{X}_2)^{(p^\ell)}$ is reduced [Liu 2002, 4.3.8, p. 137]. Recall that $(\mathscr{X}_2)^{(p^\ell)}$ stands for the base-change of $\mathscr{X}_2$ by $F_S^{\ell,*}$. Notice that we have a commutative diagram

$$\begin{tikzcd}
\mathscr{X}_2 \arrow[r, "\mathrm{Frob}^\ell_{\mathscr{X}_2/S}"] \arrow[d, hook] & (\mathscr{X}_2)^{(p^\ell)} \arrow[d, hook] \\
\mathscr{A} \arrow[r, "\mathrm{Frob}^\ell_{\mathscr{A}/S}"] & \mathscr{A}^{(p^\ell)}
\end{tikzcd}$$

and that $\mathrm{Frob}^\ell_{\mathscr{X}_2/S}$ is bijective. Hence, $(\mathscr{X}_2)^{(p^\ell)}$ is isomorphic to $\mathscr{X}_1$. Now, since F_S is faithfully flat and $\mathscr{X}_2$ is flat over S, we see that $\mathscr{X}_2$ is actually smooth over S because

$\mathscr{X}_1$ is smooth over S. Hence, every point of $\mathscr{X}_2(\bar{k})$ can be lifted to a point in $\mathscr{X}_2(S^{\text{sh}})$ (see for instance [Liu 2002, Corollary 6.2.13, p. 224]). Since the morphism $[p^\ell]$ is finite and flat and the scheme $\mathscr{X}^{+Q}$ is integral, we see that the map $\mathscr{X}_2 \to \mathscr{X}^{+Q}$ is surjective. This implies that the map $\mathscr{X}_2(\bar{k}) \to \mathscr{X}^{+Q}(\bar{k})$ is surjective. We conclude that *every element of* $\mathscr{X}^{+Q}(\bar{k})$ *is liftable to an element in* $\mathscr{X}^{+Q}(S^{\text{sh}}) \cap p^\ell \cdot \mathscr{A}(S^{\text{sh}})$. Since ℓ was arbitrary, this contradicts Theorem 4.1. □

Now we want to conclude by:

Remark 4.8. Buium [1992] also introduces an "exceptional set", which is very similar to the set Exc considered here, and he makes a similar use of it (catching rational points). There is nevertheless one important difference between Buium's and our methods: the proof of Theorem 3.3, which is crucial in our study of the structure of Exc, uses "Galois equations" and not differential equations. In this sense, our techniques also differ from the techniques employed in [Hrushovski 1996], which is close in spirit to [Buium 1992] and where the Galois-theoretic language is not used either.

Remark 4.9. Although Corollary 1.4 shows that the Mordell–Lang conjecture may be reduced to the Manin–Mumford conjecture under the assumptions of Theorem 1.3, the difficulty of circumventing the fact that the underlying abelian variety might not be ordinary (which was a hurdle for some time) is not thus removed. Indeed, the most difficult part of the algebraic-geometric proof of the Manin–Mumford conjecture given in [Pink and Rössler 2004] concerns the analysis of endomorphisms of abelian varieties, which are not globally the composition of a separable isogeny with a power of a relative Frobenius morphism.

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rossler@math.univ-toulouse.fr Institut de Mathématiques, Equipe Emile Picard, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse cedex 9, France
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