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We investigate some general questions in algebraic dynamics in the case of generic endomorphisms of projective spaces over a field of characteristic zero. The main results that we prove are that a generic endomorphism has no nontrivial preperiodic subvarieties, any infinite set of preperiodic points is Zariski-dense and any infinite subset of a single orbit is also Zariski-dense, thereby verifying the dynamical “Manin–Mumford” conjecture of Zhang and the dynamical “Mordell–Lang” conjecture of Denis and Ghioca and Tucker in this case.

1. Introduction

The goal of this article is to study some aspects of the algebraic dynamics of generic endomorphisms¹ of \mathbb{P}^n of degree $d > 1$ over a field K of characteristic zero. Properties of algebraic varieties, for example smooth projective curves or abelian varieties, are often easier to derive for generic varieties than for arbitrary varieties, the main reason being that one has a great deal of freedom in choosing specialisations. It is natural to expect that the same holds for algebraic dynamical systems; we show that this is indeed the case for generic endomorphisms of \mathbb{P}^n . We prove three results for such endomorphisms: two of them have analogues expected to hold much more generally, though at present this is far from being known.

Our main result is:

Theorem 1.1. *Let $f : \mathbb{P}_K^n \rightarrow \mathbb{P}_K^n$ be a generic endomorphism of degree $d > 1$ over an algebraically closed field K of characteristic zero. For each $x \in \mathbb{P}^n(K)$, every infinite subset of $O_f(x)$, the f -orbit of x , is Zariski-dense in \mathbb{P}_K^n .*

This implies the dynamical “Mordell–Lang” conjecture of [Denis 1994; Ghioca and Tucker 2009] for generic endomorphisms. This conjecture has been proved for étale endomorphisms of arbitrary varieties by Bell, Ghioca and Tucker [Bell et al.

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¹The precise meaning of generic endomorphism is given in Definition 3.1, but we note here that when $K = \mathbb{C}$ this means we consider endomorphisms in the complement of a countable union of proper subvarieties of the natural parameter variety of endomorphisms of degree d .

2010], but there are only a few other cases where it is known. The proof of this theorem is based on two other results. The first is:

Theorem 1.2. *Let $f : \mathbb{P}_K^n \rightarrow \mathbb{P}_K^n$ be a generic endomorphism of degree $d > 1$ over an algebraically closed field K of characteristic zero. If $X \subset \mathbb{P}_K^n$ is an irreducible subvariety such that $f^r(X) = X$ for some $r > 0$, then X is a point or $X = \mathbb{P}_K^n$.*

This is a rather straightforward consequence of the transitivity of the monodromy action on the set of periodic points of a fixed period of a generic endomorphism, which we prove (Proposition 3.3) using a result of Bousch [1992], Lau and Schleicher [1994] and Morton [1998] for polynomials in one variable of the form $z \mapsto z^d + c$.

We then extend Proposition 3.3 to prove transitivity of the monodromy action on the set of preperiodic points of fixed period and preperiod of a generic endomorphism (Proposition 4.6); this does not hold for the 1-parameter family of polynomials mentioned above, and we use a 2-parameter family containing this. This allows us to prove Zhang’s “Manin–Mumford” conjecture [Ghioca et al. 2011] in the case of generic endomorphisms in the following strong form:

Theorem 1.3. *For $f : \mathbb{P}_K^n \rightarrow \mathbb{P}_K^n$ a generic endomorphism of degree $d > 1$ over an algebraically closed field K of characteristic zero, any infinite subset of $\mathbb{P}^n(K)$ consisting of f -preperiodic points is Zariski-dense in \mathbb{P}_K^n .*

We prove Theorem 1.1 by combining Theorems 1.2 and 1.3 with some p -adic as well as mod p arguments. Note that the statement does not involve (pre)periodic points in any way. However, using a lifting argument for periodic points, we show that any subvariety Y containing an infinite subset of $O_f(x)$ must contain infinitely many periodic points, or x can be specialised in such a way that one may apply the p -adic interpolation argument used in [Bell et al. 2010] to prove the conjecture for étale endomorphisms. Theorems 1.3 and 1.2 then force Y to be equal to \mathbb{P}_K^n in either of these cases.

2. Preliminaries

Let X be a set and $f : X \rightarrow X$ any map. By f^n , we shall mean the n -fold composite of f with itself. For $x \in X$, we denote by $O_f(x)$ its orbit under f , i.e., the set $\{f^n(x)\}_{n \geq 0}$. A point $x \in X$ is said to be f -periodic if $f^n(x) = x$ for some $n > 0$. The smallest such integer is called the *period* of x . We denote the set of all periodic points of period b by $\text{Per}_f(b)$. A point $x \in X$ is said to be f -preperiodic if $O_f(x)$ is finite. The *preperiod* of f is the smallest nonnegative integer a such that $f^a(x)$ is periodic, and the *period* of x is the period of any periodic point in its orbit. We denote by $\text{Prep}_f(a, b)$ the set of all such points. Let $\text{Orb}_f(b)$ denote the set of orbits of f -periodic points of period b . If this is finite, then $|\text{Per}_f(b)| = b \cdot |\text{Orb}_f(b)|$. We drop f from any of the notation introduced above if there is no scope for confusion.

If X is an algebraic variety over a field K and $f : X \rightarrow X$ is a morphism over K , we use the same notation as above for the induced map on the set of L -rational points of X for any extension field L of K .

Let S be a smooth irreducible variety over a field k , and let $g : Z \rightarrow S$ be a finite flat morphism. By the *monodromy* or *Galois action* of g , we shall mean the action of $\text{Gal}(\overline{k(S)}/k(S))$ on (the points of) a geometric generic fibre of g . If g is generically smooth — this is always true if $\text{char}(k) = 0$ and Z is reduced — there is a Zariski open subset U of S such that g induces a finite étale morphism $g^{-1}(U) \rightarrow U$, and then the monodromy may be interpreted as an action of $\pi_1^{\text{ét}}(U, *)$, where $*$ is a geometric point of U . If $k = \mathbb{C}$, it may be interpreted as an action of $\pi_1^{\text{top}}(U, *)$.

The monodromy action is transitive if and only if Z is irreducible. If Z is generically smooth, this is equivalent to Z^0 , the smooth locus of Z , being connected or, if $k = \mathbb{C}$, path-connected.

Definition 2.1. Let $\pi : \mathcal{X} \rightarrow S$ be a projective morphism and $f : \mathcal{X} \rightarrow \mathcal{X}$ a surjective morphism over S . We say that f is *quasipolarised* if there exists a line bundle \mathcal{L} on \mathcal{X} such that $f^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$ is S -ample.

For any morphism $\pi : \mathcal{X} \rightarrow S$ and $f : \mathcal{X} \rightarrow \mathcal{X}$ a morphism over S , we denote by Γ_f the graph of f in $\mathcal{X} \times_S \mathcal{X}$. Let $P_f(n)$ be the closed subscheme of \mathcal{X} defined by the intersection of Γ_{f^n} with the diagonal. A geometric point of the fibre of $P_f(n)$ over any point $s \in S$ is a periodic point of period dividing n of the map f_s of \mathcal{X}_s induced by f . Similarly, let $P_f(m, n)$ be the intersection of Γ_{f^m} and Γ_{f^n} , which we view as a subscheme of \mathcal{X} via the first projection.

Lemma 2.2. *Let $\pi : \mathcal{X} \rightarrow S$ be a smooth projective morphism with S a regular irreducible finite-dimensional scheme, and let $f : \mathcal{X} \rightarrow \mathcal{X}$ be a finite quasipolarised morphism. Then:*

- (1) *For any $m, n \geq 0, m \neq n, P_f(m, n)$ is finite and flat over S .*
- (2) *For any s_1 and s_2 in S with s_2 a specialisation of s_1 , any element of $\text{Prep}_{f_{s_2}}(a, b)$ can be lifted to an element of $\text{Prep}_{f_{s_1}}(a, b)$.*

Proof. Since f is proper, $P_f(m, n)$, being a closed subscheme of \mathcal{X} , is also proper over S . The dimension of each irreducible component of $P_f(m, n)$ is at least equal to $\dim(S)$ since the codimension of Γ_{f^i} in $\mathcal{X} \times_S \mathcal{X}$ is the relative dimension of \mathcal{X} over S . To prove $P_f(m, n)$ is finite over S , it suffices to show that the fibres of this map are finite since proper quasifinite morphisms are finite. Furthermore, the finiteness of the fibres implies that the dimension of each component of $P_f(m, n)$ is exactly $\dim(S)$; hence, $P_f(m, n)$ is a local complete intersection in $\mathcal{X} \times_S \mathcal{X}$ that is regular, which implies that $P_f(m, n)$ is Cohen–Macaulay. Since the dimension of each irreducible component of $P_f(m, n)$ is at least $\dim(S)$, each such component

dominates S if all the fibres are finite. It then follows from the fibrewise flatness criterion that all of (1) is a consequence of the finiteness of the fibres.

Let \mathcal{L} be a line bundle on \mathcal{X} so that $M = f^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$ is ample. Then $(f^m)^*(\mathcal{L}) = \mathcal{L} \otimes M \otimes f^*(M) \otimes \cdots \otimes (f^{m-1})^*(M)$ and similarly for n . By the construction of $P_f(m, n)$, $(f^m)^*(\mathcal{L})$ and $(f^n)^*(\mathcal{L})$ restrict to the same line bundle on it, so assuming $m > n$ without loss of generality, we get that $(f^{m-1})^*(M) \otimes \cdots \otimes (f^n)^*(M)$ is trivial on $P_f(m, n)$. But M is ample; hence, so is $(f^i)^*(M)$ for all $i \geq 0$ since f is finite. There is at least one factor in the above tensor product of line bundles, so this is only possible if all the fibres are finite.

If $x \in \text{Prep}_{f_{s_2}}(a, b)$, then x occurs in the fibre of $P_f(a+b, a)$ over s_1 and does not occur in the fibre of any $P_f(m, n)$ for $m < a+b$ or $n < a$. By (1), there is a point \tilde{x} in the fibre of $P_f(a+b, a)$ specialising to x . By the definition of $P_f(m, n)$, it follows that $f_{s_1}^{a+b}(\tilde{x}) = f_{s_1}^a(\tilde{x})$, so \tilde{x} is preperiodic with preperiod $\leq a$ and period $\leq b$. Since neither the preperiod nor the period can increase under specialisation and \tilde{x} specialises to x , (2) follows. \square

3. Periodic points and periodic subvarieties

Let $\text{Mor}_{n,d}$ be the scheme over \mathbb{Z} representing morphisms of $\mathbb{P}_{\mathbb{Z}}^n$ to itself of algebraic degree d . Its k -valued points, for any field k , consist of $(n+1)$ -tuples of homogeneous polynomials of degree d over k without common zeros in \mathbb{P}_k^n up to a scalar. It is smooth and of finite type over \mathbb{Z} and has geometrically irreducible fibres. For any field L , we denote $\text{Mor}_{n,d} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(L)$ by $\text{Mor}_{n,d/L}$.

Definition 3.1. If K is an algebraically closed field, we say that an endomorphism f of \mathbb{P}_K^n is *generic* if the image of the induced map $\text{Spec}(K) \rightarrow \text{Mor}_{n,d}$ corresponding to some conjugate of f by an element of $\text{PGL}_{n+1}(K) = \text{Aut}(\mathbb{P}_K^n)$ is the generic point of a fibre of the structure morphism $\text{Mor}_{n,d} \rightarrow \text{Spec}(\mathbb{Z})$.

If $K = \mathbb{C}$, the set of points in $\text{Mor}_{n,d}(\mathbb{C})$ corresponding to generic morphisms is the complement of a countable union of proper subvarieties.

We recall the theorem of Bousch [1992, Chapitre 3, Théorème 4], Lau and Schleicher [1994, Theorem 4.1] and Morton [1998, Theorem D] alluded to earlier; the statement below is [Morton 1998, Theorem 10] except that we have replaced the field $\overline{\mathbb{Q}}$ there by k .

Theorem 3.2. *Let k be a field of characteristic zero, and let $f(z) = z^d + t$ with t transcendental over k and $d \geq 2$. For any $b \geq 1$, the Galois group of the polynomial $f^b(z) - z$ over $k(t)$ is the direct product $\prod_{e|b} (\mathbb{Z}/e\mathbb{Z} \text{ wr } S_{r_e})$, where wr denotes the wreath product and $e \cdot r_e$ is the number of periodic points of period e over $\overline{k(t)}$.*

The theorem can be interpreted as saying that the Galois action is as large as possible given that it must commute with the action of f . One may expect that a

similar result holds for generic endomorphisms of projective spaces of arbitrary dimension; however, the following proposition, for the proof of which we will use only the transitivity of the Galois action in the above theorem, suffices for our applications:

Proposition 3.3. *Let k be a field of characteristic zero, and let $k_{n,d}$ be the function field of $\text{Mor}_{n,d/k}$. Let $f_{n,d}$ be the endomorphism of $\mathbb{P}_{k_{n,d}}^n$ corresponding to the generic point of $\text{Mor}_{n,d/k}$, and let b be any positive integer. Then $\text{Gal}(\overline{k_{n,d}}/k_{n,d})$ acts transitively on $\text{Per}_{f_{n,d}}(b)$.*

For a field k and any element $\lambda \in k$, let $\phi_\lambda : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ be the map given by $z \mapsto z^d + \lambda$; the integer d will be assumed to be fixed whenever we use this notation. The periodic points of ϕ_0 are 0 and the roots of unity of order prime to d : if ζ is a primitive n -th root of unity with $(n, d) = 1$, then the period of ζ is the order of d in $(\mathbb{Z}/n\mathbb{Z})^\times$.

We shall need the following simple lemma for the proof of [Proposition 3.3](#):

Lemma 3.4. *Let $d > 1$ and $m, m' \geq 1$ be integers such that $(m, d) = (m', d) = 1$. Assume that the highest powers of 2 dividing m and m' are unequal or are both equal to 1. Let a and a' be the orders of d in $(\mathbb{Z}/m\mathbb{Z})^\times$ and $(\mathbb{Z}/m'\mathbb{Z})^\times$, respectively.*

- (1) *The order of d in $(\mathbb{Z}/\text{lcm}(m, m')\mathbb{Z})^\times$ is divisible by $\text{lcm}(a, a')$.*
- (2) *There exist roots of unity ζ and ζ' of orders m and m' , respectively, so that $\zeta \zeta'^{-1}$ is of order $\text{lcm}(m, m')$.*
- (3) *For ζ and ζ' as above, there exists a primitive $\text{lcm}(m, m')$ -th root of unity η so that $\eta \zeta$ is a primitive $\text{lcm}(m, m')$ -th root of unity.*

Proof. The natural quotient maps from $\mathbb{Z}/\text{lcm}(m, m')\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/m'\mathbb{Z}$ induce group homomorphisms from $(\mathbb{Z}/\text{lcm}(m, m')\mathbb{Z})^\times$ to $(\mathbb{Z}/m\mathbb{Z})^\times$ and $(\mathbb{Z}/m'\mathbb{Z})^\times$, respectively. This implies that the order of d in $(\mathbb{Z}/\text{lcm}(m, m')\mathbb{Z})^\times$ is divisible by $\text{lcm}(a, a')$.

To prove (2) and (3), we may reduce to the case that m and m' are powers of the same prime p . Let $P \subset \mathbb{Z}/p^r\mathbb{Z}$ be the set of generators, so $|P| = p^r - p^{r-1}$. If $p > 2$, then $|P| > p^{r-1}$, so the translate of P by any element of $\mathbb{Z}/p^r\mathbb{Z}$ has a nonempty intersection with P . If $p = 2$, the claim follows from the extra condition since the translate of P by an element not in P always intersects P nontrivially. \square

Proof of Proposition 3.3. If $n = 1$ and $b > 1$, the proposition follows immediately from [Theorem 3.2](#). If $b = 1$, a much simpler version of the argument below shows transitivity; since we do not use this later, we leave the details to the reader.

We now assume $n > 1$. Consider the morphism $g_{n,d} : \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$ given by $[x_0, \dots, x_n] \mapsto [x_0^d, \dots, x_n^d]$. The set $\text{Per}_{g_{n,d}}(b)$ consists of points that have a representative $[\xi_0, \xi_1, \dots, \xi_n]$ with each ξ_i equal to 0 or a $(d^b - 1)$ -th root of unity.

The standard affine charts of \mathbb{P}_k^n given by the locus where a fixed coordinate is nonzero are preserved by $g_{n,d}$. A simple computation on each such chart shows that the eigenvalues of the differential of $g_{n,d}^b$ at a fixed point are equal to $d^b \xi$, where ξ is a root of unity or 0. This is never equal to 1 since $d > 1$, so $\Gamma_{g_{n,d}^b}$ and the diagonal intersect transversely in $\mathbb{P}_k^n \times \mathbb{P}_k^n$ for all $b > 0$. Consequently, all periodic points of $g_{n,d}$ have multiplicity 1, so we may use $g_{n,d}$ as a basepoint in $\text{Mor}_{n,d}$ in order to compute the Galois action on $\text{Per}_{f_{n,d}}$.

For $0 \leq i \leq n$, consider the family of endomorphisms $f_i : \mathbb{P}_k^n \times \mathbb{A}_k^n \rightarrow \mathbb{P}_k^n \times \mathbb{A}_k^n$ given by

$$f_i((x_0, \dots, x_i, \dots, x_n], (c_1, c_2, \dots, c_n))) \\ = ([x_0^d + c_1 x_i^d, \dots, x_{i-1}^d + c_i x_i^d, x_i^d, x_{i+1}^d + c_{i+1} x_i^d, \dots, x_n^d + c_n x_i^d], (c_1, c_2, \dots, c_n)).$$

On the open affine U_i given by $x_i \neq 0$, f_i is the product of the n polynomials ϕ_{c_j} . On the complement of this affine, i.e., on the subvariety given by $x_i = 0$ (which is also preserved by f_i), the maps do not depend on c_j , so the monodromy action of this family on the periodic points in this locus is trivial. Let G_i be the subgroup of the monodromy group acting on $\text{Per}_{g_{n,d}}$ corresponding to this family; by applying [Theorem 3.2](#), one gets a complete description of this group. We let G be the subgroup of the monodromy group generated by all the G_i .

Let $P = [\zeta, 1, \dots, 1, 1]$ where ζ is in $\text{Per}_{\phi_0}(b)$, and let $Q = [\xi_0, \xi_1, \dots, \xi_n]$ be any other element of $\text{Per}_{g_{n,d}}(b)$. We may assume that some $\xi_i = 1$, so each $\xi_j \in \text{Per}_{\phi_0}(b')$ for some $b' \mid b$, and also the lcm of the periods of all the ξ_j is b . We prove the transitivity of the monodromy action by showing that there exists an element in the monodromy that sends P to Q .

From the transitivity of the Galois action in [Theorem 3.2](#), it follows that for $\xi_j \in \text{Per}_{\phi_0}(b')$, as long as some $\xi_i = 1$ with $i \neq j$, we can find an element of G that fixes all coordinates of Q except that it replaces ξ_j with any other $\xi_j' \in \text{Per}_{\phi_0}(b')$. Since $0, 1 \in \text{Per}_{\phi_0}(0)$, we may use this to assume that all $\xi_i \neq 0$ and then also, by dividing through by ξ_n , that $\xi_n = 1$.

We now show that we may also assume that $\xi_0 \in \text{Per}_{\phi_0}(b)$. Suppose ξ_0 is a primitive m -th root of unity, ξ_1 is a primitive m' -th root of unity and $\xi_0 \in \text{Per}_{\phi_0}(a)$ and $\xi_1 \in \text{Per}_{\phi_0}(a')$. By using the action of G , we may change ξ_0 and ξ_1 so that $m = d^a - 1$ and $m' = d^{a'} - 1$. If the highest powers of 2 dividing m and m' are equal and greater than 1, we may change ξ_0 to a primitive $(d^a - 1)/2$ -th root of unity; the period a remains unchanged. By [Lemma 3.4](#), we may then assume that $\xi_0 \xi_1^{-1}$ is a primitive $\text{lcm}(m, m')$ -th root of unity. We multiply all coordinates of Q by ξ_1^{-1} , so the zeroth coordinate becomes $\xi_0 \xi_1^{-1}$, and the second becomes 1. Using the action of G , we then replace the zeroth coordinate by η as in [Lemma 3.4](#) while keeping all other coordinates fixed. We then multiply all coordinates by ξ_1 . The resulting point

has all coordinates except for the zeroth equal to the corresponding coordinates of Q while the zeroth coordinate is now in $\text{Per}_{\phi_0}(\text{lcm}(a, a'))$. Repeating this procedure with ξ_1 replaced by ξ_2 , then ξ_3 and so on, since the lcm of the periods of all the ξ_i is b , it follows that eventually we have that $\xi_0 \in \text{Per}_{\phi_0}(b)$.

We now inductively transform P into Q using the action of G . If ξ_i has period a_i as an element of Per_{ϕ_0} , we may use the action of G to replace it by a primitive $(d^{a_i} - 1)$ -th root of unity. If the highest power of 2 dividing $d^{a_i} - 1$ is equal to the highest power of 2 dividing $d^b - 1$ and d is odd, then we use a primitive $(d^{a_i} - 1/2)$ -th root of unity instead if $i > 0$.

Let $P_0 = P$, and suppose we have constructed $P_i = [\xi_{0,i}, \xi_{1,i}, \dots, \xi_{n-1,i}, 1]$ in $\text{Per}_{\phi_0}(b)$ by induction, with the following properties:

- (1) $\xi_{0,i} \in \text{Per}_{\phi_0}(b)$.
- (2) $\xi_{j,i} = \xi_j$ for $0 < j \leq i$.
- (3) $\xi_{j,1} = 1$ for $j > i$.

Clearly P_0 satisfies these properties; we will show that given P_i with $i < n$ we can find an element of G that transforms it into a point P_{i+1} with the required properties.

So suppose P_i has been constructed. By Lemma 3.4, there exists $\eta \in \text{Per}_{\phi_0}(b)$ so that $\eta^{-1}\xi_{i+1} \in \text{Per}_{\phi_0}(b)$. Since $\xi_{0,i} \in \text{Per}_{\phi_0}(b)$, we may use the action of G to replace $\xi_{0,i}$ with η while keeping the other coordinates fixed. We then multiply all coordinates by η^{-1} , so the zeroth coordinate becomes 1, and the $(i + 1)$ -th coordinate becomes η^{-1} . Since $\eta^{-1}\xi_{i+1} \in \text{Per}_{\phi_0}(b)$, we may use the action of G to replace the $(i + 1)$ -th coordinate by $\eta^{-1}\xi_{i+1}$ while keeping all the other coordinates fixed. If we now multiply all coordinates by η , we obtain a point P_{i+1} with the property that all coordinates except the zeroth and $(i + 1)$ -th of P_i and P_{i+1} are equal, the zeroth coordinate is $\eta \in \text{Per}_{\phi_0}(b)$ and the $(i + 1)$ -th is ξ_{i+1} , so fulfilling the requirements.

We thus obtain a point P_n with the property that all coordinates of P_n and Q are equal except possibly for the first. Since the zeroth coordinate of both is in $\text{Per}_{\phi_0}(b)$ and the n -th is 1, we may use an element of G to transform P_n into Q . It follows that the action of G , hence of the full monodromy group, is transitive on $\text{Per}_{\phi_0}(b)$. \square

Corollary 3.5. *No preperiodic point of $f_{n,d}$ lies in the ramification locus.*

Proof. The ramification locus of $f_{n,d}$ is defined over $k_{n,d}$; thus, if one preperiodic point lies in the ramification locus, so must its entire Galois orbit.

The Galois orbit must map onto the Galois orbit of the corresponding periodic point, i.e., the periodic point y such that $f_{n,d}^r(x) = y$ and $f_{n,d}^s(x)$ is not periodic for any $s < r$. But this orbit consists of all periodic points of a fixed period b by Proposition 3.3. By specialisation to the d -power map $g_{n,d}$, we see that this is not possible:

the ramification locus of this map is actually preserved by the map and for no period b are all the periodic points of period b contained in the ramification locus. \square

Theorem 3.6. *Let $f : \mathbb{P}_K^n \rightarrow \mathbb{P}_K^n$ be a generic endomorphism of degree $d > 1$ over an algebraically closed field K of characteristic zero. Then:*

- (1) *If $X \subset \mathbb{P}_K^n$ is an irreducible subvariety such that $f^r(X) = X$ for some $r > 0$, then X is a point or $X = \mathbb{P}_K^n$.*
- (2) *If $O_f(x)$ is infinite for $x \in \mathbb{P}^n(K)$, then $O_f(x)$ is Zariski-dense in \mathbb{P}_K^n .*

Proof. Since f is generic, we may identify f with $f_{n,d}$ defined above. Any specialisation of X to a subvariety of \mathbb{P}^n defined over a finite extension of $k_{n,d}$ satisfies the same property as X , so we may assume that X is defined over a finite extension of degree m of $k_{n,d}$. Replacing X by the union of its Galois conjugates, we may assume that X is defined over $k_{n,d}$. Since $\dim(X) > 0$, it follows by [Fakhruddin 2003, Theorem 5.1] applied to f^r that X contains infinitely many periodic points of f . Since X is defined over $k_{n,d}$, it then follows from Proposition 3.3 that there is an infinite sequence of distinct integers $b_i, i = 1, 2, \dots$, such that X contains all the periodic points of $f_{n,d}$ of period b_i for all i . Let $g_{n,d} : \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n$ be, as before, the map that raises each coordinate to its d -th power. By specialisation and Lemma 2.2(2), we obtain a subvariety X' of \mathbb{P}_k^n defined over k that contains all the periodic points of $g_{n,d}$ of period b_i for all i .

For $Z \subset \mathbb{A}_k^n$ any subvariety (with k any field), if there is a sequence of finite subsets $S_i \subset \mathbb{A}^n(\bar{k})$ such that $(S_i)^n \subset Z(\bar{k})$ for all i and $|S_i| \rightarrow \infty$, then $Z = \mathbb{A}_k^n$; this is well-known and follows, for example, from [Tao and Vu 2006, Theorem 9.2]. By the previous paragraph, for all i , X' contains all elements of $\mathbb{P}^n(\bar{k})$ of the form $[\xi_0, \dots, \xi_{n-1}, 1]$ with $\xi_j \in \text{Per}_{\phi_0}(b_i)$. Since $|\text{Per}_{\phi_0}(b_i)| \rightarrow \infty$ with i , we must have $X' = \mathbb{P}_k^n$ and hence $X = \mathbb{P}_{k_{n,d}}^n$, proving (1).

To prove (2), we observe that the Zariski closure of $O_f(x)$ is mapped into itself by f . If it is infinite, it must contain a positive-dimensional subvariety X such that $f^r(X) = X$ for some $r > 0$. By (1), we must have $X = \mathbb{P}_K^n$. \square

Remark 3.7. One can give a simpler proof of (1) using the Lefschetz trace formula rather than [Fakhruddin 2003, Theorem 5.1] if one only considers smooth X . Also, from Theorem 4.1, it follows that the elementary version of [Fakhruddin 2003, Theorem 5.1] asserting the density of preperiodic points suffices, but this introduces other dependencies.

4. The dynamical “Manin–Mumford” conjecture for generic endomorphisms

The original “Manin–Mumford” conjecture of Zhang asserted that, for any polarised endomorphism f of a projective variety X over a field K of characteristic 0, any subvariety Y of X containing a Zariski-dense set of preperiodic points is preperiodic.

This was known for abelian varieties and the multiplication-by- m maps but was later shown to be false in general, even if $X = \mathbb{P}_K^n$, by Ghioca and Tucker. Ghioca, Tucker and Zhang then proposed a modified conjecture that takes into account the action of f on the tangent space; see the article [Ghioca et al. 2011] for a discussion of the history, the statement of the modified version and some positive results.²

The following theorem implies that Zhang's conjecture, in its original form, holds for generic endomorphisms of \mathbb{P}_K^n :

Theorem 4.1. *For $f : \mathbb{P}_K^n \rightarrow \mathbb{P}_K^n$ a generic endomorphism of degree $d > 1$ over an algebraically closed field K of characteristic zero, any infinite subset of $\mathbb{P}^n(K)$ consisting of f -preperiodic points is Zariski-dense in \mathbb{P}_K^n .*

As in the other results of this paper, one of the key ingredients of the proof is the Galois action on the set of periodic points. However, Proposition 3.3 alone does not suffice since we also need the transitivity of the monodromy action on the set of preperiodic points. It turns out that transitivity does not hold for the monodromy action on $\text{Prep}_f(a, b)$ for f as in Theorem 3.2 and $d > 2$. Nevertheless, by considering a larger family of polynomials, we prove in Proposition 4.6 that the monodromy action on $\text{Prep}_{f_{n,d}}(a, b)$ is indeed transitive for all a and b . This then allows us to use a specialisation argument to prove Theorem 4.1.

Fix $d > 1$. For $b > 0$, consider the polynomial $P_b(c) = \phi_c^b(0)$. The roots of $P_b(c)$ are exactly the parameters c so that 0 is a periodic point of period dividing b for the polynomial $\phi_c(z)$.

Lemma 4.2 (Gleason). *All roots of P_b are multiplicity-free.*

Proof. The proof given in [Douady and Hubbard 1985, Lemma 19.1] for $d = 2$ goes through for general d simply by replacing 2 by d . \square

Lemma 4.3. *Fix $d > 1$. For $(c, \epsilon) \in \mathbb{A}^2$, let $\phi_{c,\epsilon} : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ be given by $\phi_{c,\epsilon}(z) = z(z - \epsilon)^{d-1} + c$. Then the monodromy action on $\text{Prep}_{\phi_{c,\epsilon}}(1, b)$ is transitive for all $b > 0$.*

Proof. It follows from Theorem 3.2 that the monodromy action on $\text{Per}_{\phi_{c,\epsilon}}(b)$ is transitive for all $b > 0$, so it suffices to prove that for any $b > 0$ there exists $x \in \text{Per}_{\phi_{c,\epsilon}}(b)$ and an element γ of the monodromy so that $\gamma(x) = x$ and γ cyclically permutes the $d - 1$ elements of $\phi_{c,\epsilon}^{-1}(x)$ (so if $d = 2$, there is nothing to prove).

By Lemma 4.2 and a simple counting argument, it follows that, for any $b > 0$, there exists $c_b \in \mathbb{C}$ so that $0 \in \text{Per}_{\phi_{c_b}}(b)$. Since 0 is a critical point of ϕ_{c_b} , it is of multiplicity 1 as an element of $\text{Per}_{\phi_{c_b}}(b)$. It follows that, for $|\epsilon| \ll 0$, there exists $c_{b,\epsilon}$ close to c_b so that 0 is a periodic point of $\text{Per}_{\phi_{c_{b,\epsilon},\epsilon}}(b)$ of multiplicity 1. By the

²It seems reasonable to expect the modified conjecture to hold even for quasipolarised endomorphisms.

definition of $\phi_{c,\epsilon}$, it then follows that ϵ is the unique element of $\text{Prep}_{\phi_{c_b,\epsilon,\epsilon}}(1, b)$ that is mapped to $\phi_{c_b,\epsilon,\epsilon}(0) = c_{b,\epsilon}$ by $\phi_{c_b,\epsilon,\epsilon}$.

We now consider the 1-parameter family of maps $\phi_{c,\epsilon}$ with ϵ such that $|\epsilon| \ll 0$ fixed. In a neighbourhood of $c_{b,\epsilon}$, the element $c_{b,\epsilon} \in \text{Per}_{\phi_{c_b,\epsilon,\epsilon}}(b)$ deforms uniquely as an element of $\text{Per}_{\phi_{c,\epsilon}}(b)$. However, since the critical points of $\phi_{c,\epsilon}$ are independent of c , it follows that the element $\epsilon \in \text{Prep}_{\phi_{c_b,\epsilon,\epsilon}}(1, b)$ deforms to $d-1$ distinct elements of $\text{Prep}_{\phi_{c,\epsilon}}(1, b)$ that are all mapped to the deformed periodic point above by $\phi_{c,\epsilon}$.

We claim that the monodromy action of a small loop around $c_{b,\epsilon}$ gives us the required element γ . Since $c_{b,\epsilon}$ deforms uniquely as a periodic point, the monodromy action of γ on this point is trivial as required. To prove that the second condition is satisfied, let $C \subset \mathbb{A}^2$ be the curve consisting of all points (z, c) so that z is a preperiodic point of preperiod 1 and period b of $\phi_{c,\epsilon}$ (with ϵ fixed as above). It suffices to prove that C is smooth at the point $(\epsilon, c_{b,\epsilon})$.

To see this, we consider the explicit equation for C . It is given in an neighbourhood of $(\epsilon, c_{b,\epsilon})$ by

$$\phi_{c,\epsilon}^{b+1}(z) - \phi_{c,\epsilon}(z) = 0. \quad (4-1)$$

To see that it is smooth at $(c_{b,\epsilon}, \epsilon)$, it suffices to substitute ϵ for z and check that the resulting polynomial in c , $\phi_{c,\epsilon}^{b+1}(\epsilon) - \phi_{c,\epsilon}(\epsilon)$, has $c_{b,\epsilon}$ as a root of multiplicity 1. However, from the definition of $\phi_{c,\epsilon}$ it follows that

$$\phi_{c,\epsilon}^{b+1}(\epsilon) - \phi_{c,\epsilon}(\epsilon) = \phi_{c,\epsilon}^{b+1}(0) - \phi_{c,\epsilon}(0),$$

so we may replace ϵ by 0. In a neighbourhood of the point $(0, c_{b,\epsilon})$, the curve given by (4-1) is smooth since it parametrises periodic points of period b and the periodic point 0 of $\phi_{c_b,\epsilon,\epsilon}$ is of multiplicity 1. To show that the multiplicity of $c_{b,\epsilon}$ as a root of $\phi_{c,\epsilon}^{b+1}(0) - \phi_{c,\epsilon}(0)$ is 1, we may then specialise ϵ to 0, so it suffices to consider the multiplicity of c_b as a root of the polynomial $P_b(c) = \phi_c^b$. By Lemma 4.2, this multiplicity is indeed 1 as required. \square

To prove the transitivity of the monodromy action on $\text{Prep}_{\phi_{c,\epsilon}}(a, b)$ for $a > 1$, we shall need some results about Misiurewicz points. We refer the reader to [Lau and Schleicher 1994; Eberlein 1999] for the basic facts that we use below, which generalise results proved in [Douady and Hubbard 1984] in the case $d = 2$. Recall that $c_0 \in \mathbb{C}$ is called a Misiurewicz point if c_0 is a strictly preperiodic point of the map ϕ_{c_0} . By the results of [op. cit.], for any strictly preperiodic angle $\theta \in \mathbb{Q}/\mathbb{Z}$, there is a Misiurewicz point c_0 such that the parameter ray with angle θ lands at c_0 . By [Eberlein 1999, Lemma 8.3], the preperiod of θ (with respect to multiplication by d) is equal to the preperiod of c_0 (with respect to ϕ_{c_0}) and the period of the kneading sequence of θ , $K(\theta)$, is equal to the period of c_0 (with respect to ϕ_{c_0}). For $\theta = 1/(d^a \cdot (d^b - 1))$, $a, b > 0$, the preperiod of θ is a and the period of $K(\theta)$ is b , so there exists a Misiurewicz point with any preperiod $a > 0$ and period b .

A point $\lambda \in \mathbb{C}$ is called parabolic of period b if it is the landing point of a parameter ray with angle θ that is periodic of period b . By results from [Douady and Hubbard 1984; Eberlein 1999], a parabolic point is never a Misiurewicz point.

To prove the transitivity of the monodromy action for $a > 1$, we shall need the following analogue of Lemma 4.2, due to Douady and Hubbard:

Lemma 4.4. *For a Misiurewicz point c_0 as above, the equation $\phi_c^{a+1+b}(0) - \phi_c^{a+1}(0) = 0$ has a simple root at $c = c_0$.*

Proof. The lemma is formulated and proved for $d = 2$ as Corollary 8.5 of [Douady and Hubbard 1984]; however, Proposition 8.5 of the same work holds for general d , and so the proof goes through if we substitute Theorems 8.1 and 8.2 of [Eberlein 1999] for Douady and Hubbard's Theorem 8.2. \square

Lemma 4.5. *The monodromy action of the 2-dimensional family of polynomials $\phi_{c,\epsilon}$ on $\text{Prep}_{\phi_{c,\epsilon}}(a, b)$ is transitive for all $a, b > 0$.*

Proof. We already know transitivity if $a = 1$. Thus, by induction, we may assume $a > 1$, and then it suffices to prove that for the 1-dimensional family of polynomials ϕ_c , there exists $x \in \text{Per}_{\phi_c}(a - 1, b)$ and an element γ of the monodromy such that $\gamma(x) = x$ and γ induces a cyclic permutation on the d elements of $\text{Per}_{\phi_c}(a - 1, b)$ comprising $\phi_c^{-1}(x)$.

Since 0 and ∞ are the only critical points of ϕ_λ , the preperiodic points for general λ are multiplicity-free. Let c be a Misiurewicz point of preperiod $a - 1$ and period b , and consider a small loop γ in the parameter plane around c such that all parabolic points of period b and all Misiurewicz points of preperiod a and period b are outside this loop. We note that the preperiodic points of ϕ_c of preperiod $\leq a$ and period b are multiplicity-free since none of them are critical values and a Misiurewicz point is never a parabolic point. This remains true in a neighbourhood of c , so we may assume that this holds in a neighbourhood U of γ containing its interior; in particular, c deforms uniquely as a preperiodic point $c_\lambda \in \text{Prep}_{\phi_\lambda}(a, b)$ as λ varies in this neighbourhood.

Since ϕ_c is totally ramified at 0, 0 is the unique element of $\text{Prep}_{\phi_c}(a, b)$ mapping to c by ϕ_c . By construction, for any other $\lambda \in U$, there are d points of $\text{Prep}_{\phi_\lambda}(a, b)$ mapping to c_λ and these points all come together at 0 as $\lambda \rightarrow c$. The set of these points in a neighbourhood of $(c, 0)$ is exactly the zero locus D of the polynomial $\phi_\lambda^{a+1+b}(z) - \phi_\lambda^{a+1}(z)$. By Lemma 4.4, the multiplicity of this after setting $z = 0$ is 1 at $(c, 0)$, so it follows that D must be smooth at this point. Since the inverse image of c in D is a single point, it follows that the map induced by the projection to the first factor is totally ramified of degree d at $(c, 0)$. Consequently, the monodromy around γ induces a cyclic permutation of order d on $\phi_\lambda^{-1}(c_\lambda)$. We may thus take $x = c_\lambda$ for any λ with $|\lambda| \ll 0$ to complete the proof. \square

Proposition 4.6. *The Galois action on $\text{Prep}_{f_{n,d}}(a, b)$ is transitive for all $a, b > 0$.*

Proof. Since $f_{n,d}$ is defined over $k_{n,d}$, the Galois action on $\text{Prep}_{f_{n,d}}(a, b)$, $a, b \geq 1$, is compatible with the natural surjections

$$\text{Prep}_{f_{n,d}}(a + 1, b) \xrightarrow{f_{n,d}} \text{Prep}_{f_{n,d}}(a, b).$$

Thus, by induction on a , it suffices to show that for any $a > 0$, there exists an element $x \in \text{Prep}_{f_{n,d}}(a - 1, b)$ such that, for any $y, y' \in \text{Prep}_{f_{n,d}}(a, b)$ with $f_{n,d}(y) = f_{n,d}(y')$, there exists an element γ of the monodromy such that $\gamma(y) = y'$.

If $n = 1$, the claim follows immediately from [Lemma 4.5](#), so in the following, we shall assume $n > 1$.

As before, we may assume that $k = \mathbb{C}$. The proof of transitivity is similar to that for the case of periodic points except that we replace the use of the maps ϕ_c by $\phi_{c,\epsilon}$. So for $0 \leq i \leq n$, consider the family of endomorphisms $f_i : \mathbb{P}^n \times \mathbb{A}^{2n} \rightarrow \mathbb{P}^n \times \mathbb{A}^{2n}$ defined by

$$\begin{aligned} f_i &((x_0, \dots, x_n], (c_1, \dots, c_n, \epsilon_1, \dots, \epsilon_n)) \\ &= \left([x_0(x_0 - \epsilon_1 x_1)^{d-1} + c_1 x_1^d, \dots, x_{i-1}(x_{i-1} - \epsilon_i x_i)^{d-1} + c_i x_i^d, x_i^d, \right. \\ &\quad \left. x_{i+1}(x_{i+1} - \epsilon_{i+1} x_i)^{d-1} + x_i^d, \dots, x_n(x_n - \epsilon_n x_i)^{d-1} + c_n x_i^d \right], \\ &\quad (c_1, \dots, c_n, \epsilon_1, \dots, \epsilon_n) \end{aligned}$$

On the open affine U_i given by $x_i \neq 0$, f_i is the product of the n polynomials ϕ_{c_j, ϵ_j} .

Let $g_{n,d}$ be the d -power map as before. Contrary to the case of periodic points, the map from the locus of preperiodic points to the base is not étale at preperiodic points of $g_{n,d}$ contained in its ramification locus. However, the map is étale at preperiodic points all of whose coordinates are nonzero, and this will suffice (except when $d = 2$) for our needs (compare the discussion of monodromy on page [589](#)).

Suppose $a = 1$. Let ζ be a primitive $(d^b - 1)$ -th root of unity, and let $x' \in \text{Per}_{g_{n,d}}(b)$ be the point $[\zeta, 1, \dots, 1, 1]$, and let $x = g_{n,d}(x')$. The preperiodic points y such that $g_{n,d}(y) = x$ are of the form $[\zeta \xi_1, \xi_2, \dots, \xi_n, 1]$, where all the ξ_i are d -th roots of unity and at least one of them is not equal to 1.

We now also assume that $d > 2$. Using the monodromy action of the family f_n and [Lemma 4.3](#), it follows that we may assume that $\xi_i = \xi$, where ξ is a fixed d -th root of unity or $\xi_i = 1$.

Since $d > 1$, there exists a d -th root of unity ξ' such that $\xi' \neq 1, \xi$. Let $y = [\zeta \xi, 1, \dots, 1]$ and $y' = [\zeta \xi, \xi, \dots, \xi, 1, \dots, 1]$, where there are n_1 ξ 's and n_2 1 's with $n_1, n_2 > 0$. Each step of the following sequence of transformations is given either by multiplying through by a constant or by applying the monodromy of f_i for some i such that the i -th coordinate is equal to 1:

$$\begin{aligned}
y = [\zeta \xi, 1, \dots, 1] &\rightarrow [1, \zeta^{-1} \xi^{-1}, \dots, \zeta^{-1} \xi^{-1}] \\
&\rightarrow [1, \zeta^{-1} \xi'^{-1}, \dots, \zeta^{-1} \xi'^{-1}, \zeta^{-1} \xi^{-1}, \dots, \zeta^{-1} \xi^{-1}] \\
&\rightarrow [\zeta \xi, \xi \xi'^{-1}, \dots, \xi \xi'^{-1}, 1, \dots, 1] \\
&\rightarrow [\zeta \xi, \xi, \dots, \xi, 1, \dots, 1] = y'.
\end{aligned}$$

In the last transformation, we also use the fact that $\xi \xi'^{-1}$ is a d -th root of unity not equal to 1 so, like ξ and ξ' , an element of $\text{Prep}_{\phi_0}(1, 1)$.

Let $y'' = [\zeta, \xi, \dots, \xi, 1, \dots, 1]$, where there are n_1 ξ 's and n_2 1's with $n_1, n_2 > 0$ as before. Since $d > 2$, there exists $\zeta' \in \text{Per}_{\phi_0}(b)$ such that $\zeta \zeta'^{-1}$ also has period b . We then have a similar sequence of transformations:

$$\begin{aligned}
y' = [\zeta \xi, \xi, \dots, \xi, 1, \dots, 1] &\rightarrow [1, \zeta^{-1}, \dots, \zeta^{-1}, \zeta^{-1} \xi^{-1}, \dots, \zeta^{-1} \xi^{-1}] \\
&\rightarrow [1, \zeta' \zeta^{-1}, \dots, \zeta' \zeta^{-1}, \zeta^{-1} \xi^{-1}, \dots, \zeta^{-1} \xi^{-1}] \\
&\rightarrow [\zeta \xi, \zeta' \xi, \dots, \zeta' \xi, 1, \dots, 1] \\
&\rightarrow [\zeta \xi, \zeta' \xi', \dots, \zeta' \xi', 1, \dots, 1] \\
&\rightarrow [\zeta \xi (\zeta' \xi')^{-1}, 1, \dots, 1, (\zeta' \xi')^{-1}, \dots, (\zeta' \xi')^{-1}] \\
&\rightarrow [\zeta (\zeta' \xi)^{-1}, 1, \dots, 1, (\zeta' \xi)^{-1}, \dots, (\zeta' \xi)^{-1}] \\
&\rightarrow [\zeta, \zeta' \xi, \dots, \zeta' \xi, 1, \dots, 1] \\
&\rightarrow [1, \zeta^{-1} \zeta' \xi, \dots, \zeta^{-1} \zeta' \xi, \zeta^{-1}, \dots, \zeta^{-1}] \\
&\rightarrow [1, \zeta^{-1} \xi, \dots, \zeta^{-1} \xi, \zeta^{-1}, \dots, \zeta^{-1}] \\
&\rightarrow [\zeta, \xi, \dots, \xi, 1, \dots, 1] = y''.
\end{aligned}$$

Here we have used that $\zeta, \zeta', \zeta \zeta'^{-1}$ and their inverses are in $\text{Per}_{\phi_0}(b)$ and each one of these multiplied by $\xi, \xi', \xi \xi^{-1}$ or any of their inverses is an element of $\text{Prep}_{\phi_0}(1, b)$. By symmetry, we then conclude that the desired transitivity holds in this case.

Now suppose $d = 2$. In this case, $\text{Prep}_{\phi_0}(1, 1) = \{-1\}$, a singleton, so the above argument breaks down, and we will need to consider paths passing through elements of $\text{Prep}_{g_{n,d}}(1, b)$, one of whose coordinates is 0. This is justified by [Lemma 4.7](#) below.

As before, let $y = [-\zeta, \dots, -\zeta, 1]$ with ζ a primitive $(2^b - 1)$ -th root of unity and $y' = [-\zeta, \dots, -\zeta, \zeta, \dots, \zeta, 1]$, where there is at least one $-\zeta$ and one ζ . We need to consider the cases $b = 1$ and $b > 1$ separately.

First suppose $b = 1$, so $\zeta = 1$. We then have the sequence of transformations

$$\begin{aligned}
y = [-1, \dots, -1, 1] &\rightarrow [1, \dots, 1, -1] \rightarrow [1, \dots, 1, 0, -1] \\
&\rightarrow [-1, \dots, -1, 0, 1] \rightarrow [-1, \dots, -1, 1, 1].
\end{aligned}$$

Repeating this procedure, we see that we can connect y to y' , and by symmetry, the transitivity follows in this case.

Now suppose $b > 1$, so there exists $\zeta' \in \text{Per}_{\phi_0}(b)$ so that, as before, $\zeta \zeta'^{-1} \in \text{Per}_{\phi_0}(b)$. We then have a sequence of transformations

$$\begin{aligned} y &= [-\zeta, \dots, -\zeta, 1] \rightarrow [1, \dots, 1, -\zeta^{-1}] \rightarrow [1, \dots, 1, 0, -\zeta^{-1}] \\ &\rightarrow [-\zeta, \dots, -\zeta, 0, 1] \rightarrow [-\zeta, \dots, -\zeta, 1, 1] \\ &\rightarrow [1, \dots, 1, -\zeta^{-1}, -\zeta^{-1}] \rightarrow [1, \dots, 1, -\zeta'^{-1}, -\zeta^{-1}] \\ &\rightarrow [-\zeta^{-1}, \dots, -\zeta^{-1}, \zeta \zeta'^{-1}, 1] \rightarrow [-\zeta^{-1}, \dots, -\zeta^{-1}, \zeta, 1]. \end{aligned}$$

Repeating this procedure, we see that we can connect y to y' , and then by symmetry, transitivity follows.

Finally, suppose $a > 1$. By induction, we can choose x to be an arbitrary point of preperiod $a - 1$ and period b , so we let $x' = [\zeta, \zeta, \dots, \zeta, 1]$ where ζ is an element of $\text{Prep}_{\phi_0}(a, 1)$ and $x = g_{n,d}(x)$. The points in $g_{n,d}^{-1}(x)$ are of the form $[\zeta \xi_1, \dots, \zeta \xi_n, 1]$ where ξ_1 is a d -th root of 1, so $\zeta \xi_i \in \text{Prep}_{\phi_0}(a, 1)$. One sees that the monodromy acts transitively on $g_{n,d}^{-1}(x)$ simply by considering the monodromy of the family of maps f_n and applying [Lemma 4.5](#). \square

Let $F : \mathbb{P}_k^n \times \text{Mor}_{n,d/k} \rightarrow \mathbb{P}_k^n \rightarrow \text{Mor}_{n,d/k}$ be the universal morphism of degree d , and consider $P_F(b+1, b) \subset \text{Mor}_{n,d/k}$ (the notation is defined just before [Lemma 2.2](#)). The fibre of the projection map from $P_F(b+1, b)$ to $\text{Mor}_{n,d/k}$ over any point $f \in \text{Mor}_{n,d/k}$ consists of f -preperiodic points of preperiod at most 1 and period dividing b .

Lemma 4.7. *If $d = 2$ and $\text{char}(k) \neq 2$, then $P_F(b+1, b)$ is smooth at any preperiodic point of $g_{n,2}$ of preperiod 1 and period b .*

Proof. We have $F(P_F(b+1, b)) = P_F(b)$. As we have already seen, $P_F(b)$ is smooth at all periodic points of $g_{n,d}$. Moreover, F is analytically locally at any point of $\text{Prep}_{g_{n,d}}(1, b)$ with exactly one coordinate equal to 0, a cyclic cover of degree 2. Thus, to prove smoothness, it suffices to show that the discriminant of F intersects $P_F(b)$ transversely at any point of $\text{Per}_{g_{n,d}}(b)$ with exactly one coordinate equal to 0.

To prove transversality, it then suffices to restrict to any subvariety of $\text{Mor}_{n,d}$ and prove transversality for the induced subvarieties. By considering, say, the family f_i as in the proof of [Proposition 3.3](#), where i is such that the i -th coordinate of the point under consideration is nonzero, we reduce to the case of the one parameter ϕ_c , $c \in \mathbb{A}_c^1$, and we need to prove transversality at the point $(z, c) = (0, 0)$. The discriminant locus is given by $z = 0$ (since $d = 2$) and the locus of fixed points by $z^2 + c - z = 0$, so the lemma follows. \square

Proof of Theorem 4.1. Let X be the Zariski closure of an infinite subset of preperiodic points. By Lemma 2.2(1), all preperiodic points of a quasipolarised map are defined over the algebraic closure of the base field, so we may assume without loss of generality that K is the algebraic closure of $k_{n,d}$ and X is defined over a finite extension of $k_{n,d}$. By replacing X by the union of its Galois conjugates, we may then assume that X is defined over $k_{n,d}$.

Since $\text{Prep}_{f_{n,d}}(a, b)$ is finite for all a and b , there exists an infinite sequence of tuples (a_i, b_i) , with $a_i \geq 0$ and $b_i > 0$, so that X contains a point $x_i \in \text{Prep}_{f_{n,d}}(a_i, b_i)$ for all i . Since X is defined over $k_{n,d}$, it follows from Propositions 3.3 and 4.6 that $\text{Prep}_{f_{n,d}}(a_i, b_i) \subset X$ for all i . We let X' be the Zariski closure of the specialisations, over the point in $\text{Mor}_{n,d}$ corresponding to $g_{n,d}$, of the preperiodic points in $X(K)$. Since all preperiodic points lift to the generic fibre by Lemma 2.2, it follows that $X' \subset \mathbb{P}_k^n$ has the same properties as X but with respect to $\text{Per}_{g_{n,d}}$.

For any i , the set of points in \mathbb{P}_k^n of the form $[\xi_0, \dots, \xi_{n-1}, 1]$ with $\xi_i \in \text{Prep}_{\phi_0}(a_i, b_i)$ is contained in $\text{Prep}_{g_{n,d}}(a_i, b_i)$ and hence in X' . As in the proof of Theorem 3.6, it then follows that $X' = \mathbb{P}_k^n$, so $X = \mathbb{P}_{k_{n,d}}^n$. □

Remark 4.8. Note that we do not use the full strength of the genericity hypothesis in the proofs of this section or of the previous one. It suffices to assume that the morphism under consideration corresponds to the generic point of an irreducibility subvariety of $\text{Mor}_{d,n/k}$ that contains all the families $f_{c,\epsilon}$ and is smooth at $g_{n,d}$. Since all the $f_{c,\epsilon}$ are smooth and have dimension $2n$, there exist such subvarieties for all n with dimension independent of d .

5. The dynamical “Mordell–Lang” conjecture for generic endomorphisms

Let (X, f) be an algebraic dynamical system over a field K of characteristic 0; i.e., X is an algebraic variety and $f : X \rightarrow X$ is a morphism. The conjecture of Ghioca and Tucker [2009] asserts that, if $x \in X(K)$ and Y a subvariety of X are such that $O_f(x) \cap Y(K)$ is infinite, then there is a periodic subvariety Z of X with $Z \subset Y$ and $Z(K) \cap O_f(x) \neq \emptyset$. It has been proved when f is étale by Bell et al. [2010] and in a few other cases. It is not known in general if $X = \mathbb{P}_K^n$ and $\deg(f) > 1$; this was the original case investigated by Denis [1994], who proved the assertion under the assumption that $O_f(x) \cap Y(K)$ is large in a suitable sense.

For $(X, f) = (\mathbb{P}_K^n, f)$, with f a generic endomorphism, by Theorem 3.6, there are no nontrivial f -periodic subvarieties contained in \mathbb{P}_K^n , so the conjecture in this case is equivalent to the following:

Theorem 5.1. *Let $f : \mathbb{P}_K^n \rightarrow \mathbb{P}_K^n$ be a generic endomorphism of degree $d > 1$ over an algebraically closed field K of characteristic zero. For each $x \in \mathbb{P}^n(K)$, every infinite subset of $O_f(x)$ is Zariski-dense in \mathbb{P}_K^n .*

The idea of the proof is as follows. We first use specialisation to reduce to the case that K is a finite extension of $k_{n,d}$. We then show using a p -adic argument, for a prime p dividing d , that any Y such that $O_f(x) \cap Y$ is infinite must contain infinitely many periodic points, or there exists a prime q not dividing d , such that x and f have specialisations \bar{x} and \bar{f} over \mathbb{F}_q with \bar{f} étale on the orbit of \bar{x} . Both these conditions lead to the conclusion that $Y = \mathbb{P}^n_K$, the first from [Theorem 4.1](#) and second by using a result of Bell, Ghioca and Tucker, which we recall:

Lemma 5.2. *Let L/\mathbb{Q}_p be a finite extension, $\pi : X \rightarrow \text{Spec}(R)$ a smooth scheme of finite type over the ring of integers R of L and $f : X \rightarrow X$ a morphism over $\text{Spec}(R)$. Suppose $x \in X(R)$ is such that f is étale on the orbit of x . If $Y \subset X$ is any closed subscheme with $Y \cap O_f(x)$ infinite, then Y_L contains a positive-dimensional periodic subvariety of X_L .*

Proof. If $L = \mathbb{Q}_p$, this is an immediate consequence of the results in [\[Bell et al. 2010\]](#). For general L , it follows from the methods in the same work, if one replaces [Theorem 3.3](#) therein by [Theorem 7](#) of [\[Amerik 2011\]](#). □

Lemma 5.3. *[Theorem 5.1](#) for arbitrary extensions K of $k_{n,d}$ follows from the case of finite extensions.*

Proof. Without loss of generality, we may assume that $k = \mathbb{Q}$ and K is a finitely generated extension of $k_{n,d}$. Let $x \in \mathbb{P}^n(K)$, and let Y be a subvariety of \mathbb{P}^n_K such that $O_f(x) \cap Y(K)$ is infinite.

Since K is finitely generated, there exists a smooth irreducible scheme M of finite type over \mathbb{Z} with function field K and a dominant morphism $\pi : M \rightarrow \text{Mor}_{n,d}$ inducing the inclusion $k_{n,d} \subset K$ on function fields. Let $\mathbf{f} : M \times \mathbb{P}^n \rightarrow M \times \mathbb{P}^n$ be the pullback of the universal morphism from $\text{Mor}_{n,d} \times \mathbb{P}^n$, so \mathbf{f} restricted to the generic fibre of the first projection is equal to f . Let \mathbf{x} and \mathbf{Y} be the Zariski closures of x and Y , respectively, in $M \times \mathbb{P}^n$. By shrinking M if necessary, we may assume that \mathbf{x} and \mathbf{Y} are flat over M .

Since π is a dominant finite-type morphism, there exists a point f' of M (which we think of as an endomorphism of \mathbb{P}^n using π) mapping to the generic point of $\text{Mor}_{n,d}$ and so that the residue field K' at f' is a finite extension of $k_{n,d}$. Let x' and Y' denote the fibres of \mathbf{x} and \mathbf{Y} , respectively, over f' . If $O_{f'}(x')$ is infinite, then so is $O_{f'}(x') \cap Y'$; hence, by the condition on K' , it would follow that $Y' = \mathbb{P}^n_{K'}$, which (by flatness) implies $Y = \mathbb{P}^n_K$.

If $O_{f'}(x')$ is finite, then x' is f' -preperiodic. Since f' is generic, it follows from [Corollary 3.5](#) that $O_{f'}(x')$ does not intersect the ramification locus of f' . Let $|O_{f'}(x')| = n$, and let \mathbf{Z} denote the Zariski closure of $\{x, f(x), \dots, f^{n-1}(x)\}$ in $M \times \mathbb{P}^n$. Let $\mathbf{R} \subset M \times \mathbb{P}^n$ be the ramification locus of \mathbf{f} , and consider the closed subset $\mathbf{R} \cap \mathbf{Z}$ of $M \times \mathbb{P}^n$. By the above, the fibre of this subset over f' is empty, so

by the properness of \mathbb{P}^n , its projection in M is a proper closed subset. Replacing M by the complement of this subset, we may assume that $\mathbf{R} \cap \mathbf{Z} = \emptyset$.

Now let f'' be any closed point (which we again think of as an endomorphism) of M that lies in the closure of f' . Since M is of finite type over \mathbb{Z} , the residue field of f'' is a finite field F . Let x'' and Y'' denote the fibres of \mathbf{x} and \mathbf{Y} , respectively, over f'' . Since f'' is in the closure of f' , x'' is in the closure of x' ; hence, $|O_{f''}(x'')| \leq n$. Since $\mathbf{R} \cap \mathbf{Z} = \emptyset$, it follows that f'' is unramified at all points of $O_{f''}(x'')$. Let $W(F)$ be the ring of Witt vectors of F . Since M is smooth over \mathbb{Z} , by Hensel's lemma, the set of points in $M(W(F))$ that specialise to f'' is in bijection (after choosing local coordinates) with $W(F)^n$. The subset consisting of points that lie in a proper closed subscheme of M is a countable union of nowhere dense (in the adic topology) subsets. It follows by Baire's theorem that there exists a point in $M(W(F))$ specialising to f'' and not lying in any proper closed subscheme. Letting L be the quotient field of $W(K)$, it follows that the image of the induced map from $\text{Spec}(L)$ to M must be the generic point. We thus get an inclusion of K into L , and we may apply Lemma 5.2 with $R = W(K)$ and $X = \text{Spec}(W(F)) \times \mathbb{P}^n$ to the base change of f , \mathbf{x} and \mathbf{Y} via the morphism $\text{Spec}(W(F)) \rightarrow M$ to conclude using Theorem 3.6(1) that $Y = \mathbb{P}_K^n$. \square

Lemma 5.4. *Let p be a prime and $g_{n,d,p}$ denote the endomorphism of $\mathbb{P}_{\mathbb{F}_p}^n$ given by raising each coordinate to its d -th power. Let $X \subset \mathbb{P}_{\mathbb{F}_p}^n$ be a positive-dimensional subvariety. Then the set $\bigcup_{r \geq 0} g_{n,d,p}^r(X(\overline{\mathbb{F}_p}))$ contains periodic points of infinitely many distinct periods.*

Proof. Since $g_{n,d,p}$ preserves the standard decomposition of \mathbb{P}^n as a disjoint union of affine spaces, by projecting to a suitable coordinate, we reduce to the statement for $n = 1$, in which case the statement is obvious. \square

Remark 5.5. We expect that the lemma holds with $g_{n,d,p}$ replaced by an arbitrary quasipolarised morphism — or even more generally with some extra conditions on X — defined over a finite field, but this seems much harder to prove. However, for endomorphisms of abelian varieties the corresponding statement can indeed be proved.

Lemma 5.6. *Let $d > 1$ be an integer and p a prime such that $p \mid d$. Let ϕ be the morphism $\mathbb{A}^2 \rightarrow \mathbb{A}^2$ given by $\phi(x, c) = (x^d + c, c)$ over the field $\overline{\mathbb{F}_p}$.*

- (a) *For $c \in \overline{\mathbb{F}_p}$, the monodromy action on the set of fixed points of ϕ_c is transitive.*
- (b) *Let $X \subset \mathbb{A}^2$ be an irreducible subvariety of dimension 1, mapping dominantly to \mathbb{A}^1 via the second projection p_2 . Assume that the intersection of X with the generic fibre of p_2 is not a preperiodic point of ϕ . Then the ϕ -periods of the points in $X(\overline{\mathbb{F}_p})$ (which are all preperiodic) are unbounded.*

Proof. We will use elementary intersection theory on $\mathbb{P}^1 \times Y$ with Y a smooth projective curve.

Let $X_n = \phi^n(X)$ for $n \geq 0$. By replacing X by $\phi^r(X)$ for some $r \geq 0$, we assume that the map $p_2 : X_n \rightarrow \mathbb{P}^1$ has degree e for all n .

For any integer $b > 0$, let P_b be the locus of points in \mathbb{A}^2 that are ϕ -periodic of period b . Since $p \mid d$, ϕ is inseparable on the fibres of p_2 , so the graph of $\phi^b|_{\mathbb{P}^1 \times \{c\}}$ intersects the diagonal in \mathbb{A}^2 transversely for all $c \in \mathbb{A}^1$ and all b . It follows that P_b is a finite étale cover of \mathbb{A}^1 via p_2 and $P_b \cap P_{b'} = \emptyset$ for $b \neq b'$.

Let \bar{P}_b be the closure of P_b in $\mathbb{P}^1 \times \mathbb{P}^1 \supset \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$, and let \bar{X}_n be the closure of X_n in $\mathbb{P}^1 \times \mathbb{P}^1$. The curve P_b is a subcurve of the curve Q_b in \mathbb{A}^2 with equation

$$(\dots((x^d + c)^d + c)^d \dots)^d + c - x = 0,$$

where we have b pairs of brackets. Replacing c by $1/c'$ in the above equation and multiplying through by $c'^{d(b-1)}$, we get the equation

$$(\dots((c'x^d + 1)^d + c'^{d^{b-1}-d^{b-2}})^d \dots)^d + c'^{d^{b-1}-1} - c'^{d^{b-1}}x = 0.$$

It follows that the only point on all the \bar{P}_b intersecting the fibre over $c = \infty$ is the point at infinity on this fibre and the support of $\bar{P}_b \cap \bar{P}_{b'}$ is equal to this point if $b \neq b'$. When $b = 1$, the equation is

$$c'x^d + 1 - c'x = 0.$$

One then sees that \bar{P}_1 is irreducible since the equation shows that it is smooth at the point (∞, ∞) and the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of any irreducible component of P_1 must contain this point; this proves (a).

If the ϕ -periods of all points in $X(\bar{\mathbb{F}}_p)$ are bounded, then there must exist some $b > 0$ so that $|\bar{X}_n \cap \bar{P}_b| \rightarrow \infty$ as $n \rightarrow \infty$ and, by assumption, $\bar{X}_n \not\subseteq \bar{P}_b$ for all b . Writing $[\bar{X}_n] = e[\{0\} \times \mathbb{P}^1] + a_n[\mathbb{P}^1 \times \{0\}]$ in $\text{NS}(\mathbb{P}^1 \times \mathbb{P}^1)$, it follows that $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

Let \bar{X}' be the normalisation of \bar{X} , and denote by $\eta : \bar{X}' \rightarrow \mathbb{P}^1$ the composition of the normalisation map and the projection p_2 . Let x_1, x_2, \dots, x_r be the points in \bar{X} mapping to the point (∞, ∞) in $\mathbb{P}^1 \times \mathbb{P}^1$. If $r = 0$, it follows that $X \cap P_b \neq \emptyset$ for all b , so we may assume that $r > 0$. Let b_1, b_2, \dots, b_{r+1} be any distinct integers, and let $\gamma : Y \rightarrow \bar{X}'$ be a Galois cover such that there are components S_i of $(\text{id} \times \eta\gamma)^{-1}(\bar{P}_{b_i})$ in $\mathbb{P}^1 \times Y$ that map isomorphically to Y via the second projection. Let Z be the section of this projection induced by the tautological section of $\mathbb{P}^1 \times \bar{X}'$, and let $Z_n = \psi^n(Z)$, where ψ is the map of $\mathbb{P}^1 \times Y$ induced by ϕ . We may write $[S_i] = [* \times Y] + s_i[\mathbb{P}^1 \times *]$ and $[Z_n] = [* \times Y] + a_n[\mathbb{P}^1 \times *]$ in $\text{NS}(\mathbb{P}^1 \times Y)$ where $s_i \geq 0$ and $*$ denotes any point. It follows that the intersection number $S_i \cdot Z_n = s_i + a_n \rightarrow \infty$ as $n \rightarrow \infty$ for all $i = 1, 2, \dots, r$.

Now consider the local intersection multiplicity $I_y(S_i, Z_n)$ of S_i and Z_n at a point $\infty \times y$, where $y \in Y$ is such that $\gamma(y) = x_j$ for some j . If this is bounded for all such y and all n , since $S_i \cdot Z_n \rightarrow \infty$, it would follow that for large n , S_i and Z_n must intersect in a point (z, y') such that $\eta\gamma(y') \in \mathbb{A}^1 \subset \mathbb{P}^1$, which implies that $X_n \cap P_{b_i} \neq \emptyset$.

Suppose this is not the case, so $I_y(S_i, Z_{n_l}) \rightarrow \infty$ for some infinite sequence $n_l \rightarrow \infty$. Since the S_i are all distinct smooth curves and there are only finitely many of them, it follows that $I_y(T, Z_{n_l})$ must remain bounded as $n_l \rightarrow \infty$, where T runs over all $S_{i'}$ for $i' \neq i$ and all of their Galois conjugates. Up to Galois conjugation, there are only r points y as above, so it follows that we must have that, for all large n_l , there exists $i_{n_l} \in \{1, 2, \dots, r+1\}$ so that $I_y(S_{i_{n_l}}, Z_{n_l})$ is bounded by an integer independent of n_l for all y as above. It follows that we must have $X_{n_l} \cap P_{b_{i_{n_l}}} \neq \emptyset$.

By choosing infinitely many disjoint sets of $r+1$ distinct integers $\{b_1, b_2, \dots, b_{r+1}\}$ as above, we see that $X_{n_b} \cap P_b \neq \emptyset$ for infinitely many distinct integers b (and n_b depending on b). Since all the P_b are disjoint, it follows that X contains preperiodic points of infinitely many distinct periods. \square

Remark 5.7. We also expect this lemma to hold in much greater generality, e.g., for any 1-parameter family of maps defined over $\overline{\mathbb{F}}_p$.

The following lemma is the key to our construction of a periodic point in Y under the assumption that $O(x) \cap Y(K)$ is infinite:

Lemma 5.8. *Let L/\mathbb{Q}_p be a finite extension, $\pi : X \rightarrow \text{Spec}(R)$ a smooth projective scheme over the ring of integers R of L and $f : X \rightarrow X$ a quasipolarised morphism over $\text{Spec}(R)$. Assume that the differential of f , df , is 0 on the special fibre of X . For any $x \in X(L) = X(R)$, let b be the period of the reduction \bar{x} of x in the special fibre of X . Then for any integer $a \geq 0$, the sequence of points $f^{a+bn}(x)$ converges to a periodic point of $X(L)$ of period b .*

Proof. Replacing f by f^n and x by $f^{a'}(x)$, for any integer a' greater than the preperiod of \bar{x} , we may assume that \bar{x} is a fixed point of f , and we then need to prove that $f^n(x)$ converges to a fixed point.

Since f is quasipolarised, by Lemma 2.2(2), \bar{x} lifts to a fixed point y of f defined over a finite extension of L ; by replacing L by this extension, we may assume that $y \in X(L)$.

Let A be the completion of the local ring of \bar{x} on X . Since π is smooth, $A \cong R[[z_1, z_2, \dots, z_n]]$, where $n+1 = \dim(X)$. Using any such isomorphism, the set of points of $X(L)$ that specialise to \bar{x} is identified with the set $(m_R)^n$, where m_R is the maximal ideal of R . We fix such an isomorphism, which we also assume identifies y with $(0, \dots, 0) \in (m_R)^n$.

Since \bar{x} is a fixed point of f , f induces an endomorphism of A that, with respect to the chosen isomorphism, is given by an n -tuple of elements (f_1, f_2, \dots, f_n) in

the maximal ideal of the local ring $R[[z_1, z_2, \dots, z_n]]$. Moreover, since f fixes y , it follows that the constant term of each f_i is 0. Since df is assumed to be zero on the special fibre of X , it follows that the coefficients of the linear term of each f_i lies in m_R . For any $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in (m_R)^n$, let $|\lambda| = \max_i \{|\lambda_i|\}$. The conditions on the f_i imply that, for any such λ , $|f(\lambda)| < |\lambda|$ if $\lambda \neq (0, \dots, 0)$. Since R is a discrete valuation ring, it follows that for any such λ we have that $f^n(\lambda) \rightarrow (0, 0, \dots, 0)$ as $n \rightarrow \infty$; hence, $f^n(x) \rightarrow y$ as $n \rightarrow \infty$. \square

Proof of Theorem 5.1. By Lemma 5.3, we may assume that K is a finite extension of $k_{n,d}$. Furthermore, we may assume without loss of generality that our base field $k = \mathbb{Q}$.

Let $x \in \mathbb{P}^n(K)$, and assume that Y is a subvariety defined over K such that $I = \mathcal{O}(x) \cap Y(K)$ is infinite. Let X be the Zariski closure of the image of x in $\text{Mor}_{n,d} \times \mathbb{P}^n_{\mathbb{Z}}$, and let χ denote the map $X \rightarrow \text{Mor}_{n,d}$ induced by projection to the first factor.

Let p be a prime dividing d . Since $\text{Mor}_{n,d}$ is smooth over $\text{Spec}(\mathbb{Z})$, there is a map $g : \text{Spec}(\mathbb{Z}_p) \rightarrow \text{Mor}_{n,d}$ such that the generic point of $\text{Spec}(\mathbb{Z}_p)$ maps to the generic point of $\text{Mor}_{n,d}$ and the closed point maps to the point corresponding to $g_{d,n,p}$, the d -power map over \mathbb{F}_p . Since $p \mid d$, the differential of the endomorphism of $\mathbb{P}^n_{\mathbb{Z}_p}$ corresponding to g , which we also denote by g , is zero on the special fibre. Suppose the fibre $X_{g_{d,n,p}}$ of χ over $g_{d,n,p}$ is infinite. By Lemma 5.4, the set $\bigcup_{r \geq 0} g_{d,n,p}^r(X_{g_{d,n,p}}(\overline{\mathbb{F}}_p))$ contains infinitely many periodic points. By applying Lemma 5.8, we can lift all these periodic points to periodic points of $f_{n,d}$ contained in Y . It then follows from Theorem 4.1 that $Y = \mathbb{P}^n_K$. Thus, we may assume from now on that χ is finite over an open neighbourhood of $g_{n,d,p}$.

By replacing x by $f^r(x)$ for some large r , we may assume that $X_{g_{d,n,p}}$ contains a periodic point $x' = [x'_0, x'_1, \dots, x'_n]$ with $x'_i \in \overline{\mathbb{F}}_p$. Since χ is finite in a neighbourhood of $g_{n,d,p}$, $\text{Mor}_{d,n}$ is smooth, hence normal, and X is irreducible, it follows from the going-down theorem that if none of the $x'_i = 0$ then the fibre $X_{g_{n,d}}$ of χ over $g_{n,d}$ contains a point \tilde{x}' lifting x' . By specialisation, it follows that for all large primes q the fibre of χ over $g_{n,d,q}$ contains a point all of whose coordinates are nonzero or, equivalently, not contained in the ramification locus of $g_{n,d,q}$. Since this locus is invariant under $g_{n,d,q}$, we may apply Lemma 5.2 to conclude the existence of a positive-dimensional periodic subvariety of Y , which, by Theorem 3.6, implies $Y = \mathbb{P}^n_K$.

We now use Lemma 5.6 to show that such an x' must exist, at least after replacing x by a Galois conjugate, or Y must contain infinitely many periodic points, both cases leading to the conclusion that $Y = \mathbb{P}^n_K$. Let x' be as above, and suppose that $x'_0 = 0$. Some other coordinate must be nonzero, so by symmetry, we may assume that $x'_n \neq 0$, and then by multiplying through by a scalar, we may assume $x'_n = 1$.

Consider the family of endomorphisms ψ_c of $\mathbb{P}_{\mathbb{F}_p}^n$ parametrised by \mathbb{A}^1 given by

$$\psi_c([x_0, x_1, \dots, x_n]) = [x_0^d + cx_n^d, x_1^d, \dots, x_n^d],$$

so $\psi_0 = g_{n,d,p}$. Note that, on the affine space given by the locus with $x_n \neq 0$, ψ_c is given in affine coordinates by $(z_0, z_1, \dots, z_{n-1}) \mapsto (z_0^d + c, z_1^d, \dots, z_{n-1}^d)$.

Let $S \subset \text{Mor}_{n,d}$ be the subscheme corresponding to the family ψ_c . By the going-down theorem, there is an irreducible component T of $\chi^{-1}(S)$ mapping onto S and containing the point $(g_{n,d,p}, x')$. Let T' be the image of T in $\mathbb{P}_{\mathbb{F}_p}^n$ under the projection of X to \mathbb{P}^n . Since $x'_n = 1$, T' is not contained in the locus given by $x_n = 0$, so by projecting to the first n coordinates, we get a rational map ρ from T to \mathbb{A}^n .

Suppose the composition of ρ with the i -th projection is nonconstant for some i , $0 < i \leq n-1$. Since the action of ψ_c on the i -th coordinate doesn't depend on c , it follows that $T(\overline{\mathbb{F}_p})$ must contain preperiodic points of arbitrarily large period. By [Lemma 5.8](#) as before, we obtain infinitely many periodic points in Y , forcing $Y = \mathbb{P}_K^n$.

So suppose ρ composed with all the i -th projections are constant for $i > 0$, and let $\sigma : T \rightarrow \mathbb{A}^1 \times S$ be given by $(\pi_0\rho, \chi)$. By applying [Lemma 5.6\(b\)](#), it follows that if the image of T is not contained inside a preperiodic curve for the map ϕ (using the identification of S with \mathbb{A}^1) there must be ϕ preperiodic points in the image with unbounded period. By the construction of ψ_d , it follows that there are preperiodic points on T of unbounded period. As before, this implies that $Y = \mathbb{P}^n$.

The last case we need to consider is when the image of T lies in a preperiodic curve. By replacing x by an element in its orbit if necessary, we may assume that this image lies in the periodic locus. Now 0 is a fixed point of the map $z \mapsto z^d$, and the point $(0, 0)$ is contained in the image of T by construction. By [Lemma 5.6\(a\)](#), it follows that the point $(0, 1)$ is also in the image of T . We conclude that $X_{g_{d,n,p}}$ contains the periodic point $x'' = [1, x'_1, \dots, x'_{n-1}, 1]$. By replacing x' with x'' and repeating the above argument if necessary, we conclude that Y contains infinitely many periodic points, in which case it must be \mathbb{P}_K^n , or $X_{g_{d,n,p}}$ contains a periodic point $x' = [x'_0, x'_1, \dots, x'_n]$ with $x'_i \neq 0$ for all i . As we have already seen, this also implies that $Y = \mathbb{P}_K^n$, concluding the proof. \square

Remark 5.9. Note that a statement similar to [Remark 4.8](#) holds: it suffices to consider generic points of irreducible subschemes of $\text{Mor}_{d,n}$ that contain all the families f_i and are smooth at the point $g_{n,d,p}$ for some prime p dividing d .

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