Proper triangular $\mathbb{G}_a$-actions on $\mathbb{A}^4$
are translations

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We describe the structure of geometric quotients for proper locally triangulable $\mathbb{G}_a$-actions on locally trivial $\mathbb{A}^3$-bundles over a nœtherian normal base scheme X defined over a field of characteristic 0. In the case where $\dim X = 1$, we show in particular that every such action is a translation with geometric quotient isomorphic to the total space of a vector bundle of rank 2 over X. As a consequence, every proper triangulable $\mathbb{G}_a$-action on the affine four space $\mathbb{A}_k^4$ over a field of characteristic 0 is a translation with geometric quotient isomorphic to $\mathbb{A}_k^3$.

Introduction

The study of algebraic actions of the additive group $\mathbb{G}_a = \mathbb{G}_{a,\mathbb{C}}$ on complex affine spaces $\mathbb{A}^n = \mathbb{A}_{\mathbb{C}}^n$ has a long history which began in 1968 with a pioneering result of Rentschler [1968] who established that every such action on the plane $\mathbb{A}^2$ is triangular in a suitable polynomial coordinate system. Consequently, every fixed point free $\mathbb{G}_a$-action on $\mathbb{A}^2$ is a translation, in the sense that the geometric quotient $\mathbb{A}^2/\mathbb{G}_a$ is isomorphic to $\mathbb{A}^1$ and that $\mathbb{A}^2$ is equivariantly isomorphic to $\mathbb{A}^2/\mathbb{G}_a \times \mathbb{G}_a$ where $\mathbb{G}_a$ acts by translations on the second factor.

Arbitrary $\mathbb{G}_a$-actions turn out to be no longer triangulable in higher dimensions [Bass 1984]. But the question whether a fixed point free $\mathbb{G}_a$-action on $\mathbb{A}^3$ is a translation or not was settled affirmatively, first for triangulable actions in [Snow 1988], then in [Deveney and Finston 1994] under the additional assumption that the action is proper and then in general in [Kaliman 2004]. The argument for triangulable actions depends on their explicit form in an appropriate coordinate system which is used to check that the algebraic quotient $\pi : \mathbb{A}^3 \to \mathbb{A}^3//\mathbb{G}_a = \mathrm{Spec}(\Gamma(\mathbb{A}^3, \mathcal{O}_{\mathbb{A}^3})^{\mathbb{G}_a})$ is a geometric quotient and that $\mathbb{A}^3//\mathbb{G}_a$ is isomorphic to $\mathbb{A}^2$. For proper actions, the properness implies that the geometric quotient $\mathbb{A}^3/\mathbb{G}_a$,

which a priori only exists as an algebraic space, is separated whence a scheme by virtue of Chow's Lemma. This means equivalently that the $\mathbb{G}_a$-action is not only locally equivariantly trivial in the étale topology but in fact locally trivial in the Zariski topology, that is, that $\mathbb{A}^3$ is covered by invariant Zariski affine open subsets of the form $V_i = U_i \times \mathbb{G}_a$ on which $\mathbb{G}_a$ acts by translations on the second factor. Since $\mathbb{A}^3$ is factorial, the open subsets V_i can even be chosen to be principal, which implies in turn that $\mathbb{A}^3/\mathbb{G}_a$ is a quasiaffine scheme, in fact an open subset of $\mathbb{A}^3//\mathbb{G}_a \simeq \mathbb{A}^2$ with at most finite complement. The equality $\mathbb{A}^3/\mathbb{G}_a = \mathbb{A}^3//\mathbb{G}_a$ ultimately follows by comparing Euler characteristics. Kaliman's general proof proceeds along a completely different approach, drawing on topological arguments to show directly that the algebraic quotient morphism $\pi : \mathbb{A}^3 \to \mathbb{A}^3//\mathbb{G}_a$ is a locally trivial $\mathbb{A}^1$-bundle. Similar topological methods have been also applied by Kaliman and Saveliev [2004] to conclude more generally that every fixed point free $\mathbb{G}_a$-action on a smooth complex contractible affine threefold X is a translation in the broader sense that X has the structure of a trivial $\mathbb{G}_a$-bundle over its geometric quotient $X/\mathbb{G}_a$, which is a smooth contractible affine surface.

Kaliman's result can be reinterpreted as the striking fact that the topological contractiblity of $\mathbb{A}^3$ is a strong enough constraint to guarantee that a fixed point free $\mathbb{G}_a$-action on it is automatically proper. This implication fails completely in higher dimensions where nonproper fixed point free $\mathbb{G}_a$-actions abound, even in the case of triangular actions on $\mathbb{A}^4$ as illustrated by Deveney, Finston and Gehrke [Deveney et al. 1994]. And starting from dimension 5, properness is known to be no longer enough to imply global equivariant triviality as illustrated by examples of proper triangular actions on $\mathbb{A}^5$ with strictly quasiaffine geometric quotients constructed by Winkelmann [1990].

On the other hand, a general characterization claimed by Fauntleroy and Magid [1976] asserted that proper $\mathbb{G}_a$-actions on factorial affine varieties were always locally equivariantly trivial in the Zariski topology, with quasiaffine geometric quotients. But counterexamples were constructed latter on by Deveney and Finston [1995] in the form of proper triangular actions on $\mathbb{A}^5$ whose geometric quotients exists only as separated algebraic spaces. So the question whether a proper $\mathbb{G}_a$-action on $\mathbb{A}^4$ is a translation or is at least locally equivariantly trivial in the Zariski topology is essentially the last unsettled problem concerning proper $\mathbb{G}_a$-actions on affine spaces, and very little progress had been made on the subject during the last decades.

The only existing partial results so far concern triangular actions: Deveney, van Rossum and Finston [2004] established that a Zariski locally equivariantly trivial triangular $\mathbb{G}_a$-action on $\mathbb{A}^4$ is a translation. The proof depends on the finite generation of the ring of invariants for such actions established by Daigle and Freudenburg [2001] and exploits the very particular structure of these rings.

Incidentally, it is known in general that local triviality for a proper action on $\mathbb{A}^n$ follows from the finite generation and regularity of the ring of invariants. But even knowing the former for triangular actions on $\mathbb{A}^4$, a direct proof of the latter condition remains elusive. The second positive result concerns a special type of triangular $\mathbb{G}_a$-actions generated by derivations of $\mathbb{C}[x, y, z, u]$ of the form $r(x)\partial_y + q(x, y)\partial_z + p(x, y)\partial_u$ where $r(x) \in \mathbb{C}[x]$ and $p(x, y), q(x, y) \in \mathbb{C}[x, y,]$. To insist on the fact that $p(x, y)$ belongs to $\mathbb{C}[x, y]$ and not only to $\mathbb{C}[x, y, z]$ as it would be the case for a general triangular situation, these derivations (and the $\mathbb{G}_a$-actions they generate) were named *twin-triangular* in [Deveney and Finston 2000]. The case where $r(x)$ has simple roots was first settled there by explicitly computing the invariant ring $\mathbb{C}[x, y, z, u]^{\mathbb{G}_a}$ and investigating the structure of the algebraic quotient morphism $\mathbb{A}^4 \to \mathbb{A}^4 /\!/ \mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[x, y, z_1, z_2]^{\mathbb{G}_a})$. The simplicity of the roots of $r(x)$ was crucial to achieve the computation, and the generalization of the result to arbitrary twin-triangular actions obtained in 2012 by the first two authors [Dubouloz and Finston 2014] required completely different methods which focused more on the nature of the corresponding geometric quotients $\mathbb{A}^4_{\mathbb{C}}/\mathbb{G}_a$. The latter a priori exist only as separated algebraic spaces and the crucial step in *loc. cit.* was to show that for twin-triangular actions they are in fact schemes, or, equivalently that proper twin-triangular $\mathbb{G}_a$-actions on $\mathbb{A}^4$ are not only locally equivariantly trivial in the étale topology but also in the Zariski topology. This enabled in turn the use of the aforementioned result of Deveney, Finston, and van Rossum to conclude that such actions are indeed translations.

In this article, we reconsider proper triangular actions on $\mathbb{A}^4$ in broader framework and we develop new techniques which enable to completely solve the question of global equivariant triviality for such actions. The triangularity assumption is of course a restriction, and it might look quite artificial from a geometric point of view. But its main consequence is to reduce an a priori four-dimensional problem to a relative three-dimensional one over a parameter space, a reduction which is crucial for our argument and turns out to be the natural context in which to interpret the aforementioned counterexamples to global or Zariski local equivariant triviality. A second more technical benefit is that it enables an effective characterization of the properness of a $\mathbb{G}_a$-action in terms of its associated locally nilpotent derivation, a problem which is in general much more delicate to handle than deciding the weaker property of being fixed point free.

The existence of smooth factorial affine hypersurfaces of $\mathbb{A}^5$ on which the proper triangular $\mathbb{G}_a$-actions constructed by Deveney and Finston [1995] restrict to proper actions whose geometric quotients exist only as separated algebraic spaces shows that even under appropriate triangularity assumptions, the question whether a proper $\mathbb{G}_a$-action on $\mathbb{A}^4$ is Zariski locally equivariantly trivial remains a subtle problem. It also indicates that in order to weaken these appropriate hypotheses,

additional algebrogeometric properties of $\mathbb{A}^4$ beyond factoriality, such as for instance topological contractibility, should play a role in the problem. But on the other hand, the existence of smooth contractible complex affine threefolds nonisomorphic to $\mathbb{A}^3$ shows that topological methods are not sufficient to infer that a given proper $\mathbb{G}_a$-action on $\mathbb{A}^4$ is a translation from its local or even global equivariant triviality. In particular, knowing that every such action is a translation would solve the Zariski Cancellation Problem in dimension three, for if X is a variety such that $X \times \mathbb{A}^1 \simeq \mathbb{A}^4$, the $\mathbb{G}_a$-action by translations on the second factor of $X \times \mathbb{A}^1$ is obviously proper.

In this article we embed the study of proper triangular $\mathbb{G}_a$-actions on $\mathbb{A}^4$ into the following more general setup: given a nœtherian normal scheme X defined over a field of characteristic zero, we consider Zariski locally trivial $\mathbb{A}^3$-bundles $\pi : E \to X$ equipped with proper locally triangulable actions of the additive group scheme $\mathbb{G}_{a,X}$. The local triangularity assumption means roughly that X can be covered by affine open subsets $U = \mathrm{Spec}(A)$ over which the restriction of E is equivariantly isomorphic to $\mathbb{A}^3_U = \mathrm{Spec}(A[y, z, u])$ equipped with the $\mathbb{G}_{a,U}$-action induced by a triangular A-derivation of $A[y, z, u]$. Our main result then is this:

Theorem. *Let X be a nœtherian normal scheme defined over a field of characteristic zero, let $\pi : E \to X$ be a Zariski locally trivial $\mathbb{A}^3$-bundle equipped with a proper locally triangulable $\mathbb{G}_{a,X}$-action and let $\mathrm{p} : \mathfrak{X} = E/\mathbb{G}_{a,X} \to X$ be the geometric quotient taken in the category of algebraic X-spaces. Then there exists an open subscheme U of X with $\mathrm{codim}_X(X \setminus U) \geq 2$ such that $\mathfrak{X}_U = \mathrm{p}^{-1}(U) \to U$ has the structure of a Zariski locally trivial $\mathbb{A}^2$-bundle.*

The conclusion of this theorem is essentially optimal. Indeed, in the example due to Winkelmann [1990], one has $X = \mathrm{Spec}(\mathbb{C}[x, y])$, $\pi = \mathrm{pr}_{x,y} : \mathbb{A}^3_X = \mathrm{Spec}(\mathbb{C}[x, y][u, v, w]) \to X$ equipped with the proper triangular $\mathbb{G}_{a,X}$-action generated by the $\mathbb{C}[x, y]$-derivation $\partial = x\partial_u + y\partial_v + (1+xv-yu)\partial_w$ of $\mathbb{C}[x, y][u, v, w]$, and the geometric quotient $\mathrm{p} : \mathfrak{X} = \mathbb{A}^3_X/\mathbb{G}_{a,X} \to X$ is the strictly quasiaffine complement of the closed subset $\{x = y = z = 0\}$ in the 4-dimensional smooth affine quadric $Q \subset \mathbb{A}^3_X$ with equation $xt_2 + yt_1 = z(z + 1)$. The structure morphism $\mathrm{p} : \mathfrak{X} \to X$ is easily seen to be an $\mathbb{A}^2$-fibration, which restricts to a locally trivial $\mathbb{A}^2$-bundle over the open subset $U = X \setminus \{(0, 0)\}$. However, there is no Zariski or étale open neighborhood of the origin $(0, 0) \in X$ over which $\mathrm{p} : \mathfrak{X} \to X$ restricts to a trivial $\mathbb{A}^2$-bundle for otherwise $\mathrm{p} : \mathfrak{X} \to X$ would be an affine morphism and so $\mathfrak{X}$ would be an affine scheme. The situation for the $\mathbb{C}[x, y]$-derivation $\partial = x\partial_u + y\partial_v + (1+xv^2)\partial_w$ of $\mathbb{C}[x, y][u, v, w]$ constructed by Deveney and Finston [1995] is very similar: here the geometric quotient $\mathfrak{X} = \mathbb{A}^3_X/\mathbb{G}_{a,X}$ is a separated algebraic space which is not a scheme and the structure morphism $\mathrm{p} : \mathfrak{X} \to X$ is again an $\mathbb{A}^2$-fibration restricting to a Zariski locally trivial $\mathbb{A}^2$-bundle over $U = X \setminus \{(0, 0)\}$ but whose restriction to any Zariski or étale open neighborhood of the origin $(0, 0) \in X$ is nontrivial.

In contrast, in the case of a 1-dimensional affine base, we can immediately derive the following corollaries:

Corollary. *Let* $\pi : E \to S$ *be a rank* 3 *vector bundle over an affine Dedekind scheme* $S = \mathrm{Spec}(A)$ *defined over a field* k *of characteristic* 0. *Then every proper locally triangulable* $\mathbb{G}_{a,S}$-*action on* E *is equivariantly trivial with geometric quotient* $E/\mathbb{G}_{a,S}$ *isomorphic to a vector bundle of rank* 2 *over* S, *stably isomorphic to* E.

Proof. By the previous theorem, the geometric quotient $\mathrm{p} : E/\mathbb{G}_{a,S} \to S$ has the structure of a Zariski locally trivial $\mathbb{A}^2$-bundle, hence is a vector bundle of rank 2 by [Bass et al. 1977]. In particular, $E/\mathbb{G}_{a,S}$ is affine, which implies in turn that $\rho : E \to E/\mathbb{G}_{a,S}$ is a trivial $\mathbb{G}_{a,S}$-bundle. So E is isomorphic to $E/\mathbb{G}_{a,S} \times_S \mathbb{A}^1_S$ as vector bundles over S. $\qquad\square$

Corollary. *Let* $S = \mathrm{Spec}(A)$ *be an affine Dedekind scheme defined over a field of characteristic* 0. *Then every proper triangular* $\mathbb{G}_{a,S}$-*action on* $\mathbb{A}^3_S$ *is a translation.*

Proof. By the previous corollary, $\mathbb{A}^3_S/\mathbb{G}_{a,S}$ is a stably trivial vector bundle of rank 2 over S, whence is isomorphic to the trivial bundle $\mathbb{A}^2_S$ over S by virtue of [Bass 1968, Chapter IV, Corollary 3.5]. $\qquad\square$

Coming back to the question of proper triangular $\mathbb{G}_a$-actions on $\mathbb{A}^4$, the observation that such actions preserve a variable in a appropriate coordinate system and hence can be considered as proper triangular actions of the additive group scheme $\mathbb{G}_{a,S}$ on the affine 3-space $\mathbb{A}^3_S$ over the affine Dedekind base $S = \mathbb{A}^1$ suffices to settle the problem:

Corollary. *If* k *is a field of characteristic* 0, *then every proper triangular* $\mathbb{G}_{a,k}$-*action on* $\mathbb{A}^4_k$ *is a translation.*

It is worth mentioning that our Main Theorem and an appeal to the aforementioned result [Deveney et al. 2004] would already be enough to conclude that every proper triangular $\mathbb{G}_{a,k}$-action on $\mathbb{A}^4_k$ is a translation, but our results do actually eliminate the need for *loc. cit.* hence the a priori dependency on the fact that the corresponding rings of invariants are finitely generated.

Let us now briefly explain the general philosophy behind the proof. After localizing at codimension 1 points of X, the Main Theorem reduces to the statement that a proper $\mathbb{G}_{a,S}$-action $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^3_S$ on the affine space $\mathbb{A}^3_S = \mathrm{Spec}(A[y, z, u])$ over the spectrum of a discrete valuation ring, generated by a triangular A-derivation $\partial = a\partial_y + q(y)\partial_z + p(y, z)\partial_u$ of $A[y, z, u]$, where $a \in A \setminus \{0\}$, $q(y) \in A[y]$ and $p(y, z) \in A[y, z]$, is a translation. Triangularity immediately implies that the restriction of σ to the generic fiber of $\mathrm{pr}_S : \mathbb{A}^3_S \to S$ is a translation with $a^{-1}y$ as a global slice. This reduces the problem to the study of neighborhoods of points of the geometric quotient $\mathfrak{X} = \mathbb{A}^3_S/\mathbb{G}_{a,S}$ supported on the closed fiber of the structure morphism $\mathrm{p} : \mathfrak{X} \to S$. A second feature of triangularity is that σ commutes with

the action $\tau : \mathbb{G}_{a,S} \times_S \mathbb{A}_S^3 \to \mathbb{A}_S^3$ generated by the A-derivation ∂_u which therefore descends to a $\mathbb{G}_{a,S}$-action $\bar{\tau}$ on the geometric quotient $\mathfrak{X} = \mathbb{A}_S^3/\mathbb{G}_{a,S}$. On the other hand, σ descends via the projection $\mathrm{pr}_{y,z} : \mathbb{A}_S^3 \to \mathbb{A}_S^2 = \mathrm{Spec}(A[y,z])$ to the action $\bar{\sigma}$ on $\mathbb{A}_S^2$ generated by the A-derivation $\bar{\bar{\partial}} = a\partial_y + q(y)\partial_z$ of $A[y,z]$. Even though $\bar{\sigma}$ and $\bar{\tau}$ are no longer fixed point free in general, if we take the quotient of $\mathbb{A}_S^2$ by the action $\bar{\sigma}$ as an algebraic stack $[\mathbb{A}_S^2/\mathbb{G}_{a,S}]$ we obtain a cartesian square

$$
\begin{array}{ccc}
\mathbb{A}_S^3 & \longrightarrow & \mathfrak{X} = \mathbb{A}_S^3/\mathbb{G}_{a,S} \\
{\scriptstyle \mathrm{pr}_{y,z}}\downarrow & & \downarrow \\
\mathbb{A}_S^2 & \longrightarrow & [\mathbb{A}_S^2/\mathbb{G}_{a,S}]
\end{array}
$$

which simultaneously identifies the quotient stacks $[\mathbb{A}_S^2/\mathbb{G}_{a,S}]$ for the action $\bar{\sigma}$ and $[\mathfrak{X}/\mathbb{G}_{a,S}]$ for the action $\bar{\tau}$ with the quotient stack of $\mathbb{A}_S^3$ for the $\mathbb{G}_{a,S}^2$-action defined by the commuting actions σ and τ. In this setting, the global equivariant triviality of the action σ becomes equivalent to the statement that a separated algebraic S-space $\mathfrak{X}$ admitting a $\mathbb{G}_{a,S}$-action whose algebraic stack quotient $[\mathfrak{X}/\mathbb{G}_{a,S}]$ is isomorphic to that of a triangular $\mathbb{G}_{a,S}$-action on $\mathbb{A}_S^2$ is an affine scheme.

While a direct proof of this reformulation seems totally out of reach with existing methods, it turns out that its conclusion holds over a certain $\mathbb{G}_{a,S}$-invariant principal open subset V of $\mathbb{A}_S^2$ which dominates S and for which the algebraic stack quotient $[V/\mathbb{G}_{a,S}]$ is in fact represented by a locally separated algebraic subspace of $[\mathbb{A}_S^2/\mathbb{G}_{a,S}]$. This provides at least an affine open subscheme $V \times_S \mathbb{A}_S^1/\mathbb{G}_{a,S}$ of $\mathfrak{X}$ dominating S, and leaves us with a closed subset of codimension at most 2 of $\mathfrak{X}$, supported on the closed fiber of $\mathrm{p} : \mathfrak{X} \to S$, in a neighborhood of which no further information is a priori available to decide even the schemeness of $\mathfrak{X}$. But similar to the argument in [Dubouloz and Finston 2014], this situation can be rescued for twin-triangular actions: the fact that for such actions $\partial u = p(y,z)$ is actually a polynomial in y only enables the same reasoning with respect to the other projection $\mathrm{pr}_{y,u} : \mathbb{A}_S^3 \to \mathbb{A}_S^2 = \mathrm{Spec}(A[y,u])$, yielding a second affine open subscheme $V' \times_S \mathbb{A}_S^1/\mathbb{G}_{a,S}$ of $\mathfrak{X}$ dominating S. This implies at least the schemeness of $\mathfrak{X}$, provided that the open subsets V and V' can be chosen so that the union of the corresponding open subschemes of $\mathfrak{X}$ covers the closed fiber of $\mathrm{p} : \mathfrak{X} \to S$.

The scheme of the article is the following. The first two sections recall basic notions and discuss a couple of preliminary technical reductions. The third section is devoted to establishing an effective criterion for nonproperness of fixed point free triangular actions from which we deduce the intermediate fact that every proper triangular action is twin-triangulable. Then in the next section, we establish that proper twin-triangular actions are indeed translations. Here, in contrast with the proof for the complex case given in [Dubouloz and Finston 2014], our argument

is independent of finite generation of rings of invariants and reduces the systematic study of algebraic spaces quotients to a minimum thanks to an appropriate "Sheshadri cover trick" [Seshadri 1972].

1. Recollection on proper, fixed point free and locally triangulable $\mathbb{G}_a$-actions

1A. *Proper versus fixed point free actions.* Recall that an action $\sigma : \mathbb{G}_{a,S} \times_S E \to E$ of the additive group scheme $\mathbb{G}_{a,S} = \mathrm{Spec}_S(\mathcal{O}_S[t]) = S \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}[t])$ on an S-scheme E is called proper if the morphism $\Phi = (\mathrm{pr}_2, \sigma) : \mathbb{G}_{a,S} \times_S E \to E \times_S E$ is proper.

1A1. If S is moreover defined over a field k of characteristic zero, then the fact that $\mathbb{G}_{a,k}$ is affine and has no nontrivial algebraic subgroups implies that properness is equivalent to Φ being a closed immersion. In particular, a proper $\mathbb{G}_{a,S}$-action is in this case fixed point free and as such, is equivariantly locally trivial in the étale topology on E. That is, there exists an affine S-scheme U and a surjective étale morphism $f : V = U \times_S \mathbb{G}_{a,S} \to E$ which is equivariant for the action of $\mathbb{G}_{a,S}$ on $U \times_S \mathbb{G}_{a,S}$ by translations on the second factor. This implies in turn the existence of a geometric quotient $\rho : E \to \mathfrak{X} = E/\mathbb{G}_{a,S}$ in the form of an étale locally trivial principal $\mathbb{G}_{a,S}$-bundle over an algebraic S-space $\mathrm{p} : \mathfrak{X} \to S$ (see, for example, [Laumon and Moret-Bailly 2000, Corollary 10.4]). Informally, $\mathfrak{X}$ is the quotient of U by the étale equivalence relation which identifies two points $u, u' \in U$ whenever there exists $t, t' \in \mathbb{G}_{a,S}$ such that $f(u, t) = f(u', t')$.

1A2. Conversely, a fixed point free $\mathbb{G}_{a,S}$-action is proper if and only if the geometric quotient $\mathfrak{X} = E/\mathbb{G}_{a,S}$ is a separated S-space. Indeed, by definition $\mathrm{p} : \mathfrak{X} \to S$ is separated if and only if the diagonal morphism $\Delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is a closed immersion, a property which is local on the target with respect to the fpqc topology [Knutson 1971, II, Extension 3.8; SGA1 1971, VIII, Corollaire 5.5]. Since $\rho : E \to \mathfrak{X}$ is a $\mathbb{G}_{a,S}$-bundle, taking the fpqc base change by $\rho \times \rho : E \times_S E \to \mathfrak{X} \times_S \mathfrak{X}$ yields a cartesian square

$$
\begin{CD}
\mathbb{G}_{a,S} \times_S E @>\Phi>> E \times_S E \\
@V{\rho \circ \mathrm{pr}_2}VV @VV{\rho \times \rho}V \\
\mathfrak{X} @>\Delta>> \mathfrak{X} \times_S \mathfrak{X}
\end{CD}
$$

from which we see that Δ is a closed immersion if and only if Φ is.

1B. *Locally triangulable actions.* Given an affine scheme $S = \mathrm{Spec}(A)$ defined over a field of characteristic zero, an action $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^n_S \to \mathbb{A}^n_S$ generated by a locally nilpotent A-derivation ∂ of $\Gamma(\mathbb{A}^n_S, \mathcal{O}_{\mathbb{A}^n_S})$ is called *triangulable* if there exists an isomorphism of A-algebras $\tau : \Gamma(\mathbb{A}^n_A, \mathcal{O}_{\mathbb{A}^n_A}) \xrightarrow{\sim} A[x_1, \ldots, x_n]$ such that the

conjugate $\delta = \tau \circ \partial \circ \tau^{-1}$ of ∂ is triangular with respect to the ordered coordinate system $(x_1, \ldots, x_n)$, that is, has the form

$$\delta = p_0 \frac{\partial}{\partial x_1} + \sum_{i=1}^{n} p_{i-1}(x_1, \ldots, x_{i-1}) \frac{\partial}{\partial x_i}$$

where $p_0 \in A$ and for every $i = 1, \ldots, n$, $p_{i-1}(x_1, \ldots, x_{i-1}) \in A[x_1, \ldots, x_{i-1}] \subset A[x_1, \ldots, x_n]$. By localizing this notion over the base S, we arrive at the following definition:

Definition 1.1. Let X be a scheme defined over a field of characteristic zero and let $\pi : E \to X$ be a Zariski locally trivial $\mathbb{A}^n$-bundle over X. An action $\sigma : \mathbb{G}_{a,X} \times_X E \to E$ of $\mathbb{G}_{a,X}$ on E is called *locally triangulable* if there exists a covering of $\mathrm{Spec}(A)$ by affine open subschemes $S_i = \mathrm{Spec}(A_i)$, $i \in I$, such that $E|_{S_i}$ is isomorphic to $\mathbb{A}^n_{S_i}$ and such that the $\mathbb{G}_{a,S_i}$-action $\sigma_i : \mathbb{G}_{a,S_i} \times_{S_i} \mathbb{A}^n_{S_i} \to \mathbb{A}^n_{S_i}$ on $\mathbb{A}^n_{S_i}$ induced by σ is triangulable.

A Zariski locally trivial $\mathbb{A}^1$-bundle $\pi : E \to X$ equipped with a fixed point free $\mathbb{G}_{a,X}$-action is nothing but a principal $\mathbb{G}_{a,X}$-bundle. As mentioned in the introduction, the nature of fixed point free locally triangulable $\mathbb{G}_{a,X}$-actions on Zariski locally trivial $\mathbb{A}^2$-bundles $\pi : E \to X$ is classically known. Namely, we have the following generalization of the main theorem of [Snow 1988]:

Proposition 1.2. *Let X be a nœtherian normal scheme defined over a field of characteristic 0 and let $\pi : E \to X$ be a Zariski locally trivial $\mathbb{A}^2$-bundle equipped with a fixed point free locally triangulable $\mathbb{G}_{a,X}$-action. Then the geometric quotient $p : E/\mathbb{G}_{a,X} \to X$ has the structure of a Zariski locally trivial $\mathbb{A}^1$-bundle over X.*

Proof. The assertion being local on the base X, we may assume that $X = \mathrm{Spec}(A)$ is the spectrum of a normal local domain containing a field of characteristic 0 and that $E = \mathbb{A}^2_X = \mathrm{Spec}(A[y, z])$ is equipped with the $\mathbb{G}_{a,X}$-action generated by a triangular derivation $\partial = a\partial_y + q(y)\partial_z$ of $A[y, z]$, where $a \in A$ and $q(y) \in A[y]$. The fixed point freeness hypothesis is equivalent to the property that a and $q(y)$ generate the unit ideal in $A[y, z]$. So $q(y)$ has the form $q(y) = b + c\tilde{q}(y)$ where $b \in A$ is relatively prime with a, $c \in \sqrt{aA}$ and $\tilde{q}(y) \in A[y]$. Letting $Q(y) = \int_0^y q(\tau) \, d\tau = by + c \int_0^y \tilde{q}(\tau) \, d\tau$, the polynomial $v = az - Q(y) \in A[y, z]$ belongs to the kernel $\mathrm{Ker}\,\partial$ of ∂ hence defines a $\mathbb{G}_{a,X}$-invariant morphism $v : E \to \mathbb{A}^1_X = \mathrm{Spec}(A[t])$. Since a and b generate the unit ideal in A, it follows from the Jacobian criterion that $v : E \to \mathbb{A}^1_X$ is a smooth morphism. Furthermore, the fibers of v coincide precisely with the $\mathbb{G}_{a,X}$-orbits on E. Indeed, over the principal open subset $X_a = \mathrm{Spec}(A_a)$ of X, ∂ admits $a^{-1}y$ as a slice and we have an equivariant isomorphism $E|_{X_a} \simeq \mathrm{Spec}(A[a^{-1}v, a^{-1}y]) \simeq \mathbb{A}^1_{X_a} \times_X \mathbb{G}_{a,X}$ where $\mathbb{G}_{a,X}$ acts by translations on the second factor. On the other hand, the

restriction $E|_Z$ of E over the closed subset $Z \subset X$ with defining ideal $\sqrt{aA} \subset A$ is equivariantly isomorphic to $\mathbb{A}^2_Z$ equipped with the $\mathbb{G}_{a,Z}$-action generated by the derivation $\bar{\partial} = \bar{b}\partial_z$ of $(A/\sqrt{aA})[y, z]$, where $\bar{b} \in (A/\sqrt{aA})^*$ denotes the residue class of b. The restriction of v to $E|_Z$ coincides via this isomorphism to the morphism $\mathbb{A}^2_Z \to \mathbb{A}^1_Z$ defined by the polynomial $\bar{v} = \bar{b}y \in (A/\sqrt{aA})[y, z]$ which is obviously a geometric quotient. The above properties imply that the morphism $\tilde{v} : E/\mathbb{G}_{a,X} \to \mathbb{A}^1_X$ induced by v is smooth and bijective. Since it admits étale quasisections, $\tilde{v}$ is then an isomorphism locally in the étale topology on $\mathbb{A}^1_X$ whence an isomorphism. $\qquad\square$

2. Preliminary reductions

2A. *Reduction to a local base.* The statement of the Main Theorem can be re-phrased equivalently as the fact that a proper locally triangulable $\mathbb{G}_{a,S}$-action on a Zariski locally trivial $\mathbb{A}^3$-bundle $\pi : E \to S$ is a translation in codimension 1. This means that for every point $s \in S$ of codimension 1 with local ring $\mathcal{O}_{S,s}$, the fiber product $E \times_S S' \simeq \mathbb{A}^3_{S'}$ of $E \to S$ with the canonical immersion $S' = \mathrm{Spec}(\mathcal{O}_{S,s}) \hookrightarrow S$ equipped with the induced proper triangular action of $\mathbb{G}_{a,S'} = \mathbb{G}_{a,S} \times_S S'$ is equivariantly isomorphic to the trivial bundle $\mathbb{A}^2_{S'} \times_{S'} \mathbb{G}_{a,S'}$ over S' equipped with the action of $\mathbb{G}_{a,S'}$ by translations on the second factor.

2A1. So we are reduced to the case where S is the spectrum of a discrete valuation ring A containing a field of characteristic 0, say with maximal ideal $\mathfrak{m}$ and residue field $\kappa = A/\mathfrak{m}$, and where $\pi = \mathrm{pr}_S : E = \mathbb{A}^3_S = \mathrm{Spec}(A[y, z, u]) \to S = \mathrm{Spec}(A)$ is equipped with a proper triangulable $\mathbb{G}_{a,S}$-action $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^3_S$. Letting $x \in \mathfrak{m}$ be uniformizing parameter, every such action is equivalent to one generated by an A-derivation ∂ of $A[y, z, u]$ of the form

$$\partial = x^n \partial_y + q(y)\partial_z + p(y, z)\partial_u$$

where $n \geq 0$, $q(y) \in A[y]$ and $p(y, z) = \sum_{r=0}^{\ell} p_r(y)z^r \in A[y, z]$, the fixed point freeness of σ being equivalent to the property that x^n, $q(y)$ and $p(y, z)$ generate the unit ideal in $A[y, z, u]$.

2B. *Reduction to proving the affineness of the geometric quotient.* With the nota-tion of Section 2A1, we can already observe that if $n = 0$ then y is an obvious global slice for ∂ and hence that the action is globally equivariantly trivial with geometric quotient $\mathfrak{X} = \mathbb{A}^3_S/\mathbb{G}_{a,S} \simeq \mathbb{A}^2_S$. Similarly, if the residue class of $q(y)$ in $\kappa[y]$ is a nonzero constant then the action σ is a translation. Indeed, in this case, the $\mathbb{G}_{a,S}$-action $\bar{\sigma} : \mathbb{G}_{a,S} \times_S \mathbb{A}^2_S \to \mathbb{A}^2_S$ on $\mathbb{A}^2_S = \mathrm{Spec}(A[y, z])$ generated by the A-derivation $\bar{\partial} = x^n \partial_y + q(y)\partial_z$ of $A[y, z]$ is fixed point free hence globally equivariantly trivial with geometric quotient $\mathbb{A}^2_S/\mathbb{G}_{a,S} \simeq \mathbb{A}^1_S$ by virtue of Proposition 1.2. On the

other hand, the $\mathbb{G}_{a,S}$-equivariant projection $\mathrm{pr}_{y,z} : \mathbb{A}^3_S \to \mathbb{A}^2_S$ descends to a locally trivial $\mathbb{A}^1$-bundle between the geometric quotients $\mathbb{A}^3_S / \mathbb{G}_{a,S}$ and $\mathbb{A}^2_S / \mathbb{G}_{a,S}$, and since $\mathbb{A}^2_S / \mathbb{G}_{a,S} \simeq \mathbb{A}^1_S$ is affine and factorial, it follows that $\mathbb{A}^3_S / \mathbb{G}_{a,S} \simeq \mathbb{A}^2_S / \mathbb{G}_{a,S} \times_S \mathbb{A}^1_S \simeq \mathbb{A}^2_S$. The affineness of $\mathbb{A}^2_S$ implies in turn that the quotient morphism $\mathbb{A}^3_S \to \mathbb{A}^3_S / \mathbb{G}_{a,S}$ is the trivial $\mathbb{G}_{a,S}$-bundle whence that $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^3_S$ is a translation. Alternatively, one can observe that a global slice $s \in A[y,z]$ for the action $\bar{\sigma}$ is also a global slice for σ via the inclusion $A[y,z] \subset A[y,z,u]$

More generally, the following lemma reduces the question of global equivariant triviality with geometric quotient $\mathfrak{X} = \mathbb{A}^3_S / \mathbb{G}_{a,S}$ isomorphic to $\mathbb{A}^2_S$ to showing that $\mathfrak{X}$, which a priori only exists as an algebraic S-space, is an affine S-scheme:

Lemma 2.1. *A fixed point free triangular action $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^3_S$ is a translation if and only if its geometric quotient $\mathfrak{X} = \mathbb{A}^3_S / \mathbb{G}_{a,S}$ is an affine S-scheme.*

Proof. One direction is clear, so assume that $\mathfrak{X}$ is an affine S-scheme. It suffices to show that the structure morphism $\mathrm{p} : \mathfrak{X} \to S$ is an $\mathbb{A}^2$-fibration, that is, a faithfully flat morphism with all its fibers isomorphic to affine planes over the corresponding residue fields. Indeed, if so, the affineness of $\mathfrak{X}$ implies on the one hand that $\mathfrak{X}$ is isomorphic to the trivial $\mathbb{A}^2$-bundle $\mathbb{A}^2_S$ by virtue of [Sathaye 1983] and on the other hand that $\rho : \mathbb{A}^3_S \to \mathfrak{X}$ is isomorphic to the trivial $\mathbb{G}_{a,S}$-bundle $\mathfrak{X} \times_S \mathbb{G}_{a,S}$ over S, which yields $\mathbb{G}_{a,S}$-equivariant isomorphisms $\mathbb{A}^3_S \simeq \mathfrak{X} \times_S \mathbb{G}_{a,S} \simeq \mathbb{A}^2_S \times_S \mathbb{G}_{a,S}$.

To see that $\mathrm{p} : \mathfrak{X} \to S$ is an $\mathbb{A}^2$-fibration, recall that $\mathrm{pr}_S : \mathbb{A}^3_S \to S$ and the quotient morphism $\rho : \mathbb{A}^3_S \to \mathfrak{X} = \mathbb{A}^3_S / \mathbb{G}_{a,S}$ are both faithfully flat, so that $\mathrm{p} : \mathfrak{X} \to S$ is faithfully flat too [Knutson 1971, II.3.2; EGA 1965, IV$_2$, Corollaire 2.2.13(iii)]. Letting $\mathfrak{m}$ and ξ be the closed and generic points of S respectively, the fibers $\mathrm{pr}_S^{-1}(\mathfrak{m}) \simeq \mathbb{A}^3_\kappa$ and $\mathrm{pr}_S^{-1}(\xi) \simeq \mathbb{A}^3_{\kappa(\xi)}$ coincide with the total spaces of the restriction of the $\mathbb{G}_{a,S}$-bundle $\rho : \mathbb{A}^3_S \to \mathfrak{X}$ over the fibers $\mathfrak{X}_\mathfrak{m} = \mathrm{p}^{-1}(\mathfrak{m})$ and $\mathfrak{X}_\xi = \mathrm{p}^{-1}(\xi)$ respectively. Since the $\mathbb{G}_{a,\kappa(\xi)}$-action induced by σ on $\mathrm{pr}_S^{-1}(\xi)$ admits $x^{-n}y$ as a global slice, it is a translation with geometric quotient $\mathbb{A}^3_{\kappa(\xi)} / \mathbb{G}_{a,\kappa(\xi)} \simeq \mathbb{A}^2_{\kappa(\xi)}$ and so $\mathfrak{X}_\xi \simeq \mathbb{A}^2_{\kappa(\xi)}$. On the other hand, we may assume in view of the above discussion that $n \geq 1$ so that the $\mathbb{G}_{a,\kappa}$-action on $\mathrm{pr}_S^{-1}(\mathfrak{m}) \simeq \mathbb{A}^3_\kappa$ induced by σ coincides with the fixed point free action generated by the $\kappa[y]$-derivation $\bar{\partial} = \bar{q}(y)\partial_z + \bar{p}(y,z)\partial_u$ of $\kappa[y][z,u]$, where $\bar{q}(y)$ and $\bar{p}(y,z)$ denote the respective residue classes of $q(y)$ and $p(y,z)$ modulo x. By virtue of Proposition 1.2, the geometric quotient $\mathbb{A}^3_\kappa / \mathbb{G}_{a,\kappa}$ has the structure of a Zariski locally trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^1_\kappa = \mathrm{Spec}(\kappa[y])$ hence is isomorphic to $\mathbb{A}^2_\kappa$. This implies that $\mathfrak{X}_\mathfrak{m} \simeq \mathbb{A}^3_\kappa / \mathbb{G}_{a,\kappa} \simeq \mathbb{A}^2_\kappa$, as desired. $\square$

Note that the above characterization holds independently of the a priori knowledge that the corresponding rings of invariants are finitely generated. But on the other hand, by exploiting the more general fact that arbitrary $\mathbb{G}_{a,S}$-actions on the affine 3-space $\mathbb{A}^3_S$ over the spectrum S of a discrete valuation ring A containing a field of

characteristic 0 have finitely generated rings of invariants [Bhatwadekar and Daigle 2009], one can derive the following stronger alternative:

Proposition 2.2. *A fixed point free action* $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^3_S$ *is either a translation or its geometric quotient* $\mathfrak{X} = \mathbb{A}^3_S / \mathbb{G}_{a,S}$ *is an algebraic space which is not a scheme.*

Proof. Indeed, the quotient morphism $\rho : \mathbb{A}^3_S \to \mathfrak{X}$ is again an $\mathbb{A}^2$-fibration thanks to [Daigle and Kaliman 2009, Theorem 3.2] which asserts that for every field κ of characteristic 0 a fixed point free action of $\mathbb{G}_{a,\kappa}$-action on $\mathbb{A}^3_\kappa$ is a translation, and so the assertion is equivalent to the fact that a Zariski locally equivariantly trivial action σ has affine geometric quotient $\mathfrak{X}$. This can be seen in a similar way as in the proof of [Deveney et al. 2004, Theorem 2.1]. Namely, by hypothesis we can find an open covering of $\mathbb{A}^3_S$ by finitely many invariant affine open subsets U_i on which the induced $\mathbb{G}_{a,S}$-action is a translation with affine geometric quotient $U_i / \mathbb{G}_{a,S}$, $i = 1, \dots, n$. Since U_i and $\mathbb{A}^3_S$ are affine, $\mathbb{A}^3_S \setminus U_i$ is a $\mathbb{G}_{a,S}$-invariant Weil divisor on $\mathbb{A}^3_S$ which is in fact principal as A, whence $A[y, z, u]$, is factorial. It follows that there exists invariant regular functions $f_i \in A[y, z, u]^{\mathbb{G}_a} \simeq \Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ such that $U_i = \mathrm{Spec}(A[x, y, z]_{f_i})$ coincides with the inverse image by the quotient morphism $\rho : \mathbb{A}^3_S \to \mathfrak{X}$ of the principal open subset $\mathfrak{X}_{f_i}$ of $\mathfrak{X}$, $i = 1, \dots, n$. Since $\rho : \mathbb{A}^3_S \to \mathfrak{X}$ is a $\mathbb{G}_{a,S}$-bundle and U_i is isomorphic to $U_i / \mathbb{G}_{a,S} \times_S \mathbb{G}_{a,S}$ by assumption, we conclude that $\mathfrak{X}$ is covered by the principal affine open subsets $\mathfrak{X}_{f_i} \simeq U_i / \mathbb{G}_{a,S}$, $i = 1, \dots, n$, whence is quasiaffine. Now since by the aforementioned result [Bhatwadekar and Daigle 2009], $A[y, z, u]^{\mathbb{G}_a}$ is an integrally closed finitely generated A-algebra, it is enough to check that the canonical open immersion $j : \mathfrak{X} \to X = \mathrm{Spec}(\Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})) \simeq \mathrm{Spec}(A[y, z, u]^{\mathbb{G}_a})$ is surjective. The surjectivity over the generic point of S follows immediately from the fact the kernel of a locally nilpotent derivation of a polynomial ring in three variables over a field K of characteristic 0 is isomorphic to a polynomial ring in two variables over K (see, for example, [Miyanishi 1986]). So it remains to show that the induced open immersion $j_\mathfrak{m} : \mathfrak{X}_m \simeq \mathbb{A}^2_\kappa \hookrightarrow X_\mathfrak{m} = \mathrm{Spec}(A[y, z, u]^{\mathbb{G}_a} \otimes_A A/\mathfrak{m})$ between the corresponding fibers over the closed point $\mathfrak{m}$ of S is surjective, in fact, an isomorphism. Since $x \in A[y, z, u]^{\mathbb{G}_a}$ is prime, $X_\mathfrak{m} \simeq \mathrm{Spec}(A[y, z, u]^{\mathbb{G}_a}/(x))$ is an integral κ-scheme of finite type and [Bhatwadekar and Daigle 2009, Corollary 4.10] can be interpreted more precisely as the fact that $X_\mathfrak{m}$ is isomorphic to $C \times_\kappa \mathbb{A}^1_\kappa$ for a certain 1-dimensional affine κ-scheme C. This implies in turn that $j_\mathfrak{m}$ is an isomorphism. Indeed, since C is dominated via $j_\mathfrak{m}$ by a general affine line $\mathbb{A}^1_\kappa \subset \mathbb{A}^2_\kappa$, its normalization $\tilde{C}$ is isomorphic to $\mathbb{A}^1_\kappa$ and so $j_\mathfrak{m}$ factors through an open immersion $\tilde{j}_\mathfrak{m} : \mathbb{A}^2_\kappa \hookrightarrow \tilde{C} \times_\kappa \mathbb{A}^1_\kappa \simeq \mathbb{A}^2_\kappa$. The latter is surjective for otherwise the complement of its image would be of pure codimension 1 hence a principal divisor $\mathrm{div}(f)$ for a nonconstant regular function f on $\tilde{C} \times_\kappa \mathbb{A}^1_\kappa$. But then f would restrict to a nonconstant invertible function on the image of $\mathbb{A}^2_\kappa$ which is absurd. Thus

$\tilde{j}_{\mathfrak{m}} : \mathbb{A}_\kappa^2 \hookrightarrow \tilde{C} \times_\kappa \mathbb{A}_\kappa^1 \simeq \mathbb{A}_\kappa^2$ is an isomorphism and since the normalization morphism $\tilde{C} \times_\kappa \mathbb{A}_\kappa^1 \to C \times_\kappa \mathbb{A}_\kappa^1$ is finite whence closed it follows that $j_{\mathfrak{m}} : \mathbb{A}_\kappa^2 \hookrightarrow C \times_\kappa \mathbb{A}_\kappa^1$ is an open and closed immersion hence an isomorphism. □

2C. *Reduction to extensions of irreducible derivations.* In view of the discussion at the beginning of Section 2B, we may assume for the A-derivation

$$\partial = x^n \partial_y + q(y)\partial_z + p(y,z)\partial_u$$

that $n > 0$ and that the residue class of $q(y)$ in $\kappa[y]$ is either zero or not constant. In the first case, $q(y) \in \mathfrak{m}A[y]$ has the form $q(y) = x^\mu q_0(y)$ where $\mu > 0$ and where $q_0(y) \in A[y]$ has nonzero residue class modulo $\mathfrak{m}$, so that the derivation $\bar{\partial} = x^n \partial_y + q(y)\partial_z$ induced by ∂ on the subring $A[y,z]$ is reducible. On the other hand, the fixed point freeness of the $\mathbb{G}_{a,S}$-action σ generated by ∂ implies that up to multiplying u by an invertible element in A, one has $p(y,z) = 1 + x^\nu p_0(y,z)$ for some $\nu > 0$ and $p_0(y,z) \in A[y,z]$.

If $\mu \geq n$, then letting $Q_0(y) = \int_0^y q_0(\tau)\,d\tau \in A[y]$, the $\mathbb{G}_{a,S}$-invariant polynomial $z_1 = z - x^{\mu-n}Q_0(y)$ is a variable of $A[y,z,u]$ over $A[y,u]$, and so ∂ is conjugate to the derivation $x^n \partial_y + p(y, z_1 + x^{\mu-n}Q_0(y))\partial_u$ of the polynomial ring in two variables $A[z_1][y,u]$ over $A[z_1]$. Since σ is fixed point free, Proposition 1.2 implies that it is equivariantly trivial with geometric quotient isomorphic to the total space of the trivial $\mathbb{A}^1$-bundle over $\mathbb{A}_S^1 = \operatorname{Spec}(A[z_1])$ whence to $\mathbb{A}_S^2$.

Otherwise, if $\mu < n$, then the $\mathbb{G}_{a,S}$-action $\tilde{\sigma} : \mathbb{G}_{a,S} \times_S \mathbb{A}_S^3 \to \mathbb{A}_S^3$ on $\mathbb{A}_S^3 = \operatorname{Spec}(A[\tilde{y}, \tilde{z}, \tilde{u}])$ generated by the A-derivation

$$\tilde{\partial} = x^{n-\mu}\partial_{\tilde{y}} + q_0(\tilde{y})\partial_{\tilde{z}} + (1 + x^\nu p_0(\tilde{y}, \tilde{z}))\partial_{\tilde{u}}$$

is again fixed point free, hence admits a geometric quotient $\tilde{\rho} : \mathbb{A}_S^3 \to \tilde{\mathfrak{X}} = \mathbb{A}_S^3/\mathbb{G}_{a,S}$ in the form of an étale locally trivial $\mathbb{G}_{a,S}$-bundle over a certain algebraic S-space $\tilde{\mathfrak{X}}$.

Lemma 2.3. *The quotient spaces $\mathfrak{X} = \mathbb{A}_S^3/\mathbb{G}_{a,S}$ and $\tilde{\mathfrak{X}} = \mathbb{A}_S^3/\mathbb{G}_{a,S}$ for the $\mathbb{G}_{a,S}$-actions σ and $\tilde{\sigma}$ on $\mathbb{A}_S^3$ generated by ∂ and $\tilde{\partial}$ respectively are isomorphic. In particular σ is proper (resp. equivariantly trivial) if and only if $\tilde{\sigma}$ is proper (resp. equivariantly trivial).*

Proof. Letting $\tilde{\rho}_i : V_i = \mathbb{A}_S^3 \to \tilde{\mathfrak{X}}_i = V_i/\mathbb{G}_{a,S}$, $i = 0, \dots, \mu$, denote the geometric quotient of $V_i = \operatorname{Spec}(A[\tilde{y}_i, \tilde{z}_i, \tilde{u}_i])$ for the fixed point free $\mathbb{G}_{a,S}$-action $\tilde{\sigma}_i$ generated by the A-derivation
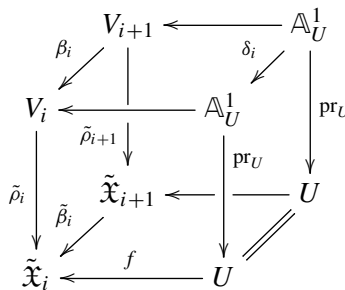
$$\tilde{\partial}_i = (1 + x^\nu p_0(\tilde{y}_i, \tilde{z}_i))\partial_{\tilde{u}_i} + x^{\mu-i}q_0(\tilde{y}_i)\partial_{\tilde{z}_i} + x^{n-i}\partial_{\tilde{y}_i},$$

the first assertion will follow from the more general fact that $\tilde{\mathfrak{X}}_i \simeq \tilde{\mathfrak{X}}_{i+1}$ for every $i = 0, \dots, \mu - 1$. Indeed, we first observe that since $\tilde{u}_i$ is a slice for $\tilde{\partial}_i$ modulo x,

$\tilde{\mathfrak{X}}_{i,\mathfrak{m}} = \tilde{\mathfrak{X}}_i \times_S \mathrm{Spec}(\kappa)$ is isomorphic to $\mathbb{A}^2_\kappa = \mathrm{Spec}((A/\mathfrak{m})[\tilde{y}_i, \tilde{z}_i])$ and the restriction of $\tilde{\rho}_i$ over $\tilde{\mathfrak{X}}_{i,\mathfrak{m}}$ is isomorphic to the trivial bundle $\mathrm{pr}_1 : \tilde{\mathfrak{X}}_{i,\mathfrak{m}} \times_\kappa \mathrm{Spec}(\kappa[\tilde{u}_i]) \to \tilde{\mathfrak{X}}_{i,\mathfrak{m}}$. Now let $\beta_i : V_{i+1} \to V_i$ be the affine modification of the total space of $\tilde{\rho}_i : \mathbb{A}^3_S \to \tilde{\mathfrak{X}}_i$ with center at the zero section of the induced bundle $\mathrm{pr}_1 : \tilde{\mathfrak{X}}_{i,\mathfrak{m}} \times_\kappa \mathrm{Spec}(\kappa[\tilde{u}_i]) \to \tilde{\mathfrak{X}}_{i,\mathfrak{m}}$ and with principal divisor x. In view of the previous description, $\beta_i : V_{i+1} \to V_i$ coincides with the affine modification of $\mathrm{Spec}(A[\tilde{y}_i, \tilde{z}_i, \tilde{u}_i])$ with center at the ideal $(x, \tilde{u}_i)$ and principal divisor x, that is, with the birational S-morphism induced by the homomorphism of A-algebra

$$\beta_i^* : A[\tilde{y}_{i+1}, \tilde{z}_{i+1}, \tilde{u}_{i+1}] \to A[\tilde{y}_i, \tilde{z}_i, \tilde{u}_i],$$

$$(\tilde{y}_{i+1}, \tilde{z}_{i+1}, \tilde{u}_{i+1}) \mapsto (\tilde{y}_i, \tilde{z}_i, x\tilde{u}_i).$$

By construction, β_i is equivariant for the $\mathbb{G}_{a,S}$-actions $\tilde{\sigma}_{i+1}$ and $\bar{\sigma}_i$ generated respectively by the locally nilpotent A-derivations $\tilde{\partial}_{i+1}$ of $A[\tilde{y}_{i+1}, \tilde{z}_{i+1}, \tilde{u}_{i+1}]$ and $\bar{\partial}_i = x\tilde{\partial}_i$ of $A[\tilde{y}_i, \tilde{z}_i, \tilde{u}_i]$. Furthermore, since $\tilde{\rho}_i : V_i \to \tilde{\mathfrak{X}}_i$ is also $\mathbb{G}_{a,S}$-invariant for the action $\bar{\sigma}_i$, the morphism $\tilde{\rho}_i \circ \beta_i : V_{i+1} \to \tilde{\mathfrak{X}}_i$ is $\mathbb{G}_{a,S}$-invariant, whence descends to a morphism $\tilde{\beta}_i : \tilde{\mathfrak{X}}_{i+1} \to \tilde{\mathfrak{X}}_i$. Since the latter restricts to an isomorphism over the generic point of S, it remains to check that it is also an isomorphism in a neighborhood of every point $p \in \tilde{\mathfrak{X}}_i$ lying over the closed point $\mathfrak{m}$ of S. Let $f : U = \mathrm{Spec}(B) \to \tilde{\mathfrak{X}}_i$ be an affine étale neighborhood of such a point $p \in \tilde{\mathfrak{X}}_i$ over which $\tilde{\rho}_i : V_i \to \tilde{\mathfrak{X}}_i$ becomes trivial, say $V_i \times_{\tilde{\mathfrak{X}}_i} U$ is isomorphic to $\mathbb{A}^1_U = \mathrm{Spec}(B[\tilde{v}_i])$. The $\mathbb{G}_{a,S}$-action on V_i generated by $\bar{\partial}_i$ lifts to the $\mathbb{G}_{a,U}$-action on $\mathbb{A}^1_U$ generated by the locally nilpotent B-derivation $x\partial_{\tilde{v}_i}$ and since $\beta_i : V_{i+1} \to V_i$ is the affine modification of V_i with center at the zero section of the restriction of $\tilde{\rho}_i : V_i \to \tilde{\mathfrak{X}}_i$ over the closed point of S, we have a commutative diagram

$$
\begin{array}{ccc}
V_{i+1} & \xleftarrow{\quad} & \mathbb{A}^1_U \\
\beta_i \swarrow \quad \big\downarrow & \quad \delta_i \swarrow \quad & \big\downarrow \mathrm{pr}_U \\
V_i \xleftarrow{\tilde{\rho}_{i+1}} \mathbb{A}^1_U & & \\
\tilde{\rho}_i \big\downarrow \quad \tilde{\mathfrak{X}}_{i+1} \xleftarrow{\quad} & \big\downarrow \mathrm{pr}_U & \\
\quad \tilde{\beta}_i \swarrow \quad & U & \\
\tilde{\mathfrak{X}}_i \xleftarrow{f} U & &
\end{array}
$$

in which the top and front squares are cartesian, and where the morphism $\delta_i : \mathbb{A}^1_U = \mathrm{Spec}(B[\tilde{v}_{i+1}]) \to \mathbb{A}^1_U = \mathrm{Spec}(B[\tilde{v}_i])$ is defined by the B-algebras homomorphism $B[\tilde{v}_i] \to B[\tilde{v}_{i+1}]$, $\tilde{v}_i \mapsto x\tilde{v}_{i+1}$. The latter is equivariant for the action on $\mathrm{Spec}(B[\tilde{v}_{i+1}])$ generated by the locally nilpotent B-derivation $\partial_{\tilde{v}_{i+1}}$ and we conclude that $\mathrm{pr}_2 : \mathbb{A}^1_U \simeq \mathbb{A}^1_U \times_{V_i} V_{i+1} \to V_{i+1}$ is an étale trivialization of the $\mathbb{G}_{a,S}$-action induced by $\tilde{\sigma}_{i+1}$ on the open subscheme $(\tilde{\rho}_i \circ \beta_i)^{-1}(f(U))$ of V_{i+1}. This implies

in turn that $U \times_{\tilde{\mathfrak{X}}_i} \tilde{\mathfrak{X}}_{i+1} \simeq U$, whence that $\tilde{\beta}_i : \tilde{\mathfrak{X}}_{i+1} \to \tilde{\mathfrak{X}}_i$ is an isomorphism in a neighborhood of $p \in \tilde{\mathfrak{X}}_i$ as desired.

The second assertion is a direct consequence of the fact that properness and global equivariant triviality of σ and $\tilde{\sigma}$ are respectively equivalent to the separatedness and the affineness of the geometric quotients $\mathfrak{X} \simeq \tilde{\mathfrak{X}}$.   □

2C1. Summing up, we are now reduced to proving that a proper $\mathbb{G}_{a,S}$-action on $\mathbb{A}^3_S$ generated by an A-derivation

$$\partial = x^n \partial_y + q(y)\partial_z + p(y, z)\partial_u$$

of $A[y, z, u]$, such that $n > 0$ and $q(y) \in A[y]$ has nonconstant residue class in $\kappa[y]$, has affine geometric quotient $\mathfrak{X} = \mathbb{A}^3_S / \mathbb{G}_{a,S}$. This will be done in two steps in the next sections: we will first establish that a proper $\mathbb{G}_{a,S}$-action as above is conjugate to one generated by a special type of A-derivation called *twin-triangular*. Then we will prove in Section 4 that proper twin-triangular $\mathbb{G}_{a,S}$-actions on $\mathbb{A}^3_S$ do indeed have affine geometric quotients.

3. Reduction to twin-triangular actions

We keep the same notation as in Section 2A1 above, namely A is a discrete valuation ring containing a field of characteristic 0, with maximal ideal $\mathfrak{m}$, residue field $\kappa = A/\mathfrak{m}$, and uniformizing parameter $x \in \mathfrak{m}$. We let again $S = \mathrm{Spec}(A)$.

We call an A-derivation ∂ of $A[y, z, u]$ *twin-triangulable* if there exists a coordinate system (y, z_+, z_-) of $A[y, z, u]$ over $A[y]$ in which the conjugate of ∂ is *twin-triangular*, that is, has the form $x^n \partial_y + p_+(y)\partial_{z_+} + p_-(y)\partial_{z_-}$ for certain polynomials $p_{\pm}(y) \in A[y]$. This section is devoted to the proof of the following intermediate characterization of proper triangular $\mathbb{G}_{a,S}$-actions:

Proposition 3.1. *With the notation above, let ∂ be an A-derivation of $A[y, z, u]$ of the form*

$$\partial = x^n \partial_y + q(y)\partial_z + p(y, z)\partial_u$$

where $n > 0$ and where $q(y) \in A[y]$ has nonconstant residue class in $\kappa[y]$. If the $\mathbb{G}_{a,S}$-action on $\mathbb{A}^3_S = \mathrm{Spec}(A[y, z, u])$ generated by ∂ is proper, then ∂ is twin-triangulable.

The proof given below proceeds in two steps: we first construct a coordinate $\tilde{u}$ of $A[y, z, u]$ over $A[y, z]$ with the property that $\partial\tilde{u} = \tilde{p}(y, z)$ is either a polynomial in y only or its leading term $\tilde{p}_\ell(y)$ as a polynomial in z has a very particular form. In the second case, we exploit the properties of $\tilde{p}_\ell(y)$ to show that the $\mathbb{G}_{a,S}$-action generated by ∂ is not proper.

3A. *The $\sharp$-reduction of a triangular A-derivation.* The conjugate of an A-derivation $\partial = x^n\partial_y + q(y)\partial_z + p(y,z)\partial_u$ of $A[y,z,u]$, as in Proposition 3.1, by an isomorphism of $A[y,z]$-algebras $\psi : A[y,z][\tilde{u}] \xrightarrow{\sim} A[y,z][u]$ is again triangular of the form

$$\psi^{-1}\partial\psi = x^n\partial_y + q(y)\partial_z + \tilde{p}(y,z)\partial_{\tilde{u}}$$

for some polynomial $\tilde{p}(y,z) \in A[y,z]$. In particular, we may choose from the very beginning a coordinate system of $A[y,z,u]$ over $A[y,z]$ with the property that the degree of $\partial u \in A[y,z]$ with respect to z is minimal among all possible conjugates $\psi^{-1}\partial\psi$ of ∂ as above. In what follows, we will say for short that such a derivation ∂ is $\sharp$-*reduced* with respect to the coordinate system (y,z,u). Letting $Q(y) = \int_0^y q(\tau)\,d\tau \in A[y]$, this property can be characterized effectively as follows:

Lemma 3.2. *Let* $\partial = x^n\partial_y + q(y)\partial_z + p(y,z)\partial_u$ *be a $\sharp$-reduced derivation of* $A[y,z,u]$ *as in Proposition 3.1. If ∂ is not twin-triangular (i.e. $p(y,z) = p_0(y) \in A[y]$) then the leading term $p_\ell(y)$, $\ell \geq 1$, of $p(y,z)$ as a polynomial in z is not congruent modulo x^n to a polynomial of the form $q(y)f(Q(y))$ for some $f(\tau) \in A[\tau]$.*

Proof. Suppose that $p(y,z) = \sum_{r=0}^{\ell} p_r(y)z^r$ with $\ell \geq 1$ and that

$$p_\ell(y) = q(y)f(Q(y)) + x^n g(y)$$

for some polynomials $f(\tau), g(\tau) \in A[\tau]$. Then letting $G(y) = \int_0^y g(\tau)\,d\tau$ and

$$\tilde{u} = u - G(y)z^\ell - \sum_{k=0}^{\deg f} \frac{(-1)^k}{\prod_{j=0}^k (\ell + 1 + j)} f^{(k)}(Q(y))x^{kn}z^{\ell+1+k},$$

one checks by direct computation that

$$\partial\tilde{u} = \sum_{r=0}^{\ell-2} p_r(y)z^r + (p_{\ell-1}(y) - G(y)q(y))z^{\ell-1}.$$

Thus $(y,z,\tilde{u})$ is a coordinate system of $A[y,z,u]$ over $A[y,z]$ in which the image of $\tilde{u}$ by the conjugate of ∂ has degree $\leq \ell - 1$, a contradiction to the $\sharp$-reducedness of ∂. $\qquad\square$

To prove Proposition 3.1, it remains to show that a proper $\mathbb{G}_{a,S}$-action on $\mathbb{A}_S^3$ generated by a $\sharp$-reduced A-derivation of $A[y,z,u]$ is twin-triangular. This is done in the next subsection.

3B. *A nonvaluative criterion for nonproperness.* To disprove the properness of an algebraic action $\sigma : \mathbb{G}_{a,S} \times_S E \to E$ of $\mathbb{G}_{a,S}$ on an S-scheme E, it suffices in principle to check that the image of $\Phi = (\text{pr}_2, \sigma) : \mathbb{G}_a \times_S E \to E \times_S E$ is not closed. However, this image turns out to be complicated to determine in general, and it is more

convenient for our purpose to consider the following auxiliary construction: letting $j : \mathbb{G}_{a,S} \simeq \mathrm{Spec}(\mathcal{O}_S[t]) \hookrightarrow \mathbb{P}^1_S = \mathrm{Proj}(\mathcal{O}_S[w_0, w_1])$, $t \mapsto [t : 1]$ be the natural open immersion, the properness of the projection $\mathrm{pr}_{E \times_S E} : \mathbb{P}^1_S \times_S E \times_S E \to E \times_S E$ implies that (p_2, σ) is proper if and only if $\varphi = (j \circ \mathrm{pr}_1, \mathrm{pr}_2, \sigma) : \mathbb{G}_{a,S} \times_S E \to \mathbb{P}^1_S \times_S E \times_S E$ is proper, hence a closed immersion. Therefore the nonproperness of σ is equivalent to the fact that the closure of $\mathrm{Im}(\varphi)$ in $\mathbb{P}^1_S \times_S E \times_S E$ intersects the "boundary" $\{w_1 = 0\}$ in a nontrivial way.

3B1. Now let $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^3_S$ be the $\mathbb{G}_{a,S}$-action generated by a non-twin-triangular $\sharp$-reduced A-derivation $\partial = x^n \partial_y + q(y)\partial_z + p(y, z)\partial_u$ of $A[y, z, u]$ and let

$$\varphi = (j \circ \mathrm{pr}_1, \mathrm{pr}_2, \mu) : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S = \mathrm{Spec}(A[t][y, z, u]) \to \mathbb{P}^1_S \times_S \mathbb{A}^3_S \times_S \mathbb{A}^3_S$$

be the corresponding immersion. To disprove the properness of σ, it is enough to check that the image by φ of the closed subscheme $H = \{z = 0\} \simeq \mathrm{Spec}(A[t][y, u])$ of $\mathbb{G}_{a,S} \times_S \mathbb{A}^3_S$ is not closed in $\mathbb{P}^1_S \times_S \mathbb{A}^3_S \times_S \mathbb{A}^3_S$. After identifying $A[y, z, u] \otimes_A A[y, z, u]$ with the polynomial ring $A[y_1, y_2, z_1, z_2, u_1, u_2]$ in the obvious way, the image of H by $(\mathrm{pr}_1, \mathrm{pr}_2, \sigma) : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^1_S \times_S \mathbb{A}^3_S \times_S \mathbb{A}^3_S$ is equal to the closed subscheme of $\mathrm{Spec}(A[t][y_1, y_2, z_1, z_2, u_1, u_2])$ defined by the following system of equations:

$$y_2 = y_1 + x^n t,$$
$$z_1 = 0,$$
$$z_2 = x^{-n}(Q(y_1 + x^n t) - Q(y_1)) = (y_1 - y_2)^{-1}(Q(y_2) - Q(y_1))t,$$
$$u_2 = u_1 + x^{-n} \int_0^t p(y_1 + x^n \tau)(Q(y_1 + x^n \tau) - Q(y_1)) \, d\tau.$$

Letting $p(y, z) = \sum_{r=0}^{\ell} p_r(y)z^r$ with $\ell \geq 1$ and

$$\Gamma_r(y_1, y_2) = \int_{y_1}^{y_2} p_r(\xi)(Q(\xi) - Q(y_1))^r \, d\xi \in A[y_1, y_2], \quad r = 0, \ldots, \ell,$$

the last equality can be rewritten modulo the first ones in the form

$$u_2 = u_1 + \sum_{r=0}^{\ell} x^{-nr} \int_0^t p_r(y_1 + x^n \tau)(Q(y_1 + x^n \tau) - Q(y_1))^r \, d\tau$$

$$= u_1 + t(y_2 - y_1)^{-1} \sum_{r=0}^{\ell} x^{-nr} \int_{y_1}^{y_2} p_r(\xi)(Q(\xi) - Q(y_1))^r \, d\xi$$

$$= u_1 + \sum_{r=0}^{\ell} ((y_2 - y_1)^{-r-1} \Gamma_r(y_1, y_2))t^{r+1}.$$

It follows that the closure V of $\varphi(H)$ is contained in the closed subscheme W of $\mathbb{P}^1_S \times_S \mathbb{A}^3_S \times_S \mathbb{A}^3_S$ defined by the equations $z_1 = 0$ and

$$(y_2 - y_1)w_1 - x^n w_0 = 0,$$
$$w_1 z_2 - (y_2 - y_1)^{-1}(Q(y_2) - Q(y_1))w_0 = 0,$$
$$w_1^{\ell+1}(u_2 - u_1) - \sum_{r=0}^{\ell}\big((y_2 - y_1)^{-r-1}\Gamma_r(y_1, y_2)\big)w_0^{r+1}w_1^{\ell-r} = 0.$$

We further observe that W is irreducible, whence equal to V, given that $\Gamma_\ell(y_1, y_2) \in A[y_1, y_2]$ does not belong to the ideal generated by x^n and $Q(y_2) - Q(y_1)$. If so, then $W = V$ intersects $\{w_1 = 0\}$ along a closed subscheme Z isomorphic to the spectrum of the algebra

$$\big(A[y_1, y_2]/(x^n, (y_2 - y_1)^{-1}(Q(y_2) - Q(y_1)), (y_2 - y_1)^{-\ell-1}\Gamma_\ell(y_1, y_2))\big)[z_2, u_1, u_2].$$

By virtue of the $\sharp$-reducedness assumption $p_\ell(y)$ is not of the form $q(y)f(Q(y)) + x^n g(y)$, so the nonproperness of $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^3_S$ is then a consequence of the following lemma which guarantees precisely that $\Gamma_\ell(y_1, y_2) \notin (x^n, Q(y_2) - Q(y_1))A[y_1, y_2]$ and that Z is not empty.

Lemma 3.3. *Let $q(y) \in A[y]$ be a polynomial with nonconstant residue class in $\kappa[y]$ and let $Q(y) = \int_0^y q(\tau)\,d\tau$. For a polynomial $p(y) \in A[y]$ and an integer $\ell \geq 1$, the following holds*:

(a) *The polynomial $\Gamma_\ell(y_1, y_2) = \int_{y_1}^{y_2} p(y)(Q(y) - Q(y_1))^\ell\,dy$ belongs to the ideal $(x^n, Q(y_2) - Q(y_1))$ if and only if $p(y)$ can be written in the form $q(y)f(Q(y)) + x^n g(y)$ for certain polynomials $f(\tau), g(\tau) \in A[\tau]$.*

(b) *The polynomial $(y_2 - y_1)^{-\ell-1}\Gamma_\ell(y_1, y_2)$ is not invertible modulo the ideal $(x^n, (y_2 - y_1)^{-1}(Q(y_2) - Q(y_1)))$.*

Proof. For the first assertion, a sequence of ℓ successive integrations by parts shows that

$$\Gamma_\ell(y_1, y_2) = \big[E_1(y)(Q(y) - Q(y_1))^\ell\big]_{y_1}^{y_2} - \ell\int_{y_1}^{y_2} E_1(y)q(y)(Q(y) - Q(y_1))^{\ell-1}\,dy$$
$$= S(y_1, y_2) + (-1)^\ell \ell! \int_{y_1}^{y_2} E_\ell(y)q(y)\,dy$$
$$= S(y_1, y_2) + (-1)^\ell \ell!(E_{\ell+1}(y_2) - E_{\ell+1}(y_1)),$$

where E_k is defined recursively by $E_1(y) = \int_0^y p(\tau)\,d\tau$, $E_{k+1}(y) = \int_0^y E_k(\tau)q(\tau)\,d\tau$, and where $S(y_1, y_2) \in (Q(y_2) - Q(y_1))A[y_1, y_2]$. So $\int_{y_1}^{y_2} p(y)(Q(y) - Q(y_1))^r\,dy$ belongs to $(x^n, Q(y_2) - Q(y_1))A[y_1, y_2]$ if and only if $E_{\ell+1}(y_2) - E_{\ell+1}(y_1)$ belongs to this ideal.

Since the residue class of $Q(y) \in A[y]$ in $\kappa[y]$ is not constant, it follows from the local criterion for flatness that $A[y]$ is a faithfully flat algebra over $A[Q(y)]$.

By faithfully flat descent, this implies in turn that the sequence

$$A[Q(y)] \hookrightarrow A[y] \xrightarrow{\cdot \otimes 1 - 1 \otimes \cdot} A[y] \otimes_{A[\tau]} A[y]$$

is exact whence, with the natural identification

$$A[y] \otimes_{A[\tau]} A[y] \simeq A[y_1, y_2]/(Q(y_2) - Q(y_1)),$$

that a polynomial $F \in A[y]$ with $F(y_2) - F(y_1)$ belonging to the ideal

$$(Q(y_2) - Q(y_1))A[y_1, y_2]$$

has the form $F(y) = G(Q(y))$ for a certain polynomial $G(\tau) \in A[\tau]$. Thus $E_{\ell+1}(y_2) - E_{\ell+1}(y_1)$ belongs to $(x^n, Q(y_2) - Q(y_1))A[y_1, y_2]$, if and only if $E_{\ell+1}(y)$ is of the form $G(Q(y)) + x^n R_{\ell+1}(y)$ for some $G(\tau), R_{\ell+1}(\tau) \in A[\tau]$. This implies in turn that $E_\ell(y)q(y) = G'(Q(y))q(y) + x^n R'_{\ell+1}(y)$ whence, since $q(y) \in A[y] \setminus \mathfrak{m}A[y]$ is not a zero divisor modulo x^n, that $E_\ell(y) = G'(Q(y)) + x^n R_\ell(y)$ for a certain $R_\ell(\tau) \in A[\tau]$. We conclude by induction that $E_1(y) = G^{(\ell+1)}(Q(y)) + x^n R_1(y)$ and finally that $p(y) = G^{(\ell+2)}(Q(y))q(y) + x^n R(y)$ for a certain $R(\tau) \in A[\tau]$. This proves (a).

The second assertion is clear in the case where $p(y) \in \mathfrak{m}A[y]$. Otherwise, if $p(y) \in A[y] \setminus \mathfrak{m}A[y]$ then reducing modulo x and passing to the algebraic closure $\bar{\kappa}$ of κ, it is enough to show that if $q(y) \in \bar{\kappa}[y]$ is not constant and $p(y) \in \bar{\kappa}[y]$ is a nonzero polynomial then for every $\ell \geq 1$, the affine curves C and D in $\mathbb{A}^2_{\bar{\kappa}} = \mathrm{Spec}(\bar{\kappa}[y_1, y_2])$ defined by the vanishing of the polynomials $\Theta(y_1, y_2) = (y_2 - y_1)^{-\ell-1} \int_{y_1}^{y_2} p(y)(Q(y) - Q(y_1))^\ell \, dy$ and $R(y_1, y_2) = (y_2 - y_1)^{-1} \int_{y_1}^{y_2} q(y) \, dy$ respectively always intersect each other. Suppose on the contrary that $C \cap D = \varnothing$ and let $m = \deg q \geq 1$ and $d = \deg p \geq 0$. Then the closures $\bar{C}$ and $\bar{D}$ of C and D respectively in $\mathbb{P}^2_{\bar{\kappa}} = \mathrm{Proj}(\bar{\kappa}[y_1, y_2, y_3])$ intersect each others along a closed subscheme Y of length $\deg \bar{C} \cdot \deg \bar{D} = m(d + \ell m)$ supported on the line $\{y_3 = 0\} \simeq \mathrm{Proj}(\bar{\kappa}[y_1, y_2])$. By definition, up to multiplication by a nonzero scalar, the top homogeneous components of R and Θ have the form $\prod_{i=1}^m (y_2 - \zeta^i y_1)$, where $\zeta \in \bar{\kappa}$ is a primitive $(m+1)$-th root of unity, and $(y_2 - y_1)^{\ell-1} \int_{y_1}^{y_2} y^d (y^{m+1} - y_1^{m+1})^\ell \, dy$ respectively. But on the other hand, we have for every $i = 1, \ldots, m$

$$\bar{\kappa}[y_2] \Big/ \left(y_2 - \zeta^i, (y_2 - 1)^{-r-1} \int_1^{y_2} y^d (y^{m+1} - 1)^r \, dy \right)$$

$$\simeq \bar{\kappa}[y_2] \Big/ \left(y_2 - \zeta^i, (\zeta^i - 1)^{-r-1} \int_1^{\zeta^i} \tau^d (\tau^{m+1} - 1)^r \, d\tau \right),$$

and hence the length of the above algebra is either 1 or 0 depending on whether $\int_1^{\zeta^i} \tau^d (\tau^{m+1} - 1) \, d\tau \in \bar{\kappa}$ is zero or not. This implies that the length of Y is at most equal to m and so the only possibility would be that $d = 0$ and $\ell = m = 1$, in other

words C and D are parallel lines in $\mathbb{A}^2_\kappa$. But since $\int_1^{-1}(\tau^2 - 1)\,d\tau \neq 0$, this last possibility is also excluded. $\qquad\qquad\square$

4. Global equivariant triviality of twin-triangular actions

By virtue of Proposition 3.1, every proper triangular $\mathbb{G}_{a,S}$-action on $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}^3_S \to \mathbb{A}^3_S$ on $\mathbb{A}^3_S$ is conjugate to one generated by a twin-triangular A-derivation ∂ of $A[y, z_+, z_-]$ of the form

$$\partial = x^n \partial_y + p_+(y)\partial_{z_+} + p_-(y)\partial_{z_-}$$

for certain polynomials $p_\pm(y) \in A[y]$. So to complete the proof of the Main Theorem, it remains to show the following generalization of the main result in [Dubouloz and Finston 2014]:

Proposition 4.1. *Let S be the spectrum of discrete valuation ring A containing a field of characteristic 0. Then a proper twin-triangular $\mathbb{G}_{a,S}$-action on $\mathbb{A}^3_S$ has affine geometric quotient $\mathfrak{X} = \mathbb{A}^3_S/\mathbb{G}_{a,S}$.*

4A1. The principle of the proof given below is the following: we exploit the twin triangularity to construct two $\mathbb{G}_{a,S}$-invariant principal open subsets W_{Γ_+} and W_{Γ_-} in $\mathbb{A}^3_S$ with the property that the union of corresponding principal open subspaces $\mathfrak{X}_{\Gamma_\pm} = W_{\Gamma_\pm}/\mathbb{G}_{a,S}$ of $\mathfrak{X}$ covers the closed fiber of the structure morphism $p : \mathfrak{X} \to S$. We then show that $\mathfrak{X}_{\Gamma_+}$ and $\mathfrak{X}_{\Gamma_-}$ are in fact affine subschemes of $\mathfrak{X}$. On the other hand, since ∂ admits $x^{-n}y$ as a global slice over A_x, the generic fiber of p is isomorphic to the affine plane over the function field A_x of S. So it follows that $\mathfrak{X}$ is covered by three principal affine open subschemes $\mathfrak{X}_{\Gamma_+}$, $\mathfrak{X}_{\Gamma_-}$ and $\mathfrak{X}_x$ corresponding to regular functions x, Γ_+, Γ_- which generate the unit ideal in $\Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \simeq A[y, z_+, z_-]^{\mathbb{G}_{a,S}} \subset A[y, z_+, z_-]$, whence is an affine scheme.

4A2. The fact that the affineness of $p : \mathfrak{X} = \mathbb{A}^3_S/\mathbb{G}_{a,S} \to S = \operatorname{Spec}(A)$ is a local property with respect to the fpqc topology on S [SGA1 1971, VIII, Corollaire 5.6] enables a reduction to the case where the discrete valuation ring A is Henselian or complete. Since it contains a field of characteristic zero, an elementary application of Hensel's Lemma implies that a maximal subfield of such a local ring A is a field of representatives, that is, a subfield which is mapped isomorphically by the quotient projection $A \mapsto A/\mathfrak{m}$ onto the residue field $\kappa = A/\mathfrak{m}$. This is in fact the only property of A that we will use in the sequel. So from now on, $(A, \mathfrak{m}, \kappa)$ is a discrete valuation ring containing a field κ of characteristic 0 and with residue field $A/\mathfrak{m} \simeq \kappa$.

4B. *Twin-triangular actions in general position and associated invariant covering.* Here we construct a pair of principal $\mathbb{G}_{a,S}$-invariant open subsets $W_\pm = W_{\Gamma_\pm}$ of $\mathbb{A}^3_S$ associated with a twin-triangular A-derivation of $A[y, z_+, z_-]$ whose geometric

quotients will be studied in the next subsection. We begin with a technical condition which will be used to guarantee that the union of W_+ and W_- covers the closed fiber of the projection $\mathrm{pr}_S : \mathbb{A}^3_S \to S$.

Definition 4.2. Let $(A, \mathfrak{m}, \kappa)$ be a discrete valuation ring containing a field of characteristic 0 and let $x \in \mathfrak{m}$ be a uniformizing parameter. A twin-triangular A-derivation $\partial = x^n \partial_y + p_+(y)\partial_{z_+} + p_-(y)\partial_{z_-}$ of $A[y, z_+, z_-]$ is said to be in *general position* if it satisfies the following properties:

(a) The residue classes $\bar{p}_\pm \in \kappa[y]$ of the polynomials $p_\pm \in A[y]$ modulo $\mathfrak{m}$ are both nonzero and relatively prime.

(b) There exist integrals $\bar{P}_\pm \in A[y]$ of $\bar{p}_\pm$ with respect to y for which the inverse images of the branch loci of the morphisms $\bar{P}_+ : \mathbb{A}^1_\kappa \to \mathbb{A}^1_\kappa$ and $\bar{P}_- : \mathbb{A}^1_\kappa \to \mathbb{A}^1_\kappa$ are disjoint.

Lemma 4.3. *With the notation above, every twin-triangular A-derivation ∂ of $A[y, z_+, z_-]$ generating a fixed point free $\mathbb{G}_{a,S}$-action on $\mathbb{A}^3_S$ is conjugate to one in general position.*

Proof. A twin-triangular derivation $\partial = x^n \partial_y + p_+(y)\partial_{z_+} + p_-(y)\partial_{z_-}$ generates a fixed point free $\mathbb{G}_{a,S}$-action if and only if x^n, $p_+(y)$ and $p_-(y)$ generate the unit ideal in $A[y, z_+, z_-]$. So the residue classes $\bar{p}_+$ and $\bar{p}_-$ of p_+ and p_- are relatively prime and at least one of them, say $\bar{p}_-$, is nonzero. If $\bar{p}_+ = 0$ then p_- is necessarily of the form $p_-(y) = c + x\tilde{p}_-(y)$ for some $c \in A^*$ and so changing z_+ for $z_+ + z_-$ yields a twin-triangular derivation conjugate to ∂ for which the corresponding polynomials $p_\pm(y)$ both have nonzero residue classes modulo x. More generally, changing z_- for $az_- + bz_+$ for general $a \in A^*$ and $b \in A$ yields a twin-triangular derivation conjugate to ∂ and still satisfying condition (a) in Definition 4.2. So it remains to show that up to such a coordinate change, condition (b) in the definition can be achieved.

This can be seen as follows : we consider $\mathbb{A}^2_\kappa$ embedded in $\mathbb{P}^2_\kappa = \mathrm{Proj}(\kappa[u, v, w])$ as the complement of the line $L_\infty = \{w = 0\}$ so that the coordinate system (u, v) on $\mathbb{A}^2$ is induced by the projections from the κ-rational points $[0 : 1 : 0]$ and $[1 : 0 : 0]$ respectively. We let C be the closure in $\mathbb{P}^2$ of the image of the morphism $j = (\bar{P}_+, \bar{P}_-) : \mathbb{A}^1_\kappa = \mathrm{Spec}(\kappa[y]) \to \mathbb{A}^2_\kappa$ defined by the residue classes $\bar{P}_+$ and $\bar{P}_-$ in $\kappa[y]$ of integrals $P_\pm(y) \in A[y]$ of $p_\pm(y)$, and we denote by $Z \subset C$ the image by j of the inverse image of the branch locus of $\bar{P}_+ : \mathbb{A}^1_\kappa \to \mathbb{A}^1_\kappa$. Note that Z is a finite subset of C defined over k, and therefore the set of lines in $\mathbb{P}^2_k$ passing through a point of Z and tangent to a local analytic branch of C at some point is finite. This follows from the fact that the set of lines in $\mathbb{P}^2_k$ intersecting transversely a fixed curve is Zariski open. Therefore, the complement of the finitely many intersection points of these lines with L_∞ is a Zariski open subset U of L_∞

with the property that for every $q \in U$, the line through q and every arbitrary point of Z intersects every local analytic branch of C transversally at every point. By construction, the rational projections from $[0:1:0]$ and an arbitrary κ-rational point in $U \setminus \{[0:1:0]\}$ induce a new coordinate system on $\mathbb{A}_\kappa^2$ of the form $(u, av + bu)$, $a \neq 0$, with the property that Z is not contained in the inverse image of the branch locus of the morphism $a \bar{P}_- + b \bar{P}_+ : \mathbb{A}_\kappa^1 \to \mathbb{A}_\kappa^1$. Changing z_- for $az_- + bz_+$ for a pair $(a, b) \in \kappa^* \times \kappa \subset A^* \times A$ corresponding to a general point in U yields a twin-triangular derivation conjugate to ∂ and satisfying simultaneously conditions (a) and (b) in Definition 4.2. $\qquad\square$

4B1. Now let $\partial = x^n \partial_y + p_+(y)\partial_{z_+} + p_-(y)\partial_{z_-}$ be a twin-triangular A-derivation of $A[y, z_+, z_-]$ generating a proper whence fixed point free $\mathbb{G}_{a,S}$-action $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}_S^3 \to \mathbb{A}_S^3$. By virtue of Lemma 4.3 above, we may assume up to a coordinate change preserving twin-triangularity that ∂ is in general position. Property (a) in Definition 4.2 then guarantees in particular that the triangular derivations $\partial_\pm = x^n \partial_y + p_\pm(y)\partial_{z_\pm}$ of $A[y, z_\pm]$ are both irreducible. Furthermore, given any integral $P_\pm(y) \in A[y]$ of $p_\pm(y)$, the morphism $\bar{P}_\pm : \mathbb{A}_\kappa^1 \to \mathbb{A}_\kappa^1$ obtained by restricting $P_\pm : \mathbb{A}_S^1 = \mathrm{Spec}(A[y]) \to \mathbb{A}_S^1 = \mathrm{Spec}(A[t])$ to the closed fiber of $\mathrm{pr}_S : \mathbb{A}_S^3 \to S$ is not constant. The branch locus of $\bar{P}_\pm$ is then a principal divisor $\mathrm{div}(\alpha_\pm(t))$ for a certain polynomial $\alpha_\pm(t) \in \kappa[t] \subset A[t]$ generating the kernel of the homomorphism $\kappa[t] \to \kappa[y]/(\bar{p}_\pm(y))$, $t \mapsto \bar{P}_\pm(y) + (\bar{p}_\pm(y))$. Property (b) in Definition 4.2 guarantees that we can choose P_+ and P_- in such a way that the polynomial $\alpha_+(\bar{P}_+(y))$ and $\alpha_-(\bar{P}_-(y))$ generate the unit ideal in $\kappa[y]$. Up to a diagonal change of coordinates on $A[y, z_+, z_-]$, we may further assume without loss of generality that $\bar{P}_+$ and $\bar{P}_-$ are monic.

4B2. We let $R_\pm = A[t]_{\alpha_\pm}$ and we let $U_\pm = \mathrm{Spec}(R_\pm)$ be the principal open subset of $\mathbb{A}_S^1 = \mathrm{Spec}(A[t])$ where $\alpha_\pm$ does not vanish. The polynomial $\Phi_\pm = -x^n z_\pm + P_\pm(y) \in A[y, z_+, z_-]$ belongs to the kernel of ∂ hence defines a $\mathbb{G}_{a,S}$-invariant morphism $\Phi_\pm : \mathbb{A}_S^3 = \mathrm{Spec}(A[y, z_+, z_-]) \to \mathbb{A}_S^1 = \mathrm{Spec}(A[t])$. We let

$$W_\pm = \Phi_\pm^{-1}(U_\pm) \simeq \mathrm{Spec}\big(R_\pm[y, z_+, z_-]/(-x^n z_\pm + P_\pm(y) - t)\big)$$

Note that $W_\pm$ is a $\mathbb{G}_{a,S}$-invariant open subset of $\mathbb{A}_S^3$ which can be identified with the principal open subset where the $\mathbb{G}_{a,S}$-invariant regular function $\Gamma_\pm = \alpha_\pm \circ \Phi_\pm$ does not vanish. Since $\alpha_+(\bar{P}_+(y))$ and $\alpha_-(\bar{P}_-(y))$ generate the unit ideal in $\kappa[y]$, it follows that the union of W_+ and W_- covers the closed fiber of the projection $\mathrm{pr}_S : \mathbb{A}_S^3 \to S$.

4C. *Affineness of geometric quotients.* With the notation of Section 4B2 above, the geometric quotient $\mathfrak{X}_\pm = W_\pm / \mathbb{G}_{a,S}$ for the action induced by $\sigma : \mathbb{G}_{a,S} \times_S \mathbb{A}_S^3 \to \mathbb{A}_S^3$ can be identified with the principal open subspace $\mathfrak{X}_{\Gamma_\pm}$ of $\mathfrak{X} = \mathbb{A}_S^3 / \mathbb{G}_{a,S}$ where

the invariant function $\Gamma_\pm \in A[y, z_+, z_-]^{\mathbb{G}_{a,S}} \simeq \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ does not vanish. The properness of σ implies that $\mathfrak{X}$, whence $\mathfrak{X}_+$ and $\mathfrak{X}_-$, are separated algebraic spaces, and the construction of W_+ and W_- guarantees that the closed fiber of the structure morphism $p : \mathfrak{X} \to S$ is contained in the union of $\mathfrak{X}_+$ and $\mathfrak{X}_-$. So to complete the proof of Proposition 4.1, it remains to show that $\mathfrak{X}_\pm$ is an affine scheme. In fact, since $\mathfrak{X}_\pm$ is by construction an algebraic space over the affine scheme $U_\pm = \mathrm{Spec}(R_\pm)$, its affineness is equivalent to that of the structure morphism $q_\pm : \mathfrak{X}_\pm \to U_\pm$, a property which can be checked locally with respect to the étale topology on $U_\pm$.

4C1. In our situation, there is a natural finite étale base change $\varphi_\pm : \tilde{U}_\pm \to U_\pm$ which is obtained as follows: By construction, see Section 4B1 above, the morphism $\bar{P}_\pm : \mathbb{A}^1_\kappa = \mathrm{Spec}(\kappa[y]) \to \mathrm{Spec}(\kappa[t])$, restricts to a finite étale covering $h_{0,\pm} : C_{1,\pm} = \mathrm{Spec}(\kappa[y]_{\alpha_\pm(\bar{P}_\pm(y))}) \to C_\pm = \mathrm{Spec}(\kappa[t]_{\alpha_\pm(t)})$ of degree $r_\pm = \deg_y(\bar{P}_\pm(y))$. Letting $\tilde{C}_\pm = \mathrm{Spec}(B_\pm)$ be the normalization of $C_\pm$ in the Galois closure $L_\pm$ of the field extension $i_\pm : \kappa(t) \hookrightarrow \kappa(y)$, the induced morphism $h_\pm : \tilde{C}_\pm \to C_\pm$ is an étale Galois cover with Galois group $G_\pm = \mathrm{Gal}(L_\pm / \kappa(t))$, which factors as

$$h_\pm : \tilde{C}_\pm = \mathrm{Spec}(B_\pm) \xrightarrow{h_{1,\pm}} C_{1,\pm} = \mathrm{Spec}(\kappa[y]_{\alpha_\pm(\bar{P}_\pm(y))}) \xrightarrow{h_{0,\pm}} C_\pm = \mathrm{Spec}(\kappa[t]_{\alpha_\pm(t)})$$

where $h_{1,\pm} : \tilde{C}_\pm \to C_{1,\pm}$ is an étale Galois cover for a certain subgroup $H_\pm$ of $G_\pm$ of index $r_\pm$. Letting $\tilde{R}_\pm = A \otimes_\kappa B_\pm \simeq A[t]_{\alpha_\pm(t)} \otimes_{\kappa[t]_{\alpha_\pm(t)}} B_\pm$ and $\tilde{U}_\pm = \mathrm{Spec}(\tilde{R}_\pm)$, the morphism $\varphi_\pm = \mathrm{pr}_1 : \tilde{U}_\pm \simeq U_\pm \times_{C_\pm} \tilde{C}_\pm \to U_\pm$ is an étale Galois cover with Galois group $G_\pm$, in particular a finite morphism. Since $\mathfrak{X}_\pm$ is separated, the algebraic space $\tilde{\mathfrak{X}}_\pm = \mathfrak{X}_\pm \times_{U_\pm} \tilde{U}_\pm$ is separated and, by construction, isomorphic to the geometric quotient of the scheme

$$\tilde{W}_\pm = W_\pm \times_{U_\pm} \tilde{U}_\pm \simeq \mathrm{Spec}\big(\tilde{R}_\pm[y, z_+, z_-]/(-x^n z_\pm + P_\pm(y) - t)\big)$$

by the proper $\mathbb{G}_{a,\tilde{U}_\pm}$-action generated by the locally nilpotent $\tilde{R}_\pm$-derivation $x^n \partial_y + p_+(y)\partial_{z_+} + p_-(y)\partial_{z_-}$ of $\tilde{R}_\pm[y, z_+, z_-]//(-x^n z_\pm + P_\pm(y) - t)$, which commutes with the action of $G_\pm$. The following lemma completes the proof of Proposition 4.1 whence of the Main Theorem.

Lemma 4.4. *The geometric quotient $\tilde{\mathfrak{X}}_\pm = \tilde{W}_\pm / \mathbb{G}_{a,\tilde{U}_\pm}$ is an affine $\tilde{U}_\pm$-scheme.*

Proof. Since $\tilde{U}_\pm$ is affine, the assertion is equivalent to the affineness of $\tilde{\mathfrak{X}}_\pm$. From now on, we only consider the case of $\tilde{\mathfrak{X}}_+ = \tilde{W}_+ / \mathbb{G}_{a,\tilde{U}_+}$, the case of $\tilde{\mathfrak{X}}_-$ being similar. To simplify the notation, we drop the corresponding subscript "+", writing simply $\tilde{W} = \mathrm{Spec}(\tilde{R}[y, z, z_-]/(-x^n z + P(y) - t))$. We denote $x \otimes 1 \in \tilde{R} = A \otimes_\kappa B$ by x and we further identify B with a sub-κ-algebra of $\tilde{R}$ via the homomorphism $1 \otimes \mathrm{id}_B : B \to \tilde{R}$ and with the quotient $\tilde{R}/x\tilde{R}$ via the composition $1 \otimes \mathrm{id}_B : B \to A \otimes_\kappa B \to A \otimes_\kappa B/((x \otimes 1)A \otimes_\kappa B) = \kappa \otimes_\kappa B \simeq B$.

By construction of B, the monic polynomial $\bar{P}(y) - t \in B[y]$ splits as $\bar{P}(y) - t = \prod_{\bar{g} \in G/H}(y - t_{\bar{g}})$ for certain elements $t_{\bar{g}} \in B$, $\bar{g} \in G/H$, on which the Galois group G

acts by permutation $g' \cdot t_{\bar{g}} = t_{\overline{(g')^{-1} \cdot g}}$. Furthermore, since $h_0 : C_1 \to C$ is étale, it follows that for distinct $\bar{g}, \bar{g}' \in G/H$, $t_{\bar{g}} - t_{\bar{g}'} \in B$ is an invertible regular function on $\tilde{C}$ whence on $\tilde{U} = S \times_{\mathrm{Spec}(\kappa)} \tilde{C}$ via the identifications made above. This implies in turn that there exists a collection of elements $\sigma_{\bar{g}} \in \tilde{R}$ with respective residue classes $t_{\bar{g}} \in B = \tilde{R}/x\tilde{R}$ modulo x, $\bar{g} \in G/H$, on which G acts by permutation, a G-invariant polynomial $S_1 \in \tilde{R}[y]$ with invertible residue class modulo x and a G-invariant polynomial $S_2 \in \tilde{R}[y]$ such that in $\tilde{R}[y]$ one can write

$$P(y) - t = S_1(y) \prod_{\bar{g} \in G/H} (y - \sigma_{\bar{g}}) + x^n S_2(y).$$

Concretely, the elements $\sigma_{\bar{g}} = \sigma_{\bar{g}, n-1} \in \tilde{R}$, $\bar{g} \in G/H$, can be constructed by induction via a sequence of elements $\sigma_{\bar{g}, m} \in \tilde{R}$, $\bar{g} \in G/H$, $m = 0, \dots, n-1$, starting with $\sigma_{\bar{g}, 0} = t_{\bar{g}} \in B \subset \tilde{R}$ and culminating in $\sigma_{\bar{g}, n-1} = \sigma_{\bar{g}}$, and characterized by the property that for every $m = 0, \dots, n-1$, there exists $\mu_{\bar{g}, m} \in \tilde{R}$ such that $P(\sigma_{\bar{g}, m}) - t = x^{m+1} \mu_{\bar{g}, m}$, $\bar{g} \in G/H$. Indeed, writing $P(y) - t = \prod_{\bar{g} \in G/H} (y - t_{\bar{g}}) + x \tilde{P}(y)$ for a certain $\tilde{P}(y) \in \tilde{R}[y]$ and assuming that the $\sigma_{\bar{g}, m}$, $\bar{g} \in G/H$, have been constructed up to a certain index $m < n-1$, we look for elements $\sigma_{\bar{g}, m+1} \in \tilde{R}$ written in the form $\sigma_{\bar{g}, m} + x^{m+1} \lambda_{\bar{g}}$ for some $\lambda_{\bar{g}} \in \tilde{R}$. For a fixed $\bar{g}_0 \in G/H$, the conditions impose that

$$
\begin{aligned}
P(\sigma_{\bar{g}_0, m+1}) - t &= \prod_{\bar{g} \in G/H} (\sigma_{\bar{g}_0, m} + x^{m+1} \lambda_{\bar{g}_0} - t_{\bar{g}}) + x \tilde{P}(\sigma_{\bar{g}_0, m} + x^{m+1} \lambda_{\bar{g}_0}) \\
&= x^{m+1} \lambda_{\bar{g}_0} \prod_{\bar{g} \in (G/H) \setminus \{\bar{g}_0\}} (t_{\bar{g}_0} - t_{\bar{g}}) + P(\sigma_{\bar{g}_0, m}) - t + x^{m+2} v_{\bar{g}_0, m} \\
&= x^{m+1} \lambda_{\bar{g}_0} \prod_{\bar{g} \in (G/H) \setminus \{\bar{g}_0\}} (t_{\bar{g}_0} - t_{\bar{g}}) + x^{m+1} \mu_{\bar{g}_0, m} + x^{m+2} v_{\bar{g}_0, m}
\end{aligned}
$$

for some $v_{\bar{g}_0, m} \in \tilde{R}$, and since $\prod_{\bar{g} \in (G/H) \setminus \{\bar{g}_0\}} (t_{\bar{g}_0} - t_{\bar{g}}) \in \tilde{R}^*$, we conclude that

$$\lambda_{\bar{g}_0} = \frac{\mu_{\bar{g}_0, m}}{\prod_{\bar{g} \in (G/H) \setminus \{\bar{g}_0\}} (t_{\bar{g}_0} - t_{\bar{g}})} \quad \text{and} \quad \mu_{\bar{g}_0, m+1} = v_{\bar{g}_0, m}.$$

A direct computation shows further that $g' \cdot \sigma_{\bar{g}, m+1} = \sigma_{\overline{(g')^{-1} \cdot g}, m+1}$ and that $g' \cdot \mu_{\bar{g}, m+1} = \mu_{\overline{(g')^{-1} \cdot g}, m+1}$. Iterating this procedure $n-1$ times yields the desired collection of elements $\sigma_{\bar{g}} = \sigma_{\bar{g}, n-1} \in \tilde{R}$. By construction, $\prod_{\bar{g} \in G/H} (y - \sigma_{\bar{g}}) \in \tilde{R}[y]$ is then an invariant polynomial which divides $P(y) - t$ modulo $x^n \tilde{R}$, which implies in turn the existence of the G-invariant polynomials $S_1(y), S_2(y) \in \tilde{R}[y]$.

The closed fiber of the induced morphism $\tilde{W} \to S$ consists of a disjoint union of closed subschemes $D_{\bar{g}} \simeq \mathrm{Spec}(\tilde{R}[z, z_-]) \simeq \mathbb{A}^2_{\tilde{C}}$ with defining ideals $(x, y - \sigma_{\bar{g}})$, $\bar{g} \in G/H$. The open subscheme $\tilde{W}_{\bar{g}} = \tilde{W} \setminus \bigcup_{\bar{g}' \in (G/H) \setminus \{\bar{g}\}} D_{\bar{g}'}$ of $\tilde{W}$ is $\mathbb{G}_{a, \tilde{U}}$-invariant

and one checks using the above expression for $P(y) - t$ that the rational map
$$\tilde{W} \dashrightarrow \mathrm{Spec}(\tilde{R}[u_{\bar{g}}, z_-]),$$
$$(y, z, z_-) \mapsto (u_{\bar{g}}, z_-) = \left(\frac{y - \sigma_{\bar{g}}}{x^n} = \frac{z - S_2(y)}{S_1(y) \prod_{\bar{g}' \in (G/H) \setminus \{\bar{g}\}} (y - \sigma_{\bar{g}'})}, z_- \right)$$
induces a $\mathbb{G}_{a, \tilde{U}}$-equivariant isomorphism $\tau_g : \tilde{W}_{\bar{g}} \xrightarrow{\sim} \mathbb{A}^2_{\tilde{U}} = \mathrm{Spec}(\tilde{R}[u_{\bar{g}}, z_-])$ for the $\mathbb{G}_{a, \tilde{U}}$-action on $\mathbb{A}^2_{\tilde{U}}$ generated by the locally nilpotent $\tilde{R}$-derivation $\partial_{u_{\bar{g}}} + p_-(x^n u_{\bar{g}} + \sigma_{\bar{g}}) \partial_{z_-}$ of $\tilde{R}[u_{\bar{g}}, z_-]$. The latter is a translation with $u_{\bar{g}}$ as a global slice and with geometric quotient $\tilde{W}_{\bar{g}} / \mathbb{G}_{a, \tilde{U}}$ isomorphic to $\mathrm{Spec}(\tilde{R}[v_{\bar{g}}])$ where

$$v_{\bar{g}} = z_- - x^{-n}(P_-(x^n u_{\bar{g}} + \sigma_{\bar{g}}) - P_-(\sigma_{\bar{g}})) \in \tilde{R}[u_{\bar{g}}, z_-]^{\mathbb{G}_{a, \tilde{U}}}.$$

By construction, for distinct $\bar{g}, \bar{g}' \in G/H$, the rational functions $\tau_{\bar{g}}^* v_{\bar{g}}$ and $\tau_{\bar{g}'}^* v_{\bar{g}'}$ on $\tilde{W}$ differ by the addition of the element

$$f_{\bar{g}, \bar{g}'} = x^{-n}(P_-(\sigma_{\bar{g}}) - P_-(\sigma_{\bar{g}'})) \in \tilde{R}_x \in \Gamma(\tilde{W}_{\bar{g}} \cap \tilde{W}_{\bar{g}'}, \mathcal{O}_{\tilde{W}}).$$

This implies that $\tilde{\mathfrak{X}} = \tilde{W} / \mathbb{G}_{a, \tilde{U}}$ is isomorphic to the $\tilde{U}$-scheme obtained by gluing r copies $\tilde{\mathfrak{X}}_{\bar{g}} = \mathrm{Spec}(\tilde{R}[v_{\bar{g}}])$ of $\mathbb{A}^1_{\tilde{U}}$ along the principal open subsets $\tilde{\mathfrak{X}}_{\bar{g}, x} \simeq \mathrm{Spec}(\tilde{R}_x[v_{\bar{g}}])$ via the isomorphisms induced by the $\tilde{R}_x$-algebra isomorphisms

$$\xi_{\bar{g}, \bar{g}'}^* : \tilde{R}_x[v_{\bar{g}}] \to \tilde{R}_x[v_{\bar{g}'}], \ v_{\bar{g}} \mapsto v_{\bar{g}'} + f_{\bar{g}, \bar{g}'}, \quad \bar{g}, \bar{g}' \in G/H, \ \bar{g} \neq \bar{g}'.$$

Since by assumption $\tilde{\mathfrak{X}}$ is separated, it follows from [EGA 1960, I, Proposition (5.5.6)] that for every pair of distinct elements $\bar{g}, \bar{g}' \in G/H$, the subring $\tilde{R}[v_{\bar{g}'}, f_{\bar{g}, \bar{g}'}]$ of $\tilde{R}_x[v_{\bar{g}'}]$ generated by the union of $\tilde{R}[v_{\bar{g}'}]$ and $\xi_{\bar{g}, \bar{g}'}^*(\tilde{R}[v_{\bar{g}}])$ is equal to $\tilde{R}_x[v_{\bar{g}'}]$. This holds if and only if $\tilde{R}[f_{\bar{g}, \bar{g}'}] = \tilde{R}_x$ whence if and only if $f_{\bar{g}, \bar{g}'} \in \tilde{R}_x$ has the form $f_{\bar{g}, \bar{g}'} = x^{-m_{\bar{g}, \bar{g}'}} F_{\bar{g}, \bar{g}'}$ for a certain $m_{\bar{g}, \bar{g}'} > 1$ and an element $F_{\bar{g}, \bar{g}'} \in \tilde{R}$ with invertible residue class modulo x.

This additional information enables a proof of the affineness of $\tilde{\mathfrak{X}}$ by induction on r as follows: given a pair of distinct elements $\bar{g}, \bar{g}' \in G/H$ such that $m_{\bar{g}, \bar{g}'} = m > 0$ is maximal, we let $\theta_{\bar{g}} = 0$ and $\theta_{\bar{g}''} = x^{m - m_{\bar{g}, \bar{g}''}} F_{\bar{g}, \bar{g}''} \in \tilde{R}$ for every $\bar{g}'' \in (G/H) \setminus \{\bar{g}\}$. The choice of the elements $\theta_{\bar{g}''} \in \tilde{R}$ guarantees that the local sections

$$\psi_{\bar{g}''} = x^m v_{\bar{g}''} + \theta_{\bar{g}''} \in \Gamma(\tilde{\mathfrak{X}}_{\bar{g}''}, \mathcal{O}_{\tilde{\mathfrak{X}}}), \quad \bar{g}'' \in G/H,$$

glue to a global regular function $\psi \in \Gamma(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}})$. Since $\theta_{\bar{g}'} = F_{\bar{g}, \bar{g}'}$ is invertible modulo x, the regular functions x, ψ and $\psi - \theta_{\bar{g}'}$ generate the unit ideal in $\Gamma(\tilde{\mathfrak{X}}, \mathcal{O}_{\tilde{\mathfrak{X}}})$. The principal open subset $\tilde{\mathfrak{X}}_x$ of $\tilde{\mathfrak{X}}$ is isomorphic to $\tilde{\mathfrak{X}}_{\bar{g}, x} \simeq \mathrm{Spec}(\tilde{R}_x[v_{\bar{g}}])$ for every $\bar{g} \in G/H$, hence is affine. On the other hand, $\tilde{\mathfrak{X}}_{\psi}$ and $\tilde{\mathfrak{X}}_{\psi - \theta_{\bar{g}'}}$ are contained respectively in the open subschemes $\tilde{\mathfrak{X}}(\bar{g})$ and $\tilde{\mathfrak{X}}(\bar{g}')$ obtained by gluing only the $r - 1$ open subsets $\tilde{\mathfrak{X}}_{\bar{g}''}$ corresponding to the elements $\bar{g}''$ in $(G/H) \setminus \{\bar{g}\}$ and $(G/H) \setminus \{\bar{g}'\}$ respectively. By the induction hypothesis, the latter are both affine

and hence $\tilde{\mathfrak{X}}_\psi$ and $\tilde{\mathfrak{X}}_{\psi-\theta_{\bar{g}'}}$ are affine as well. This shows that $\tilde{\mathfrak{X}}$ is an affine scheme and completes the proof. $\qquad\square$

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