

# *Algebra & Number Theory*

Volume 9

2015

No. 1

**Local Beilinson–Tate operators**

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In 1968 Tate introduced a new approach to residues on algebraic curves, based on a certain ring of operators that acts on the completion at a point of the function field of the curve. This approach was generalized to higher-dimensional algebraic varieties by Beilinson in 1980. However, Beilinson’s paper had very few details, and his operator-theoretic construction remained cryptic for many years. Currently there is a renewed interest in the Beilinson–Tate approach to residues in higher dimensions.

Our paper presents a variant of Beilinson’s operator-theoretic construction. We consider an  $n$ -dimensional topological local field  $K$ , and define a ring of operators  $E(K)$  that acts on  $K$ , which we call the ring of *local Beilinson–Tate operators*. Our definition is of an analytic nature (as opposed to the original geometric definition of Beilinson). We study various properties of the ring  $E(K)$ . In particular we show that  $E(K)$  has an  *$n$ -dimensional cubical decomposition*, and this gives rise to a *residue functional* in the style of Beilinson and Tate. Presumably this residue functional coincides with the residue functional that we had constructed in 1992; but we leave this as a conjecture.

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## Introduction

Let  $X$  be a smooth curve over a perfect base field  $\mathbb{k}$ , with function field  $\mathbf{k}(X)$ , and let  $x \in X$  be a closed point. The completion  $K := \mathbf{k}(X)_{(x)}$  of  $\mathbf{k}(X)$  at  $x$  is a local field. Tate [1968] introduced a ring  $E(K) \subset \text{End}_{\mathbb{k}}(K)$ , and two-sided ideals

*MSC2010*: primary 12J25; secondary 32A27, 13J05, 11R56, 46A13, 46H30.

*Keywords*: topological local fields, residues, Tate residue, Beilinson adeles.

$E(K)_1, E(K)_2 \subset E(K)$ . These new objects were defined using the valuation ring of  $K$ . Let us call the elements of  $E(K)$  *local Tate operators*. Heuristically, elements of  $E(K)_1$  are “compact operators”, and elements of  $E(K)_2$  are “discrete operators”. An operator  $\phi \in \text{End}_{\mathbb{k}}(K)$  is called *finite potent* if for some positive integer  $m$  the operator  $\phi^m$  has finite rank. Tate proved that each  $\phi \in E(K)_1 \cap E(K)_2$  is finite potent, and that  $E(K)_1 + E(K)_2 = E(K)$ . Using some algebraic manipulations of the structure  $(E(K), \{E(K)_j\})$ , Tate constructed a *residue functional*

$$\text{Res}_{\mathbf{k}(X)/\mathbb{k}, x}^T : \Omega_{\mathbf{k}(X)/\mathbb{k}}^1 \rightarrow \mathbb{k}.$$

Here  $\Omega_{\mathbf{k}(X)/\mathbb{k}}^1$  is the module of Kähler 1-forms of  $\mathbf{k}(X)$ . Then he showed that his residue functional is the same as the one gotten by Laurent series expansion at  $x$ .

Finally, Tate gave a global variant of this residue functional, using the adèles of  $X$  instead of the completion  $\mathbf{k}(X)_{(x)}$ . He related the local and global residues, and proved that, when the curve  $X$  is proper, the sum of the local residues of any form  $\alpha \in \Omega_{\mathbf{k}(X)/\mathbb{k}}^1$  is zero. The Tate construction gave a totally new way of looking at residues and duality for curves.

This circle of ideas was extended by Beilinson [1980] to higher dimensions in his extremely brief paper (that did not contain any proofs). Actually Beilinson’s paper had in it several important innovations, related to a finite type  $\mathbb{k}$ -scheme  $X$ . By a chain of points in  $X$  of length  $n$  we mean a sequence  $\xi = (x_0, \dots, x_n)$  of points such that  $x_i$  is a specialization of  $x_{i-1}$ . The chain  $\xi$  is *saturated* if each  $x_i$  is an immediate specialization of  $x_{i-1}$ . Beilinson said that:

- (1) For a chain  $\xi$  of length  $n$  and a quasi-coherent sheaf  $\mathcal{M}$  there is an  $\mathcal{O}_X$ -module  $\mathcal{M}_\xi$ , gotten by an  $n$ -fold zigzag inverse and direct limit process. When  $\mathcal{M}$  is coherent and  $n = 0$ , this is the  $\mathfrak{m}_{x_0}$ -adic completion  $\widehat{\mathcal{M}}_{x_0}$  of the stalk  $\mathcal{M}_{x_0}$ . (Let us call  $\mathcal{M}_\xi$  the *Beilinson completion* of  $\mathcal{M}$  along  $\xi$ .)
- (2) For every  $n \in \mathbb{N}$  and quasi-coherent sheaf  $\mathcal{M}$ , there is a sheaf  $\mathbb{A}^n(\mathcal{M})$  called the *sheaf of adèles of degree  $n$  with values in  $\mathcal{M}$* . It is a restricted product of the Beilinson completions  $\mathcal{M}_\xi$  along length  $n$  chains. The sheaves  $\mathbb{A}^n(\mathcal{M})$  assemble into a flasque resolution of  $\mathcal{M}$ . When  $X$  is a curve,  $\mathbb{A}^1(\mathcal{O}_X)$  is the usual sheaf of adèles of  $X$ .
- (3) For a saturated chain  $\xi = (x_0, \dots, x_n)$ , the completion  $\mathbf{k}(x_0)_\xi$  of the residue field  $\mathbf{k}(x_0)$  is a finite product of  $n$ -dimensional local fields.
- (4) Let  $A := \mathbf{k}(x_0)_\xi$  as in (3). Then there is a ring  $E(A) \subset \text{End}_{\mathbb{k}}(A)$ , with an  *$n$ -dimensional cubical decomposition* (see Definition 0.3 below), from which a Tate-style residue functional can be obtained.
- (5) The higher adèles in (2) and the cubically decomposed ring of operators  $E(A)$  in (4) can be combined to prove a global residue theorem when  $X$  is proper.

The adelic resolution (2) was clarified, and all claims proved (for any noetherian scheme  $X$ ), by Huber [1991]. The assertion about higher local fields (3) was proved in [Yekutieli 1992] (for any excellent noetherian scheme  $X$ ); see Theorem 6.1.

For a long time assertions (4) and (5) were essentially neglected and remained cryptic. Very recently we heard about renewed interest in the work of Beilinson, mainly by Braunling, Groechenig and Wolfson [Braunling 2014a; 2014b; Braunling et al. 2014]. Item (4) above is discussed in [Braunling 2014a; 2014b]. A long-term goal of this team is to understand and make precise the global aspect (5) of Beilinson’s construction, and then to apply this construction in various directions. Independently, Osipov [2005; 2007] has been studying higher adeles and higher local fields.

Discussions with Wolfson and Braunling led us to the realization that the topological aspects of higher local fields, and their implications on item (4) above, are not sufficiently understood. *The purpose of this paper is to present an analytic variant of the Beilinson–Tate construction for topological local fields and to study its properties.* Presumably our analytic construction agrees with the geometric construction of [Beilinson 1980; Braunling 2014b], and the resulting residue functional is the same as the residue functional from [Yekutieli 1992] — see Conjectures 0.9 and 0.12 below.

Throughout the introduction we keep the assumption that  $\mathbb{k}$  is a perfect base field. An  $n$ -dimensional topological local field over  $\mathbb{k}$  is —roughly speaking— a field extension  $K$  of  $\mathbb{k}$ , with a rank  $n$  valuation, and with a topology compatible with the valuation. An example is the field of iterated Laurent series  $K = \mathbb{k}((t_2))((t_1))$ , which is of dimension 2. See Definitions 3.1 and 3.8 for details. It is important to mention that a topological local field  $K$  of dimension  $n \geq 2$  is not a topological ring, but only a *semi-topological ring*: multiplication is continuous only in one argument. We abbreviate “topological local field” to “TLF” and “semi-topological” to “ST”. The theory of ST rings and modules is reviewed in Section 1.

A TLF  $K$  of dimension  $n$  has discrete valuation rings  $\mathcal{O}_i(K)$  and residue fields  $\mathbf{k}_i(K)$  for  $i = 1, \dots, n$ . They are related as follows:  $\mathbf{k}_i(K)$  is the residue field of  $\mathcal{O}_i(K)$  and the fraction field of  $\mathcal{O}_{i+1}(K)$ ; and  $K$  is the fraction field of  $\mathcal{O}_1(K)$ . By a *system of liftings* for  $K$  we mean a sequence  $\sigma = (\sigma_1, \dots, \sigma_n)$ , where each  $\sigma_i : \mathbf{k}_i(K) \rightarrow \mathcal{O}_i(K)$  is a continuous lifting of the canonical surjection  $\mathcal{O}_i(K) \twoheadrightarrow \mathbf{k}_i(K)$ . Such systems of liftings always exist; see Proposition 3.19.

Consider a TLF  $K$  equipped with a system of liftings  $\sigma$ . We define a ring of operators  $E_\sigma(K) \subset \text{End}_{\mathbb{k}}(K)$ , and ideals  $E_\sigma(K)_{i,j} \subset E_\sigma(K)$ . Our definition (Definitions 4.5 and 4.14 in the body of the paper) is a modification of Beilinson’s original definition from [Beilinson 1980]. But whereas Beilinson’s original definition was geometric in nature (and pertained only to a TLF arising as a factor of a completion  $\mathbf{k}(x_0)_\xi$ ), our definition is of an analytic nature. (We saw a similar definition in a private communication from Braunling.)

Here is our first main result. It is repeated as [Corollary 4.22](#).

**Theorem 0.1.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , and let  $\sigma$  and  $\sigma'$  be two systems of liftings for  $K$ .*

- (1) *There is equality  $E_\sigma(K) = E_{\sigma'}(K)$  of these subrings of  $\text{End}_{\mathbb{k}}(K)$ .*
- (2) *For any  $i = 1, \dots, n$  and  $j = 1, 2$  there is equality  $E_\sigma(K)_{i,j} = E_{\sigma'}(K)_{i,j}$  of these ideals of  $E_\sigma(K)$ .*

The theorem justifies the next definition.

**Definition 0.2.** Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ . Define  $E(K) := E_\sigma(K)$  and  $E(K)_{i,j} := E_\sigma(K)_{i,j}$ , where  $\sigma$  is any system of liftings for  $K$ . We call  $E(K)$  the ring of *local Beilinson–Tate operators* on  $K$ .

Here is a definition from [\[Braunling 2014b\]](#), which distills Beilinson’s definition [\[1980\]](#). The notation we use is closer to the original notation of Tate.

**Definition 0.3.** Let  $A$  be a commutative  $\mathbb{k}$ -ring. An  *$n$ -dimensional cubically decomposed ring of operators on  $A$*  is data  $(E, \{E_{i,j}\})$  consisting of:

- A  $\mathbb{k}$ -subring  $E \subset \text{End}_{\mathbb{k}}(A)$  containing  $A$ .
- Two-sided ideals  $E_{i,j} \subset E$ , indexed by  $i \in \{1, \dots, n\}$  and  $j \in \{1, 2\}$ .

These are the conditions:

- (i) For every  $i \in \{1, \dots, n\}$  we have  $E = E_{i,1} + E_{i,2}$ .
- (ii) Every operator  $\phi \in \bigcap_{i=1}^n \bigcap_{j=1}^2 E_{i,j}$  is finite potent.

The next result is [Theorem 4.24](#)(1) in the body of the paper.

**Theorem 0.4.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ . The data*

$$(E(K), \{E(K)_{i,j}\})$$

*of local Beilinson–Tate operators is an  $n$ -dimensional cubically decomposed ring of operators on  $K$ .*

Let  $A$  be any commutative ST  $\mathbb{k}$ -ring. We can talk about the ring  $\text{End}_{\mathbb{k}}^{\text{cont}}(A)$  of continuous  $\mathbb{k}$ -linear operators on  $A$ . There is also the ring  $\mathcal{D}_{A/\mathbb{k}}^{\text{cont}}$  of *continuous differential operators*; see the review in [Section 2](#). There are inclusions of  $\mathbb{k}$ -rings

$$A \subset \mathcal{D}_{A/\mathbb{k}}^{\text{cont}} \subset \text{End}_{\mathbb{k}}^{\text{cont}}(A) \subset \text{End}_{\mathbb{k}}(A).$$

**Theorem 0.5.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ . The ring  $E(K)$  of local Beilinson–Tate operators satisfies*

$$\mathcal{D}_{K/\mathbb{k}}^{\text{cont}} \subset E(K) \subset \text{End}_{\mathbb{k}}^{\text{cont}}(K).$$

This is repeated as [Theorem 4.24\(2\)](#). Actually [Theorem 0.5](#) is used in the proofs of [Theorems 0.1](#) and [0.4](#).

In [[Yekutieli 1992](#)] we developed a theory of residues for TLFs. For every  $n$ -dimensional TLF  $K$ , we consider the module of top-degree separated differential forms  $\Omega_{K/\mathbb{k}}^{n,\text{sep}}$ . It is a rank 1 free  $K$ -module, and it has the fine  $K$ -module topology. This means that, for any nonzero form  $\alpha \in \Omega_{K/\mathbb{k}}^{n,\text{sep}}$ , the corresponding homomorphism  $K \rightarrow \Omega_{K/\mathbb{k}}^{n,\text{sep}}$  is a topological isomorphism. (We will say more about the fine topology later in the introduction.) The residue functional

$$\text{Res}_{K/\mathbb{k}}^{\text{TLF}} : \Omega_{K/\mathbb{k}}^{n,\text{sep}} \rightarrow \mathbb{k} \tag{0-6}$$

constructed in [[Yekutieli 1992](#)] is a continuous  $\mathbb{k}$ -linear homomorphism, enjoying several important properties. See [Theorem 5.4](#) for details.

Beilinson [[1980](#)] claimed that an  $n$ -dimensional cubically decomposed ring of operators  $(E, \{E_{i,j}\})$  on a commutative  $\mathbb{k}$ -ring  $A$  determines a residue functional

$$\text{Res}_{A/\mathbb{k};E}^{\text{BT}} : \Omega_{A/\mathbb{k}}^n \rightarrow \mathbb{k}. \tag{0-7}$$

For  $n = 1$  this is the original abstract residue of [[Tate 1968](#)]. For  $n \geq 2$  this was worked out in [[Braunling 2014b](#)], using Lie algebra homology and Hochschild homology.

Now consider an  $n$ -dimensional TLF  $K$  over  $\mathbb{k}$ . By [Theorem 0.4](#),  $K$  is equipped with an  $n$ -dimensional cubically decomposed ring of operators  $E(K)$ ; and we let

$$\text{Res}_{K/\mathbb{k}}^{\text{BT}} : \Omega_{K/\mathbb{k}}^n \rightarrow \mathbb{k} \tag{0-8}$$

denote the corresponding residue functional.

Let  $A$  be any commutative ST  $\mathbb{k}$ -ring. For any  $q$  the module of Kähler differentials  $\Omega_{A/\mathbb{k}}^q$  has a canonical topology (this is recalled at the end of [Section 2](#)). There is a canonical continuous surjection  $\Omega_{A/\mathbb{k}}^q \twoheadrightarrow \Omega_{A/\mathbb{k}}^{q,\text{sep}}$  to the separated module of differentials. Often (e.g., when  $A = K$  is a TLF of dimension at least 1 and  $\text{char}(\mathbb{k}) = 0$ ) the kernel of this canonical surjection is very big.

**Conjecture 0.9.** Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ . The following diagram of  $\mathbb{k}$ -linear homomorphisms is commutative:

$$\begin{array}{ccc} \Omega_{K/\mathbb{k}}^n & \xrightarrow{\text{can}} & \Omega_{K/\mathbb{k}}^{n,\text{sep}} \\ & \searrow \text{Res}_{K/\mathbb{k}}^{\text{BT}} & \downarrow \text{Res}_{K/\mathbb{k}}^{\text{TLF}} \\ & & \mathbb{k} \end{array}$$

When  $n \leq 1$  we know the conjecture is true. For  $n = 0$  it is trivial, and for  $n = 1$  it is proved in [[Tate 1968](#)]. In order to help in proving this conjecture in higher dimensions we have included a review of the residue functional  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$  and its

properties. This is [Section 5](#) of the paper. We also state [Conjecture 5.7](#), which is closely related to [Conjecture 0.9](#).

Suppose  $A$  is a finite product of  $n$ -dimensional TLFs over  $\mathbb{k}$ ; say  $A = \prod_{l=1}^r K_l$ . Let us define  $E(A) := \prod_l E(K_l)$  and  $E(A)_{i,j} := \prod_l E(K_l)_{i,j}$ . It is not hard to see that the data

$$(E(A), \{E(A)_{i,j}\}) \tag{0-10}$$

is an  $n$ -dimensional cubically decomposed ring of operators on  $A$ .

Let  $X$  be a finite type  $\mathbb{k}$ -scheme, let  $\xi = (x_0, \dots, x_n)$  be a saturated chain of points in  $X$  such that  $x_n$  is a closed point, and let  $K := \mathbf{k}(x_0)$ . According to [\[Yekutieli 1992\]](#) (see [Theorem 6.1](#)), the Beilinson completion  $A := K_\xi$  is a finite product of  $n$ -dimensional TLFs over  $\mathbb{k}$ . Beilinson’s construction [\[1980\]](#), worked out in detail in [\[Braunling 2014b\]](#), gives rise to an  $n$ -dimensional cubically decomposed ring of operators

$$(E_{X,\xi}(K), \{E_{X,\xi}(K)_{i,j}\}) \tag{0-11}$$

on  $K_\xi$ . (This is our notation.) Note that by definition both  $E_{X,\xi}(K)$  and  $E(K_\xi)$  are subrings of  $\text{End}_{\mathbb{k}}(K_\xi)$ .

**Conjecture 0.12.** Let  $X$  be a finite type  $\mathbb{k}$ -scheme, let  $\xi = (x_0, \dots, x_n)$  be a saturated chain of points in  $X$  such that  $x_n$  is a closed point, and let  $K := \mathbf{k}(x_0)$ . Then the  $n$ -dimensional cubically decomposed rings of operators  $E(K_\xi)$  and  $E_{X,\xi}(K)$  are equal.

To help in proving this conjecture we have included [Section 6](#), in which we recall some facts from [\[Yekutieli 1992\]](#) about the Beilinson completions  $\mathbf{k}(x_0)_\xi$ , and provide our interpretation of the geometric definition of  $E_{X,\xi}(K)$ . In [Remark 6.11](#) we explain the geometric significance of these conjectures.

To finish the introduction we wish to discuss a technical result that is used in the proof of [Theorem 0.1](#). This result is of a very general nature, and could possibly find other applications.

We work in the category  $\text{STRing}_{\mathbb{c}} \mathbb{k}$  of commutative ST  $\mathbb{k}$ -rings. The morphisms are continuous  $\mathbb{k}$ -ring homomorphisms. Let  $A \in \text{STRing}_{\mathbb{c}} \mathbb{k}$ . The *fine  $A$ -module topology* on an  $A$ -module  $M$  is the finest topology that makes  $M$  into a ST  $A$ -module. For example, if  $M \cong A^r$  for  $r \in \mathbb{N}$ , then the product topology is the fine  $A$ -module topology on  $M$ . For more see [Section 1](#).

Consider an artinian local ring  $A$  in  $\text{STRing}_{\mathbb{c}} \mathbb{k}$ , with residue field  $K$ . Give  $K$  the fine  $A$ -module topology relative to the canonical surjection  $\pi : A \rightarrow K$ ; so  $\pi$  becomes a homomorphism in  $\text{STRing}_{\mathbb{c}} \mathbb{k}$ . Suppose  $\sigma : K \rightarrow A$  is a homomorphism in  $\text{STRing}_{\mathbb{c}} \mathbb{k}$  such that  $\pi \circ \sigma$  is the identity of  $K$ . We call  $\sigma$  a *lifting of  $K$  in  $\text{STRing}_{\mathbb{c}} \mathbb{k}$* . The lifting  $\sigma$  is called a *precise lifting* if the topology on  $A$  coincides with the fine  $K$ -module topology on it (via  $\sigma$ ). The ring  $A$  is called a *precise artinian local*

ring if it admits some precise lifting. There are examples of artinian local rings in  $\text{STRing}_c \mathbb{k}$  that are not precise, like in [Example 1.27](#). However, the rings that we are interested in (such as quotients of  $\mathcal{O}_1(K)$  for a TLF  $K$  — see [Lemma 3.14](#), and Beilinson completions of artinian local rings — see [Example 1.26](#)) are precise. The reader might wonder if all continuous liftings  $\sigma : K \rightarrow A$  for a precise artinian local ring  $A$  are precise. This is answered affirmatively in [Corollary 0.14](#) below. It is a consequence of the following technical result:

Given a lifting  $\sigma : K \rightarrow A$  and an  $A$ -module  $M$ , we denote by  $\text{rest}_\sigma(M)$  the  $K$ -module whose underlying  $\mathbb{k}$ -module is  $M$ , and  $K$  acts via  $\sigma$ .

**Theorem 0.13.** *Let  $A$  be a precise artinian local ring in  $\text{STRing}_c \mathbb{k}$ , with residue field  $K$ . Put on  $K$  the fine  $A$ -module topology. Let  $\sigma_1, \sigma_2 : K \rightarrow A$  be liftings in  $\text{STRing}_c \mathbb{k}$  of the canonical surjection  $\pi : A \rightarrow K$ , and assume that  $\sigma_2$  is a precise lifting.*

*Let  $M_1$  and  $M_2$  be finite  $A$ -modules, and let  $\phi : M_1 \rightarrow M_2$  be an  $A$ -linear homomorphism. For  $l = 1, 2$  choose  $K$ -linear isomorphisms  $\psi_l : K^{r_l} \xrightarrow{\sim} \text{rest}_{\sigma_l}(M_l)$ . Let  $\bar{\phi} \in \text{Mat}_{r_2 \times r_1}(\text{End}_{\mathbb{k}}(K))$  be the matrix representing the  $\mathbb{k}$ -linear homomorphism*

$$\psi_2^{-1} \circ \phi \circ \psi_1 : K^{r_1} \rightarrow K^{r_2}.$$

*Then:*

- (1) *The matrix  $\bar{\phi}$  belongs to  $\text{Mat}_{r_2 \times r_1}(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ .*
- (2) *Assume that  $M_1 = M_2$  and  $\phi$  is the identity automorphism. Write  $r := r_1$ . Then the matrix  $\bar{\phi}$  belongs to  $\text{GL}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ .*

This is repeated as [Theorem 2.8](#) in the body of the paper. From it we deduce the next result, which is [Corollary 2.12](#).

**Corollary 0.14.** *Let  $A$  be a precise artinian local ring in  $\text{STRing}_c \mathbb{k}$ , with residue field  $K$ . Give  $K$  the fine  $A$ -module topology. Then any lifting  $\sigma : K \rightarrow A$  in  $\text{STRing}_c \mathbb{k}$  is a precise lifting.*

## 1. Semi-topological rings and modules

We begin with a general discussion of various categories of rings. The notation introduced here will make some of our more delicate definitions possible.

Let  $\text{Ring}$  be the category of rings (not necessarily commutative). The morphisms are unit-preserving ring homomorphisms. Inside it there is the full subcategory  $\text{Ring}_c$  of commutative rings.

Now let us fix a nonzero commutative base ring  $\mathbb{k}$ . A ring homomorphism  $f : \mathbb{k} \rightarrow A$  is called *central* if  $f(\mathbb{k})$  is contained in the center of  $A$ . In this case we call  $A$  a *central  $\mathbb{k}$ -ring*. (A more common name for  $A$  is an associative unital  $\mathbb{k}$ -algebra.) The central  $\mathbb{k}$ -rings form a category  $\text{Ring } \mathbb{k}$ , in which a morphism is a



ring homomorphism  $A \rightarrow B$  that respects the given central homomorphisms  $\mathbb{k} \rightarrow A$  and  $\mathbb{k} \rightarrow B$ . Inside  $\text{Ring } \mathbb{k}$  there is the full subcategory  $\text{Ring}_{\mathbb{C}} \mathbb{k}$  of commutative  $\mathbb{k}$ -rings. Of course, if  $\mathbb{k} = \mathbb{Z}$  then  $\text{Ring } \mathbb{Z} = \text{Ring}$ .

Let  $A$  be a local ring in  $\text{Ring}_{\mathbb{C}} \mathbb{k}$ , with maximal ideal  $\mathfrak{m}$  and residue field  $K = A/\mathfrak{m}$ . Recall that  $A$  is called a complete local ring if the canonical homomorphism  $A \rightarrow \lim_{\leftarrow i} A/\mathfrak{m}^i$  is bijective. The canonical surjection  $\pi : A \twoheadrightarrow K$  makes  $K$  into an object of  $\text{Ring}_{\mathbb{C}} \mathbb{k}$ . A *lifting* of the canonical surjection  $\pi : A \twoheadrightarrow K$  in  $\text{Ring}_{\mathbb{C}} \mathbb{k}$ , or a *coefficient field* for  $A$  in  $\text{Ring}_{\mathbb{C}} \mathbb{k}$ , is a homomorphism  $\sigma : K \rightarrow A$  in  $\text{Ring}_{\mathbb{C}} \mathbb{k}$  such that the composition  $\pi \circ \sigma$  is the identity of  $K$ .

The next result is part of the Cohen structure theorem. We will repeat its proof, because the proof itself will feature in some of our constructions.

**Theorem 1.1** (Cohen). *Assume  $\mathbb{k}$  is a perfect field. Let  $A$  be a complete local ring in  $\text{Ring}_{\mathbb{C}} \mathbb{k}$ , with residue field  $K$ . Then there exists a lifting  $\sigma : K \rightarrow A$  in  $\text{Ring}_{\mathbb{C}} \mathbb{k}$  of the canonical surjection  $\pi : A \twoheadrightarrow K$ . Moreover, if  $\mathbb{k} \rightarrow K$  is finite, then this lifting  $\sigma$  is unique.*

*Proof.* Consider the  $K$ -module  $\Omega_{K/\mathbb{k}}^1$  of Kähler differential forms. Choose a collection  $\{b_x\}_{x \in X}$  of elements of  $K$  so that the collection of forms  $\{d(b_x)\}_{x \in X}$  is a basis of the  $K$ -module  $\Omega_{K/\mathbb{k}}^1$ . According to [Matsumura 1986, Theorems 26.5 and 26.8], the collection of elements  $\{b_x\}_{x \in X}$  is algebraically independent over  $\mathbb{k}$ , and  $K$  is formally étale over the subfield  $\mathbb{k}(\{b_x\})$  generated by this collection. (Actually, if either  $\text{char } \mathbb{k} = 0$  or  $K$  is finitely generated over  $\mathbb{k}$ , then the field  $K$  is separable algebraic over  $\mathbb{k}(\{b_x\})$ . See [Matsumura 1986, Theorem 26.2].)

For any  $x \in X$  choose an arbitrary lifting  $\sigma_b(b_x) \in A$  of the element  $b_x$ ; thus  $\pi(\sigma_b(b_x)) = b_x$ . Since the collection  $\{b_x\}_{x \in X}$  is algebraically independent over  $\mathbb{k}$ , the subring  $\mathbb{k}[\{b_x\}]$  of  $K$  is a polynomial ring. Therefore the function  $\sigma_b : X \rightarrow A$  extends uniquely to a homomorphism  $\sigma_p : \mathbb{k}[\{b_x\}] \rightarrow A$  in  $\text{Ring}_{\mathbb{C}} \mathbb{k}$ . Because  $A$  is a local ring, for any nonzero element  $b \in \mathbb{k}[\{b_x\}]$  its lift  $\sigma_p(b)$  is invertible in  $A$ . Thus  $\sigma_p$  extends uniquely to a homomorphism  $\sigma_r : \mathbb{k}(\{b_x\}) \rightarrow A$ . (The subscripts  $b$ ,  $p$ ,  $r$  refer to “basis”, “polynomial” and “rational”.)

Let  $A_i := A/\mathfrak{m}^{i+1}$ , with surjection  $\pi_i : A \rightarrow A_i$ . Because  $\mathbb{k}(\{b_x\}) \rightarrow K$  is formally étale, the homomorphism  $\pi_i \circ \sigma_r : \mathbb{k}(\{b_x\}) \rightarrow A_i$  extends uniquely to a homomorphism  $\sigma_i : K \rightarrow A_i$ , which lifts  $A_i \twoheadrightarrow K = A_0$ . We get an inverse system of liftings, and thus a lifting  $\sigma : K \rightarrow \lim_{\leftarrow i} A/\mathfrak{m}^i = A$ ,  $\sigma := \lim_{\leftarrow i} \sigma_i$ . The restriction of  $\sigma$  to  $\mathbb{k}(\{b_x\})$  equals  $\sigma_r$ , and in particular we see that  $\sigma$  is a homomorphism in  $\text{Ring}_{\mathbb{C}} \mathbb{k}$ .

If  $\mathbb{k} \rightarrow K$  is finite then  $X = \emptyset$ , and hence  $\sigma$  is unique. □

**Remark 1.2.** Liftings exist whenever they can exist, namely if and only if  $A$  contains a field. This is called the equal characteristics case. Indeed, if  $A$  contains a field then it contains some perfect field  $\mathbb{k}$  (e.g.,  $\mathbb{Q}$  or  $F_p$ ). Now [Theorem 1.1](#) can

be applied. Note that, if the residue field  $K$  contains  $\mathbb{Q}$ , then  $A$  also contains  $\mathbb{Q}$ .

The complication arises when the residue field  $K = A/\mathfrak{m}$  contains  $\mathbb{F}_p$ , but  $A$  does not contain it (i.e.,  $p \neq 0$  in  $A$ ). This is called the mixed characteristics case. In this case the notion of lifting has to be modified. First the base ring  $\mathbb{k}$  is replaced by two rings: a perfect field  $\mathbb{k}$  of characteristic  $p$ , and a complete DVR  $\tilde{\mathbb{k}}$  whose maximal ideal is generated by  $p$  and whose residue field is  $\mathbb{k}$ . The ring  $\tilde{\mathbb{k}}$  is called the ring of Witt vectors of  $\mathbb{k}$ . (e.g., when  $\mathbb{k} = \mathbb{F}_p$ , its ring of Witt vectors is  $\tilde{\mathbb{k}} = \hat{\mathbb{Z}}_{(p)}$ , the  $p$ -adic integers.) A homomorphism  $\mathbb{k} \rightarrow K$  lifts canonically to a homomorphism  $\tilde{\mathbb{k}} \rightarrow A$ . Next there is a complete DVR  $\tilde{K}$ , whose maximal ideal is generated by  $p$  and whose residue field is  $K$ , and  $\tilde{\mathbb{k}} \rightarrow \tilde{K}$  is  $p$ -adically formally smooth. Therefore there exists a lifting  $\sigma : \tilde{K} \rightarrow A$  over  $\tilde{\mathbb{k}}$ . Moreover, all such liftings are controlled by  $\Omega_{K/\mathbb{k}}^1$ , just as in the proof of [Theorem 1.1](#).

In this paper we shall deal exclusively with the equal characteristics case.

We are going to look at a more subtle lifting situation, involving topologies on  $A$  and  $K$ .

We consider the base ring  $\mathbb{k}$  as a topological ring with the discrete topology. Recall that a topological  $\mathbb{k}$ -module is a  $\mathbb{k}$ -module  $M$  endowed with a topology such that addition and multiplication are continuous functions  $M \times M \rightarrow M$  and  $\mathbb{k} \times M \rightarrow M$  respectively. We say that the topology on  $M$  is  $\mathbb{k}$ -linear, and that  $M$  is a *linearly topologized  $\mathbb{k}$ -module*, if the element  $0 \in M$  has a basis of open neighborhoods consisting of open  $\mathbb{k}$ -submodules.

In order to define a  $\mathbb{k}$ -linear topology on a  $\mathbb{k}$ -module  $M$ , all we have to do is to provide a collection  $\{U_i\}_{i \in I}$  of  $\mathbb{k}$ -submodules of  $M$  that is cofiltered under inclusion; namely, for any  $i, j \in I$ , there exists  $k \in I$  such that  $U_k \subset U_i \cap U_j$ . The resulting topology on  $M$ , in which the collection  $\{U_i\}_{i \in I}$  is a basis of open neighborhoods of  $0 \in M$ , is called the  $\mathbb{k}$ -linear topology generated by this collection.

**Definition 1.3.** Let  $M_1, \dots, M_p, N$  be linearly topologized  $\mathbb{k}$ -modules, and let  $\mu : \prod_{i=1}^p M_i \rightarrow N$  be a  $\mathbb{k}$ -multilinear function. We say that  $\mu$  is *semi-continuous* if, for every  $\mathbf{m} = (m_1, \dots, m_p) \in \prod_{i=1}^p M_i$  and every  $i \in \{1, \dots, p\}$ , the homomorphism

$$\mu_{\mathbf{m},i} : M_i \rightarrow N, \quad \mu_{\mathbf{m},i}(m'_i) := \mu(m_1, \dots, m_{i-1}, m'_i, m_{i+1}, \dots, m_p),$$

is continuous.

**Definition 1.4** [[Yekutieli 1992](#)]. A *semi-topological  $\mathbb{k}$ -ring* is a  $\mathbb{k}$ -ring  $A$  with a  $\mathbb{k}$ -linear topology on it (so the underlying  $\mathbb{k}$ -module of  $A$  is a linearly topologized  $\mathbb{k}$ -module) such that multiplication  $\mu : A \times A \rightarrow A$  is a semi-continuous bilinear function.

The semi-topological  $\mathbb{k}$ -rings form a category  $\text{STRing } \mathbb{k}$ , in which the morphisms are the continuous  $\mathbb{k}$ -ring homomorphisms.

We use the abbreviation “ST” for “semi-topological”. The ring  $\mathbb{k}$  with its discrete topology is the initial object of  $\text{STRing } \mathbb{k}$ . Inside  $\text{STRing } \mathbb{k}$  there is the full subcategory  $\text{STRing}_{\text{c}} \mathbb{k}$  of commutative ST  $\mathbb{k}$ -rings.

**Example 1.5.** Suppose  $A$  is a commutative  $\mathbb{k}$ -ring, and  $\mathfrak{a} \subset A$  is an ideal. Give  $A$  the  $\mathfrak{a}$ -adic topology. Then  $A$  is a ST  $\mathbb{k}$ -ring. (The ring  $A$  is actually a topological ring, because multiplication  $A \times A \rightarrow A$  is continuous.) The ring of Laurent series  $A((t))$  — see [Definition 1.17](#) — is a ST  $\mathbb{k}$ -ring, but it is usually not a topological ring.

**Definition 1.6.** Let  $A$  be a ST  $\mathbb{k}$ -ring. A *left ST  $A$ -module* is a left  $A$ -module  $M$  endowed with a  $\mathbb{k}$ -linear topology on it (so  $M$  is a linearly topologized  $\mathbb{k}$ -module) such that the bilinear function  $\mu : A \times M \rightarrow M$ ,  $\mu(a, m) := a \cdot m$ , is semi-continuous.

The ST left  $A$ -modules form a category, in which the morphisms are the continuous  $A$ -linear homomorphisms. We denote this category by  $\text{STMod } A$ .

There is a similar right module version, denoted by  $\text{STMod } A^{\text{op}}$ .

**Remark 1.7.** If  $A$  is a discrete ST  $\mathbb{k}$ -ring (e.g.,  $A = \mathbb{k}$ ), then a ST  $A$ -module  $M$  is also a topological  $A$ -module, because the multiplication function  $A \times M \rightarrow M$  is continuous. We will usually ignore this fact.

**Proposition 1.8.** *Let  $A$  be a ST  $\mathbb{k}$ -ring. The category  $\text{STMod } A$  has these properties:*

- (1) *It is a  $\mathbb{k}$ -linear additive category.*
- (2) *It has limits and colimits (of arbitrary cardinality). In particular there are coproducts, products, kernels and cokernels.*

*Proof.* This is all essentially in [[Yekutieli 1992](#), Section 1.2]. The fact that  $\text{STMod } A$  is  $\mathbb{k}$ -linear is clear.

Given a collection  $\{M_x\}_{x \in X}$  of ST  $A$ -modules, indexed by a set  $X$ , let  $M := \bigoplus_{x \in X} M_x$  be the direct sum in  $\text{Mod } A$ . Given any collection  $\{U_x\}_{x \in X}$ , where  $U_x \subset M_x$  is an open  $\mathbb{k}$ -submodule, let  $U := \bigoplus_{x \in X} U_x$ , which is a  $\mathbb{k}$ -submodule of  $M$ . Give  $M$  the  $\mathbb{k}$ -linear topology generated by these  $\mathbb{k}$ -submodules  $U$ . This makes  $M$  into a ST  $A$ -module, and together with the embeddings  $M_x \hookrightarrow M$  it becomes a coproduct in  $\text{STMod } A$ . Likewise, the product  $\prod_{x \in X} M_x$  in  $\text{Mod } A$ , with the product topology, becomes a product in  $\text{STMod } A$ .

Let  $\phi : M \rightarrow N$  be a homomorphism in  $\text{STMod } A$ . Then  $\text{Ker}(\phi)$ , with the topology induced on it from  $M$  (the subspace topology), is a kernel of  $\phi$ . The module  $\text{Coker}(\phi)$ , with the topology induced on it from  $N$  (the quotient topology), is a cokernel of  $\phi$ .

Now that we have coproducts, products, kernels and cokernels, any limit and colimit can be produced in  $\text{STMod } A$ . □

Let  $A \in \text{STRing } \mathbb{k}$ . We often use the notation  $\text{Hom}_A^{\text{cont}}(M, N)$  to denote the  $\mathbb{k}$ -module of continuous  $A$ -linear homomorphism between two ST left  $A$ -modules  $M$  and  $N$ . This is just another way to refer to the  $\mathbb{k}$ -module  $\text{Hom}_{\text{STMod } A}(M, N)$ .

**Remark 1.9.** The concept of ST module is very close to the concept of *smooth representation* from the theory of representations of topological groups. Perhaps some of our work here can be used in that area.

**Definition 1.10.** Let  $A$  be a ST ring, and let  $M$  be a left  $A$ -module. The *fine  $A$ -module topology* on  $M$  is the finest linear topology on  $M$  that makes it into a ST  $A$ -module.

It is not clear at first whether such a topology exists; but it does — see [Yekutieli 1992, Section 1.2]. The fine topology can be characterized as follows: a ST  $A$ -module  $M$  has the fine  $A$ -module topology if and only if, for any  $N \in \text{STMod } A$ , the canonical function

$$\text{Hom}_A^{\text{cont}}(M, N) \rightarrow \text{Hom}_A(M, N)$$

is bijective [Yekutieli 1992, Proposition 1.2.4]. (So we get a left adjoint to the forgetful functor  $\text{STMod } A \rightarrow \text{Mod } A$ .)

The fine  $A$ -module topology can be described quite explicitly. First consider a free module  $F = \bigoplus_{x \in X} A$ . The direct sum (i.e., coproduct) topology on it is the fine topology. Now take any  $A$ -module  $M$ , and let  $F \twoheadrightarrow M$  be some  $A$ -linear surjection from a free module  $F$ . Then the quotient topology on  $M$  coincides with its fine topology.

**Definition 1.11.** Let  $\phi : M \rightarrow N$  be a homomorphism in  $\text{STMod } A$ .

- (1)  $\phi$  is called a *strict monomorphism* if it is injective and the topology on  $M$  equals the subspace topology on it induced by  $\phi$  and  $N$ .
- (2)  $\phi$  is called a *strict epimorphism* if it is surjective and the topology on  $N$  equals the quotient topology on it induced by  $\phi$  and  $M$ .

**Example 1.12.** Let  $\phi : M \rightarrow N$  be a homomorphism in  $\text{STMod } A$ , and assume both modules have the fine  $A$ -module topologies. If  $\phi$  is a surjection, then it is a strict epimorphism. If  $\phi : M \rightarrow N$  a split injection in  $\text{STMod } A$ , then it is a strict monomorphism.

**Definition 1.13.** Let  $f : A \rightarrow B$  be a homomorphism in  $\text{STRing } \mathbb{k}$ . We say that  $f$  is a *strict monomorphism* (resp. *strict epimorphism*) in  $\text{STRing } \mathbb{k}$  if it is so in  $\text{STMod } \mathbb{k}$ .

**Definition 1.14.** Let  $A \in \text{STRing}_c \mathbb{k}$ , let  $f : A \rightarrow B$  be a homomorphism in  $\text{Ring}_c \mathbb{k}$ , and let  $M \in \text{Mod } B$ . We view  $M$  as an  $A$ -module via  $f$ . The fine  $A$ -module topology on  $M$  is called the *fine  $(A, f)$ -module topology*.

**Lemma 1.15.** *In the situation of Definition 1.14:*

- (1) *The ring  $B$ , with the fine  $(A, f)$ -module topology, becomes an object of  $\text{STRing}_{\mathbb{C}} \mathbb{k}$ ; and  $f : A \rightarrow B$  becomes a morphism in  $\text{STRing}_{\mathbb{C}} \mathbb{k}$ .*
- (2) *Give  $B$  the fine  $(A, f)$ -module topology. Then the fine  $B$ -module topology on  $M$  coincides with the fine  $(A, f)$ -module topology on it. Therefore  $M$ , endowed with the fine  $(A, f)$ -module topology, is an object of  $\text{STMod } B$ .*

*Proof.* This is easy using [Yekutieli 1992, Proposition 1.2.4].  $\square$

**Lemma 1.16.** *Let  $\{B_i\}_{i \in \mathbb{N}}$  be an inverse system in  $\text{STRing } \mathbb{k}$ . The ring  $B := \varprojlim_i B_i$ , with its inverse limit topology (see Proposition 1.8(2)), is a ST  $\mathbb{k}$ -ring.*

*Proof.* This follows almost immediately from the definitions.  $\square$

Here is the most important construction of ST rings in our context. This is [Yekutieli 1992, Definition 1.3.3]. Lemmas 1.15 and 1.16 justify it.

**Definition 1.17.** Let  $A$  be a commutative ST  $\mathbb{k}$ -ring.

- (1) Let  $t$  be a variable.
  - (a) For any  $i \in \mathbb{N}$  put on the truncated polynomial ring  $A[t]/(t^{i+1})$  the fine  $A$ -module topology. This makes  $A[t]/(t^{i+1})$  a ST  $\mathbb{k}$ -ring.
  - (b) Give the ring of formal power series  $A[[t]] := \varprojlim_{\leftarrow i} A[t]/(t^{i+1})$  the inverse limit topology. In this way  $A[[t]]$  is a ST  $\mathbb{k}$ -ring.
  - (c) Give the ring of formal Laurent series  $A((t)) := A[[t]][t^{-1}]$  the fine  $A[[t]]$ -module topology. In this way  $A((t))$  is a ST  $\mathbb{k}$ -ring.
- (2) Let  $\mathbf{t} = (t_1, \dots, t_n)$  be a sequence of variables. The *ring of iterated Laurent series*

$$A((\mathbf{t})) = A((t_1, \dots, t_n))$$

is the commutative ST  $\mathbb{k}$ -ring defined recursively on  $n$  by

$$A((t_1, \dots, t_n)) := A((t_2, \dots, t_n))((t_1)),$$

using part (1).

Note that as ST  $A$ -modules there is an isomorphism

$$A((\mathbf{t})) = \left( \prod_{i \geq 0} A \cdot t^i \right) \oplus \left( \bigoplus_{i < 0} A \cdot t^i \right) \cong \left( \prod_{i \in \mathbb{N}} A \right) \oplus \left( \bigoplus_{i \in \mathbb{N}} A \right).$$

**Remark 1.18.** Strangely, for  $n \geq 2$  (and when  $A$  is nonzero), the ring  $B := A((t_1, \dots, t_n))$  is *not topological*; namely, multiplication is not a continuous function  $B \times B \rightarrow B$ . This is the reason for introducing the semi-topological apparatus.

Furthermore, the topology on  $B$  is *not metrizable*. Still,  $B$  is *complete*, in the sense that the canonical homomorphism  $B \rightarrow \varprojlim_{\leftarrow U} B/U$ , where  $U$  runs over all open  $\mathbb{k}$ -submodules of  $B$ , is bijective.

**Exercise 1.19.** Let  $K := \mathbb{k}((\mathbf{t}))$ , the ring of Laurent series in the sequence of variables  $\mathbf{t} = (t_1, \dots, t_n)$ , with its topology from [Definition 1.17](#). Let  $F(\mathbb{Z}^n, \mathbb{k})$  be the set of functions  $a : \mathbb{Z}^n \rightarrow \mathbb{k}$ , written in subscript notation; namely for  $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$  the value of  $a$  is  $a_{\mathbf{i}} \in \mathbb{k}$ . The notation for monomials in  $\mathbf{t}$  is  $\mathbf{t}^{\mathbf{i}} := t_1^{i_1} \cdots t_n^{i_n}$ . We say that a collection  $\{a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^n}$  of elements of  $K$  is a *Cauchy collection* if, for every open  $\mathbb{k}$ -submodule  $U \subset K$ , there is a finite subset  $I \subset \mathbb{Z}^n$  such that  $a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \in U$  for all  $\mathbf{i} \notin I$ . A function  $a \in F(\mathbb{Z}^n, \mathbb{k})$  is called Cauchy if the collection  $\{a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^n}$  is Cauchy. The set of Cauchy functions is denoted by  $F_c(\mathbb{Z}^n, \mathbb{k})$ . The exercise is to show that for any  $a \in F_c(\mathbb{Z}^n, \mathbb{k})$  the series  $\sum_{\mathbf{i} \in \mathbb{Z}^n} a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}$  converges in  $K$ , and that the resulting function  $F_c(\mathbb{Z}^n, \mathbb{k}) \rightarrow K$  is a  $\mathbb{k}$ -linear bijection. (For a slightly more general assertion see the end of [\[Yekutieli 1992, Section 1.3\]](#).)

**Definition 1.20.** Let  $f : A \rightarrow B$  be a homomorphism in  $\text{STRing } \mathbb{k}$ . Given  $M$  in  $\text{STMod } B$ , we denote by  $\text{rest}_f(M)$  the ST  $A$ -module whose underlying ST  $\mathbb{k}$ -module is  $M$ , and  $A$  acts via  $f$ .

In this way we get a functor

$$\text{rest}_f : \text{STMod } B \rightarrow \text{STMod } A.$$

We now return to liftings.

**Definition 1.21.** Let  $A$  be a local ring in  $\text{STRing}_c \mathbb{k}$ , with residue field  $K$ . We put on  $K$  the fine  $A$ -module topology, so that the canonical surjection  $\pi : A \twoheadrightarrow K$  is a morphism in  $\text{STRing}_c \mathbb{k}$ . A *lifting of  $K$*  in  $\text{STRing}_c \mathbb{k}$  is a homomorphism  $\sigma : K \rightarrow A$  in  $\text{STRing}_c \mathbb{k}$  such that the composition  $\pi \circ \sigma$  is the identity of  $K$ .

The important thing to remember is that  $\sigma : K \rightarrow A$  has to be continuous.

**Example 1.22.** Assume  $\mathbb{k}$  is a field of characteristic 0. Let  $K := \mathbb{k}((t_2))$  and  $A := K[[t_1]]$ , with topologies from [Definition 1.17](#). So we are in the situation of [Definition 1.21](#). Consider the lifting  $\sigma : K \rightarrow A$  from [Example 3.13](#). If at least one of the elements  $c_i$  is nonzero, the lifting  $\sigma$  is not continuous.

**Remark 1.23.** Let  $A$  be a local ring in  $\text{STRing}_c \mathbb{k}$ , with maximal ideal  $\mathfrak{m}$ . We do not assume any relation between the given topology of  $A$  and its  $\mathfrak{m}$ -adic topology. For instance,  $A$  could have the discrete topology, which is finer than any other topology.

On the other hand, in [Example 1.22](#) above, where  $A = \mathbb{k}((t_2))[[t_1]]$  and  $\mathfrak{m} = (t_1)$ , the  $\mathfrak{m}$ -adic topology on  $A$  is finer than the given topology on it (since the discrete topology on  $K = \mathbb{k}((t_2))$  is finer than its  $t_2$ -adic topology).

The next definition is a generalization of [\[Yekutieli 1992, Definition 2.2.1\]](#).

**Definition 1.24.** Let  $A$  be an artinian local ring in  $\text{STRing}_c \mathbb{k}$ , with residue field  $K$ . We put on  $K$  the fine  $A$ -module topology, so that the canonical surjection  $\pi : A \twoheadrightarrow K$  is a strict epimorphism in  $\text{STRing}_c \mathbb{k}$ .

- (1) A lifting  $\sigma : K \rightarrow A$  in  $\text{STRing}_c \mathbb{k}$  is called a *precise lifting* if the original topology of  $A$  equals the fine  $(K, \sigma)$ -module topology on it.
- (2) The topology on  $A$  is called a *precise topology*, and  $A$  is called a *precise artinian local ring in  $\text{STRing}_c \mathbb{k}$* , if there exists some precise lifting  $\sigma : K \rightarrow A$  in  $\text{STRing}_c \mathbb{k}$ .

Here are examples:

**Example 1.25.** Start with an artinian local ring  $A$  in  $\text{Ring}_c \mathbb{k}$ , and with a given lifting  $\sigma : K \rightarrow A$  of the residue field. Put any topology on  $K$  that makes it into an object of  $\text{STRing}_c \mathbb{k}$ . Next give  $A$  the fine  $(K, \sigma)$ -module topology. According to [Lemma 1.15\(2\)](#), the fine  $A$ -module topology on  $K$  equals its original topology. We see that  $\sigma : K \rightarrow A$  is a precise lifting, and hence  $A$  is a precise artinian local ring in  $\text{STRing}_c \mathbb{k}$ .

[Definition 1.24](#) makes sense also for an artinian *semi-local* ring  $A$ , with Jacobson radical  $\mathfrak{r}$  and residue ring  $K := A/\mathfrak{r}$ . Of course, here  $K$  is a finite product of fields. This is used in the next example.

**Example 1.26.** We use the Beilinson completion that is explained in [Section 6](#). Assume  $\mathbb{k}$  is a perfect field, and let  $X$  be a finite type  $\mathbb{k}$ -scheme. Take a saturated chain of points  $\xi = (x_0, \dots, x_n)$  in  $X$ , and let  $A := \mathcal{O}_{X, x_0}/\mathfrak{m}_{x_0}^{l+1}$  for some  $l \in \mathbb{N}$ . So  $A$  is an artinian local ring, and its residue field is  $K := \mathbb{k}(x_0)$ . Let  $\sigma : K \rightarrow A$  be a lifting in  $\text{Ring}_c \mathbb{k}$ .

We view  $A$  and  $K$  as quasi-coherent sheaves on  $X$ , constant on the closed set  $\overline{\{x_0\}}$ . The lifting  $\sigma$  is a differential operator of  $\mathcal{O}_X$ -modules, and hence, according to [\[Yekutieli 1992, Propositions 3.1.10 and 3.2.2\]](#), there is a homomorphism  $\sigma_\xi : K_\xi \rightarrow A_\xi$  in  $\text{STRing}_c \mathbb{k}$  that lifts the canonical surjection  $\pi_\xi : A_\xi \rightarrow K_\xi$ . The arguments in the proof of [\[Yekutieli 1992, Proposition 3.2.5\]](#) show that  $K_\xi$  has the fine  $A_\xi$ -module topology, and vice versa.

By [Theorem 6.1](#) the ring  $K_\xi$  is a finite product of fields. Therefore  $A_\xi$  is an artinian semi-local ring, with residue ring  $K_\xi$ . We see that the lifting  $\sigma_\xi : K_\xi \rightarrow A_\xi$  is a precise lifting, and  $A_\xi$  is a precise artinian semi-local ring in  $\text{STRing}_c \mathbb{k}$ .

**Example 1.27.** Assume  $\mathbb{k}$  is a field. Let  $K := \mathbb{k}((t))$  with the discrete topology, and let  $\mathfrak{m} := \mathbb{k}((t))$  with the  $t$ -adic topology. We view  $\mathfrak{m}$  as a ST  $K$ -module. Define  $A := K \oplus \mathfrak{m}$ , the trivial extension of  $K$  by  $\mathfrak{m}$  (so  $\mathfrak{m}^2 = 0$ ). For any lifting  $\sigma : K \rightarrow A$ , the  $(K, \sigma)$ -module topology on  $A$  is the discrete topology. Therefore there is no precise lifting, and  $A$  is not a precise artinian local ring in  $\text{STRing}_c \mathbb{k}$ .

A question that immediately comes to mind is this: If  $A$  is a precise artinian local ring in  $\text{STRing}_c \mathbb{k}$ , are all liftings  $\sigma : K \rightarrow A$  in  $\text{STRing}_c \mathbb{k}$  precise? This is answered affirmatively in [Corollary 2.12](#) in the next section.

Let  $A$  be a ST  $\mathbb{k}$ -ring and let  $M$  be a ST  $A$ -module. The closure  $\overline{\{0\}}$  of the zero submodule  $\{0\}$  is an  $A$ -submodule of  $M$ .

**Definition 1.28.** Let  $A$  be a ST  $\mathbb{k}$ -ring and let  $M$  be a ST  $A$ -module.

- (1) If  $\{0\}$  is closed in  $M$ , then  $M$  is called a *separated ST module*.
- (2) Define  $M^{\text{sep}} := M/\overline{\{0\}}$ . This is a ST  $A$ -module with the quotient topology from  $M$ , and we call it *the separated ST module associated to  $M$* .

Of course,  $M$  is a separated ST module if and only if it is a Hausdorff topological space.

The assignment  $M \mapsto M^{\text{sep}}$  is a  $\mathbb{k}$ -linear functor from  $\text{STMod } A$  to itself. There is a functorial strict epimorphism  $\tau_M : M \rightarrow M^{\text{sep}}$ . The ST module  $M^{\text{sep}}$  is separated, and it is easy to see that for any separated ST  $A$ -module  $N$  the homomorphism

$$\text{Hom}_A^{\text{cont}}(M^{\text{sep}}, N) \rightarrow \text{Hom}_A^{\text{cont}}(M, N)$$

induced by  $\tau_M$  is bijective.

**Remark 1.29.** The reader might wonder why we work with separated modules and not with complete modules. The reason is that, for a ST  $A$ -module  $M$ , its completion  $\widehat{M} := \varprojlim_U M/U$ , where  $U$  runs over all open subgroups of  $M$ , could fail to be an  $A$ -module!

However, in many important instances (such as the module of differentials of a topological local field), the ST  $A$ -module  $M^{\text{sep}}$  turns out to be complete.

We end this section with a discussion of ST tensor products.

**Definition 1.30** [Yekutieli 1992, Definition 1.2.11]. Suppose  $A$  is a commutative ST  $\mathbb{k}$ -ring, and  $M_1, \dots, M_p$  are ST  $A$ -modules. The tensor product topology on the  $A$ -module

$$\bigotimes_{i=1}^p M_i := M_1 \otimes_A \cdots \otimes_A M_p$$

is the finest linear topology such that the canonical multilinear function

$$\prod_{i=1}^p M_i \rightarrow \bigotimes_{i=1}^p M_i$$

is semi-continuous.

With this topology,  $\bigotimes_{i=1}^p M_i$  is a ST  $A$ -module. Given any semi-continuous  $A$ -multilinear function  $\beta : \prod_{i=1}^p M_i \rightarrow N$ , where  $N$  is a ST  $A$ -module, the corresponding  $A$ -linear homomorphism  $\bigotimes_{i=1}^p M_i \rightarrow N$  is continuous. For more details see [Yekutieli 1992, Section 1.2].



**Example 1.31.** Let  $f : A \rightarrow B$  be a homomorphism in  $\text{STRing}_c \mathbb{k}$ , and let  $M$  be in  $\text{STMod } A$ . Then  $B \otimes_A M$ , with the tensor product topology, is a ST  $B$ -module. We get an adjoint to the forgetful functor  $\text{rest}_f$ . If  $M$  has the fine  $A$ -module topology, then  $B \otimes_A M$  has the fine  $B$ -module topology. See [Yekutieli 1992, Proposition 1.2.14 and Corollary 1.2.15].

**Remark 1.32.** Assume the base ring  $\mathbb{k}$  is a field. Let  $M$  and  $N$  be ST  $\mathbb{k}$ -modules (i.e., linearly topologized  $\mathbb{k}$ -modules). Beilinson [2008] talks about three topologies on the tensor product  $M \otimes_{\mathbb{k}} N$ .

In our paper we encounter two topologies on  $M \otimes_{\mathbb{k}} N$ . The first is the tensor product topology from Definition 1.30. It is symmetric: the automorphism  $m_1 \otimes m_2 \mapsto m_2 \otimes m_1$  of  $M \otimes_{\mathbb{k}} M$  is a homeomorphism.

For the second kind of tensor product topology consider  $M := \mathbb{k}(t_2)$  with the  $t_2$ -adic topology, and  $N := \mathbb{k}(t_1)$  the  $t_1$ -adic topology. So  $M \cong N$  in  $\text{STMod } \mathbb{k}$ . Let  $K := \mathbb{k}((t_1, t_2))$  be the field of iterated Laurent series, with the topology of Definition 1.17, starting from the discrete topology on  $\mathbb{k}$ . The embedding  $M \otimes_{\mathbb{k}} N \subset K$  induces a topology on it, making it into a ST  $\mathbb{k}$ -module. Presumably this topology on  $M \otimes_{\mathbb{k}} N$  can be described in terms of the topologies of  $M$  and  $N$ . Now  $K$  is complete, and  $M \otimes_{\mathbb{k}} N$  is dense in it. Since the roles of the two variables in  $K$  are different (e.g., the series  $\sum_{i \in \mathbb{N}} t_1^i \cdot t_2^{-i}$  is summable, but the series  $\sum_{i \in \mathbb{N}} t_1^{-i} \cdot t_2^i$  is not summable), we see that this topology on  $M \otimes_{\mathbb{k}} N$  is not symmetric.

It should be interesting to compare our two tensor product topologies to the three discussed in [Beilinson 2008].

## 2. Continuous differential operators

Our approach to continuous differential operators is an adaptation to the ST context of the definitions from [EGA IV 1967]. We are following [Yekutieli 1992; 1995]. Recall that the base ring  $\mathbb{k}$  is a nonzero commutative ring, and it has the discrete topology.

Let  $A$  be a commutative  $\mathbb{k}$ -ring. Any  $\mathbb{k}$ -central  $A$ -bimodule  $P$  has an increasing filtration  $\{F_i(P)\}_{i \in \mathbb{Z}}$  by  $A$ -sub-bimodules, called the differential filtration. This filtration is defined inductively. For  $i \leq -1$  we define  $F_i(P) := 0$ . For  $i \geq 0$  the elements of  $F_i(P)$  are the elements  $p \in P$  such that  $a \cdot p - p \cdot a \in F_{i-1}(P)$  for every  $a \in A$ .

Now assume  $A$  is a commutative ST  $\mathbb{k}$ -ring, and let  $M, N$  be ST  $A$ -modules. The set  $\text{Hom}_{\mathbb{k}}^{\text{cont}}(M, N)$  of continuous  $\mathbb{k}$ -linear homomorphisms is a  $\mathbb{k}$ -central  $A$ -bimodule, so it has a differential filtration. We define

$$\text{Diff}_{A/\mathbb{k}}^{\text{cont}}(M, N) := \bigcup_i F_i(\text{Hom}_{\mathbb{k}}^{\text{cont}}(M, N)) \subset \text{Hom}_{\mathbb{k}}^{\text{cont}}(M, N).$$

The elements of

$$F_i(\mathrm{Diff}_{A/\mathbb{k}}^{\mathrm{cont}}(M, N)) := F_i(\mathrm{Hom}_{\mathbb{k}}^{\mathrm{cont}}(M, N))$$

are by definition continuous differential operators of order at most  $i$ . Note that

$$F_0(\mathrm{Diff}_{A/\mathbb{k}}^{\mathrm{cont}}(M, N)) = \mathrm{Hom}_A^{\mathrm{cont}}(M, N).$$

When  $N = M$  we write

$$\mathrm{Diff}_{A/\mathbb{k}}^{\mathrm{cont}}(M) := \mathrm{Diff}_{A/\mathbb{k}}^{\mathrm{cont}}(M, M).$$

This is a subring of  $\mathrm{End}_{\mathbb{k}}^{\mathrm{cont}}(M)$ . Let  $\mathrm{Der}_{A/\mathbb{k}}^{\mathrm{cont}}(M)$  be the  $A$ -module of continuous  $\mathbb{k}$ -linear derivations  $A \rightarrow M$ . Then

$$F_1(\mathrm{Diff}_{A/\mathbb{k}}^{\mathrm{cont}}(A, M)) = M \oplus \mathrm{Der}_{A/\mathbb{k}}^{\mathrm{cont}}(M)$$

as left  $A$ -modules.

If  $M = A$  then we write

$$\mathcal{D}_{A/\mathbb{k}}^{\mathrm{cont}} := \mathrm{Diff}_{A/\mathbb{k}}^{\mathrm{cont}}(A). \quad (2-1)$$

This is the ring of continuous differential operators of  $A$  (relative to  $\mathbb{k}$ ). Let us write  $\mathcal{T}_{A/\mathbb{k}}^{\mathrm{cont}} := \mathrm{Der}_{A/\mathbb{k}}^{\mathrm{cont}}(A)$ , the Lie algebra of continuous derivations of  $A$ . Then

$$F_1(\mathcal{D}_{A/\mathbb{k}}^{\mathrm{cont}}) = A \oplus \mathcal{T}_{A/\mathbb{k}}^{\mathrm{cont}}$$

as left  $A$ -modules.

If  $A$  is discrete, then  $\mathcal{D}_{A/\mathbb{k}}^{\mathrm{cont}} = \mathcal{D}_{A/\mathbb{k}}$ , the usual ring of differential operators from [EGA IV 1967]; and  $\mathcal{T}_{A/\mathbb{k}}^{\mathrm{cont}} = \mathcal{T}_{A/\mathbb{k}}$ , the usual Lie algebra of derivations.

**Remark 2.2.** There is a canonical topology on  $\mathrm{Hom}_{\mathbb{k}}^{\mathrm{cont}}(M, N)$ , called the *Hom topology*, making it a ST  $A$ -module; see [Yekutieli 1995, Definition 1.1]. However, in this paper we shall not need this topology, and hence we consider  $\mathrm{Hom}_{\mathbb{k}}^{\mathrm{cont}}(M, N)$  as an untopologized object (or as a discrete ST  $\mathbb{k}$ -module).

**Example 2.3.** Let  $\mathbf{t} = (t_1, \dots, t_n)$  be a sequence of variables of length  $n \geq 1$ . In Definition 1.17 we saw how to make the ring of iterated Laurent series  $\mathbb{k}((\mathbf{t})) := \mathbb{k}((t_1, \dots, t_n))$  into a ST  $\mathbb{k}$ -ring. This is a separated ST ring, i.e.,  $\mathbb{k}((\mathbf{t})) = \mathbb{k}((\mathbf{t}))^{\mathrm{sep}}$ . Let  $\mathbb{k}[\mathbf{t}]$  be the polynomial ring, with discrete topology. According to [Yekutieli 1992, Corollary 1.5.19] the ring homomorphism  $\mathbb{k}[\mathbf{t}] \rightarrow \mathbb{k}((\mathbf{t}))$  is *topologically étale relative to  $\mathbb{k}$* . This implies that any  $\mathbb{k}$ -linear differential operator  $\phi$  on  $\mathbb{k}[\mathbf{t}]$  extends uniquely to a continuous  $\mathbb{k}$ -linear differential operator  $\hat{\phi}$  on  $\mathbb{k}((\mathbf{t}))$ . This gives us a ring homomorphism  $\mathcal{D}_{\mathbb{k}[\mathbf{t}]/\mathbb{k}} \rightarrow \mathcal{D}_{\mathbb{k}((\mathbf{t}))/\mathbb{k}}^{\mathrm{cont}}$  that respects the differential filtrations, and such that the induced homomorphism

$$\mathbb{k}((\mathbf{t})) \otimes_{\mathbb{k}[\mathbf{t}]} \mathcal{D}_{\mathbb{k}[\mathbf{t}]/\mathbb{k}} \rightarrow \mathcal{D}_{\mathbb{k}((\mathbf{t}))/\mathbb{k}}^{\mathrm{cont}} \quad (2-4)$$

is bijective.

If  $\mathbb{k}$  has characteristic 0 (i.e.,  $\mathbb{Q} \subset \mathbb{k}$ ), then by (2-4) any  $\hat{\phi} \in F_l(\mathcal{D}_{\mathbb{k}((\mathbf{t}))/\mathbb{k}}^{\text{cont}})$  can be expressed uniquely as a finite sum

$$\hat{\phi} = \sum_{(i_1, \dots, i_n)} a_{(i_1, \dots, i_n)} \cdot \partial_1^{i_1} \cdots \partial_n^{i_n}, \tag{2-5}$$

where  $i_k \in \mathbb{N}$ ,  $\sum_k i_k \leq l$ ,  $a_{(i_1, \dots, i_n)} \in \mathbb{k}((\mathbf{t}))$  and  $\partial_i := \partial/\partial t_i$ .

On the other hand, if  $\mathbb{k}$  has characteristic  $p > 0$  (i.e.,  $\mathbb{F}_p \subset \mathbb{k}$ ), then the structure of  $\mathcal{D}_{\mathbb{k}((\mathbf{t}))/\mathbb{k}}^{\text{cont}}$  is totally different. For every  $m \geq 0$  let  $\mathbb{k}((\mathbf{t}^{p^m})) := \mathbb{k}((t_1^{p^m}, \dots, t_n^{p^m}))$ , which is a subring of  $\mathbb{k}((\mathbf{t}))$ . The ring  $\mathbb{k}((\mathbf{t}))$  is a free module over  $\mathbb{k}((\mathbf{t}^{p^m}))$  of rank  $p^{nm}$ , and the topology on  $\mathbb{k}((\mathbf{t}))$  is the fine  $\mathbb{k}((\mathbf{t}^{p^m}))$ -module topology. According to [Yekutieli 1992, Theorem 1.4.9 and Corollary 2.1.18] we have

$$\mathcal{D}_{\mathbb{k}((\mathbf{t}))/\mathbb{k}}^{\text{cont}} = \mathcal{D}_{\mathbb{k}((\mathbf{t}))/\mathbb{k}} = \bigcup_{m \geq 0} \text{End}_{\mathbb{k}((\mathbf{t}^{p^m}))}(\mathbb{k}((\mathbf{t}))).$$

Let  $B$  be a  $\mathbb{k}$ -ring (not necessarily commutative). For any  $r_1, r_2 \in \mathbb{N}$  let  $\text{Mat}_{r_2 \times r_1}(B)$  be the set of  $r_2 \times r_1$  matrices with entries in  $B$ . The set of matrices  $\text{Mat}_r(B) := \text{Mat}_{r \times r}(B)$  is a  $\mathbb{k}$ -ring with matrix multiplication, and  $\text{Mat}_{r_2 \times r_1}(B)$  is a  $\mathbb{k}$ -central  $\text{Mat}_{r_2}(B)$ - $\text{Mat}_{r_1}(B)$ -bimodule. The group of invertible elements of  $\text{Mat}_r(B)$  is denoted by  $\text{GL}_r(B)$ .

Now consider some  $M \in \text{Mod } \mathbb{k}$ . The  $\mathbb{k}$ -ring  $B := \text{End}_{\mathbb{k}}(M)$  acts on  $M$  from the left. We view  $M^{r_1}$  as a column module, namely we make the identification  $M^{r_1} = \text{Mat}_{r_1 \times 1}(M)$ . Then, for any  $\phi \in \text{Mat}_{r_2 \times r_1}(B)$  and  $m \in M^{r_1}$ , the matrix product  $\phi \cdot m$  is an element of  $M^{r_2}$ . In this way we obtain a canonical isomorphism

$$\text{Hom}_{\mathbb{k}}(M^{r_1}, M^{r_2}) \cong \text{Mat}_{r_2 \times r_1}(\text{End}_{\mathbb{k}}(M)) = \text{Mat}_{r_2 \times r_1}(B) \tag{2-6}$$

of left  $\text{Mat}_{r_2}(B)$ -modules and right  $\text{Mat}_{r_1}(B)$ -modules.

The next lemma shows that this also happens in the topological and differential contexts.

**Lemma 2.7.** *Let  $A \in \text{STRing}_{\mathbb{C}} \mathbb{k}$  and  $M \in \text{STMod } A$ . For any natural numbers  $r_1$  and  $r_2$ , matrix multiplication gives rise to bijections*

$$\text{Mat}_{r_2 \times r_1}(\text{End}_{\mathbb{k}}^{\text{cont}}(M)) \cong \text{Hom}_{\mathbb{k}}^{\text{cont}}(M^{r_1}, M^{r_2})$$

and

$$\text{Mat}_{r_2 \times r_1}(\text{Diff}_{A/\mathbb{k}}^{\text{cont}}(M)) \cong \text{Diff}_{A/\mathbb{k}}^{\text{cont}}(M^{r_1}, M^{r_2}).$$

*In particular, a homomorphism  $\phi : M^r \rightarrow M^r$  in  $\text{STMod } \mathbb{k}$  is an isomorphism if and only if the corresponding matrix belongs to  $\text{GL}_r(\text{End}_{\mathbb{k}}^{\text{cont}}(M))$ .*

*Proof.* This is a straightforward consequence of the definitions. □

Lifting, precise liftings and precise artinian local rings in  $\text{STRing}_{\mathbb{C}} \mathbb{k}$  were introduced in Definitions 1.21 and 1.24. The main result of this section is the next theorem.

**Theorem 2.8.** *Let  $A$  be a precise artinian local ring in  $\text{STRing}_{\mathbb{C}} \mathbb{k}$ , with residue field  $K$ . Give  $K$  the fine  $A$ -module topology. Let  $\sigma_1, \sigma_2 : K \rightarrow A$  be liftings in  $\text{STRing}_{\mathbb{C}} \mathbb{k}$  of the canonical surjection  $A \twoheadrightarrow K$ , and assume that  $\sigma_2$  is a precise lifting.*

*Let  $M_1$  and  $M_2$  be finite  $A$ -modules, and let  $\phi : M_1 \rightarrow M_2$  be an  $A$ -linear homomorphism. For  $l = 1, 2$  choose  $K$ -linear isomorphisms  $\psi_l : K^{r_l} \xrightarrow{\sim} \text{rest}_{\sigma_l}(M_l)$ . Let  $\bar{\phi} \in \text{Mat}_{r_2 \times r_1}(\text{End}_{\mathbb{k}}(K))$  be the matrix such that the diagram*

$$\begin{array}{ccc} M_1 & \xrightarrow{\phi} & M_2 \\ \psi_1 \uparrow & & \uparrow \psi_2 \\ K^{r_1} & \xrightarrow{\bar{\phi}} & K^{r_2} \end{array}$$

in  $\text{Mod } \mathbb{k}$  is commutative. Then the following hold:

- (1) The matrix  $\bar{\phi}$  belongs to  $\text{Mat}_{r_2 \times r_1}(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ .
- (2) Assume that  $M_1 = M_2$  and  $\phi$  is the identity automorphism. Write  $r := r_1$ . Then the matrix  $\bar{\phi}$  belongs to  $\text{GL}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ .

*Proof.* (1) Give  $M_1$  and  $M_2$  the fine  $A$ -module topologies. Let us write  $\bar{M}_l := K^{r_l}$ ; these are ST  $K$ -modules with the fine  $K$ -module topologies. Since  $\sigma_1, \sigma_2 : K \rightarrow A$  are continuous, it follows that both  $\psi_l : \bar{M}_l \rightarrow M_l$  are continuous, namely are homomorphisms in  $\text{STMod } \mathbb{k}$ . Furthermore, because  $\sigma_2$  is a precise lifting, it follows that  $\psi_2 : \bar{M}_2 \rightarrow M_2$  is a homeomorphism, so it is an isomorphism in  $\text{STMod } \mathbb{k}$ . We conclude that  $\bar{\phi} = \psi_2^{-1} \circ \phi \circ \psi_1$  is a homomorphism in  $\text{STMod } \mathbb{k}$ , namely it is continuous.

Next, let us view  $A$  as a  $K$ -ring via  $\sigma_1$ . (There is no topology in this paragraph.) The canonical surjection  $A \twoheadrightarrow K$  makes  $A$  into an augmented  $K$ -ring. Let us view  $\bar{M}_1$  as an  $A$ -module via this augmentation. Now both  $\bar{M}_1$  and  $M_1$  are finite length  $A$ -modules, and  $\psi_1 : \bar{M}_1 \rightarrow M_1$  is  $K$ -linear. According to [Yekutieli 1992, Proposition 1.4.4],  $\psi_1$  is a differential operator over  $A$ . (The order of this operator is bounded by  $r_1 - 1$ .)

Similarly, we can view  $A$  as an augmented  $K$ -ring via  $\sigma_2$ . (There is no topology in this paragraph either.) Now both  $\psi_2 : \bar{M}_2 \rightarrow M_2$  and its inverse  $\psi_2^{-1} : M_2 \rightarrow \bar{M}_2$  are  $K$ -linear, and therefore they are differential operators over  $A$ . We conclude that the composition

$$\bar{\phi} = \psi_2^{-1} \circ \phi \circ \psi_1 : \bar{M}_1 \rightarrow \bar{M}_2$$

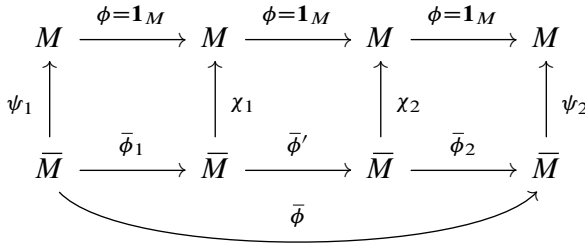
is a differential operator over  $A$ . Here the liftings  $\sigma_1, \sigma_2$  stop playing a role. Now  $A$  acts on  $\bar{M}_1$  and  $\bar{M}_2$  via the canonical surjection  $A \twoheadrightarrow K$ , and this implies that  $\bar{\phi}$  is a differential operator over  $K$ .

Combining the two results above we conclude that  $\bar{\phi} : \bar{M}_1 \rightarrow \bar{M}_2$  is a continuous differential operator over  $K$ . Using Lemma 2.7 we see that the matrix  $\bar{\phi}$  belongs to  $\text{Mat}_{r_2 \times r_1}(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ . This establishes (1).

(2) The proof of this part is very similar to that of [Yekutieli 1995, Lemma 6.6]

Let  $\mathfrak{m}$  be the maximal ideal of  $A$ , and write  $M := M_1$ . Consider the  $\mathfrak{m}$ -adic filtration on  $M$ . The associated graded module  $\text{gr}_{\mathfrak{m}}(M)$  is a  $K$ -module of length  $r$  (regardless of any lifting). By a *filtered  $K$ -basis* of  $M$  we mean a collection  $\{m_i\}_{i=1}^r$  of elements of  $M$  such that the collection of symbols  $\{\bar{m}_i\}_{i=1}^r$  is a  $K$ -basis of  $\text{gr}_{\mathfrak{m}}(M)$  and such that  $\deg(\bar{m}_i) \leq \deg(\bar{m}_{i+1})$ . Such bases exist: simply choose a graded basis of  $\text{gr}_{\mathfrak{m}}(M)$ , suitably ordered, and lift it to  $M$ .

Choose a filtered  $K$ -basis  $\{m_i\}_{i=1}^r$  of  $M$ . For  $l = 1, 2$  let  $\chi_l : \bar{M} \rightarrow \text{rest}_{\sigma_l}(M)$  be the  $K$ -linear isomorphism corresponding to this filtered basis. We get a commutative diagram



in  $\text{Mod } \mathbb{k}$ . By what we already know from (1), the matrices in the bottom row belong to  $\text{Mat}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ ; and they satisfy  $\bar{\phi} = \bar{\phi}_2 \circ \bar{\phi}' \circ \bar{\phi}_1$ . Moreover  $\bar{\phi}_1, \bar{\phi}_2$  are in  $\text{GL}_r(K) \subset \text{GL}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ . Thus it suffices to prove that  $\bar{\phi}' \in \text{GL}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ .

Write  $\bar{\phi}' = [\gamma_{i,j}]$  with  $\gamma_{i,j} \in \mathcal{D}_{K/\mathbb{k}}^{\text{cont}}$ . These operators satisfy

$$\sum_{i=1}^r \sigma_1(a_i) \cdot m_i = \sum_{i,j=1}^r \sigma_2(\gamma_{i,j}(a_i)) \cdot m_j \tag{2-9}$$

for any column  $[a_i] \in K^r$ . Therefore, for any  $i$  and any  $a \in K$ , taking  $a_i := a$  and  $a_j := 0$  for  $j \neq i$ , formula (2-9) gives

$$\sigma_1(a) \cdot m_i = \sum_{j=1}^r \sigma_2(\gamma_{i,j}(a)) \cdot m_j.$$

But the basis  $\{m_i\}_{i=1}^r$  is filtered, and this implies that  $\gamma_{i,j}(a) = 0$  for  $j < i$  and  $\gamma_{i,i}(a) = a$ . As elements of the ring  $\mathcal{D}_{K/\mathbb{k}}^{\text{cont}}$  we get  $\gamma_{i,j} = 0$  for  $j < i$  and  $\gamma_{i,i} = 1$ .

So the matrix  $\bar{\phi}'$  is upper triangular with 1 on the diagonal:

$$\bar{\phi}' = [\gamma_{i,j}] = \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \text{Mat}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}}).$$

The matrix  $\epsilon := 1 - \bar{\phi}' \in \text{Mat}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$  is nilpotent, and hence the matrix

$$\theta := \sum_{i=0}^r \epsilon^i \in \text{Mat}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$$

satisfies  $\theta \circ \bar{\phi}' = \bar{\phi}' \circ \theta = 1$ . We conclude that  $\bar{\phi}' \in \text{GL}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ .  $\square$

**Remark 2.10.** An attempt to deduce assertion (2) of the theorem from assertion (1) by functoriality will not work. This is because (a priori) there is no symmetry between the two liftings  $\sigma_1$  and  $\sigma_2$ : only the lifting  $\sigma_2$  is assumed to be precise.

Eventually we show ([Corollary 2.12](#)) that the lifting  $\sigma_1$  is also precise. But this relies on [Theorem 2.8](#)!

**Corollary 2.11.** *In the situation of part (2) of the theorem, the fine  $(K, \sigma_1)$ -module topology on  $M_1 = M_2$  equals the fine  $(K, \sigma_2)$ -module topology on it.*

*Proof.* For  $l = 1, 2$  let us denote by  $M_l^{\text{st}}$  the  $\mathbb{k}$ -module  $M_l$  endowed with the fine  $(K, \sigma_l)$ -module topology. We want to prove that  $M_1^{\text{st}} = M_2^{\text{st}}$ ; or, equivalently, we want to prove that the identity automorphism  $\phi : M_1 \rightarrow M_2$  in  $\text{Mod } \mathbb{k}$  becomes an isomorphism  $\phi^{\text{st}} : M_1^{\text{st}} \rightarrow M_2^{\text{st}}$  in  $\text{STMod } \mathbb{k}$ .

We have equality  $\phi = \psi_2 \circ \bar{\phi} \circ \psi_1^{-1}$  of isomorphisms  $M_1 \rightarrow M_2$  in  $\text{Mod } \mathbb{k}$ . By definition of the fine topology,  $\psi_l : K^{r_l} \xrightarrow{\sim} M_l^{\text{st}}$  are isomorphisms in  $\text{STMod } \mathbb{k}$ . Therefore it suffices to prove that  $\bar{\phi} : K^{r_1} \rightarrow K^{r_2}$  is an isomorphism in  $\text{STMod } \mathbb{k}$ . This is true by [Lemma 2.7](#) and part (2) of the theorem.  $\square$

The next corollary is a generalization of [[Yekutieli 1992](#), Proposition 2.2.2(a)].

**Corollary 2.12.** *Let  $A$  be a precise artinian local ring in  $\text{STRing}_c \mathbb{k}$ , with residue field  $K$ . Give  $K$  the fine  $A$ -module topology. Then any lifting  $\sigma : K \rightarrow A$  in  $\text{STRing}_c \mathbb{k}$  is a precise lifting.*

*Proof.* Write  $\sigma_1 := \sigma$ . By definition there exists some precise lifting  $\sigma_2 : K \rightarrow A$ ; so the topology on  $A$  equals the fine  $(K, \sigma_2)$ -module topology. Now apply [Corollary 2.11](#) with  $M := A$ .  $\square$

Here is another corollary, pointed out to us by Wolfson:

**Corollary 2.13.** *Let  $A$  be a precise artinian local ring in  $\text{STRing}_c \mathbb{k}$ , with residue field  $K$ , and let  $\sigma : K \rightarrow A$  be a precise lifting. Let  $M$  be a finite  $A$ -module,*

and choose a  $K$ -linear isomorphism  $K^r \xrightarrow{\sim} \text{rest}_\sigma(M)$ . Then  $A$  acts on  $M \cong K^r$  via  $\text{Mat}_r(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ .

*Proof.* In the theorem, take  $M_I := M$ . For  $a \in A$  we get an  $A$ -linear homomorphism  $\phi : M \rightarrow M$ ,  $\phi(m) := a \cdot m$ . □

**Remark 2.14.** If we only wanted to know that  $\bar{\phi} \in \text{Mat}_{r_2 \times r_1}(\text{End}_{\mathbb{k}}^{\text{cont}}(K))$  in [Theorem 2.8\(1\)](#), and that in part (2)  $\bar{\phi} \in \text{GL}_r(\text{End}_{\mathbb{k}}^{\text{cont}}(K))$ , then we did not have to talk about differential operators at all, and the proof could have been included in [Section 1](#). The reason for placing the proof here is twofold. First, it is more economical to prove the full result at once.

The second reason is more delicate. Sometimes in characteristic  $p > 0$ , differential operators are automatically continuous. See [Example 2.3](#). In such cases all liftings  $\sigma : K \rightarrow A$  are continuous. This says that [Theorem 2.8](#) could hold without assuming a priori that the liftings  $\sigma_1, \sigma_2 : K \rightarrow A$  are continuous.

We finish this section with a discussion of differential forms. This will be needed in [Section 5](#). Recall that for  $A \in \text{Ring}_c \mathbb{k}$  we have the de Rham complex, or the DG ring of Kähler differentials,  $\Omega_{A/\mathbb{k}} = \bigoplus_{i \geq 0} \Omega_{A/\mathbb{k}}^i$ , with its differential  $d$ . In degree 0 we have  $\Omega_{A/\mathbb{k}}^0 = A$ , and the  $\mathbb{k}$ -linear derivation  $d : A \rightarrow \Omega_{A/\mathbb{k}}^1$  is universal, in the sense that for any  $\mathbb{k}$ -linear derivation  $\partial : A \rightarrow M$  there is a unique  $A$ -linear homomorphism  $\phi : \Omega_{A/\mathbb{k}}^1 \rightarrow M$  such that  $\partial = \phi \circ d$ . The  $A$ -module  $\Omega_{A/\mathbb{k}}^i$  is the  $i$ -th exterior power of the  $A$ -module  $\Omega_{A/\mathbb{k}}^1$ , and the operator  $d$  on  $\Omega_{A/\mathbb{k}}$  is the unique extension of  $d : \Omega_{A/\mathbb{k}}^0 \rightarrow \Omega_{A/\mathbb{k}}^1$  to an odd derivation.

Now consider  $A \in \text{STRing}_c \mathbb{k}$ . The abstract DG ring  $\Omega_{A/\mathbb{k}}$  is too big (at least in characteristic 0). However the DG ring  $\Omega_{A/\mathbb{k}}$  has a canonical ST structure. For every  $i$  consider the  $(i + 1)$ -st tensor power  $T_{\mathbb{k}}^{i+1}(A) := A \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} A$ , with its tensor product topology ([Definition 1.30](#)). There is a surjection  $T_{\mathbb{k}}^{i+1}(A) \twoheadrightarrow \Omega_{A/\mathbb{k}}^i$ ,

$$a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto a_0 \cdot d(a_1) \cdots d(a_i),$$

and we use it to give  $\Omega_{A/\mathbb{k}}^i$  the quotient topology. Then  $\Omega_{A/\mathbb{k}} = \bigoplus_{i \geq 0} \Omega_{A/\mathbb{k}}^i$  gets the direct sum topology. It turns out that  $\Omega_{A/\mathbb{k}}$  becomes a DG ST ring. In particular the differential  $d$  is continuous. For any  $i$  let  $\Omega_{A/\mathbb{k}}^{i, \text{sep}} := (\Omega_{A/\mathbb{k}}^i)^{\text{sep}}$ , the associated separated ST module. Let  $\Omega_{A/\mathbb{k}}^{\text{sep}} := \bigoplus_{i \geq 0} \Omega_{A/\mathbb{k}}^{i, \text{sep}}$ , with the direct sum topology. Note that  $\Omega_{A/\mathbb{k}}^{0, \text{sep}} = A^{\text{sep}}$ .

**Proposition 2.15** [[Yekutieli 1992](#)]. *Let  $A$  be a commutative ST  $\mathbb{k}$ -ring.*

- (1) *The ST  $\mathbb{k}$ -module  $\Omega_{A/\mathbb{k}}^{\text{sep}}$  has a DG ST  $\mathbb{k}$ -ring structure such the canonical surjection  $\tau_A : \Omega_{A/\mathbb{k}}^{\text{sep}} \twoheadrightarrow \Omega_{A/\mathbb{k}}^{\text{sep}}$  is a homomorphism of DG ST  $\mathbb{k}$ -rings.*
- (2) *Let  $M$  be a separated ST  $A$ -module. The derivation  $d : A \rightarrow \Omega_{A/\mathbb{k}}^{1, \text{sep}}$  induces a bijection*

$$\text{Hom}_A^{\text{cont}}(\Omega_{A/\mathbb{k}}^{1, \text{sep}}, M) \xrightarrow{\sim} \text{Der}_{A/\mathbb{k}}^{\text{cont}}(M).$$

For a proof and full details see [Yekutieli 1992, Section 1.5].

**Example 2.16.** Let  $K := \mathbb{k}((t_1, \dots, t_n))$  be as in Example 2.3, and let  $\mathbb{k}[t] := \mathbb{k}[t_1, \dots, t_n]$ . Since  $K$  is a separated ST  $\mathbb{k}$ -ring, we see that  $\Omega_{K/\mathbb{k}}^{0, \text{sep}} = K$ . Because the homomorphism  $\mathbb{k}[t] \rightarrow K$  is topologically étale in  $\text{STRing}_{\mathbb{k}}$ , it follows that  $\Omega_{K/\mathbb{k}}^{1, \text{sep}}$  is a free  $K$ -module with basis the sequence  $(d(t_1), \dots, d(t_n))$ . For every  $i$  we have

$$\Omega_{K/\mathbb{k}}^{i, \text{sep}} = \bigwedge^i_K \Omega_{K/\mathbb{k}}^{1, \text{sep}},$$

a free  $K$ -module of rank  $\binom{n}{i}$ , with the fine  $K$ -module topology. For proofs see [Yekutieli 1992, Corollaries 1.5.19 and 1.5.13].

Note that if  $\mathbb{k}$  is a field of characteristic 0, then the  $K$ -module  $\Omega_{K/\mathbb{k}}^1$  is a free  $K$ -module of rank equal to  $\text{tr.deg}_{\mathbb{k}}(K)$ , which is uncountably infinite. Thus the kernel of the canonical surjection  $\tau_K : \Omega_{K/\mathbb{k}}^1 \rightarrow \Omega_{K/\mathbb{k}}^{1, \text{sep}}$  is gigantic.

### 3. Topological local fields

In this section we review definitions and results from [Yekutieli 1992, Section 2.1]. We start with a definition due to Parshin [1976; 1983] and Kato [1979]. See also [Fesenko and Kurihara 2000].

**Definition 3.1.** Let  $K$  be a field. An  $n$ -dimensional local field structure on  $K$ , for  $n \geq 1$ , is a sequence  $\mathcal{O}_1(K), \dots, \mathcal{O}_n(K)$  of complete discrete valuation rings, such that:

- $K$  is the fraction field of  $\mathcal{O}_1(K)$ .
- For  $1 \leq i \leq n-1$ , the residue field of  $\mathcal{O}_i(K)$  is the fraction field of  $\mathcal{O}_{i+1}(K)$ .

The data  $(K, \{\mathcal{O}_i(K)\}_{i=1}^n)$  is an  $n$ -dimensional local field. We refer to  $\mathcal{O}_i(K)$  as the  $i$ -th valuation ring of  $K$ . The residue field of  $\mathcal{O}_i(K)$  is denoted by  $\mathbf{k}_i(K)$ , and its maximal ideal is denoted by  $\mathfrak{m}_i(K)$ . We also write  $\mathbf{k}_0(K) := K$ .

Let  $K$  be an  $n$ -dimensional local field. A system of uniformizers in  $K$  (called a regular system of parameters in [Yekutieli 1992]) is a sequence  $(a_1, \dots, a_n)$  of elements of  $\mathcal{O}_1(K)$  such that  $a_1$  generates the maximal ideal  $\mathfrak{m}_1(K)$  of  $\mathcal{O}_1(K)$  and, if  $n \geq 2$ , the sequence  $(\bar{a}_2, \dots, \bar{a}_n)$ , which is the image of  $(a_2, \dots, a_n)$  under the canonical surjection  $\mathcal{O}_1(K) \twoheadrightarrow \mathbf{k}_1(K)$ , is a system of uniformizers in  $\mathbf{k}_1(K)$ . A system of uniformizers  $\mathbf{a} = (a_1, \dots, a_n)$  in  $K$  determines a valuation on  $K$ , with values in the group  $\mathbb{Z}^n$  ordered lexicographically.

It is easy to find a system of uniformizers in an  $n$ -dimensional local field  $K$ . Say  $(\bar{a}_2, \dots, \bar{a}_n)$  is a system of uniformizers in  $\mathbf{k}_1(K)$ . Choose an arbitrary lifting to a sequence  $(a_2, \dots, a_n)$  in  $\mathcal{O}_1(K)$ , and append to it any uniformizer  $a_1$  of  $\mathcal{O}_1(K)$ .

Let  $\mathcal{O}(K)$  be the subring of  $K$  defined by

$$\mathcal{O}(K) := \mathcal{O}_1(K) \times_{\mathbf{k}_1(K)} \mathcal{O}_2(K) \cdots \times_{\mathbf{k}_{n-1}(K)} \mathcal{O}_n(K). \quad (3-2)$$



This is a local ring, whose residue field is  $\mathbf{k}_n(K)$ . We call  $\mathcal{O}(K)$  the ring of integers of  $K$ . The ring  $\mathcal{O}(K)$  is integrally closed in its field of fractions  $K$ ; but unless  $n = 1$  (in which case  $\mathcal{O}(K) = \mathcal{O}_1(K)$ ),  $\mathcal{O}(K)$  is not noetherian.

A 0-dimensional local field is just a field; there are no valuations. Its ring of integers is  $\mathcal{O}(K) := K$ , and  $\mathbf{k}_0(K) := K$  too.

**Definition 3.3.** Let  $\mathbb{k}$  be a nonzero commutative ring. An  $n$ -dimensional local field over  $\mathbb{k}$ , for  $n \in \mathbb{N}$ , is an  $n$ -dimensional local field  $(K, \{\mathcal{O}_i(K)\}_{i=1}^n)$  together with a ring homomorphism  $\mathbb{k} \rightarrow \mathcal{O}(K)$  such that the induced ring homomorphism  $\mathbb{k} \rightarrow \mathbf{k}_n(K)$  is finite.

In other words, the  $n$ -dimensional local field structure of  $K$  lives in the category  $\text{Ring}_{\mathbb{c}} \mathbb{k}$  of commutative  $\mathbb{k}$ -rings. If  $\mathbb{k}$  is a field, then  $\mathbf{k}_n(K)$  is a finite field extension of  $\mathbb{k}$ .

By abuse of notation, we usually call  $K$  an  $n$ -dimensional local field over  $\mathbb{k}$ , and keep the data  $\{\mathcal{O}_i(K)\}_{i=1}^n$  implicit.

**Remark 3.4.** Some authors insist that the base ring be  $\mathbb{k} = \mathbb{Z}$ ; this forces  $\mathbf{k}_n(K)$  to be a finite field. We do not impose such a restriction.

**Definition 3.5.** Let  $K$  and  $L$  be  $n$ -dimensional local fields over  $\mathbb{k}$ , for  $n \geq 0$ . A morphism of  $n$ -dimensional local fields over  $\mathbb{k}$  is a  $\mathbb{k}$ -ring homomorphism  $f : K \rightarrow L$  such that the following conditions hold when  $n \geq 1$ :

- $f(\mathcal{O}_1(K)) \subset \mathcal{O}_1(L)$ .
- The induced  $\mathbb{k}$ -ring homomorphism  $f : \mathcal{O}_1(K) \rightarrow \mathcal{O}_1(L)$  is a local homomorphism.
- The induced  $\mathbb{k}$ -ring homomorphism  $\bar{f} : \mathbf{k}_1(K) \rightarrow \mathbf{k}_1(L)$  is a morphism of  $(n-1)$ -dimensional local fields over  $\mathbb{k}$ .

The category of  $n$ -dimensional local fields over  $\mathbb{k}$  is denoted by  $\text{LF}^n \mathbb{k}$ . Note that any morphism in  $\text{LF}^n \mathbb{k}$  is finite. Cf. [Remark 3.11](#) below regarding more general morphisms between local fields.

**Remark 3.6.** It can be shown that a field  $K$  in  $\text{Ring}_{\mathbb{c}} \mathbb{k}$  admits at most one structure of an  $n$ -dimensional local field (see, e.g., [\[Morrow 2013, Remark 2.3\]](#)). This implies that the forgetful functor  $\text{LF}^n \mathbb{k} \rightarrow \text{Ring}_{\mathbb{c}} \mathbb{k}$  is fully faithful.

From here on we assume that the base ring  $\mathbb{k}$  is a *perfect field*. This implies that all our local fields are of equal characteristic.

**Definition 3.7.** Let  $\mathbb{k}$  be a perfect field. Given a finite field extension  $\mathbb{k}'$  of  $\mathbb{k}$ , the *standard  $n$ -dimensional topological local field* over  $\mathbb{k}$  with last residue field  $\mathbb{k}'$  is the field of iterated Laurent series

$$\mathbb{k}'((t_1, \dots, t_n)) := \mathbb{k}'((t_n)) \cdots ((t_1)).$$

Let us write  $K := \mathbb{k}'((t_1, \dots, t_n))$ . The field  $K$  comes equipped with these two structures:

- (1) A structure of an  $n$ -dimensional local field, in which the valuation rings are

$$\mathcal{O}_i(K) := \mathbb{k}'((t_{i+1}, \dots, t_n))[[t_i]],$$

and the residue fields are

$$\mathbf{k}_i(K) := \mathbb{k}'((t_{i+1}, \dots, t_n)).$$

- (2) A structure of an ST  $\mathbb{k}$ -ring, with the topology from [Definition 1.17](#), starting from the discrete topology on  $\mathbb{k}'$ .

For  $n = 0$  we have  $K = \mathbb{k}'$ , a finite extension of  $\mathbb{k}$  with the discrete topology.

**Definition 3.8** [[Yekutieli 1992](#), Section 2.1]. Let  $\mathbb{k}$  be a perfect field. An  $n$ -dimensional topological local field over  $\mathbb{k}$ , for  $n \geq 0$ , is a field  $K$  together with:

- (a) A structure  $\{\mathcal{O}_i(K)\}_{i=1}^n$  of an  $n$ -dimensional local field on  $K$ .
- (b) A ring homomorphism  $\mathbb{k} \rightarrow \mathcal{O}(K)$  such that  $\mathbb{k} \rightarrow \mathbf{k}_n(K)$  is finite.
- (c) A topology on  $K$ , making it a semi-topological  $\mathbb{k}$ -ring.

The condition is this:

- (P) There is a bijection

$$f : \mathbb{k}'((t_1, \dots, t_n)) \xrightarrow{\sim} K$$

from the standard  $n$ -dimensional topological local field with last residue field  $\mathbb{k}' := \mathbf{k}_n(K)$ . The bijection  $f$  must have these two properties:

- (i)  $f$  is an isomorphism in  $\text{LF}^n / \mathbb{k}$  (i.e., it respects the valuations).
- (ii)  $f$  is an isomorphism in  $\text{STRing}_{\mathbb{c}} \mathbb{k}$  (i.e., it respects the topologies).

Such a bijection  $f$  is called a *parametrization* of  $K$ .

The parametrization  $f$  is not part of the structure of  $K$ ; it is required to exist, but (as we shall see) there are many distinct parametrizations. We use the abbreviation “TLF” for “topological local field”.

**Definition 3.9.** Let  $K$  and  $L$  be  $n$ -dimensional TLFs over  $\mathbb{k}$ . A *morphism of TLFs*  $f : K \rightarrow L$  is a homomorphism of  $\mathbb{k}$ -rings satisfying these two conditions:

- (i)  $f$  is a morphism of  $n$ -dimensional local fields (i.e., it respects the valuations; see [Definition 3.5](#)).
- (ii)  $f$  is a homomorphism of ST  $\mathbb{k}$ -rings (i.e., it is continuous).

The category of  $n$ -dimensional TLFs over  $\mathbb{k}$  is denoted by  $\text{TLF}^n / \mathbb{k}$ .

There are forgetful functors  $\text{TLF}^n \mathbb{k} \rightarrow \text{LF}^n \mathbb{k}$  and  $\text{TLF}^n \mathbb{k} \rightarrow \text{STRing}_{\mathbb{c}} \mathbb{k}$ .

**Remark 3.10.** The conditions of [Definition 3.3](#) and [3.8](#) are more restrictive than those of [[Yekutieli 1992](#), Definition 2.1.10], in this respect: here we require that the last residue field  $\mathbb{k}' := \mathbf{k}_n(K)$  is finite over the base field  $\mathbb{k}$ , whereas in [[loc. cit.](#)] we only required that  $\Omega_{\mathbb{k}'/\mathbb{k}}^1$  should be a finite  $\mathbb{k}'$ -module (which allows  $\mathbb{k}'$  to be a finitely generated extension field of  $\mathbb{k}$  with transcendence degree greater than 0).

If the TLF  $K$  arises as a local factor of a Beilinson completion  $\mathbf{k}(x_0)_\xi$ , as in [Theorem 6.1](#), then the last residue field  $\mathbf{k}_n(K)$  is finite over  $\mathbb{k}$ . So this fits into [Definition 3.8](#).

**Remark 3.11.** In [[Yekutieli 1992](#), Section 2.1] we also allow the much more general possibility of a morphism of TLFs  $f : K \rightarrow L$  where  $\dim(K) < \dim(L)$ . For instance, the inclusions  $\mathbb{k} \rightarrow \mathbb{k}((t_2)) \rightarrow \mathbb{k}((t_1, t_2))$  are morphisms. In this way we get a category  $\text{TLF } \mathbb{k}$ , which contains each  $\text{TLF}^n \mathbb{k}$  as a full subcategory.

**Remark 3.12.** The papers on higher local fields from the Parshin school (prior to 1992) did not have a correct treatment of the topology on higher local fields. Some papers (e.g., [[Parshin 1976](#); [1983](#); [Beilinson 1980](#)]) ignored it. Others — most notably [[Lomadze 1981](#)] — erroneously claimed that the topology of a local field is intrinsic, namely that it is determined by the valuations. This is correct in dimension 1; but it is *false* when  $\text{char}(\mathbb{k}) = 0$  and the dimension is 2 or higher. We gave a counterexample in [[Yekutieli 1992](#), Example 2.1.22] that we reproduce in an expanded form as [Example 3.13](#) below.

It is a deep fact, also proved in [[Yekutieli 1992](#)], that in characteristic  $p > 0$  the topology is determined by the valuation, so that the forgetful functor  $\text{TLF}^n \mathbb{k} \rightarrow \text{LF}^n \mathbb{k}$  is an equivalence. The proof relies on the structure of the ring of differential operators  $\mathcal{D}_{K/\mathbb{k}}$  in characteristic  $p > 0$  (see [[Yekutieli 1992](#), Theorem 2.1.14 and Proposition 2.1.21]).

**Example 3.13.** This is a slightly expanded version of [[Yekutieli 1992](#), Example 2.1.22]. Let  $\mathbb{k}$  be a field of characteristic 0, and let  $K := \mathbb{k}((t_1, t_2))$ , the standard TLF of dimension 2. We choose a collection  $\{b_i\}_{i \in I}$  of elements in  $\mathbf{k}_1(K) = \mathbb{k}((t_2))$  that is a transcendence basis over the subfield  $\mathbb{k}(t_2)$ . For any  $i \in I$  we choose some element  $c_i \in \mathcal{O}_1(K)$ . As explained in the proof of [Theorem 1.1](#), there is a unique lifting

$$\sigma : \mathbb{k}((t_2)) \rightarrow \mathcal{O}_1(K) = \mathbb{k}((t_2))\llbracket t_1 \rrbracket$$

in  $\text{Ring}_c \mathbb{k}$  such that  $\sigma(t_2) = t_2$  and  $\sigma(b_i) = b_i + t_1 c_i$  for all  $i \in I$ . Next we extend  $\sigma$  to a  $\mathbb{k}$ -ring automorphism  $f : \mathcal{O}_1(K) \rightarrow \mathcal{O}_1(K)$  by setting  $f(t_1) := t_1$ . By localization this extends to a  $\mathbb{k}$ -ring automorphism  $f : K \rightarrow K$ .

It is easy to check that  $f$  is an automorphism of  $K$  in the category  $\text{LF}^2 \mathbb{k}$  of local fields. However, since  $f$  is the identity on the subfield  $\mathbb{k}(t_1, t_2) \subset K$ , and this subfield is a dense subset of  $K$ , it follows that  $f$  is continuous if and only if it is

the identity automorphism of  $K$ , which occurs if and only if  $c_i = 0$  for all  $i$ . Thus, if we choose at least one  $c_i \neq 0$ ,  $f$  is not a morphism in  $\mathrm{TLF}^2 \mathbb{k}$ .

Let  $K$  be a TLF of dimension  $n \geq 1$  over  $\mathbb{k}$ . The inclusion  $\mathcal{O}_1(K) \hookrightarrow K$  gives  $\mathcal{O}_1(K)$  an induced structure of ST  $\mathbb{k}$ -ring (it is the subspace topology). Then the surjection  $\mathcal{O}_1(K) \twoheadrightarrow \mathbf{k}_1(K)$  gives  $\mathbf{k}_1(K)$  an induced structure of ST  $\mathbb{k}$ -ring (it is the quotient topology). And so on all the way to  $\mathbf{k}_n(K)$ . In other words, the topologies are such that each  $\mathcal{O}_i(K) \hookrightarrow \mathbf{k}_{i-1}(K)$  is a strict monomorphism in  $\mathrm{STRing}_{\mathbb{c}} \mathbb{k}$ , and each  $\mathcal{O}_i(K) \twoheadrightarrow \mathbf{k}_i(K)$  is a strict epimorphism.

If we choose a parametrization  $K \cong \mathbb{k}'((t_1, \dots, t_n))$ , then the induced ring isomorphisms

$$\mathbb{k}'((t_{i+1}, \dots, t_n))[[t_i]] \cong \mathcal{O}_i(K)$$

and

$$\mathbb{k}'((t_{i+1}, \dots, t_n)) \cong \mathbf{k}_i(K)$$

are also isomorphisms of ST  $\mathbb{k}$ -rings. This follows from [Yekutieli 1992, Proposition 1.3.5]. In particular, each  $\mathbf{k}_i(K)$  is an  $(n-i)$ -dimensional TLF over  $\mathbb{k}$ .

Recall the notions of precise lifting and precise artinian local ring from Definition 1.24.

**Lemma 3.14.** *Let  $K$  be a TLF of dimension  $n \geq 1$  over  $\mathbb{k}$ ; let  $l \geq 0$ . Then the ST ring  $A_l := \mathcal{O}_1(K)/\mathfrak{m}_1(K)^{l+1}$ , with the quotient topology from  $\mathcal{O}_1(K)$ , is a precise artinian local ring in  $\mathrm{STRing}_{\mathbb{c}} \mathbb{k}$ .*

*Proof.* Choose a parametrization  $K \cong \mathbb{k}'((t_1, \dots, t_n))$ , and let  $\bar{K} := \mathbb{k}'((t_2, \dots, t_n))$ . Then  $\bar{K} \cong \mathbf{k}_1(K)$  and  $\bar{K}[[t_1]] \cong \mathcal{O}_1(K)$  as ST  $\mathbb{k}$ -rings; and the inclusion  $\bar{K} \rightarrow \bar{K}[[t_1]]$  represents a lifting  $\sigma_1 : \mathbf{k}_1(K) \rightarrow \mathcal{O}_1(K)$ . As ST  $\bar{K}$ -modules,  $\mathcal{O}_1(K) \cong \prod_{i=0}^{\infty} \bar{K}$  and  $A_l \cong \prod_{i=0}^l \bar{K}$ . This shows that the quotient topology on  $A_l$  coincides with the fine  $\bar{K}$ -module topology on it. So  $\sigma_1$  is a precise lifting.  $\square$

**Lemma 3.15.** *Let  $K \in \mathrm{TLF}^n \mathbb{k}$ , with last residue field  $\mathbb{k}' := \mathbf{k}_n(K)$ . There is a unique lifting  $\sigma : \mathbb{k}' \rightarrow \mathcal{O}(K)$  in  $\mathrm{STRing}_{\mathbb{c}} \mathbb{k}$  of the canonical surjection  $\mathcal{O}(K) \twoheadrightarrow \mathbb{k}'$ .*

*Proof.* Since  $\mathbb{k}'$  is discrete, we do not have to worry about continuity. We use induction on  $n$ . Let  $\bar{\sigma} : \mathbb{k}' \rightarrow \mathcal{O}(\mathbf{k}_1(K)) \subset \mathbf{k}_1(K)$  be the unique lifting for this  $(n-1)$ -dimensional TLF. Consider the canonical surjection  $\pi : \mathcal{O}_1(K) \twoheadrightarrow \mathbf{k}_1(K)$ . By Theorem 1.1 there is a unique  $\mathbb{k}$ -ring homomorphism  $\sigma : \mathbb{k}' \rightarrow \mathcal{O}_1(K)$  such that  $\pi \circ \sigma = \bar{\sigma}$ . It is trivial to see that  $\sigma(\mathbb{k}')$  is inside  $\mathcal{O}(K)$ .  $\square$

The construction and classification of parametrizations of a TLF (condition (P) in Definition 3.8) is made clear by the next theorem (which is a special case of [Yekutieli 1992, Corollary 2.1.19]).

**Theorem 3.16** [Yekutieli 1992]. *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , let  $(a_1, \dots, a_n)$  be a system of uniformizers in  $K$ , let  $\mathbb{k}' := \mathbf{k}_n(K)$ , and let  $\sigma : \mathbb{k}' \rightarrow$*

$\mathcal{O}(K)$  be the unique lifting over  $\mathbb{k}$ . Then  $\sigma$  extends uniquely to an isomorphism of TLFs

$$f : \mathbb{k}'((t_1, \dots, t_n)) \rightarrow K$$

such that  $f(t_i) = a_i$ .

**Definition 3.17.** Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ . By a *system of liftings* for  $K$  we mean a sequence  $\sigma = (\sigma_1, \dots, \sigma_n)$ , where for each  $i$

$$\sigma_i : \mathbf{k}_i(K) \rightarrow \mathcal{O}_i(K)$$

is a homomorphism of ST  $\mathbb{k}$ -rings that lifts the canonical surjection  $\mathcal{O}_i(K) \twoheadrightarrow \mathbf{k}_i(K)$ .

The important thing to remember is that each lifting  $\sigma_i : \mathbf{k}_i(K) \rightarrow \mathcal{O}_i(K)$  is continuous. When  $n = 0$  the only system of liftings is the empty system  $\sigma = ()$ .

**Example 3.18.** Take a standard TLF  $K := \mathbb{k}'((t_1, \dots, t_n))$ . It comes equipped with a standard system of liftings

$$\sigma_i : \mathbf{k}_i(K) \rightarrow \mathcal{O}_i(K),$$

namely the inclusions

$$\sigma_i : \mathbb{k}'((t_{i+1}, \dots, t_n)) \rightarrow \mathbb{k}'((t_{i+1}, \dots, t_n))[[t_i]].$$

**Proposition 3.19.** Any  $n$ -dimensional TLF  $K$  over  $\mathbb{k}$  admits a system of liftings.

*Proof.* Take a parametrization  $f : \mathbb{k}'((t_1, \dots, t_n)) \rightarrow K$ . The standard system of liftings of  $\mathbb{k}'((t_1, \dots, t_n))$  induces a system of liftings on  $K$ .  $\square$

#### 4. Lattices and BT operators

As before,  $\mathbb{k}$  is a perfect base field.

**Definition 4.1.** Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , and let  $M$  be a finite  $K$ -module. An  $\mathcal{O}_1(K)$ -lattice in  $M$  is a finite  $\mathcal{O}_1(K)$ -submodule  $L$  of  $M$  such that  $M = K \cdot L$ . We denote by  $\text{Lat}(M)$  the set of  $\mathcal{O}_1(K)$ -lattices in  $M$ .

Let  $L$  be an  $\mathcal{O}_1(K)$ -lattice in  $M$ . Recall that  $\mathcal{O}_1(K)$  is a DVR. This implies that  $L$  is a free  $\mathcal{O}_1(K)$ -module, of rank equal to that of  $M$ .

**Example 4.2.** Consider a TLF  $K$ , and take  $M := K^r$ . Choose a uniformizer  $a \in \mathcal{O}_1(K)$ . For any  $i \in \mathbb{Z}$  there is a lattice  $L_i := a^i \cdot \mathcal{O}_1(K)^r \subset K^r$ . Let us call these standard lattices. They do not depend on the choice of uniformizer.

When  $r = 1$ , all the  $\mathcal{O}_1(K)$ -lattices in  $M$  are standard. When  $r > 1$ ,  $M$  has many more lattices. However any  $\mathcal{O}_1(K)$ -lattice  $L$  in  $M$  can be sandwiched between two standard lattices:  $L_i \subset L \subset L_{-j}$  for  $i, j \gg 0$ .

Suppose  $M$  is a finite  $K$ -module, and  $L, L' \in \text{Lat}(M)$  with  $L \subset L'$ . Then the quotient  $L'/L$  is a finite length  $\mathcal{O}_1(K)$ -module. If we are given a lifting  $\sigma_1 : \mathbf{k}_1(K) \rightarrow \mathcal{O}_1(K)$ , then  $L'/L$  becomes a finite module over the TLF  $\mathbf{k}_1(K)$ , which we denote by  $\text{rest}_{\sigma_1}(L'/L)$ ; see [Definition 1.20](#).

**Lemma 4.3.** *Let  $M$  be a finite  $K$ -module, let  $L$  be an  $\mathcal{O}_1(K)$ -lattice in  $M$ , and let  $a \in \mathcal{O}_1(K)$  be a uniformizer. Give  $M$  the fine  $K$ -module topology. For every  $i \in \mathbb{Z}$  give the lattice  $L_i := a^i \cdot L$  the fine  $\mathcal{O}_1(K)$ -module topology. For every  $i \in \mathbb{N}$  give the quotient  $L/L_i$  the fine  $\mathcal{O}_1(K)$ -module topology.*

- (1) *The topology on  $M$  equals the fine  $\mathcal{O}_1(K)$ -module topology on it.*
- (2) *The inclusions  $L_i \rightarrow M$ , for  $i \in \mathbb{Z}$ , are strict monomorphisms in  $\text{STMod } \mathbb{k}$ .*
- (3) *Consider the direct system  $\{L_{-j}\}_{j \in \mathbb{N}}$  in  $\text{STMod } \mathbb{k}$ . Give  $\lim_{j \rightarrow} L_{-j}$  the direct limit topology. Then the canonical bijection  $\lim_{j \rightarrow} L_{-j} \rightarrow M$  is an isomorphism in  $\text{STMod } \mathbb{k}$ .*
- (4) *The canonical surjections  $L \rightarrow L/L_i$ , for  $i \in \mathbb{N}$ , are strict epimorphisms in  $\text{STMod } \mathbb{k}$ .*
- (5) *Let  $\sigma_1 : \mathbf{k}_1(K) \rightarrow \mathcal{O}_1(K)$  be a lifting in  $\text{STRing}_c \mathbb{k}$  of the canonical surjection. Then for every  $i \in \mathbb{N}$  the topology on  $L/L_i$  equals the fine  $(\mathbf{k}_1(K), \sigma_1)$ -module topology on it.*
- (6) *Consider the inverse system  $\{L/L_i\}_{i \in \mathbb{N}}$  in  $\text{STMod } \mathbb{k}$ . Give  $\lim_{\leftarrow i} (L/L_i)$  the inverse limit topology. Then the canonical bijection  $L \rightarrow \lim_{\leftarrow i} (L/L_i)$  is an isomorphism in  $\text{STMod } \mathbb{k}$ .*

*Proof.* All these assertions become clear after we choose an  $\mathcal{O}_1(K)$ -linear isomorphism  $L \cong \mathcal{O}_1(K)^r$  and a ST  $\mathbb{k}$ -ring isomorphism  $\mathcal{O}_1(K) \cong \mathbf{k}_1(K)[[t]]$ . See [\[Yekutieli 1992, Proposition 1.3.5\]](#).  $\square$

Let  $K$  be a TLF of dimension  $n \geq 1$  over  $\mathbb{k}$ . If  $\sigma = (\sigma_1, \dots, \sigma_n)$  is a system of liftings for  $K$ , then we write  $d_1(\sigma) := (\sigma_2, \dots, \sigma_n)$ . This is a system of liftings for the TLF  $\mathbf{k}_1(K)$ .

**Definition 4.4.** Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , and let  $(M_1, M_2)$  be a pair of finite  $K$ -modules.

- (1) By a pair of  $\mathcal{O}_1(K)$ -lattices in  $(M_1, M_2)$  we mean a pair  $(L_1, L_2)$ , where  $L_i \in \text{Lat}(M_i)$ . The set of such pairs is denoted by  $\text{Lat}(M_1, M_2)$ .
- (2) Let  $\phi : M_1 \rightarrow M_2$  be a  $\mathbb{k}$ -linear homomorphism, and let  $(L_1, L_2), (L'_1, L'_2)$  be in  $\text{Lat}(M_1, M_2)$ . We say that  $(L'_1, L'_2)$  is a  $\phi$ -refinement of  $(L_1, L_2)$  if  $L'_1 \subset L_1, L_2 \subset L'_2, \phi(L'_1) \subset L_2$  and  $\phi(L_1) \subset L'_2$ . In this case we write

$$(L'_1, L'_2) \prec_{\phi} (L_1, L_2),$$

and refer to it as a  $\phi$ -refinement in  $\text{Lat}(M_1, M_2)$ .

The relation  $\prec_\phi$  is a partial ordering on  $\text{Lat}(M_1, M_2)$ . If  $(L'_1, L'_2) \prec_\phi (L_1, L_2)$ , then there is an induced  $\mathbb{k}$ -linear homomorphism  $\bar{\phi} : L_1/L'_1 \rightarrow L'_2/L_2$ .

The next two definitions are variations of the original definitions in [Beilinson 1980], which are themselves generalizations to  $n \geq 2$  of the definitions in [Tate 1968]. We saw similar definitions in the more recent papers [Osipov 2005; 2007; Braunling 2014a; 2014b]. The notation we use is close to that of Tate.

**Definition 4.5.** Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , let  $\sigma = (\sigma_1, \dots, \sigma_n)$  be a system of liftings for  $K$ , and let  $(M_1, M_2)$  be a pair of finite  $K$ -modules. We define the subset

$$E_\sigma^K(M_1, M_2) \subset \text{Hom}_{\mathbb{k}}(M_1, M_2)$$

as follows:

- (1) If  $n = 0$ , any  $\mathbb{k}$ -linear homomorphism  $\phi : M_1 \rightarrow M_2$  belongs to  $E_\sigma^K(M_1, M_2)$ .
- (2) If  $n \geq 1$ , then a  $\mathbb{k}$ -linear homomorphism  $\phi : M_1 \rightarrow M_2$  belongs to  $E_\sigma^K(M_1, M_2)$  if it satisfies these two conditions:
  - (i) Every  $(L_1, L_2) \in \text{Lat}(M_1, M_2)$  has some  $\phi$ -refinement  $(L'_1, L'_2)$ .
  - (ii) For every  $\phi$ -refinement  $(L'_1, L'_2) \prec_\phi (L_1, L_2)$  in  $\text{Lat}(M_1, M_2)$  the induced homomorphism

$$\bar{\phi} : L_1/L'_1 \rightarrow L'_2/L_2$$

belongs to

$$E_{d_1(\sigma)}^{k_1(K)}(\text{rest}_{\sigma_1}(L_1/L'_1), \text{rest}_{\sigma_1}(L'_2/L_2)).$$

A homomorphism  $\phi : M_1 \rightarrow M_2$  that belongs to  $E_\sigma^K(M_1, M_2)$  is called a *local Beilinson–Tate operator* relative to  $\sigma$ , or a BT operator for short.

Let  $K$  be a TLF over  $\mathbb{k}$  of dimension at least 1. We denote by  $\mathcal{O}_1(K) \parallel_{\mathfrak{m}_1(K)}$  the ST  $\mathbb{k}$ -ring which is the ring  $\mathcal{O}_1(K)$  with its  $\mathfrak{m}_1(K)$ -adic topology. Given an  $\mathcal{O}_1(K)$ -module  $M$ , the fine  $\mathcal{O}_1(K) \parallel_{\mathfrak{m}_1(K)}$ -module topology on  $M$  is called the fine  $\mathfrak{m}_1(K)$ -adic topology. Now suppose  $\phi : M_1 \rightarrow M_2$  is a  $\mathbb{k}$ -linear homomorphism. We say that  $\phi$  is  *$\mathfrak{m}_1(K)$ -adically continuous* if it is continuous for the fine  $\mathfrak{m}_1(K)$ -adic topologies on  $M_1$  and  $M_2$ .

**Example 4.6.** If  $n = 1$  then the usual topology on  $\mathcal{O}_1(K)$  equals the  $\mathfrak{m}_1(K)$ -adic topology. Thus  $K$  has the fine  $\mathfrak{m}_1(K)$ -adic topology. If  $n > 1$  then the fine  $\mathfrak{m}_1(K)$ -adic topology is finer than, and not equal to, the usual topology on  $\mathcal{O}_1(K)$  and  $K$ .

**Lemma 4.7.** *In the situation of Definition 4.5, the homomorphism  $\phi : M_1 \rightarrow M_2$  satisfies condition (2.i) if and only if it is  $\mathfrak{m}_1(K)$ -adically continuous.*

*Proof.* Let  $\bar{K}$  be the field  $\mathbf{k}_1(K)$ , but with the discrete topology. The lifting  $\sigma_1$  induces an isomorphism of ST rings  $\bar{K}[[t]] \xrightarrow{\sim} \mathcal{O}_1(K) \parallel_{\mathfrak{m}_1(K)}$ . Thus the field  $K$ , with the fine  $\mathfrak{m}_1(K)$ -adic topology, is isomorphic to  $\bar{K}((t))$  as ST  $\mathbb{k}$ -rings. But we know that

$$\bar{K}((t)) \cong \left( \prod_{i \geq 0} \bar{K} \cdot t^i \right) \oplus \left( \bigoplus_{i < 0} \bar{K} \cdot t^i \right)$$

as ST  $\mathbb{k}$ -modules, where  $\bar{K} \cdot t^i \cong \bar{K}$  is discrete; cf. the proof of [Yekutieli 1992, Proposition 1.3.5]. It is now an exercise in quantifiers to compare  $t$ -adic continuity to condition (2.i). Cf. [Braunling 2014b, Remark 1 in Section 1.1], where this is also mentioned.  $\square$

**Lemma 4.8.** *In the situation of Definition 4.5, give  $M_1$  and  $M_2$  the fine  $K$ -module topologies. Then every  $\phi \in E_{\sigma}^K(M_1, M_2)$  is continuous.*

*Proof.* The proof is by induction on  $n$ . For  $n = 0$  there is nothing to prove, since these are discrete modules. So assume  $n \geq 1$ . (Actually for  $n = 1$  this was proved in Lemma 4.7.) In view of Lemma 4.3, it suffices to prove that, for every  $(L'_1, L'_2) \prec_{\phi} (L_1, L_2)$  in  $\text{Lat}(M_1, M_2)$ , the induced homomorphism  $\bar{\phi} : L_1/L'_1 \rightarrow L_2/L'_2$  is continuous. But  $\bar{\phi}$  is a BT operator in dimension  $n - 1$ , so by induction it is continuous.  $\square$

**Lemma 4.9.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , and let  $\sigma$  be a system of liftings for  $K$ . For  $l = 1, 2, 3, 4$  let  $M_l$  be a finite  $K$ -module, and for  $l = 1, 2, 3$  let  $\phi_l : M_l \rightarrow M_{l+1}$  be a  $\mathbb{k}$ -linear homomorphism.*

- (1) *If  $\phi_1$  is  $K$ -linear then it is a BT operator.*
- (2) *If  $\phi_1$  and  $\phi_2$  are BT operators, then  $\phi_2 \circ \phi_1$  is a BT operator.*
- (3) *Assume that  $\phi_1$  is surjective and  $K$ -linear,  $\phi_3$  is injective and  $K$ -linear, and  $\phi_3 \circ \phi_2 \circ \phi_1$  is a BT operator. Then  $\phi_2$  is a BT operator.*

Here is a diagram depicting the situation:

$$M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} M_3 \xrightarrow{\phi_3} M_4.$$

*Proof.* We prove all three assertions by induction on  $n$  and on their sequential order. For  $n = 0$  all assertions are trivial, so let us assume that  $n \geq 1$ . The conditions mentioned below are those in Definition 4.5.

(1) For this we assume that assertion (1) is true in dimension  $n - 1$ . Condition (2.i), namely the  $\mathfrak{m}_1(K)$ -adic continuity of  $\phi_1$ , is clear. Consider any  $\phi$ -refinement  $(L'_1, L'_2) \prec_{\phi_1} (L_1, L_2)$  in  $\text{Lat}(M_1, M_2)$ . Since the induced homomorphism  $\bar{\phi} : L_1/L'_1 \rightarrow L_2/L'_2$  is  $\mathcal{O}_1(K)$ -linear, it is also  $(\mathbf{k}_1(K), \sigma_1)$ -linear. By induction on  $n$ ,  $\bar{\phi}$  is a BT operator. So condition (2.ii) holds.



(2) Here we assume that assertions (2) and (3) are true in dimension  $n - 1$ . Write  $\psi := \phi_2 \circ \phi_1$ . The  $m_1(K)$ -adic continuity of  $\psi$ , that is, condition (2.i), is clear. Consider any  $\psi$ -refinement  $(L'_1, L'_3) \prec_{\psi} (L_1, L_3)$  in  $\text{Lat}(M_1, M_3)$ . To satisfy condition (2.ii) we have to prove that  $\bar{\psi} : L_1/L'_1 \rightarrow L'_3/L_3$  is a BT operator in dimension  $n - 1$ . Let  $L_2^{\diamond} \in \text{Lat}(M_2)$  be a lattice that contains  $\phi_1(L_1)$ , and let  $L_3^{\diamond} \in \text{Lat}(M_3)$  be a lattice that contains both  $L'_3$  and  $\phi_2(L_2^{\diamond})$ . Let  $L_2^{\heartsuit} \in \text{Lat}(M_2)$  be a lattice that is contained in  $L_2^{\diamond}$ . Let  $L_1^{\heartsuit} \in \text{Lat}(M_1)$  be such that  $L_1^{\heartsuit} \subset L'_1$  and  $\phi_1(L_1^{\heartsuit}) \subset L_2^{\heartsuit}$ . All these choices are possible because condition (2.i) is satisfied by  $\phi_1$  and  $\phi_2$ . Consider the commutative diagram

$$\begin{array}{ccccccc} \frac{L_1}{L_1^{\heartsuit}} & \xrightarrow{\alpha} & \frac{L_1}{L'_1} & \xrightarrow{\bar{\psi}} & \frac{L'_3}{L_3} & \xrightarrow{\beta} & \frac{L_3^{\diamond}}{L_3} \\ & \searrow \bar{\phi}_1 & & & & \nearrow \bar{\phi}_2 & \\ & & & & \frac{L_2^{\diamond}}{L_2^{\heartsuit}} & & \end{array}$$

in  $\text{Mod } \mathbb{k}$ . Since  $\phi_1$  and  $\phi_2$  are BT operators, condition (2.ii) says that  $\bar{\phi}_1$  and  $\bar{\phi}_2$  are BT operators (in dimension  $n - 1$ ). By part (2) the composition  $\bar{\phi}_2 \circ \bar{\phi}_1$  is a BT operator. The homomorphisms  $\alpha$  and  $\beta$  are  $k_1(K)$ -linear. Therefore by part (3) the homomorphism  $\bar{\psi}$  is a BT operator.

(3) For this we assume that assertions (1) and (2) are true in dimension  $n$ . Let  $\psi := \phi_3 \circ \phi_2 \circ \phi_1$ . Choose  $K$ -linear homomorphisms  $\psi_1 : M_2 \rightarrow M_1$  and  $\psi_3 : M_4 \rightarrow M_3$  that split  $\phi_1$  and  $\phi_3$  respectively. Then  $\phi_2 = \psi_3 \circ \psi \circ \psi_1$ . By assertions (1) and (2), we see that  $\phi_2$  is a BT operator.  $\square$

**Lemma 4.10.** *In the situation of Definition 4.5, the set  $E_{\sigma}^K(M_1, M_2)$  is a  $\mathbb{k}$ -submodule of  $\text{Hom}_{\mathbb{k}}(M_1, M_2)$ .*

*Proof.* The proof is by induction on  $n$ , and we can assume that  $n \geq 1$ . Take any  $\phi_1, \phi_2 \in E_{\sigma}^K(M_1, M_2)$  and any  $a \in \mathbb{k}$ . Let  $\psi := a \cdot \phi_1 + \phi_2$ ; we have to show that  $\psi \in E_{\sigma}^K(M_1, M_2)$ . Since condition (2.i) of Definition 4.5 is about  $m_1(K)$ -adic continuity (by Lemma 4.7), we see that  $\psi$  satisfies it.

We need to check condition (2.ii) of that definition. So let  $(L'_1, L'_2)$  be a  $\psi$ -refinement of  $(L_1, L_2)$ . By  $m_1(K)$ -adic continuity there are lattices  $L_1^{\diamond} \subset L'_1$  and  $L_2^{\diamond} \subset L'_2$  such that  $\phi_i(L_1^{\diamond}) \subset L_2$  and  $\phi_i(L_1) \subset L_2^{\diamond}$ . Consider the commutative diagram

$$\begin{array}{ccccccc} \frac{L_1}{L_1^{\diamond}} & \xrightarrow{\alpha} & \frac{L_1}{L'_1} & \xrightarrow{\bar{\psi}} & \frac{L'_2}{L_2} & \xrightarrow{\beta} & \frac{L_2^{\diamond}}{L_2} \\ & \searrow & & & & \nearrow & \\ & & & & & & \\ & & & & \xrightarrow{a \cdot \bar{\phi}_1 + \bar{\phi}_2} & & \end{array}$$

in  $\text{Mod } \mathbb{k}$ . The induction hypothesis tells us that  $a \cdot \bar{\phi}_1 + \bar{\phi}_2$  is a BT operator. The homomorphisms  $\alpha$  and  $\beta$  are  $\mathbf{k}_1(K)$ -linear. Therefore, according to [Lemma 4.9\(3\)](#), the homomorphism  $\bar{\psi}$  is a BT operator.  $\square$

**Lemma 4.11.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , let  $\sigma$  be a system of liftings for  $K$ , and let  $(M_1, M_2)$  be a pair of finite  $K$ -modules. Then*

$$\text{Diff}_{K/\mathbb{k}}^{\text{cont}}(M_1, M_2) \subset E_{\sigma}^K(M_1, M_2).$$

*Proof.* We use induction on  $n$ . For  $n = 0$  there is nothing to prove, so let's assume that  $n \geq 1$ . (Actually, for  $n = 1$  there is nothing to prove either; see [Example 4.12](#).)

Let  $\phi : M_1 \rightarrow M_2$  be a continuous differential operator. Choose  $K$ -linear isomorphisms  $M_l \cong K^{r_l}$  for  $l = 1, 2$ ; so we may view  $\phi$  as a matrix  $[\phi_{i,j}]$  in  $\text{Mat}_{r_2 \times r_1}(\mathcal{D}_{K/\mathbb{k}}^{\text{cont}})$ . According to (1) and (2) of [Lemma 4.9](#), it suffices to prove that each  $\phi_{i,j}$  is a BT operator. Therefore we can assume that  $M_1 = M_2 = K$  and  $\phi \in \mathcal{D}_{K/\mathbb{k}}^{\text{cont}}$ .

Choose a uniformizer  $a \in \mathcal{O}_1(K)$ . If  $\text{char}(\mathbb{k}) = 0$  then by formula (2-5) there is an integer  $d$ , depending on the coefficients of the operator  $\phi$  in that expansion, such that  $\phi(a^i \cdot \mathcal{O}_1(K)) \subset a^{i-d} \cdot \mathcal{O}_1(K)$  for all  $i$ . Hence  $\phi$  is  $\mathfrak{m}_1(K)$ -adically continuous.

If  $\text{char}(\mathbb{k}) = p > 0$ , then by [[Yekutieli 1992](#), Theorem 1.4.9] the operator  $\phi$  is linear over the subfield  $K' := \mathbb{k} \cdot K^{p^d} \subset K$  for a sufficiently large natural number  $d$ . The field  $K'$  is also an  $n$ -dimensional TLF,  $K' \rightarrow K$  is a morphism of TLFs, and  $\mathcal{O}_1(K') \rightarrow \mathcal{O}_1(K)$  is a finite homomorphism. So the  $\mathfrak{m}_1(K)$ -adic topology on  $\mathcal{O}_1(K)$  coincides with its  $\mathfrak{m}_1(K')$ -adic topology. Since  $\phi$  is  $\mathcal{O}_1(K')$ -linear, it follows that  $\phi$  is  $\mathfrak{m}_1(K')$ -adically continuous. Using [Lemma 4.7](#), we see that in both cases ( $\text{char}(\mathbb{k}) = 0$  and  $\text{char}(\mathbb{k}) > 0$ ) condition (2.i) of [Definition 4.5](#) holds.

Now take a  $\phi$ -refinement  $(L'_1, L'_2) \prec_{\phi} (L_1, L_2)$  in  $\text{Lat}(M_1, M_2)$ . Write  $\bar{M}_1 := \text{rest}_{\sigma_1}(L_1/L'_1)$  and  $\bar{M}_2 := \text{rest}_{\sigma_1}(L'_2/L_2)$ . We must prove that  $\bar{\phi} : \bar{M}_1 \rightarrow \bar{M}_2$  is a BT operator between these  $\mathbf{k}_1(K)$ -modules. We know that  $\bar{\phi}$  is a differential operator over  $\mathcal{O}_1(K)$ , and therefore it is also a differential operator over  $\mathbf{k}_1(K)$ . Choose some  $\mathbf{k}_1(K)$ -linear isomorphisms  $\psi_l : \mathbf{k}_1(K)^{r_l} \xrightarrow{\sim} \bar{M}_l$ . Then

$$\psi := \psi_2^{-1} \circ \bar{\phi} \circ \psi_1 : \mathbf{k}_1(K)^{r_1} \rightarrow \mathbf{k}_1(K)^{r_2}$$

is a differential operator over  $\mathbf{k}_1(K)$ . By the induction hypothesis,  $\psi$  is a BT operator. Finally, by (1) and (2) of [Lemma 4.9](#), the homomorphism  $\bar{\phi} = \psi_2 \circ \psi \circ \psi_1^{-1}$  is a BT operator.  $\square$

**Example 4.12.** If  $n = 0$ , then by definition

$$E_{\sigma}^K(M_1, M_2) = \text{Hom}_{\mathbb{k}}(M_1, M_2).$$

This is a finite  $\mathbb{k}$ -module.

If  $n = 1$  then condition (2.ii) of [Definition 4.5](#) is trivially satisfied. [Lemma 4.7](#) and [Example 4.12](#) show that

$$E_{\sigma}^K(M_1, M_2) = \text{Hom}_{\mathbb{k}}^{\text{cont}}(M_1, M_2),$$

the module of continuous  $\mathbb{k}$ -linear homomorphisms. This was already noticed in [[Osipov 2005; 2007; Braunling 2014b](#), Section 1.1].

The equalities above indicate that the choice of  $\sigma$  is irrelevant. However in dimensions 0 and 1 there is only one lifting, so in fact there is no news here. Later, in [Theorem 4.20](#), we will prove that in any dimension the system of liftings  $\sigma$  is not relevant.

**Example 4.13.** For  $n \geq 2$  the inclusion

$$E_{\sigma}^K(M_1, M_2) \subset \text{Hom}_{\mathbb{k}}^{\text{cont}}(M_1, M_2)$$

is usually proper (i.e., it is not an equality). Here is a calculation demonstrating this: Let  $K := \mathbb{k}((t_1, t_2))$ , the standard TLF with its standard system of liftings  $\sigma$ . Take  $M_1 = M_2 := K$ . Any  $a \in K$  is a series  $a = \sum_{i \in \mathbb{Z}} a_i(t_2) \cdot t_1^i$ , where  $a_i(t_2) \in \mathbb{k}((t_2))$  and  $a_i(t_2) = 0$  for  $i \ll 0$ . We let  $\psi \in \text{End}_{\mathbb{k}}(K)$  be

$$\psi \left( \sum_{i \in \mathbb{Z}} a_i(t_2) \cdot t_1^i \right) := a_0(t_1).$$

To see that this is continuous we use the continuous decomposition

$$K = \mathbb{k}((t_2))((t_1)) \cong \mathbb{k}((t_2))[[t_1]] \oplus \left( \bigoplus_{i < 0} \mathbb{k}((t_2)) \cdot t_1^i \right).$$

This gives a continuous function  $\psi_1 : K \rightarrow \mathbb{k}((t_2))$ , sending  $\sum_{i \in \mathbb{Z}} a_i(t_2) \cdot t_1^i$  to  $a_0(t_2)$ . Next there is an isomorphism  $\psi_2 : \mathbb{k}((t_2)) \rightarrow \mathbb{k}((t_1))$ ,  $a_0(t_2) \mapsto a_0(t_1)$ . Finally the inclusion  $\psi_3 : \mathbb{k}((t_1)) \rightarrow \mathbb{k}((t_2))((t_1))$  is continuous. The function  $\psi$  is  $\psi = \psi_3 \circ \psi_2 \circ \psi_1$ , so it is continuous.

Take the standard lattices  $L_i = t_1^i \cdot \mathbb{k}((t_2))[[t_1]]$  in  $K$ . For every  $j$  the element  $a_j := t_2^j$  belongs to  $L_0$ , yet the element  $\psi(a_j) = t_1^j$  does not belong to  $L_{j+1}$ . Thus  $\psi(L_0)$  is not contained in any lattice, and requirement (2.i) of [Definition 4.5](#) is violated, so  $\psi$  does not belong to  $E_{\sigma}^K(K, K)$ .

Recall that for an  $n$ -dimensional TLF  $K$ , with  $n \geq 2$ , and a system of liftings  $\sigma = (\sigma_1, \dots, \sigma_n)$ , the truncation  $d_1(\sigma) = (\sigma_2, \dots, \sigma_n)$  is a system of liftings for the first residue field  $\mathbf{k}_1(K)$ .

**Definition 4.14.** Let  $K$  be a TLF over  $\mathbb{k}$  of dimension  $n \geq 1$ , let  $\sigma = (\sigma_1, \dots, \sigma_n)$  be a system of liftings for  $K$ , and let  $(M_1, M_2)$  be a pair of finite  $K$ -modules. For

integers  $i \in \{1, \dots, n\}$  and  $j \in \{1, 2\}$ , we define the subset

$$E_{\sigma}^K(M_1, M_2)_{i,j} \subset E_{\sigma}^K(M_1, M_2)$$

to be the set of BT operators  $\phi : M_1 \rightarrow M_2$  that satisfy the conditions:

- (i) The operator  $\phi$  belongs to  $E_{\sigma}^K(M_1, M_2)_{1,1}$  if there exists some  $L_2 \in \text{Lat}(M_2)$  such that  $\phi(M_1) \subset L_2$ .
- (ii) The operator  $\phi$  belongs to  $E_{\sigma}^K(M_1, M_2)_{1,2}$  if there exists some  $L_1 \in \text{Lat}(M_1)$  such that  $\phi(L_1) = 0$ .
- (iii) Let  $n \geq 2$ . For  $i \in \{2, \dots, n\}$  and  $j \in \{1, 2\}$ , the operator  $\phi$  belongs to  $E_{\sigma}^K(M_1, M_2)_{i,j}$  if for any  $\phi$ -refinement  $(L'_1, L'_2) \prec_{\phi} (L_1, L_2)$  in  $\text{Lat}(M_1, M_2)$  the induced homomorphism

$$\bar{\phi} : L_1/L'_1 \rightarrow L'_2/L_2$$

belongs to

$$E_{d_1(\sigma)}^{k_1(K)}(\text{rest}_{\sigma_1}(L_1/L'_1), \text{rest}_{\sigma_1}(L'_2/L_2))_{i-1,j}.$$

**Definition 4.15.** Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , and let  $\sigma$  be a system of liftings for  $K$ . We define

$$E_{\sigma}(K) := E_{\sigma}^K(K, K).$$

If  $n \geq 1$  we define

$$E_{\sigma}(K)_{i,j} := E_{\sigma}^K(K, K)_{i,j}.$$

**Lemma 4.16.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , with  $n \geq 1$ , and let  $\sigma$  be a system of liftings for  $K$ . For  $l = 1, 2, 3, 4$  let  $M_l$  be a finite  $K$ -module, and for  $l = 1, 2, 3$  let  $\phi_l \in E_{\sigma}^K(M_l, M_{l+1})$ . Take any  $j \in \{1, 2\}$  and  $i \in \{1, \dots, n\}$ .*

- (1) *The set  $E_{\sigma}^K(M_1, M_2)_{i,j}$  is a  $\mathbb{k}$ -submodule of  $E_{\sigma}^K(M_1, M_2)$ .*
- (2) *If  $\phi_2 \in E_{\sigma}^K(M_2, M_3)_{i,j}$ , then  $\phi_3 \circ \phi_2 \circ \phi_1 \in E_{\sigma}^K(M_1, M_4)_{i,j}$ .*
- (3) *Assume that  $\phi_1$  is surjective and  $K$ -linear,  $\phi_3$  is injective and  $K$ -linear, and  $\phi_3 \circ \phi_2 \circ \phi_1 \in E_{\sigma}^K(M_1, M_4)_{i,j}$ . Then  $\phi_2 \in E_{\sigma}^K(M_2, M_3)_{i,j}$ .*

*Proof.* We use induction on  $n$  and on the sequential order of the assertions.

(1) For  $i = 1$  this is clear. Now assume  $i \geq 2$  (and hence also  $n \geq 2$ ). For this we use the same strategy as in the proof of [Lemma 4.10](#). We are allowed to make use of assertion (3) in dimension  $n - 1$ .

(2) For  $i = 1$  this is clear. Now assume  $i \geq 2$  (and hence also  $n \geq 2$ ). Here we use the same proof as of [Lemma 4.9\(2\)](#), relying on assertions (2) and (3) in dimension  $n - 1$ .

(3) Same as the proof of [Lemma 4.9\(3\)](#). We rely on assertion (2) in dimension  $n$ .  $\square$

**Lemma 4.17.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , with  $n \geq 1$ , and let  $\sigma$  be a system of liftings for  $K$ . Let  $M_1$  and  $M_2$  be finite  $K$ -modules. For any  $i$  there is the equality*

$$E_{\sigma}^K(M_1, M_2) = E_{\sigma}^K(M_1, M_2)_{i,1} + E_{\sigma}^K(M_1, M_2)_{i,2}.$$

*Proof.* For  $i = 1$  this is clear. (It is Tate’s original observation [1968].)

Assume  $i \geq 2$  (and hence also  $n \geq 2$ ). For this we use induction on  $n$ . Choose  $K$ -linear isomorphisms  $K^{r_l} \cong M_l$  for  $l = 1, 2$ . According to Lemmas 4.9 and 4.16 there are  $\mathbb{k}$ -linear isomorphisms

$$E_{\sigma}^K(M_1, M_2) \cong \text{Mat}_{r_2 \times r_1}(E_{\sigma}(K))$$

and

$$E_{\sigma}^K(M_1, M_2)_{i,j} \cong \text{Mat}_{r_2 \times r_1}(E_{\sigma}(K)_{i,j}).$$

Therefore we can assume that  $M_1 = M_2 = K$ .

The induction hypothesis says that the identity automorphism  $\mathbf{1}_{\mathbf{k}_1(K)}$  of the TLF  $\mathbf{k}_1(K)$  is a sum  $\mathbf{1}_{\mathbf{k}_1(K)} = \bar{\phi}_1 + \bar{\phi}_2$ , where  $\bar{\phi}_j \in E_{d_1(\sigma)}(\mathbf{k}_1(K))_{i-1,j}$ . Choose a uniformizer  $a \in \mathcal{O}_1(K)$ . Any element of  $K$  has a unique expansion as a series  $\sum_{q \in \mathbb{Z}} \sigma_1(b_q) \cdot a^q$ , where  $b_q \in \mathbf{k}_1(K)$  and  $b_q = 0$  for  $q \ll 0$ . Define  $\phi_j \in \text{End}_{\mathbb{k}}(K)$  by the formula

$$\phi_j \left( \sum_{q \in \mathbb{Z}} \sigma_1(b_q) \cdot a^q \right) := \sum_{q \in \mathbb{Z}} \sigma_1(\bar{\phi}_j(b_q)) \cdot a^q.$$

A little calculation shows that  $\phi_j \in E_{\sigma}(K)_{i,j}$ ; and clearly  $\phi_1 + \phi_2 = \mathbf{1}_K$ . □

**Definition 4.18** [Tate 1968]. Let  $M$  be a  $\mathbb{k}$ -module. An operator  $\phi \in \text{End}_{\mathbb{k}}(M)$  is called *finite potent* if, for some positive integer  $q$ , the operator  $\phi^q$  has finite rank, i.e., the  $\mathbb{k}$ -module  $\phi^q(M)$  is finite.

**Lemma 4.19.** *Let  $K$  be a TLF over  $\mathbb{k}$  of dimension  $n \geq 1$ , let  $\sigma$  be a system of liftings for  $K$ , and let  $M$  be a finite  $K$ -module. Then any operator*

$$\phi \in \bigcap_{\substack{i=1, \dots, n \\ j=1, 2}} E_{\sigma}^K(M, M)_{i,j}$$

*is finite potent.*

*Proof.* The proof is by induction on  $n$ . (For  $n = 1$  this is Tate’s original observation.)

Since  $\phi \in E_{\sigma}^K(M, M)_{1,1}$ , there is a lattice  $L_2 \in \text{Lat}(M)$  such that  $\phi(M) \subset L_2$ . Since  $\phi \in E_{\sigma}^K(M, M)_{1,2}$ , there is a lattice  $L_1 \in \text{Lat}(M)$  such that  $\phi(L_1) = 0$ . After replacing  $L_1$  by a smaller lattice, we can assume that  $L_1 \subset L_2$ . Consider the

commutative diagram

$$\begin{array}{ccccccc}
 0 & \xrightarrow{\subset} & L_1 & \xrightarrow{\subset} & L_2 & \xrightarrow{\subset} & M \\
 \phi \downarrow & \nearrow & \downarrow \phi & & \downarrow \phi & \nearrow \phi & \downarrow \phi \\
 0 & \xrightarrow{\subset} & L_1 & \xrightarrow{\subset} & L_2 & \xrightarrow{\subset} & M
 \end{array}$$

in  $\text{Mod } \mathbb{k}$ . Define  $\bar{M} := L_2/L_1$ . If we can prove that the induced homomorphism  $\bar{\phi} : \bar{M} \rightarrow \bar{M}$  is finite potent, then it will follow, by a simple linear algebra argument based on the diagram above, that  $\phi$  is finite potent.

If  $n = 1$  then  $\bar{M}$  is finite over  $\mathbb{k}$ , so we are done. If  $n \geq 2$ , then by definition

$$\bar{\phi} \in \bigcap_{\substack{i=1, \dots, n-1 \\ j=1, 2}} E_{d_1(\sigma)}^K(\bar{M}, \bar{M})_{i,j}.$$

The induction hypothesis says that  $\bar{\phi}$  is finite potent. □

**Theorem 4.20.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , and let  $(M_1, M_2)$  be a pair of finite  $K$ -modules. Suppose  $\sigma$  and  $\sigma'$  are two systems of liftings for  $K$ .*

(1) *There is equality*

$$E_{\sigma}^K(M_1, M_2) = E_{\sigma'}^K(M_1, M_2)$$

inside  $\text{Hom}_{\mathbb{k}}(M_1, M_2)$

(2) *If  $n \geq 1$ , there is equality*

$$E_{\sigma}^K(M_1, M_2)_{i,j} = E_{\sigma'}^K(M_1, M_2)_{i,j}$$

for all  $i = 1, \dots, n$  and  $j = 1, 2$ .

*Proof.* (1) By symmetry it is enough to prove the inclusion “ $\subset$ ”. The proof is by induction on  $n$ . For  $n = 0$  there is nothing to prove.

Now assume  $n \geq 1$ . Let  $\phi \in E_{\sigma}^K(M_1, M_2)$ . We have to prove that  $\phi$  is in  $E_{\sigma'}^K(M_1, M_2)$ . Since condition (2.i) of [Definition 4.5](#) does not involve the liftings, there is nothing to check.

Next we consider condition (2.ii). Take some  $\phi$ -refinement  $(L'_1, L'_2) \prec_{\phi} (L_1, L_2)$  in  $\text{Lat}(M_1, M_2)$ . Define  $\bar{M}_1 := L_1/L'_1$  and  $\bar{M}_2 := L'_2/L_2$ , and let  $\bar{\phi} : \bar{M}_1 \rightarrow \bar{M}_2$  be the induced homomorphism. Let us write  $\bar{K} := k_1(K)$ ,  $\bar{\sigma} := d_1(\sigma)$  and  $\bar{\sigma}' := d_1(\sigma')$ . We know that

$$\bar{\phi} \in E_{\bar{\sigma}}^{\bar{K}}(\text{rest}_{\sigma_1}(\bar{M}_1), \text{rest}_{\sigma_1}(\bar{M}_2)). \quad (4-21)$$

The induction hypothesis says that  $E_{\bar{\sigma}}(\bar{K}) = E_{\bar{\sigma}'}(\bar{K})$ .

Choose  $\bar{K}$ -linear isomorphisms  $\chi_l: \bar{K}^{r_l} \xrightarrow{\sim} \text{rest}_{\sigma_l}(\bar{M}_l)$  and  $\chi'_l: \bar{K}^{r_l} \xrightarrow{\sim} \text{rest}_{\sigma'_l}(\bar{M}_l)$ . This gives rise to a commutative diagram

$$\begin{array}{ccccccc}
 \bar{M}_1 & \xrightarrow{=} & \bar{M}_1 & \xrightarrow{\bar{\phi}} & \bar{M}_2 & \xrightarrow{=} & \bar{M}_2 \\
 \uparrow \chi'_1 & & \uparrow \chi_1 & & \uparrow \chi_2 & & \uparrow \chi'_2 \\
 \bar{K}^{r_1} & \xrightarrow{\psi_1} & \bar{K}^{r_1} & \xrightarrow{\psi} & \bar{K}^{r_2} & \xrightarrow{\psi_2} & \bar{K}^{r_2}
 \end{array}$$

in  $\text{Mod } \mathbb{k}$ . According to formula (4-21) and Lemma 4.9, the operator  $\psi$  is in  $\text{Mat}_{r_2 \times r_1}(\mathbb{E}_{\bar{\sigma}}(\bar{K}))$ . Combining Lemma 3.14 and Theorem 2.8 we see that the operators  $\psi_l$  belong to  $\text{GL}_{r_l}(\mathcal{D}_{\bar{K}/\mathbb{k}}^{\text{cont}})$ . Therefore, by Lemma 4.11,  $\psi_l \in \text{GL}_{r_l}(\mathbb{E}_{\bar{\sigma}}(\bar{K}))$ . We conclude that  $\psi' := \psi_2 \circ \psi \circ \psi_1$  is in

$$\text{Mat}_{r_2 \times r_1}(\mathbb{E}_{\bar{\sigma}}(\bar{K})) = \text{Mat}_{r_2 \times r_1}(\mathbb{E}_{\bar{\sigma}'}(\bar{K})).$$

So by Lemma 4.9 we have

$$\bar{\phi} = \chi'_2 \circ \psi' \circ \chi_1^{-1} \in \mathbb{E}_{\bar{\sigma}'}^{\bar{K}}(\text{rest}_{\sigma'_1}(\bar{M}_1), \text{rest}_{\sigma'_1}(\bar{M}_2)).$$

This is what we had to prove.

(2) Again we only prove the inclusion “ $\subset$ ”, and the proof is by induction on  $n$ . For  $i = 1$  the conditions do not involve the liftings, so there is nothing to check. Now consider  $i \geq 2$  (and hence  $n \geq 2$ ). We assume that the theorem is true for dimension  $n - 1$ . Take some  $\phi \in \mathbb{E}_{\bar{\sigma}}^K(M_1, M_2)_{i,j}$ , and let  $(L'_1, L'_2) \prec_{\phi} (L_1, L_2)$  be a  $\phi$ -refinement in  $\text{Lat}(M_1, M_2)$ . In the notation of the proof of part (1) above, the operator  $\psi$  is inside  $\text{Mat}_{r_2 \times r_1}(\mathbb{E}_{\bar{\sigma}}(\bar{K})_{i,j})$ . This is because

$$\bar{\phi} \in \mathbb{E}_{\bar{\sigma}}^{\bar{K}}(\text{rest}_{\sigma_1}(\bar{M}_1), \text{rest}_{\sigma_1}(\bar{M}_2))_{i,j},$$

and  $\mathbb{E}_{\bar{\sigma}}(\bar{K})_{i,j}$  is a two-sided ideal in the ring  $\mathbb{E}_{\bar{\sigma}}(\bar{K})$ . The induction hypothesis tells us that  $\mathbb{E}_{\bar{\sigma}}(\bar{K})_{i,j} = \mathbb{E}_{\bar{\sigma}'}(\bar{K})_{i,j}$ . Therefore the same calculations as above yield

$$\bar{\phi} \in \mathbb{E}_{\bar{\sigma}'}^{\bar{K}}(\text{rest}_{\sigma'_1}(\bar{M}_1), \text{rest}_{\sigma'_1}(\bar{M}_2))_{i,j}$$

as required.  $\square$

Taking  $M_1 = M_2 := K$  in the theorem we obtain:

**Corollary 4.22.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , and let  $\sigma$  and  $\sigma'$  be two systems of liftings for  $K$ . Then  $\mathbb{E}_{\sigma}(K) = \mathbb{E}_{\sigma'}(K)$ . If  $n \geq 1$  then  $\mathbb{E}_{\sigma}(K)_{i,j} = \mathbb{E}_{\sigma'}(K)_{i,j}$  for all  $i, j$ .*

The corollary justifies the next definition.

**Definition 4.23.** Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ .

(1) We define

$$E(K) := E_{\sigma}(K),$$

where  $\sigma$  is any system of liftings for  $K$ . Elements of  $E(K)$  are called *local Beilinson–Tate operators* on  $K$ .

(2) Assume  $n \geq 1$ . For  $i \in \{1, \dots, n\}$  and  $j \in \{1, 2\}$  we define

$$E(K)_{i,j} := E_{\sigma}^K(K, K)_{i,j},$$

where  $\sigma$  is any system of liftings for  $K$ .

Of course, when  $n = 0$  we have  $E(K) = \text{End}_{\mathbb{k}}(K)$ , which is not interesting. The next theorem summarizes what we know about BT operators in dimensions 1 and above. Recall the notion of an  $n$ -dimensional cubically decomposed ring of operators on a commutative  $\mathbb{k}$ -ring  $A$ , from [Definition 0.3](#).

**Theorem 4.24.** *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , with  $n \geq 1$ .*

- (1) *The ring of BT operators  $E(K)$ , with its collection of ideals  $\{E(K)_{i,j}\}$ , is an  $n$ -dimensional cubically decomposed ring of operators on  $K$ .*
- (2) *There are inclusions of rings*

$$K \subset \mathcal{D}_{K/\mathbb{k}}^{\text{cont}} \subset E(K) \subset \text{End}_{\mathbb{k}}^{\text{cont}}(K) \subset \text{End}_{\mathbb{k}}(K).$$

*Proof.* Assertion (1) is a combination of [Lemmas 4.16](#), [4.17](#) and [4.19](#). Assertion (2) is a combination of [Lemmas 4.9](#), [4.10](#) and [4.11](#).  $\square$

**Remark 4.25.** It would be good to have a structural understanding of the objects  $E(K)$  and  $E(K)_{i,j}$  associated to a TLF  $K$ . For instance, does  $E(K)$  carry a canonical structure of an ST ring, or perhaps some “higher semi-topological structure”? Such a structure could help in proving [Conjecture 5.7](#).

**Remark 4.26.** [Osipov \[2007\]](#) introduced the categories  $C_n$ ,  $n \in \mathbb{N}$ , that fiber over  $\text{Mod } \mathbb{k}$ . These categories are defined inductively, in a way that closely resembles Beilinson’s definitions [\[1980\]](#). The paper [\[Braunling et al. 2014\]](#) introduced the categories  $\text{Tate}_n$  of  $n$ -Tate spaces, also fibered over  $\text{Mod } \mathbb{k}$ . Presumably these two concepts coincide.

Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ . It seems likely that  $K$  should have a canonical  $C_n$ -structure, or a canonical  $\text{Tate}_n$ -structure. Moreover, the subrings  $\text{End}_{C_n}(K)$ ,  $\text{End}_{\text{Tate}_n}(K)$  and  $E(K)$  of  $\text{End}_{\mathbb{k}}(K)$  should coincide.

If that is the case, then some of our statements above become similar or equivalent to some results in [\[Osipov 2007\]](#). For instance, our [Lemma 4.8](#) corresponds to [\[Osipov 2007, Proposition 3\]](#).



### 5. Residues

In this section we provide background for [Conjecture 0.9](#) in the Introduction. The base ring  $\mathbb{k}$  is a perfect field, and it has the discrete topology.

Recall the way the DG ring of separated differential forms  $\Omega_{A/\mathbb{k}}^{\text{sep}} = \bigoplus_{i \geq 0} \Omega_{A/\mathbb{k}}^{i,\text{sep}}$  was defined in [Section 2](#) for any commutative ST  $\mathbb{k}$ -ring  $A$ . The usual module of Kähler differentials  $\Omega_{A/\mathbb{k}}^i$  is a ST  $\mathbb{k}$ -module, with topology induced from the surjection  $T_{\mathbb{k}}^{i+1}(A) \twoheadrightarrow \Omega_{A/\mathbb{k}}^i$ . Then  $\Omega_{A/\mathbb{k}}^{i,\text{sep}} := (\Omega_{A/\mathbb{k}}^i)^{\text{sep}}$  is the associated separated ST module. In degree 0 we have  $\Omega_{A/\mathbb{k}}^{0,\text{sep}} = A^{\text{sep}}$ . There is a canonical surjection of DG ST  $\mathbb{k}$ -rings

$$\tau_A : \Omega_{A/\mathbb{k}} \twoheadrightarrow \Omega_{A/\mathbb{k}}^{\text{sep}}, \tag{5-1}$$

which is a topological strict epimorphism. Given any homomorphism  $f : A \rightarrow B$  in the category  $\text{STRing}_{\mathbb{k}}$ , there is an induced commutative diagram of DG ST  $\mathbb{k}$ -rings

$$\begin{array}{ccc} \Omega_{A/\mathbb{k}} & \xrightarrow{\Omega(f)} & \Omega_{B/\mathbb{k}} \\ \tau_A \downarrow & & \downarrow \tau_B \\ \Omega_{A/\mathbb{k}}^{\text{sep}} & \xrightarrow{\Omega^{\text{sep}}(f)} & \Omega_{B/\mathbb{k}}^{\text{sep}} \end{array}$$

Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ , with its DG ST ring of separated differential forms  $\Omega_{K/\mathbb{k}}^{\text{sep}} = \bigoplus_{i=0}^n \Omega_{K/\mathbb{k}}^{i,\text{sep}}$ . In degree 0 we have  $\Omega_{K/\mathbb{k}}^{0,\text{sep}} = K$ , since  $K$  is separated (in fact it is a complete ST  $\mathbb{k}$ -module). In degree  $n$  the  $K$ -module  $\Omega_{K/\mathbb{k}}^{n,\text{sep}}$  is free of rank 1 with the fine  $K$ -module topology. If  $\mathbf{a} = (a_1, \dots, a_n)$  is a system of uniformizers for  $K$ , then the element

$$\text{dlog}(\mathbf{a}) := a_1^{-1} \cdot d(a_1) \cdots a_n^{-1} \cdot d(a_n) \tag{5-2}$$

is a basis of  $\Omega_{K/\mathbb{k}}^{n,\text{sep}}$ . See [Theorem 3.16](#) and [Example 2.16](#).

There is a theory of trace homomorphisms for separated differential forms. For any morphism  $f : K \rightarrow L$  in  $\text{TLF}^n \mathbb{k}$ , there is a homomorphism

$$\text{Tr}_{L/K}^{\text{TLF}} : \Omega_{L/\mathbb{k}}^{\text{sep}} \rightarrow \Omega_{K/\mathbb{k}}^{\text{sep}}. \tag{5-3}$$

This is a degree 0 homomorphism of DG ST  $\Omega_{K/\mathbb{k}}^{\text{sep}}$ -modules. It is uniquely characterized by these properties: it is functorial; in degree 0 it coincides with the usual trace  $\text{tr}_{L/K} : L \rightarrow K$ ; and

$$\text{Tr}_{L/K}^{\text{TLF}} \circ \text{dlog} = \text{dlog} \circ n_{L/K}$$

as functions  $L^\times \rightarrow \Omega_{K/\mathbb{k}}^{1,\text{sep}}$ , where  $n_{L/K} : L^\times \rightarrow K^\times$  is the usual norm. The homomorphism  $\text{Tr}_{L/K}^{\text{TLF}}$  is nondegenerate in top degree, in the sense that the induced

homomorphism

$$\Omega_{L/\mathbb{k}}^{n,\text{sep}} \rightarrow \text{Hom}_K(L, \Omega_{K/\mathbb{k}}^{n,\text{sep}})$$

is bijective. See [Yekutieli 1992, Section 2.3].

In [Yekutieli 1992, Section 2.4] we introduced the residue functional for TLFs. Its properties are summarized in the following theorem:

**Theorem 5.4** [Yekutieli 1992]. *Let  $K$  be an  $n$ -dimensional TLF over  $\mathbb{k}$ . There is a  $\mathbb{k}$ -linear homomorphism*

$$\text{Res}_{K/\mathbb{k}}^{\text{TLF}} : \Omega_{K/\mathbb{k}}^{n,\text{sep}} \rightarrow \mathbb{k}$$

with these properties:

- (1) *Continuity: the homomorphism  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$  is continuous.*
- (2) *Uniformization: let  $\mathbf{a} = (a_1, \dots, a_n)$  be a system of uniformizers for  $K$ , and let  $\mathbb{k}' \rightarrow \mathcal{O}(K)$  be the unique  $\mathbb{k}$ -ring lifting of the last residue field  $\mathbb{k}' := \mathbf{k}_n(K)$  into the ring of integers  $\mathcal{O}(K)$  of  $K$ . Then, for any  $b \in \mathbb{k}'$  and any  $i_1, \dots, i_n \in \mathbb{Z}$ , we have*

$$\text{Res}_{K/\mathbb{k}}^{\text{TLF}}(b \cdot a_1^{i_1} \cdots a_n^{i_n} \cdot \text{dlog}(\mathbf{a})) = \begin{cases} \text{tr}_{\mathbb{k}'/\mathbb{k}}(b) & \text{if } i_1 = \cdots = i_n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

- (3) *Functoriality: let  $f : K \rightarrow L$  be a morphism in the category  $\text{TLF}^n \mathbb{k}$ . Then*

$$\text{Res}_{L/\mathbb{k}}^{\text{TLF}} = \text{Res}_{K/\mathbb{k}}^{\text{TLF}} \circ \text{Tr}_{L/K}^{\text{TLF}}.$$

- (4) *Nondegeneracy: the residue pairing*

$$\langle -, - \rangle_{\text{res}} : K \times \Omega_{K/\mathbb{k}}^{n,\text{sep}} \rightarrow \mathbb{k}, \quad \langle a, \alpha \rangle_{\text{res}} := \text{Res}_{K/\mathbb{k}}^{\text{TLF}}(a \cdot \alpha)$$

is a topological perfect pairing.

Furthermore, the function  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$  is the uniquely determined by properties (1) and (2).

**Remark 5.5.** Actually the residue homomorphism  $\text{Res}_{-/-}^{\text{TLF}}$  exists in much greater generality. Recall from Remark 3.11 that there is a category  $\text{TLF } \mathbb{k}$  whose objects are TLFs of all dimensions, and there are morphisms  $f : K \rightarrow L$  for  $\dim(K) < \dim(L)$ . The category  $\text{TLF}^n \mathbb{k}$  is a full subcategory of  $\text{TLF } \mathbb{k}$ . In [Yekutieli 1992, Section 2.4] we construct a residue homomorphism

$$\text{Res}_{L/K}^{\text{TLF}} : \Omega_{L/\mathbb{k}}^{\text{sep}} \rightarrow \Omega_{K/\mathbb{k}}^{\text{sep}}$$

for any morphism  $K \rightarrow L$  in  $\text{TLF } \mathbb{k}$ . This is a DG ST  $\Omega_{K/\mathbb{k}}^{\text{sep}}$ -linear homomorphism of degree  $-m$ , where  $m := \dim(L) - \dim(K)$ , and it has properties like those in Theorem 5.4. When  $K = \mathbb{k}$  this is the residue homomorphism  $\text{Res}_{L/\mathbb{k}}^{\text{TLF}}$  above; and when  $m = 0$  this is the trace homomorphism:  $\text{Res}_{L/K}^{\text{TLF}} = \text{Tr}_{L/K}^{\text{TLF}}$ .

Another remark is a sign change: the uniformization formula above differs from that of [Yekutieli 1992, Theorem 2.4.3] by a factor of  $(-1)^{\binom{n}{2}}$ . This is disguised as a permutation of the factors of the differential form  $d\log(t_1, \dots, t_n)$ . Cf. also [Yekutieli 1992, Remark 2.4.4]. Our better acquaintance recently with DG conventions dictates the current formula.

Let  $K$  be a TLF over  $\mathbb{k}$  of dimension  $n \geq 1$ . The homological algebra and Lie algebra construction of [Beilinson 1980], as explained in [Braunling 2014b, Section 3.1], takes as input the cubically decomposed ring of BT operators  $E(K)$  from Definition 4.23, and produces the *Beilinson–Tate residue functional*

$$\text{Res}_{K/\mathbb{k}}^{\text{BT}} : \Omega_{K/\mathbb{k}}^n \rightarrow \mathbb{k}. \tag{5-6}$$

Not much is known about this residue functional when  $n \geq 2$ . We have already posed Conjecture 0.9, comparing  $\text{Res}_{K/\mathbb{k}}^{\text{BT}}$  to  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$ . Here is another conjecture:

**Conjecture 5.7.** Let  $K$  be a TLF over  $\mathbb{k}$ . The  $\mathbb{k}$ -linear functional  $\text{Res}_{K/\mathbb{k}}^{\text{BT}}$  is continuous.

It is closely related to the first conjecture. Indeed:

**Proposition 5.8.** (1) *Conjecture 0.9 implies Conjecture 5.7.*

(2) *Conjectures 5.7 and 0.12 together imply Conjecture 0.9.*

*Proof.* (1) We know that  $\tau_K : \Omega_{K/\mathbb{k}}^n \rightarrow \Omega_{K/\mathbb{k}}^{n,\text{sep}}$  and  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$  are continuous; see Theorem 5.4(1).

(2) Assume  $\text{Res}_{K/\mathbb{k}}^{\text{BT}}$  is continuous. Then, since  $\mathbb{k}$  is separated, the homomorphism  $\text{Res}_{K/\mathbb{k}}^{\text{BT}}$  factors via  $\tau_K$ . It remains to compare the continuous functionals

$$\text{Res}_{K/\mathbb{k}}^{\text{BT}}, \text{Res}_{K/\mathbb{k}}^{\text{TLF}} : \Omega_{K/\mathbb{k}}^{n,\text{sep}} \rightarrow \mathbb{k}.$$

Conjecture 0.12 says that we can use the results of [Braunling 2014b]. Now according to [Braunling 2014b, Theorem 26(3)], the functional  $\text{Res}_{K/\mathbb{k}}^{\text{BT}}$  satisfies the uniformization condition (2) of Theorem 5.4. Since the  $\mathbb{k}$ -module spanned by the forms

$$b \cdot a_1^{i_1} \cdots a_n^{i_n} \cdot d\log(\mathbf{a})$$

is dense inside  $\Omega_{K/\mathbb{k}}^{n,\text{sep}}$ , and both functionals  $\text{Res}_{K/\mathbb{k}}^{\text{BT}}$  and  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$  agree on it, these functionals must be equal.  $\square$

To end this section here are some remarks and examples related to the TLF residue:

**Remark 5.9.** The uniqueness of the residue functional  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$  has several other expressions, besides properties (1)–(2) of Theorem 5.4. For simplicity let us assume that  $\mathbb{k}$  is infinite and  $k_n(K) = \mathbb{k}$ .

Here is one alternative characterization: Let  $G$  be the “Galois group” of  $K/\mathbb{k}$ , namely  $G := \text{Aut}_{\text{TLF}^n \mathbb{k}}(K)$ . The group  $G$  acts on  $\Omega_{K/\mathbb{k}}^{n,\text{sep}}$  by continuous  $\mathbb{k}$ -linear isomorphisms, and hence it acts on  $\text{Hom}_{\mathbb{k}}^{\text{cont}}(\Omega_{K/\mathbb{k}}^{n,\text{sep}}, \mathbb{k})$ . It is not hard to show that  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$  is the only  $G$ -invariant element  $\rho \in \text{Hom}_{\mathbb{k}}^{\text{cont}}(\Omega_{K/\mathbb{k}}^{n,\text{sep}}, \mathbb{k})$  that also satisfies  $\rho(\text{dlog}(\mathbf{a})) = 1$ , where  $\mathbf{a}$  is any system of uniformizers of  $K$ .

For the second characterization of the residue functional, let us assume that  $\text{char}(\mathbb{k}) = 0$ . (This also works in  $\text{char}(\mathbb{k}) = p > 0$ , but in a more complicated way — see [Yekutieli 1992, Digression 2.4.28].) Define  $H_{\text{DR}}^n(K) := H^n(\Omega_{K/\mathbb{k}}^{\text{sep}})$ . This is a rank 1  $\mathbb{k}$ -module generated by the cohomology class of  $\text{dlog}(\mathbf{a})$ . A calculation shows that  $\text{Res}_{K/\mathbb{k}}^{\text{TLF}}$  is the only  $\mathbb{k}$ -linear homomorphism  $\rho : \Omega_{K/\mathbb{k}}^{n,\text{sep}} \rightarrow \mathbb{k}$  that factors through  $H_{\text{DR}}^n(K)$  (i.e., it vanishes on  $n$ -coboundaries) and also satisfies  $\rho(\text{dlog}(\mathbf{a})) = 1$ .

**Remark 5.10.** In dimension 1 the residue functional on local fields (with its topological aspects) was understood a long time ago (see [Serre 1988]).

The first attempt to extend the residue functional to local fields of dimension  $n \geq 2$  was by Parshin and his school [1976; 1978; 1983; Beilinson 1980; Lomadze 1981]. In [Parshin 1976] the case of a surface is discussed, without attempt to isolate the resulting 2-dimensional local field from its geometric origin. In [Parshin 1983] there is a brief mention of a residue functional on a standalone  $n$ -dimensional local field, but without any details whatsoever. Beilinson [1980], quoting [Parshin 1976; 1978], incorrectly states that the residue functional on an  $n$ -dimensional local field  $K$  is independent of the parametrization of  $K$  (which, according to Theorem 3.16, means independent of the topology on  $K$ ).

Lomadze [1981] studied the setup of a standalone  $n$ -dimensional local field in great detail. However, since he misunderstood the role of the topology in local fields of dimension  $n \geq 2$  (see Remark 3.12), the residue functional he proposed was not well defined. To be specific, [Lomadze 1981] claimed that for a local field  $K \in \text{LF}^n \mathbb{k}$  there is a  $\mathbb{k}$ -linear homomorphism, let us denote it by  $\text{res} : \Omega_{K/\mathbb{k}}^n \rightarrow \mathbb{k}$ , which satisfies continuity, uniformization (property (2) of Theorem 5.4), and invariance under automorphisms of  $K$  in  $\text{LF}^n \mathbb{k}$ . However this is false for  $n \geq 2$  and  $\text{char}(\mathbb{k}) = 0$ , as was shown by a counterexample in [Yekutieli 1992]. We reproduce this counterexample, in an expanded form, in Examples 5.11 and 5.12 below.

In characteristic  $p > 0$  the residue functional is indeed well defined on the category  $\text{LF}^n \mathbb{k}$ . But this is due to the fact, discovered in [Yekutieli 1992], that the forgetful functor  $\text{TLF}^n \mathbb{k} \rightarrow \text{LF}^n \mathbb{k}$  is an equivalence when  $\text{char}(\mathbb{k}) = p > 0$ . See Remark 3.12.

**Example 5.11.** This is an expanded version of [Yekutieli 1992, Example 2.1.24]. It shows that, when  $\text{char}(\mathbb{k}) = 0$  and  $n \geq 2$ , there cannot be a  $\mathbb{k}$ -linear homomorphism

$\text{res} : \Omega_{K/\mathbb{k}}^n \rightarrow \mathbb{k}$  for a local field  $K \in \text{LF}^n \mathbb{k}$  which satisfies continuity, uniformization, and invariance under automorphisms of  $K$  in  $\text{LF}^n \mathbb{k}$ .

Let  $A$  be any commutative ST  $\mathbb{k}$ -ring. In order to distinguish between an “abstract” differential form  $\alpha \in \Omega_{A/\mathbb{k}}^i$  and the “separated” differential form  $\tau_A(\alpha) \in \Omega_{A/\mathbb{k}}^{i,\text{sep}}$ , we shall write  $\bar{\alpha} := \tau_A(\alpha)$ . Also we denote by  $\bar{d}$  the differential operator in the DG ring  $\Omega_{A/\mathbb{k}}^{\text{sep}}$ . So  $\tau_A \circ d = \bar{d} \circ \tau_A$  as  $\mathbb{k}$ -linear homomorphisms  $\Omega_{A/\mathbb{k}}^i \rightarrow \Omega_{A/\mathbb{k}}^{i+1,\text{sep}}$ . Note that when  $A$  itself is separated we have  $\Omega_{A/\mathbb{k}}^{0,\text{sep}} = \Omega_{A/\mathbb{k}}^0 = A$ .

Since the homomorphism  $\text{res} : \Omega_{K/\mathbb{k}}^n \rightarrow \mathbb{k}$  is assumed to be continuous, and  $\mathbb{k}$  is separated (because it is discrete), it follows that  $\text{res}$  factors through  $\Omega_{K/\mathbb{k}}^{n,\text{sep}}$ , and  $\text{res}(\alpha) = \text{res}(\bar{\alpha})$  for any  $\alpha \in \Omega_{K/\mathbb{k}}^n$ .

We shall use the setup of [Example 3.13](#). So  $\text{char}(\mathbb{k}) = 0$ ,  $n = 2$ , and  $K = \mathbb{k}((t_1, t_2)) = \mathbb{k}((t_2))((t_1))$ , the standard 2-dimensional TLF with last residue field  $\mathbb{k}$ . We choose a collection  $\{b_i\}_{i \in I}$  in  $\mathbb{k}((t_2))$  that is a transcendence basis over the subfield  $\mathbb{k}(t_2)$ . We single out one element of the indexing set, say  $i_0 \in I$ , and define  $\sigma(b_{i_0}) := b_{i_0} + t_1$ . For  $i \neq i_0$  we let  $\sigma(b_i) := b_i$ . This determines an automorphism  $f$  of  $K$  in the category  $\text{LF}^2 \mathbb{k}$ . (We already observed in [Example 3.13](#) that  $f$  is not continuous). Let us write  $b := b_{i_0}$ ; so  $f(t_1) = t_1$ ,  $f(t_2) = t_2$  and  $f(b) = b + t_1$ .

Define the differential forms

$$\alpha := t_1^{-1} \cdot d(b) \cdot t_2^{-1} \cdot d(t_2), \quad \beta := t_1^{-1} \cdot d(b + t_1) \cdot t_2^{-1} \cdot d(t_2)$$

and

$$\gamma := t_1^{-1} \cdot d(t_1) \cdot t_2^{-1} \cdot d(t_2) = \text{dlog}(t_1, t_2)$$

in  $\Omega_{K/\mathbb{k}}^2$ . Note that  $\beta = \alpha + \gamma$  and  $\beta = f(\alpha)$ .

Consider the continuous  $\mathbb{k}$ -linear derivation  $\partial/\partial t_2$  of  $\mathbb{k}((t_2))$ . It is dual to the differential form  $\bar{d}(t_2) \in \Omega_{\mathbb{k}((t_2))/\mathbb{k}}^{1,\text{sep}}$ . Hence, letting  $b' := \partial(b)/\partial t_2 \in \mathbb{k}((t_2))$ , we have  $\bar{d}(b) = b' \cdot \bar{d}(t_2)$  in  $\Omega_{\mathbb{k}((t_2))/\mathbb{k}}^{1,\text{sep}}$ . Since the inclusion  $\mathbb{k}((t_2)) \rightarrow K$  is continuous, it follows that  $\bar{d}(b) = b' \cdot \bar{d}(t_2)$  in  $\Omega_{K/\mathbb{k}}^{1,\text{sep}}$ . But then  $\bar{d}(b) \cdot \bar{d}(t_2) = 0$  in  $\Omega_{K/\mathbb{k}}^{2,\text{sep}}$ , from which we deduce that  $\bar{\alpha} = 0$  in  $\Omega_{K/\mathbb{k}}^{2,\text{sep}}$ . Therefore  $\text{res}(\alpha) = \text{res}(\bar{\alpha}) = 0$ . On the other hand,  $\bar{\beta} = \bar{\alpha} + \bar{\gamma} = \bar{\gamma}$ . And hence

$$\text{res}(\beta) = \text{res}(\bar{\beta}) = \text{res}(\bar{\gamma}) = \text{res}(\gamma) = 1$$

by the uniformization property. We see that  $\beta = f(\alpha)$ ,  $\text{res}(\alpha) = 0$  and  $\text{res}(\beta) = 1$ .

**Example 5.12.** Here is another way to view the previous example. Again  $\mathbb{k}$  has characteristic 0. Let  $K$  be the local field  $\mathbb{k}((t_1, t_2))$ . We consider various topologies on  $K$  that make it into a TLF; namely we are looking at the objects in the fiber above  $K$  of the forgetful functor  $F : \text{TLF}^n \mathbb{k} \rightarrow \text{LF}^n \mathbb{k}$ . [Theorem 3.16](#) shows that the group  $\text{Aut}_{\text{LF}^n \mathbb{k}}(K)$  acts transitively on the objects in this fiber.

The first topology on the local field  $K$  is the standard topology of  $\mathbb{k}((t_1, t_2))$ , and we denote the resulting TLF by  $K_{\text{st}}$ . For the second topology we use the

automorphism  $f$  from [Example 5.11](#). We take the fine  $(K_{\text{st}}, f^{-1})$ -module topology on  $K$ , and call the resulting TLF  $K_{\text{nt}}$ . Thus  $f : K_{\text{nt}} \rightarrow K_{\text{st}}$  is an isomorphism in  $\text{TLF}^n \mathbb{k}$ , and  $F(K_{\text{nt}}) = F(K_{\text{st}}) = K$  in  $\text{LF}^n \mathbb{k}$ .

Let  $K_t$  be any TLF such that  $F(K_t) = K$  (for instance the standard TLF  $K_{\text{st}}$  and the nonstandard TLF  $K_{\text{nt}}$ ). There is a surjection

$$\tau_t = \tau_{K_t} : \Omega_{K/\mathbb{k}}^2 = \Omega_{K_t/\mathbb{k}}^2 \twoheadrightarrow \Omega_{K_t/\mathbb{k}}^{2,\text{sep}},$$

and thus a residue homomorphism  $\text{res}_t : \Omega_{K/\mathbb{k}}^2 \rightarrow \mathbb{k}$  defined by  $\text{res}_t := \text{Res}_{K_t/\mathbb{k}}^{\text{TLF}} \circ \tau_t$ .

Consider the differential forms  $\alpha, \beta, \gamma \in \Omega_{K/\mathbb{k}}^2$  from [Example 5.11](#). The calculation there shows that  $\tau_{\text{st}}(\alpha) = 0$ . On the other hand, since  $f \circ \tau_{\text{nt}} = \tau_{\text{st}} \circ f$  and  $f(\gamma) = \gamma$ , we have  $f(\tau_{\text{nt}}(\alpha)) = \tau_{\text{st}}(f(\alpha)) = \tau_{\text{st}}(\beta) = \tau_{\text{st}}(\alpha) + \tau_{\text{st}}(\gamma) = \tau_{\text{st}}(\gamma) = f(\tau_{\text{nt}}(\gamma))$ , and therefore  $\tau_{\text{nt}}(\alpha) = \tau_{\text{nt}}(\gamma)$ . We conclude that, for the differential form  $\alpha \in \Omega_{K/\mathbb{k}}^2$ , we have  $\text{res}_{\text{st}}(\alpha) = 0$ , but  $\text{res}_{\text{nt}}(\alpha) = \text{res}_{\text{nt}}(\gamma) = 1$ .

**Question 5.13.** Take any  $n \geq 2$ . Consider the local field  $K := \mathbb{k}((t_1, \dots, t_n))$ , and the various TLFs  $K_t$  lying above it in  $\text{TLF}^n \mathbb{k}$ , as in the previous example. We know that the residue  $\text{res}_t(\alpha)$ , for  $\alpha \in \Omega_{K/\mathbb{k}}^n$ , could change as we change the topology. However our counterexample involved transcendentals (the element  $b$ ).

What about the subfield  $\mathbb{k}(t_1, \dots, t_n) \in K$ ? Is it true that for a form  $\alpha$  in  $\Omega_{\mathbb{k}(t_1, \dots, t_n)/\mathbb{k}}^n$  the residue  $\text{res}_t(\alpha)$  is independent of the topology  $K_t$  on  $K$ ?

## 6. Geometry: completions

In this section we give background for [Conjecture 0.12](#) in the introduction. We recall some facts on the Beilinson completion operation, and reproduce Beilinson’s geometric definition of the BT operators.

Throughout this section  $\mathbb{k}$  is a noetherian commutative ring, and  $X$  is a finite type  $\mathbb{k}$ -scheme. By a chain of points of length  $n$  in  $X$  we mean a sequence  $\xi = (x_0, \dots, x_n)$  of points in  $X$  such that  $x_i$  is a specialization of  $x_{i-1}$  for all  $i$ . The chain  $\xi$  is called a *saturated chain* if every  $x_i$  is an immediate specialization of  $x_{i-1}$ , namely the closed set  $\overline{\{x_i\}}$  has codimension 1 in  $\overline{\{x_{i-1}\}}$ . If  $n \geq 1$ , we denote by  $d_0(\xi)$  the chain obtained from  $\xi$  by deleting the point  $x_0$ .

Let  $\mathcal{M}$  be a quasi-coherent  $\mathcal{O}_X$ -module. Beilinson [1980] introduced the completion  $\mathcal{M}_\xi$  of  $\mathcal{M}$  along  $\xi$ , which we refer to as the *Beilinson completion*. This is a very special case of his higher adeles. The definition of  $\mathcal{M}_\xi$  is inductive on  $n$ , by an  $n$ -fold zigzag of inverse and direct limits. For a detailed account see [Yekutieli 1992, Section 3] or [Morrow 2013]. A basic geometric fact used in the definition is that, for any coherent sheaf  $\mathcal{M}$ , point  $x \in X$  and number  $i \in \mathbb{N}$ , the truncated localization  $\mathcal{M}_x/\mathfrak{m}_x^{i+1}\mathcal{M}_x$ , when viewed as an  $\mathcal{O}_X$ -module supported on the closed set  $\overline{\{x\}}$ , is quasi-coherent. An important instance of this is when  $\mathcal{M} = \mathcal{O}_X$  and  $i = 0$ , which gives the residue field  $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ .

Here are some important properties of the Beilinson completion operation. Let  $\mathcal{M}$  be some quasi-coherent  $\mathcal{O}_X$ -module and let  $\xi = (x_0, \dots, x_n)$  be a chain in  $X$ . We can view the completion  $\mathcal{M}_\xi$  either as a module over the local ring  $\mathcal{O}_{X, x_n}$  or as a constant  $\mathcal{O}_X$ -module supported on the closed set  $\overline{\{x_n\}}$ . Warning:  $\mathcal{M}_\xi$  is usually not quasi-coherent. For any subchain  $\xi' \subset \xi$  there is a canonical homomorphism  $\mathcal{M}_{\xi'} \rightarrow \mathcal{M}_\xi$ . When  $n = 0$ , so  $\xi = (x_0)$ , there is a canonical homomorphism  $\mathcal{M}_{x_0} \rightarrow \mathcal{M}_{(x_0)}$ , where the former is the stalk at the point. If  $\mathcal{M}$  is coherent, then the homomorphism  $\widehat{\mathcal{M}}_{x_0} \rightarrow \mathcal{M}_{(x_0)}$  from the  $\mathfrak{m}_{x_0}$ -adic completion is an isomorphism.

The completion  $\mathcal{O}_{X, \xi}$  of the structure sheaf  $\mathcal{O}_X$  is a commutative ring, the canonical sheaf homomorphism  $\mathcal{O}_X \rightarrow \mathcal{O}_{X, \xi}$  is flat, and  $\mathcal{M}_\xi$  is an  $\mathcal{O}_{X, \xi}$ -module. The sheaf homomorphism  $\mathcal{O}_{X, \xi} \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow \mathcal{M}_\xi$  is an isomorphism. Thus the functor  $\mathcal{M} \mapsto \mathcal{M}_\xi$  is exact. If  $\mathcal{M}$  is coherent,  $\xi$  is saturated, and  $n \geq 1$ , then the canonical homomorphism

$$\mathcal{O}_{X, x_0} \otimes_{\mathcal{O}_X} \mathcal{M}_{d_0(\xi)} \rightarrow \mathcal{M}_\xi$$

is an isomorphism.

The zigzag completion operation endows  $\mathcal{M}_\xi$  with a  $\mathbb{k}$ -linear topology, similar to the iterated Laurent series construction in Definition 1.17. The ring  $\mathcal{O}_{X, \xi}$  becomes a ST  $\mathbb{k}$ -ring, and  $\mathcal{M}_\xi$  is a ST  $\mathcal{O}_{X, \xi}$ -module.

Let  $A$  be a semi-local commutative ring, with Jacobson radical  $\mathfrak{r}$ . We say that  $A$  is a *complete semi-local ring* if the canonical homomorphism  $A \rightarrow \lim_{\leftarrow i} A/\mathfrak{r}^i$  is bijective. The *residue ring* of  $A$  is the ring  $A/\mathfrak{r}$ , which is a finite product of fields.

**Theorem 6.1** [Parshin 1976; Beilinson 1980; Yekutieli 1992]. *Let  $\mathbb{k}$  be an excellent noetherian ring, let  $X$  be a finite type  $\mathbb{k}$ -scheme, and let  $\xi = (x_0, \dots, x_n)$  be a saturated chain in  $X$  of length  $n \geq 1$  such that  $x_n$  is a closed point. Then the Beilinson completions  $\mathcal{O}_{X, \xi}$  and  $\mathbf{k}(x_0)_\xi$  have these algebraic properties:*

- (1) *The ring  $\mathbf{k}(x_0)_\xi$  is a finite product of  $n$ -dimensional local fields over  $\mathbb{k}$ .*
- (2) *The ring  $\mathcal{O}_{X, \xi}$  is a complete semi-local commutative  $\mathbb{k}$ -ring, with Jacobson radical  $\mathfrak{r} = \mathcal{O}_{X, \xi} \otimes_{\mathcal{O}_{X, x_0}} \mathfrak{m}_{x_0}$  and residue ring  $\mathbf{k}(x_0)_\xi$ .*
- (3) *Let  $K$  be one of the factors of the reduced artinian semi-local ring  $\mathbf{k}(x_0)_\xi$ , which by (1) is an  $n$ -dimensional local field. The DVR  $\mathcal{O}_1(K)$  is the integral closure in  $K$  of the ring  $\mathcal{O}_{X, d_0(\xi)}$ .*

*If the base ring  $\mathbb{k}$  is a perfect field, then the completion  $\mathbf{k}(x_0)_\xi$  also has these topological properties:*

- (4) *Let  $K$  be one of the factors of the ring  $\mathbf{k}(x_0)_\xi$ . Then  $K$ , with the induced topology from  $\mathbf{k}(x_0)_\xi$ , is an  $n$ -dimensional TLF over  $\mathbb{k}$ .*
- (5) *The image of the field  $\mathbf{k}(x_0)$  in the ST ring  $\mathbf{k}(x_0)_\xi$  is dense.*

*Proof.* (1–3) For  $n = 1$  this is classical. For  $n = 2$  this is in [Parshin 1976]. For  $n \geq 3$  these assertions appear in [Beilinson 1980] without a proof. The proofs are [Yekutieli 1992, Theorem 3.3.2 and Corollary 3.3.5].

(4–5) For  $n = 1$  this is classical. For  $n \geq 2$  these assertions are [Yekutieli 1992, Proposition 3.3.6 and Corollary 3.3.7].  $\square$

**Remark 6.2.** The condition that  $x_n$  is a closed point is only important to ensure that the last residue fields  $k_n(K)$  are finite over  $\mathbb{k}$ . Cf. Remark 3.10. The results in [Yekutieli 1992] quoted in the proof above only require the chain  $\xi$  to be saturated.

Suppose  $\xi = (x_0, \dots, x_n)$  is a saturated chain in  $X$ . We have seen that there is a commutative diagram of flat ring homomorphisms

$$\begin{array}{ccccc} \mathcal{O}_{X,x_n} & \longrightarrow & \mathcal{O}_{X,(x_n)} & & \\ \downarrow & & \searrow & & \\ \mathcal{O}_{X,x_0} & \longrightarrow & \mathcal{O}_{X,(x_0)} & \longrightarrow & \mathcal{O}_{X,\xi} \end{array}$$

**Definition 6.3.** Let  $\xi = (x_0, \dots, x_n)$  be a saturated chain in  $X$  of length  $n \geq 1$ , and let  $M$  be a finite length  $\mathcal{O}_{X,x_0}$ -module. An  $\mathcal{O}_{X,x_1}$ -lattice in  $M$  is a finite  $\mathcal{O}_{X,x_1}$ -submodule  $L$  of  $M$  such that  $M = \mathcal{O}_{X,x_0} \cdot L$ . We denote by  $\text{Lat}_{X,\xi}(M)$  the set of all such lattices.

Of course the points  $x_2, \dots, x_n$  have no influence on  $\text{Lat}_{X,\xi}(M)$ . Note that if  $\xi$  has length 0 then  $M_\xi = M$  for any finite length  $\mathcal{O}_{X,x_0}$ -module  $M$ .

**Lemma 6.4.** Let  $\xi = (x_0, \dots, x_n)$  be a saturated chain in  $X$ , and let  $M$  be a finite length  $\mathcal{O}_{X,x_0}$ -module. If  $L, L' \in \text{Lat}_{X,\xi}(M)$  and  $L \subset L'$ , then  $L'/L$  is a finite length  $\mathcal{O}_{X,x_1}$ -module.

*Proof.* We can assume that  $M \neq 0$ . Let  $Z$  be the support in  $\text{Spec } \mathcal{O}_{X,x_1}$  of  $L$ . Then  $Z$  is a 1-dimensional scheme, with only two points: the closed point  $x_1$  and the generic point  $x_0$ . The finite  $\mathcal{O}_{X,x_1}$ -module  $L'/L$  satisfies

$$(L'/L)_{x_0} \cong \mathcal{O}_{X,x_0} \otimes_{\mathcal{O}_{X,x_1}} (L'/L) = 0,$$

and hence it is supported on  $\{x_1\}$ .  $\square$

Let  $\xi = (x_0, \dots, x_n)$  be a saturated chain in  $X$ , and let  $M$  be a finite length  $\mathcal{O}_{X,x_0}$ -module. We can view  $M$  as a quasi-coherent sheaf on  $X$ , constant on the closed set  $\overline{\{x_0\}}$ . The canonical homomorphism  $M_{d_0(\xi)} \rightarrow M_\xi$  is bijective. (If  $n = 0$  then  $d_0(\xi)$  is empty, and we define  $M_0 := M$ .) Note that  $M_\xi$  is a finite length  $\mathcal{O}_{X,\xi}$ -module.



Suppose we are given  $\mathcal{O}_{X,x_1}$ -lattices  $L \subset L'$  in  $M$ . By the exactness of completion there are inclusions

$$L_{d_0(\xi)} \subset L'_{d_0(\xi)} \subset M_{d_0(\xi)} = M_\xi,$$

and there is a canonical isomorphism of finite length  $\mathcal{O}_{X,d_0(\xi)}$ -modules

$$(L'/L)_{d_0(\xi)} \cong L'_{d_0(\xi)}/L_{d_0(\xi)}.$$

Let  $(M_1, M_2)$  be a pair of finite length  $\mathcal{O}_{X,x_0}$ -modules. Let  $\text{Lat}_{X,\xi}(M_1, M_2)$  be the set of pairs  $(L_1, L_2)$ , where  $L_i \in \text{Lat}_{X,\xi}(M_i)$ . We write  $M_{i,\xi} := (M_i)_\xi$ . Suppose  $\phi : M_{1,\xi} \rightarrow M_{2,\xi}$  is a  $\mathbb{k}$ -linear operator. Like in [Definition 4.4](#), we say that  $(L'_1, L'_2)$  is a  $\phi$ -refinement of  $(L_1, L_2)$ , and that  $(L'_1, L'_2) \prec_\phi (L_1, L_2)$  is a  $\phi$ -refinement in  $\text{Lat}_{X,\xi}(M_1, M_2)$ , if  $L'_1 \subset L_1$ ,  $L_2 \subset L'_2$ ,  $\phi(L_{1,d_0(\xi)}) \subset L'_{2,d_0(\xi)}$  and  $\phi(L'_{1,d_0(\xi)}) \subset L_{2,d_0(\xi)}$ .

Suppose  $A$  is a semi-local ring, with residue ring  $K$ . Any finite length  $A$ -module  $M$  has a canonical decomposition  $M = \bigoplus_n M_n$ , where  $n$  runs over the finite set of maximal ideals of  $A$ , which of course coincides with the set  $\text{Spec } K$ .

**Definition 6.5.** Let  $A$  be a semi-local ring in  $\text{Ring}_c \mathbb{k}$ , with residue ring  $K$ . Let  $M_1, M_2$  be finite length  $A$ -modules, and let  $\phi : M_1 \rightarrow M_2$  be a  $\mathbb{k}$ -linear homomorphism. We say that  $\phi$  is *local on Spec  $K$*  if  $\phi(M_{1,n}) \subset M_{2,n}$  for every  $n \in \text{Spec } K$ .

Here is a slight enhancement of the original definition found in [\[Beilinson 1980\]](#); see [Remark 6.7](#) below and [Definition 4.5](#).

**Definition 6.6** [\[Beilinson 1980\]](#). Let  $\xi = (x_0, \dots, x_n)$  be a saturated chain of points in  $X$  such that  $x_n$  is a closed point. Let  $(M_1, M_2)$  be a pair of finite length modules over the ring  $\mathcal{O}_{X,x_0}$ . We define the subset

$$E_{X,\xi}(M_1, M_2) \subset \text{Hom}_{\mathbb{k}}(M_{1,\xi}, M_{2,\xi})$$

as follows:

- (1) If  $n = 0$ , then any  $\mathbb{k}$ -linear homomorphism  $\phi : M_{1,\xi} \rightarrow M_{2,\xi}$  belongs to  $E_{X,\xi}(M_1, M_2)$ .
- (2) If  $n \geq 1$ , a  $\mathbb{k}$ -linear homomorphism  $\phi : M_{1,\xi} \rightarrow M_{2,\xi}$  belongs to  $E_{X,\xi}(M_1, M_2)$  if it satisfies these three conditions:
  - (i) Every  $(L_1, L_2) \in \text{Lat}_{X,\xi}(M_1, M_2)$  has some  $\phi$ -refinement  $(L'_1, L'_2)$ .
  - (ii) For every  $\phi$ -refinement  $(L'_1, L'_2) \prec_\phi (L_1, L_2)$  in  $\text{Lat}_{X,\xi}(M_1, M_2)$  the induced homomorphism

$$\bar{\phi} : (L_1/L'_1)_{d_0(\xi)} \rightarrow (L'_2/L_2)_{d_0(\xi)}$$

belongs to

$$E_{X,d_0(\xi)}(L_1/L'_1, L'_2/L_2).$$

- (iii) The homomorphism  $\phi$  is local on  $\text{Spec } \mathbf{k}(x_0)_\xi$ .

**Remark 6.7.** Condition (2.iii) of [Definition 6.6](#) is not part of the original definition in [\[Beilinson 1980\]](#). Note that Tate [\[1968\]](#) only considered smooth curves, for which the completion is always a single local field, and there is no issue of locality.

The same locality condition eventually appears in Brauning’s treatment — see the definition of the ring  $E_j$  in [\[Brauning 2014b, Theorem 26\(1\)\]](#).

The next definition uses notation like that of Tate. It can of course be rewritten using the notations of [\[Beilinson 1980\]](#) or of [\[Brauning 2014a; 2014b\]](#). Compare to [Definition 4.14](#) above.

**Definition 6.8** [\[Beilinson 1980\]](#). Let  $\xi = (x_0, \dots, x_n)$  be a saturated chain of points in  $X$  of length  $n \geq 1$  such that  $x_n$  is a closed point. Let  $(M_1, M_2)$  be a pair of finite length modules over the ring  $\mathcal{O}_{X, x_0}$ . For any  $i \in \{1, \dots, n\}$  and  $j \in \{1, 2\}$  we define the subset

$$E_{X, \xi}(M_1, M_2)_{i, j} \subset E_{X, \xi}(M_1, M_2)$$

to be the set of operators  $\phi : M_{1, \xi} \rightarrow M_{2, \xi}$  in  $E_{X, \xi}(M_1, M_2)$  that satisfy the conditions:

- (i)  $\phi$  belongs to  $E_{X, \xi}(M_1, M_2)_{1, 1}$  if there exists some  $L_2 \in \text{Lat}_{X, \xi}(M_2)$  such that  $\phi(M_{1, \xi}) \subset L_{2, d_0(\xi)}$ .
- (ii)  $\phi$  belongs to  $E_{X, \xi}(M_1, M_2)_{1, 2}$  if there exists some  $L_1 \in \text{Lat}_{X, \xi}(M_1)$  such that  $\phi(L_{1, d_0(\xi)}) = 0$ .
- (iii) Let  $n \geq 2$ . For  $i \in \{2, \dots, n\}$  and  $j \in \{1, 2\}$ ,  $\phi$  belongs to  $E_{X, \xi}(M_1, M_2)_{i, j}$  if, for any  $\phi$ -refinement  $(L'_1, L'_2) \prec_\phi (L_1, L_2)$  in  $\text{Lat}_{X, \xi}(M_1, M_2)$ , the induced homomorphism

$$\bar{\phi} : (L_1/L'_1)_{d_0(\xi)} \rightarrow (L_2/L'_2)_{d_0(\xi)}$$

belongs to

$$E_{X, d_0(\xi)}(L_1/L'_1, L'_2/L_2)_{i-1, j}.$$

**Definition 6.9.** Let  $\xi = (x_0, \dots, x_n)$  be a saturated chain of points in  $X$ , of length  $n \geq 1$ , such that  $x_n$  is a closed point. Consider the residue field  $K := \mathbf{k}(x_0)$ .

- (1) We define  $E_{X, \xi}(K) := E_{X, \xi}(K, K)$ .
- (2) If  $n \geq 1$  we define  $E_{X, \xi}(K)_{i, j} := E_{X, \xi}(K, K)_{i, j}$ .

By definition there are inclusions

$$E_{X, \xi}(K)_{i, j} \subset E_{X, \xi}(K) \subset \text{End}_{\mathbb{k}}(K_\xi).$$

**Theorem 6.10** [\[Beilinson 1980; Brauning 2014b, Proposition 13\]](#). Assume  $\mathbb{k}$  is a perfect field. The data

$$(E_{X, \xi}(K), \{E_{X, \xi}(K)_{i, j}\})$$

from [Definition 6.9](#) is an  $n$ -dimensional cubically decomposed ring of operators on  $K_\xi$ , in the sense of [Definition 0.3](#).

[Conjecture 0.12](#) asserts that this  $n$ -dimensional cubically decomposed ring of operators on  $K_\xi$  coincides with the cubically decomposed ring of operators

$$(E(K_\xi), \{E(K_\xi)_{i,j}\})$$

from [Definition 4.23](#), modified as in formula (0-10).

**Remark 6.11.** Consider an integral finite type  $\mathbb{k}$ -scheme  $X$  of dimension  $n$ . Let  $\xi = (x_0, \dots, x_n)$  be a maximal chain in  $X$ ; so  $x_0$  is the generic point. Write  $K := k(X) = k(x_0)$ . According to [Theorem 6.10](#) there is a cubically decomposed ring of operators  $E_{X,\xi}(K)$  on  $K_\xi$ . Applying the abstract BT residue of formula (0-7) with  $E := E_{X,\xi}(K)$ , we obtain the functional

$$\text{Res}_{X,\xi}^{\text{BT}} := \text{Res}_{K_\xi/\mathbb{k};E}^{\text{BT}} : \Omega_{K/\mathbb{k}}^n \rightarrow \mathbb{k}.$$

This is the residue functional that Beilinson [1980] had.

Beilinson [1980] claimed that the functionals  $\text{Res}_{X,\xi}^{\text{BT}}$  satisfy several geometric properties. Most notably, when  $X$  is a proper scheme, then for any  $\alpha \in \Omega_{K/\mathbb{k}}^n$  there is a global residue formula:

$$\sum_{\xi} \text{Res}_{X,\xi}^{\text{BT}}(\alpha) = 0. \tag{6-12}$$

The sum is on all maximal chains  $\xi$  in  $X$ .

[Conjectures 0.9](#) and [0.12](#), combined with our results in [Yekutieli 1992] regarding the residue functionals  $\text{Res}_{K_\xi/\mathbb{k}}^{\text{TLF}}$ , imply most of the geometric properties of the residue functionals  $\text{Res}_{X,\xi}^{\text{BT}}$  stated in [Beilinson 1980], including formula (6-12).

Conversely, as noted by Beilinson (private communication), a direct proof of the geometric properties of the functionals  $\text{Res}_{X,\xi}^{\text{BT}}$  (perhaps by generalizing Tate’s original idea to higher dimensions), together with [Conjecture 0.12](#), would imply [Conjecture 0.9](#).

### Acknowledgments

I wish to thank Jesse Wolfson and Oliver Braunling for bringing their work to my attention and for many illuminating discussions. Thanks also to Matthew Morrow, John Tate and Alexander Beilinson for their input. Lastly, I want to thank the three anonymous referees for reading my paper carefully and suggesting many improvements.

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Communicated by Edward Frenkel

Received 2014-06-25

Revised 2014-10-20

Accepted 2014-12-25

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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ANT peer review and production are managed by EditFLOW® from MSP.

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# Algebra & Number Theory

Volume 9    No. 1    2015

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