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Sagnik Chakraborty, Rajendra Vasant Gurjar and Masayoshi Miyanishi

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Factorially closed subrings of commutative rings

Sagnik Chakraborty, Rajendra Vasant Gurjar and Masayoshi Miyanishi

We prove some new results about factorially closed subrings of commutative rings. We generalize this notion to quasifactorially closed subrings of commutative rings and prove some results about them from algebraic and geometric viewpoints. We show that quasifactorially closed subrings of polynomial and power series rings of dimension at most three are again polynomial (resp. power series) rings in a smaller number of variables. As an application of our results, we give a short proof of a result of Lê Dũng Tráng in connection with the Jacobian problem.

Introduction

We assume throughout the article that the base field k is an algebraically closed field of characteristic 0. Whenever we use topological arguments, k is tacitly assumed to be the field of complex numbers $\mathbb{C}$. By assuming naturally that k is embedded into $\mathbb{C}$, we can see that the results proved over $\mathbb{C}$ can be proved over k. For an integral domain S, the field of fractions of S is denoted by $Q(S)$, and the multiplicative group of units by S^*.

The present article grew out of the discussions we had during the workshop *Automorphisms of affine varieties*, held at the Kerala School of Mathematics, India (February 17–22, 2014). In particular, a part of our discussion was inspired by a talk given by Neena Gupta [2014] and a question asked by A. Kanel-Belov.

Let $A \subseteq B$ be integral domains. Then A is said to be *factorially closed*, or *fc*, in B if for any two nonzero elements $b_1, b_2 \in B$, $b_1 b_2 \in A$ implies that $b_1, b_2 \in A$. In some papers an fc subring is also called an *inert* subring. Factorially closed subrings appear naturally as the rings of invariants of the action of the additive group G_a, or a connected semisimple group on a polynomial ring.

The notion of fc subring is not well-behaved in the case of local rings due to the existence of too many units. Hence we have introduced a weaker notion: *quasifactorially closed* subrings. For any integral domains $A \subseteq B$, A is said to be quasifactorially closed, or qfc, in B if, for any nonzero $b \in B$, if there exists

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some nonzero $b' \in B$ such that $bb' \in A$, then there exists a unit $u \in B$ such that $bu \in A$. It turns out that quasifactorial closedness is more geometric and has several interesting applications. For example, we have proved that a qfc subring of a power series ring in at most three variables is again isomorphic to a power series ring in a smaller number of variables.

The fc property is also related to the property of the existence of nonconstant invertible regular functions on general fibers of the corresponding morphism of schemes.

We now mention the main results proved in this paper (with some hypothesis):

(1) An inclusion of graded domains $A \subseteq B$ is fc if and only if it is graded fc. Further, A is fc in B if and only if the localization of A at its irrelevant maximal ideal is fc in the corresponding localization of B (Theorems 2 and 3).

(2) For an inclusion of affine normal domains $A \subseteq B$ the fc locus is always open (Corollary 4.1).

(3) For an inclusion of affine UFDs $A \subseteq B$ the qfc locus is open if at most finitely many prime elements of A split in B (Theorem 5). (An example in Section 3 shows that the reverse implication is false.)

(4) If an inclusion of complete local normal domains $A \subseteq B$ over k is qfc then A is algebraically closed in B. Further, any irreducible element of A is irreducible in B (Theorem 6 and its corollaries).

(5) An fc subring of a polynomial ring in at most three variables is again a polynomial ring (Theorem 1). Similarly, a complete qfc subring of a power series ring in at most three variables is again a power series ring (Theorem 8).

(6) If an inclusion of affine normal domains $A \subseteq B$ (with a suitable hypothesis) is fc, then a general fiber of the corresponding morphism of affine varieties does not have any nonconstant invertible regular functions (Theorem 11).

Using this we give a new short proof of a result proved by many authors (M. Razar, R. Heitmann, S. Friedland, L. D. Tráng, C. Weber, W. Neumann, P. Norbury) in connection with the Jacobian problem [Neumann and Norbury 1998; Tráng 2008].

In Section 3 we give some examples of ring extensions which shed more light on the fc (and qfc) property.

In Section 4 we have listed some open problems about fc and qfc extensions.

1. Factorially closed subrings

We start with some basic properties of factorial closedness. Some easy proofs have been omitted.

Lemma 1 (local properties of factorial closedness). *The following statements are equivalent for an inclusion of integral domains $A \subseteq B$:*

(1) *The ring A is fc in B.*

(2) *For any multiplicatively closed set S in A, $S^{-1}A \subseteq S^{-1}B$ is fc.*

(3) *For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is fc.*

(4) *For any maximal ideal $\mathfrak{m} \in \operatorname{Max} A$, $A_{\mathfrak{m}} \subseteq B_{\mathfrak{m}}$ is fc.*

(5) *There exist finitely many nonzero elements $a_1, a_2, \dots, a_n \in A$, generating the unit ideal, such that A_{a_i} is fc in B_{a_i} for each $i = 1, 2, \dots, n$.*

Moreover, if A is normal, the above statements are equivalent to the following one:

(7) *For each prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ of height 1, $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is fc.*

Proof. Omitted. $\square$

Lemma 2 (transitive and sandwich properties of factorial closedness). *Let $A \subseteq B \subseteq C$ be integral domains.*

(1) *If the ring A is fc in B and B is fc in C, then A is fc in C.*

(2) *If A is fc in C then it is fc in B. However, in this case B need not be fc in C.*

The example $k \subseteq k[t^2] \subseteq k[t]$ shows that B need not be fc in C.

Lemma 3. *Let A be an fc subring of B.*

(1) *The ring A is algebraically closed in B.*

(2) *If $Q(A)$ is the field of fractions of A, then $Q(A) \cap B = A$. This is the same thing as saying that each principal ideal of A is a contracted ideal.*

(3) *If B is integrally closed (or a UFD), then so is A. In fact, in the case of Krull domains, the natural homomorphism of divisor class groups $\operatorname{Cl}(A) \to \operatorname{Cl}(B)$ is an injection whenever it is defined.*

(4) *Any unit of B is in A.*

Proof. The first assertion follows from the slightly more general fact that, for a pair of integral domains $A \subseteq B$, if $B \setminus A$ is closed under multiplication then A is algebraically closed in B.

The other three statements follow from the first one and the next observation. $\square$

Remark. For an inclusion of Krull domains $A \subset B$, there is a natural homomorphism $\operatorname{Cl}(A) \to \operatorname{Cl}(B)$ if and only if no height 1 prime ideal of B contracts to a prime ideal of height > 1 in A [Samuel 1964].

If I is an ideal of A such that the ideal IB is principal, then, since A is fc in B, I itself must be principal. This observation will be implicitly used later.

Lemma 4. *Let A be an fc subring of B. Then the Jacobson radical of B, $\operatorname{Jac} B$, is contained in A. Moreover, if $\operatorname{Jac} B \neq 0$ then $A = B$. In particular, if B is semilocal then $A = B$.*

Proof. If $b \in \operatorname{Jac} B$, $1 + b \in B^*$. So, by Lemma 3(4), $1 + b \in A$ implies that $b \in A$. Now, if b is a nonzero element in $\operatorname{Jac} B$, for any $x \in B$, xb is also in $\operatorname{Jac} B$ and consequently in A. So $x \in A$. If B is semilocal, and not a field, then $\operatorname{Jac} B \neq 0$ and the rest of the assertion follows. If B is a field then every nonzero element in B is a unit, and since A is fc in B we again get $A = B$. □

Lemma 5. *If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ are two sequences of integral domains, such that $A_i \subseteq B_i$ is an fc subring for each i, then $\bigcup_i A_i \subseteq \bigcup_i B_i$ is also factorially closed.*

Lemma 6. *If $A \subseteq B \subseteq C$ are integral domains with A an fc subring of B, then, for any subring D of C, $D \cap A \subseteq D \cap B$ is also factorially closed.*

Before looking into the ring-theoretic properties of factorial closedness, we would like to describe the structure of a factorially closed subalgebra of the polynomial ring $R = k[x_1, \ldots, x_n]$. This question can be answered if $n \leq 3$, and the answer is simply a polynomial subalgebra. We consider only the case where $n = 3$. The case $n = 2$ has a similar answer and is easier.

Theorem 1. *Let A be a factorially closed subring of $R = k[x, y, z]$.*

(1) *If $\dim A = 3$, then $A = R$.*

(2) *If $\dim A = 2$, then A is a polynomial ring in two variables.*

(3) *If $\dim A = 1$, then $A = k[f]$, where $f - c$ is irreducible in $k[x, y, z]$ for every $c \in k$.*

Proof. (1) In this case, the transcendence degree of A over k is 3. So A, being algebraically closed in $k[x, y, z]$ by Lemma 3(1), must be equal to $k[x, y, z]$.

In the other two cases, since A is a UFD (by Lemma 3(3)) of transcendence degree ≤ 2, by a result of Zariski [Nagata 1965, p. 52, Theorem 4] A is affine.

So the assertion (3) follows from the fact that, when A has dimension 1, A is an affine PID with trivial units.

(2) Note that A is a normal affine domain of dimension 2. We assume for simplicity that $k = \mathbb{C}$. Let $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} A$. The inclusion $A \hookrightarrow R$ defines a dominant morphism $p : Y \to X$. Then every fiber of p is either the empty set or is 1-dimensional. For, if there exists a fiber component D of dimension 2, let it be defined by $f = 0$ with $f \in R$. Since $p(D)$ is a closed point of X corresponding to a maximal ideal $\mathfrak{m}$ of A, we have $\mathfrak{m} \subseteq \mathfrak{m}R \subseteq fR$. This implies that any nonzero

element of $\mathfrak{m}$ is divisible by f, whence $f \in A$. This is a contradiction since A is 2-dimensional. Furthermore, a general fiber of p is irreducible since A is factorially closed in R. By [Miyanishi 1986, Theorem 3], X is isomorphic to either $\mathbb{A}^2$ or an affine hypersurface $x_1^2 + x_2^3 + x_3^5 = 0$ in $\mathbb{A}^3$. But, arguing as in the proof of [Miyanishi 1986, Theorem 4], we can show that the latter case cannot occur. □

Let $B := \bigoplus_i B_i$ be a $\mathbb{Z}$-graded domain and $A := \bigoplus_i A_i$ a graded subring of B, i.e., $A_i \subseteq B_i$ for each i. We say that A is graded factorially closed or gfc, in short, in B if, given any two nonzero homogeneous elements $b_i, b_j \in B$, $b_i b_j \in A$ implies that $b_i, b_j \in A$. First, the following lemma shows that gfc is a local property:

Lemma 7. *Let $A \subseteq B$ be $\mathbb{Z}$-graded domains. Then the following statements are equivalent*:

(1) *The ring A is gfc in B.*

(2) *For any multiplicative set S in A, generated by homogeneous elements, $S^{-1}A \subseteq S^{-1}B$ is gfc.*

(3) *For any homogeneous prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, $A_{(\mathfrak{p})} \subseteq B_{(\mathfrak{p})}$, where $A_{(\mathfrak{p})}$ and $B_{(\mathfrak{p})}$ denote the localizations of A and B respectively at the multiplicative set consisting of all homogeneous elements of A not contained in $\mathfrak{p}$, is gfc.*

If, moreover, A happens to be positively graded, the above statements are equivalent to the following:

(4) *For any homogeneous maximal ideal $\mathfrak{m} \in \operatorname{Max} A$, $A_{(\mathfrak{m})} \subseteq B_{(\mathfrak{m})}$ is graded factorially closed.*

Proof. We only show (3) $\Longrightarrow$ (1). Let $x, y \in B$ be homogeneous elements such that $xy \in A$. So, $x, y \in A_{(\mathfrak{p})}$ for every homogeneous prime ideal $\mathfrak{p}$. But the set $(A : x) := \{a \in A \mid ax \in A\}$ is a homogeneous ideal in A. So, if $x \notin A$, then $(A : x)$ must be proper ideal and hence contained in a homogeneous prime ideal, leading to a contradiction. For positively graded rings, note that any homogeneous ideal is actually contained in a homogeneous maximal ideal. □

Note that properties analogous to the fc property as expressed in Lemmas 1, 2 and 3 also hold for graded factorially closed subrings. The reader is invited to come up with the precise formulations and their proofs.

Next we take our first step in building a bridge between factorial closedness and graded factorial closedness.

Lemma 8. *Let $A \subseteq B$ be $\mathbb{Z}$-graded domains and $\mathfrak{p}$ a homogeneous prime ideal of A. If $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is fc then $A_{(\mathfrak{p})} \subseteq B_{(\mathfrak{p})}$ is gfc.*

Proof. Let $x, y \in B_{(\mathfrak{p})}$ be homogeneous elements such that $xy \in A_{(\mathfrak{p})}$. Let $x = x'/s$ and $y = y'/t$ with $x', y' \in B$ and $s, t \in \bigcup_i A_i - \mathfrak{p}$. Since $A_\mathfrak{p} \subseteq B_\mathfrak{p}$ is factorially closed, $x, y \in A_\mathfrak{p}$. So there exist nonzero elements in A, $a = \sum_j a_j$ and $\alpha = \sum_j \alpha_j$ with $\alpha \notin \mathfrak{p}$ such that $x'/s = a/\alpha$. Again, α being outside $\mathfrak{p}$ implies that $\alpha_{j^*} \notin \mathfrak{p}$ for some j^*. So $x'\alpha_{j^*} \in A$ and consequently $x \in A_{(\mathfrak{p})}$. Similarly $y \in A_{(\mathfrak{p})}$, and this finishes the proof. □

It is natural to ask whether gfc implies fc. Our next few results show that this is indeed true. We first treat the easy case of polynomial ring extensions and then show that the general case, under minor restrictions, reduces to this special case.

Lemma 9. *Let A be a factorially closed subring of an integral domain B. Then the polynomial ring $A[x]$ is also factorially closed in $B[x]$.*

Proof. Let $f(x), g(x) \in B[x] - \{0\}$ be such that $f(x)g(x) \in A[x]$. It is enough to show that $f(x) \in A[x]$. We consider the following two possible cases:

Case 1: *A is infinite.* Since f and g can have only finitely many roots, there exist infinitely many $a \in A$ such that $f(a), g(a)$ are nonzero elements of B and $f(a)g(a) \in A$, and consequently $f(a) \in A$, since A is factorially closed in B. In particular, if f has degree n, there exist $n+1$ distinct values in A, say $a_1, a_2, \ldots, a_{n+1}$, such that $f(a_i) \in A$ for each $i = 1, 2, \ldots, n+1$. So, treating the coefficients of f as variables, and plugging in the values a_i, we get $n+1$ linear equations in $n+1$ variables. The simultaneous linear equations have a solution in B. If we look at the corresponding Vandermonde matrix, it is obvious that the solution actually lies in $Q(A)$. Since $Q(A) \cap B = A$ by Lemma 3(2), $f(x) \in A[x]$.

Case 2: *A is a field.* Without any loss of generality we may assume that f and g are monic polynomials. Let L be a splitting field of fg over $Q(B)$. The roots of fg, and hence in particular the roots of f, are integral over $Q(A)$. Consequently, the coefficients of f, being symmetric functions of the roots, are integral over $Q(A)$ and hence algebraic over A. But since A is algebraically closed in B by Lemma 3(1), the coefficients are actually in A, and hence $f(x) \in A[x]$. □

For the general case, let A be a graded factorially closed subring of a $\mathbb{Z}$-graded domain B such that $A_i \neq 0$ and $A_{i+1} \neq 0$ for some integer i. We want to show that $A \subseteq B$ is factorially closed. Let S be the multiplicative set consisting of all nonzero homogeneous elements of A. Note that if $S^{-1}A \subseteq S^{-1}B$ is factorially closed then so is $A \subseteq B$. If $K := (S^{-1}A)_0$ and $\widetilde{B} := (S^{-1}B)_0$, then K is a field which is factorially closed in $\widetilde{B}$. Choose any nonzero elements $a_i \in A_i$, $a_{i+1} \in A_{i+1}$, and let $t := a_{i+1}/a_i$. Then $t \in (S^{-1}A)_1$ and $S^{-1}A = K[t, t^{-1}]$. To show that $K[t, t^{-1}]$ is factorially closed in $S^{-1}B$, let $b := b_{i_0} + b_{i_1} + \cdots + b_{i_r}$ and $c := c_{j_0} + c_{j_1} + \cdots + c_{j_s}$, with b_{i_0}, b_{i_r} and c_{j_0}, c_{j_s} nonzero, be elements of $S^{-1}B$ such that $bc \in S^{-1}A$. Writing $\overline{b_{i_\alpha}} := b_{i_\alpha} t^{-i_\alpha}$ for $\alpha = 0, 1, \ldots, r$ and $\overline{c_{j_\beta}} := c_{j_\beta} t^{-j_\beta}$ for $\beta = 0, 1, \ldots, s$, we get that

$b=\overline{b_{i_0}}t^{i_0}+\overline{b_{i_1}}t^{i_1}+\cdots+\overline{b_{i_r}}t^{i_r}$, $c=\overline{c_{j_0}}t^{j_0}+\overline{c_{j_1}}t^{j_1}+\cdots+\overline{c_{j_s}}t^{j_s}\in\widetilde{B}[t,t^{-1}]$. But since K is factorially closed in $\widetilde{B}$, so is $K[t]\subseteq\widetilde{B}[t]$, and consequently, by Lemmas 1 and 2, $K[t,t^{-1}]$ is factorially closed in $\widetilde{B}[t,t^{-1}]$. So $b,c\in S^{-1}A$, proving that $S^{-1}A\subseteq S^{-1}B$ is factorially closed, and hence $A\subseteq B$ is also factorially closed.

Therefore, we have proved the following result:

Theorem 2. *Let $A\subseteq B$ be $\mathbb{Z}$-graded domains with $A_i\neq 0$ and $A_{i+1}\neq 0$ for some integer i. Then A is factorially closed in B if it is graded factorially closed.*

Question. Is the hypothesis that A_i, A_{i+1} are nonzero for some i necessary?

We do not know the answer to the above question in general. But we sketch below a different proof of Theorem 2 without assuming the condition that A_i and A_{i+1} are nonzero for some i. However it works only when B is a UFD.

Let $A\subseteq B$ be $\mathbb{Z}$-graded domains with B a UFD. Now, assuming that A is graded factorially closed in B, we would like to show that A is factorially closed in B. After inverting all nonzero homogeneous elements of A, we may assume that A is of the form $k[t,t^{-1}]$, where t is a homogeneous prime element of positive degree in B. Further, since $k[t]$ is factorially closed in $B_0[t]$ by Theorem 9, it suffices to prove that $B_0[t]$ is factorially closed in $B^+:=\bigoplus_{i\geq 0}B_i$. So, if $f,g\in B^+$ are such that $fg\in B_0[t]$, we want to show that $f,g\in B_0[t]$. Let $f=f_0+f_1+\cdots+f_m$ and $g=g_0+g_1+\cdots+g_n$, with f_m and g_n nonzero. We can write f and g as

$$f=f_0't^{\alpha_0}+f_1't^{\alpha_1}+\cdots+f_m't^{\alpha_m}\quad\text{and}\quad g=g_0't^{\beta_0}+g_1't^{\beta_1}+\cdots+g_m't^{\beta_m},$$

where $f_i=f_i't^{\alpha_i}$ and t does not divide f_i' unless it is zero, in which case we also take α_i to be zero, and similarly for g. If either $f_i'\in B_0$ for each i or $g_j'\in B_0$ for each j, we are done. Otherwise, we define α_* and β_* to be the minimums of the α_i for $f_i\neq 0$ and the β_j for $g_j\neq 0$, respectively. Let us also define $i^*:=\max\{i\mid\alpha_i=\alpha_*\}$ and $j^*:=\max\{j\mid\beta_j=\beta_*\}$. Note that $i^*<m$ and $j^*<n$. Looking at the homogeneous component of degree i^*+j^* in fg, we get

$$(fg)_{i^*+j^*}=f_{i^*}'g_{j^*}'t^{\alpha_*+\beta_*}+\text{(elements divisible by }t^{\alpha_*+\beta_*+1}).$$

But that means $f_{i^*}'g_{j^*}'$ must be divisible by t, which is a contradiction.

Finally, we put together the results connecting factorial closedness and graded factorial closedness in the form of the following theorem:

Theorem 3. *For positively graded domains $A\subseteq B$, if we assume that $A_1\neq 0$, then the following statements are equivalent*:

(1) *$A\subseteq B$ is fc.*

(2) *For any homogeneous prime ideal $\mathfrak{p}$, $A_{(\mathfrak{p})}\subseteq B_{(\mathfrak{p})}$ is gfc.*

(3) *For any homogeneous prime ideal $\mathfrak{p}$, $A_{\mathfrak{p}}\subseteq B_{\mathfrak{p}}$ is fc.*

(4) *For any homogeneous maximal ideal $\mathfrak{m}$, $A_{(\mathfrak{m})}\subseteq B_{(\mathfrak{m})}$ is gfc.*

(5) *For any homogeneous maximal ideal* $\mathfrak{m}$, $A_{\mathfrak{m}} \subseteq B_{\mathfrak{m}}$ *is fc.*

Proof. Follows directly from Lemmas 7 and 8 and Theorem 2. □

In particular, if A_0 is a field, A is positively graded, $A_1 \neq (0)$ and $\mathfrak{m}$ is the irrelevant maximal ideal of A, then $A \subseteq B$ will be factorially closed if $A_{\mathfrak{m}} \subseteq B_{\mathfrak{m}}$ is factorially closed.

Given a subring A of an integral domain B, we define the factorially closed locus or fc locus of A in B to be $\mathfrak{FC}(A : B) := \{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}} \text{ is factorially closed}\}$. We intend to investigate the nature of this fc locus in the Zariski topology. We start with a few definitions: let $A/B := \{b \in B \mid bb' \in A \text{ for some } b' \in B - \{0\}\}$, which is an A-module. If $\bar{A}$ denotes the algebraic closure of A in B, it is easy to see that $A \subseteq Q(A) \cap B \subseteq \bar{A} \subseteq A/B \subseteq B$. Taking A to be the ring of integers $\mathbb{Z}$ and B to be $\mathbb{Z}[\sqrt{2}, x, y, 1/2y]$, one can see that the inclusions can be proper at each stage. By Lemma 1(2), factorial closedness is preserved under localization. So $\mathfrak{FC}(A : B)$ is closed under generalization. The following lemma gives a necessary and sufficient condition for the nonemptiness of the fc locus.

Lemma 10. *With notation as above, the following statements are equivalent*:

(1) *There exists a prime ideal* $\mathfrak{p} \in \operatorname{Spec} A$ *such that* $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ *is fc.*

(2) *The inclusion* $Q(A) \subseteq S^{-1}B$ *is fc, where* $S := A - \{0\}$.

(3) *There is an equality* $Q(A)^* = (S^{-1}B)^*$.

(4) *There is an equality* $A/B = Q(A) \cap B$.

Proof. We will only give the proof that (4) implies (2). The other implications are similar and easy.

Assume that $A/B = Q(A) \cap B$. We will show that $Q(A)$ is fc in $S^{-1}B$.

Let $(b_1/s_1) \cdot (b_2/s_2) \in Q(A)$, where the s_i are nonzero elements of A. Then there is a nonzero element $\alpha \in A$ such that $b_1 b_2 \alpha \in A$. This implies that $b_i \in A/B$, and hence $b_i \in Q(A)$. □

Note that $Q(A)^* = (S^{-1}B)^*$ implies that $B^* \subseteq Q(A)$. But the converse is false as the example $K[xz, yz] \subseteq K[x, y, z]$ with K a field shows.

To give conditions for the openness of the fc locus, we need a few auxiliary lemmas.

Lemma 11. *Let* $A \subseteq B$ *be integral domains. If* $\mathfrak{p} \in \operatorname{Spec} A$ *is a prime ideal of height* 1 *which is not in the image of* $\operatorname{Spec} B$, *then* $V(\mathfrak{p}) := \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{p} \subseteq \mathfrak{q}\}$ *does not meet* $\mathfrak{FC}(A : B)$.

Proof. Otherwise, if $\mathfrak{q} \in V(\mathfrak{p}) \cap \mathfrak{FC}(A:B)$, then $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is factorially closed by Lemma 1. But, since $\mathfrak{p}$ is not in the image of Spec B, each prime ideal of $B_{\mathfrak{p}}$ contracts to (0) in $A_{\mathfrak{p}}$. Consequently, every nonzero element of $A_{\mathfrak{p}}$ is a unit in $B_{\mathfrak{p}}$ which is a contradiction by Lemma 3. Hence we must have that $V(\mathfrak{p}) \cap \mathfrak{FC}(A:B) = \varnothing$. □

Lemma 12. *Let $A \subseteq B$ be integral domains with A noetherian and normal. If the image of* Spec B *contains all prime ideals* $\mathfrak{p} \in$ Spec A *of height* 1, *then either the fc locus $\mathfrak{FC}(A:B)$ is empty or A is factorially closed in B.*

Proof. If $A/B \neq Q(A) \cap B$, we know that the factorially closed locus will be empty. So we are interested in showing that, if $A/B = Q(A) \cap B$, then A is factorially closed in B. In order to prove factorial closedness, first note that it suffices to prove that any principal ideal of A is contracted from some ideal of B, or, equivalently, that $xB \cap A = xA$ for any $x \in A$. For, if it is true, let us consider $b_1, b_2 \in B - \{0\}$ such that $b_1 b_2 = a \in A$. Since $A/B = Q(A) \cap B$, $b_1 \in Q(A)$. Let $b_1 = \alpha/\beta$, with $\alpha, \beta \in A - \{0\}$. Now $\alpha \in \beta B \cap A = \beta A$, implying that $b_1 \in A$, and consequently A is factorially closed in B. So all we need to show is that any principal ideal of A is contracted from some ideal of B. But since A is a noetherian normal domain, any prime ideal associated to a principal ideal has height 1, and, as a result, using primary decomposition any principal ideal of A can be written as a finite intersection of primary ideals of height 1. So it is enough to prove that any height-1 primary ideal of A is a contracted ideal. Now, given any prime ideal $\mathfrak{p} \in$ Spec A of height 1, let us consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, hook] & B_{\mathfrak{p}} \\ A \arrow[u, hook] \arrow[r, hook] & A_{\mathfrak{p}} \arrow[u, hook] \end{tikzcd}$$

The local ring $A_{\mathfrak{p}}$ is a DVR and $\mathfrak{p}A_{\mathfrak{p}}$ is a contracted ideal. So each $\mathfrak{p}A_{\mathfrak{p}}$-primary ideal is also contracted. Again, $\mathfrak{p}$-primary ideals of A are in a one-to-one correspondence with the $\mathfrak{p}A_{\mathfrak{p}}$-primary ideals of $A_{\mathfrak{p}}$. So we conclude that each $\mathfrak{p}$-primary ideal of A is contracted from some ideal in B, and this completes the proof. □

Note. Let $A \subseteq B$ be integral domains with A noetherian. We have proved that if $A_{\mathfrak{p}}$ is a DVR for some $\mathfrak{p} \in$ Spec A then $\mathfrak{p} \in \mathfrak{FC}(A:B)$ if and only if it is in the image of Spec B.

Now we are in a position to characterize the openness of the fc locus:

Theorem 4. *Let $A \subseteq B$ be integral domains with A noetherian and normal. Then $\mathfrak{FC}(A:B)$, if nonempty, is open in* Spec A *if and only if the image of* Spec B *misses only finitely many height-*1 *prime ideals.*

Proof. If $\mathfrak{FC}(A:B)$ is open, its complement contains at most finitely many prime ideals of height 1. So, by Lemma 12, the image of Spec B misses at most finitely many height-1 prime ideals. Conversely, assume that $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ are the only prime ideals of height 1 lying outside the image of Spec B. In view of Lemma 12, it is enough to show that any $\mathfrak{q} \in \operatorname{Spec} A$ not contained in $\bigcup_{i=1}^n V(\mathfrak{p}_i)$ is in $\mathfrak{FC}(A:B)$. So, let us choose any $\mathfrak{q} \in \operatorname{Spec} A - \bigcup_{i=1}^n V(\mathfrak{p}_i)$. We can find $x \in \bigcap_{i=1}^n \mathfrak{p}_i - \mathfrak{q}$. Considering the inclusion $A_x \subseteq B_x$, all height-1 prime ideals of A_x are in the image of Spec B_x. Since we are only interested in the case when $\mathfrak{FC}(A:B) \neq \varnothing$, we may assume by Lemma 10 that $A/B = Q(A) \cap B$. Consequently $A_x/B_x = Q(A_x) \cap B_x$. Therefore, Lemma 11 applies to show that A_x is factorially closed in B_x. Hence $\mathfrak{FC}(A:B) = \operatorname{Spec} A - \left(\bigcup_{i=1}^n V(\mathfrak{p}_i)\right)$ is open. □

Corollary 4.1. *With notation as in Theorem 4, if B is a finitely generated algebra over A then $\mathfrak{FC}(A:B)$ is always open.*

Proof. This follows from Theorem 4, since the corresponding dominant morphism of affine schemes Spec $B \to$ Spec A always contains a nonempty open set in its image, and consequently the image of Spec B can miss at most finitely many height-1 primes of Spec A. □

2. Quasifactorially closed subrings

If we attempt to generalize Lemma 9 to the case of formal power series rings $A[\![x]\!] \subsetneq B[\![x]\!]$ the attempt fails quite badly. For, suppose that $A \subsetneq B$ is factorially closed. If $b \in B - A$, then the element $1 + bx + x^2 + x^3 + x^4 + \cdots$ is a unit in $B[\![x]\!]$ which is not in $A[\![x]\!]$. So the extension $A[\![x]\!] \subseteq B[\![x]\!]$ is never factorially closed unless $A = B$. The presence of 'extra units' in the bigger ring turns out to be an obvious obstruction. To rectify this problem, we come up with a weaker notion of quasifactorially closedness.

Recall that, given an inclusion of integral domains $A \subseteq B$, A is said to be quasifactorially closed, or qfc for short, in B if, for any nonzero $b \in B$, if there exists some nonzero $b' \in B$ such that $bb' \in A$, then there exists a unit $u \in B$ such that $bu \in A$.

If A and B have the same units then the notions of factorial closedness and quasifactorial closedness coincide. Also note that $A \subseteq B$ is quasifactorially closed whenever either A or B is a field. But quasifactorial closedness, in general, is more of a geometric notion and does not behave well with algebraic operations. For example, although it is closed under localization, we are not yet sure if global information can be retrieved from local data as in Lemma 1. The *sandwich property*, as in Lemma 2, also fails, as any integral domain is always quasifactorially closed in any field containing it. The following example shows that the *transitive property* need not hold true either.

Example. Factorial closedness holds for $K[x] \subseteq K[x, y, z, w]/(xy - zw)$, and $A := K[x, y, z, w]/(xy - zw)$ is qfc in $A[1/y]$. But $K[x]$ is not qfc in $A[1/y]$, as there is no unit u in $A[1/y]$ such that $uz \in K[x]$.

Note that the above example also shows that in the definition of quasifactorial closedness it may not be possible to get a unit $u \in B$ such that $bu, b'u^{-1} \in A$. In fact, if it were true then one can check that, for integral domains $A \subseteq B \subseteq C$, if $A \subseteq B$ is fc and $B \subseteq C$ is qfc then $A \subseteq C$ would also be qfc, which is clearly not true, as the above example shows.

The following example shows that $A \subseteq B$ being qfc does not imply that $A[x] \subseteq B[x]$ is qfc:

Example. Take any nontrivial algebraic field extension L/K. Then $K \subseteq L$ is qfc but $K[x] \subseteq L[x]$ is not qfc. If $K = \mathbb{R}$ and $L = \mathbb{C}$, take $b = ix - 1$ and $b' = ix + 1$. Then $bb' \in \mathbb{R}[x]$, but there is no unit $u \in \mathbb{C}[x]$ such that $bu \in \mathbb{R}[x]$.

Let $A \subseteq B$ be fc. Then any irreducible element of A remains irreducible in B. If A is a UFD but B is not, then prime elements of A need not remain prime in B, as the example $k[x] \subseteq k[x, y, z]/(xy - z^2 - 1)$ shows, where the prime element x of $k[x]$ does not remain a prime in $k[x, y, z]/(xy - z^2 - 1)$. But if B is also a UFD then $A \subseteq B$ is fc if and only if each prime element of A remains a prime in B and $A^* = B^*$. For UFDs $A \subseteq B$, primes of A remaining primes in B is a sufficient condition for qfc. But it is not necessary, as the first example in Section 4 will show. However, it follows from Theorem 6 and its corollaries that the converse is also true in the case of complete local UFDs.

For integral domains $A \subseteq B$, we define the qfc locus of A in B by $\mathfrak{QFC}(A : B) := \{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}} \text{ is qfc}\}$. Just like the fc locus, the qfc locus is also closed under generalization. Note that $\mathfrak{QFC}(A : B)$ is always nonempty since $(0) \in \mathfrak{QFC}(A : B)$. For, let $S = A - \{0\}$. If $(b_1/s_1).(b_2/c_2) \in Q(A)$ then b_i/s_i are units in $S^{-1}B$. Then $(b_i/s_i).(s_i/b_i) \in Q(A)$, implying that $Q(A)$ is qfc in $S^{-1}B$.

Next, we prove an openness criterion, analogous to Theorem 4, for the qfc locus, albeit for a somewhat restricted class of rings.

Theorem 5. *Let $A \subseteq B$ be affine UFDs. Assume that A and B have the same group of units and $Q(A)$ is algebraically closed in $Q(B)$. Then $\mathfrak{QFC}(A : B)$ is a nonempty open set if one, and hence all, of the following equivalent conditions hold*:

(1) *Given any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ of height ≥ 2, $\mathfrak{p}B$ has height ≥ 2.*

(2) *No prime element of B divides two distinct prime elements of A.*

(3) *Any two coprime elements of A continue to be coprime in B.*

(4) *There are only finitely many prime elements of A which either split in B or are units in B.*

Proof. Since A and B are UFDs, the proof of the equivalence of (1), (2) and (3) in the above statement is easy. This part does not need openness of $\mathfrak{QFC}(A:B)$.

Let V and W denote the irreducible affine varieties corresponding to A and B respectively, and let $f : W \to V$ be the induced morphism.

First we consider the case when $\dim A = 1$.

By assumption, $Q(A)$ is algebraically closed in $Q(B)$. Then it is well-known (by a suitable application of Bertini's theorem) that only finitely many scheme-theoretic fibers of the morphism $W \to V$ are either empty or not reduced and irreducible. This shows that (4) is also always true, so that conditions (1)–(4) are equivalent.

Now we will assume that $\dim A \geq 2$.

Assume now that the equivalent conditions (1), (2) and (3) hold. We will show that (4) holds.

Again, since $Q(A)$ is algebraically closed in $Q(B)$, there is a proper closed subvariety $S \subset V$ such that the inverse image of any point $p \notin S$ is scheme-theoretically reduced and irreducible. By (1), the inverse image of any closed subvariety of V of codim ≥ 2 does not contain any divisor in W. Now we can see that the only possible irreducible divisors $D \subset V$ which split in W are those contained in S.

The image $f(W)$ contains a nonempty Zariski-open subset since f is dominant. Hence $f(W)$ can miss at most finitely many divisors in V. This shows that (4) is true.

Next, we will show that (4) implies (1). Suppose that this is not true. Then there is a closed irreducible subvariety $S \subset V$ of codimension > 1 such that the inverse image of S in W contains an irreducible divisor Δ, defined by a prime element q. Now, if D is any irreducible divisor in V which contains S then the prime element defining D will split in B. Since $\dim B > 1$, there are infinitely many such prime elements in A.

This proves the equivalence of (1)–(4).

Now we will assume that the equivalent conditions (1)–(4) hold. We will show that $\mathfrak{QFC}(A:B)$ is a nonempty open set.

Let $p_1, \dots, p_r$ be the prime elements in A such that p_i is a non-unit in B and not a prime element in B.

We will show that $\mathfrak{QFC}(A:B) = \operatorname{Spec} A \setminus \bigcup_{i=1}^{r} V(p_i A)$, and hence $\mathfrak{QFC}(A:B)$ is nonempty and open.

First we will show that if a prime element $p \in A$ is not a unit in B and does not remain a prime element in B, then $V(pA) \cap \mathfrak{QFC}(A:B) = \varnothing$.

So, let $p \in A$ be such a prime element and let $\mathfrak{q} \in V(pA)$. If $A_{\mathfrak{q}} \subseteq B_{\mathfrak{q}}$ is qfc then so is $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$, where $\mathfrak{p} := pA$. Since p is not a prime element in B, there exists $b_1, b_2 \in B - B^*$ such that $p = b_1 b_2$. But $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ being qfc implies that either b_1 or b_2 must be a unit in $B_{\mathfrak{p}}$. Without any loss of generality, let us assume that $b_1 \in B_{\mathfrak{p}}^*$. So b_1 divides some element $s \in A - \mathfrak{p}$. But s and p are coprime in A, and hence in B by (3). So b_1 must be a unit in B, which is a contradiction.

By assumption, there are only finitely many prime elements $p_i \in A$ such that p_i is a non-unit in B and not a prime element in B. We have already seen that $\bigcup_{i=1}^{n} V(p_i A) \subseteq \mathfrak{QFC}(A:B)^c$. So it suffices to show that any prime ideal of A which does not contain any of the p_i is in $\mathfrak{QFC}(A:B)$. Let $\mathfrak{q} \in \operatorname{Spec} A$ be such a prime ideal. Choose any $a \in \bigcap_{i=1}^{n} p_i A - \mathfrak{q}$. Then $A_a \subseteq B_a$ is qfc since each prime element in A_a continues to be a prime element in B_a. Consequently, $A_{\mathfrak{q}} \subseteq B_{\mathfrak{q}}$ is also qfc, and this completes the proof. □

Remark. In Section 3, we will give an example to show that $\mathfrak{QFC}(A:B)$ can be open and nonempty even when there are infinitely many prime elements in A which are not units in B and are not prime elements in B.

The following theorem shows that the qfc property has some nice consequences in the case of complete local domains:

Theorem 6. *Let $(A, \mathfrak{m}_A) \subseteq (B, \mathfrak{m}_B)$ be local domains such that $\mathfrak{m}_A = \mathfrak{m}_B \cap A$ and $A/\mathfrak{m}_A = B/\mathfrak{m}_B$. Moreover, assume that A is complete in the $\mathfrak{m}_A$-adic topology and $\bigcap_{n=1}^{\infty} \mathfrak{m}_B^n = (0)$. If A is qfc in B then A is algebraically closed in B.*

Proof. Let $b \in B$ be algebraic over A. We will construct a sequence $(a_n) \in A^{\mathbb{N}}$ such that, for each n, $a_{n+1} = a_n + \alpha_1\alpha_2\cdots\alpha_n\alpha_{n+1}$ and $b = a_n + \alpha_1\alpha_2\cdots\alpha_n\beta_n$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1} \in \mathfrak{m}_A$ and $\beta_n \in \mathfrak{m}_B$. Any such sequence will be a Cauchy sequence in the $\mathfrak{m}_A$-adic topology of A which converges to b in the $\mathfrak{m}_B$-adic topology of B, implying that $b \in A$.

Since b is algebraic over A, there exist elements $c_0, c_1, \ldots, c_r \in A$, with c_0 and c_r nonzero, such that

$$c_0 + c_1 b + \cdots + c_r b^r = 0,$$

implying that $b(c_1 + c_2 b + \cdots + c_r b^{r-1}) \in A$. Since A is qfc in B, there exists a unit $u \in B^*$ such that $bu = a \in A$ or, equivalently, $b = au^{-1}$. We can write $u^{-1} = u_1' + b_1$ for some $u_1' \in A^*$ and $b_1 \in \mathfrak{m}_B$, so that $b = a(u_1' + b_1)$. Setting $a_1 := au_1'$, the induction hypothesis is satisfied for $n = 1$.

Next, suppose that we have already found elements $a_1, a_2, \ldots, a_n \in A$ satisfying the required conditions. To find a_{n+1}, note that $b = a_n + \alpha_1\alpha_2\cdots\alpha_n\beta_n$, implying that $\alpha_1\alpha_2\cdots\alpha_n\beta_n$ is also algebraic over A, and consequently there exists a unit $u_{n+1} \in B^*$ such that $\beta_n = \alpha_{n+1}u_{n+1}$ for some $\alpha_{n+1} \in \mathfrak{m}_A$. Writing u_{n+1} as $u_{n+1} = u_{n+1}' + \beta_{n+1}$, where $u_{n+1}' \in A^*$ and $\beta_{n+1} \in \mathfrak{m}_B$, we get

$$b = a_n + \alpha_1\alpha_2\cdots\alpha_n\alpha_{n+1}(u_{n+1}' + \beta_{n+1}).$$

It is obvious that $a_{n+1} := a_n + \alpha_1\alpha_2\cdots\alpha_n\alpha_{n+1}u_{n+1}'$ satisfies the required properties. This, together with induction, completes the proof. □

With notation as in Theorem 6, we have the following easy corollaries:

Corollary 6.1. *There is an equality $Q(A) \cap B = A$.*

Corollary 6.2. *If B is normal, then so is A.*

Corollary 6.3. *If B satisfies the ascending chain condition for principal ideals, then so does A.*

Corollaries 6.1, 6.2 and 6.3 are immediate consequence of Theorem 6.

Corollary 6.4. *Any irreducible element of A remains irreducible in B.*

This can be proved in the same way as the proof of Theorem 6. We leave the details to the reader.

Corollary 6.5. *If $a \in A$ is a prime element of B, then it is already a prime element in A.*

Corollary 6.6. *Let $a, a' \in A$. Then $aA = a'A$ if and only if $aB = a'B$. In particular, if two elements of A are not associates in A, they cannot become associates in B.*

This follows easily from Corollary 6.1. For, if $a' \in aB$, then, writing $a' = ab$ with $b \in B$, by Corollary 6.1 we have $b \in Q(A) \cap B = A$.

Corollary 6.7. *If B is a UFD, then so is A.*

Proof. Corollary 6.3 shows that any element in A can be written as a product of irreducible elements in A. By Corollary 6.4, any irreducible element in A remains irreducible and hence a prime in B. Now Corollary 6.5 finishes the proof. □

Corollary 6.8. *If two elements of A have no common factor in A, they cannot have a common factor in B.*

Proof. If $b \in \mathfrak{m}_B$ is a common factor of $a, a' \in \mathfrak{m}_A$, then there is a unit $u \in B^*$ such that $bu \in \mathfrak{m}_A$. But then, by Corollary 6.1, bu is a common factor of a and a' in A, leading to a contradiction. □

Corollary 6.9. *If B is a UFD, then, for any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ of height ≥ 2, $\mathfrak{p}B$ has height ≥ 2.*

Now we prove an analogue of Theorem 1 for power series rings. First, we consider the 2-dimensional case.

Theorem 7. *Let $k \subsetneq A \subsetneq B := k[[x, y]]$ be a noetherian complete (with respect to its maximal ideal) local qfc subring of B, the power series ring in two variables. Then A is isomorphic to a power series ring in one variable over k.*

Proof. By Corollary 6.7, A is a UFD. Let $p \in A$ be a prime element. By Corollary 6.4, p is a prime element in B, and since A is qfc in B we have $pB \cap A = pA$. This gives an inclusion of integral domains $A/pA \subseteq B/pB$. Now $\dim B/pB = 1$, and hence any two elements in B/pB are analytically dependent. Thus, $\dim A/pA \leq 1$. Now $\dim A \leq 2$. If $\dim A = 1$, then A is clearly isomorphic to a power series ring in one variable over k.

Now assume that $\dim A = 2$. We will show that $A = k[[x, y]]$. For that, choose any two relatively prime elements u, v from the maximal ideal of A. By Corollaries 6.4 and 6.6, u and v are nonassociate prime elements of $k[[x, y]]$. So, in particular, the extension of the maximal ideal of A to B is (x, y)-primary. Since A, B are complete, we infer that B is integral over A. But then by Theorem 6 A must be equal to $k[[x, y]]$. □

Next we consider the case when $\dim B = 3$.

Theorem 8. *Let $k \subseteq A \subseteq B := k[[X, Y, Z]]$, where A is a 2-dimensional noetherian complete (with respect to its maximal ideal) local qfc subring of B. Then A is isomorphic to a power series ring in two variables over k.*

Proof. The proof is similar to the proof of Theorem 7.

Note that A is a UFD by Corollary 6.7.

By Brieskorn's theorem [1968], either A is isomorphic to a power series in two variables over k, or $A \cong k[[u, v, w]]/(u^2 + v^3 + w^5)$. We have to show that A cannot be isomorphic to $k[[u, v, w]]/(u^2 + v^3 + w^5)$. By the argument in the proof of Theorem 7, the extended ideal $(\bar{u}, \bar{v}, \bar{w})B$ has height > 1. We know that A is the ring of invariants of the binary icosahedral group of order 120 acting on a power series ring $k[[s, t]]$. The morphism $\operatorname{Spec} k[[s, t]] \setminus \{(s, t)\} \to \operatorname{Spec} A \setminus \{(\bar{u}, \bar{v}, \bar{w})\}$ is finite unramified. Since $\operatorname{Spec} B \setminus V((\bar{u}, \bar{v}, \bar{w}))$ is simply connected, by covering space theory we have a factorization

$$\operatorname{Spec} B \setminus V((\bar{u}, \bar{v}, \bar{w})) \to \operatorname{Spec}\ k[[s, t]] \setminus \{(s, t)\} \to \operatorname{Spec} A \setminus \{(\bar{u}, \bar{v}, \bar{w})\}.$$

By Hartog's theorem, we have $A \subset k[[s, t]] \subseteq B$. But then A is not algebraically closed in B, contradicting Theorem 6. This shows that A is isomorphic to a power series ring in two variables over k. □

Question. In Theorem 8, is the assumption $\dim A = 2$ necessary, i.e., can a proper qfc subring of $k[[x, y, z]]$ have dimension > 2?

It is well-known that, if $A \subseteq B$ are affine normal domains over an algebraically closed field of characteristic 0 such that $Q(A)$ is algebraically closed in $Q(B)$, then a general fiber of the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is irreducible. By Theorem 6, if $A \subseteq B$ are complete normal domains over an algebraically closed field of characteristic 0 such that A is qfc in B, then $Q(A)$ is algebraically closed in $Q(B)$. In view of the above observation we can ask the following question:

Question. Let (V, p), (W, q) be normal complex analytic germs and $f : (W, q) \to (V, p)$ a complex analytic morphism such that the analytic local ring of V is algebraically closed in that of W. Is a general fiber of f irreducible?

We have the following modest result as an affirmative answer to this question:

Theorem 9. *Let (W, q) be a normal complex analytic germ and $f : W \to \mathbb{C}$ a complex analytic morphism of germs. Assume that the ring $\mathbb{C}\{f\} \subset \mathcal{O}_{W,q}$ is qfc. Then a general fiber of f is connected.*

Proof. We will use a result of Tráng [1977] on the topology of singular points, which generalizes Milnor's results.

Tráng [1977] proved that there are positive numbers $0 < \delta \ll \epsilon \ll 1$ such that if D is a disc of radius δ in $\mathbb{C}$ then the morphism $B_\epsilon \cap W \cap f^{-1}(D - \{0\}) \to D - \{0\}$ is a topological fiber bundle, where B_ϵ is a ball of radius ϵ with center q in $\mathbb{C}^n$, such that $(W, q) \subseteq (\mathbb{C}^n, 0)$ is a closed embedding of germs. Since $\mathbb{C}\{f\} \subset \mathcal{O}_{W,q}$ is qfc, the fiber $\{f = 0\}$ is irreducible by Corollary 6.4. We have a long exact sequence of homotopy groups

$$\pi_1(F) \longrightarrow \pi_1(B_\epsilon \cap W \cap f^{-1}(D - \{0\})) \longrightarrow \pi_1(D - \{0\}) \longrightarrow \pi_0(F) \\ \longrightarrow \pi_0(B_\epsilon \cap W \cap f^{-1}(D - \{0\})) \longrightarrow \pi_0(D - \{0\}) \longrightarrow (1).$$

Here F is a general fiber of $B_\epsilon \cap W \cap f^{-1}(D - \{0\}) \to D - \{0\}$. Both $B_\epsilon \cap W \cap f^{-1}(D - \{0\})$ and $D - \{0\}$ are connected. Since $\{f = 0\}$ is reduced and irreducible, a small transverse loop in $B_\epsilon \cap W \cap f^{-1}(D - \{0\})$ maps onto the generator of the fundamental group of $D - \{0\}$, hence the homomorphism

$$\pi_1(B_\epsilon \cap W \cap f^{-1}(D - \{0\})) \to \pi_1(D - \{0\})$$

is surjective. It follows that F is connected, and this proves the result. □

The next result is an interesting consequence of the property of being factorially closed. To state the result, we need a definition. Let $f : Y \to X$ be a dominant morphism of smooth algebraic varieties such that the general fibers are irreducible and reduced. Then there exist an open immersion $\iota : Y \hookrightarrow W$ and a projective morphism $\overline{f} : W \to X$ such that $f = \overline{f} \circ \iota$, where W is a smooth algebraic variety and $D := W \setminus Y$ is a divisor with simple normal crossings. Let $D = D_1 + \cdots + D_r$ be the irreducible decomposition. We further assume that D intersects transversally the fiber $F_P = \overline{f}^{-1}(P)$ for every closed point $P \in X$. We say that $\overline{f}$ is an SNC-completion of f. Suppose that for every $P \in X$ and every $1 \le i \le r$ the intersection $D_i \cdot F_P$ is irreducible and reduced. If there exists such an SNC-completion of f, we say that f is *fiberwise integral at infinity*. If there is an open set U of X such that $f : f^{-1}(U) \to U$ has a completion which is fiberwise integral at infinity, then we say that f is *generically fiberwise integral at infinity*. This condition is equivalent to saying that the generic fiber Y_η, with η the generic point of X, can be embedded into a projective smooth variety W_η defined over the field $k(\eta)$ in such a way that $D_\eta = W_\eta \setminus Y_\eta$ is a divisor consisting of geometrically integral smooth components with simple normal crossings.

Theorem 10. *Let A be an affine domain of dimension* 1 *over k. Assume that A is fc in a regular affine domain R over k. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} R$ and $f : Y \to X$ be the induced morphism. Assume that f has an SNC-completion which is generically fiberwise integral at infinity. Then there is a maximal ideal $\mathfrak{m}$ of A such that the affine domain $R/\mathfrak{m}R$ is regular and has no nontrivial units.*

Proof. Since A is fc in R and R is normal, it follows that A is normal, and hence regular as $\dim A = 1$. A general fiber of the morphism f is reduced and irreducible. In particular, by Bertini's theorem, $R/\mathfrak{m}R$ is a regular affine domain for all but finitely many maximal ideals in A. Removing from X the closed points corresponding to these maximal ideals, we may assume that f is a smooth morphism. Let $\overline{f} : W \to X$ be an SNC-completion which is fiberwise integral at infinity. Here we may have to replace X by a suitable open set. Let $D = W \setminus Y$ be the divisor at infinity and let $D = D_1 + D_2 + \cdots + D_r$ be the irreducible decomposition of D. If the result is not true, then we may assume that $R/\mathfrak{m}R$ has a nontrivial unit for every maximal ideal $\mathfrak{m}$ of A. Note that, by definition, each D_i meets each fiber F_P of $\overline{f}$ transversally and the intersection $D_i \cdot F_P$ is integral, i.e., irreducible and reduced.

Let P be a closed point of X. The fiber $f^{-1}(P)$ has a nonconstant unit u_P, and the divisor (u_P) in $F_P := \overline{f}^{-1}(P)$ has the form $\sum_i a(P)_i D_i|_{F_P}$ with $a(P)_i \in \mathbb{Z}$. Note that the subgroup $\sum_i \mathbb{Z} D_i$ of $\operatorname{Pic}(W)$, which is generated by the irreducible components of D, is a countable group. Choosing a nonconstant unit u_P for every $P \in X(k)$, we have a mapping $P \mapsto (u_P)$ from $X(k)$ to the group $\sum_i \mathbb{Z} D_i$, where $X(k)$ is the set of closed points of X and (u_P) is identified with $\sum_i a(P)_i D_i$. Since each $D_i \cap F_P$ is irreducible and reduced for each i, such an identification is possible. Then we can find a fixed divisor $D_0 = \sum_i a_i D_i$ and an infinite set Λ of $X(k)$ such that $D_0 \cdot F_P = (u_P)$ for each P of Λ. This means that the line bundle $\mathbb{O}(D_0)$ on W restricts to a trivial line bundle on F_P for each $P \in \Lambda$. By the upper-semicontinuity theorem [Hartshorne 1977, Chapter III, Theorem 12.8], the set of points in X such that the restriction of D_0 to F_P is trivial is a closed subvariety T of X containing the infinite set Λ. (Use the theorem for $\mathscr{L}$ and $\mathscr{L}^{-1}$ so that $\dim_k H^0(F_P, \mathscr{L}|_{F_P}) \geq 0$ and $\dim_k H^0(F_P, \mathscr{L}^{-1}|_{F_P}) \geq 0$.) Since $\dim X = 1$, $T = X$ and D_0 restricts to a trivial line bundle on every fiber of $\overline{f}$. By [Hartshorne 1977, Chapter III, Exercise 12.4], D_0 is linearly equivalent to the pullback by $\overline{f}$ of a divisor of the form $\sum_{j=1}^{s} b_j Q_j$ on X. Thus, the restriction of D_0 to $\overline{f}^{-1}(X \setminus \{Q_1, \ldots, Q_s\})$ is linearly equivalent to zero. Write D_0 as the divisor of a rational function (φ) on $\overline{f}^{-1}(X \setminus \{Q_1, \ldots, Q_s\})$. Then φ gives a nonconstant unit of $f^{-1}(X \setminus \{Q_1, \ldots, Q_s\})$. Since the units on $f^{-1}(X \setminus \{Q_1, \ldots, Q_s\})$ and $X \setminus \{Q_1, \ldots, Q_s\}$ are the same by the assumption of factorial closedness, φ is constant on each fiber of f. However, φ restricts onto the unit u_P up to a nonzero constant for every $P \in \Lambda$. This is a contradiction because u_P is not a constant. □

Without the assumption that f has an SNC-completion which is fiberwise integral at infinity, Theorem 10 does not hold.

Example. Let W be the Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$ with vertical and horizontal $\mathbb{P}^1$-fibrations. Let $p_1 : W \to \mathbb{P}^1$ be the vertical one with a fiber L, and let the horizontal one p_2 be given by a linear system $|M|$. Let D_1 be an irreducible curve such that $D_1 \sim 2M + L$. Then the restriction $p_1|_{D_1} : D_1 \to \mathbb{P}^1$, being a double covering, has two branch points. Let L_1, L_2 be two fibers of p_1 over these branch points. Let $D = D_1 + L_1 + L_2$, and let $Y := W \setminus D$ and $X := \mathbb{P}^1 - \{\text{two branch points}\}$. Let $f = p_1|_Y$. Then every fiber of $f : Y \to X$ is irreducible; hence $k(X)$ is algebraically closed in $k(Y)$ and it is easy to see that A is factorially closed in R, where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} R$, because $A = k[t, t^{-1}]$ and every prime element of A is $t - c$ with some nonzero constant $c \in k$. Then the fiber over $t = c$ is irreducible. Hence $t - c$ is a prime element in R. Furthermore, the units of R are the same as the units of A because the only linear relation among the components of D is the one between L_1 and L_2. But every closed fiber of f has a nontrivial unit because it is isomorphic to $\mathbb{A}^1_*$. Note that R is not factorial since $\operatorname{Pic}(R) = \mathbb{Z}/2\mathbb{Z}$.

Remark. In this example, the fibration $f : Y \to X$ is a twisted $\mathbb{A}^1_*$-fibration. Let X' be the curve D_1 with two ramifying points for $p_1|_{D_1}$ removed, and let $f' : Y' \to X'$ be the base change of f by $X' \to X$. Then $Y' \cong A^1_* \times \mathbb{A}^1_*$. Write $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} A'$. Then A is fc in R, but A' is not fc in $R' := R \otimes_A A'$. In fact, $R'^*/k^* \cong \mathbb{Z} \times \mathbb{Z}$ and $A'^*/k^* \cong \mathbb{Z}$. If A' were fc in R', then we must have $R'^* = A'^*$. Note that A'/A is a finite étale extension. Hence the factorial closedness is not preserved even by an étale base change.

Using Theorem 10 we can now give a very short proof of a result of [Neumann and Norbury 1998].

Theorem 11. *Let $f, g \in \mathbb{C}[X, Y]$ be a pair of polynomials in two variables with nonzero constant Jacobian determinant. Suppose that the following conditions are satisfied*:

(a) *For all $c \in \mathbb{C}$, the polynomial $f - c$ is irreducible and defines a rational curve.*

(b) *Let $\mathbb{C}^2 \subset Y$ be an open embedding in a smooth quasiprojective surface such that $f : \mathbb{C}^2 \to \mathbb{C}$ extends to a proper morphism $Y \to \mathbb{C}$ and $Y \setminus \mathbb{C}^2$ is a simple normal crossing divisor such that each irreducible component of $Y \setminus \mathbb{C}^2$ is a cross-section of the morphism $Y \to \mathbb{C}$.*

Then $\{f = 0\} \cong \mathbb{C}$, and hence the Jacobian Conjecture is true for the pair (f, g).

Remark. In [Neumann and Norbury 1998] f is called a simple rational polynomial.

Proof. By assumption, $f-c$ is an irreducible polynomial for all constants c. Also, $\mathbb{C}^2$ has no nonconstant invertible regular functions. Hence the morphism $\mathbb{C}^2 \to \mathbb{C}$ is an fc morphism. Condition (b) implies that $f : \mathbb{C}^2 \to \mathbb{C}$ is generically fiberwise integral at infinity. By Theorem 10 and condition (a), for all but finitely many $c \in \mathbb{C}$ the affine curve $\{f = c\}$ is smooth rational irreducible with no nonconstant invertible regular functions. Hence it is isomorphic to $\mathbb{C}$. By the Abhyankar–Moh–Suzuki theorem, after a suitable automorphism of $\mathbb{C}[X, Y]$ the polynomial f is mapped onto X. It is well-known that this implies that the Jacobian Conjecture is true for (f, g). □

The next result is another interesting example of qfc subrings.

Theorem 12. *Let $A = \mathbb{C}\{z_1, z_2, \dots, z_n\}/P$ be an analytic local domain which is a UFD. Then A is qfc in $\hat{A}$.*

Proof. We use Artin's approximation theorem [1968].

It is known that $\hat{A}$ is also a UFD. To show that A is qfc in $\hat{A}$, it follows easily from the definition of a qfc subring that it is enough to show that any prime element of A remains a prime element in $\hat{A}$.

Suppose that $f \in A$ is a prime element. Assume that there are non-units g, h in $\hat{A}$ such that $f = gh$. Let P be generated by $f_1, f_2, \dots, f_r$. Let $w_1, w_2, \dots, w_r$, w_{r+1}, w_{r+2} be new indeterminates. Consider the system of equations in the variables $z_1, \dots, z_n, w_1, w_2, \dots, w_{r+2}$

$$f_1 - w_1 = 0 = f_2 - w_2 = \cdots = f_r - w_r = f - w_{r+1}w_{r+2}.$$

This system has solutions $w_1 = f_1, \dots, w_r = f_r$, $w_{r+1} = g$, $w_{r+2} = h$ in $\mathbb{C}[[z_1, \dots, z_n]]$. By Artin's theorem, we can find solutions g_0, h_0 in $\mathbb{C}\{z_1, \dots, z_n\}$ such that $f = g_0h_0$ modulo P and g_0, h_0 approximate g, h to any order. In particular, g_0 and h_0 cannot be units. Thus, every prime element in A remains a prime element in $\hat{A}$, and consequently A is qfc in $\hat{A}$. □

Corollary 12.1. *Any element of $\mathbb{C}[[z_1, \dots, z_n]]$ which is algebraic over $\mathbb{C}\{z_1, \dots, z_n\}$ is itself convergent.* (Here, $z_1, z_2, \dots, z_n$ are indeterminates over $\mathbb{C}$.)

Question. Is Theorem 12 valid without assuming that A is a UFD?

3. Examples

We give some examples which shed more light on fc and qfc extensions.

(1) The following example shows that a qfc extension $A \subseteq B$ of local domains can be quite strange if A is not complete.

Let k be any field. Consider $A := k[x, e^x - 1]_{(x, e^x - 1)}$ and $B := k[[x]]$. Then $A \subseteq B$ is a local inclusion of local UFDs. Note that A is a regular local ring of dimension 2, whereas B has only one nonzero prime ideal, namely (x). From these observations we can easily deduce the following properties:

(a) Infinitely many primes of A split in B.

(b) Infinitely many distinct primes of A become associates in B.

(c) The dimension of A is bigger than the dimension of B.

(d) A prime element of B, namely x, divides infinitely many distinct prime elements of A.

(2) The properties of being a qfc extension and a flat extension are independent.

For, a ring of invariants A of a semisimple group acting on a polynomial ring B is fc in B, but the extension is not flat in general.

On the other hand, the extension $k[t^2] \subseteq k[t]$ is flat but not qfc.

(3) For an extension of normal affine domains $A \subseteq B$, the set of points $\mathfrak{m} \in \operatorname{Max} B$ such that $A_{\mathfrak{m}\cap A} \subseteq B_{\mathfrak{m}}$ is qfc is in general not Zariski-open in $\operatorname{Max} B$.

An example of this is the inclusion $A := k[x] \subseteq B := k[x, y, z]/(xy - z^2)$. If $\mathfrak{m}_0$ is the maximal ideal corresponding to the origin $(0, 0, 0)$ in B, then $A_{(x)}$ is qfc in $B_{\mathfrak{m}_0}$, but for maximal ideals corresponding to nearby points $(0, \lambda, 0)$ this is not true.

Remark. One may ask a similar question for fc extensions. But at least in the case of affine domains it is not very interesting, for then by Lemma 4 $A_{\mathfrak{m}\cap A} = B_{\mathfrak{m}}$. So A and B must be birational where the set of such points is clearly open.

(4) The ring extension $A := k[xy] \subseteq B := k[x, y]$ is such that A is algebraically closed in B and $A^* = B^*$, but the fc locus $\mathfrak{FC}(A : B)$ is empty.

(5) The ring extension $A := k[x] \subset B := k[x, y, z]/(x^2 + y^2 + z^2 - 1)$ has the property that any irreducible element of A remains irreducible in B, the extension is faithfully flat and both rings have same units, but A is not fc in B. Note that B is not factorial.

(6) Let $A := k[x, xy] \subset B := k[x, y]$, where x, y are indeterminates. Then $Q(A) \subset Q(B)$ is maximally algebraic. Any element of the form $x + axy$ is a prime element in A but not a prime element in B, where $a \in k^*$.

If $\mathfrak{q}$ is any prime ideal in A other than (x, xy), then either x or xy is a unit in $A_{\mathfrak{q}}$. Hence both x, y are units in $B_{\mathfrak{q}}$. It follows that any prime element in $A_{\mathfrak{q}}$ is either a prime element in $B_{\mathfrak{q}}$ or a unit in $B_{\mathfrak{q}}$. This shows that $\mathfrak{FC}(A : B) = \operatorname{Spec} A \setminus \{(x, xy)\}$ is nonempty and open, but infinitely many prime elements in A are non-units in B and are not prime elements in B.

If $\mathfrak{p}$ is any height-1 prime ideal in A, then at least one of x, xy does not lie in $\mathfrak{p}$. Hence, in $B_{\mathfrak{p}}$, x is always a prime element. From this we see that $A_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ is qfc.

Clearly A is not fc in B. Since A, B are UFDs and have the same units, A is not qfc in B.

This shows that the local analogue of Lemma 1(6) does not hold for the qfc property.

4. Open problems

(1) Let $A \subseteq B$ be normal complete local domains over k such that A is qfc in B. Is the power series ring in one variable $A[[x]]$ qfc in $B[[x]]$?

(2) Suppose that $A \subseteq B$ are normal affine domains such that for any maximal ideal $\mathfrak{m} \subset A$ the extension $A_\mathfrak{m} \subseteq B_\mathfrak{m}$ is qfc. Is A qfc in B?

(3) Let $A \subseteq B$ be an qfc inclusion of normal complete domains over k. Is $\dim A \leq \dim B$?

(4) Is any fc subring of a PID also a PID?

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sagnik@math.tifr.res.in *School of Mathematics, Tata Institute of Fundamental Research, Homi-Bhabha Road, Navy Nagar, Mumbai 400005, India*

gurjar@math.tifr.res.in *School of Mathematics, Tata Institute of Fundamental Research, Homi-Bhabha Road, Navy Nagar, Mumbai 400005, India*

miyanisi@amber.plala.or.jp *Research Center for Mathematical Sciences, Kwansei Gakuin University, 2-1, Gakuen, Sanda 669-1337, Japan*

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