Algebra & Number Theory

Volume 9
2015
No. 8

G-valued crystalline representations with minuscule p-adic Hodge type

Brandon Levin

msp

ALGEBRA AND NUMBER THEORY 9:8 (2015)
dx.doi.org/10.2140/ant.2015.9.1741

G-valued crystalline representations with minuscule p-adic Hodge type

Brandon Levin

We study G-valued semistable Galois deformation rings, where G is a reductive group. We develop a theory of Kisin modules with G-structure and use this to identify the connected components of crystalline deformation rings of minuscule p-adic Hodge type with the connected components of moduli of "finite flat models with G-structure". The main ingredients are a construction in integral p-adic Hodge theory using Liu's theory of $(\varphi, \hat{G})$-modules and the local models constructed by Pappas and Zhu.

1. Introduction

1.1. ***Overview.*** One of the principal challenges in the study of modularity lifting or, more generally, automorphy lifting via the techniques introduced in [Taylor and Wiles 1995] is understanding local deformation conditions at $\ell = p$. Kisin [2009] introduced a ground-breaking new technique for studying one such condition, flat deformations, which led to better modularity lifting theorems. Kisin [2008] extended those techniques to construct potentially semistable deformation rings with specified Hodge–Tate weights. In this paper, we study Galois deformations valued in a reductive group G and extend Kisin's techniques to this setting. In particular, we define and prove structural results about "flat" G-valued deformations.

Let G be a reductive group over a $\mathbb{Z}_p$-finite flat local domain Λ with connected fibers. Let $\mathbb{F}$ be the residue field of Λ and $F := \Lambda[1/p]$. Let $K/\mathbb{Q}_p$ be a finite extension with absolute Galois group Γ_K and fix a representation $\bar{\eta}: \Gamma_K \to G(\mathbb{F})$. The

MSC2010: primary 11S20; secondary 14L15, 14F30.
Keywords: Galois representation, p-adic Hodge theory, finite flat group scheme, local model.

(framed) G-valued deformation functor is represented by a complete local Noetherian Λ-algebra $R^{\square}_{G,\bar{\eta}}$. For any geometric cocharacter μ of $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p}F)/F}G_F$, there exists a quotient $R^{\mathrm{st},\mu}_{\bar{\eta}}$ (resp. $R^{\mathrm{cris},\mu}_{\bar{\eta}}$) of $R^{\square}_{G,\bar{\eta}}$ whose points over finite extensions F'/F are semistable (resp. crystalline) representations with p-adic Hodge type μ (see [Balaji 2012, Theorem 3.0.12]).

When $G=\mathrm{GL}_n$ and μ is minuscule, $R^{\mathrm{cris},\mu}_{\bar{\eta}}$ it appears the thesis changed is a quotient of a flat deformation ring. For modularity lifting, it is important to know the connected components of $\mathrm{Spec}\, R^{\mathrm{cris},\mu}_{\bar{\eta}}[1/p]$. Intuitively, Kisin's [2009] technique is to resolve the flat deformation ring by "moduli of finite flat models" of deformations of $\bar{\eta}$. When $K/\mathbb{Q}_p$ is ramified, the resolution is not smooth, but its singularities are relatively mild, which allowed for the determination of the connected components in many instances when $G=\mathrm{GL}_2$ [Kisin 2009, Propositions 2.5.6 and 2.5.15]. Kisin's technique extends beyond the flat setting (for μ arbitrary), where one resolves deformation rings by moduli spaces of integral p-adic Hodge theory data called $\mathfrak{S}$-modules of finite height, also known as *Kisin modules*.

In this paper, we define a notion of Kisin module with G-structure or, as we call them, *G-Kisin modules* (Definition 2.2.7) and we construct a resolution

$$\Theta : X^{\mathrm{cris},\mu}_{\bar{\eta}} \to \mathrm{Spec}\, R^{\mathrm{cris},\mu}_{\bar{\eta}},$$

where Θ is a projective morphism and $\Theta[1/p]$ is an isomorphism (see Propositions 2.3.3 and 2.3.9). The same construction works for $R^{\mathrm{st},\mu}_{\bar{\eta}}$ as well. The goal then is to understand the singularities of $X^{\mathrm{cris},\mu}_{\bar{\eta}}$. The natural generalization of the flat condition for GL_n to an arbitrary group G is *minuscule* p-adic Hodge type μ. A cocharacter μ of a reductive group H is minuscule if its weights when acting on $\mathrm{Lie}\, H$ lie in $\{-1,0,1\}$ (see Definition 4.1.1 and discussion afterward). Our main theorem is a generalization of the main result of [Kisin 2009] on the geometry of $X^{\mathrm{cris},\mu}_{\bar{\eta}}$ for G reductive and μ minuscule:

Theorem 4.4.1. *Assume $p\nmid \pi_1(G^{\mathrm{der}})$, where G^{der} is the derived subgroup of G. Let μ be a minuscule geometric cocharacter of $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p}F)/F}G_F$. Then $X^{\mathrm{cris},\mu}_{\bar{\eta}}$ is normal and $X^{\mathrm{cris},\mu}_{\bar{\eta}}\otimes_{\Lambda_{[\mu]}}\mathbb{F}_{[\mu]}$ is reduced, where $\Lambda_{[\mu]}$ is the ring of integers of the reflex field of μ.*

When $G=\mathrm{GSp}_{2g}$, this is a result of Broshi [2008]; also, this is a stronger result than in [Levin 2013], where we placed a more restrictive hypothesis on μ (see Remark 1.1.1). The significance of Theorem 4.4.1 is that it allows one to identify the connected components of $\mathrm{Spec}\, R^{\mathrm{cris},\mu}_{\bar{\eta}}[1/p]$ with the connected components of the fiber in $X^{\mathrm{cris},\mu}_{\bar{\eta}}$ over the closed point of $\mathrm{Spec}\, R^{\mathrm{cris},\mu}_{\bar{\eta}}$, a projective scheme over $\mathbb{F}_{[\mu]}$ (see Corollary 4.4.2). This identification led to the successful determination of the connected components of $\mathrm{Spec}\, R^{\mathrm{cris},\mu}_{\bar{\eta}}[1/p]$ in the case when $G=\mathrm{GL}_2$ [Kisin 2009; Gee 2006; Imai 2010; 2012; Hellmann 2011]. Outside of GL_2,

relatively little is known about the connected components of these deformations rings without restricting the ramification in K.

When $K/\mathbb{Q}_p$ is unramified, we have a stronger result:

Theorem 4.4.6. *Assume $K/\mathbb{Q}_p$ is unramified, $p > 3$, and $p \nmid \pi_1(G^{\mathrm{ad}})$. Then the universal crystalline deformation ring $R_{\bar{\eta}}^{\mathrm{cris},\mu}$ is formally smooth over $\Lambda_{[\mu]}$. In particular*, Spec $R_{\bar{\eta}}^{\mathrm{cris},\mu}[1/p]$ *is connected.*

Remark 1.1.1. In [Levin 2013], we made the assumption on the cocharacter μ that there exists a representation $\rho : G \to \mathrm{GL}(V)$ such that $\rho \circ \mu$ is minuscule. This extra hypothesis on μ excluded most adjoint groups like PGL_n as well as exceptional types like E_6 and E_7, both of which have minuscule cocharacters. One can weaken the assumptions in Theorem 4.4.6 if one assumes this stronger condition on μ.

Remark 1.1.2. The groups $\pi_1(G^{\mathrm{der}})$ and $\pi_1(G^{\mathrm{ad}})$ appearing in Theorems 4.4.1 and 4.4.6 are the fundamental groups in the sense of semisimple groups. Note that $\pi_1(G^{\mathrm{der}})$ is a subgroup of $\pi_1(G^{\mathrm{ad}})$. The assumption that $p \nmid \pi_1(G^{\mathrm{der}})$ insures that the local models we use have nice geometric properties. The stronger assumption in Theorem 4.4.6 that $p \nmid \pi_1(G^{\mathrm{ad}})$ is probably not necessary and is a byproduct of the argument, which involves reduction to the adjoint group.

There are two main ingredients in the proof of Theorem 4.4.1 and its applications, one coming from integral p-adic Hodge theory and the other from local models of Shimura varieties. In Kisin's original construction, a key input was an advance in integral p-adic Hodge theory, building on work of Breuil, which allows one to describe finite flat group schemes over $\mathcal{O}_K$ in terms of certain linear algebra objects called *Kisin modules* of height in $[0, 1]$ [Kisin 2006; 2009]. More precisely, then, $X_{\bar{\eta}}^{\mathrm{cris},\mu}$ is a moduli space of G-Kisin modules with "type" μ. Intuitively, one can imagine $X_{\bar{\eta}}^{\mathrm{cris},\mu}$ as a moduli of finite flat models with additional structure.

The proof of Theorem 4.4.1 uses a recent advance of Liu [2010] in integral p-adic Hodge theory to overcome a difficulty in identifying the local structure of $X_{\bar{\eta}}^{\mathrm{cris},\mu}$. Heuristically, the difficulty arises because for a general group G one cannot work only in the setting of Kisin modules of height in $[0, 1]$, where one has a nice equivalence of categories between that category and the category of finite flat group schemes. Beyond the height-in-$[0, 1]$ situation, the Kisin module only remembers the Galois action of the subgroup $\Gamma_\infty \subset \Gamma_K$ which fixes the field $K(\pi^{1/p}, \pi^{1/p^2}, \dots)$ for some compatible system of p-power roots of a uniformizer π of K.

Liu [2010] introduced a more complicated linear algebra structure on a Kisin module, called a $(\varphi, \widehat{G})$-module, which captures the action of Γ_K, the full Galois group. We call them $(\varphi, \widehat{\Gamma})$-modules to avoid confusion with the group G. Let A be a finite local Λ-algebra which is either Artinian or flat. Our principal result

(Theorem 4.3.6) says roughly that, if $\rho : \Gamma_\infty \to G(A)$ has "type" μ, i.e., comes from a G-Kisin module $(\mathfrak{P}_A, \phi_A)$ over A of type μ with μ minuscule, then there exists a canonical extension $\tilde{\rho} : \Gamma_K \to G(A)$ and, furthermore, if A is flat over $\mathbb{Z}_p$ then $\tilde{\rho}[1/p]$ is crystalline. This is rough in the sense that what we actually prove is an isomorphism of certain deformation functors. As a consequence, we get that the local structure of $X_{\bar{\eta}}^{\mathrm{cris},\mu}$ at a point $(\mathfrak{P}_{\mathbb{F}'}, \phi_{\mathbb{F}'}) \in X_{\bar{\eta}}^{\mathrm{cris},\mu}(\mathbb{F}')$ is smoothly equivalent to the deformation groupoid $D^{\mu}_{\mathfrak{P}_{\mathbb{F}'}}$ of $\mathfrak{P}_{\mathbb{F}'}$ with type μ.

To prove Theorem 4.4.1, one studies the geometry of $D^{\mu}_{\mathfrak{P}_{\mathbb{F}'}}$. Here, the key input comes from the theory of local models of Shimura varieties. A *local model* is a projective scheme X over the ring of integers of a p-adic field F such that X is supposed to étale-locally model the integral structure of a Shimura variety. Classically, local models were built out of moduli spaces of linear algebra structures. Rapoport and Zink [1996] formalized the theory of local models for Shimura varieties of PEL type. Subsequent refinements of these local models were studied mostly on a case by case basis by Faltings, Görtz, Haines, Pappas, and Rapoport, among others.

Pappas and Zhu [2013] define, for any triple (G, P, μ), where G is a reductive group over F (which splits over a tame extension), P is a parahoric subgroup, and μ is any cocharacter of G, a local model $M(\mu)$ over the ring of integers of the reflex field of μ. Their construction, unlike previous constructions, is purely group-theoretic, i.e., it does not rely on any particular representation of G. They build their local models inside degenerations of affine Grassmannians extending constructions of Beilinson, Drinfeld, Gaitsgory, and Zhu to mixed characteristic. The geometric fact we will use is that $M(\mu)$ is normal with special fiber reduced [Pappas and Zhu 2013, Theorem 0.1].

The significance of local models in this paper is that the singularities of $X_{\bar{\eta}}^{\mathrm{cris},\mu}$ are smoothly equivalent to those of a local model $M(\mu)$ for the Weil-restricted group $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p} F)/F} G_F$. This equivalence comes from a diagram of formally smooth morphisms (3-3-9-2):

$$\begin{array}{ccc} & \widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_{\mathbb{F}}} & \\ \swarrow & & \searrow \\ D^{\mu}_{\mathfrak{P}_{\mathbb{F}}} & & \overline{D}^{\mu}_{Q_{\mathbb{F}}}, \end{array} \tag{1-1-2-1}$$

which generalizes constructions from [Kisin 2009, Proposition 2.2.11; Pappas and Rapoport 2009, §3]. The deformation functor $\overline{D}^{\mu}_{Q_{\mathbb{F}}}$ is represented by the completed local ring at an $\mathbb{F}$-point of $M(\mu)$. Intuitively, the above modification corresponds to adding a trivialization to the G-Kisin module and then taking the "image of Frobenius". We construct the diagram (1-1-2-1) in Section 3 with no assumptions on the cocharacter μ (to be precise, $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$ is deformations of type less than or equal

to μ in general). It is intriguing to wonder whether $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$ and diagram (1-1-2-1) have any relevance to studying higher-weight Galois deformation rings, i.e., when μ is not minuscule.

As a remark, we usually cannot apply [Pappas and Zhu 2013] directly, since the group $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p}F)/F}G$ will generally not split over a tame extension. In [Levin 2013], we develop a theory of local models following Pappas and Zhu's approach but adapted to these Weil-restricted groups (for maximal special parahoric level). These results are reviewed in Section 3.2 and are studied in more generality in [Levin 2014].

We now a give brief outline of the article. In Section 2, we define and develop the theory of G-Kisin modules and construct resolutions of semistable and crystalline G-valued deformation rings (Propositions 2.3.3 and 2.3.9). This closely follows the approach of [Kisin 2008]. The proof that "semistable implies finite height" (Proposition 2.3.13) requires an extra argument not present in the GL_n case (Lemma 2.3.6). In Section 3, we study the relationship between deformations of G-Kisin modules and local models. We construct the big diagram (Theorem 3.3.3) and then impose the μ-type condition to arrive at the diagram (3-3-9-2). We also give an initial description of the local structure of $X^{\mathrm{cris},\mu}_{\bar{\eta}}$ in Corollary 3.3.15. Section 4.2 develops the theory of $(\varphi,\widehat{\Gamma})$-modules with G-structure and Section 4.3 is devoted to the proof of our key result (Theorem 4.3.6) in integral p-adic Hodge theory. In the last section, Section 4.4, we prove Theorems 4.4.1 and 4.4.6, which follow relatively formally from the results of Sections 3.3 and 4.3.

1.2. ***Notations and conventions.*** We take F to be our coefficient field, a finite extension of $\mathbb{Q}_p$. Let Λ be the ring of integers of F with residue field $\mathbb{F}$. Let G be a reductive group scheme over Λ with connected fibers and ${}^f\mathrm{Rep}_{\Lambda}(G)$ the category of representations of G on finite free Λ-modules. We will use V to denote a fixed faithful representation of G, i.e., $V\in {}^f\mathrm{Rep}_{\Lambda}(G)$ such that $G\to \mathrm{GL}(V)$ is a closed immersion. The derived subgroup of G will be denoted by G^{der} and its adjoint quotient by G^{ad}.

All G-bundles will be with respect to the fppf topology. If X is a Λ-scheme, then $\mathrm{GBun}(X)$ will denote the category of G-bundles on X. We will denote the trivial G-bundle by $\mathscr{E}^0$. For any G-bundle P on a Λ-scheme X and any $W\in {}^f\mathrm{Rep}_{\Lambda}(G)$, $P(W)$ will denote the pushout of P with respect to W (see the discussion before Theorem 2.1.1). Let $\overline{F}$ be an algebraic closure of F. For a linear algebraic F-group H, $X_*(H)$ will denote the group $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, H_{\overline{F}})$ of geometric cocharacters. For $\mu\in X_*(H)$, $[\mu]$ will denote its conjugacy class. The reflex field $F_{[\mu]}$ of $[\mu]$ is the smallest subfield of $\overline{F}$ over which the conjugacy class $[\mu]$ is defined.

If Γ is a profinite group and B is a finite Λ-algebra, then ${}^f\mathrm{Rep}_{B}(\Gamma)$ will be the category of continuous representations of Γ on finite projective B-modules

where B is given the p-adic topology. More generally, $\mathrm{GRep}_B(\Gamma)$ will denote the category of pairs (P, η) where P is a G-bundle over $\mathrm{Spec}\, B$ and $\eta : \Gamma \to \mathrm{Aut}_G(P)$ is a continuous homomorphism.

Let K be a p-adic field with ring of integers $\mathcal{O}_K$ and residue field k. Denote its absolute Galois group by Γ_K. We furthermore take $W := W(k)$ and $K_0 := W[1/p]$. We fix a uniformizer π of K and let $E(u)$ the minimal polynomial of π over K_0. Our convention will be to work with covariant p-adic Hodge theory functors, so we take the p-adic cyclotomic character to have Hodge–Tate weight -1.

For any local ring R, we let m_R denote the maximal ideal. We will denote the completion of B with respect to a specified topology by $\widehat{B}$.

2. Kisin modules with G-structure

In this section, we construct resolutions of Galois deformation rings by moduli spaces of Kisin modules (i.e., $\mathfrak{S}$-modules) with G-structure. For GL_n, this technique was introduced in [Kisin 2009] to study flat deformation rings. In [Kisin 2008], the same technique is used to construct potentially semistable deformation rings for GL_n. Here we develop a theory of G-Kisin modules (Definition 2.2.7). In particular, in Section 2.4, we show the existence of a universal G-Kisin module over these deformation rings (Theorem 2.4.2) and relate the filtration defined by a G-Kisin module to p-adic Hodge type. One can construct G-valued semistable and crystalline deformation rings with fixed p-adic Hodge type without G-Kisin modules [Balaji 2012]. However, the existence of a resolution by a moduli space of Kisin modules allows for finer analysis of the deformation rings; see Section 4.

2.1. ***Background on G-bundles.*** All bundles will be for the fppf topology. For any G-bundle P on a Λ-scheme X and any $W \in {}^f\mathrm{Rep}_\Lambda(G)$, define

$$P(W) := P \times^G W = (P \times W)/\sim$$

to be the pushout of P with respect to W. This is a vector bundle on X. This defines a functor from ${}^f\mathrm{Rep}_\Lambda(G)$ to the category Vec_X of vector bundles on X.

Theorem 2.1.1. *Let G be a flat affine group scheme of finite type over* $\mathrm{Spec}\,\Lambda$ *with connected fibers. Let X be a Λ-scheme. The functor $P \mapsto \{P(W)\}$ from the category of G-bundles on X to the category of fiber functors (i.e., faithful exact tensor functors) from ${}^f\mathrm{Rep}_\Lambda(G)$ to Vec_X is an equivalence of categories.*

Proof. When the base is a field, this is a well-known result [Deligne and Milne 1982, Theorem 3.2] in Tannakian theory. When the base is a Dedekind domain, see [Broshi 2013, Theorem 4.8] or [Levin 2013, Theorem 2.5.2]. □

We will also need the following gluing lemma for G-bundles:

Lemma 2.1.2. *Let B be any Λ-algebra. Let $f \in B$ be a non-zero-divisor and G be a flat affine group scheme of finite type over Λ. The category of triples $(P_f, \widehat{P}, \alpha)$, where $P_f \in \mathrm{GBun}(\mathrm{Spec}\, B_f)$, $\widehat{P} \in \mathrm{GBun}(\mathrm{Spec}\, \widehat{B})$, and α is an isomorphism between P_f and $\widehat{P}$ over $\mathrm{Spec}\, \widehat{B}_f$, is equivalent to the category of G-bundles on B.*

Proof. This is a generalization of the Beauville–Laszlo formal gluing lemma for vector bundles. See [Pappas and Zhu 2013, Lemma 5.1] or [Levin 2013, Theorem 3.1.8]. □

Let $i : H \subset G$ be a flat closed Λ-subgroup. We are interested in the "fibers" of the pushout map

$$i_* : \mathrm{HBun} \to \mathrm{GBun}$$

carrying an H-bundle Y to the G-bundle $Y \times^H G$. Let Q be a G-bundle on a Λ-scheme S. For any S-scheme X, define $\mathrm{Fib}_Q(X)$ to be the category of pairs (P, α), where $P \in \mathrm{HBun}(X)$ and $\alpha : i_*(P) \cong Q_X$ is an isomorphism in $\mathrm{GBun}(X)$. A morphism $(P, \alpha) \to (P', \alpha')$ is a map $f : P \to P'$ of H-bundles such that $\alpha' \circ i_*(f) \circ \alpha^{-1}$ is the identity.

Proposition 2.1.3. *The category $\mathrm{Fib}_Q(X)$ has no nontrivial automorphisms for any S-scheme X. Furthermore, the underlying functor $|\mathrm{Fib}_Q|$ is represented by the pushout $Q \times^G (G/H)$. In particular, if G/H is affine (resp. quasiaffine) over S then $|\mathrm{Fib}_Q|$ is affine (resp. quasiaffine) over X.*

Proof. See [Serre 1958, Proposition 9] or [Levin 2013, Lemma 2.2.3]. □

Proposition 2.1.4. *Let G be a smooth affine group scheme of finite type over* Spec Λ *with connected fibers.*

(1) *Let R any Λ-algebra and I a nilpotent ideal of R. For any G-bundle P on* Spec R, *P is trivial if and only if $P \otimes_R R/I$ is trivial.*

(2) *Let R be any complete local Λ-algebra with finite residue field. Any G-bundle on* Spec R *is trivial.*

Proof. For (1), because G is smooth, P is also smooth. Thus, $P(R) \to P(R/I)$ is surjective. A G-bundle is trivial if and only if it admits a section.

Part (2) reduces to the case of $R = \mathbb{F}$ using part (1). Lang's theorem says that $H^1_{\text{ét}}(\mathbb{F}, G)$ is trivial for any smooth connected algebraic group over $\mathbb{F}$ (see [Springer 1998, Theorem 4.4.17]) □

2.2. ***Definitions and first properties.*** Let K be a p-adic field with ring of integers $\mathcal{O}_K$ and residue field k. Set $W := W(k)$ and $K_0 := W[1/p]$. Recall Breuil and Kisin's ring $\mathfrak{S} := W[\![u]\!]$ and let $E(u) \in W[u]$ be the Eisenstein polynomial associated to a choice of uniformizer π of K that generates K over K_0. Fix a compatible system $\{\pi^{1/p}, \pi^{1/p^2}, \dots\}$ of p-power roots of π and let $K_\infty = K(\pi^{1/p}, \pi^{1/p^2}, \dots)$. Set $\Gamma_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$.

Let $\mathbb{O}_{\mathscr{E}}$ denote the p-adic completion of $\mathfrak{S}[1/u]$. We equip both $\mathbb{O}_{\mathscr{E}}$ and $\mathfrak{S}$ with a Frobenius endomorphism φ defined by taking the ordinary Frobenius lift on W and $u \mapsto u^p$. For any $\mathbb{Z}_p$-algebra B, let $\mathbb{O}_{\mathscr{E},B} := \mathbb{O}_{\mathscr{E}} \otimes_{\mathbb{Z}_p} B$ and $\mathfrak{S}_B := \mathfrak{S} \otimes_{\mathbb{Z}_p} B$. We equip both of these rings with Frobenii having trivial action on B. Note that all tensor products are over $\mathbb{Z}_p$ even though the group G may only be defined over the Λ.

Definition 2.2.1. Let B be any Λ-algebra. For any G-bundle on $\operatorname{Spec} \mathbb{O}_{\mathscr{E},B}$, we let $\varphi^*(P) := P \otimes_{\mathbb{O}_{\mathscr{E},B},\varphi} \mathbb{O}_{\mathscr{E},B}$ be the pullback under Frobenius. An $(\mathbb{O}_{\mathscr{E},B}, \varphi)$-*module with G-structure* is a pair (P, ϕ_P), where P is a G-bundle on $\operatorname{Spec} \mathbb{O}_{\mathscr{E},B}$ and $\phi_P : \varphi^*(P) \cong P$ is an isomorphism. Let $\mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$ be the category of such pairs.

Remark 2.2.2. When $G = \mathrm{GL}_d$, $\mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$ is equivalent to the category of rank-d étale $(\mathbb{O}_{\mathscr{E},B}, \varphi)$-modules via the usual equivalence between GL_d-bundles and rank-d vector bundles.

When B is $\mathbb{Z}_p$-finite and Artinian, the functor T_B defined by

$$T_B(M, \phi) = (M \otimes_{\mathbb{O}_{\mathscr{E}}} \mathbb{O}_{\widehat{\mathscr{E}}^{\mathrm{un}}})^{\phi=1}$$

induces an equivalence of categories between étale $(\mathbb{O}_{\mathscr{E},B}, \varphi)$-modules (which are $\mathbb{O}_{\mathscr{E},B}$-projective) and the category of representations of Γ_∞ on finite projective B-modules (see [Kisin 2009, Lemma 1.2.7]). A quasi-inverse is given by

$$\underline{M}_B(V) := (V \otimes_{\mathbb{Z}_p} \mathbb{O}_{\widehat{\mathscr{E}}^{\mathrm{un}}})^{\Gamma_\infty}.$$

This equivalence extends to algebras which are finite flat over $\mathbb{Z}_p$.

Definition 2.2.3. For any profinite group Γ and Λ-algebra B, define $\mathrm{GRep}_B(\Gamma)$ to be the category of pairs (P, η) where P is a G-bundle over $\operatorname{Spec} B$ and, with B given the p-adic topology, $\eta : \Gamma \to \mathrm{Aut}_G(P)$ is a continuous homomorphism.

In the G-setting, $\mathrm{GRep}_B(\Gamma)$ will play the role of representation of Γ on finite projective B-modules. We have the following generalization of T_B:

Proposition 2.2.4. *Let B be any Λ-algebra which is $\mathbb{Z}_p$-finite and either Artinian or $\mathbb{Z}_p$-flat. There exists an equivalence of categories*

$$T_{G,B} : \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}} \to \mathrm{GRep}_B(\Gamma_\infty)$$

with a quasi-inverse $\underline{M}_{G,B}$. Furthermore, for any finite map $B \to B'$ and any $(P, \phi_P) \in \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$, there is a natural isomorphism

$$T_{G,B'}(P \otimes_B B') \cong T_{G,B}(P) \otimes_B B'.$$

Proof. Using Theorem 2.1.1, we can give Tannakian interpretations of $\mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$ and $\mathrm{GRep}_B(\Gamma_\infty)$. The former is equivalent to the category

$$[{}^f \mathrm{Rep}_\Lambda(G), \mathrm{Mod}^{\varphi,\text{ét}}_{\mathbb{O}_{\mathscr{E},B}}]^{\otimes}$$

of faithful exact tensor functors. The latter is equivalent to the category of faithful exact tensor functors from ${}^f\mathrm{Rep}_\Lambda(G)$ to ${}^f\mathrm{Rep}_B(\Gamma_\infty)$. We define $T_{G,B}(P,\phi_P)$ to be the functor which assigns to any $W \in {}^f\mathrm{Rep}_\Lambda(G)$ the Γ_∞-representation $T_B(P(W),\phi_{P(W)})$. This is an object of $\mathrm{GRep}_B(\Gamma_\infty)$ because T_B is a tensor exact functor (see [Broshi 2008, Lemma 3.4.1.6] or [Levin 2013, Theorem 4.1.3]). Similarly, one can define $\underline{M}_{G,B}$ which is quasi-inverse to $T_{G,B}$. Compatibility with extending the coefficients follows from [Kisin 2009, Lemma 1.2.7(3)]. □

Definition 2.2.5. Let B be any $\mathbb{Z}_p$-algebra. A *Kisin module with bounded height* over B is a finitely generated projective $\mathfrak{S}_B$-module $\mathfrak{M}_B$ together with an isomorphism $\phi_{\mathfrak{M}_B}:\varphi^*(\mathfrak{M}_B)[1/E(u)] \cong \mathfrak{M}_B[1/E(u)]$. We say that $(\mathfrak{M}_B,\phi_{\mathfrak{M}_B})$ has *height in* $[a,b]$ if

$$E(u)^a\mathfrak{M}_B \supset \phi_{\mathfrak{M}_B}(\varphi^*(\mathfrak{M}_B)) \supset E(u)^b\mathfrak{M}_B$$

as submodules of $\mathfrak{M}_B[1/E(u)]$.

Let $\mathrm{Mod}_{\mathfrak{S}_B}^{\varphi,\mathrm{bh}}$ (resp. $\mathrm{Mod}_{\mathfrak{S}_B}^{\varphi,[a,b]}$) be the category of Kisin modules with bounded height (resp. height in $[a,b]$) with morphisms being $\mathfrak{S}_B$-module maps respecting Frobenii. Then $\mathrm{Mod}_{\mathfrak{S}_B}^{\varphi,[0,h]}$ is the usual category of Kisin modules with height at most h, as in [Brinon and Conrad 2009; Kisin 2006; 2009].

Example 2.2.6. Let $\mathfrak{S}(1)$ be the Kisin module whose underlying module is $\mathfrak{S}$ and whose Frobenius is given by $c_0^{-1}E(u)\varphi_{\mathfrak{S}}$ where $E(0)=c_0p$. For any $\mathbb{Z}_p$-algebra, we define $\mathfrak{S}_B(1)$ by base change from $\mathbb{Z}_p$ and define $\mathbb{O}_{\mathscr{E},B}(1) := \mathfrak{S}_B(1)\otimes_{\mathfrak{S}_B}\mathbb{O}_{\mathscr{E},B}$, an étale $(\mathbb{O}_{\mathscr{E},B},\varphi)$-module.

In order to reduce to the effective case (height in $[0,h]$), it is often useful to "twist" by tensoring with $\mathfrak{S}_B(1)$. For any $\mathfrak{M}_B \in \mathrm{Mod}_{\mathfrak{S}_B}^{\varphi,\mathrm{bh}}$ and any $n\in\mathbb{Z}$, define $\mathfrak{M}_B(n)$ by n-fold tensor product with $\mathfrak{S}_B(1)$ (negative n being tensoring with the dual). It is not hard to see that if $\mathfrak{M}_B \in \mathrm{Mod}_{\mathfrak{S}_B}^{\varphi,[a,b]}$ then $\mathfrak{M}_B(n) \in \mathrm{Mod}_{\mathfrak{S}_B}^{\varphi,[a+n,b+n]}$.

Definition 2.2.7. Let B be any Λ-algebra. A *G-Kisin module* over B is a pair $(\mathfrak{P}_B,\phi_{\mathfrak{P}_B})$, where $\mathfrak{P}_B$ is a G-bundle on $\mathfrak{S}_B$ and

$$\phi_{\mathfrak{P}_B}:\varphi^*(\mathfrak{P}_B)[1/E(u)] \cong \mathfrak{P}_B[1/E(u)]$$

is an isomorphism of G-bundles. Denote the category of such objects by $\mathrm{GMod}_{\mathfrak{S}_B}^{\varphi,\mathrm{bh}}$.

Remark 2.2.8. Unlike the Kisin module for GL_n, G-bundles do not have endomorphisms. Additionally, there is no reasonable notion of effective G-Kisin module. The Frobenius on a G-Kisin module is only ever defined after inverting $E(u)$. Later, we use auxiliary representations of G to impose height conditions.

The category $\mathrm{Mod}_{\mathfrak{S}_B}^{\varphi,\mathrm{bh}}$ is a tensor exact category, where a sequence of Kisin modules

$$0\to\mathfrak{M}'_B\to\mathfrak{M}_B\to\mathfrak{M}''_B\to 0$$

is *exact* if the underlying sequence of $\mathfrak{S}_B$-modules is exact. For any representation $W \in {}^f\mathrm{Rep}_\Lambda(G)$, the pushout $(\mathfrak{P}_B(W), \phi_{\mathfrak{P}_B}(W))$ is a Kisin module with bounded height. Using Theorem 2.1.1, one can interpret $\mathrm{GMod}^{\varphi,\mathrm{bh}}_{\mathfrak{S}_B}$ as the category of faithful exact tensor functors from ${}^f\mathrm{Rep}_\Lambda(G)$ to $\mathrm{Mod}^{\varphi,\mathrm{bh}}_{\mathfrak{S}_B}$.

Since $E(u)$ is invertible in $\mathbb{O}_{\mathscr{E}}$, there is a natural map $\mathfrak{S}_B[1/E(u)] \to \mathbb{O}_{\mathscr{E},B}$ for any $\mathbb{Z}_p$-algebra B. This induces a functor

$$\Upsilon_G : \mathrm{GMod}^{\varphi,\mathrm{bh}}_{\mathfrak{S}_B} \to \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$$

for any Λ-algebra B.

Definition 2.2.9. Let B be any Λ-algebra and let $P_B \in \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$. A *$G$-Kisin lattice of P_B* is a pair $(\mathfrak{P}_B, \alpha)$ where $\mathfrak{P}_B \in \mathrm{GMod}^{\varphi,\mathrm{bh}}_{\mathfrak{S}_B}$ and $\alpha : \Upsilon_G(\mathfrak{P}_B) \cong P_B$ is an isomorphism.

From the Tannakian perspective, a G-Kisin lattice of P is equivalent to Kisin lattices $\mathfrak{M}_W$ in $P(W)$ for each $W \in {}^f\mathrm{Rep}_\Lambda(G)$ functorial in W and compatible with tensor products. Furthermore, we have the following, which says that the bounded height condition can be checked on a single faithful representation.

Proposition 2.2.10. *Let $P_B \in \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$. A G-Kisin lattice of P_B is equivalent to an extension $\mathfrak{P}_B$ of the bundle P_B to $\mathrm{Spec}\,\mathfrak{S}_B$ such that, for a single faithful representation $V \in {}^f\mathrm{Rep}_\Lambda(G)$,*

$$\mathfrak{P}_B(V) \subset P_B(V)$$

is a Kisin lattice of bounded height.

Proof. The only claim which does not follow from unwinding definitions is that, if we have an extension $\mathfrak{P}_B$ such that $\mathfrak{P}_B(V) \subset P_B(V)$ is a Kisin lattice for a single faithful representation V, then $\mathfrak{P}_B(W) \subset P_B(W)$ is a Kisin lattice for all representations W of G.

By [Levin 2013, Theorem C.1.7], any $W \in {}^f\mathrm{Rep}_\Lambda(G)$ can be written as a subquotient of direct sums of tensor products of V and the dual of V. It suffices then to prove that bounded height is stable under duals, tensor products, quotients, and saturated subrepresentations.

Duals and tensor products are easy to check. For subquotients, let

$$0 \to M_B \to N_B \to L_B \to 0$$

be an exact sequence of étale $(\mathbb{O}_{\mathscr{E},B}, \varphi)$-modules. Suppose that the sequence is induced by an exact sequence

$$0 \to \mathfrak{M}_B \to \mathfrak{N}_B \to \mathfrak{L}_B \to 0$$

of projective $\mathfrak{S}_B$-lattices. Assume $\mathfrak{N}_B$ has bounded height with respect to ϕ_{N_B}. By twisting, we can assume $\mathfrak{N}_B$ has height in $[0, h]$.

Since $\mathfrak{M}_B = M_B \cap \mathfrak{N}_B$, $\mathfrak{M}_B$ is ϕ_{M_B}-stable. Similarly, $\mathfrak{L}_B$ is ϕ_{L_B}-stable. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varphi^*(\mathfrak{M}_B) & \longrightarrow & \varphi^*(\mathfrak{N}_B) & \longrightarrow & \varphi^*(\mathfrak{L}_B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_{M_B}} & & \downarrow{\scriptstyle \phi_{N_B}} & & \downarrow{\scriptstyle \phi_{L_B}} & & \\ 0 & \longrightarrow & \mathfrak{M}_B & \longrightarrow & \mathfrak{N}_B & \longrightarrow & \mathfrak{L}_B & \longrightarrow & 0. \end{array}$$

All the linearizations are injective because they are isomorphisms at the level of $\mathbb{O}_{\mathscr{E},B}$-modules. By the snake lemma, the sequence of cokernels is exact. If $E(u)^h$ kills $\mathrm{Coker}(\phi_{N_B})$, then it kills $\mathrm{Coker}(\phi_{M_B})$ and $\mathrm{Coker}(\phi_{P_B})$ as well. Thus, $\mathfrak{M}_B$ and $\mathfrak{P}_B$ both have height in $[0,h]$ whenever $\mathfrak{N}_B$ does. □

Definition 2.2.11. For any B as in Proposition 2.2.4, define

$$T_{G,\mathfrak{S}_B} : \mathrm{GMod}^{\varphi,\mathrm{bh}}_{\mathfrak{S}_B} \to \mathrm{GRep}_B(\Gamma_\infty)$$

to be the composition $T_{G,\mathfrak{S}_B} := T_{G,B} \circ \Upsilon_G$.

We end this section with an important full faithfulness result:

Proposition 2.2.12. *Assume B is finite flat over Λ. Then the natural extension map*

$$\Upsilon_G : \mathrm{GMod}^{\varphi,\mathrm{bh}}_{\mathfrak{S}_B} \to \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$$

is fully faithful.

Proof. This follows from the full faithfulness of Υ_{GL_n} for all $n \geq 1$ by considering a faithful representation of G. When $B = \mathbb{Z}_p$, this is [Brinon and Conrad 2009, Proposition 11.2.7]. One can reduce to this case by forgetting coefficients, since any finitely generated projective $\mathfrak{S}_B$-module is finite free over $\mathfrak{S}$. □

2.3. ***Resolutions of G-valued deformations rings.*** Fix a faithful representation V of G over Λ and integers a, b with $a \leq b$. We will use V and a, b to impose finiteness conditions on our moduli space.

Definition 2.3.1. Let B be any Λ-algebra. We say that a G-Kisin lattice $\mathfrak{P}_B$ in $(P_B, \phi_{P_B}) \in \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},B}}$ has *height in* $[a,b]$ if $\mathfrak{P}_B(V)$ in $P_B(V)$ has height in $[a,b]$.

For any finite local Artinian Λ-algebra A and any $(P_A, \phi_{P_A}) \in \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},A}}$, consider the moduli problem over $\mathrm{Spec}\, A$, for any A-algebra B,

$$X^{[a,b]}_{P_A}(B) := \{G\text{-Kisin lattices in } P_A \otimes_{\mathbb{O}_{\mathscr{E},A}} \mathbb{O}_{\mathscr{E},B} \text{ with height in } [a,b]\}/\cong .$$

Theorem 2.3.2. *Assume that P_A is a trivial bundle over* $\mathrm{Spec}\, \mathbb{O}_{\mathscr{E},A}$. *The functor $X^{[a,b]}_{P_A}$ is represented by a closed finite-type subscheme of the affine Grassmannian* $\mathrm{Gr}_{G'}$ *over* $\mathrm{Spec}\, A$, *where G' is the Weil restriction* $\mathrm{Res}_{(W\otimes_{\mathbb{Z}_p}\Lambda)/\Lambda} G$.

Proof. By Proposition 2.2.10, $X_{P_A}^{[a,b]}(B)$ is the set of bundles over $\mathfrak{S}_B$ extending $P_B := P_A \otimes_{\mathbb{O}_{\mathscr{E},A}} \mathbb{O}_{\mathscr{E},B}$ with height in $[a,b]$ with respect to V. We want to identify this set with a subset of $\mathrm{Gr}_{G'}(B)$.

Consider the diagram

$$\begin{array}{ccc} \mathfrak{S} \otimes_{\mathbb{Z}_p} B & \longrightarrow & (W \otimes_{\mathbb{Z}_p} B)[\![u]\!] \\ \downarrow & & \downarrow \\ \mathbb{O}_{\mathscr{E},B} & \longrightarrow & (W \otimes_{\mathbb{Z}_p} B)((u)), \end{array}$$

where the vertical arrows are localization at u and the top horizontal arrow is u-adic completion. The Beauville–Laszlo gluing lemma, Lemma 2.1.2, says that the set of extensions of P_B to $\mathfrak{S}_B$ is in bijection with the set of extensions of $\widehat{P}_B$ to $W_B[\![u]\!]$, where $\widehat{P}_B$ is the u-adic completion. This second set is in bijection with the B-points of the Weil restriction $\mathrm{Res}_{(W\otimes_{\mathbb{Z}_p}\Lambda)/\Lambda}\mathrm{Gr}_G$, which is isomorphic to $\mathrm{Gr}_{G'}$ by [Richarz 2015, Lemma 1.16] or [Levin 2013, Proposition 3.4.2].

Set $M_A := P_A(V)$. By [Kisin 2008, Proposition 1.3], the functor $X_{M_A}^{[a,b]}$ of Kisin lattices in M_A with height in $[a,b]$ is represented by a closed subscheme of $\mathrm{Gr}_{\mathrm{Res}_{(W\otimes_{\mathbb{Z}_p}\Lambda)/\Lambda}\mathrm{GL}(V)}$. Evaluation at V induces a map of functors

$$X_{P_A}^{[a,b]} \to X_{M_A}^{[a,b]}. \tag{2-3-2-1}$$

By Proposition 2.2.10, the subset $X_{P_A}^{[a,b]}(B) \subset \mathrm{Gr}_{G'}(B)$ is exactly the preimage of $X_{M_A}^{[a,b]}(B)$. □

We now extend the construction beyond the Artinian setting by passing to the limit. Let R be a complete local Noetherian Λ-algebra with residue field $\mathbb{F}$. Let $\eta : \Gamma_\infty \to G(R)$ be a continuous representation.

Proposition 2.3.3. *For any $n \geq 1$, let $\eta_n : \Gamma_\infty \to G(R/m_R^n)$ denote the reduction modulo m_R^n. From $\{\eta_n\}$, we construct a system $\underline{M}_{G,R/m_R^n}(\eta_n) =: (P_{\eta_n}, \phi_n)$ in $\mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},R/m_R^n}}$. Assume that P_{η_1} is a trivial G-bundle. There exists a projective R-scheme*

$$\Theta : X_\eta^{[a,b]} \to \mathrm{Spec}\, R$$

whose reduction modulo m_R^n is $X_{P_{\eta_n}}^{[a,b]}$ for any $n \geq 1$.

Proof. By Proposition 2.2.4, there are natural isomorphisms

$$P_{\eta_{n+1}} \otimes_{\mathbb{O}_{\mathscr{E},R/m_R^{n+1}}} \mathbb{O}_{\mathscr{E},R/m_R^n} \cong P_{\eta_n}$$

for all $n \geq 1$. Since P_{η_1} is a trivial G-bundle, all P_{η_n} are trivial, by Proposition 2.1.4(1), so we can apply Theorem 2.3.2. Consider then the system

$$\{X_{P_{\eta_n}}^{[a,b]}\}$$

of schemes over $\{R/m_R^n\}$. Since G' is reductive, the affine Grassmannian $\mathrm{Gr}_{G'}$ is ind-projective [Levin 2013, Theorem 3.3.11]. In particular, any ample line bundle on $\mathrm{Gr}_{G'}$ will restrict to a compatible system of ample line bundles on $\{X_{P_{\eta n}}^{[a,b]}\}$. By formal GAGA [EGA III$_1$ 1961, Théorème (5.4.5)], there exists a projective R-scheme $X_\eta^{[a,b]}$ whose reductions modulo m_R^n are $X_{P_{\eta n}}^{[a,b]}$. □

Remark 2.3.4. Unlike for GL_n, there are nontrivial G-bundles over $\mathrm{Spec}\,\mathbb{F}((u))$, which is why we need the assumption in Proposition 2.3.3. If P_{η_1} admits any G-Kisin lattice $\mathfrak{P}_{\eta_1}$, then by Proposition 2.1.4(2) the G-bundle $\mathfrak{P}_{\eta_1}$ is trivial, since $\mathfrak{S}_\mathbb{F}$ is a semilocal ring with finite residue fields. Thus, the assumption in Proposition 2.3.3 is natural if you are interested in studying Γ_∞-representations of finite height. By Steinberg's theorem, one can always make P_{η_1} trivial by passing to a finite extension $\mathbb{F}'$ of $\mathbb{F}$.

We record for reference the following compatibility with base change:

Proposition 2.3.5. *Let $f : R \to S$ be a local map of complete local Noetherian Λ-algebras with finite residue fields of characteristic p. Let η_S be the induced map $\Gamma_\infty \to G(S)$. Then there is a natural map $f' : X_{\eta_S}^{[a,b]} \to X_\eta^{[a,b]}$ which makes the following diagram Cartesian*:

$$\begin{array}{ccc} X_{\eta_S}^{[a,b]} & \xrightarrow{f'} & X_\eta^{[a,b]} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, S & \xrightarrow{f} & \mathrm{Spec}\, R. \end{array}$$

In particular, if $R \to S$ is surjective then f' is a closed immersion.

We will now study the projective F-morphism

$$\Theta[1/p] : X_\eta^{[a,b]}[1/p] \to \mathrm{Spec}\, R[1/p].$$

We show it is a *closed immersion* (this is essentially a consequence of Proposition 2.2.12) and that the closed points of the image are G-valued representations with height in $[a, b]$ in a suitable sense; see Proposition 2.3.9. Next, we show that, if η is the restriction of $\eta' : \Gamma_K \to G(R)$, then the image of $\Theta[1/p]$ contains all semistable representations with $\eta'(V)$ having Hodge–Tate weights in $[a, b]$. These are generalizations of results from [Kisin 2008].

The following lemma will be useful at several key points:

Lemma 2.3.6 (extension lemma). *Let G be a smooth affine group scheme over Λ. Let C be a finite flat Λ-algebra and let U be the open complement of the finite set of closed points of* $\mathrm{Spec}\,\mathfrak{S}_C$.

(1) *There is an equivalence of categories between G-bundles Q on U and the category of triples $(\mathfrak{P}^*, P, \gamma)$ where $\mathfrak{P}^*$ is a G-bundle on* $\operatorname{Spec} \mathfrak{S}_C[1/p]$, *$P$ is a G-bundle on* $\operatorname{Spec} \mathbb{O}_{\mathscr{E},C}$, *and γ is an isomorphism of their restrictions to* $\operatorname{Spec} \mathbb{O}_{\mathscr{E},C}[1/p]$.

(2) *Assume G is a reductive group scheme with connected fibers. Let V be a faithful representation of G over Λ. If Q is a G-bundle on U such that the locally free coherent sheaf $Q(V)$ on U extends to a projective $\mathfrak{S}_C$-module $\mathfrak{M}_C$, then there exists a unique (up to unique isomorphism) G-bundle $\widetilde{Q}$ over* $\operatorname{Spec} \mathfrak{S}_C$ *such that $\widetilde{Q}|_U \cong Q$ and $\widetilde{Q}(V) = \mathfrak{M}_C$.*

Proof. Note that we can write U as the union of $\operatorname{Spec} \mathfrak{S}_C[1/u]$ and $\operatorname{Spec} \mathfrak{S}_C[1/p]$. Recall also that $\mathbb{O}_{\mathscr{E},C}$ is the p-adic completion of $\mathfrak{S}_C[1/u]$. Since p is a non-zero-divisor in $\mathfrak{S}_C[1/u]$, we can apply the gluing lemma, Lemma 2.1.2, to P and $\mathfrak{P}^*[1/u]$ to construct a G-bundle Q' on $\operatorname{Spec} \mathfrak{S}_C[1/u]$ which, by construction, is isomorphic to $\mathfrak{P}^*$ along $\operatorname{Spec} \mathfrak{S}_C[1/u, 1/p]$. The G-bundles $\mathfrak{P}^*$ and Q' glue to give a bundle Q over U. Each step in the construction is a categorical equivalence.

For part (2), consider the functor $|\mathrm{Fib}_{\mathfrak{M}_C}|$, which by Proposition 2.1.3 and [Levin 2013, Theorem C.2.5] is represented by an affine scheme Y. $\mathfrak{M}_C$ defines a U-point of $\mathrm{Fib}_{\mathfrak{M}_C}$. Since $\Gamma(U, \mathbb{O}_U) = \mathfrak{S}_C$, we deduce that

$$\mathrm{Hom}_{\mathfrak{S}_C}(\operatorname{Spec} \mathfrak{S}_C, \mathrm{Fib}_{\mathfrak{M}_C}) = \mathrm{Hom}_{\mathfrak{S}_C}(U, \mathrm{Fib}_{\mathfrak{M}_C}).$$

A $\mathfrak{S}_C$-point of $\mathrm{Fib}_{\mathfrak{M}_C}$ is exactly a bundle $\widetilde{Q}$ extending Q and mapping to $\mathfrak{M}_C$.

A similar argument, using that the Isom-scheme between G-bundles is representable by an affine scheme, shows that if an extension exists it is unique up to unique isomorphism (without any reductivity hypotheses). □

Let B be any finite local F-algebra with residue field F'. Define B^0 to be the subring of elements which map to $\mathbb{O}_{F'}$ modulo the maximal ideal of B. Let Int_B denote the set of finitely generated $\mathbb{O}_{F'}$-subalgebras C of B^0 such that $C[1/p] = B$.

Definition 2.3.7. A continuous homomorphism $\eta : \Gamma_\infty \to G(B)$ has *bounded height* if there exists a $C \in \mathrm{Int}_B$ and $g \in G(B)$ such that

(1) $\eta'_C := g\eta g^{-1}$ factors through $G(C)$;

(2) $\underline{M}_{G,C}(\eta'_C) \in \mathrm{GMod}^{\varphi}_{\mathbb{O}_{\mathscr{E},C}}$ admits a G-Kisin lattice of bounded height.

We define *height in* $[a, b]$ with respect to the chosen faithful representation V by replacing bounded height in (2) with height in $[a, b]$.

Lemma 2.3.8. *Let B be a finite local $\mathbb{Q}_p$-algebra and choose $C \in \mathrm{Int}_B$ and $M_C \in \mathrm{Mod}^{\varphi,ét}_{\mathbb{O}_{\mathscr{E},C}}$. If M_C, considered as an $\mathbb{O}_{\mathscr{E}}$-module, has bounded height (resp. height in $[a, b]$), then there exists some $C' \supset C$ in Int_B, such that $M_C \otimes_C C'$ has bounded height (resp. height in $[a, b]$).*

Proof. This is the main content in the proof of part (2) of Proposition 1.6.4 in [Kisin 2008]. If F' is the residue field of B, then one first constructs a Kisin lattice $\mathfrak{M}_{\mathbb{O}_{F'}}$ in $M_C \otimes_C \mathbb{O}_{F'}$. The Kisin lattice in $M_C \otimes_C C'$ is constructed by lifting $\mathfrak{M}_{\mathbb{O}_{F'}}$ (the extension to C' is required to insure that the lift is ϕ-stable). □

Proposition 2.3.9. *The morphism Θ becomes a closed immersion after inverting p. Furthermore, if* $\operatorname{Spec} R_\eta^{[a,b]} \subset \operatorname{Spec} R$ *is the scheme-theoretic image of Θ, then, for any finite F-algebra B, a Λ-algebra map $x : R \to B$ factors through $R_\eta^{[a,b]}$ if and only if $\eta \otimes_{R,x} B$ has height in $[a, b]$.*

Proof. The map Θ is injective on C-points for any finite flat Λ-algebra C, by Proposition 2.2.12. The proof of the first assertion is then the same as in [Kisin 2008, Proposition 1.6.4].

For the second assertion, say $x : R \to B$ factors through $R_\eta^{[a,b]}$. Because $\Theta[1/p]$ is a closed immersion, $x : R \to B$ comes from a B-point y of $X_\eta^{[a,b]}$. Any such x is induced by $x_C : R \to C$ for some $C \in \mathrm{Int}_B$. By properness of Θ, there exists $y_C \in X_\eta^{[a,b]}(C)$ such that $\Theta(y_C) = x_C$. This implies that $\eta \otimes_{R,x_C} C$ has height in $[a, b]$ as a G-valued representation and hence $\eta \otimes_{R,x} B$ also has height in $[a, b]$ (see Definition 2.3.7).

Now, let $x : R \to B$ be a homomorphism such that $\eta_B := \eta \otimes_{R,x} B$ has height in $[a, b]$ as a G-valued representation. Any homomorphism $R \to B$ factors through some $C \in \mathrm{Int}_B$, so that η_B has image in $G(C)$; call this map η_C. We claim that there exists some $C' \supset C$ in Int_B such that $\eta_{C'} = \eta_C \otimes_C C'$ has height in $[a, b]$ and hence x is in the image of $X_\eta^{[a,b]}(B)$. Essentially, we have to show that if one Galois stable "lattice" in η_B has finite height then all "lattices" do. For GL_n, this is Lemma 2.1.15 in [Kisin 2006]. We invoke the GL_n result below.

Since η_B has height in $[a, b]$, there exists $C' \in \mathrm{Int}_B$ and $g \in G(B)$ such that $\eta' = g\eta_B g^{-1}$ factors through $G(C')$ and has height in $[a, b]$. Enlarging C if necessary, we assume both η_C and η' are valued in $G(C)$. Let $P_\eta := \underline{M}_{G,C}(\eta)$ and $P_{\eta'} := \underline{M}_{G,C}(\eta')$. Then g induces an isomorphism

$$P_{\eta'}[1/p] \cong P_{\eta_C}[1/p].$$

Since $P_{\eta'}$ has a G-Kisin lattice with height in $[a, b]$, we get a bundle $\mathfrak{Q}_C$ over $\mathfrak{S}_C[1/p]$ extending $P_{\eta_C}[1/p]$. By Lemma 2.3.6(1), $P_{\eta'}$ and $\mathfrak{Q}_C$ glue to give a bundle Q_C over the complement of the closed points of $\operatorname{Spec} \mathfrak{S}_C$.

We would like to apply Lemma 2.3.6(2). $P_{\eta_C}(V)$ has height in $[a, b]$ as an $\mathbb{O}_\mathscr{E}$-module by [Kisin 2006, Lemma 2.1.15] since it corresponds to a lattice in $\eta_C(V)[1/p] \cong \eta'(V)[1/p]$. By Lemma 2.3.8, there exists $\widetilde{C} \supset C$ in Int_B such that $P_{\eta_C}(V) \otimes_C \widetilde{C}$ has height in $[a, b]$ as an $\mathbb{O}_{\mathscr{E},\widetilde{C}}$-module. Replace C by $\widetilde{C}$. Then, if $\mathfrak{M}_C$ is the unique Kisin lattice in $P_{\eta_C}(V)$, we have

$$\mathfrak{M}'_C[1/p] \cap P_{\eta_C}(V) = \mathfrak{M}_C,$$

where $\mathfrak{M}'_C$ is the unique Kisin lattice in $P_{\eta'}(V)$. This shows that $Q_C(V)$ extends across the closed points, so we can apply Lemma 2.3.6(2) to construct a G-Kisin lattice of P_{η_C}. □

Now, assume that η is the restriction to Γ_∞ of a continuous representation of Γ_K, which we continue to call η. Recall the definition of semistable for a G-valued representation:

Definition 2.3.10. If B is a finite F-algebra, a continuous representation $\eta_B : \Gamma_K \to G_F(B)$ is *semistable* (resp. *crystalline*) if, for all representations W in $\mathrm{Rep}_F(G_F)$, the induced representation $\eta_B(W)$ on $W \otimes_F B$ is semistable (resp. crystalline).

Note that because the semistable and crystalline conditions are stable under tensor products and subquotients, it suffices to check these conditions on a single faithful representation of G_F.

Remark 2.3.11. Since we are working with covariant functors, our convention will be that the cyclotomic character has Hodge–Tate weight -1. This is, unfortunately, opposite to the convention in [Kisin 2008].

The following theorem generalizes [Kisin 2008, Theorem 2.5.5]:

Theorem 2.3.12. *Let R be a complete local Noetherian Λ-algebra with finite residue field and $\eta : \Gamma_K \to G(R)$ a continuous representation. Given any a, b integers with $a < b$, there exists a quotient $R_\eta^{[a,b],\mathrm{st}}$ (resp. $R_\eta^{[a,b],\mathrm{cris}}$) of $R_\eta^{[a,b]}$ with the property that, if B is any finite F-algebra and $x : R \to B$ a map of Λ-algebras, then x factors through $R_\eta^{[a,b],\mathrm{st}}$ (resp. $R_\eta^{[a,b],\mathrm{cris}}$) if and only if $\eta_x : \Gamma_K \to G(B)$ is semistable (resp. crystalline) and $\eta_x(V)$ has Hodge–Tate weights in $[a, b]$.*

Since the semistable and crystalline properties can be checked on a single faithful representation, the quotients $R_{\eta(V)}^{[a,b],\mathrm{st}}$ and $R_{\eta(V)}^{[a,b],\mathrm{cris}}$ of R constructed by applying [Kisin 2008, Theorem 2.5.5] to $\eta(V)$ satisfy the universal property in Theorem 2.3.12 with respect to maps $x : R \to B$, where B is a finite F-algebra. What remains is to show that $R_\eta^{[a,b],\mathrm{st}} := R_{\eta(V)}^{[a,b],\mathrm{st}}$ is a quotient of $R_\eta^{[a,b]}$, i.e., that "semistable implies finite height".

Proposition 2.3.13. *Let R and η be as in* 2.3.12*. For any map $x : R \to B$ with B a finite local F-algebra, if the representation η_x is semistable and $\eta_x(V)$ has Hodge–Tate weights in $[a, b]$, then x factors through $R_\eta^{[a,b]}$.*

Proof. By Lemma 2.3.8, there exists $C \in \mathrm{Int}_B$ such that η_x factors through $\mathrm{GL}(V_C)$, hence $G(C)$, and that $M_C := P_{\eta_x}(V)$ admits a Kisin lattice $\mathfrak{M}_C$ with height in $[a, b]$. By Proposition 2.2.10, it suffices to extend the bundle P_{η_x} to $\mathrm{Spec}\, \mathfrak{S}_C$ such that $\mathfrak{P}_{\eta_x}(V) = \mathfrak{M}_C$.

We will apply Lemma 2.3.6. Consider a candidate fiber functor $\mathfrak{F}_{\eta_x}$ for $\mathfrak{P}_{\eta_x}$ which assigns to any $W \in {}^f\mathrm{Rep}_\Lambda(G)$ the unique Kisin lattice of bounded height in $\mathfrak{M}_W \subset P_{\eta_x}(W) = M_W$ (as an $\mathbb{O}_\mathscr{E}$-module, not as an $\mathbb{O}_{\mathscr{E},C'}$-module). Such a lattice exists since $\eta_x(W)$ is semistable. The difficulties are that $\mathfrak{M}_W$ may not be $\mathbb{O}_{\mathscr{E},C'}$-projective and that it is not obvious whether $\mathfrak{F}_{\eta_x}$ is exact. It can happen that a nonexact sequence of $\mathfrak{S}$-modules can map under $T_\mathfrak{S}$ to an exact sequence of Γ_∞-representations (see [Liu 2012, Example 2.5.6]).

Let $B = C[1/p]$. By [Kisin 2008, Corollary 1.6.3], $\mathfrak{M}_W[1/p]$ is finite projective over $\mathfrak{S}_C[1/p] = \mathfrak{S}_B$ for all W. We claim furthermore that $\mathfrak{F}_{\eta_x} \otimes_{\mathfrak{S}_C} \mathfrak{S}_B$ is exact. For any exact sequence $0 \to W'' \to W \to W' \to 0$ in ${}^f\mathrm{Rep}_\Lambda(G)$, we have a left-exact sequence

$$0 \to \mathfrak{M}_{W''}[1/p] \to \mathfrak{M}_W[1/p] \to \mathfrak{M}_{W'}[1/p].$$

Exactness on the right follows from [Levin 2013, Lemma 4.2.22] on the behavior of exactness for sequences of $\mathfrak{S}$-modules. Thus, $\mathfrak{F}_{\eta_x} \otimes_{\mathfrak{S}_C} \mathfrak{S}_B$ defines a bundle $\mathfrak{P}_B$ over $\mathfrak{S}_B$. Clearly, $\mathfrak{P}_B \otimes_{\mathfrak{S}_B} \mathbb{O}_{\mathscr{E},B} \cong P_{\eta_x} \otimes_{\mathbb{O}_{\mathscr{E},C}} \mathbb{O}_{\mathscr{E},B}$. By Lemma 2.3.6(1), we get a bundle Q over U such that $Q(W) = \mathfrak{M}_W|_U$. Since $\mathfrak{M}_V$ is a projective $\mathfrak{S}_C$-module by our choice of C, Q extends to a bundle $\widetilde{Q}$ over $\mathfrak{S}_C$ by Lemma 2.3.6(2). □

2.4. ***Universal G-Kisin module and filtrations.*** For this section, we make a small change in notation. Let R_0 be a complete local Noetherian Λ-algebra with finite residue field and let $R = R_0[1/p]$.

Define $\widehat{\mathfrak{S}}_{R_0}$ to be the m_{R_0}-adic completion of $\mathfrak{S} \otimes_{\mathbb{Z}_p} R_0$. The Frobenius on $\mathfrak{S} \otimes_{\mathbb{Z}_p} R_0$ extends to a Frobenius on $\widehat{\mathfrak{S}}_{R_0}$.

Definition 2.4.1. A $(\widehat{\mathfrak{S}}_{R_0}[1/p], \varphi)$-module of *bounded height* is a finitely generated projective $\widehat{\mathfrak{S}}_{R_0}[1/p]$-module $\mathfrak{M}_R$ together with an isomorphism

$$\phi_R : \varphi^*(\mathfrak{M}_R)[1/E(u)] \cong \mathfrak{M}_R[1/E(u)].$$

Let $\eta : \Gamma_\infty \to G(R_0)$ be continuous representation. If $\widehat{\mathbb{O}}_{\mathscr{E},R_0}$ is the m_{R_0}-adic completion of $\mathbb{O}_{\mathscr{E},R_0}$, then the inverse limit $\varprojlim \underline{M}_{G,R_0/m_{R_0}^n}(\eta_n)$ defines a pair (P_η, ϕ_η) over $\widehat{\mathbb{O}}_{\mathscr{E},R_0}$ [Levin 2013, Corollary 2.3.5]. Assume $R_0 = R_{0,\eta}^{[a,b]}$. For any finite F-algebra B and any homomorphism $x : R_0 \to B$, there is a unique G-Kisin lattice in $P_\eta \otimes_{\widehat{\mathbb{O}}_{\mathscr{E},R_0},x} \mathbb{O}_{\mathscr{E},B}$ by Proposition 2.2.12; call it $(\mathfrak{P}_x, \phi_x)$. In the following theorem, we construct a universal G-bundle over $\widehat{\mathfrak{S}}_{R_0}[1/p]$ with a Frobenius which specializes to $(\mathfrak{P}_x, \phi_x)$ at every x.

Theorem 2.4.2. *Assume that $R_0 = R_{0,\eta}^{[a,b]}$. Let B be a finite F-algebra. The pair $(P_\eta[1/p], \phi_\eta[1/p])$ extends to a G-bundle $\widetilde{\mathfrak{P}}_\eta$ over $\widehat{\mathfrak{S}}_{R_0}[1/p]$ together with a Frobenius*

$$\phi_{\widetilde{\mathfrak{P}}_\eta} : \varphi^*(\widetilde{\mathfrak{P}}_\eta)[1/E(u)] \cong \widetilde{\mathfrak{P}}_\eta[1/E(u)]$$

such that, for any $x : R_0[1/p] \to B$, *the base change*

$$\big(\widetilde{\mathfrak{P}}_\eta \otimes_{\widehat{\mathfrak{S}}_{R_0}[1/p]} \mathfrak{S}_B, \phi_{\widetilde{\mathfrak{P}}_\eta} \otimes_{\widehat{\mathfrak{S}}_{R_0}[1/p,1/E(u)]} \mathfrak{S}_B[1/E(u)]\big)$$

is $(\mathfrak{P}_x, \phi_x)$.

Proof. Let $X_n := X_{\eta_n}^{[a,b]}$ be the projective $R_0/m_{R_0}^n$-scheme as in Section 4.3. Take $Y_n := X_n \times_{\operatorname{Spec} R_0/m_{R_0}^n} \operatorname{Spec} \mathfrak{S}_{R_0/m_{R_0}^n}$, a projective $\mathfrak{S}_{R_0/m_{R_0}^n}$-scheme. Let $X_\eta^{[a,b]} \to \operatorname{Spec} R_0$ be the algebraization of $\varprojlim X_n$ as before. The base change Y of $X_\eta^{[a,b]}$ along the map $R_0 \to \widehat{\mathfrak{S}}_{R_0}$ has the property that

$$Y \bmod m_{R_0}^n \cong Y_n.$$

Furthermore, Y is a proper $\widehat{\mathfrak{S}}_{R_0}$-scheme.

Over each Y_n, we have a universal G-Kisin lattice $(\mathfrak{P}_n, \phi_n)$ with height in $[a, b]$. By [Levin 2013, Corollary 2.3.5], there exists a G-bundle $\mathfrak{P}_\eta$ on Y such that $\mathfrak{P}_\eta \bmod m_{R_0}^n = \mathfrak{P}_n$. We would like to construct a Frobenius ϕ over $Y[1/E(u)]$ which reduces to ϕ_n modulo m_R^n for each $n \geq 1$. A priori, the Frobenius is only defined over the m_{R_0}-adic completion of $\widehat{\mathfrak{S}}_{R_0}[1/E(u)]$, which we denote by $\widehat{S}$.

We have a projective morphism

$$Y_{\widehat{S}} \to \operatorname{Spec} \widehat{S},$$

where $Y_{\widehat{S}}$ is the base change of $Y[1/E(u)]$ along $\operatorname{Spec} \widehat{S} \to \operatorname{Spec} \widehat{\mathfrak{S}}_{R_0}[1/E(u)]$. $Y_{\widehat{S}}$ is faithfully flat over $Y[1/E(u)]$, since $\widehat{\mathfrak{S}}_{R_0}[1/E(u)]$ is Noetherian. Define $\operatorname{Isom}_G := \operatorname{Isom}_G(\varphi^*(\mathfrak{P}_\eta), \mathfrak{P}_\eta)$ to be the affine finite-type Y-scheme of G-bundle isomorphisms. The compatible system $\{\phi_n\}$ lifts to an element

$$\widehat{\phi} \in \operatorname{Isom}_G(Y_{\widehat{S}}).$$

We would like to descend $\widehat{\phi}$ to a $Y[1/E(u)]$-point of Isom_G. Let $i : G \hookrightarrow \mathrm{GL}(V)$ be our chosen faithful representation. Consider the closed immersion

$$i_* : \operatorname{Isom}_G \hookrightarrow \operatorname{Isom}_{\mathrm{GL}(V)}(\varphi^*(\mathfrak{P}_\eta)(V), \mathfrak{P}_\eta(V)).$$

The image $i_*(\widehat{\phi})$ descends to a $Y[1/E(u)]$-point of $\operatorname{Isom}_{\mathrm{GL}(V)}(\varphi^*(\mathfrak{P}_\eta)(V), \mathfrak{P}_\eta(V))$ (twist to reduce to the effective case). Since $Y_{\widehat{S}}$ is faithfully flat over $Y[1/E(u)]$, for any closed immersion $Z \subset Z'$ of Y-schemes we have

$$Z\big(Y[1/E(u)]\big) = Z(Y_{\widehat{S}}) \cap Z'\big(Y[1/E(u)]\big).$$

Applying this with $Z' = \operatorname{Isom}_G$ and $Z = \operatorname{Isom}_{\mathrm{GL}(V)}(\varphi^*(\mathfrak{P}_\eta)(V), \mathfrak{P}_\eta(V))$, we get a universal pair $(\mathfrak{P}_\eta, \phi_\eta)$ over Y and $Y[1/E(u)]$, respectively. Since $R_0 = R_{0,\eta}^{[a,b]}$, $\Theta[1/p] : X_\eta^{[a,b]}[1/p] \to R_0[1/p]$ is an isomorphism and the pair $\widetilde{\mathfrak{P}}_\eta := \mathfrak{P}_\eta[1/p]$ and $\phi_{\mathfrak{P}_\eta}[1/p]$ over $\widehat{\mathfrak{S}}_{R_0}[1/p]$ has the desired properties. □

We now discuss the notion of p-adic Hodge type for G-valued representation and relate this to a filtration associated to a G-Kisin module.

Let B be any finite F-algebra. For any representation of Γ_K on a finite free B-module V_B, set

$$D_{\mathrm{dR}}(V_B) := (V_B \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K},$$

a filtered $(K \otimes_{\mathbb{Q}_p} B)$-module whose associated graded is projective (see [Balaji 2012, Definition 2.1.6, Lemma 2.4.2]). Furthermore, D_{dR} defines a tensor exact functor from the category of de Rham representations on projective B-modules to the category $\mathrm{Fil}_{K\otimes_{\mathbb{Q}_p} B}$ of filtered $(K \otimes_{\mathbb{Q}_p} B)$-modules (see [Balaji 2012, Lemma 2.4.2]). For any field κ, Fil_κ will be the tensor category of $\mathbb{Z}$-filtered vectors spaces $(V, \{\mathrm{Fil}^i V\})$, where $\mathrm{Fil}^i(V) \supset \mathrm{Fil}^{i+1}(V)$.

We recall a few facts from the Tannakian theory of filtrations:

Definition 2.4.3. Let H be any reductive group over a field κ. For any extension $\kappa' \supset \kappa$, an *H-filtration over κ'* is a tensor exact functor from $\mathrm{Rep}_\kappa(H)$ to $\mathrm{Fil}_{\kappa'}$.

Associated to any cocharacter $\nu : \mathbb{G}_{\mathrm{m}} \to H_{\kappa'}$ is a tensor exact functor from $\mathrm{Rep}_\kappa(H)$ to graded κ'-vector spaces which assigns to each representation W the vector space $W_{\kappa'}$, with its weight grading defined by the $\mathbb{G}_{\mathrm{m}}$-action through ν, which we denote by ω_ν (see [Deligne and Milne 1982, Example 2.30]).

Definition 2.4.4. For any H-filtration $\mathscr{F}$ over κ', a *splitting* of $\mathscr{F}$ is an isomorphism between $\mathrm{gr}(\mathscr{F})$ and ω_ν for some $\nu : \mathbb{G}_{\mathrm{m}} \to H_{\kappa'}$.

By [Saavedra Rivano 1972, Proposition IV.2.2.5], all H-filtrations over κ' are splittable. For any given $\mathscr{F}$, the cocharacters ν for which there exists an isomorphism $\mathrm{gr}(\mathscr{F}) \cong \omega_\nu$ lie in the common $H(\kappa')$-conjugacy class. If κ' is a finite extension of κ contained in $\bar{\kappa}$, then the *type* $[\nu_{\mathscr{F}}]$ of the filtration $\mathscr{F}$ is the geometric conjugacy class of ν for any splitting ω_ν over κ'. For any conjugacy class $[\nu]$ of geometric cocharacters of H, there is a smallest field of definition, contained in a chosen separable closure of κ, called the *reflex field* of $[\nu]$. We denote this by $\kappa_{[\nu]}$.

Let G be as before, so that G_F is a (connected) reductive group over F, and let $\eta : \Gamma_K \to G(B)$ be a continuous representation which is de Rham. Then D_{dR} defines a tensor exact functor from $\mathrm{Rep}_F(G_F)$ to $\mathrm{Fil}_{K\otimes_{\mathbb{Q}_p} B}$ (see Proposition 2.4.2 in [Balaji 2012]), which we denote by $\mathscr{F}_\eta^{\mathrm{dR}}$.

Fix a geometric cocharacter $\mu \in X_*((\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p} F)/F} G)_{\overline{F}})$ and denote its conjugacy class by $[\mu]$. The cocharacter μ is equivalent to a set $(\mu_\psi)_{\psi: K\to\overline{F}}$ of cocharacters μ_ψ of $G_{\overline{F}}$ indexed by $\mathbb{Q}_p$-embeddings of K into $\overline{F}$.

Definition 2.4.5. Let $F_{[\mu]}$ be the reflex field of $[\mu]$. For any embedding $\psi : K \to \overline{F}$ over $\mathbb{Q}_p$, let $\mathrm{pr}_\psi : K \otimes_{\mathbb{Q}_p} \overline{F} \to \overline{F}$ denote the projection. If F' is a finite extension of $F_{[\mu]}$, a G-filtration $\mathscr{F}$ over $K \otimes_{\mathbb{Q}_p} F'$ has *type* $[\mu]$ if $\mathrm{pr}_\psi^*(\mathscr{F} \otimes_{F',i} \overline{F})$

has type $[\mu_\psi]$ for any $F_{[\mu]}$-embedding $i : F' \hookrightarrow \overline{F}$. A de Rham representation $\eta : \Gamma_K \to G(F')$ has *p-adic Hodge type* μ if $\mathcal{F}_\eta^{\mathrm{dR}}$ has type $[\mu]$.

Let $\Lambda_{[\mu]}$ denote the ring of integers of $F_{[\mu]}$. For any μ in the conjugacy class $[\mu]$, $\mathbb{G}_{\mathrm{m}}$ acts on $V \otimes_\Lambda \overline{F}$ through μ_ψ for each $\psi : K \to \overline{F}$. We take a and b be the minimal and maximal weights taken over all μ_ψ.

Theorem 2.4.6. *Let R_0 be a complete local Noetherian $\Lambda_{[\mu]}$-algebra with finite residue field and $\eta : \Gamma_K \to G(R_0)$ a continuous homomorphism. Let $R_{0,\eta}^{[a,b],\mathrm{st}}$ be as in Theorem 2.3.12. There exists a quotient $R_{0,\eta}^{\mathrm{st},\mu}$ of $R_{0,\eta}^{[a,b],\mathrm{st}}$ such that, for any finite extension F' of $F_{[\mu]}$, a homomorphism $\zeta : R_0 \to F'$ factors through $R_{0,\eta}^{\mathrm{st},\mu}$ if and only if the $G(F')$-valued representation corresponding to ζ is semistable with p-adic Hodge type $[\mu]$.*

Proof. See [Balaji 2012, Proposition 3.0.9]. □

Remark 2.4.7. One can deduce from the construction in [Balaji 2012, Proposition 3.0.9] or by other arguments [Levin 2013, Theorem 6.1.19] that the p-adic Hodge type on the generic fiber of the semistable deformation ring $R_{0,\eta}^{[a,b],\mathrm{st}}$ is locally constant so that $\operatorname{Spec} R_{0,\eta}^{\mathrm{st},\mu}[1/p]$ is a union of connected components of $\operatorname{Spec} R_{0,\eta}^{[a,b],\mathrm{st}}[1/p]$.

Finally, we recall how the de Rham filtration is obtained from the Kisin module.

Definition 2.4.8. Let B be a finite $\mathbb{Q}_p$-algebra. Let $(\mathfrak{M}_B, \phi_B)$ be a Kisin module over B with bounded height. Define

$$\mathrm{Fil}^i(\varphi^*(\mathfrak{M}_B)) := \phi_B^{-1}(E(u)^i \mathfrak{M}_B) \cap \varphi^*(\mathfrak{M}_B).$$

Set $\mathfrak{D}_B := \varphi^*(\mathfrak{M}_B)/E(u)\varphi^*(\mathfrak{M}_B)$, a finite projective $(K \otimes_{\mathbb{Q}_p} B)$-module. Define $\mathrm{Fil}^i(\mathfrak{D}_B)$ to be the image of $\mathrm{Fil}^i(\varphi^*(\mathfrak{M}_B))$ in $\mathfrak{D}_B$.

Proposition 2.4.9. *Let B be a finite $\mathbb{Q}_p$-algebra and V_B a finite free B-module with an action of Γ_K which is semistable with Hodge–Tate weights in $[a, b]$. Any $\mathbb{Z}_p$-stable lattice in V_B has finite height. If $\mathfrak{M}_B$ is the $(\mathfrak{S}_B, \varphi)$-module of bounded height attached to V_B, then there is a natural isomorphism $\mathfrak{D}_B \cong D_{\mathrm{dR}}(V_B)$ of filtered $(K \otimes_{\mathbb{Q}_p} B)$-modules.*

Proof. The relevant results are in the proofs of Corollary 2.6.2 and Theorem 2.5.5(2) in [Kisin 2008]. Since Kisin works with contravariant functors, one has to do a small translation. Under Kisin's conventions, $\mathfrak{M}_B$ would be associated to the B-dual V_B^*, and it is shown there that $D_B \cong D_{\mathrm{dR}}^*(V_B^*)$ as filtered $K \otimes_{\mathbb{Q}_p} B$-modules in the case where $[a, b] = [0, h]$. By compatibility with duality [Balaji 2012, Proposition 2.2.9], $D_{\mathrm{dR}}^*(V_B^*) \cong D_{\mathrm{dR}}(V_B)$. The general case follows by twisting. □

3. Deformations of G-Kisin modules

In this section, we study the local structure of the "moduli space" of G-Kisin modules. This generalizes results of [Kisin 2009; Pappas and Rapoport 2009]. G-Kisin modules may have nontrivial automorphisms and so it is more natural, as was done in [Kisin 2009, §2.2], to work with *groupoids*. The goal of the section is to smoothly relate the deformation theory of a G-Kisin module to the local structure of a local model for the group $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p}F)/F}G_F$.

Intuitively, the smooth modification (a chain of formally smooth morphisms) corresponds to adding a trivialization to the G-Kisin module and then taking the "image of Frobenius" similar to Proposition 2.2.11 of [Kisin 2009]. The target of the modification is a deformation functor for the moduli space $\mathrm{Gr}_G^{E(u),W}$ discussed in Section 3.3, which is a version of the affine Grassmannian that appears in the work of Pappas and Zhu [2013] on local models. Finally, we show that the condition of having p-adic Hodge type μ is related to a (generalized) local model $M(\mu)\subset \mathrm{Gr}_G^{E(u),W}$. In this section, there are no conditions on the cocharacter μ. We will impose conditions on μ only in the next section when we study the analogue of flat deformations.

3.1. ***Definitions and representability results.*** Let $\mathbb{F}$ be the residue field of Λ. Define the categories

$$\mathscr{C}_\Lambda = \{\text{Artin local } \Lambda\text{-algebras with residue field } \mathbb{F}\}$$

and

$$\widehat{\mathscr{C}}_\Lambda = \{\text{complete local Noetherian } \Lambda\text{-algebras with residue field } \mathbb{F}\}.$$

Morphisms are local Λ-algebra maps. Recall that fiber products in the category $\widehat{\mathscr{C}}_\Lambda$ exist and are represented by completed tensor products. A *groupoid* over $\mathscr{C}_\Lambda$ (or $\widehat{\mathscr{C}}_\Lambda$) will be in the sense of Definition A.2.2 of [Kisin 2009]; this is also known as a category cofibered in groupoids over $\mathscr{C}_\Lambda$ (or $\widehat{\mathscr{C}}_\Lambda$). Recall also the notion of a 2-fiber product of groupoids from (A.4) in [Kisin 2009]. See [Kim 2009, §10] for more details related to groupoids.

Choose a bounded-height G-Kisin module $(\mathfrak{P}_\mathbb{F},\phi_\mathbb{F})\in \mathrm{GMod}_{\mathfrak{S}_\mathbb{F}}^{\varphi,\mathrm{bh}}$. Define $D_{\mathfrak{P}_\mathbb{F}} = \bigcup_{a<b} D_{\mathfrak{P}_\mathbb{F}}^{[a,b]}$ to be the deformation groupoid of $\mathfrak{P}_\mathbb{F}$ as a G-Kisin module of bounded height over $\widehat{\mathscr{C}}_\Lambda$. The morphisms $D_{\mathfrak{P}_\mathbb{F}}^{[a,b]}\subset D_{\mathfrak{P}_\mathbb{F}}$ are relatively representable closed immersions, so intuitively $D_{\mathfrak{P}_\mathbb{F}}$ is an ind-object built out of the finite-height pieces.

Let $\mathscr{E}^0$ denote the trivial G-bundle over Λ. Throughout we will be choosing various trivializations of the G-bundle $\mathfrak{P}_\mathbb{F}$ and other related bundles. This is always possible because $\mathfrak{S}_\mathbb{F}$ is a complete semilocal ring with all residue fields finite (see Proposition 2.1.4(2)).

Proposition 3.1.1. *For any $\mathfrak{P}_{\mathbb{F}}$ with height in $[a,b]$, the deformation groupoid $D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}$ admits a formally smooth morphism $\pi : \operatorname{Spf} R \to D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}$ for some $R \in \widehat{\mathscr{C}}_{\Lambda}$ (i.e., has a versal formal object in the sense of* [SGA 7_{I} 1972]).

Proof. One can check the abstract Schlessinger's criterion in [SGA 7_{I} 1972, Theorem 1.11]. However, it will be useful to have an explicit versal formal object. Fix a trivialization $\beta_{\mathbb{F}}$ of $\mathfrak{P}_{\mathbb{F}}$ mod $E(u)^N$ for any $N \geq 1$, and define

$$\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}}(A) := \{(\mathfrak{P}_A, \beta_A) \mid \mathfrak{P}_A \in D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}(A),\ \beta_A : \mathfrak{P}_A \cong \mathscr{C}^0_{\mathfrak{S}_A} \bmod E(u)^N\},$$

where β_A lifts $\beta_{\mathbb{F}}$. Since G is smooth, the forgetful morphism

$$\pi^{(N)} : \widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}} \to D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}$$

is formally smooth for any N.

If $N > (b-a)/(p-1)$, then $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}}$ is prorepresentable by a complete local Noetherian Λ-algebra. The proof uses Schlessinger's criterion. The two key points are that objects in $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}}(A)$ have no nontrivial automorphisms, for which one inducts on the power of p which kills A (see [Levin 2013, Proposition 8.1.6]), and that the tangent space of the underlying functor is finite-dimensional, which uses a successive approximation argument (see [Levin 2013, Proposition 8.1.8]). □

It will also be useful to have an infinite version of $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}}$. Fix a trivialization $\beta_{\mathbb{F}} : \mathfrak{P}_{\mathbb{F}} \cong \mathscr{C}^0_{\mathfrak{S}_{\mathbb{F}}}$. Define a groupoid on $\mathscr{C}_{\Lambda}$ by

$$\widetilde{D}^{[a,b],(\infty)}_{\mathfrak{P}_{\mathbb{F}}}(A) := \{(\mathfrak{P}_A, \beta_A) \mid \mathfrak{P}_A \in D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}(A),\ \beta_A : \mathfrak{P}_A \cong \mathscr{C}^0_{\mathfrak{S}_A}\},$$

where β_A lifts $\beta_{\mathbb{F}}$. Define $\widetilde{D}^{(\infty)}_{\mathfrak{P}_{\mathbb{F}}} := \bigcup_{a<b} \widetilde{D}^{[a,b],(\infty)}_{\mathfrak{P}_{\mathbb{F}}}$.

3.2. ***Local models for Weil-restricted groups.*** In this section, we associate to any geometric conjugacy class $[\mu]$ of cocharacters of $\operatorname{Res}_{(K\otimes_{\mathbb{Q}_p} F)/F} G_F$ a local model $M(\mu)$ (Definition 3.2.3) over the ring of integers $\Lambda_{[\mu]}$ of the reflex field $F_{[\mu]}$ of $[\mu]$ (the relevant parahoric here is $\operatorname{Res}_{(\mathbb{O}_K\otimes_{\mathbb{Z}_p}\Lambda)/\Lambda} G$). By construction, $M(\mu)$ is a flat projective $\Lambda_{[\mu]}$-scheme. The principal result (Theorem 3.2.4) says that $M(\mu)$ is normal and its special fiber is reduced.

The details of the proof of Theorem 3.2.4 are in [Levin 2013, §10], where we follow the strategy introduced in [Pappas and Zhu 2013]. We cannot apply Pappas and Zhu's result directly because the group $\operatorname{Res}_{(K\otimes_{\mathbb{Q}_p} F)/F} G_F$ usually does not split over a tame extension of F. In [Levin 2014], we generalize [Levin 2013, §10] and [Pappas and Zhu 2013] to groups of the form $\operatorname{Res}_{L/F} H$, where H is reductive group over L which splits over a tame extension of L, and allow arbitrary parahoric level structure. Here we recall the relevant definitions and results, leaving the details to [Levin 2013; 2014].

For any Λ-algebra R, set $R_W := R \otimes_{\mathbb{Z}_p} W$. Our local models are constructed inside the following moduli space:

Definition 3.2.1. For any Λ-algebra R, let $\widehat{R_W[u]}_{(E(u))}$ denote the $E(u)$-adic completion of $R_W[u]$. Define

$$\mathrm{Gr}_G^{E(u),W}(R) := \{\text{isomorphism classes of pairs } (\mathscr{E}, \alpha)\},$$

where $\mathscr{E}$ is a G-bundle on $\widehat{R_W[u]}_{(E(u))}$ and

$$\alpha : \mathscr{E}|_{\widehat{R_W[u]}_{(E(u))}[E(u)^{-1}]} \cong \mathscr{E}^0_{\widehat{R_W[u]}_{(E(u))}[E(u)^{-1}]}.$$

Proposition 3.2.2. *The functor* $\mathrm{Gr}_G^{E(u),W}$ *is an ind-scheme which is ind-projective over* Λ. *Furthermore*:

(1) *The generic fiber* $\mathrm{Gr}_G^{E(u),W}[1/p]$ *is naturally isomorphic to the affine Grassmannian of* $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p} F)/F} G_F$ *over the field* F.

(2) *If* k_0 *is the residue field of* W, *then the special fiber* $\mathrm{Gr}_G^{E(u),W} \otimes_\Lambda \mathbb{F}$ *is naturally isomorphic to the affine Grassmannian of* $\mathrm{Res}_{(k_0\otimes_{\mathbb{F}_p}\mathbb{F})/\mathbb{F}}(G_{\mathbb{F}})$.

Proof. See §10.1 in [Levin 2013]. □

Let H be any reductive group over F and Gr_H be the affine Grassmannian of H. Associated to any geometric conjugacy class $[\mu]$ of cocharacters, there is an affine Schubert variety $S(\mu)$ in $(\mathrm{Gr}_H)_{F_{[\mu]}}$, where $F_{[\mu]}$ is the reflex field of $[\mu]$. These are the closures of orbits for the positive loop group L^+H.

The geometric conjugacy classes of cocharacters of H can be identified with the set of dominant cocharacters for a choice of maximal torus and Borel subgroup over $\overline{F}$. The dominant cocharacters have partial ordering defined by $\mu \geq \lambda$ if and only if $\mu - \lambda$ is a nonnegative sum of positive coroots. Then $S(\mu)_{\overline{F}}$ is the union of the locally closed affine Schubert cells for all $\mu' \leq \mu$ [Richarz 2013, Proposition 2.8].

Definition 3.2.3. Let $F_{[\mu]}/F$ be the reflex field of $[\mu]$ with ring of integers $\Lambda_{[\mu]}$. If $S(\mu) \subset \mathrm{Gr}_{\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p} F)/F} G_F} \otimes_F F_{[\mu]}$ is the closed affine Schubert variety associated to μ, then the *local model* $M(\mu)$ associated to μ is the flat closure of $S(\mu)$ in $\mathrm{Gr}_G^{E(u),W} \otimes_\Lambda \Lambda_{[\mu]}$. It is a flat projective scheme over $\mathrm{Spec}\, \Lambda_{[\mu]}$.

The main theorem on the geometry of local models is:

Theorem 3.2.4. *Suppose that* $p \nmid |\pi_1(G^{\mathrm{der}})|$, *where* G^{der} *is the derived subgroup of* G. *Then* $M(\mu)$ *is normal. The special fiber* $M(\mu) \otimes_{\Lambda_\mu} \overline{\mathbb{F}}$ *is reduced, irreducible, normal, Cohen–Macaulay and Frobenius-split.*

For the next subsection, it will useful to recall a group which acts on $\mathrm{Gr}_G^{E(u),W}$ and $M(\mu)$. Define

$$L^{+,E(u)}G(R) := G(\widehat{R_W[u]}_{(E(u))}) = \varprojlim_{i\geq 1} G(R_W[u]/(E(u)^i))$$

for all Λ-algebras R. $L^{+,E(u)}G$ is represented by a group scheme that is the projective limit of the affine, flat, finite-type group schemes $\mathrm{Res}_{((\Lambda\otimes_{\mathbb{Z}_p} W)[u]/E(u)^i)/\Lambda}G$.

The group $L^{+,E(u)}G$ acts on $\mathrm{Gr}_G^{E(u),W}$ by changing the trivialization. This action is *nice* in the sense of [Gaitsgory 2001, A.3], i.e., $\mathrm{Gr}_G^{E(u),W} \cong \varinjlim_i Z_i$, where Z_i are $L^{+,E(u)}G$-stable closed subschemes on which $L^{+,E(u)}G$ acts through the quotient $\mathrm{Res}_{((\Lambda\otimes_{\mathbb{Z}_p} W)[u]/E(u)^i)/\Lambda}G$.

Corollary 3.2.5. *For any μ, the local model $M(\mu)$ is stable under the action of $L^{+,E(u)}G$.*

Proof. Since everything is flat, it suffices to show that $M(\mu)[1/p]$ is stable under $L^{+,E(u)}G[1/p]$. The functor $L^{+,E(u)}G[1/p]$ on F-algebras is naturally isomorphic to the positive loop group $L^+\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p} F)/F}(G)$, so that the isomorphism in Proposition 3.2.2(1) is equivariant. $M(\mu)[1/p]$ is the closed affine Schubert variety $S(\mu)$ which is stable under the action of this group. □

3.3. ***Smooth modification.*** We begin by defining the deformation functor which will be the target of our modification.

Definition 3.3.1. Choose a G-bundle $Q_{\mathbb{F}}$ over $\mathfrak{S}_{\mathbb{F}}$ together with a trivialization δ_0 of $Q_{\mathbb{F}}$ over $\mathfrak{S}_{\mathbb{F}}[1/E(u)]$. Define a deformation functor on $\mathscr{C}_\Lambda$ by

$$\overline{D}_{Q_{\mathbb{F}}}(A) := \{\text{isomorphism classes of triples } (\mathscr{E},\delta,\psi)\},$$

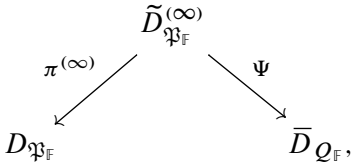
where $\mathscr{E}$ is a G-bundle on $\mathfrak{S}_A$, $\delta : \mathscr{E}|_{\mathfrak{S}_A[E(u)^{-1}]} \cong \mathscr{E}^0_{\mathfrak{S}_A[E(u)^{-1}]}$, and the map $\psi : \mathscr{E}\otimes_{\mathfrak{S}_A}\mathfrak{S}_{\mathbb{F}} \cong Q_{\mathbb{F}}$ is compatible with δ and δ_0.

Example 3.3.2. Let $G = \mathrm{GL}(V)$. For any $(Q_A,\delta_A) \in \overline{D}_{Q_{\mathbb{F}}}(A)$, δ_A identifies Q_A with a "lattice" in $(V\otimes_\Lambda \mathfrak{S}_A)[1/E(u)]$, that is, a finitely generated projective $\mathfrak{S}_A$-module L_A such that $L_A[1/E(u)] = (V\otimes_\Lambda \mathfrak{S}_A)[1/E(u)]$.

The main result of this section is the following:

Theorem 3.3.3. *Let Λ be a $\mathbb{Z}_p$-finite, flat, local domain with residue field $\mathbb{F}$. Let G be a connected reductive group over Λ and $\mathfrak{P}_{\mathbb{F}}$ a G-Kisin module with coefficients in $\mathbb{F}$. Fix a trivialization $\beta_{\mathbb{F}}$ of $\mathfrak{P}_{\mathbb{F}}$ as a G-bundle. There exists a diagram of*

groupoids over $\mathscr{C}_\Lambda$,



where $Q_\mathbb{F} := (\varphi^*(\mathfrak{P}_\mathbb{F}), \beta_\mathbb{F}[1/E(u)] \circ \phi_{\mathfrak{P}_\mathbb{F}})$. *Both* $\pi^{(\infty)}$ *and* Ψ *are formally smooth.*

Later in the section, we will refine this modification by imposing appropriate conditions on both sides. Intuitively, the above modification corresponds to adding a trivialization to the G-Kisin module and then taking the "image of Frobenius". The groupoid $\widetilde{D}^{(\infty)}_{\mathfrak{P}_\mathbb{F}}$ is defined at the end of Section 3.1 and $\pi^{(\infty)}$ is formally smooth since G is smooth. Next, we construct the morphism Ψ and show that it is formally smooth. To avoid excess notation, we sometimes omit the data of the residual isomorphisms modulo m_A. One can check that the everything is compatible with such isomorphisms.

Definition 3.3.4. For any $(\mathfrak{P}_A, \phi_{\mathfrak{P}_A}, \beta_A) \in \widetilde{D}^{(\infty)}_{\mathfrak{P}_\mathbb{F}}(A)$, we set

$$\Psi((\mathfrak{P}_A, \phi_{\mathfrak{P}_A}, \beta_A)) = (\varphi^*(\mathfrak{P}_A), \delta_A),$$

where δ_A is the composite

$$\varphi^*(\mathfrak{P}_A)[1/E(u)] \xrightarrow{\phi_{\mathfrak{P}_A}} \mathfrak{P}_A[1/E(u)] \xrightarrow{\beta_A[1/(E(u))]} \mathscr{E}^0_{\mathfrak{S}_A}[1/E(u)].$$

Proposition 3.3.5. *The morphism* Ψ *of groupoids is formally smooth.*

Proof. Choose $A \in \mathscr{C}_\Lambda$ and an ideal I of A. Consider a pair $(Q_A, \delta_A) \in \overline{D}_{Q_\mathbb{F}}(A)$ over a pair $(Q_{A/I}, \delta_{A/I})$. Let $(\mathfrak{P}_{A/I}, \phi_{A/I}, \beta_{A/I})$ be an element in the fiber over $(Q_{A/I}, \delta_{A/I})$. The triple $(\mathfrak{P}_{A/I}, \phi_{A/I}, \beta_{A/I})$ is isomorphic to a triple of the form $(\mathscr{E}^0_{\mathfrak{S}_{A/I}}, \phi'_{A/I}, \mathrm{Id}_{A/I})$. Let $\gamma_{A/I}$ be the isomorphism between $\varphi^*(\mathscr{E}^0_{\mathfrak{S}_{A/I}})$ and $Q_{A/I}$. We want to construct a lift $(\mathfrak{P}_A, \phi_A, \beta_A)$ such that $\Psi(\mathfrak{P}_A, \phi_A, \beta_A) = (Q_A, \delta_A)$. Take $\mathfrak{P}_A = \mathscr{E}^0_{\mathfrak{S}_A}$ to be the trivial bundle and β_A to be the identity.

Now, pick any lift $\gamma_A : \varphi^*(\mathscr{E}^0_{\mathfrak{S}_A}) \cong Q_A$, of $\gamma_{A/I}$ which exists since G is smooth. We can define the Frobenius by

$$\phi_A = \delta_A \circ \gamma_A[1/E(u)].$$

It is easy to check that $\Psi(\mathfrak{P}_A, \phi_A, \beta_A) \cong (Q_A, \delta_A)$. □

We would now like to relate $\overline{D}_{Q_\mathbb{F}}$ to $\mathrm{Gr}_G^{E(u),W}$ from the previous section.

Proposition 3.3.6. *A pair* $(Q_\mathbb{F}, \delta_0)$ *as in Definition 3.3.1 defines a point* $x_\mathbb{F}$ *in* $\mathrm{Gr}_G^{E(u),W}(\mathbb{F})$. *Furthermore, for any* $A \in \mathscr{C}_\Lambda$, *there is a natural functorial bijection between* $\overline{D}_{Q_\mathbb{F}}(A)$ *and the set of* $x_A \in \mathrm{Gr}_G^{E(u),W}(A)$ *such that* $x_A \bmod m_A = x_\mathbb{F}$.

Proof. Recall that $\mathfrak{S}_A = (W \otimes_{\mathbb{Z}_p} A)[\![u]\!]$ because A is finite over $\mathbb{Z}_p$. Also, $\mathrm{Gr}_G^{E(u),W}(A)$ is the set of isomorphism classes of bundles on the $E(u)$-adic completion of $(W \otimes_{\mathbb{Z}_p} A)[u]$ together with a trivialization after inverting $E(u)$. Since p is nilpotent in A, we can identify $(W \otimes_{\mathbb{Z}_p} A)[\![u]\!]$ and the $E(u)$-adic completion $\widehat{(W \otimes_{\mathbb{Z}_p} A)[u]}_{(E(u))}$. This identifies $\overline{D}_{Q_{\mathbb{F}}}(A)$ with the desired subset of $\mathrm{Gr}_G^{E(u),W}(A)$. □

For any $\mathbb{Z}_p$-algebra A, let $\widehat{S}_A$ denote the $E(u)$-adic completion of $(W \otimes_{\mathbb{Z}_p} A)[u]$.

Lemma 3.3.7. *For any finite flat $\mathbb{Z}_p$-algebra Λ', there is a $(W \otimes_{\mathbb{Z}_p} \Lambda')[u]$-algebra isomorphism*

$$\mathfrak{S}_{\Lambda'} \to \widehat{S}_{\Lambda'}.$$

Proof. For any $n \geq 1$, we have an isomorphism

$$\mathfrak{S}_{\Lambda'}/p^n \cong \widehat{S}_{\Lambda'}/p^n$$

since $(E(u))$ and u define the same adic topologies modulo p^n. Passing to the limit, we get an isomorphism of their p-adic completions. Both $\mathfrak{S}_{\Lambda'}$ and $\widehat{S}_{\Lambda'}$ are already p-adically complete and separated. □

Fix a geometric cocharacter μ of $\mathrm{Res}_{(K \otimes_{\mathbb{Q}_p} F)/F} G_F$, which we can write as $\mu = (\mu_\psi)_{\psi: K \to \overline{F}}$, where the μ_ψ are cocharacters of $G_{\overline{F}}$. *Assume* that $F = F_{[\mu]}$, so that the generalized local model $M(\mu)$ is a closed subscheme of $\mathrm{Gr}_G^{E(u),W}$ over Λ; see Definition 3.2.3. Recall that V is a fixed faithful representation of G. For each ψ, μ_ψ induces an action of $\mathbb{G}_{\mathrm{m}}$ on $V_{\overline{F}}$. Define a (resp. b) to be the smallest (resp. largest) weight appearing in $V_{\overline{F}}$ over all μ_ψ.

Definition 3.3.8. Define a closed subfunctor $\overline{D}^{\mu}_{Q_{\mathbb{F}}}$ of $\overline{D}_{Q_{\mathbb{F}}}$ by

$$\overline{D}^{\mu}_{Q_{\mathbb{F}}}(A) := \{(Q_A, \delta_A) \in \overline{D}_{Q_{\mathbb{F}}}(A) \mid (Q_A, \delta_A) \in M(\mu)(A)\}$$

under the identification in Proposition 3.3.6. Define $\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_{\mathbb{F}}}$ to be the base change of $\overline{D}^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$ along Ψ. It is a closed subgroupoid of $\widetilde{D}^{(\infty)}_{\mathfrak{P}_{\mathbb{F}}}$.

The following proposition says that $\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_{\mathbb{F}}}$ descends to a closed subgroupoid $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$ of $D_{\mathfrak{P}_{\mathbb{F}}}$:

Proposition 3.3.9. *Let a and b be as in the discussion before Definition 3.3.8. There is a closed subgroupoid $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}} \subset D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}} \subset D_{\mathfrak{P}_{\mathbb{F}}}$ such that $\pi^{(\infty)}|_{\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_{\mathbb{F}}}}$ factors through $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$ and*

$$\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_{\mathbb{F}}} \to D^{\mu}_{\mathfrak{P}_{\mathbb{F}}} \times_{D_{\mathfrak{P}_{\mathbb{F}}}} \widetilde{D}^{(\infty)}_{\mathfrak{P}_{\mathbb{F}}}$$

is an equivalence of closed subgroupoids. Furthermore, $\pi^{\mu} : \widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_{\mathbb{F}}} \to D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$ is formally smooth.

Proof. For any $A \in \mathscr{C}_\Lambda$ define $D^\mu_{\mathfrak{P}_\mathbb{F}}(A)$ to be the full subcategory whose objects are $\pi^{(\infty)}(\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}}(A))$. Observe that for any $x \in D^\mu_{\mathfrak{P}_\mathbb{F}}(A)$ the group $G(\mathfrak{S}_A)$ acts transitively on the fiber $(\pi^{(\infty)})^{-1}(x) \subset \widetilde{D}^{(\infty)}_{\mathfrak{P}_\mathbb{F}}(A)$ by changing the trivialization. The key point is that $\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}}(A)$ is stable under $G(\mathfrak{S}_A)$, by Corollary 3.2.5. Hence,

$$(\pi^{(\infty)})^{-1}(x) \subset \widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}}(A). \tag{3-3-9-1}$$

It is not hard to see then that the map to the fiber product is an isomorphism and that π^μ is formally smooth.

It remains to show that $D^\mu_{\mathfrak{P}_\mathbb{F}} \to D_{\mathfrak{P}_\mathbb{F}}$ is closed. Let $\mathfrak{P}_A \in D_{\mathfrak{P}_\mathbb{F}}(A)$ and choose a trivialization β_A of $\mathfrak{P}_A$, i.e., a lift to $\widetilde{D}^{(\infty)}_{\mathfrak{P}_\mathbb{F}}(A)$. We want a quotient $A \to A'$ such that, for any $f : A \to B$, $\mathfrak{P}_A \otimes_{A,f} B \in D^\mu_{\mathfrak{P}_\mathbb{F}}(B)$ if and only if f factors through A'. Let $A \to A'$ represent the closed condition $\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}} \subset \widetilde{D}^{(\infty)}_{\mathfrak{P}_\mathbb{F}}$. Clearly, $\mathfrak{P}_A \otimes_A A' \in D^\mu_{\mathfrak{P}_\mathbb{F}}(A')$ and so any further base change is as well. Now, let $f : A \to B$ be such that $\mathfrak{P}_A \otimes_{A,f} B \in D^\mu_{\mathfrak{P}_\mathbb{F}}(B)$. The trivialization β_A induces a trivialization β_B on $\mathfrak{P}_B$. The pair $(\mathfrak{P}_B, \beta_B)$ lies in $\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}}(B)$ by (3-3-9-1). □

We have constructed a diagram of formally smooth morphisms

$$\begin{array}{ccc} & \widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}} & \\ {\scriptstyle \pi^\mu}\swarrow & & \searrow{\scriptstyle \Psi^\mu} \\ D^\mu_{\mathfrak{P}_\mathbb{F}} & & \overline{D}^\mu_{Q_\mathbb{F}}, \end{array} \tag{3-3-9-2}$$

where $\overline{D}^\mu_{Q_\mathbb{F}}$ is represented by the completed local ring at the $\mathbb{F}$-point of $M(\mu)$ corresponding to $(Q_\mathbb{F}, \delta_\mathbb{F})$. Next, we would like to replace $\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}}$ by a "smaller" groupoid which is representable.

Let a and b be as in the discussion before Definition 3.3.8 and choose $N > b-a$. Recall the representable groupoid $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_\mathbb{F}}$ (Proposition 3.1.1). Define a closed subgroupoid

$$\widetilde{D}^{(N),\mu}_{\mathfrak{P}_\mathbb{F}} := D^\mu_{\mathfrak{P}_\mathbb{F}} \times_{D_{\mathfrak{P}_\mathbb{F}}} \widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_\mathbb{F}}$$

of $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_\mathbb{F}}$. By Proposition 3.3.9, the morphism $\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}} \to D^{(N),\mu}_{\mathfrak{P}_\mathbb{F}}$ is formally smooth.

Proposition 3.3.10. *For any $N > b-a$, the morphism $\Psi^\mu : \widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_\mathbb{F}} \to \overline{D}^\mu_{\mathfrak{P}_\mathbb{F}}$ factors through $\widetilde{D}^{(N),\mu}_{\mathfrak{P}_\mathbb{F}}$. Furthermore, $\widetilde{D}^{(N),\mu}_{\mathfrak{P}_\mathbb{F}}$ is formally smooth over $\overline{D}^\mu_{Q_\mathbb{F}}$.*

Proof. By our assumption on N, $\widetilde{D}^{(N),\mu}_{\mathfrak{P}_\mathbb{F}}$ is representable, so it suffices to define the factorization $\Psi^\mu_N : \widetilde{D}^{(N),\mu}_{\mathfrak{P}_\mathbb{F}} \to \overline{D}^\mu_{\mathfrak{P}_\mathbb{F}}$ on underlying functors. For any $x \in \widetilde{D}^{(N),\mu}_{\mathfrak{P}_\mathbb{F}}(A)$, set

$$\Psi^{(N),\mu}(x) := \Psi^\mu(\tilde{x})$$

for any lift $\tilde{x}$ of x to $\widetilde{D}^{(\infty),\mu}_{\mathfrak{P}_{\mathbb{F}}}(A)$. The image is independent of the choice of lift by Corollary 3.2.5. The map $\Psi^{(N),\mu}$ is formally smooth since Ψ^{μ} is. □

In the remainder of this section, we discuss the relationship between $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$ and p-adic Hodge type μ. For this, it will useful to work in a larger category than $\widehat{\mathscr{C}}_{\Lambda}$. All of our deformation problems can be extended to the category of complete local Noetherian Λ-algebras R with finite residue field. For any such R, we define $D^{\star}_{\mathfrak{P}_{\mathbb{F}}}(R)$ (and, similarly, $\widetilde{D}^{\star}_{\mathfrak{P}_{\mathbb{F}}}(R)$, $\overline{D}^{\star}_{Q_{\mathbb{F}}}(R)$) to be the category of deformations to R of $\mathfrak{P}_{\mathbb{F}} \otimes_{\mathbb{F}} R/m_R$ with condition $\star$, where $\star$ is any of our various conditions. For any finite local Λ-algebra Λ', the category $\widehat{\mathscr{C}}_{\Lambda'}$ is a subcategory of the category of complete local Noetherian Λ-algebras with finite residue field.

The functors $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}}$, $\widetilde{D}^{(N),\mu}_{\mathfrak{P}_{\mathbb{F}}}$ and $\overline{D}^{\mu}_{Q_{\mathbb{F}}}$ are all representable on $\widehat{\mathscr{C}}_{\Lambda}$. It is easy to check, using the criterion in [Chai et al. 2014, Proposition 1.4.3.6], that these functors commute with change in coefficients, i.e., if $\widetilde{R}^{[a,b],(N)}$ represents $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}}$ over $\mathscr{C}_{\Lambda}$ then $\widetilde{R}^{[a,b],(N)} \otimes_{\Lambda} \Lambda'$ represents the extension of $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}}$ restricted to the category $\widehat{\mathscr{C}}_{\Lambda'}$, and similarly for $\widetilde{D}^{(N),\mu}_{\mathfrak{P}_{\mathbb{F}}}$ and $\overline{D}^{\mu}_{Q_{\mathbb{F}}}$.

An argument as in Theorem 2.4.2 shows that, in $D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}(R)$, an object of $D^{[a,b]}$ is the same as a G-bundle $\mathfrak{P}_R$ on $\widehat{\mathfrak{S}}_R$ together with a Frobenius

$$\phi_{\mathfrak{P}_R} : \varphi^*(\mathfrak{P}_R)[1/E(u)] \cong \mathfrak{P}_R[1/E(u)]$$

deforming $\mathfrak{P}_{\mathbb{F}} \otimes_{\mathbb{F}} R/m_R$ and having height in $[a, b]$. The condition on the height is essential in order to define the Frobenius over R. We would like to give a criterion that says when $(\mathfrak{P}_R, \phi_{\mathfrak{P}_R})$ lies in $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}(R)$.

Choose $(\mathfrak{P}_R, \phi_{\mathfrak{P}_R}) \in D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}(R)$. For any finite extension F' of F and any homomorphism $x : R \to F'$, denote the base change of $\mathfrak{P}_R$ to $\mathfrak{S}_{F'}$ by $(\mathfrak{P}_x, \phi_x)$. Associated to $(\mathfrak{P}_x, \phi_x)$ is a functor $\mathfrak{D}_x$ from $\mathrm{Rep}_F(G_F)$ to filtered $(K \otimes_{\mathbb{Q}_p} F')$-modules given by $\mathfrak{D}_x(W) = \varphi^*(\mathfrak{P}_x)(W)/E(u)\varphi^*(\mathfrak{P}_x)(W)$ with the filtration defined as in Definition 2.4.8.

Lemma 3.3.11. *For any finite extension F' of F and any $x : R \to F'$, the functor $\mathfrak{D}_x$ is a tensor exact functor.*

Proof. Any such x factors through the ring of integers Λ' of F', so that $(\mathfrak{P}_x, \phi_x)$ comes from a pair $(\mathfrak{P}_{x_0}, \phi_{x_0})$ over $\mathfrak{S}_{\Lambda'}$. Let $\widehat{S}_{\Lambda'}$ and $\widehat{S}_{F'}$ be the $E(u)$-adic completions of $(W \otimes_{\mathbb{Z}_p} \Lambda')[u]$ and $(W \otimes_{\mathbb{Z}_p} F')[u]$, respectively. By Lemma 3.3.7, we can equivalently think of $(\mathfrak{P}_{x_0}, \phi_{x_0})$ as a pair over $\widehat{S}_{\Lambda'}$.

Choose a trivialization β_0 of $\mathfrak{P}_{x_0}$ and set $Q_{x_0} := \varphi^*(\mathfrak{P}_{x_0})$ with trivialization $\delta_{x_0} := \beta_0[1/E(u)] \circ \phi_{x_0}$. Define (Q_x, δ_x) to be $(Q_{x_0}, \delta_{x_0}) \otimes_{\widehat{S}_{\Lambda'}} \widehat{S}_{F'}$ and define a filtration on $\mathfrak{D}_{Q_x} := Q_x \bmod E(u)$ by

$$\mathrm{Fil}^i(\mathfrak{D}_{Q_x}(W)) = (Q_x(W) \cap E(u)^i(W \otimes \widehat{S}_{F'}))/(E(u)Q_x(W) \cap E(u)^i(W \otimes \widehat{S}_{F'}))$$

for any $W \in \mathrm{Rep}_F(G_F)$. Since $\widehat{S}_{\Lambda'}[1/p]/(E(u)) = \widehat{S}_{F'}/(E(u))$, there is an isomorphism

$$\mathfrak{D}_x \cong \mathfrak{D}_{Q_x}$$

of tensor exact functors to $\mathrm{Mod}_{K\otimes_{\mathbb{Q}_p} F'}$ identifying the filtrations.

It suffices then to show that $\mathfrak{D}_{Q_x}$ is a tensor exact functor to the category of filtered $(K\otimes_{\mathbb{Q}_p} F')$-modules. Without loss of generality, we assume that F' contains a Galois closure of K. Then

$$\widehat{S}_{F'} \cong \prod_{\psi} F'[\![u-\psi(\pi)]\!]$$

over embeddings $\psi : K \to F'$ (first decompose $W \otimes_{\mathbb{Z}_p} F'$ and then decompose $E(u)$ in each factor). Thus, (Q_x, δ_x) decomposes as a product $\prod_\psi (Q_x^\psi, \delta_x^\psi)$, where each pair defines a point z_ψ of the affine Grassmannian of $G_{F'}$. The quotient $\mathfrak{D}_{Q_x}$ decomposes compatibly as $\prod_\psi \mathfrak{D}_{Q_x^\psi}$. We are reduced then to a computation for a point $z_\psi \in \mathrm{Gr}_{G_{F'}}(F')$. Without loss of generality, we can assume $G_{F'}$ is split. Up to translation by the positive loop group (which induces an isomorphism on filtrations), z_ψ is the image $[g]$ for some $g \in T(F'((t)))$ where T is maximal split torus of $G_{F'}$. Using the weight space decomposition for T on any representation W, one can compute directly that $\mathfrak{D}_{Q_x^\psi}$ is a tensor exact functor. For more details, see [Levin 2013, Proposition 3.5.11, Lemma 8.2.15]. □

Definition 3.3.12. Let F' be any finite extension of F with ring of integers Λ'. We say a G-Kisin module $(\mathfrak{P}_{\Lambda'}, \phi_{\Lambda'})$ over Λ' has *p-adic Hodge type* μ if the G_F-filtration associated to $\mathfrak{P}_{\Lambda'}[1/p]$ as above has type μ.

Theorem 3.3.13. *Assume that $F = F_{[\mu]}$. Let R be any complete local Noetherian Λ-algebra with finite residue field which is Λ-flat and reduced. Then $\mathfrak{P}_R \in D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}(R)$ lies in $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}(R)$ if and only if, for all finite extensions F'/F and all homomorphisms $x : R \to F'$, the G_F-filtration $\mathfrak{D}_x$ has type less than or equal to $[\mu]$.*

Proof. Choose a lift $\tilde{y}$ of $\mathfrak{P}_R$ to $\widetilde{D}^{[a,b],(N)}_{\mathfrak{P}_{\mathbb{F}}}(R)$. Clearly, $\mathfrak{P}_R \in D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}(R)$ if and only if $\tilde{y} \in \widetilde{D}^{(N),\mu}_{\mathfrak{P}_{\mathbb{F}}}(R)$, which happens if and only $\Psi(\tilde{y}) \in \overline{D}^{\mu}_{Q_{\mathbb{F}}}(R)$. Let R^μ be the quotient of R representing the fiber product

$$\mathrm{Spf}\, R \times_{\overline{D}^{[a,b]}_{Q_{\mathbb{F}}}} \overline{D}^{\mu}_{Q_{\mathbb{F}}}.$$

To show that $R^\mu = R$, it suffices to show that $\mathrm{Spec}\, R^\mu[1/p]$ contains all closed points of $\mathrm{Spec}\, R[1/p]$, since R is flat and $R[1/p]$ is reduced and Jacobson.

The groupoid $\overline{D}^{\mu}_{Q_{\mathbb{F}}}$ is represented by a completed stalk on the local model $M(\mu) \subset \mathrm{Gr}_G^{E(u),W}$, so that, for any $x : R \to F'$, $\Psi(\tilde{y})[1/p]$ defines an F'-point (Q_x, δ_x) of $\mathrm{Gr}_G^{E(u),W}$. Since $M(\mu)(F') = S(\mu)(F')$, $(Q_x, \delta_x) \in S(\mu)(F')$ if and

only if the filtration $\mathfrak{D}_{Q_x}$ has type less than or equal to $[\mu]$ [Levin 2013, Proposition 3.5.11]. The proof of Lemma 3.3.11 shows that the two filtrations agree, i.e.,

$$\mathfrak{D}_x \cong \mathfrak{D}_{Q_x}.$$

Thus, x factors through R^μ exactly when the type of the filtration $\mathfrak{D}_x$ is less than or equal to $[\mu]$. □

Fix a continuous representation $\bar\eta : \Gamma_K \to G(\mathbb{F})$. Let $R_{\bar\eta}^{[a,b],\mathrm{cris}}$ be the universal framed G-valued crystalline deformation ring with Hodge–Tate weights in $[a, b]$, and let $\Theta : X_{\bar\eta}^{[a,b],\mathrm{cris}} \to \operatorname{Spec} R_{\bar\eta}^{[a,b],\mathrm{cris}}$ be as in Proposition 2.3.3.

Definition 3.3.14. Assume $F = F_{[\mu]}$. Define $R_{\bar\eta}^{\mathrm{cris},\le\mu}$ to be the flat closure of the connected components of

$$\operatorname{Spec} R_{\bar\eta}^{[a,b],\mathrm{cris}}[1/p]$$

with type less than or equal to μ (see Theorem 2.4.6). Define $X_{\bar\eta}^{\mathrm{cris},\le\mu}$ to be the flat closure in $X_{\bar\eta}^{[a,b],\mathrm{cris}}$ of the same connected components (since $\Theta[1/p]$ is an isomorphism).

Corollary 3.3.15. *Let $X_{\bar\eta}^{\mathrm{cris},\le\mu}$ be as in Definition 3.3.14. A point $\bar x \in X_{\bar\eta}^{\mathrm{cris},\le\mu}(\mathbb{F}')$ corresponds to a G-Kisin lattice $\mathfrak{P}_{\mathbb{F}'}$ over $\mathfrak{S}_{\mathbb{F}'}$. The deformation problem $D_{\bar x}^{\mathrm{cris},\mu}$ which assigns to any $A \in \mathscr{C}_{\Lambda\otimes_{\mathbb{Z}_p} W(\mathbb{F}')}$ the set of isomorphisms classes of triples*

$$\{(y, \mathfrak{P}_A, \delta_A) \mid y : R_{\bar\eta}^{\mathrm{cris},\le\mu} \to A,\ \mathfrak{P}_A \in D^\mu_{\mathfrak{P}_{\mathbb{F}'}}(A),\ \delta_A : T_{G,\mathfrak{S}_A}(\mathfrak{P}_A) \cong \eta_y|_{\Gamma_\infty}\}$$

is representable. Furthermore, if $\widehat{\mathbb{O}}_{\bar x}^\mu$ is the completed local ring of $X_{\bar\eta}^{\mathrm{cris},\le\mu}$ at $\bar x$, then the natural map $\operatorname{Spf}\widehat{\mathbb{O}}_{\bar x}^\mu \to D_{\bar x}^{\mathrm{cris},\mu}$ is a closed immersion which is an isomorphism modulo p-power torsion.

Proof. Without loss of generality, we can replace Λ by $\Lambda \otimes_{W(\mathbb{F})} W(\mathbb{F}')$. By construction and Proposition 2.3.5, for any $A \in \mathscr{C}_\Lambda$, the deformation functor

$$D_{\bar x}^{\mathrm{cris},\mu,\mathrm{bc}}(A) = \{y : R_{\bar\eta}^{\mathrm{cris},\le\mu} \to A, \mathfrak{P}_A \in D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}'}}(A), \delta_A : T_{G,\mathfrak{S}_A}(\mathfrak{P}_A) \cong \eta_y|_{\Gamma_\infty}\}/\cong$$

is representable. That is, $D_{\bar x}^{\mathrm{cris},\mu,\mathrm{bc}}$ represents the completed stalk at a point of the fiber product $X_{\bar\eta}^{[a,b],\mathrm{cris}} \times_{\operatorname{Spec} R_{\bar\eta}^{[a,b],\mathrm{cris}}} \operatorname{Spec} R_{\bar\eta}^{\mathrm{cris},\le\mu}$. Since $D^\mu_{\mathfrak{P}_{\mathbb{F}'}} \subset D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}'}}$ is closed, so is $D_{\bar x}^{\mathrm{cris},\mu} \subset D_{\bar x}^{\mathrm{cris},\mu,\mathrm{bc}}$ and hence $D_{\bar x}^{\mathrm{cris},\mu}$ is representable by $R_{\bar x}^{\mathrm{cris},\mu}$. To see that the closed immersion $\operatorname{Spf}\widehat{\mathbb{O}}_{\bar x}^\mu \to D_{\bar x}^{\mathrm{cris},\mu,\mathrm{bc}}$ factors through $D_{\bar x}^{\mathrm{cris},\mu}$, it suffices to show that the "universal" lattice $\mathfrak{P}_{\widehat{\mathbb{O}}_{\bar x}^\mu} \in D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}'}}(\widehat{\mathbb{O}}_{\bar x}^\mu)$ lies in $D^\mu_{\mathfrak{P}_{\mathbb{F}'}}(\widehat{\mathbb{O}}_{\bar x}^\mu)$.

By Proposition 2.3.9 and Theorem 2.3.12, $\Theta[1/p]$ is an isomorphism. Furthermore, by [Balaji 2012, Proposition 4.1.5], $R_{\bar\eta}^{[a,b],\mathrm{cris}}[1/p]$ and $R_{\bar\eta}^{\mathrm{cris},\le\mu}[1/p]$ are formally smooth over F. Hence, $\widehat{\mathbb{O}}_{\bar x}^\mu$ satisfies the hypotheses of Theorem 3.3.13.

By Theorem 3.3.13, we are reduced to showing that for any finite F'/F and any homomorphism $x : \widehat{\mathbb{O}}_{\bar x}^\mu \to F'$ the filtration $\mathfrak{D}_x$ corresponding to the base change

$\mathfrak{P}_x := \mathfrak{P}_{\widehat{\mathbb{O}}^{\mu}_{\bar{x}}} \otimes_x F'$ has type less than or equal to μ. The homomorphism x corresponds to a closed point of Spec $R^{\mathrm{cris},\leq\mu}_{\bar{\eta}}[1/p]$, i.e., a crystalline representation ρ_x with p-adic Hodge type less than or equal to μ. Furthermore, $\mathfrak{P}_x$ is the unique $(\mathfrak{S}_{F'}, \varphi)$-module of bounded height associated to ρ_x. By Proposition 2.4.9, the de Rham $\mathscr{F}^{\mathrm{dR}}_{\rho_x}$ filtration associated to ρ_x is isomorphic to the filtration $\mathfrak{D}_x$ associated to $(\mathfrak{P}_x, \phi_x)$. Thus, $\mathfrak{D}_x$ has type less than or equal to μ for all points x and so $\mathfrak{P}_{\widehat{\mathbb{O}}^{\mu}_{\bar{x}}} \in D^{\mu}_{\mathfrak{P}_{\mathbb{F}'}}(\widehat{\mathbb{O}}^{\mu}_{\bar{x}})$, by Theorem 3.3.13.

By the argument above, Spec $\widehat{\mathbb{O}}^{\mu}_{\bar{x}}$ and Spec $R^{\mathrm{cris},\mu}_{\bar{x}}$ have the same F'-points for any finite extension of F. Since $R^{\mathrm{cris},\leq\mu}_{\bar{\eta}}[1/p]$ is formally smooth over F, the kernel of $R^{\mathrm{cris},\mu}_{\bar{x}} \to \widehat{\mathbb{O}}^{\mu}_{\bar{x}}$ is p-power torsion. □

Remark 3.3.16. In fact, Corollary 3.3.15 holds as well for semistable deformation rings with p-adic Hodge type less than or equal to μ. To apply Theorem 3.3.13 and make the final deduction, we needed that the generic fiber of the crystalline deformation ring was reduced (to argue at closed points). This is true for G-valued semistable deformation rings by the main result of [Bellovin 2014].

4. Local analysis

In this section, we analyze finer properties of crystalline G-valued deformation rings with minuscule p-adic Hodge type. The techniques in this section are inspired by [Kisin 2009; Liu 2013]. We develop a theory of $(\varphi, \widehat{\Gamma})$-modules with G-structure and our main result, Theorem 4.3.6, is stated in these terms. However, the idea is the following: given a G-Kisin module $(\mathfrak{P}_A, \phi_A)$ over some finite Λ-algebra A, we get a representation of Γ_∞ via the functor $T_{G,\mathfrak{S}_A}$. In general, this representation need not extend (and certainly not in a canonical way) to a representation of the full Galois group Γ_K. When $G = \mathrm{GL}_n$ and $\mathfrak{P}_A$ has height in $[0, 1]$ then, via the equivalence between Kisin modules with height in $[0, 1]$ and finite flat group schemes [Kisin 2006, Theorem 2.3.5], one has a canonical extension to Γ_K which is flat. We show (at least when A is a Λ-flat domain) that the same holds for G-Kisin modules of minuscule type: there exists a canonical extension to Γ_K which is crystalline. This is stated precisely in Corollary 4.3.8. We end by applying this result to identify the connected components of G-valued crystalline deformation rings with the connected components of a moduli space of G-Kisin modules (Corollary 4.4.2).

4.1. ***Minuscule cocharacters.*** We begin with some preliminaries on minuscule cocharacters and adjoint representations which we use in our finer analysis with $(\varphi, \widehat{\Gamma})$-modules in the subsequent sections.

Let H be a reductive group over field κ. The conjugation action of H on itself gives a representation

$$\mathrm{Ad} : H \to \mathrm{GL}(\mathrm{Lie}(H)). \tag{4-1-0-1}$$

This is algebraic, so, for any κ-algebra R, $H(R)$ acts on $\mathrm{Lie}(H_R) = \mathrm{Lie}\, H \otimes_\kappa R$. We will use Ad to denote these actions as well. We can define Ad for G over $\mathrm{Spec}\,\Lambda$ in the same way.

Definition 4.1.1. Any cocharacter $\lambda : \mathbb{G}_m \to H$ gives a grading on $\mathrm{Lie}\, H$ defined by

$$\mathrm{Lie}\, H(i) := \{Y \in \mathrm{Lie}\, H \mid \mathrm{Ad}(\lambda(a))Y = a^i Y\}.$$

A cocharacter λ is called *minuscule* if $\mathrm{Lie}\, H(i) = 0$ for $i \notin \{-1, 0, 1\}$.

Minuscule cocharacters were studied by Deligne [1979] in connection with the theory of Shimura varieties. A detailed exposition of their main properties can be found in §1 of [Gross 2000].

Assume now that H is split and fix a maximal split torus T contained in a Borel subgroup B. This gives rise to a set of simple roots Δ and a set of simple coroots $\Delta^\vee$. In each conjugacy class of cocharacters, there is a unique dominant cocharacter valued in T. The set of dominant cocharacters is denoted by $X_*(T)^+$.

Recall the Bruhat (partial) ordering on $X_*(T)^+$: given dominant cocharacters $\mu, \mu' : \mathbb{G}_m \to T$, we say $\mu' \leq \mu$ if $\mu - \mu' = \sum_{\alpha \in \Delta^\vee} n_\alpha \alpha$ with $n_\alpha \geq 0$.

Proposition 4.1.2. *Let μ be a dominant minuscule cocharacter. Then there is no dominant μ' such that $\mu' < \mu$ in the Bruhat order.*

Proof. See Exercise 24 from Chapter IV.1 of [Bourbaki 2002]. □

Proposition 4.1.3. *If μ is a minuscule cocharacter, then the (open) affine Schubert variety $S^0(\mu)$ is equal to $S(\mu)$. Furthermore, $S(\mu)$ is smooth and projective. In fact, $S(\mu) \cong H/P(\mu)$, where $P(\mu)$ is a parabolic subgroup associated to the cocharacter μ.*

Proof. Since the closure $S(\mu) = \bigcup_{\mu' \leq \mu} S^0(\mu')$ [Richarz 2013, Proposition 2.8], the first part follows from Proposition 4.1.2. For the remaining facts, we refer to discussion after [Pappas et al. 2013, Definition 1.3.5] and [Levin 2013, Proposition 3.5.7]. □

For any $\mu : \mathbb{G}_m \to T$, we get an induced map $\mathbb{G}_m(\kappa((t))) \to T(\kappa((t))) \subset H(\kappa((t)))$ on loop groups. We let $\mu(t)$ denote the image of $t \in \kappa((t))^\times$.

Proposition 4.1.4. *For any $X \in \mathrm{Lie}\, H \otimes_\kappa \kappa[\![t]\!]$, we have*

$$\mathrm{Ad}(\mu(t))(X) \in \frac{1}{t}(\mathrm{Lie}\, H \otimes_\kappa \kappa[\![t]\!]).$$

Proof. As in Definition 4.1.1, we can decompose

$$\mathrm{Lie}\, H = \mathrm{Lie}\, H(-1) \oplus \mathrm{Lie}\, H \oplus \mathrm{Lie}\, H(1).$$

Then $\mathrm{Ad}(\mu(t))$ acts on $\mathrm{Lie}\, H(i) \otimes \kappa((t))$ by multiplication by t^i. The largest denominator is then t^{-1}. □

4.2. ***Theory of $(\varphi, \widehat{\Gamma})$-modules with G-structure.*** We review Liu's theory [2010; Caruso and Liu 2011] of $(\varphi, \widehat{G})$. We call them $(\varphi, \widehat{\Gamma})$-modules to avoid confusion with the algebraic group G. The theory of $(\varphi, \widehat{\Gamma})$-modules is an adaptation of the theory of (φ, Γ)-modules to the non-Galois extension $K_\infty = K(\pi^{1/p}, \pi^{1/p^2}, \dots)$. The $\widehat{\Gamma}$ refers to an additional structure added to a Kisin module which captures the full action of Γ_K as opposed to just the subgroup $\Gamma_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$. The main theorem in [Liu 2010] is an equivalence of categories between (torsion-free) $(\varphi, \widehat{\Gamma})$-modules and Γ_K-stable lattices in semistable $\mathbb{Q}_p$-representations.

Let $\widetilde{\boldsymbol{E}}^+$ denote the perfection of $\mathcal{O}_{\overline{K}}/(p)$. There is a unique surjective map

$$\Theta : W(\widetilde{\boldsymbol{E}}^+) \to \widehat{\mathcal{O}}_{\overline{K}}$$

which lifts the projection $\widetilde{\boldsymbol{E}}^+ \to \mathcal{O}_{\overline{K}}/(p)$. The compatible system $(\pi^{1/p^n})_{n\geq 0}$ of the p^n-th roots of π defines an element $\underline{\pi}$ of $\widetilde{\boldsymbol{E}}^+$. Let $[\underline{\pi}]$ denote the Teichmüller representative in $W(\widetilde{\boldsymbol{E}}^+)$. There is an embedding

$$\mathfrak{S} \hookrightarrow W(\widetilde{\boldsymbol{E}}^+),$$

defined by $u \mapsto [\underline{\pi}]$, which is compatible with the Frobenii. If $\widetilde{\boldsymbol{E}}$ is the fraction field of $\widetilde{\boldsymbol{E}}^+$, then $W(\widetilde{\boldsymbol{E}}^+) \subset W(\widetilde{\boldsymbol{E}})$. The embedding $\mathfrak{S} \hookrightarrow W(\widetilde{\boldsymbol{E}}^+)$ extends to an embedding

$$\mathcal{O}_{\mathscr{E}} \hookrightarrow W(\widetilde{\boldsymbol{E}}).$$

As before, let $K_\infty = \bigcup K(\pi^{1/p^n})$. Set $K_{p^\infty} := \bigcup K(\zeta_{p^n})$, where ζ_{p^n} is a primitive p^n-th root of unity. Denote the compositum of K_∞ and K_{p^∞} by K_{∞,p^∞}; K_{∞,p^∞} is Galois over K.

Definition 4.2.1. Define

$$\widehat{\Gamma} := \mathrm{Gal}(K_{\infty,p^\infty}/K) \quad \text{and} \quad \widehat{\Gamma}_\infty := \mathrm{Gal}(K_{\infty,p^\infty}/K_\infty).$$

There is a subring $\widehat{R} \subset W(\widetilde{\boldsymbol{E}}^+)$ which plays a central role in the theory of $(\varphi, \widehat{\Gamma})$-modules. The definition can be found on p. 5 of [Liu 2010]. The relevant properties of $\widehat{R}$ are

(1) $\widehat{R}$ is stable by the Frobenius on $W(\widetilde{\boldsymbol{E}}^+)$;

(2) $\widehat{R}$ contains $\mathfrak{S}$;

(3) $\widehat{R}$ is stable under the action of the Galois group Γ_K and Γ_K acts through the quotient $\widehat{\Gamma}$.

For any $\mathbb{Z}_p$-algebra A, set $\widehat{R}_A := \widehat{R} \otimes_{\mathbb{Z}_p} A$ with a Frobenius induced by the Frobenius on $\widehat{R}$. Similarly, define $W(\widetilde{\boldsymbol{E}}^+)_A := W(\widetilde{\boldsymbol{E}}^+) \otimes_{\mathbb{Z}_p} A$ and $W(\widetilde{\boldsymbol{E}})_A := W(\widetilde{\boldsymbol{E}}) \otimes_{\mathbb{Z}_p} A$. For any $\mathfrak{S}_A$-module $\mathfrak{M}_A$, define

$$\widehat{\mathfrak{M}}_A := \widehat{R}_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M}_A = \widehat{R}_A \otimes_{\mathfrak{S}_A} \varphi^*(\mathfrak{M}_A)$$

and

$$\widetilde{\mathfrak{M}}_A := W(\widetilde{\boldsymbol{E}}^+)_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M}_A = W(\widetilde{\boldsymbol{E}}^+)_A \otimes_{\widehat{R}_A} \widehat{\mathfrak{M}}_A.$$

Recall that $\varphi^*(\mathfrak{M}_A) := \mathfrak{S}_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M}_A$ and that the linearized Frobenius is a map $\phi_{\mathfrak{M}_A} : \varphi^*(\mathfrak{M}_A) \to \mathfrak{M}_A$ (when $\mathfrak{M}_A$ has height in $[0,\infty)$).

If $\mathfrak{M}_A$ is a projective $\mathfrak{S}_A$-module then, by Lemma 3.1.1 in [Caruso and Liu 2011], $\varphi^*(\mathfrak{M}_A) \subset \widehat{\mathfrak{M}}_A \subset \widetilde{\mathfrak{M}}_A$. Although the map $m \mapsto 1 \otimes m$ from $\mathfrak{M}_A$ to $\widehat{\mathfrak{M}}_A$ is not $\mathfrak{S}_A$-linear, it is injective when $\mathfrak{M}_A$ is $\mathfrak{S}_A$-projective. The image is a $\varphi(\mathfrak{S}_A)$-submodule of $\widehat{\mathfrak{M}}_A$. We will think of $\mathfrak{M}_A$ inside of $\widehat{\mathfrak{M}}_A$ in this way. Finally, for any étale $(\mathbb{O}_{\mathscr{E},A}, \varphi)$-module $\mathcal{M}_A$, we define

$$\widetilde{\mathcal{M}}_A := W(\widetilde{\boldsymbol{E}})_A \otimes_{\varphi,\mathbb{O}_{\mathscr{E},A}} \mathcal{M}_A = W(\widetilde{\boldsymbol{E}})_A \otimes_{\mathbb{O}_{\mathscr{E},A}} \varphi^*(\mathcal{M}_A)$$

with semilinear Frobenius extending the Frobenius on $\mathcal{M}_A$. To summarize, for any Kisin module $(\mathfrak{M}_A, \phi_A)$, we have the diagram

$$\begin{array}{ccccc} (\mathfrak{M}_A, \phi_A) & \rightsquigarrow & \widehat{\mathfrak{M}}_A & \rightsquigarrow & \widetilde{\mathfrak{M}}_A \\ \wr\downarrow & & & & \wr\downarrow \\ (\mathcal{M}_A, \phi_A) & & \rightsquigarrow & & (\widetilde{\mathcal{M}}_A, \tilde{\phi}_A). \end{array}$$

Now, let $\gamma \in \hat{\Gamma}$ and let $\widehat{\mathfrak{M}}_A$ be an $\hat{R}_A$-module. A map $g : \widehat{\mathfrak{M}}_A \to \widehat{\mathfrak{M}}_A$ is *γ-semilinear* if

$$g(am) = \gamma(a)g(m)$$

for any $a \in \hat{R}_A$, $m \in \widehat{\mathfrak{M}}_A$. A (semilinear) $\hat{\Gamma}$-action on $\widehat{\mathfrak{M}}_A$ is a γ-semilinear map g_γ for each $\gamma \in \hat{\Gamma}$ such that

$$g_{\gamma'} \circ g_\gamma = g_{\gamma'\gamma}$$

as $(\gamma'\gamma)$-semilinear morphisms. A (semilinear) $\hat{\Gamma}$-action on $\widehat{\mathfrak{M}}_A$ extends in the natural way to a (semilinear) Γ_K-action on $\widetilde{\mathfrak{M}}_A$ and on $\widetilde{\mathcal{M}}_A$.

For any local Artinian $\mathbb{Z}_p$-algebra A, choose a $\mathbb{Z}_p$-module isomorphism $A \cong \bigoplus \mathbb{Z}/p^{n_i}\mathbb{Z}$ so that, as a $W(\widetilde{\boldsymbol{E}})$-module, $W(\widetilde{\boldsymbol{E}})_A \cong \bigoplus W_{n_i}(\widetilde{\boldsymbol{E}})$. We equip $W(\widetilde{\boldsymbol{E}})_A$ with the product topology, where $W_{n_i}(\widetilde{\boldsymbol{E}})$ has a topology induced by the isomorphism $W_{n_i}(\widetilde{\boldsymbol{E}}) \cong \widetilde{\boldsymbol{E}}^{n_i}$ given by Witt components (see §4.3 of [Brinon and Conrad 2009] for more details on the topology of $\widetilde{\boldsymbol{E}}$). We can similarly define a topology on $W(\widetilde{\boldsymbol{E}}^+)_A$ using the topology on $\widetilde{\boldsymbol{E}}^+$, and it is clear that this is the same as the subspace topology from the inclusion $W(\widetilde{\boldsymbol{E}}^+)_A \subset W(\widetilde{\boldsymbol{E}})_A$. Finally, we give $\hat{R}_A$ the subspace topology from the inclusion $\hat{R}_A \subset W(\widetilde{\boldsymbol{E}}^+)_A$. The same procedure works for A finite flat over $\mathbb{Z}_p$.

A $\hat{\Gamma}$-action on $\widehat{\mathfrak{M}}_A$ is *continuous* if, for any basis (equivalently for all bases) of $\widehat{\mathfrak{M}}_A$, the induced map $\hat{\Gamma} \to \mathrm{GL}_r(\hat{R}_A)$ is continuous, where r is the rank of $\widehat{\mathfrak{M}}_A$ (such a basis exists by [Kisin 2009, Lemma 1.2.2(4)]).

Definition 4.2.2. Let A be a finite $\mathbb{Z}_p$-algebra. A $(\varphi, \hat{\Gamma})$-*module with height in* $[a, b]$ over A is a triple $(\mathfrak{M}_A, \phi_{\mathfrak{M}_A}, \hat{\Gamma})$, where

(1) $(\mathfrak{M}_A, \phi_{\mathfrak{M}_A}) \in \mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,[a,b]}$;

(2) $\hat{\Gamma}$ is a continuous (semilinear) $\hat{\Gamma}$-action on $\widehat{\mathfrak{M}}_A$;

(3) the Γ_K-action on $\widetilde{\mathfrak{M}}_A$ commutes with $\tilde{\phi}_{\mathfrak{M}_A}$ (as endomorphisms of $\widetilde{\mathcal{M}}_A$);

(4) regarding $\mathfrak{M}_A$ as a $\varphi(\mathfrak{S}_A)$-submodule of $\widehat{\mathfrak{M}}_A$, we have $\mathfrak{M}_A \subset \widehat{\mathfrak{M}}_A^{\hat{\Gamma}\infty}$;

(5) $\hat{\Gamma}$ acts trivially on $\widehat{\mathfrak{M}}_A / I_+(\widehat{\mathfrak{M}}_A)$ (see §3.1 of [Caruso and Liu 2011] for the definition of $I_+(\widehat{\mathfrak{M}}_A)$).

We often refer to the additional data of a $(\varphi, \hat{\Gamma})$-module on a Kisin module as a $\hat{\Gamma}$-*structure*.

Remark 4.2.3. Although we allow arbitrary height $[a, b]$ (in particular, negative height), the ring $\hat{R}$ is still sufficient for defining the $\hat{\Gamma}$-action. This follows from the fact that the $\hat{\Gamma}$-action on $\mathfrak{S}(1)$ is given by $\hat{c}$ (see [Liu 2010, Example 3.2.3]), which is a unit in $\hat{R}$. See also [Levin 2013, Example 9.1.9].

Proposition 4.2.4. *Choose* $(\mathfrak{M}_A, \phi_{\mathfrak{M}_A}) \in \mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,[a,b]}$ *of rank* r. *Fix a basis* $\{f_i\}$ *of* $\mathfrak{M}_A$. *Let* C' *be the matrix for* $\phi_{\mathfrak{M}_A}$ *with respect to* $\{1 \otimes_\varphi f_i\}$. *Then a* $\hat{\Gamma}$-*structure on* $\mathfrak{M}_A$ *is the same as a continuous map*

$$B_\bullet : \hat{\Gamma} \to \mathrm{GL}_r(\hat{R}_A)$$

such that

(i) $C' \cdot \varphi(B_\gamma) = B_\gamma \cdot \gamma(C')$ *in* $\mathrm{Mat}(W(\tilde{\boldsymbol{E}})_A)$ *for all* $\gamma \in \hat{\Gamma}$;

(ii) $B_\gamma = \mathrm{Id}$ *for all* $\gamma \in \hat{\Gamma}_\infty$;

(iii) $B_\gamma \equiv \mathrm{Id} \bmod I_+(\hat{R})_A$ *for all* $\gamma \in \hat{\Gamma}$;

(iv) $B_{\gamma\gamma'} = B_\gamma \cdot \gamma(B_{\gamma'})$ *for all* $\gamma, \gamma' \in \hat{\Gamma}$.

Let $\mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,[a,b],\hat{\Gamma}}$ denote the category of $(\varphi, \hat{\Gamma})$-modules with height in $[a, b]$ over A. A morphism between $(\varphi, \hat{\Gamma})$-modules is a morphism in $\mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,[a,b]}$ that is $\hat{\Gamma}$-equivariant when extended to $\hat{R}_A$.

Let $\mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,\mathrm{bh},\hat{\Gamma}} := \bigcup_{h>0} \mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,[-h,h],\hat{\Gamma}}$, so $\mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,\mathrm{bh},\hat{\Gamma}}$ has a natural tensor product operation which at the level of $\mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,\mathrm{bh}}$ is the tensor product of bounded height Kisin modules. The $\hat{\Gamma}$-structure on the tensor product is defined via

$$\hat{R}_A \otimes_{\varphi,\mathfrak{S}_A} (\mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{N}_A) \cong (\hat{R}_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M}_A) \otimes_{\hat{R}_A} (\hat{R}_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{N}_A) = \widehat{\mathfrak{M}}_A \otimes_{\hat{R}_A} \widehat{\mathfrak{N}}_A.$$

One also defines a $\widehat{\Gamma}$-structure on the dual $\mathfrak{M}_A^\vee := \mathrm{Hom}_{\mathfrak{S}_A}(\mathfrak{M}_A, \mathfrak{S}_A)$ in the natural way (see the discussion after [Levin 2013, Proposition 9.1.5]). Note that, unlike in other references (for example [Ozeki 2013]), we do not include any Tate twist in our definition of duals.

We will now relate these $(\varphi, \widehat{\Gamma})$-modules to Γ_K-representations. For this, we require that A be $\mathbb{Z}_p$-finite and either $\mathbb{Z}_p$-flat or Artinian. Define a functor $\widehat{T}_A$ from $\mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,\mathrm{bh},\widehat{\Gamma}}$ to Galois representations by

$$\widehat{T}_A(\widehat{\mathfrak{M}}_A) := (W(\widetilde{\boldsymbol{E}}) \otimes_{\widehat{R}} \widehat{\mathfrak{M}}_A)^{\tilde{\phi}_A = 1} = (\widetilde{\mathcal{M}}_A)^{\tilde{\phi}_A = 1}.$$

The semilinear Γ_K-action on $\widetilde{\mathcal{M}}_A$ commutes with $\tilde{\phi}_A$, so $\widehat{T}_A(\widehat{\mathfrak{M}}_A)$ is a Γ_K-stable A-submodule of $W(\widetilde{\boldsymbol{E}}) \otimes_{\widehat{R}} \widehat{\mathfrak{M}}_A$.

We now recall the basic facts we will need about $\widehat{T}_A$:

Proposition 4.2.5. *Let A be $\mathbb{Z}_p$-finite and either $\mathbb{Z}_p$-flat or Artinian.*

(1) *If $\widehat{\mathfrak{M}}_A \in \mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,\mathrm{bh},\widehat{\Gamma}}$, then there is a natural $A[\Gamma_\infty]$-module isomorphism*

$$\theta_A : T_{\mathfrak{S}_A}(\mathfrak{M}_A) \to \widehat{T}_A(\widehat{\mathfrak{M}}_A).$$

Furthermore, θ_A is functorial with respect to morphisms in $\mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,\mathrm{bh},\widehat{\Gamma}}$.

(2) *$\widehat{T}_A$ is an exact tensor functor from $\mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,\mathrm{bh},\widehat{\Gamma}}$ to $\mathrm{Rep}_A(\Gamma_K)$ which is compatible with duals.*

Proof. See [Levin 2013, Propositions 9.1.6 and 9.1.7]. □

We are now ready to add G-structure to $(\varphi, \widehat{\Gamma})$-modules. Let G be a connected reductive group over a $\mathbb{Z}_p$-finite and flat local domain Λ as in previous sections.

Definition 4.2.6. Define $\mathrm{GMod}_{\mathfrak{S}_A}^{\varphi,\widehat{\Gamma}}$ to be the category of faithful exact tensor functors $[^f \mathrm{Rep}_\Lambda(G), \mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,\mathrm{bh},\widehat{\Gamma}}]^\otimes$. We will refer to these as $(\varphi, \widehat{\Gamma})$-*modules with G-structure*.

Recall the category $\mathrm{GRep}_A(\Gamma_K)$ from Definition 2.2.3. By Proposition 4.2.5(2), $\widehat{T}_A$ induces a functor

$$\widehat{T}_{G,A} : \mathrm{GMod}_{\mathfrak{S}_A}^{\varphi,\widehat{\Gamma}} \to \mathrm{GRep}_A(\Gamma_K).$$

Furthermore, if $\omega_{\Gamma_\infty} : \mathrm{GRep}_A(\Gamma_K) \to \mathrm{GRep}_A(\Gamma_\infty)$ is the forgetful functor then there is an natural isomorphism

$$T_{G,\mathfrak{S}_A} \cong \omega_{\Gamma_\infty} \circ \widehat{T}_{G,A}.$$

The functor $\widehat{T}_{G,A}$ behaves well with respect to base change along finite maps $A \to A'$ by the same argument as in Proposition 2.2.4.

We end this section by adding G-structure to the main result of [Liu 2010]. For A finite flat over Λ, an element (P_A, ρ_A) of $\mathrm{GRep}_A(\Gamma_K)$ is *semistable* (resp. *crystalline*) if $\rho_A[1/p] : \Gamma_K \to \mathrm{Aut}_G(P_A)(A[1/p])$ is semistable (resp. crystalline). For A a local domain and ρ_A semistable, we say ρ_A has *p-adic Hodge type μ* if $\rho_A[1/p]$ does for any trivialization of P_A (see Definition 2.4.5).

Theorem 4.2.7. *Let F'/F be a finite extension with ring of integers Λ'. The functor $\widehat{T}_{G,\Lambda'}$ induces an equivalence of categories between $\mathrm{GMod}^{\varphi,\widehat{\Gamma}}_{\mathfrak{S}_{\Lambda'}}$ and the full subcategory of semistable representations of $\mathrm{GRep}_{\Lambda'}(\Gamma_K)$.*

Proof. Using the Tannakian description of both categories, it suffices to show that $\widehat{T}_{\Lambda'}$ defines a tensor equivalence between $\mathrm{Mod}^{\varphi,\mathrm{bh},\widehat{\Gamma}}_{\mathfrak{S}_{\Lambda'}}$ and semistable representations of Γ_K on finite free Λ'-modules. When $F = \mathbb{Q}_p$ and the Hodge–Tate weights are negative (in our convention), this is Theorem 2.3.1 in [Liu 2010]. Note that Liu uses contravariant functors, so that our $\widehat{T}_{\Lambda'}$ is obtained by taking duals. The restriction on Hodge–Tate weights can be removed by twisting by $\widehat{\mathfrak{S}}(1)$, the $(\varphi, \widehat{\Gamma})$-module corresponding to the inverse of the p-adic cyclotomic character.

To define a quasi-inverse to $\widehat{T}_{\Lambda'}$, let L be a semistable Γ_K-representation on a finite free Λ'-module. Forgetting the coefficients, Liu [2010] constructs a $\widehat{\Gamma}$-structure $\widehat{T}^{-1}(L)$ on the unique Kisin lattice in $\underline{M}(L)$. This $(\varphi, \widehat{\Gamma})$-module over $\mathbb{Z}_p$ has an action of Λ', by functoriality of the construction. By an argument as in [Kisin 2008, Proposition 1.6.4(2)], the resulting $\mathfrak{S}_{\Lambda'}$-module is projective, so this defines an object of $\mathrm{Mod}^{\varphi,\mathrm{bh},\widehat{\Gamma}}_{\mathfrak{S}_{\Lambda'}}$, which we call $\widehat{T}^{-1}_{\Lambda'}(L)$.

Finally, we appeal to Proposition I.4.4.2 in [Saavedra Rivano 1972] to conclude that $\widehat{T}_{\Lambda'}$ and $\widehat{T}^{-1}_{\Lambda'}$ define a tensor equivalence of categories given that $\widehat{T}_{\Lambda'}$ respects tensor products (Proposition 4.2.5). □

4.3. ***Faithfulness and existence result.*** Fix an element $\tau \in \widehat{\Gamma}$ such that $\tau(\underline{\pi}) = \underline{\varepsilon} \cdot \underline{\pi}$, where $\underline{\varepsilon}$ is a compatible system of primitive p^n-th roots of unity. If $p \neq 2$, then τ is a topological generator for $\widehat{\Gamma}_{p^\infty} := \mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty})$. If $p = 2$, then some power of τ will generate $\widehat{\Gamma}_{p^\infty}$. In both cases, τ together with $\widehat{\Gamma}_\infty$ topologically generate $\widehat{\Gamma}$ (see [Liu 2010, §4.1]). Given condition (4) in Definition 4.2.2, the $\widehat{\Gamma}$-action is determined by the action of τ.

Recall the element $\mathfrak{t} \in W(\widetilde{\boldsymbol{E}}^+)$, which is the period for $\mathfrak{S}(1)$ in the sense that $\varphi(\mathfrak{t}) = c_0^{-1} E(u)\mathfrak{t}$. We will need a few structural results about $W(\widetilde{\boldsymbol{E}}^+)$.

Lemma 4.3.1. *For any $\widetilde{\gamma} \in \Gamma_K$, we have the following divisibilities in $W(\widetilde{\boldsymbol{E}}^+)$:*

$$\widetilde{\gamma}(u) \mid u, \quad \widetilde{\gamma}(\varphi(\mathfrak{t})) \mid \varphi(\mathfrak{t}), \quad \textit{and} \quad \widetilde{\gamma}(E(u)) \mid E(u).$$

Proof. See [Levin 2013, Lemma 9.3.1]. □

The $(\varphi, \widehat{\Gamma})$-modules which give rise to crystalline representations satisfy an extra divisibility condition on the action of τ [Gee et al. 2014, Corollary 4.10; Levin 2013, Proposition 9.3.4]. We call this the *crystalline condition*.

Definition 4.3.2. An object $\widehat{\mathfrak{M}}_A \in \mathrm{Mod}_{\mathfrak{S}_A}^{\varphi,[a,b],\widehat{\Gamma}}$ is *crystalline* if, for any $x \in \mathfrak{M}_A$, there exists $y \in \widetilde{\mathfrak{M}}_A$ such that $\tau(x) - x = \varphi(\mathfrak{t}) u^p y$.

Proposition 4.3.3. *If $\widehat{\mathfrak{M}}_A$ is crystalline then, for all $x \in \mathfrak{M}_A$ and $\gamma \in \widehat{\Gamma}$, there exists $y \in \widetilde{\mathfrak{M}}_A$ such that $\gamma(x) - x = \varphi(\mathfrak{t}) u^p y$.*

Proof. This is an easy calculation using that $\widehat{\Gamma}$ is topologically generated by $\widehat{\Gamma}_\infty$ and τ [Levin 2013, Proposition 9.3.3]. □

Definition 4.3.4. We say an object $\widehat{\mathfrak{P}}_A \in \mathrm{GMod}_{\mathfrak{S}_A}^{\varphi,[a,b],\widehat{\Gamma}}$ is *crystalline* if $\widehat{\mathfrak{P}}_A(W)$ is crystalline for all $W \in {}^f\mathrm{Rep}_\Lambda(G)$. For an object $\widehat{\mathfrak{P}}_\mathbb{F} \in \mathrm{GMod}_{\mathfrak{S}_\mathbb{F}}^{\varphi,[a,b],\widehat{\Gamma}}$, define the *crystalline* $(\varphi, \widehat{\Gamma})$*-module deformation groupoid* over $\mathscr{C}_\Lambda$ by

$$D_{\widehat{\mathfrak{P}}_\mathbb{F}}^{\mathrm{cris},[a,b]}(A) = \{(\widehat{\mathfrak{P}}_A, \psi_0) \in D_{\widehat{\mathfrak{P}}_\mathbb{F}}^{[a,b]}(A) \mid \widehat{\mathfrak{P}}_A \text{ is crystalline}\}$$

for any $A \in \mathscr{C}_\Lambda$.

Proposition 4.3.5. *Let F' be a finite extension of F with ring of integers Λ'. The equivalence from Theorem 4.2.7 induces an equivalence between the full subcategory of crystalline objects in* $\mathrm{GMod}_{\mathfrak{S}_{\Lambda'}}^{\varphi,\widehat{\Gamma}}$ *with the category of crystalline representations in* $\mathrm{GRep}_{\Lambda'}(\Gamma_K)$.

Proof. It suffices to show that if $\widehat{T}_A(\widehat{\mathfrak{P}}_A(W))$ is a lattice in a crystalline representation then $\widehat{\mathfrak{P}}_A(W)$ satisfies the crystalline condition. This only depends on the underlying $(\varphi, \widehat{\Gamma})$-module so we can take $A = \mathbb{Z}_p$. When $p > 2$, this is proven in Corollary 4.10 in [Gee et al. 2014]. The argument for $p = 2$ is essentially the same and was omitted only because in [Gee et al. 2014] they need further divisibilities on $(\tau - 1)^n$, for which $p = 2$ becomes more complicated. Details can be found in [Levin 2013, Proposition 9.3.4]. □

Choose a crystalline object $\widehat{\mathfrak{P}}_\mathbb{F} \in \mathrm{GMod}_{\mathfrak{S}_\mathbb{F}}^{\varphi,[a,b],\widehat{\Gamma}}$. If $\mathfrak{P}_\mathbb{F}$ is the underlying G-Kisin module of $\widehat{\mathfrak{P}}_\mathbb{F}$, then we would like to study the forgetful functor

$$\widehat{\Delta} : D_{\widehat{\mathfrak{P}}_\mathbb{F}}^{\mathrm{cris},[a,b]} \to D_{\mathfrak{P}_\mathbb{F}}^{[a,b]}.$$

More specifically, if μ and a, b are as in the discussion before Definition 3.3.8, and $F = F_{[\mu]}$, we consider

$$\widehat{\Delta}^\mu : D_{\widehat{\mathfrak{P}}_\mathbb{F}}^{\mathrm{cris},\mu} := D_{\widehat{\mathfrak{P}}_\mathbb{F}}^{\mathrm{cris},[a,b]} \times_{D_{\mathfrak{P}_\mathbb{F}}^{[a,b]}} D_{\mathfrak{P}_\mathbb{F}}^{\mu} \to D_{\mathfrak{P}_\mathbb{F}}^{\mu}.$$

We can now state our main theorem:

Theorem 4.3.6. *Assume that p does not divide $\pi_1(G^{\mathrm{der}})$, where G^{der} is the derived group of G, and that $F = F_{[\mu]}$. If μ is a minuscule geometric cocharacter of $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p}F)/F}G_F$ then*

$$\widehat{\Delta}^{\mu} : D^{\mathrm{cris},\mu}_{\widehat{\mathfrak{P}}_{\mathbb{F}}} \to D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$$

is an equivalence of groupoids over $\mathscr{C}_{\Lambda}$.

Remark 4.3.7. This generalizes [Levin 2013, Theorem 9.3.13], where we worked with G-Kisin modules with height in $[0, 1]$. See Remark 1.1.1 for more information.

Corollary 4.3.8. *Assume $F = F_{[\mu]}$ and that μ is minuscule. Let F' be a finite extension of F with ring of integers Λ'. There is an equivalence of categories between G-Kisin modules over $\mathfrak{S}_{\Lambda'}$ with p-adic Hodge type μ and the subcategory of $\mathrm{GRep}_{\Lambda'}(\Gamma_K)$ consisting of crystalline representations with p-adic Hodge type μ.*

Corollary 4.3.8 follows from the proof of Theorem 4.3.6. It generalizes the equivalence between Kisin modules of Barsotti–Tate type and lattices in crystalline representations with Hodge–Tate weights in $\{-1, 0\}$ [Kisin 2006, Theorem 2.2.7]. Note that we do not require $p \nmid |\pi_1(G^{\mathrm{der}})|$ here. For the relevant definitions, see Definition 3.3.12 and the discussion before Theorem 4.2.7. Before proving Theorem 4.3.6 and Corollary 4.3.8, we begin with some preliminaries on crystalline $(\varphi, \widehat{\Gamma})$-modules with G-structure.

Definition 4.3.9. Define $G(u^{p^i})$ to be the kernel of the reduction map

$$G(W(\widetilde{\boldsymbol{E}}^+)_A) \to G(W(\widetilde{\boldsymbol{E}}^+)_A/(\varphi(\mathfrak{t})u^{p^i})).$$

Proposition 4.3.10. *Choose $(\mathfrak{P}_A, \phi_{\mathfrak{P}_A}) \in \mathrm{GMod}^{\varphi,\mathrm{bh}}_{\mathfrak{S}_A}$. Fix a trivialization β_A of $\mathfrak{P}_A$. Let $C' \in G\big(\mathfrak{S}_A[1/\varphi(E(u))]\big)$ be $\phi_{\mathfrak{P}_A}$ with respect to the trivialization $1 \otimes_{\varphi} \beta_A$. Then a crystalline $\widehat{\Gamma}$-structure on $\mathfrak{P}_A$ is the same as a continuous map*

$$B_{\bullet} : \widehat{\Gamma} \to G(\widehat{R}_A)$$

satisfying the following properties:

(a) $C' \cdot \varphi(B_\gamma) = B_\gamma \cdot \gamma(C')$ *in* $G(W(\widetilde{\boldsymbol{E}})_A)$ *for all* $\gamma \in \widehat{\Gamma}$;

(b) $B_\gamma = \mathrm{Id}$ *for all* $\gamma \in \widehat{\Gamma}_\infty$;

(c) $B_\gamma \in G(u^p)$ *for all* $\gamma \in \widehat{\Gamma}$;

(d) $B_{\gamma\gamma'} = B_\gamma \cdot \gamma(B_{\gamma'})$ *for all* $\gamma, \gamma' \in \widehat{\Gamma}$.

Proof. Everything follows directly from Proposition 4.2.4. The only point to note is that $(u^p\varphi(\mathfrak{t})) \subset I_+(\widehat{R})_A$ because $u \in I_+(\widehat{R})$. Hence, the crystalline condition, which is equivalent to condition (c), implies condition (5) of Definition 4.2.2. □

Before we begin the proof of Theorem 4.3.6, we have two important lemmas.

Lemma 4.3.11. *Let $\mathfrak{P}_A \in D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}(A)$ and choose a trivialization β_A of the bundle $\mathfrak{P}_A$. If $C \in G\big(\mathfrak{S}_A[1/E(u)]\big)$ is the Frobenius with respect to β_A then, for any $Y \in G(u^{p^i})$,*

$$\varphi(C)\varphi(Y)\varphi(C)^{-1} \in G(u^{p^{i+1}}),$$

where $\varphi(C) = C' \in G(W(\widetilde{\boldsymbol{E}})_A)$ is the Frobenius with respect to $1 \otimes_\varphi \beta_A$.

Proof. Let $\mathcal{O}_G$ denote the coordinate ring of G and let I_e be the ideal defining the identity, so that $\mathcal{O}_G/I_e = \Lambda$ and $I_e/I_e^2 \cong (\mathrm{Lie}(G))^\vee$. Then $G(u^{p^i})$ is identified with

$$\{Y \in \mathrm{Hom}_\Lambda(\mathcal{O}_G, W(\widetilde{\boldsymbol{E}}^+)_A) \mid Y(I_e) \subset (\varphi(\mathfrak{t})u^{p^i})\}.$$

Conjugation by C induces an automorphism of $G_{\mathfrak{S}_A[1/E(u)]}$. Let

$$\mathrm{Ad}_{\mathcal{O}_G}(C)^* : \mathcal{O}_G \otimes_\Lambda \mathfrak{S}_A[1/E(u)] \to \mathcal{O}_G \otimes_\Lambda \mathfrak{S}_A[1/E(u)]$$

be the corresponding map on coordinate rings. The key observation is that

$$\mathrm{Ad}_{\mathcal{O}_G}(C)^*(I_e \otimes 1) \subset \sum_{j \geq 1} I_e^j \otimes_\Lambda E(u)^{-j}\mathfrak{S}_A. \tag{4-3-11-1}$$

By successive approximation, one is reduced to studying the induced automorphism of

$$\bigoplus_{j \geq 0}\big(I_e^j/I_e^{j+1} \otimes_\Lambda \mathfrak{S}_A[1/E(u)]\big).$$

The j-th graded piece is $\mathrm{Sym}^j(\mathrm{Lie}(G)^\vee) \otimes_\Lambda \mathfrak{S}_A[1/E(u)]$ as a representation of $G\big(\mathfrak{S}_A[1/E(u)]\big)$. Since μ is minuscule, $\mathrm{Lie}(G) \otimes_\Lambda \mathfrak{S}_A$ has height in $[-1, 1]$ and so $\mathrm{Sym}^j(\mathrm{Lie}(G)^\vee \otimes_\Lambda \mathfrak{S}_A)$ has height in $[-j, j]$. Thus,

$$\mathrm{Ad}_{\mathcal{O}_G}(C)^*\big(\mathrm{Sym}^j(\mathrm{Lie}(G)^\vee \otimes_\Lambda \mathfrak{S}_A)\big) \subset E(u)^{-j}(\mathrm{Sym}^j(\mathrm{Lie}(G)^\vee) \otimes_\Lambda \mathfrak{S}_A),$$

from which one deduces (4-3-11-1).

Let $Y \in G(u^{p^i})$. Then $\varphi(Y)(I_e) \subset \varphi(\varphi(\mathfrak{t})u^{p^i}) \subset \big(\varphi(E(u))\varphi(\mathfrak{t})u^{p^{i+1}}\big)$. For any $x \in I_e$,

$$(\varphi(C)\varphi(Y)\varphi(C)^{-1})(x) = (\varphi(Y) \otimes 1)\big((1 \otimes \varphi)(\mathrm{Ad}_{\mathcal{O}_G}(C)^*(x))\big),$$

which is a priori only in $W(\widetilde{\boldsymbol{E}})_A$. But since for any $b \in I_e^j$, $\varphi(Y)(b)$ is divisible by $\varphi(E(u))^j\varphi(\mathfrak{t})^j u^{jp^{i+1}}$, we have $\mathrm{Ad}(\varphi(C))(\varphi(Y))(x) \in (\varphi(\mathfrak{t})u^{p^{i+1}})$ so $\varphi(C)\varphi(Y)\varphi(C)^{-1}$ lies in $G(u^{p^{i+1}})$. □

By [Kisin 2006, Corollary 1.3.15], a Γ_∞-representation coming from a finite-height, torsion-free Kisin module $\mathfrak{M}$ extends to a crystalline Γ_K-representation if and only if the canonical Frobenius equivariant connection on $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}[1/\lambda]$ has at most logarithmic poles. Kisin [2006, Proposition 2.2.2] states furthermore that if $\mathfrak{M}$ has height in $[0, 1]$ then the condition of logarithmic poles is always satisfied.

The following lemma is a version of that proposition for G-Kisin modules with minuscule type:

Lemma 4.3.12. *Let F'/F be a finite extension containing $F_{[\mu]}$ and let $(\mathfrak{P}_{F'}, \phi_{F'})$ be a G-Kisin module over F'. Fix a trivialization of $\mathfrak{P}_{F'}$; let $C \in G\big(\mathfrak{S}_{F'}[1/E(u)]\big)$ be the Frobenius with respect to this trivialization. If the G-filtration $\mathfrak{D}_{\mathfrak{P}_{F'}}$ over $K \otimes_{\mathbb{Q}_p} F'$ defined before Lemma* 3.3.11 *has type μ, then the right logarithmic derivative $(dC/du)\cdot C^{-1} \in \big(\mathrm{Lie}\, G \otimes \mathfrak{S}_{F'}[1/E(u)]\big)$ has at most logarithmic poles along $E(u)$, i.e., lies in $E(u)^{-1}(\mathrm{Lie}\, G \otimes \mathfrak{S}_{F'})$.*

Proof. Choose an embedding $\sigma : K_0 \to F'$. Without loss of generality, we assume that $\sigma(E(u))$ splits in F' and write $\sigma(E(u)) = \prod_{i=1}^{e}(u - \psi_i(\pi))$ over embeddings $\psi_i : K \to F'$ which extend σ. Let C_σ denote the σ-component of C under the decomposition of $\mathfrak{S}_{F'}[1/E(u)]$ as a $W \otimes_{\mathbb{Z}_p} F' \cong \prod_{K_0 \to F'} F'$-algebra. We can furthermore compute the "pole" at $\psi_i(\pi)$ by working in the completion at $u - \psi_i(\pi)$, which is isomorphic to $F'[\![t]\!]$ with $t = u - \psi_i(\pi)$.

Let $\mu_{\psi_i} \in X_*(G_F)$ be the ψ_i-component of μ. Fix a maximal torus T of $G_{F'}$ such that μ_{ψ_i} factors through T. The Cartan decomposition for $G(F'((t)))$ combined with the assumption that $\mathfrak{D}_{\mathfrak{P}_{F'}}$ has type μ implies that

$$C_\sigma = B_i \mu_{\psi_i}(t) D_i,$$

where B_i and D_i are in $G(F'[\![t]\!])$ (see the discussion before Proposition 4.1.4 for the definition of $\mu_{\psi_i}(t)$). Finally, we compute that $(dC_\sigma/du)C_\sigma^{-1}$ equals

$$\frac{dB_i}{dt}B_i^{-1} + \mathrm{Ad}(B_i)\left(\frac{d\mu_{\psi_i}(t)}{dt}\mu_{\psi_i}(t)^{-1}\right) + \mathrm{Ad}(B_i)\left(\mathrm{Ad}(\mu_{\psi_i}(t))\left(\frac{dD_i}{dt}D_i^{-1}\right)\right).$$

We have $(dB_i/dt)B_i^{-1} \in (\mathrm{Lie}\, G \otimes F'[\![t]\!])$. Using a faithful representation on which T acts diagonally, we have $(d\mu_{\psi_i}(t)/dt)\mu_{\psi_i}(t)^{-1} \in (1/t)(\mathrm{Lie}\, G \otimes F'[\![t]\!])$. Finally, since μ_{ψ_i} is minuscule, $\mathrm{Ad}(\mu_{\psi_i}(t))(X) \in (1/t)(\mathrm{Lie}\, G \otimes F'[\![t]\!])$ for any $X \in \mathrm{Lie}\, G$ so in particular for $(dD_i/dt)D_i^{-1}$, by Proposition 4.1.4. □

Proof of Theorem 4.3.6. The faithfulness of $\widehat{\Delta}^\mu$ is clear. For fullness, let $\widehat{\mathfrak{P}}_A$ and $\widehat{\mathfrak{P}}'_A$ be in $D^{\mathrm{cris},\mu}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}(A)$ and let $\psi : \mathfrak{P}_A \cong \mathfrak{P}'_A$ be an isomorphism of underlying G-Kisin modules. To show ψ is equivariant for the $\widehat{\Gamma}$-actions, we can identify $\mathfrak{P}_A$ and $\mathfrak{P}'_A$ using ψ and choose a trivialization of $\mathfrak{P}_A$. Then it suffices to show that $(\mathfrak{P}_A, \phi_{\mathfrak{P}_A})$ has at most one crystalline $\widehat{\Gamma}$-structure. Let B_τ and B'_τ in $G(W(\widetilde{\boldsymbol{E}}^+)_A)$ define the action of τ with respect to the chosen trivialization of $\varphi^*(\mathfrak{P}_A)$ for the two $\widehat{\Gamma}$-structures. By the crystalline property, $B_\tau(B'_\tau)^{-1} \in G(u^p)$. By Proposition 4.2.4, if Frobenius is given by C' with respect to the trivialization, then

$$B_\tau(B'_\tau)^{-1} = C'\varphi(B_\tau(B'_\tau)^{-1})(C')^{-1}.$$

But then, by Lemma 4.3.11, $B_\tau(B'_\tau)^{-1} = I$ since it is in $G(u^{p^i})$ for all $i \geq 1$.

We next attempt to construct a crystalline $\widehat{\Gamma}$-structure on any $\mathfrak{P}_A \in D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}(A)$. Along the way, we will have to impose certain closed conditions on $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$ to make our construction work. In the end, we will reduce to A flat over $\mathbb{Z}_p$ to show that these conditions are always satisfied. Fix a trivialization β_A of $\mathfrak{P}_A$. We want elements $\{B_\gamma\} \in G(\widehat{R}_A)$ for all $\gamma \in \widehat{\Gamma}$ satisfying the conditions of Proposition 4.3.10. Choose an element $\gamma \in \widehat{\Gamma}$. Let C denote the Frobenius with respect to β_A and let $C' = \varphi(C)$ be the Frobenius with respect to $1 \otimes_\varphi \beta_A$.

We use the topology on $G(W(\widetilde{\boldsymbol{E}})_A)$ induced from the topology on $W(\widetilde{\boldsymbol{E}})_A$ (see the discussion before Definition 4.2.2). Take $B_0 = I$. For all $i \geq 1$, define

$$B_i := C'\varphi(B_{i-1})\gamma(C')^{-1} \in G(W(\widetilde{\boldsymbol{E}})_A). \tag{4-3-12-1}$$

If $\mathfrak{P}_A$ admits a $\widehat{\Gamma}$-structure, then the B_i converge to B_γ in $G(\widehat{R}_A)$ or, equivalently, in $G(W(\widetilde{\boldsymbol{E}})_A)$.

Base case: $B_1 = C'\gamma(C')^{-1} \in G(u^p)$. Let V be a faithful n-dimensional representation of G such that $\mathfrak{P}_A(V)$ has height in $[a, b]$. Set $r = b - a$. Consider C as an element of $\mathrm{GL}_n\big(\mathfrak{S}_A[1/E(u)]\big)$ such that

$$C'' := E(u)^{-a}C \in \mathrm{Mat}_n(\mathfrak{S}_A) \quad \text{and} \quad D'' := E(u)^b C^{-1} \in \mathrm{Mat}_n(\mathfrak{S}_A)$$

with $C''D'' = E(u)^r I$. Working in $\mathrm{Mat}_n(W(\widetilde{\boldsymbol{E}})_A)$, we compute that

$$C'\gamma(C')^{-1} - I = \varphi\bigg(\frac{1}{E(u)^{-a}\gamma(E(u))^b}(C''\gamma(D'') - E(u)^{-a}\gamma(E(u))^b I)\bigg).$$

It would thus suffice to show $u\varphi(\mathfrak{t})E(u)^{r-1}$ divides $C''\gamma(D'') - E(u)^{-a}\gamma(E(u))^b I$ in $\mathrm{Mat}_n(W(\widetilde{\boldsymbol{E}}^+)_A)$, as then $u\,\mathfrak{t}$ divides

$$\frac{1}{E(u)^{-a}\gamma(E(u))^b}\big(C''\gamma(D'') - E(u)^{-a}\gamma(E(u))^b I\big)$$

using Lemma 4.3.1.

Consider $P(u_1, u_2) = C''(u_1)D''(u_2)$, where we replace u by u_1 in C'', which is in $\mathrm{Mat}_n(\mathfrak{S}_A)$, and u by u_2 in D''. Let $P_{ij}(u_1, u_2) = \sum_{k\geq 0} c_k^{ij}(u_1)u_2^k$ be the (i, j)-th entry, where $c_k^{ij}(u_1)$ is a power series in u_1 with coefficients in $W \otimes_{\mathbb{Z}_p} A$. We have that $P_{ij}(u, u) = \delta_{ij}E(u)^r$. The (i, j)-th entry of $C''\gamma(D'')$ is

$$P_{ij}(u, [\underline{\varepsilon}]u) = \sum_{k\geq 0}[\underline{\varepsilon}]^k c_k^{ij}(u)u^k,$$

where $\underline{\varepsilon} = (\zeta_{p^i})_{i\geq 0}$ is the sequence of p^n-th roots of unity such that $\gamma(\pi^{1/p^n}) = \zeta_{p^n}\pi^{1/p^n}$. Note that $\varphi(\mathfrak{t})$ divides $[\underline{\varepsilon}] - 1$ since $[\underline{\varepsilon}] - 1 \in I^{[1]}W(\widetilde{\boldsymbol{E}}^+)$ (see [Fontaine 1994, Proposition 5.1.3]) and $\varphi(\mathfrak{t})$ is a generator for this ideal. Then

$$P_{ij}(u, [\underline{\varepsilon}]u) = \sum_{k\geq 0}([\underline{\varepsilon}]^k - 1)c_k^{ij}(u)u^k + \delta_{ij}E(u)^r.$$

Since $u([\underline{\varepsilon}]-1)E(u)^{r-1}$ divides $E(u)^r - E(u)^{-a}\gamma(E(u))^b$, it suffices to show that $u([\underline{\varepsilon}]-1)E(u)^{r-1}$ divides $\sum_k([\underline{\varepsilon}]^k-1)c_k^{ij}(u)u^k$. Using the Taylor expansion for x^k-1 at $x=1$, we have

$$[\underline{\varepsilon}]^k - 1 = \sum_{\ell=1}^{k}\binom{k}{\ell}([\underline{\varepsilon}]-1)^\ell,$$

from which we deduce that

$$\sum_{k\geq 0}([\underline{\varepsilon}]^k-1)c_k^{ij}(u)u^k = u([\underline{\varepsilon}]-1)\Bigg(\sum_{\ell\geq 1}([\underline{\varepsilon}]-1)^{\ell-1}u^{\ell-1}\sum_{k\geq 0}\binom{k+\ell}{\ell}c_{k+\ell}^{ij}(u)u^k\Bigg)$$

Since $E(u)$ divides $[\underline{\varepsilon}]-1$, we are reducing to showing that

$$E(u)^{r-\ell}\,\Bigg|\, u^{\ell-1}\sum_{k\geq 0}\binom{k+\ell}{\ell}c_{k+\ell}^{ij}(u)u^k$$

for $1\leq \ell\leq r-1$ where the expression on the right is exactly

$$\frac{u^{\ell-1}}{\ell!}\left(\frac{d^\ell P_{ij}(u_1,u_2)}{du_2^\ell}\bigg|_{(u,u)}\right).$$

Let $(\star_1)$ be the condition that $E(u)^{r-\ell}$ divides $d^\ell P_{ij}(u_1,u_2)/du_2^\ell|_{(u,u)}$ for all (i,j) and $1\leq\ell\leq r-1$. This is a closed condition on $D^\mu_{\mathfrak{P}_\mathbb{F}}$.

Induction step: Let $\mathfrak{P}_A\in D^\mu_{\mathfrak{P}_\mathbb{F}}(A)$ satisfy $(\star_1)$ with trivialization as above, so that $B_1 = C'\gamma(C')^{-1}\in G(u^p)$. We have

$$B_{i+1}B_i^{-1} = C\varphi(B_iB_{i-1}^{-1})C^{-1}.$$

As $C=\varphi(C')$, we can apply Lemma 4.3.11 to conclude that $B_{i+1}B_i^{-1}\in G(u^{p^{i+1}})$, i.e., $B_{i+1}B_i^{-1}\equiv I \bmod \varphi(\mathrm{t})u^{p^{i+1}}W(\widetilde{\boldsymbol{E}}^+)_A$. Since $W(\widetilde{\boldsymbol{E}}^+)_A$ is separated and complete, $\varinjlim B_i = B_\gamma\in G(W(\widetilde{\boldsymbol{E}}^+)_A)$ and B_γ satisfies $B_\gamma\gamma(C)=C\varphi(B_\gamma)$. It is easy to see that $B_\gamma\gamma'(B_\gamma)=B_{\gamma\gamma'}$ for any γ and γ', by continuity, so we have a $\widehat{\Gamma}$-action. If $\gamma\in\widehat{\Gamma}_\infty$, then γ acts trivially on $\mathfrak{S}_A$ and so on C as well, so $B_\gamma=I$.

Let $(\star_2)$ denote the condition that $B_\gamma\in G(\widehat{R}_A)$ for all $\gamma\in\widehat{\Gamma}$. We claim this is also a closed condition on $D^\mu_{\mathfrak{P}_\mathbb{F}}$. Since $W(\widetilde{\boldsymbol{E}}^+)/\widehat{R}$ is $\mathbb{Z}_p$-flat, the sequence

$$0\to\widehat{R}_A\to W(\widetilde{\boldsymbol{E}}^+)_A\to(W(\widetilde{\boldsymbol{E}}^+)/\widehat{R})\otimes_{\mathbb{Z}_p}A\to 0$$

is exact for any A. Any flat module over an Artinian ring is free, so the vanishing of an element $f\in(W(\widetilde{\boldsymbol{E}}^+)/\widehat{R})\otimes_{\mathbb{Z}_p}A$ is a closed condition on $\operatorname{Spec}A$.

We have shown that any element $\mathfrak{P}_A\in D^\mu_{\mathfrak{P}_\mathbb{F}}(A)$ which satisfies $(\star_1)$ and $(\star_2)$ admits a crystalline $\widehat{\Gamma}$-structure and so lies in $D^{\mathrm{cris},\mu}_{\widehat{\mathfrak{P}}_\mathbb{F}}(A)$. It suffices then to show that the closed subgroupoid defined by the conditions $(\star_1)$ and $(\star_2)$ is all of $D^\mu_{\mathfrak{P}_\mathbb{F}}$. Recall that $D^\mu_{\mathfrak{P}_\mathbb{F}}$ admits a formally smooth representable hull $D^{(N),\mu}_{\mathfrak{P}_\mathbb{F}}=\operatorname{Spf}R^{(N),\mu}_{\mathfrak{P}_\mathbb{F}}$,

where $R^{(N),\mu}_{\mathfrak{P}_{\mathbb{F}}}$ is flat and reduced by Theorem 3.2.4 and Proposition 3.3.10. Since $R^{(N),\mu}_{\mathfrak{P}_{\mathbb{F}}}$ is flat and $R^{(N),\mu}_{\mathfrak{P}_{\mathbb{F}}}[1/p]$ is reduced and Jacobson, any closed subscheme of $\operatorname{Spec} R^{(N),\mu}_{\mathfrak{P}_{\mathbb{F}}}$ which contains $\operatorname{Hom}_\Lambda(R^{(N),\mu}_{\mathfrak{P}_{\mathbb{F}}}, F')$ for all F'/F finite is the whole space. It suffices then to show that, for any F'/F finite and Λ' the ring of integers of F', every object of $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}(\Lambda')$ satisfies $(\star_1)$ and $(\star_2)$.

First, for $(\star_1)$, choose $\gamma \in \widehat{\Gamma}$. Then set $Q_\ell(u) := \big(d^\ell P_{ij}(u_1,u_2)/du_2^\ell\big|_{(u,u)}\big)$, which is in $\operatorname{Mat}_n(\mathfrak{S}_{\Lambda'})$ (we ignore $u^{\ell-1}/\ell!$ since we are in the torsion-free setting). We can check that $E(u)^{r-\ell} \mid Q_\ell(u)$, working over $F' = \Lambda'[1/p]$ or any finite extension thereof. In particular, we can put ourselves in the situation of Lemma 4.3.12. We compute then that

$$\begin{aligned} Q_\ell(u) &= (E(u)^{-a}C)\frac{d^\ell}{du^\ell}(E(u)^b C^{-1}) \\ &= (E(u)^{-a}C)\sum_{m=0}^{\ell}\binom{\ell}{m}\frac{d^m E(u)^b}{du^m}\frac{d^{\ell-m}C^{-1}}{du^{\ell-m}} \\ &= \sum_{m=0}^{\ell}\binom{\ell}{m}\left(E(u)^{-a}\frac{d^m E(u)^b}{du^m}\right)\left(C\frac{d^{\ell-m}C^{-1}}{du^{\ell-m}}\right). \end{aligned}$$

Since $E(u)^{r-m}$ divides $E(u)^{-a}\, d^m E(u)^b/du^m$, it suffices to show that

$$Y_k := E(u)^k\left(C\frac{d^k C^{-1}}{du^k}\right) \in \operatorname{Mat}_n(\mathfrak{S}_{F'})$$

for all $k \geq 0$ (applied with $k = \ell - m$). The case $k = 0$ is trivial. By Lemma 4.3.12, $X_C := E(u)(dC/du)C^{-1} = -E(u)C\, d(C^{-1})/du$ is an element of $\operatorname{Lie} G \otimes \mathfrak{S}_{F'}$ considered as subset of $\operatorname{Lie}(\mathrm{GL}(V)) \otimes \mathfrak{S}_{F'}$ so, in particular, $Y_1 \in \operatorname{Mat}_n(\mathfrak{S}_{F'})$. The product rule applied to $(d/du)(E(u)^k C\, d^{k-1}C^{-1}/du)$ implies that

$$Y_k = \frac{d}{du}(E(u)Y_{k-1}) - k\frac{dE(u)}{du}Y_{k-1} + Y_1 Y_{k-1}$$

so, by induction on k, $Y_k \in \operatorname{Mat}_n(\mathfrak{S}_{F'})$ for all $k \geq 0$.

For $(\star_2)$, recall that $\widehat{R} = R_{K_0} \cap W(\widetilde{\boldsymbol{E}}^+)$ (see p. 5 of [Liu 2010]) so it suffices to show that $B_\gamma \in G(R_{K_0} \otimes_{\mathbb{Z}_p} \Lambda')$ or, equivalently, $B_\gamma \in \mathrm{GL}_n(R_{K_0} \otimes_{\mathbb{Z}_p} \Lambda')$ with respect to V. Denote by $\mathfrak{M}_V$ the Kisin module $\mathfrak{P}_{\Lambda'}(V)$ of rank n. Since $\varphi(E(u))$ is invertible in S_{K_0}, C' lies in $\mathrm{GL}_n(S_{K_0} \otimes_{\mathbb{Z}_p} \Lambda')$ and defines a Frobenius on the Breuil module $\mathcal{M}_V := S_{K_0} \otimes_{\mathfrak{S},\varphi} \mathfrak{M}_V$. Using a similar argument to above, one can construct the monodromy operator $N_{\mathcal{M}_V}$ on $\mathcal{M}_V$ inductively, taking $N_0 = 0$ and setting

$$N_{i+1} := pC'\varphi(N_i)(C')^{-1} + u\frac{dC'}{du}(C')^{-1}. \tag{4-3-12-2}$$

The sequence $\{N_i\}$ converges to an element of $\mathrm{Mat}_n(u^p S_{K_0})$. For each N_i, let $\widetilde{N}_i$ be the induced derivation on $\mathcal{M}_V$ over $-u\,d/du$ which, on the chosen basis, is given by N_i. Equation (4-3-12-2) is equivalent to

$$\widetilde{N}_{i+1}\phi_{\mathcal{M}_V} = p\phi_{\mathcal{M}_V}\widetilde{N}_i. \tag{4-3-12-3}$$

Let $\underline{\varepsilon}(\gamma) := \gamma([\underline{\pi}])/[\underline{\pi}]$. Define a γ-semilinear map $\widetilde{B}_i$ on $R_{K_0}\otimes_{S_{K_0}}\mathcal{M}_V$ by

$$\widetilde{B}_i(x) = \sum_{j\geq 0}\frac{(-\log \underline{\varepsilon}(\gamma))^j}{j!}\otimes(\widetilde{N}_i)^j(x)$$

for all $x\in\mathcal{M}_V$. Equation (4-3-12-3) implies that

$$\widetilde{B}_{i+1}\phi_{\mathcal{M}_V} = \phi_{\mathcal{M}_V}\widetilde{B}_i.$$

By induction on i, one deduces that $\widetilde{B}_i$ is exactly the γ-semilinear morphism induced by the matrix B_i defined in (4-3-12-1).

If $N_{\mathcal{M}_V}$ is the limit of the $\widetilde{N}_i$ and $\widetilde{B}_\gamma$ is the γ-semilinear morphism induced by B_γ, then we have the formula

$$\widetilde{B}_\gamma(x) := \sum_{j\geq 0}\frac{(-\log \underline{\varepsilon}(\gamma))^j}{j!}\otimes N^j_{\mathcal{M}}(x)$$

for all $x\in\mathcal{M}_V$. Working with respect to the chosen basis for $\mathcal{M}_V$, we deduce that $B_\gamma\in\mathrm{GL}_n(R_{K_0}\otimes_{\mathbb{Z}_p}\Lambda')$, as desired. □

4.4. ***Applications to G-valued deformation rings.*** Let $\bar\eta:\Gamma_K\to G(\mathbb{F})$ be a continuous representation. As before, μ is a minuscule geometric cocharacter of $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p}F)/F}G_F$. Let $R^{\mathrm{cris},\mu}_{\bar\eta}$ be the universal G-valued framed crystalline deformation ring with p-adic Hodge type μ over $\Lambda_{[\mu]}$. Let $X^{\mathrm{cris},\mu}_{\bar\eta}$ be the projective $R^{\mathrm{cris},\mu}_{\bar\eta}$-scheme as in Corollary 3.3.15. The following theorem on the geometry of $X^{\mathrm{cris},\mu}_{\bar\eta}$ has a number of important corollaries. The proof uses the main results from Sections 3.2 and 4.2. We can say more about the connected components when K is unramified over $\mathbb{Q}_p$ (see Theorem 4.4.6).

Theorem 4.4.1. *Assume $p\nmid\pi_1(G^{\mathrm{der}})$. Let μ be a minuscule geometric cocharacter of $\mathrm{Res}_{(K\otimes_{\mathbb{Q}_p}F)/F}G_F$. Then $X^{\mathrm{cris},\mu}_{\bar\eta}$ is normal and $X^{\mathrm{cris},\mu}_{\bar\eta}\otimes_{\Lambda_{[\mu]}}\mathbb{F}_{[\mu]}$ is reduced.*

Corollary 4.4.2. *Assume $p\nmid\pi_1(G^{\mathrm{der}})$. Let $X^{\mathrm{cris},\mu}_{\bar\eta,\mathbf{0}}$ denote the fiber of $X^{\mathrm{cris},\mu}_{\bar\eta}$ over the closed point of* $\mathrm{Spec}\,R^{\mathrm{cris},\mu}_{\bar\eta}$. *The connected components of* $\mathrm{Spec}\,R^{\mathrm{cris},\mu}_{\bar\eta}[1/p]$ *are in bijection with the connected components of $X^{\mathrm{cris},\mu}_{\bar\eta,\mathbf{0}}$.*

Proof. By Theorem 2.3.12, $\mathrm{Spec}\,R^{\mathrm{cris},\mu}_{\bar\eta}[1/p]=X^{\mathrm{cris},\mu}_{\bar\eta}[1/p]$. Since $X^{\mathrm{cris},\mu}_{\bar\eta}\otimes_\Lambda\mathbb{F}$ is reduced by Theorem 4.4.1, the bijection between $\pi_0(X^{\mathrm{cris},\mu}_{\bar\eta}[1/p])$ and $\pi_0(X^{\mathrm{cris},\mu}_{\bar\eta,\mathbf{0}})$ follows from the "reduced fiber trick" [Kisin 2009, Corollary (2.4.10)]. □

Remark 4.4.3. Theorem 4.4.1 and Corollary 4.4.2 hold for unframed G-valued crystalline deformation functors when they are representable, by exactly the same arguments.

Before we begin the proof, we introduce a few auxiliary deformation groupoids. The relationship between the various deformation spaces is described in (4-4-5-1). Let $D^{\square}_{\bar\eta}$ be the deformation functor of $\bar\eta$, so $D^{\square}_{\bar\eta}(A)$ is the set of homomorphisms $\eta : \Gamma_K \to G(A)$ lifting $\bar\eta$. Let $\mathfrak{P}_{\mathbb{F}}$ be the G-Kisin module associated to a $\mathbb{F}$-point $\bar x$ of $X^{\mathrm{cris},\mu}_{\bar\eta}$.

Definition 4.4.4. Define $D^{[a,b]}_{\bar x}(A)$ to be the category of triples

$$\{(\eta_A, \mathfrak{P}_A, \delta_A) \mid \eta_A \in D^{\square}_{\bar\eta}(A),\ \mathfrak{P}_A \in D^{[a,b]}_{\mathfrak{P}_{\mathbb{F}}}(A),\ \delta_A : T_{G,\mathfrak{S}_A}(\mathfrak{P}_A) \cong \eta_A|_{\Gamma_\infty}\}.$$

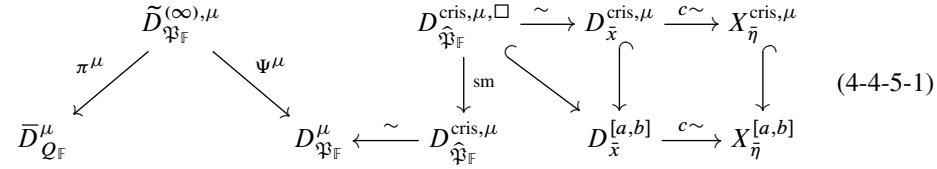
Let $\widehat{\mathfrak{P}}_{\mathbb{F}}$ denote a crystalline $\widehat\Gamma$-structure on $\mathfrak{P}_{\mathbb{F}}$ together with an isomorphism $\widehat T_{G,\mathbb{F}}(\widehat{\mathfrak{P}}_{\mathbb{F}}) \cong \bar\eta$. Define $D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}(A)$ to be the category of triples

$$\{(\eta_A, \widehat{\mathfrak{P}}_A, \delta_A) \mid \eta_A \in D^{\square}_{\bar\eta}(A),\ \widehat{\mathfrak{P}}_A \in D^{\mathrm{cris},\mu}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}(A),\ \delta_A : \widehat T_{G,A}(\widehat{\mathfrak{P}}_A) \cong \eta_A\}.$$

Proposition 4.4.5. *For any $\widehat{\mathfrak{P}}_{\mathbb{F}}$, the forgetful functor from $D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}$ to $D^{[a,b]}_{\bar x}$ is fully faithful.*

Proof. One reduces immediately to the case of GL_n and then we have the following more general fact: Choose any $\widehat{\mathfrak{M}}'_A, \widehat{\mathfrak{M}}_A \in \mathrm{Mod}^{\varphi,\mathrm{bh},\widehat\Gamma}_{\mathfrak{S}_A}$. Let $f : \mathfrak{M}'_A \to \mathfrak{M}_A$ be a map of underlying Kisin modules such that $T_{\mathfrak{S}_A}(f)$ is Γ_K-equivariant (under the identification $\widehat T_{\mathfrak{S}_A} \cong T_{\mathfrak{S}_A}$). Then f is a map of $(\varphi,\widehat\Gamma)$-modules. This is proven in [Ozeki 2013, Corollary 4.3] when height is in $[0,h]$, but can be easily extended to bounded height. The key input is a weak form of Liu's comparison isomorphism [2007, Proposition 3.2.1], which is also in [Levin 2013, Proposition 9.2.1]. □

The diagram below illustrates some of the relationships between the different deformation problems. The diagonal maps on the left and the map labeled sm are formally smooth. Maps labeled with $c\sim$ indicate that the complete stalk at a point of the target represents that deformation functor. The horizontal equivalences are consequences of Theorem 4.3.6 and the proof of Theorem 4.4.1, respectively.



(4-4-5-1)

Proof of Theorem 4.4.1. Let $\bar x$ be a point of the special fiber of $X^{\mathrm{cris},\mu}_{\bar\eta}$ defined over a finite field $\mathbb{F}'$. Since $X^{\mathrm{cris},\mu}_{\bar\eta}[1/p] = \mathrm{Spec}\, R^{\mathrm{cris},\mu}_{\bar\eta}[1/p]$ is formally smooth over F [Balaji 2012, Proposition 4.1.5], it suffices to show that the completed stalk $\widehat{\mathcal{O}}^{\mu}_{\bar x}$

at $\bar{x}$ is normal and that $\widehat{\mathbb{O}}^{\mu}_{\bar{x}} \otimes_{\Lambda_{[\mu]}} \mathbb{F}_{[\mu]}$ is reduced. To accomplish this, we compare $\widehat{\mathbb{O}}^{\mu}_{\bar{x}}$ with $\overline{D}^{\mu}_{Q_{\mathbb{F}'}}$ from Section 3.3 and then use as input the corresponding results for the local model $M(\mu)$.

These properties can be checked after an étale extension of $\Lambda_{[\mu]}$. $R^{\mathrm{cris},\mu}_{\bar{\eta}}$ commutes with changing coefficients using the abstract criterion in [Chai et al. 2014, Proposition 1.4.3.6] as does the formation of $X^{\mathrm{cris},\mu}_{\bar{\eta}}$ by Proposition 2.3.5. We can assume then, without loss of generality, that $\Lambda = \Lambda_{[\mu]}$ and $\mathbb{F}' = \mathbb{F}$. Let $\mathfrak{P}_{\mathbb{F}}$ be the G-Kisin module defined by $\bar{x}$. Since μ is minuscule, $X^{\mathrm{cris},\mu}_{\bar{\eta}} = X^{\mathrm{cris},\leq\mu}_{\bar{\eta}}$ (see Proposition 4.1.3).

Since $\widehat{\mathbb{O}}^{\mu}_{\bar{x}}$ is nonempty and Λ-flat (assuming that $R^{\mathrm{cris},\mu}_{\bar{\eta}}$ is nonempty), it has an F'-point for some finite extension F'/F. Any such point gives rise to a crystalline lift ρ of $\bar{x}$ to $\mathbb{O}_{F'}$ such that the unique Kisin lattice in $\underline{M}_{G,\mathbb{O}_{F'}}(\rho)$ reduces to $\mathfrak{P}_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. Replace $\mathbb{F}'$ by $\mathbb{F}$. Then, by Proposition 4.3.5, the corresponding $G(\mathbb{O}_{F'})$-valued representation is isomorphic to $\widehat{T}_{G,\mathbb{O}_{F'}}(\widehat{\mathfrak{P}}_{\mathbb{O}_{F'}})$ for some crystalline $(\varphi, \widehat{\Gamma})$-module with G-structure. Reducing modulo the maximal ideal, we obtain a crystalline $\widehat{\Gamma}$-structure $\widehat{\mathfrak{P}}_{\mathbb{F}}$ on $\mathfrak{P}_{\mathbb{F}}$. By Proposition 4.4.5, this is the unique such structure.

Recall the deformation problem $D^{\mathrm{cris},\mu}_{\bar{x}}$ from Corollary 3.3.15 and $D^{[a,b]}_{\bar{x}}$ from Definition 4.4.4. The natural map

$$D^{\mathrm{cris},\mu}_{\bar{x}} \to D^{[a,b]}_{\bar{x}}$$

is a closed immersion (by Theorem 2.3.12). By Corollary 3.3.15, $\operatorname{Spf} \mathbb{O}^{\mu}_{\bar{x}}$ is closed in $D^{\mathrm{cris},\mu}_{\bar{x}}$.

Fix the isomorphism $\beta_{\mathbb{F}} : \widehat{T}_{G,\mathbb{F}}(\widehat{\mathfrak{P}}_{\mathbb{F}}) \cong \bar{\eta}$. Consider the groupoid $D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}$ in Definition 4.4.4. There is a natural morphism from $D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}$ to $D^{[a,b]}_{\bar{x}}$, given by forgetting the $\widehat{\Gamma}$-structure. By Proposition 4.4.5, this morphism is fully faithful, hence a closed immersion by considering tangent spaces.

We claim that

$$D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}} = \operatorname{Spf} \mathbb{O}^{\mu}_{\bar{x}}$$

as closed subfunctors of $D^{[a,b]}_{\bar{x}}$. Since they are both representable, we look at their F'-points for any finite extension F' of F. By Theorem 4.2.7 and Corollary 3.3.15,

$$D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}(F') = D^{\mathrm{cris},\mu}_{\bar{x}}(F') = \operatorname{Spf} \mathbb{O}^{\mu}_{\bar{x}}(F').$$

Since $\mathbb{O}^{\mu}_{\bar{x}}$ is Λ-flat and $\mathbb{O}^{\mu}_{\bar{x}}[1/p]$ is formally smooth over F, we deduce that

$$\operatorname{Spf} \mathbb{O}^{\mu}_{\bar{x}} \subset D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}.$$

Finally, $D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}$ is formally smooth over $D^{\mu}_{\mathfrak{P}_{\mathbb{F}}}$, by Theorem 4.3.6. By (3-3-9-2), there is a diagram

$$\begin{array}{ccc} & \operatorname{Spf} S^{\mu} & \\ \swarrow & & \searrow \\ D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}} & & \overline{D}^{\mu}_{Q_{\mathbb{F}}}, \end{array}$$

where $S^{\mu} \in \widehat{\mathscr{C}}_{\Lambda}$ and both morphisms are formally smooth ($Q_{\mathbb{F}}$ is as in Section 3.2). The functor $\overline{D}^{\mu}_{Q_{\mathbb{F}}}$ is represented by a completed stalk $R^{\mu}_{Q_{\mathbb{F}}}$ on $M(\mu)$. In particular, $R^{\mu}_{Q_{\mathbb{F}}}$ is Λ-flat so the same is true of $D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}}$. Thus,

$$D^{\mathrm{cris},\mu,\square}_{\widehat{\mathfrak{P}}_{\mathbb{F}}} = \operatorname{Spf} \mathbb{O}^{\mu}_{\tilde{x}}.$$

By Theorem 3.2.4, $R^{\mu}_{Q_{\mathbb{F}}}$ is normal and Cohen–Macaulay, and $R^{\mu}_{Q_{\mathbb{F}}} \otimes_{\Lambda} \mathbb{F}$ is reduced, so the same holds true for $\widehat{\mathbb{O}}^{\mu}_{x}$.

□

Theorem 4.4.6. *Assume $K/\mathbb{Q}_p$ is unramified, $p > 3$, and $p \nmid \pi_1(G^{\mathrm{ad}})$. Then the universal crystalline deformation ring $R^{\mathrm{cris},\mu}_{\bar{\eta}}$ is formally smooth over $\Lambda_{[\mu]}$.*

Proof. First, replace Λ by $\Lambda_{[\mu]}$. Without loss of generality, we can assume that F contains all embeddings of K, since this can be arranged by a finite étale base change. When $K/\mathbb{Q}_p$ is unramified, $\mathrm{Gr}_G^{E(u),W}$ is a product of $[K : \mathbb{Q}_p]$ copies of the affine Grassmannian Gr_G (see [Levin 2013, Proposition 10.1.11]). If $\mu = (\mu_\psi)_{\psi : K \to F}$ then $M(\mu)_F = \prod_\psi S(\mu_\psi)$, where $S(\mu_\psi)$ are affine Schubert varieties of Gr_{G_F}. Under the assumption that $p \nmid \pi_1(G^{\mathrm{der}})$, there is a flat closed Λ-subscheme of Gr_G which, abusing notation, we denote by $S(\mu_\psi)$, whose fibers are the affine Schubert varieties for μ_ψ (see Theorem 8.4 in [Pappas and Rapoport 2008], especially the discussions in §§8.e.3–8.e.4). Thus,

$$M(\mu) = \prod_{\psi : K \to F} S(\mu_\psi).$$

Since μ_ψ is minuscule, $S(\mu_\psi)$ is isomorphic to a flag variety for G, hence $M(\mu)$ is smooth (see Proposition 4.1.3). The proof of Theorem 4.4.1 shows that the local structure of $X^{\mathrm{cris},\mu}_{\bar{\eta}}$ is smoothly equivalent to the local structure of $M(\mu)$. Thus, $X^{\mathrm{cris},\mu}_{\bar{\eta}}$ is formally smooth over Λ.

Finally, we have to show that

$$\Theta : X^{\mathrm{cris},\mu}_{\bar{\eta}} \to \operatorname{Spec} R^{\mathrm{cris},\mu}_{\bar{\eta}}$$

is an isomorphism. Since $\Theta[1/p]$ is an isomorphism and $R^{\mathrm{cris},\mu}_{\bar{\eta}}$ is Λ-flat, it suffices to show that Θ is a closed immersion. Let m_R be the maximal ideal of $R^{\mathrm{cris},\mu}_{\bar{\eta}}$.

Consider the reductions

$$\Theta_n : X^{\mathrm{cris},\mu}_{\bar{\eta},n} \to \operatorname{Spec} R^{\mathrm{cris},\mu}_{\bar{\eta}}/m_R^n.$$

We appeal to an analogue of Raynaud's uniqueness result [1974, Theorem 3.3.3] for finite flat models. For any Artin local $\mathbb{Z}_p$-algebra A and any finite A-algebra B, let $\mathfrak{P}_B$ and $\mathfrak{P}'_B$ be two distinct points in the fiber of Θ_n over $x : R^{\mathrm{cris},\mu}_{\bar{\eta}} \to A$, i.e., G-Kisin lattices in $P_x \otimes_A B$. Let V^{ad} denote the adjoint representation of G. Under the assumption that $p > 3$, [Liu 2007, Theorem 2.4.2] (which generalizes Raynaud's result) implies that $\mathfrak{P}_A(V^{\mathrm{ad}}) = \mathfrak{P}'_A(V^{\mathrm{ad}})$ as Kisin lattices in $(P_x \otimes_A B)(V^{\mathrm{ad}})$, using that μ is minuscule.

Since B is Artinian, without loss of generality we can assume it is local ring. Choose a trivialization of $\mathfrak{P}_B$. There exists $g \in G(\mathbb{O}_{\mathscr{E},B})$ such that $\mathfrak{P}'_B = g \cdot \mathfrak{P}_B$ (working inside the affine Grassmannian as in Theorem 2.3.2). The results above implies that $\mathrm{Ad}(g) \in G^{\mathrm{ad}}(\mathfrak{S}_A)$. By assumption, $Z := \ker(G \to G^{\mathrm{ad}})$ is étale so, after possibly extending the residue field $\mathbb{F}$, we can lift $\mathrm{Ad}(g)$ to an element $\tilde{g} \in G(\mathfrak{S}_A)$ such that $g = \tilde{g}z$, where $z \in Z(\mathbb{O}_{\mathscr{E},A})$. We want to show that $z \in Z(\mathfrak{S}_A)$. We can write Z as a product $Z_{\mathrm{tors}} \times (\mathbb{G}_{\mathrm{m}})^s$ for some $s \geq 0$. Since Z_{tors} has order prime to p by assumption, $Z_{\mathrm{tors}}(\mathbb{O}_{\mathscr{E},A}) = Z_{\mathrm{tors}}(\mathfrak{S}_A)$, so we can assume

$$z \in (\mathbb{G}_{\mathrm{m}}(\mathbb{O}_{\mathscr{E},A}))^s = \big((A \otimes_{\mathbb{Z}_p} W)((u))^\times\big)^s.$$

For any embedding $\psi : W \to \mathbb{O}_F$, we associate to z the s-tuple λ_ψ of integers of the degrees of the leading terms of each component base changed by ψ. To show that $\lambda_\psi = 0$, we can work over $A/m_A = \mathbb{F}$. We think of λ_ψ as a cocharacter of Z. Consider the quotient of G by its derived group $Z' := G/G^{\mathrm{der}}$. The map $X_*(Z) \to X_*(Z')$ is injective. Any character χ of Z' defines a one-dimensional representation L_χ of G so, in particular, we can consider $\mathfrak{P}_B(L_\chi)$ and $\mathfrak{P}'_B(L_\chi)$ as Kisin lattices in $P_x(L_\chi)$. Writing $\mathfrak{S}_{\mathbb{F}} \cong \bigoplus_{\psi : W \to \mathbb{O}_F} \mathbb{F}[\![u_\psi]\!]$, a Kisin lattice of $P_x(L_\chi)$ has type (h_ψ) exactly when $\phi_{P_x}(e) = (a_\psi u^{h_\psi})e$ for a basis element e and $a_\psi \in \mathbb{F}$. Since both $\mathfrak{P}_B$ and $\mathfrak{P}'_B$ have type μ, $\mathfrak{P}_B(L_\chi)$ and $\mathfrak{P}'_B(L_\chi)$ both have type $h_\psi := \langle \chi, \mu_\psi \rangle$. However, a direct computation shows that $\mathfrak{P}'_B(L_\chi)$ has type $h_\psi + \langle \chi, p\lambda_{\psi'} - \lambda_\psi \rangle$, where $\psi' = \varphi \circ \psi$. Thus, $\lambda_\psi = p\lambda_{\psi'}$. We deduce that $p^{[K:\mathbb{Q}_p]}\lambda_\psi = \lambda_\psi$ and so $\lambda_\psi = 0$.

We are reduced to the following general situation: $X \to \operatorname{Spec} A$ is proper morphism which is injective on B-points for all A-finite algebras B, where A is a local Artinian ring. By consideration of the one geometric fiber, $X \to \operatorname{Spec} A$ is quasifinite, hence finite. Thus, $X = \operatorname{Spec} A'$. By Nakayama, it suffices to show $A/m_A \to A'/(m_A)A'$ is surjective so we can assume $A = k$ is a field. Surjectivity follows from considering the two morphisms $A' \rightrightarrows A' \otimes_k A'$, which agree by injectivity of $X \to \operatorname{Spec} A$ on A-finite points. □

Acknowledgments

The enormous influence of the work of Mark Kisin and Tong Liu in this paper will be evident to the reader. This paper is based on the author's Stanford Ph.D. thesis advised by Brian Conrad, to whom the author owes a debt of gratitude for his generous guidance and his feedback on multiple drafts of the thesis. It is a pleasure to thank Tong Liu, Xinwen Zhu, George Pappas, Bhargav Bhatt, and Mark Kisin for many helpful discussions and exchanges related to this project. The author is additionally grateful for the support of the National Science Foundation and the Department of Defense in the form of NSF and NDSEG fellowships. Part of this work was completed while the author was a visitor at the Institute for Advanced Study, supported by the National Science Foundation grant DMS-1128155. The author is grateful to the IAS for its support and hospitality. Finally, we would like to thank the referee for a number of valuable suggestions.

References

[Balaji 2012] S. Balaji, *G-valued potentially semi-stable deformation rings*, Ph.D. thesis, The University of Chicago, Ann Arbor, MI, 2012, http://search.proquest.com/docview/1346024943. MR 3152673

[Bellovin 2014] R. Bellovin, "Generic smoothness for G-valued potentially semistable deformation rings", preprint, 2014. arXiv 1403.1411

[Bourbaki 2002] N. Bourbaki, *Lie groups and Lie algebras, Chapters 4–6*, Springer, Berlin, 2002. MR 2003a:17001 Zbl 0983.17001

[Brinon and Conrad 2009] O. Brinon and B. Conrad, "CMI summer school notes on p-adic Hodge theory notes", lecture notes, 2009, http://math.stanford.edu/∼conrad/.

[Broshi 2008] M. A. Broshi, *Moduli of finite flat group schemes with G-structure*, Ph.D. thesis, http://search.proquest.com/docview/304407562, Ann Arbor, MI, 2008, http://search.proquest.com/docview/304407562. MR 2717544

[Broshi 2013] M. Broshi, "G-torsors over a Dedekind scheme", *J. Pure Appl. Algebra* **217**:1 (2013), 11–19. MR 2965898 Zbl 1271.14062

[Caruso and Liu 2011] X. Caruso and T. Liu, "Some bounds for ramification of p^n-torsion semi-stable representations", *J. Algebra* **325** (2011), 70–96. MR 2012b:11090 Zbl 1269.14001

[Chai et al. 2014] C.-L. Chai, B. Conrad, and F. Oort, *Complex multiplication and lifting problems*, Mathematical Surveys and Monographs **195**, American Mathematical Society, Providence, RI, 2014. MR 3137398 Zbl 1298.14001

[Deligne 1979] P. Deligne, "Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques", pp. 247–289 in *Automorphic forms, representations and L-functions, II* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, R.I., 1979. MR 81i:10032 Zbl 0437.14012

[Deligne and Milne 1982] P. Deligne and J. S. Milne, "Tannakian categories", pp. 101–228 in *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics **900**, Springer, New York, 1982. MR 84m:14046 Zbl 0477.14004

[EGA III$_1$ 1961] A. Grothendieck, "Eléments de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents, I", *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 5–167. MR 0163910 Zbl 0118.36206

[Fontaine 1994] J.-M. Fontaine, *Le corps des périodes p-adiques*, Astérisque **223**, 1994. MR 95k:11086 Zbl 0940.14012

[Gaitsgory 2001] D. Gaitsgory, "Construction of central elements in the affine Hecke algebra via nearby cycles", *Invent. Math.* **144**:2 (2001), 253–280. MR 2002d:14072 Zbl 1072.14055

[Gee 2006] T. Gee, "A modularity lifting theorem for weight two Hilbert modular forms", *Math. Res. Lett.* **13**:5–6 (2006), 805–811. MR 2007m:11065 Zbl 1185.11030

[Gee et al. 2014] T. Gee, T. Liu, and D. Savitt, "The Buzzard–Diamond–Jarvis conjecture for unitary groups", *J. Amer. Math. Soc.* **27**:2 (2014), 389–435. MR 3164985 Zbl 1288.11045

[Gross 2000] B. H. Gross, "On minuscule representations and the principal SL_2", *Represent. Theory* **4** (2000), 225–244. MR 2001j:20066 Zbl 0986.22011

[Hellmann 2011] E. Hellmann, "Connectedness of Kisin varieties for GL_2", *Adv. Math.* **228** (2011), 219–240. MR 2822232 Zbl 1268.11074

[Imai 2010] N. Imai, "On the connected components of moduli spaces of finite flat models", *Amer. J. Math.* **132**:5 (2010), 1189–1204. MR 2012a:11064 Zbl 1205.14025

[Imai 2012] N. Imai, "On the connected components of moduli spaces of Kisin modules", *J. Algebra* **349** (2012), 1–7. MR 2012k:11072 Zbl 1242.14043

[Kim 2009] W. Kim, *Galois deformation theory for norm fields and its arithmetic applications*, Ph.D. thesis, University of Michigan, 2009, http://search.proquest.com/docview/304931137. MR 2713872

[Kisin 2006] M. Kisin, "Crystalline representations and F-crystals", pp. 459–496 in *Algebraic geometry and number theory*, edited by V. Ginzburg, Progr. Math. **253**, Birkhäuser, Boston, 2006. MR 2007j:11163 Zbl 1184.11052

[Kisin 2008] M. Kisin, "Potentially semi-stable deformation rings", *J. Amer. Math. Soc.* **21**:2 (2008), 513–546. MR 2009c:11194 Zbl 1205.11060

[Kisin 2009] M. Kisin, "Moduli of finite flat group schemes, and modularity", *Ann. of Math.* (2) **170**:3 (2009), 1085–1180. MR 2011g:11107 Zbl 1201.14034

[Levin 2013] B. Levin, *G-valued flat deformations and local models*, Ph.D. thesis, Stanford University, 2013, http://www.math.ias.edu/~bwlevin/LevinThesis.pdf.

[Levin 2014] B. Levin, "Local models for Weil-restricted groups", preprint, 2014. arXiv 1412.7135

[Liu 2007] T. Liu, "Torsion p-adic Galois representations and a conjecture of Fontaine", *Ann. Sci. École Norm. Sup.* (4) **40**:4 (2007), 633–674. MR 2010h:11191 Zbl 1163.11043

[Liu 2010] T. Liu, "A note on lattices in semi-stable representations", *Math. Ann.* **346**:1 (2010), 117–138. MR 2011d:11272 Zbl 1208.14017

[Liu 2012] T. Liu, "Lattices in filtered (ϕ, N)-modules", *J. Inst. Math. Jussieu* **11**:3 (2012), 659–693. MR 2931320 Zbl 1249.14006

[Liu 2013] T. Liu, "The correspondence between Barsotti–Tate groups and Kisin modules when $p = 2$", *J. Théor. Nombres Bordeaux* **25**:3 (2013), 661–676. MR 3179680 Zbl 06291371

[Ozeki 2013] Y. Ozeki, "Torsion representations arising from $(\varphi, \hat{G})$-modules", *J. Number Theory* **133**:11 (2013), 3810–3861. MR 3084302 Zbl 1297.11146

[Pappas and Rapoport 2008] G. Pappas and M. Rapoport, "Twisted loop groups and their affine flag varieties", *Adv. Math.* **219** (2008), 118–198. MR 2009g:22039 Zbl 1159.22010

[Pappas and Rapoport 2009] G. Pappas and M. Rapoport, "Φ-modules and coefficient spaces", *Mosc. Math. J.* **9**:3 (2009), 625–663, back matter. MR 2011c:14074 Zbl 1194.14032

[Pappas and Zhu 2013] G. Pappas and X. Zhu, "Local models of Shimura varieties and a conjecture of Kottwitz", *Invent. Math.* **194**:1 (2013), 147–254. MR 3103258 Zbl 1294.14012

[Pappas et al. 2013] G. Pappas, M. Rapoport, and B. Smithling, "Local models of Shimura varieties, I: Geometry and combinatorics", pp. 135–217 in *Handbook of moduli, III*, edited by G. Farkas and I. Morrison, Adv. Lect. Math. **26**, International Press, 2013. MR 3135437

[Rapoport and Zink 1996] M. Rapoport and T. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies **141**, Princeton University Press, NJ, 1996. MR 97f:14023 Zbl 0873.14039

[Raynaud 1974] M. Raynaud, "Schémas en groupes de type $(p, \dots, p)$", *Bull. Soc. Math. France* **102** (1974), 241–280. MR 54 #7488 Zbl 0325.14020

[Richarz 2013] T. Richarz, "Schubert varieties in twisted affine flag varieties and local models", *J. Algebra* **375** (2013), 121–147. MR 2998951 Zbl 1315.14066

[Richarz 2015] T. Richarz, "Affine grassmanians and geometric Satake equivalences", preprint, 2015. arXiv 1311.1008

[Saavedra Rivano 1972] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics **265**, Springer, New York, 1972. MR 49 #2769 Zbl 0241.14008

[Serre 1958] J.-P. Serre, "Espaces fibres algébriques", *Séminaire Claude Chevalley* **3** (1958), Exposé No. 1, 1–37. Zbl 03159916

[SGA 7_{I} 1972] D. S. Rim, "Formal deformation theory", pp. 32–132 in *Groupes de monodromie en géométrie algébrique, I: Exposés I–II, VI–IX* (Séminaire de Géométrie Algébrique du Bois Marie 1967–1969), Lecture Notes in Math. **288**, Springer, Berlin, 1972. MR 50 #7134 Zbl 0237.00013

[Springer 1998] T. A. Springer, *Linear algebraic groups*, 2nd ed., Progr. Math. **9**, Birkhäuser, Boston, 1998. MR 99h:20075 Zbl 0927.20024

[Taylor and Wiles 1995] R. Taylor and A. Wiles, "Ring-theoretic properties of certain Hecke algebras", *Ann. of Math.* (2) **141**:3 (1995), 553–572. MR 96d:11072 Zbl 0823.11030

Communicated by Mark Kisin
Received 2014-04-04 Revised 2015-06-01 Accepted 2015-07-15

bwlevin@math.uchicago.edu *Department of Mathematics, University of Chicago, 5734 S. University Avenue, Room 208C, Chicago, 60637, United States*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2015 is US $255/year for the electronic version, and $440/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 9 No. 8 2015

1937-0652(2015)9:8;1-1