

Algebra & Number Theory

Volume 9

2015

No. 9

An averaged form of Chowla's conjecture

Kaisa Matomäki, Maksym Radziwiłł and Terence Tao



An averaged form of Chowla's conjecture

Kaisa Matomäki, Maksym Radziwiłł and Terence Tao

Let λ denote the Liouville function. A well-known conjecture of Chowla asserts that, for any distinct natural numbers h_1, \dots, h_k , one has

$$\sum_{1 \leq n \leq X} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(X)$$

as $X \rightarrow \infty$. This conjecture remains unproven for any h_1, \dots, h_k with $k \geq 2$. Using the recent results of Matomäki and Radziwiłł on mean values of multiplicative functions in short intervals, combined with an argument of Kátai and Bourgain, Sarnak, and Ziegler, we establish an averaged version of this conjecture, namely

$$\sum_{h_1, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n + h_1) \cdots \lambda(n + h_k) \right| = o(H^k X)$$

as $X \rightarrow \infty$, whenever $H = H(X) \leq X$ goes to infinity as $X \rightarrow \infty$ and k is fixed. Related to this, we give the exponential sum estimate

$$\int_0^X \left| \sum_{x \leq n \leq x+H} \lambda(n) e(\alpha n) \right| dx = o(HX)$$

as $X \rightarrow \infty$ uniformly for all $\alpha \in \mathbb{R}$, with H as before. Our arguments in fact give quantitative bounds on the decay rate (roughly on the order of $\log \log H / \log H$) and extend to more general bounded multiplicative functions than the Liouville function, yielding an averaged form of a (corrected) conjecture of Elliott.

1. Introduction

Let $\lambda : \mathbb{N} \rightarrow \{-1, +1\}$ be the Liouville function, that is to say, the completely multiplicative function such that $\lambda(p) = -1$ for all primes p . The prime number theorem implies that¹

$$\sum_{1 \leq n \leq X} \lambda(n) = o(X)$$

MSC2010: 11P32.

Keywords: multiplicative functions, Hardy–Littlewood circle method, Chowla conjecture.

¹See page 2174 for our asymptotic notation conventions.

as $X \rightarrow \infty$. More generally, a famous conjecture of Chowla [1965] asserts that, for any distinct natural numbers h_1, \dots, h_k , one has

$$\sum_{1 \leq n \leq X} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(X) \tag{1-1}$$

as $X \rightarrow \infty$.

Chowla’s conjecture remains open for any h_1, \dots, h_k with $k \geq 2$. Our first main theorem establishes an averaged form of this conjecture:

Theorem 1.1 (Chowla’s conjecture on average). *For any natural number k , and any $10 \leq H \leq X$, we have*

$$\sum_{1 \leq h_1, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n + h_1) \cdots \lambda(n + h_k) \right| \ll k \left(\frac{\log \log H}{\log H} + \frac{1}{\log^{1/3000} X} \right) H^k X. \tag{1-2}$$

In fact, we have the slightly stronger bound

$$\sum_{1 \leq h_2, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n) \lambda(n + h_2) \cdots \lambda(n + h_k) \right| \ll k \left(\frac{\log \log H}{\log H} + \frac{1}{\log^{1/3000} X} \right) H^{k-1} X. \tag{1-3}$$

In the case $k = 2$ our result implies that

$$\sum_{1 \leq h \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n) \lambda(n + h) \right| = o(HX)$$

provided that $H \rightarrow \infty$ arbitrarily slowly with $X \rightarrow \infty$ (and $H \leq X$). Note that the $k = 2$ case of Chowla’s conjecture is equivalent to the above asymptotic holding in the case that H is bounded rather than going to infinity.

In fact, we have a more precise bound than (1-2) (or (1-3)) that gives more control on the exceptional tuples (h_1, \dots, h_k) for which the sums of the form $\sum_{1 \leq n \leq X} \lambda(n + h_1) \cdots \lambda(n + h_k)$ are large; see Remark 5.2. In particular, in the special case $k = 2$ we get the following result.

Theorem 1.2. *Let $\delta \in (0, 1]$ be fixed. There is a large but fixed $H = H(\delta)$ such that, for all large enough X ,*

$$\left| \sum_{1 \leq n \leq X} \lambda(n) \lambda(n + h) \right| \leq \delta X \tag{1-4}$$

for all but at most $H^{1-\delta/5000}$ integers $|h| \leq H$.

One can also replace the ranges $1 \leq h_j \leq H$ in [Theorem 1.1](#) by $b_j + 1 \leq h_j \leq b_j + H$ for any $b_j = O(X)$; see [Theorem 1.6](#).

The exponents $1/3000$ and $1/5000$ in the above theorems may certainly be improved, but we did not attempt to optimize the constants here. However, our methods cannot produce a gain much larger than $1/\log H$, as one would then have to somehow control λ on numbers that are not divisible by any prime less than H , at which point we are no longer able to exploit the averaging in the h_1, \dots, h_k parameters. It would be of particular interest to obtain a gain of more than $1/\log X$, as one could then potentially localize λ to primes and obtain some version of the prime tuples conjecture when the h_1, \dots, h_k parameters are averaged over short intervals, but this is well beyond the capability of our methods. (If instead one is allowed to average the h_1, \dots, h_k over long intervals of scale comparable to X , one can obtain various averaged forms of the prime tuples conjecture and its relatives, by rather different methods than those used here; see [[Balog 1990](#); [Mikawa 1992](#); [Kawada 1993](#); [1995](#); [Green and Tao 2010](#)].)

[Theorem 1.1](#) is closely related to the following averaged short exponential sum estimate, which may be of independent interest.

Theorem 1.3 (exponential sum estimate). *For any $10 \leq H \leq X$, one has*

$$\sup_{\alpha \in \mathbb{R}} \int_0^X \left| \sum_{x \leq n \leq x+H} \lambda(n) e(\alpha n) \right| dx \ll \left(\frac{\log \log H}{\log H} + \frac{1}{\log^{1/700} X} \right) HX.$$

Actually, for technical reasons it is convenient to prove a sharper version of [Theorem 1.3](#) in which the Liouville function has been restricted to those numbers that have “typical” factorization; see [Theorem 2.3](#). This sharper version will then be used to establish [Theorem 1.1](#).

The relationship between [Theorems 1.1](#) and [1.3](#) stems from the following Fourier-analytic identity:

Lemma 1.4 (Fourier identity). *For $H > 0$, if $f : \mathbb{Z} \rightarrow \mathbb{C}$ is a function supported on a finite set, then*

$$\int_{\mathbb{T}} \left(\int_{\mathbb{R}} \left| \sum_{x \leq n \leq x+H} f(n) e(\alpha n) \right|^2 dx \right)^2 d\alpha = \sum_{|h| \leq H} (H - |h|)^2 \left| \sum_n f(n) \bar{f}(n+h) \right|^2.$$

Proof. Using the Fourier identity $\int_{\mathbb{T}} e(n\alpha) d\alpha = 1_{n=0}$, we can expand the left-hand side as

$$\sum_{n, n', m, m'} f(n) \bar{f}(n') f(m) \bar{f}(m') 1_{n+m-n'-m'=0} \times \left(\int_{\mathbb{R}} 1_{x \leq n, n' \leq x+H} dx \right) \left(\int_{\mathbb{R}} 1_{y \leq m, m' \leq y+H} dy \right).$$

Writing $n' = n + h$, we see that both integrals are equal to $H - |h|$ if $|h| \leq H$ and vanish otherwise. The claim follows. \square

Theorem 1.3 may be compared with the classical estimate

$$\sup_{\alpha \in \mathbb{R}} \left| \sum_{1 \leq n \leq X} \lambda(n)e(\alpha n) \right| \ll_A X \log^{-A} X$$

of Davenport [1937], valid for any $A > 0$. Indeed, one can view **Theorem 1.3** as asserting that a weak form of Davenport’s estimate holds on average in short intervals. It would be of interest to also obtain nontrivial bounds on the larger quantity

$$\int_0^X \sup_{\alpha \in \mathbb{R}} \left| \sum_{x \leq n \leq x+H} \lambda(n)e(\alpha n) \right| dx \tag{1-5}$$

but this appears difficult to establish with our methods.

As with other applications of the circle method, our proof of **Theorem 1.3** splits into two cases, depending on whether the quantity α is on “major arc” or on “minor arc”. In the “major arc” case we are able to use the recent results of Matomäki and Radziwiłł [2015] on the average size of mean values of multiplicative functions on short intervals. Actually, in order to handle the presence of complex Dirichlet characters, we need to extend the results in [Matomäki and Radziwiłł 2015] to complex-valued multiplicative functions rather than real-valued ones; this is accomplished in an appendix to this paper (**Appendix A**). In the “minor arc” case we use a variant of the arguments of Kátai [1986] and Bourgain, Sarnak, and Ziegler [Bourgain et al. 2013] (see also the earlier works of Montgomery and Vaughan [1977] and Daboussi and Delange [1982]) to obtain the required cancellation. One innovation here is to rely on a combinatorial identity of Ramaré (also used in [Matomäki and Radziwiłł 2015]) as a substitute for the Turán–Kubilius inequality, as this leads to superior quantitative estimates (particularly if one first restricts the variable n to have a “typical” prime factorization).

Extension to more general multiplicative functions. Define a 1-bounded multiplicative function to be a multiplicative function $f : \mathbb{N} \rightarrow \mathbb{C}$ such that $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. Given two 1-bounded multiplicative functions f, g and a parameter $X \geq 1$, we define the distance $\mathbb{D}(f, g; X) \in [0, +\infty)$ by the formula

$$\mathbb{D}(f, g; X) := \left(\sum_{p \leq X} \frac{1 - \operatorname{Re}(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

This is known to give a (pseudo)metric on 1-bounded multiplicative functions; see [Granville and Soundararajan 2007, Lemma 3.1]. We also define the asymptotic

counterpart $\mathbb{D}(f, g; \infty) \in [0, +\infty]$ by the formula

$$\mathbb{D}(f, g; \infty) := \left(\sum_p \frac{1 - \operatorname{Re}(f(p)\overline{g(p)})}{p} \right)^{1/2}.$$

We informally say that f pretends to be g if $\mathbb{D}(f, g; X)$ (or $\mathbb{D}(f, g; \infty)$) is small (or finite).

For any 1-bounded multiplicative function g and real number $X > 1$, we introduce the quantity

$$M(g; X) := \inf_{|t| \leq X} \mathbb{D}(g, n \mapsto n^{it}; X)^2, \tag{1-6}$$

and then the more general quantity

$$M(g; X, Q) := \inf_{q \leq Q; \chi(q)} M(g\bar{\chi}; X) = \inf_{|t| \leq X; q \leq Q; \chi(q)} \mathbb{D}(g, n \mapsto \chi(n)n^{it}; X)^2,$$

where χ ranges over all Dirichlet characters of modulus $q \leq Q$. Informally, $M(g; X)$ is small when g pretends to be like a multiplicative character $n \mapsto n^{it}$, and $M(g; X, Q)$ is small when g pretends to be like a twisted Dirichlet character of modulus at most Q and twist of height at most X . We also define the asymptotic counterpart

$$M(g; \infty, \infty) = \inf_{\chi, t} \mathbb{D}(g, n \mapsto \chi(n)n^{it}; \infty)^2$$

where χ now ranges over all Dirichlet characters and t ranges over all real numbers.

Elliott proposed in [1992, Conjecture II] the following more general form of Chowla’s conjecture, which we phrase here in contrapositive form.

Conjecture 1.5 (Elliott’s conjecture). *Let $g_1, \dots, g_k : \mathbb{N} \rightarrow \mathbb{C}$ be 1-bounded multiplicative functions, and let $a_1, \dots, a_k, b_1, \dots, b_k$ be natural numbers such that any two of the pairs $(a_1, b_1), \dots, (a_k, b_k)$ are linearly independent in \mathbb{Q}^2 . Suppose that there is an index $1 \leq j_0 \leq k$ such that*

$$M(g_{j_0}; \infty, \infty) = \infty. \tag{1-7}$$

Then

$$\sum_{1 \leq n \leq X} \prod_{j=1}^k g_j(a_j n + b_j) = o(X) \tag{1-8}$$

as $X \rightarrow \infty$.

Informally, this conjecture asserts that for pairwise linearly independent pairs $(a_1, b_1), \dots, (a_k, b_k)$ and any 1-bounded multiplicative g_1, \dots, g_k , one has the asymptotic (1-8) as $X \rightarrow \infty$, unless each of the g_j pretends to be a twisted Dirichlet character $n \mapsto \chi_j(n)n^{it_j}$. Note that some condition of this form is necessary, since if $g(n)$ is equal to $\chi(n)n^{it}$ then $g(n)\overline{g(n+h)}$ will be biased to be positive for large n , if h is fixed and divisible by the modulus q of χ ; one also expects some bias when

h is not divisible by this modulus since the sums $\sum_{n \in \mathbb{Z}/q\mathbb{Z}} \chi(n) \overline{\chi(n+h)}$ do not vanish in general. From the prime number theorem in arithmetic progressions it follows that

$$M(\lambda; \infty, \infty) = \infty,$$

so Elliott’s conjecture implies Chowla’s conjecture (1-1).

When one allows the functions g_j to be complex-valued rather than real-valued, Elliott’s conjecture turns out to be false on a technicality; one can choose 1-bounded multiplicative functions g_j which are arbitrarily close at various scales to a sequence of functions of the form $n \mapsto n^{it}$ (which allows one to violate (1-8)) without globally pretending to be n^{it} (or $\chi(n)n^{it}$) for any fixed t ; we present this counterexample in Appendix B. However, this counterexample can be removed by replacing (1-7) with the stronger condition that

$$M(g_{j_0}; X, Q) \rightarrow \infty \tag{1-9}$$

as $X \rightarrow \infty$ for each fixed Q . In the real-valued case, (1-9) and (1-7) are equivalent by a triangle inequality argument of Granville and Soundararajan which we give in Appendix C.

As evidence for the corrected form of Conjecture 1.5 (in both the real-valued and complex-valued cases), we present the following averaged form of that conjecture:

Theorem 1.6 (Elliott’s conjecture on average). *Let $10 \leq H \leq X$ and $A \geq 1$. Let $g_1, \dots, g_k : \mathbb{N} \rightarrow \mathbb{C}$ be 1-bounded functions, and let $a_1, \dots, a_k, b_1, \dots, b_k$ be natural numbers with $a_j \leq A$ and $b_j \leq AX$ for $j = 1, \dots, k$. Let $1 \leq j_0 \leq k$, and suppose that g_{j_0} is multiplicative. Then one has*

$$\sum_{1 \leq h_1, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \prod_{j=1}^k g_j(a_j n + b_j + h_j) \right| \ll A^2 k \left(\exp(-M/80) + \frac{\log \log H}{\log H} + \frac{1}{\log^{1/3000} X} \right) H^k X \tag{1-10}$$

where

$$M := M(g_{j_0}; 10AX, Q) \quad \text{and} \quad Q := \min(\log^{1/125} X, \log^{20} H).$$

In fact, we have the slightly stronger bound

$$\sum_{1 \leq h_2, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} g_1(a_1 n + b_1) \prod_{j=2}^k g_j(a_j n + b_j + h_j) \right| \ll A^2 k \left(\exp(-M/80) + \frac{\log \log H}{\log H} + \frac{1}{\log^{1/3000} X} \right) H^{k-1} X. \tag{1-11}$$

Note that if $a_1, \dots, a_k, b_1, \dots, b_k$ are fixed, g_{j_0} is independent of X and obeys the condition (1-9) for any fixed Q , and $H = H(X)$ is chosen to go to infinity arbitrarily slowly as $X \rightarrow \infty$, then the quantity M in the above theorem goes to infinity (note that $M(g; X, Q)$ is nondecreasing in Q), and (1-11) then implies an averaged form of the asymptotic (1-8). Thus Theorem 1.6 is indeed an averaged form of the corrected form of Conjecture 1.5. (We discovered the counterexample in Appendix B while trying to interpret Theorem 1.6 as an averaged version of the original form of Conjecture 1.5.) Interestingly, only one of the functions g_1, \dots, g_k in Theorem 1.6 is required to be multiplicative;² one can use a van der Corput argument to reduce matters to obtaining cancellation for a sum roughly of the form $\sum_{h \leq H} \left| \sum_{1 \leq n \leq X} g_{j_0}(n) g_{j_0}(n+h) \right|^2$, which can then be treated using Lemma 1.4.

For $g(n) = \lambda(n)$ and X, Q, M as in the above theorem, one obtains, for every $\varepsilon > 0$, the bound

$$\begin{aligned}
 M &\geq \inf_{|t| \leq X; q \leq Q; \chi(q)} \sum_{\substack{\exp((\log X)^{2/3+\varepsilon}) \leq p \leq X \\ p}} \frac{1 + \operatorname{Re} \chi(p) p^{it}}{p} \\
 &\geq \left(\frac{1}{3} - \varepsilon\right) \log \log X + O(1).
 \end{aligned}
 \tag{1-12}$$

The last inequality is established via standard methods from the Vinogradov–Korobov type zero-free region

$$\left\{ \sigma + it : \sigma > 1 - \frac{c}{\max\{\log q, (\log(3 + |t|))^{2/3} (\log \log(3 + |t|))^{1/3}\}} \right\}$$

for $L(s, \chi)$ and some absolute constant $c > 0$, which applies since χ has conductor $q \leq (\log X)^{1/125}$ (so that there are no exceptional zeros); see [Montgomery 1994, §9.5]. Hence Theorem 1.6 implies Theorem 1.1. The same argument gives Theorem 1.1 when the Liouville function λ is replaced by the Möbius function μ . We remark that, as our arguments make no use of exceptional zeroes, all the implied constants in our theorems are effective.

We also have a generalized form of Theorem 1.3:

Theorem 1.7 (exponential sum estimate). *Let $X \geq H \geq 10$ and let g be a 1-bounded multiplicative function. Then*

$$\begin{aligned}
 \sup_{\alpha \in \mathbb{T}} \int_0^X \left| \sum_{x \leq n \leq x+H} g(n) e(\alpha n) \right| dx \\
 \ll \left(\exp(-M(g; X, Q)/20) + \frac{\log \log H}{\log H} + \frac{1}{\log^{1/700} X} \right) HX
 \end{aligned}$$

²We thank the referee for observing this fact. In a previous version of this paper, all of the g_j were required to be multiplicative.

where

$$Q := \min(\log^{1/125} X, \log^5 H).$$

By (1-12), [Theorem 1.7](#) implies [Theorem 1.3](#).

Remark 1.8. In the recent preprint [[Frantzikinakis and Host 2015](#)], a different averaged form of Elliott’s conjecture is established, in which one uses fewer averaging parameters h_i than in [Theorem 1.6](#) (indeed, one can average over just a single such parameter, provided that the linear parts of the forms are independent), but the averaging parameters range over a long range (comparable to X) rather than on the short range given here. The methods of proof are rather different (in particular, the arguments in [[Frantzikinakis and Host 2015](#)] rely on higher order Fourier analysis). In the long-range averaged situation considered in [[Frantzikinakis and Host 2015](#)], the counterexample in [Appendix B](#) does not apply, and one can use the original form of Elliott’s conjecture in place of the corrected version. It may be possible to combine the results here with those in [[Frantzikinakis and Host 2015](#)] to obtain an averaged version of Chowla’s or Elliott’s conjecture in which the number of averaging parameters is small, and the averaging is over a short range, but this seems to require nontrivial estimates on quantities such as (1-5), which we are currently unable to handle.

Notation. Our asymptotic notation conventions are as follows. We use $X \ll Y$, $Y \gg X$, or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for some absolute constant C . If x is a parameter going to infinity, we use $X = o(Y)$ to denote the claim that $|X| \leq c(x)Y$ for some quantity $c(x)$ that goes to zero as $x \rightarrow \infty$ (holding all other parameters fixed).

Unless otherwise specified, all sums are over the integers, except for sums over the variable p (or p_1, p_2 , etc.) which are understood to be over primes.

We use $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ to denote the standard unit circle and let $e : \mathbb{T} \rightarrow \mathbb{C}$ be the standard character $e(x) := e^{2\pi i x}$.

We use 1_S to denote the indicator of a predicate S , thus $1_S = 1$ when S is true and $1_S = 0$ when S is false. If A is a set, we write $1_{A(n)}$ for $1_{n \in A}$, so that 1_A is the indicator function of A .

2. Restricting to numbers with typical factorization

To prove [Theorem 1.6](#) and [Theorem 1.7](#) (and hence [Theorem 1.1](#) and [Theorem 1.3](#)), it is technically convenient (as in the previous paper [[Matomäki and Radziwiłł 2015](#)]) to restrict the support of the multiplicative functions to a certain dense set S of natural numbers that have a “typical” prime factorization in a certain specific sense, in order to fully exploit a useful combinatorial identity of Ramaré (see (3-2)).

This will lead to improved quantitative estimates in the arguments in subsequent sections of the paper.

More precisely, we introduce the following sets \mathcal{S} of numbers with typical prime factorization, which previously appeared in [Matomäki and Radziwiłł 2015].

Definition 2.1. Let $10 < P_1 < Q_1 \leq X$ and $\sqrt{X} \leq X_0 \leq X$ be quantities such that $Q_1 \leq \exp(\sqrt{\log X_0})$. We then define P_j, Q_j for $j > 1$ by the formulas

$$P_j := \exp(j^{4j}(\log Q_1)^{j-1} \log P_1), \quad Q_j := \exp(j^{4j+2}(\log Q_1)^j)$$

for $j > 1$. Note that the intervals $[P_j, Q_j]$ are disjoint and increase to infinity; indeed, one easily verifies that

$$P_1 < Q_1 < \exp(2^8 \log Q_1 \log P_1) = P_2$$

and

$$P_j < \exp(j^{4j}(\log Q_1)^j) < Q_j < \exp((j+1)^{4(j+1)}(\log Q_1)^j) < P_{j+1}$$

for all $j > 1$. Let J be the largest index such that $Q_J \leq \exp(\sqrt{\log X_0})$. Then we define $\mathcal{S}_{P_1, Q_1, X_0, X}$ to be the set of all the numbers $1 \leq n \leq X$ which have at least one prime factor in the interval $[P_j, Q_j]$ for each $1 \leq j \leq J$.

In practice, X will be taken to be slightly smaller than X_0^2 . The need to have two parameters X, X_0 instead of one is technical (we need to have the freedom later in the argument to replace X with a slightly smaller quantity X/d without altering J), but the reader may wish to pretend that $X_0 = \sqrt{X}$ for most of the argument.

This set is fairly dense if P_1 and Q_1 are widely separated:

Lemma 2.2. Let $10 < P_1 < Q_1 \leq X$ and $\sqrt{X} \leq X_0 \leq X$ be quantities such that $Q_1 \leq \exp(\sqrt{\log X_0})$. Then, for every large enough X ,

$$\#\{1 \leq n \leq X : n \notin \mathcal{S}_{P_1, Q_1, X_0, X}\} \ll \frac{\log P_1}{\log Q_1} \cdot X.$$

Proof. From the fundamental lemma of sieve theory (see, e.g., [Friedlander and Iwaniec 2010, Theorem 6.17]) we know that, for any $1 \leq j \leq J$ and large enough X , the number of $1 \leq n \leq X$ that are not divisible by any prime in $[P_j, Q_j]$ is at most

$$\ll X \prod_{P_j \leq p \leq Q_j} \left(1 - \frac{1}{p}\right) \ll \frac{\log P_j}{\log Q_j} X = \frac{1}{j^2} \frac{\log P_1}{\log Q_1} X.$$

Summing over j , we obtain the claim. □

Both Theorems 1.6 and 1.7 will be deduced from the following claim.

Theorem 2.3 (key exponential sum estimate). Let $X, H, W \geq 10$ be such that

$$(\log H)^5 \leq W \leq \min\{H^{1/250}, (\log X)^{1/125}\}$$

and let g be a 1-bounded multiplicative function such that

$$W \leq \exp(M(g; X, Q)/3). \tag{2-1}$$

Set

$$\mathcal{S} := \mathcal{S}_{P_1, Q_1, \sqrt{X}, X} \quad \text{where } P_1 := W^{200}, Q_1 := H/W^3.$$

Then, for any $\alpha \in \mathbb{T}$, one has

$$\int_{\mathbb{R}} \left| \sum_{x \leq n \leq x+H} 1_{\mathcal{S}}(n)g(n)e(\alpha n) \right| dx \ll \frac{(\log H)^{1/4} \log \log H}{W^{1/4}} HX. \tag{2-2}$$

In [Section 5](#) we will show how this theorem implies [Theorem 1.6](#). For now, let us at least see how it implies [Theorem 1.7](#):

Proof of [Theorem 1.7](#) assuming [Theorem 2.3](#). We may assume that $X, H,$ and $M(g; X, Q)$ are larger than any specified absolute constant, since if one of these expressions is bounded, then so is W . The claim [\(2-2\)](#) is then trivial with a suitable choice of implied constant (discarding the $(\log H)^{1/4} \log \log H$ factor).

Choose H_0 such that

$$\log H_0 := \min(\log^{1/700} X \log \log X, \exp(M(g; X, Q)/20)M(g; X, Q)).$$

We divide into two cases: $H \leq H_0$ and $H > H_0$.

First suppose that $H \leq H_0$. Then if we set $W := \log^5 H$, one verifies that all the hypotheses of [Theorem 2.3](#) hold, and hence

$$\int_0^X \left| \sum_{x \leq n \leq x+H} 1_{\mathcal{S}}(n)g(n)e(\alpha n) \right| dx \ll \frac{\log \log H}{\log H} HX.$$

On the other hand, from [Lemma 2.2](#), the choice of $W, P_1, Q_1,$ and the bound on $H,$ we see that

$$\#\{1 \leq n \leq X + H : n \notin \mathcal{S}\} \ll \frac{\log \log H}{\log H} X$$

and thus, by Fubini’s theorem and the triangle inequality,

$$\int_0^X \left| \sum_{x \leq n \leq x+H} (1 - 1_{\mathcal{S}}(n))g(n)e(\alpha n) \right| dx \ll \frac{\log \log H}{\log H} HX.$$

Summing, we obtain [Theorem 1.7](#) in this case.

Now suppose that $H > H_0$. Covering $[0, H]$ by $O(H/H_0)$ intervals of length $H_0,$ we see that

$$\int_0^X \left| \sum_{x \leq n \leq x+H} g(n)e(\alpha n) \right| dx \ll \frac{H}{H_0} \int_0^{X+H} \left| \sum_{x \leq n \leq x+H_0} g(n)e(\alpha n) \right| dx.$$

Also, observe from the choice of H_0 that the quantity

$$\exp(-M(g; X, Q)/20) + \frac{\log \log H}{\log H} + \frac{1}{\log^{1/700} X}$$

is unchanged up to multiplicative constants if one reduces H to H_0 . Finally, from Mertens’ theorem we see that $M(g; X + H, Q) = M(g; X, Q) + O(1)$. The claim then follows from the $H = H_0$ case (after performing the minor alteration of replacing X with $X + H$). □

We now begin the proof of [Theorem 2.3](#). The first step is to reduce to the case where g is completely multiplicative rather than multiplicative. More precisely, we will deduce [Theorem 2.3](#) from the following proposition.

Proposition 2.4 (completely multiplicative exponential sum estimate). *Assume $X, H, W \geq 10$ are such that*

$$(\log H)^5 \leq W \leq \min\{H^{1/250}, (\log X)^{1/125}\},$$

and let g be a 1-bounded completely multiplicative function such that

$$W \leq \exp(M(g; X, W)/3). \tag{2-3}$$

Let d be a natural number with $d < W$. Set

$$S := S_{P_1, Q_1, \sqrt{X}, X/d} \quad \text{where } P_1 := W^{200}, Q_1 := H/W^3.$$

Then for any $\alpha \in \mathbb{T}$ one has

$$\int_{\mathbb{R}} \left| \sum_{x/d \leq n \leq x/d + H/d} 1_S(n) g(n) e(\alpha n) \right| dx \ll \frac{1}{d^{3/4}} \frac{(\log H)^{1/4} \log \log H}{W^{1/4}} HX. \tag{2-4}$$

Let us explain why [Theorem 2.3](#) follows from [Proposition 2.4](#). Let the hypotheses and notation be as in [Theorem 2.3](#). The function g is not necessarily completely multiplicative, but we may approximate it by the 1-bounded completely multiplicative function $g_1 : \mathbb{N} \rightarrow \mathbb{C}$, defined as the completely multiplicative function with $g_1(p) = g(p)$ for all primes p . By Möbius inversion we may then write $g = g_1 * h$ where $*$ denotes Dirichlet convolution and h is the multiplicative function $h = g * \mu g_1$. Observe that, for all primes p , we have $h(p) = 0$ and $|h(p^j)| \leq 2$ for $j \geq 2$. We now write

$$\sum_{x \leq n \leq x+H} 1_{S_{P_1, Q_1, \sqrt{X}, X}} g(n) e(\alpha n) = \sum_{d=1}^{\infty} h(d) \sum_{x/d \leq m \leq x/d + H/d} 1_{S_{P_1, Q_1, \sqrt{X}, X}}(dm) g_1(m) e(d\alpha m)$$

and so by the triangle inequality we may upper bound the left-hand side of (2-2) by

$$\sum_{d=1}^{\infty} |h(d)| \int_{\mathbb{R}} \left| \sum_{x/d \leq m \leq x/d + H/d} 1_{S_{P_1, Q_1, \sqrt{x}, x}}(dm) g_1(m) e(d\alpha m) \right| dx.$$

Let us first dispose of the contribution where $d \geq W$. Here we trivially bound this contribution by

$$\sum_{d \geq W} |h(d)| \sum_{m \leq (2X+H)/d} O(H)$$

(after moving the absolute values inside the m summation and then performing the integration on x first). We can bound this in turn by

$$\ll HX \frac{1}{W^{1/4}} \sum_{d=1}^{\infty} \frac{|h(d)|}{d^{3/4}}.$$

From Euler products we see that $\sum_{d=1}^{\infty} |h(d)|/d^{3/4} = O(1)$, so the contribution of this case is acceptable.

Now we consider the contribution $d < W < P_1$. In this case we may reduce

$$1_{S_{P_1, Q_1, \sqrt{x}, x}}(dm) = 1_{S_{P_1, Q_1, \sqrt{x}, x/d}}(m)$$

and so this contribution to (2-2) can be upper bounded by

$$\sum_{1 \leq d < W} |h(d)| \int_{\mathbb{R}} \left| \sum_{x/d \leq m \leq x/d + H/d} 1_{S_{P_1, Q_1, \sqrt{x}, x/d}}(m) g_1(m) e(d\alpha m) \right| dx.$$

By Proposition 2.4, this is bounded by

$$\sum_{d=1}^{\infty} \frac{|h(d)|}{d^{3/4}} \frac{(\log H)^{1/4} \log \log H}{W^{1/4}} HX.$$

As before, we have $\sum_{d=1}^{\infty} |h(d)|/d^{3/4} = O(1)$, and Theorem 2.3 follows.

It remains to prove Proposition 2.4. For any $\alpha \in \mathbb{T}$, we know from the Dirichlet approximation theorem that there exists a rational number a/q with $(a, q) = 1$ and $1 \leq q \leq H/W$ such that

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{W}{qH} \leq \frac{1}{q^2}.$$

In the next two sections, we will apply separate arguments to prove Proposition 2.4 in the *minor arc* case $q > W$ and the *major arc* case $q \leq W$.

3. Proof of minor arc estimate

We now prove [Proposition 2.4](#) in the minor arc case $q > W$. It suffices to show that

$$\int_{\mathbb{R}} \theta(x) \sum_{x/d \leq n \leq x/d + H/d} 1_{\mathcal{S}}(n)g(n)e(\alpha n) dx \ll \frac{1}{d^{3/4}} \frac{(\log H)^{1/4} \log \log H}{W^{1/4}} HX \quad (3-1)$$

whenever $\theta : \mathbb{R} \rightarrow \mathbb{C}$ is measurable, with $|\theta(x)|$ at most 1 for all x and supported on $[0, X]$. We will now use a variant of an idea of Bourgain, Sarnak, and Ziegler [[Bourgain et al. 2013](#)] (building on earlier works of Kátai [[1986](#)], Montgomery and Vaughan [[1977](#)] and Daboussi and Delange [[1982](#)]).

Let \mathcal{P} be the set consisting of the primes lying between P_1 and Q_1 . Then notice that each $n \in \mathcal{S}$ has at least one prime factor from \mathcal{P} . This leads to the following variant of Ramaré’s identity (see [[Friedlander and Iwaniec 2010](#), Section 17.3]):

$$1_{\mathcal{S}}(n) = \sum_{p \in \mathcal{P}, m: mp=n} \frac{1_{\mathcal{S}'}(mp)}{1 + \#\{q|m : q \in \mathcal{P}\}}, \quad (3-2)$$

where \mathcal{S}' is the set of all $1 \leq n \leq X/d$ that have at least one prime factor in each of the intervals $[P_j, Q_j]$ for $j \geq 2$; the constraint $n \leq X/d$ arises from the corresponding constraint in the definition of \mathcal{S} .

Using this identity, we may write the left-hand side of (3-1) as

$$\sum_{p \in \mathcal{P}} \sum_m \frac{1_{\mathcal{S}'}(mp)g(mp)e(mp\alpha)}{1 + \#\{q|m : q \in \mathcal{P}\}} \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d} dx.$$

As g is completely multiplicative, $g(mp) = g(m)g(p)$. Thus it suffices to show that

$$\begin{aligned} \sum_{p \in \mathcal{P}} \sum_m \frac{1_{\mathcal{S}'}(mp)g(m)g(p)e(mp\alpha)}{1 + \#\{q|m : q \in \mathcal{P}\}} \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d} dx \\ \ll \frac{(\log H)^{1/4} \log \log H}{d^{3/4} W^{1/4}} HX. \end{aligned}$$

We can cover \mathcal{P} by intervals $[P, 2P]$ with $P_1 \ll P \ll Q_1$ and P a power of two, and observe that

$$\sum_{\substack{P_1 \ll P \ll Q_1 \\ P=2^j}} \frac{1}{\log P} \ll \log \log Q_1 - \log \log P_1 \ll \log \log H,$$

so by the triangle inequality it suffices to show that

$$\begin{aligned} \sum_{\substack{p \in \mathcal{P} \\ P \leq p \leq 2P}} \sum_m \frac{1_{\mathcal{S}'}(mp)g(m)g(p)e(mp\alpha)}{1 + \#\{q|m : q \in \mathcal{P}\}} \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d} dx \\ \ll \frac{(\log H)^{1/4}}{d^{3/4} W^{1/4} \log P} HX \end{aligned}$$

for each such P . We can rearrange the left-hand side as

$$\sum_{m \in \mathcal{S}'} \frac{g(m)}{1 + \#\{q|m : q \in \mathcal{P}\}} \sum_{\substack{p \in \mathcal{P} \\ P \leq p \leq 2P}} 1_{mp \leq X/d} g(p) e(mp\alpha) \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d} dx.$$

Observe that the summand vanishes unless we have $m \leq X/dP$. Crudely bounding³ $g(m)/(1 + \#\{q|m : q \in \mathcal{P}\})$ in magnitude by 1 and applying Hölder’s inequality, we may bound the previous expression in magnitude by

$$\left(\frac{X}{dP}\right)^{3/4} \left(\sum_{m \leq X/dP} \left| \sum_{\substack{p \in \mathcal{P} \\ P \leq p \leq 2P}} 1_{mp \leq X/d} g(p) e(mp\alpha) \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d} dx \right|^4 \right)^{1/4}.$$

It thus suffices to show that

$$\sum_{m \leq X/dP} \left| \sum_{\substack{p \in \mathcal{P} \\ P \leq p \leq 2P}} 1_{mp \leq X/d} g(p) e(mp\alpha) \int_{\mathbb{R}} \theta(x) 1_{x/d \leq mp \leq (x+H)/d} dx \right|^4 \ll \frac{\log H}{W \log^4 P} H^4 X P^3.$$

The left-hand side may be expanded as

$$\sum_{\substack{p_1, p_2, p_3, p_4 \in \mathcal{P} \\ P \leq p_1, p_2, p_3, p_4 \leq 2P}} \int \cdots \int g(p_1) g(p_2) \overline{g(p_3)} \overline{g(p_4)} \theta(x_1) \theta(x_2) \overline{\theta(x_3)} \overline{\theta(x_4)} \\ \times \sum_{\substack{m \leq X/(dp_i) \\ x_i/(dp_i) \leq m \leq (x_i+H)/(dp_i) \\ \forall i=1,2,3,4}} e(m(p_1 + p_2 - p_3 - p_4)\alpha) dx_1 dx_2 dx_3 dx_4.$$

From summing the geometric series, we observe that the summation over m is $O(\min(H/P, 1/\|(p_1 + p_2 - p_3 - p_4)\alpha\|))$, where $\|z\|$ denotes the distance from z to the nearest integer. Also, the sum vanishes unless we have $x_1 = O(X)$ and $x_i = x_1 p_i / p_1 + O(H)$ for $i = 2, 3, 4$, so there are only $O(XH^3)$ quadruples (x_1, x_2, x_3, x_4) which contribute. Thus we may bound the previous expression by

$$O\left(XH^3 \sum_{p_1, p_2, p_3, p_4 \leq 2P} \min\left(\frac{H}{P}, \frac{1}{\|(p_1 + p_2 - p_3 - p_4)\alpha\|}\right)\right)$$

and so we reduce to showing that

$$\sum_{p_1, p_2, p_3, p_4 \leq 2P} \min\left(\frac{H}{P}, \frac{1}{\|(p_1 + p_2 - p_3 - p_4)\alpha\|}\right) \ll \log H \frac{HP^3}{W \log^4 P}. \tag{3-3}$$

³By using the Turan–Kubilius inequality here one could save a factor of $\log \log H$, but such a gain will not make a significant impact on our final estimates.

The quantity $p_1 + p_2 - p_3 - p_4$ is clearly of size $O(P)$. Conversely, from a standard upper bound sieve,⁴ the number of representations of an integer $n = O(P)$ of the form $p_1 + p_2 - p_3 - p_4$ with $p_1, p_2, p_3, p_4 \leq 2P$ prime is $O(P^3/\log^4 P)$. Thus it suffices to show that

$$\sum_{n=O(P)} \min\left(\frac{H}{P}, \frac{1}{\|n\alpha\|}\right) \ll \frac{\log H}{W} H.$$

But from the Vinogradov lemma (see, e.g., [Iwaniec and Kowalski 2004, page 346]), the left-hand side is bounded by

$$O\left(\left(\frac{P}{q} + 1\right)\left(\frac{H}{P} + q \log q\right)\right) \ll \frac{H}{q} + P \log q + \frac{H}{P} + q \log q$$

which, since

$$W^{200} = P_1 \ll P \ll Q_1 = H/W^3 \quad \text{and} \quad W \leq q \leq H/W,$$

is bounded by $O(\log H/WH)$ as required. □

4. Proof of major arc estimate

We now prove Proposition 2.4 in the major arc case $q \leq W$. We will discard the factor $d^{1/4}(\log H)^{1/4} \log \log H$ and prove the stronger bound

$$\int_{\mathbb{R}} \left| \sum_{x/d \leq n \leq (x+H)/d} 1_S(n)g(n)e(\alpha n) \right| dx \ll \frac{HX}{dW^{1/4}}. \tag{4-1}$$

By hypothesis we have $\alpha = a/q + \theta$ with $q \leq W$ and $\theta = O(W/(Hq))$. Integrating by parts we see that

$$\begin{aligned} & \left| \sum_{x/d \leq n \leq (x+H)/d} 1_S(n)g(n)e(\alpha n) \right| \\ & \ll \left| \sum_{x/d \leq n \leq (x+H)/d} 1_S(n)g(n)e(an/q) \right| \\ & \quad + \frac{W}{Hq} \int_0^{H/d} \left| \sum_{x/d \leq n \leq x/d+H'} 1_S(n)g(n)e(an/q) \right| dH'. \end{aligned} \tag{4-2}$$

⁴For instance, from [Montgomery and Vaughan 2007, Theorem 3.13] one sees that any number $N = O(P)$ has $O((N/\phi(N))(P/\log^2 P))$ representations as the sum of two primes; since $\sum_{N=O(P)} N^2/\phi(N)^2 = O(P)$ (see, e.g., [Montgomery and Vaughan 2007, Exercise 2.1.14]), the claim then follows from the Cauchy–Schwarz inequality.

Thus let us focus on bounding

$$\int_{\mathbb{R}} \left| \sum_{x/d \leq n \leq x/d+H'} 1_S(n)g(n)e(an/q) \right| dx \tag{4-3}$$

with $0 \leq H' \leq H/d$. Splitting into residues classes we see that (4-3) is

$$\leq \sum_{b \pmod{q}} \int_{\mathbb{R}} \left| \sum_{\substack{x/d \leq n \leq x/d+H' \\ n \equiv b \pmod{q}}} 1_S(n)g(n) \right| dx.$$

For $n \equiv b \pmod{q}$ we have $d_0 := (b, q) | n$. Therefore let us write $b = d_0 b_0$, $q = d_0 q_0$ and $n = d_0 m$, so that the condition $n \equiv b \pmod{q}$ simplifies to $m \equiv b_0 \pmod{q_0}$. In addition, since g is completely multiplicative and since $d_0 \leq q \leq W \leq P_1$, we have

$$1_S(n)g(n) = g(d_0) \cdot 1_{S_{P_1, Q_1, \sqrt{X}, X/(dd_0)}}(m)g(m).$$

Finally we express $m \equiv b_0 \pmod{q_0}$ in terms of Dirichlet characters noting that

$$1_{m \equiv b_0 \pmod{q_0}}(m) = \frac{1}{\varphi(q_0)} \sum_{\chi \pmod{q_0}} \chi(b_0) \overline{\chi(m)}.$$

Putting everything together we see that (4-3) is less than

$$\sum_{b \pmod{q}} \frac{1}{\varphi(q_0)} \sum_{\chi \pmod{q_0}} \int_{\mathbb{R}} \left| \sum_{x/(dd_0) \leq m \leq x/(dd_0)+H'/d_0} 1_{S_{P_1, Q_1, \sqrt{X}, X/(dd_0)}}(m) \chi(m) \overline{\chi(m)} \right| dx.$$

In the integral we make the linear change of variable $y = x/(dd_0)$, so that the above expression becomes

$$d \sum_{b \pmod{q}} \frac{d_0}{\varphi(q_0)} \sum_{\chi \pmod{q_0}} \int_{\mathbb{R}} \left| \sum_{y \leq m \leq y+H'/d_0} 1_{S_{P_1, Q_1, \sqrt{X}, X/(dd_0)}}(m) \chi(m) \overline{\chi(m)} \right| dy. \tag{4-4}$$

We bound the part of the integral with $y \leq X/W^{10}$ trivially. This produces in (4-3) an error which is

$$\ll dq \cdot \frac{X}{W^{10}} \cdot H' \leq \frac{HX}{W^9} \ll \frac{HX}{dW^3}$$

since $q, d \leq W$ and $H' \leq H/d$. We split the remaining range $X/W^{10} \leq y \leq 2X/(dd_0)$ into dyadic blocks $X/W^{10} \leq X' \leq X/(dd_0)$ with X' running through powers of two. Thus the previous expression is

$$\ll d \sum_{X'} \sum_{b \pmod{q}} \frac{d_0}{\varphi(q_0)} \sum_{\chi \pmod{q_0}} \int_{X'} \left| \sum_{y \leq m \leq y+H'/d_0} 1_{S_{P_1, Q_1, \sqrt{X}, X/(dd_0)}}(m) \chi(m) \overline{\chi(m)} \right| dy + \frac{HX}{dW^3}.$$

At this point we apply [Theorem A.2](#) with $\eta = 1/20$ (note that $P_1 \geq (\log Q_1)^{40/\eta}$) to conclude that

$$\int_{X'}^{2X'} \left| \sum_{y \leq m \leq y+H'/d_0} 1_{S_{P_1, Q_1, \sqrt{x}, X/(dd_0)}}(m) g(m) \bar{\chi}(m) \right|^2 dy \ll \left(\exp(-M(g\bar{\chi}; X')) M(g\bar{\chi}; X') + \frac{(\log H'/d_0)^{1/3}}{P_1^{1/6-1/20}} + \frac{1}{(\log X')^{1/50}} \right) \frac{H'^2}{d_0^2} X'.$$

Since $P_1 = W^{200}$ and $H'/d_0 \leq H$ and $W \geq \log^5 H$, we have

$$\frac{(\log H'/d_0)^{1/3}}{P_1^{1/6-1/20}} \leq \frac{(\log H)^{1/3}}{P_1^{1/6-1/20}} \ll \frac{1}{W^{5/2}}$$

and certainly

$$\frac{1}{(\log X')^{1/50}} \ll \frac{1}{(\log X)^{1/50}} \ll \frac{1}{W^{5/2}}.$$

From Mertens’ theorem and definition of $M(g, X, W)$,

$$M(g\bar{\chi}; X') \geq M(g\bar{\chi}; X) - O(1) \geq M(g, X, W) - O(1)$$

and thus, by [\(2-3\)](#),

$$\exp(-M(g\bar{\chi}; X')) M(g\bar{\chi}; X') \ll \frac{1}{W^{5/2}}.$$

Putting all this together, we obtain

$$\int_{X'}^{2X'} \left| \sum_{y \leq m \leq y+H'/d_0} 1_{S_{P_1, Q_1, \sqrt{x}, X/(dd_0)}}(m) g(m) \bar{\chi}(m) \right|^2 dy \ll \frac{1}{W^{5/2}} \frac{H'^2}{d_0^2} X'.$$

It follows from Cauchy–Schwarz that

$$\int_{X'}^{2X'} \left| \sum_{y \leq m \leq y+H'/d_0} 1_{S_{P_1, Q_1, \sqrt{x}, X/(dd_0)}}(m) g(m) \overline{\chi(m)} \right| dy \ll W^{-5/4} \cdot \frac{H' X'}{d_0}.$$

Inserting this bound into [\(4-4\)](#) we see that [\(4-3\)](#) is bounded by

$$\ll dq \cdot \frac{1}{W^{5/4}} \cdot \frac{H}{d} \cdot \frac{X}{d} \ll \frac{qHX}{dW^{5/4}}.$$

Therefore using [\(4-2\)](#) and using $q \leq W$ we see that [\(4-1\)](#) is

$$\ll \frac{qHX}{dW^{5/4}} \cdot \left(1 + \frac{W}{Hq} \cdot \frac{H}{d} \right) \ll \frac{HX}{dW^{1/4}}. \quad \square$$

5. Elliott’s conjecture on the average

In this section we use [Theorem 2.3](#) to prove [Theorem 1.6](#), which will be deduced from the following result (compare also with [Theorem 2.3](#) and deduction of [Theorem 1.7](#) from it). For brevity, we write $1_{\mathcal{S}}g$ for the function $n \mapsto 1_{\mathcal{S}}(n)g(n)$.

Proposition 5.1 (truncated Elliott on the average). *Let $X, H, W, A \geq 10$ be such that*

$$\log^{20} H \leq W \leq \min\{H^{1/500}, (\log X)^{1/125}\}.$$

Let $g_1, \dots, g_k : \mathbb{N} \rightarrow \mathbb{C}$ be 1-bounded multiplicative functions, and let $a_1, \dots, a_k, b_1, \dots, b_k$ be natural numbers with $a_j \leq A$ and $b_j \leq 3AX$ for $j = 1, \dots, k$. Let $1 \leq j_0 \leq k$ be such that

$$W \leq \exp(M(g_{j_0}; 10AX, Q)/3).$$

Set

$$\mathcal{S} = \mathcal{S}_{P_1, Q_1, \sqrt{10AX}, 10AX} \quad \text{where } P_1 := W^{200}, Q_1 := H^{1/2}/W^3.$$

Then

$$\sum_{1 \leq h_2, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} 1_{\mathcal{S}}g_1(a_1n + b_1) \prod_{j=2}^k 1_{\mathcal{S}}g_j(a_jn + b_j + h_j) \right| \ll \frac{kA^2}{W^{1/20}} H^{k-1} X. \tag{5-1}$$

Proof of Theorem 1.6 assuming Proposition 5.1. We may assume that $X, H,$ and M are larger than any specified absolute constant, as the claim is trivial otherwise. We first make some initial reductions. The first estimate (1-10) of [Theorem 1.6](#) follows from the second (1-11) after shifting b_1 by h_1 in (1-11) and averaging, provided that we relax the hypotheses $b_j \leq AX$ slightly to $b_j \leq 2AX$. Thus it suffices to prove (1-11) under the relaxed hypotheses $b_j \leq 2AX$.

Let H_0 be such that

$$\log H_0 = \min\{\log^{1/3000} X \log \log X, \exp(M(g_{j_0}; 10AX, Q)/80)M(g_{j_0}; 10AX, Q)\}. \tag{5-2}$$

If $H \leq H_0$ we take $W = \log^{20} H$ and let \mathcal{S} be as in [Proposition 5.1](#). All the assumptions of [Proposition 5.1](#) hold and thus

$$\sum_{1 \leq h_2, \dots, h_k \leq H} \left| \sum_{1 \leq n \leq X} 1_{\mathcal{S}}g_1(a_1n + b_1) \prod_{j=2}^k 1_{\mathcal{S}}g_j(a_jn + b_j + h_j) \right| \ll \frac{kA^2}{\log H} H^{k-1} X.$$

Furthermore, from [Lemma 2.2](#) we have

$$\sum_{\substack{n \leq 10AX \\ n \notin \mathcal{S}}} 1 \ll AX \frac{\log W}{\log H}. \tag{5-3}$$

From this and the triangle inequality, we have

$$\begin{aligned} & \sum_{1 \leq n \leq X} g_1(a_1n + b_1) \prod_{j=2}^k g_j(a_jn + b_j + h_j) \\ &= \sum_{1 \leq n \leq X} 1_S g_1(a_1n + b_1) \prod_{j=2}^k 1_S g_j(a_jn + b_j + h_j) + O\left(kAX \frac{\log W}{\log H}\right). \end{aligned} \quad (5-4)$$

Hence the claim follows in the case when $H \leq H_0$.

If $H > H_0$, one can cover the summation over the h_j indices by intervals of length H_0 and apply [Theorem 1.6](#) to each subinterval (shifting the b_j by at most AX when doing so), and then sum, noting that the quantity

$$\exp(-M(g_{j_0}; 10AX, Q)/80) + \frac{\log \log H}{\log H} + \frac{1}{\log^{1/3000} X}$$

is essentially unchanged after replacing H with H_0 . □

Remark 5.2. By using larger choices of W , one can obtain more refined information on the large values of the correlations $\sum_{1 \leq n \leq X} g_1(a_1n + b_1) \prod_{j=2}^k g_j(a_jn + b_j + h_j)$. For instance, if we take $W = H^\delta$ for some H , δ such that $10 \leq H \leq H_0$ and $20 \log \log H / \log H \leq \delta \leq 1/500$, we see from [Proposition 5.1](#), (5-4), and Markov's inequality that

$$\sum_{1 \leq n \leq X} g_1(a_1n + b_1) \prod_{j=2}^k g_j(a_jn + b_j + h_j) \ll kA^2 \delta X$$

for all but at most $O(H^{k-1} / \delta H^{\delta/20})$ tuples (h_1, \dots, h_{k-1}) with $1 \leq h_j \leq H$ for $j = 2, \dots, k$. Thus we can obtain a power saving in the number of exceptional tuples, at the cost of only obtaining a weak bound on the individual correlations $\sum_{1 \leq n \leq X} g_1(a_1n + b_1) \prod_{j=2}^k g_j(a_jn + b_j + h_j)$.

It remains to prove [Proposition 5.1](#). We start by proving the following simpler case to which the general case will be reduced.

Proposition 5.3. *Let $X, H, W \geq 10$ be such that*

$$\log^{20} H \leq W \leq \min\{H^{1/250}, (\log X)^{1/125}\}.$$

Let $g : \mathbb{N} \rightarrow \mathbb{C}$ be a 1-bounded multiplicative function such that

$$W \leq \exp(M(g; X, W)/3).$$

Set

$$S = S_{P_1, Q_1, \sqrt{X}, X} \quad \text{where } P_1 := W^{200}, Q_1 := H/W^3.$$

Then

$$\sum_{1 \leq h \leq H} \left| \sum_{1 \leq n \leq X} 1_S g(n) 1_S \bar{g}(n+h) \right|^2 \ll \frac{HX^2}{W^{1/5}}. \tag{5-5}$$

To deduce [Theorem 1.2](#) we let \mathcal{S} be as in this proposition with $W := H^{\delta/900}$. The argument of [Lemma 2.2](#) actually gives $\#\{1 \leq n \leq X : n \notin \mathcal{S}\} \leq 2X \log P_1 / \log Q_1$ in this case, and thus the numbers n with $n \notin \mathcal{S}$ or $n+h \notin \mathcal{S}$ contribute to the left-hand side of (1-4) at most $9\delta/10$. Hence, recalling (1-12), the claim follows from the previous proposition and Markov’s inequality.

Proof of Proposition 5.3. The claim follows once we have shown

$$\sum_{|h| \leq 2H} (2H - |h|)^2 \cdot \left| \sum_n 1_S g(n) 1_S \bar{g}(n+h) \right|^2 \ll \frac{1}{W^{1/5}} H^3 X^2.$$

Applying [Lemma 1.4](#), it will suffice to show that

$$\int_{\mathbb{T}} \left(\int_{\mathbb{R}} \left| \sum_{x \leq n \leq x+2H} 1_S g(n) e(\alpha n) \right|^2 dx \right) d\alpha \ll \frac{1}{W^{1/5}} H^3 X^2.$$

From the Parseval identity we have

$$\int_{\mathbb{T}} \int_{\mathbb{R}} \left| \sum_{x \leq n \leq x+2H} 1_S g(n) e(\alpha n) \right|^2 dx d\alpha = \int_{\mathbb{R}} \sum_{x \leq n \leq x+2H} |1_S g(n)|^2 dx \ll HX$$

so it suffices to show that

$$\sup_{\alpha} \int_{\mathbb{R}} \left| \sum_{x \leq n \leq x+2H} 1_S g(n) e(\alpha n) \right|^2 dx \ll \frac{1}{W^{1/5}} H^2 X.$$

Using the trivial bound

$$\left| \sum_{x \leq n \leq x+2H} 1_S g(n) e(\alpha n) \right| \ll H$$

we thus reduce to showing

$$\sup_{\alpha} \int_{\mathbb{R}} \left| \sum_{x \leq n \leq x+2H} 1_S g(n) e(\alpha n) \right| dx \ll \frac{HX}{W^{1/5}}. \tag{5-6}$$

This follows from [Theorem 2.3](#) (using the lower bound $W \geq \log^{20} H$ in the hypotheses of [Proposition 5.3](#) to absorb the $\log^{1/4} H \log \log H$ factors in [Theorem 2.3](#)). \square

Proof of Proposition 5.1. We first remove the special treatment afforded to the g_1 factor in (5-1). Note that we may assume

$$W^{1/20} \geq kA^2 \tag{5-7}$$

and thus

$$H \geq W^{500} \geq (kA^2)^{10000}$$

since the claim is trivial otherwise.

Set $H' := \sqrt{H}$. For any $1 \leq h_1 \leq H'/A$, we may shift n by h_1 and conclude that

$$\begin{aligned} \sum_{1 \leq n \leq X} 1_S g_1(a_1 n + b_1) \prod_{j=2}^k 1_S g_j(a_j n + b_j + h_j) \\ = \sum_{1 \leq n \leq X} 1_S g_1(a_1 n + b_1 + a_1 h_1) \prod_{j=2}^k 1_S g_j(a_j n + b_j + h_j + a_j h_1) + O(H') \end{aligned}$$

and thus we may write the left-hand side of (5-1) as

$$\begin{aligned} \sum_{1 \leq h_2, \dots, h_k \leq H'} \left| \sum_{1 \leq n \leq X} 1_S g_1(a_1 n + b_1 + a_1 h_1) \prod_{j=2}^k 1_S g_j(a_j n + b_j + h_j + a_j h_1) \right| \\ + O(H^{k-1} H'). \end{aligned}$$

If one shifts each of the h_j for $j = 2, \dots, k$ in turn by $a_j h_1 = O(H')$, we may rewrite this as

$$\begin{aligned} \sum_{1 \leq h_2, \dots, h_k \leq H'} \left| \sum_{1 \leq n \leq X} 1_S g_1(a_1 n + b_1 + a_1 h_1) \prod_{j=2}^k 1_S g_j(a_j n + b_j + h_j) \right| \\ + O(H^{k-1} H') + O(kH^{k-2} H' X). \end{aligned}$$

Averaging in h_1 , and replacing h_1 by $a_1 h_1$ (crudely dropping the constraint that $a_1 h_1$ is divisible by a_1), we may thus bound the left-hand side of (5-1) by

$$\begin{aligned} \ll \frac{A}{H'} \sum_{1 \leq h_1 \leq H'} \sum_{1 \leq h_2, \dots, h_k \leq H'} \left| \sum_{1 \leq n \leq X} 1_S g_1(a_1 n + b_1 + h_1) \prod_{j=2}^k 1_S g_j(a_j n + b_j + h_j) \right| \\ + H^{k-1} H' + kH^{k-2} H' X. \end{aligned}$$

The g_1 term may now be combined with the product over the remaining g_j terms to form $\prod_{j=1}^k 1_S g_j(a_j n + b_j + h_j)$. The error term $H^{k-1} H' + kH^{k-2} H' X$ is certainly of size $O((kA^2/W^{1/20})H^{k-1} X)$, so it suffices to show that

$$\sum_{1 \leq h_1 \leq H'} \sum_{1 \leq h_2, \dots, h_k \leq H'} \left| \sum_{1 \leq n \leq X} \prod_{j=1}^k 1_S g_j(a_j n + b_j + h_j) \right| \ll \frac{A}{W^{1/20}} H^{k-1} H' X.$$

By covering the ranges $1 \leq h_j \leq H$ by intervals of length H' and averaging, it suffices (after relaxing the conditions $b_j \leq 3AX$ to $b_j \leq 4AX$) to prove that

$$\sum_{1 \leq h_1, h_2, \dots, h_k \leq H'} \left| \sum_{1 \leq n \leq X} \prod_{j=1}^k 1_S g_j(a_j n + b_j + h_j) \right| \ll \frac{A}{W^{1/20}} (H')^k X.$$

The situation is now symmetric with respect to permuting the indices $1, \dots, k$, so we may assume that the index j_0 in Proposition 5.1 is equal to 1. By the triangle inequality in h_2, \dots, h_k , it suffices to show that

$$\sum_{1 \leq h_1 \leq H'} \left| \sum_{1 \leq n \leq X} \prod_{j=1}^k 1_S g_j(a_j n + b_j + h_j) \right| \ll \frac{A}{W^{1/20}} H' X$$

for all h_2, \dots, h_k . Writing $G(n) := \prod_{j=2}^k 1_S g_j(a_j n + b_j + h_j)$, it thus suffices to show that

$$\sum_{1 \leq h_1 \leq H'} \left| \sum_{1 \leq n \leq X} 1_S g_1(a_1 n + b_1 + h_1) G(n) \right| \ll \frac{A}{W^{1/20}} H' X$$

for any 1-bounded function $G : \mathbb{Z} \rightarrow \mathbb{C}$.

We use a standard van der Corput argument. By the Cauchy–Schwarz inequality, it suffices to show that

$$\sum_{1 \leq h_1 \leq H'} \left| \sum_{1 \leq n \leq X} 1_S g_1(a_1 n + b_1 + h_1) G(n) \right|^2 \ll \frac{A^2}{W^{1/10}} (H')^2 X^2.$$

The left-hand side may be rewritten as

$$\sum_{n, n' \leq X} G(n) \overline{G(n')} \sum_{1 \leq h_1 \leq H'} 1_S g_1(a_1 n + b_1 + h_1) 1_S \overline{g_1}(a_1 n' + b_1 + h_1).$$

By the triangle inequality, it thus suffices to show that

$$\sum_{n, n' \leq X} \left| \sum_{1 \leq h_1 \leq H'} 1_S g_1(a_1 n + b_1 + h_1) 1_S \overline{g_1}(a_1 n' + b_1 + h_1) \right| \ll \frac{A^2}{W^{1/10}} H' X^2.$$

To abbreviate notation we now write $h = h_1$, $g = g_1$, $a = a_1$, $b = b_1$. By the Cauchy–Schwarz inequality, it suffices to show that

$$\sum_{n, n' \leq X} \left| \sum_{1 \leq h \leq H'} 1_S g(an + b + h) 1_S \overline{g}(an' + b + h) \right|^2 \ll \frac{A^4}{W^{1/5}} (H')^2 X^2.$$

Replacing n, n' by $an + b, an' + b$ respectively, it suffices to show that

$$\sum_{n, n'} \left| \sum_{1 \leq h \leq H'} 1_S g(n+h) 1_S \bar{g}(n'+h) \right|^2 \ll \frac{A^4}{W^{1/5}} (H')^2 X^2$$

where we have extended $1_S g$ by zero to the negative integers. The left-hand side can be rewritten as

$$\sum_{|h| < H'} (\lfloor H' \rfloor - |h|) \left| \sum_n 1_S g(n) 1_S \bar{g}(n+h) \right|^2,$$

and the claim follows from [Proposition 5.3](#). □

Appendix A: Mean values of complex multiplicative functions in short intervals

In this section we prove a complex variant of results in [\[Matomäki and Radziwiłł 2015\]](#) in the case that f is not p^{it} pretentious. In particular, we show that the mean value of a 1-bounded nonpretentious multiplicative function is small for most short intervals:

Theorem A.1. *Let f be a 1-bounded multiplicative function and let $M(f; X)$ be as in (1-6). Then, for $X \geq h \geq 10$,*

$$\frac{1}{X} \int_X^{2X} \left| \frac{1}{h} \sum_{x \leq n \leq x+h} f(n) \right|^2 dx \ll \exp(-M(f; X)) M(f; X) + \frac{(\log \log h)^2}{(\log h)^2} + \frac{1}{(\log X)^{1/50}}.$$

Actually, as in [\[Matomäki and Radziwiłł 2015\]](#) and earlier in this paper, one gets better quantitative results if one first restricts to a subset of n with a typical factorization. Let us first define such a subset S in this setting.

Let $\eta \in (0, 1/6)$, and let X_0 be a quantity with $\sqrt{X} \leq X_0 \leq X$. (The results in [\[Matomäki and Radziwiłł 2015\]](#) used the choice $X_0 = X$, but for technical reasons we will need a more flexible choice of this parameter.) Consider a sequence of increasing intervals $[P_j, Q_j]$, $j \geq 1$ such that:

- $Q_1 \leq \exp(\sqrt{\log X_0})$.
- The intervals are not too far from each other; precisely, for all $j \geq 2$,

$$\frac{\log \log Q_j}{\log P_{j-1} - 1} \leq \frac{\eta}{4j^2}. \tag{A-1}$$

- The intervals are not too close to each other; precisely, for all $j \geq 2$,

$$\frac{\eta}{j^2} \log P_j \geq 8 \log Q_{j-1} + 16 \log j. \tag{A-2}$$

For example, given $0 < \eta < 1/6$, the sequence of intervals $[P_j, Q_j]$ defined in [Definition 2.1](#) can be verified to obey the above estimates if

$$\exp(\sqrt{\log X_0}) \geq Q_1 \geq P_1 \geq (\log Q_1)^{40/\eta}$$

and if P_1 is sufficiently large.

Let S be the set of integers $X \leq n \leq 2X$ having at least one prime factor in each of the intervals $[P_j, Q_j]$ for $j \leq J$, where J is chosen to be the largest index j such that $Q_j \leq \exp((\log X_0)^{1/2})$. We will establish the following variant of [\[Matomäki and Radziwiłł 2015, Theorem 3\]](#).

Theorem A.2. *Let f be a 1-bounded multiplicative function. Let S be as above with $\eta \in (0, 1/6)$. If $[P_1, Q_1] \subset [1, h]$, then for all $X > X(\eta)$ large enough and all $h \geq 3$,*

$$\frac{1}{X} \int_X^{2X} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \in S}} f(n) \right|^2 dx \ll \exp(-M(f; X))M(f; X) + \frac{(\log h)^{1/3}}{P_1^{1/6-\eta}} + \frac{1}{(\log X)^{1/50}}.$$

The proof of [Theorem A.2](#) proceeds as the proof of [\[Matomäki and Radziwiłł 2015, Theorem 3\]](#). The first step is a Parseval bound:

$$\frac{1}{X} \int_X^{2X} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \in S}} f(n) \right|^2 dx \ll \int_1^{1+iX/h_1} |F(s)|^2 |ds| + \max_{T \geq X/h_1} \frac{X/h_1}{T} \int_{1+iT}^{1+i2T} |F(s)|^2 |ds|.$$

This follows exactly in the same way as [\[Matomäki and Radziwiłł 2015, Lemma 14\]](#) but there is no need to split the integral into two parts, and one can just work as for $V(x)$ there. [Theorem A.2](#) now follows immediately from the following variant of [\[Matomäki and Radziwiłł 2015, Proposition 1\]](#).

Proposition A.3. *Let f be a 1-bounded multiplicative function. Let S be as above, and let*

$$F(s) = \sum_{\substack{X \leq n \leq 2X \\ n \in S}} \frac{f(n)}{n^s}.$$

Then, for any T ,

$$\int_0^T |F(1+it)|^2 dt \ll \left(\frac{T}{X/Q_1} + 1 \right) \left(\frac{(\log Q_1)^{1/3}}{P_1^{1/6-\eta}} + \exp(-M(f; X))M(f; X) + \frac{1}{(\log X)^{1/50}} \right).$$

Proof. Since the mean value theorem gives the bound $O(T/X + 1)$, we can assume $T \leq X/2$ and $M(f; X) \geq 1$.

Now let t_1 be the value of t which attains the minimum in

$$M(f; X) = \inf_{|t| \leq X} \mathbb{D}(g, n \mapsto n^{it}; X)^2.$$

We split the integration into three ranges:

$$\begin{aligned} \mathcal{T}_0 &= \{0 \leq t \leq T : |t - t_1| \leq \exp(M(f; X))/M(f; X)\}, \\ \mathcal{T}_1 &= \{0 \leq t \leq T : \exp(M(f; X))/M(f; X) \leq |t - t_1| \leq (\log X)^{1/16}\}, \\ \mathcal{T}_2 &= \{0 \leq t \leq T : |t - t_1| \geq (\log X)^{1/16}\}. \end{aligned}$$

Notice that by the definition of t_1 , the triangle inequality and arguing as in (1-12), for any $|t| \leq X$ with $|t - t_1| \geq 1$, and any $\varepsilon > 0$,

$$\begin{aligned} 2\mathbb{D}(f, p^{it}; X) &\geq \mathbb{D}(f, p^{it}; X) + \mathbb{D}(f, p^{it_1}; X) \geq \mathbb{D}(1, p^{i(t-t_1)}) \\ &\geq \left(\frac{1}{\sqrt{3}} - \varepsilon\right) \sqrt{\log \log X} + O(1), \end{aligned}$$

so that by Halasz’s theorem, for every $|t| \leq T$,

$$F(1 + it) \ll (\log X)^{-1/16} + \frac{1}{1 + |t - t_1|}.$$

In the region $|t - t_1| \geq (\log X)^{1/16}$, the above implies the following in exactly the same way as [Matomäki and Radziwiłł 2015, Lemma 3].

Lemma A.4. *Let $X \geq Q \geq P \geq 2$. Let t_1 be as above and let*

$$G(s) = \sum_{X \leq n \leq 2X} \frac{f(n)}{n^s} \cdot \frac{1}{\#\{p \in [P, Q] : p|n\} + 1}.$$

Then, for any $t \in \mathcal{T}_2$,

$$|G(1 + it)| \ll \frac{\log Q}{(\log X)^{1/16} \log P} + \log X \cdot \exp\left(-\frac{\log X}{3 \log Q} \log \frac{\log X}{\log Q}\right).$$

This was the only part in the proof [Matomäki and Radziwiłł 2015, Proposition 1] that needed f to be real-valued, and thus we get

$$\int_{\mathcal{T}_2} |F(1 + it)|^2 dt \ll \left(\frac{T}{X/Q_1} + 1\right) \left(\frac{(\log Q_1)^{1/3}}{P_1^{1/6-\eta}} + \frac{1}{(\log X)^{1/50}}\right).$$

Using the estimate $F(1 + it) \ll 1/|t - t_1|$ for $t \in \mathcal{T}_1$ and, from Halasz’s theorem, the estimate $F(1 + it) \ll \exp(-M(f; X))M(f; X)$ for $t \in \mathcal{T}_0$, we obtain

$$\int_{\mathcal{T}_0 \cup \mathcal{T}_1} |F(1 + it)|^2 dt \ll \exp(-M(f; X))M(f; X),$$

and the claim follows. □

Proof of Theorem A.1. Let $\eta = 1/12$, $P_1 = (\log h)^{480}$, $Q_1 = h$, let P_j and Q_j for $j \geq 2$ be as in Definition 2.1, and let \mathcal{S} be as above. Then

$$\frac{1}{X} \int_X^{2X} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \notin \mathcal{S}}} f(n) \right|^2 dx \leq \frac{1}{X} \int_X^{2X} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \in \mathcal{S}}} f(n) \right|^2 dx + \frac{1}{X} \int_X^{2X} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \notin \mathcal{S}}} 1 \right|^2 dx.$$

The contribution from the first integral is acceptable by Theorem A.2. We rewrite the second integrand as

$$\begin{aligned} \left| \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \notin \mathcal{S}}} 1 \right| &= \left| 1 + O(1/h) - \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \in \mathcal{S}}} 1 \right| \\ &\leq \left| \frac{1}{X} \sum_{\substack{X \leq n \leq 2X \\ n \in \mathcal{S}}} 1 - \frac{1}{h} \sum_{\substack{x \leq n \leq x+h \\ n \in \mathcal{S}}} 1 \right| + \left| \frac{1}{X} \sum_{\substack{X \leq n \leq 2X \\ n \notin \mathcal{S}}} 1 \right| + O(1/h), \end{aligned}$$

and the claim follows from [Matomäki and Radziwiłł 2015, Theorem 3 with $f = 1$] and Lemma 2.2. □

Appendix B: Counterexample to the uncorrected Elliott conjecture

In this appendix we present a counterexample to Conjecture 1.5. More precisely:

Theorem B.1 (counterexample). *There exists a 1-bounded multiplicative function $g : \mathbb{N} \rightarrow \mathbb{C}$ such that*

$$\sum_p \frac{1 - \operatorname{Re}(g(p)\overline{\chi(p)}p^{-it})}{p} = \infty \tag{B-1}$$

for all Dirichlet characters χ and $t \in \mathbb{R}$ (i.e., one has $M(g; \infty, \infty) = \infty$), but such that

$$\left| \sum_{n \leq t_m} g(n)\overline{g(n+1)} \right| \gg t_m \tag{B-2}$$

for all sufficiently large m and some sequence t_m going to infinity.

Proof. For each prime p , we choose $g(p)$ from the unit circle $S^1 := \{z : |z| = 1\}$ by the following iterative procedure involving a sequence $t_1 < t_2 < t_3 < \dots$:

- (1) Initialize $t_1 := 100$ and $m := 1$, and set $g(p) := 1$ for all $p \leq t_1$.
- (2) Now suppose recursively that $g(p)$ has been chosen for all $p \leq t_m$. As the quantities $\log p$ are linearly independent over the integers, the (continuous) sequence $t \mapsto (t \log p \bmod 1)_{p \leq t_m}$ is equidistributed in the torus $\prod_{p \leq t_m} \mathbb{T}$ and, equivalently, the sequence $t \mapsto (p^{it})_{p \leq t_m}$ is equidistributed in the torus $\prod_{p \leq t_m} S^1$.

Thus one can find a quantity $s_{m+1} > \exp(t_m)$ such that, for all $p \leq t_m$,

$$p^{is_{m+1}} = g(p) \left(1 + O\left(\frac{1}{t_m^2}\right) \right). \tag{B-3}$$

(3) Set $t_{m+1} := s_{m+1}^2$, and then set

$$g(p) := p^{is_{m+1}} \tag{B-4}$$

for all $t_m < p \leq t_{m+1}$. Now increment m to $m + 1$ and return to step (2).

Clearly the t_m go to infinity, so $g(p)$ is defined for all primes p . We then define

$$g(n) := \mu(n)^2 \prod_{p|n} g(p), \tag{B-5}$$

which is clearly a 1-bounded multiplicative function.

Suppose that $n \leq t_{m+1}$ is squarefree. Then n is the product of distinct primes less than or equal to t_{m+1} , including at most t_m primes less than or equal to t_m . From (B-5) we then have

$$g(n) = n^{is_{m+1}} \left(1 + O\left(\frac{1}{t_m^2}\right) \right)^{O(t_m)} = n^{is_{m+1}} + O\left(\frac{1}{t_m}\right).$$

If n is not squarefree, then $g(n)$ of course vanishes. Thus, for $t_{m+1}^{3/4} \leq n \leq t_{m+1} - 1$, we have

$$\begin{aligned} g(n)\overline{g(n+1)} &= \mu^2(n)\mu^2(n+1) \left(\frac{n+1}{n}\right)^{is_{m+1}} + O\left(\frac{1}{t_m}\right) \\ &= \mu^2(n)\mu^2(n+1) + O\left(\frac{s_{m+1}}{t_{m+1}^{3/4}}\right) + O\left(\frac{1}{t_m}\right) \\ &= \mu^2(n)\mu^2(n+1) + O\left(\frac{1}{t_m}\right), \end{aligned}$$

and the claim (B-2) then easily follows since the sequence $\mu^2(n)\mu^2(n+1)$ has positive mean value.

Now we prove (B-1). From (B-4), we have

$$\begin{aligned} \sum_p \frac{1 - \operatorname{Re}(g(p)\overline{\chi(p)}p^{-it})}{p} &\geq \sum_{t_m < p \leq t_{m+1}} \frac{1 - \operatorname{Re}(\overline{\chi(p)}p^{i(s_{m+1}-t)})}{p} \\ &\geq \sum_{\exp((\log t_{m+1})^{5/6}) < p \leq t_{m+1}} \frac{1 - \operatorname{Re}(\overline{\chi(p)}p^{i(s_{m+1}-t)})}{p} \end{aligned}$$

since $\exp((\log t_{m+1})^{5/6}) \geq \exp((2t_m)^{5/6}) \geq t_m$. We see as in (1-12) that the right-hand side goes to infinity as $m \rightarrow \infty$ for any fixed χ, t , and the claim follows. \square

It is easy to see that the function g constructed in the above counterexample violates (1-9), and so is not a counterexample to the corrected form of Conjecture 1.5. It is also not difficult to modify the above counterexample so that the function g is completely multiplicative instead of multiplicative, using the fact that most numbers up to t_{m+1} have fewer than t_m prime factors less than t_m (counting multiplicity); we leave the details to the interested reader.

Appendix C: An argument of Granville and Soundararajan

In this appendix we show the equivalence of the hypotheses (1-7) and (1-9) for Elliott’s conjecture in the case that the multiplicative function g_{j_0} is real. The key lemma is the following estimate, essentially due to Granville and Soundararajan.

Lemma C.1. *Let $f : \mathbb{N} \rightarrow [-1, 1]$ be a multiplicative function, let $x \geq 100$, and let χ be a fixed Dirichlet character. For $1 \leq |\alpha| \leq x$, one has*

$$\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) \geq \frac{1}{4}\sqrt{\log \log x} + O_\chi(1). \tag{C-1}$$

When χ^2 is nonprincipal, this holds for all $|\alpha| \leq x$.

If χ^2 is principal (i.e., χ is a quadratic character), then, for $|\alpha| \leq 1$, one has

$$\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) \geq \frac{1}{3}\mathbb{D}(f, \chi; x) + O(1). \tag{C-2}$$

Proof. To establish (C-1), we notice that, by conjugation symmetry and the triangle inequality,

$$\begin{aligned} \mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) &= \frac{1}{2}(\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) + \mathbb{D}(f, n \mapsto \overline{\chi(n)}n^{-i\alpha}; x)) \\ &\geq \frac{1}{2}\mathbb{D}(n \mapsto \overline{\chi(n)}n^{-i\alpha}, n \mapsto \chi(n)n^{i\alpha}; x) \\ &= \frac{1}{2}\left(\sum_{p \leq x} \frac{1 - \operatorname{Re} \chi^2(p)p^{2i\alpha}}{p}\right)^{1/2} \end{aligned}$$

which implies the claim for $|\alpha| \geq 1$ or for nonprincipal χ^2 by the zero-free (and pole-free) region for Dirichlet L -functions (see (1-12) for a related argument).

To establish (C-2), notice first that since χ^2 is principal, χ is real-valued which together with the triangle inequality implies

$$\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x) = \mathbb{D}(f\chi, n \mapsto n^{i\alpha}; x) \geq \mathbb{D}(1, f\chi; x) - \mathbb{D}(1, n \mapsto n^{i\alpha}; x).$$

Now $\mathbb{D}(1, n \mapsto n^{i\alpha}; x) = \mathbb{D}(1, n \mapsto n^{2i\alpha}; x) + O(1)$ for $|\alpha| \leq 1$ since, from the prime number theorem, $\mathbb{D}(1, n \mapsto n^{i\alpha}; x)^2 = \log(1 + |\alpha| \log x) + O(1)$, so that the

claim follows unless $\mathbb{D}(1, n \mapsto n^{2i\alpha}; x) \geq (2/3)\mathbb{D}(1, f\chi; x)$. But in the latter case, the triangle inequality gives

$$\begin{aligned} \frac{2}{3}\mathbb{D}(f, \chi; x) &= \frac{2}{3}\mathbb{D}(1, f\chi; x) \\ &\leq \mathbb{D}(1, n \mapsto n^{2i\alpha}; x) \\ &= \mathbb{D}(n \mapsto n^{-i\alpha}, n \mapsto n^{i\alpha}; x) \\ &\leq \mathbb{D}(f\chi, n \mapsto n^{-i\alpha}; x) + \mathbb{D}(f\chi, n \mapsto n^{i\alpha}; x) \\ &= 2\mathbb{D}(f, n \mapsto \chi(n)n^{i\alpha}; x), \end{aligned}$$

and the claim (C-2) follows. \square

From this lemma, we see that if g_{j_0} is a real 1-bounded multiplicative function, then, for given Q , the condition (1-9) is equivalent to

$$\mathbb{D}(g_{j_0}, \chi; X) \rightarrow \infty$$

when $X \rightarrow \infty$ for all quadratic characters χ of modulus at most Q . But this follows from (1-7). The converse implication is trivial.

Acknowledgments

Tao was supported by a Simons Investigator grant, the James and Carol Collins Chair, the Mathematical Analysis & Application Research Fund Endowment, and NSF grant DMS-1266164. The authors thank Andrew Granville and the anonymous referee for useful comments and corrections.

References

- [Balog 1990] A. Balog, “The prime k -tuplets conjecture on average”, pp. 47–75 in *Analytic number theory* (Allerton Park, IL, 1989), edited by B. C. Berndt et al., Progr. Math. **85**, Birkhäuser, Boston, 1990. MR 92e:11105 Zbl 0719.11066
- [Bourgain et al. 2013] J. Bourgain, P. Sarnak, and T. Ziegler, “Disjointness of Moebius from horocycle flows”, pp. 67–83 in *From Fourier analysis and number theory to radon transforms and geometry*, edited by H. M. Farkas et al., Dev. Math. **28**, Springer, New York, 2013. MR 2986954 Zbl 06150358
- [Chowla 1965] S. Chowla, *The Riemann hypothesis and Hilbert's tenth problem*, Mathematics and Its Applications **4**, Gordon and Breach Science Publishers, New York-London-Paris, 1965. MR 31 #2201 Zbl 0133.30003
- [Daboussi and Delange 1982] H. Daboussi and H. Delange, “On multiplicative arithmetical functions whose modulus does not exceed one”, *J. London Math. Soc.* (2) **26**:2 (1982), 245–264. MR 83k:10082 Zbl 0499.10052
- [Davenport 1937] H. Davenport, “On some infinite series involving arithmetical functions, II”, *Quart. J. Math. Oxf.* **8** (1937), 313–320. Zbl 0017.39101
- [Elliott 1992] P. D. T. A. Elliott, “On the correlation of multiplicative functions”, *Notas Soc. Mat. Chile* **11**:1 (1992), 1–11. MR 95j:11076 Zbl 0810.11055

- [Frantzikinakis and Host 2015] N. Frantzikinakis and B. Host, “Asymptotics for multilinear averages of multiplicative functions”, preprint, 2015. [arXiv 1502.02646](#)
- [Friedlander and Iwaniec 2010] J. Friedlander and H. Iwaniec, *Opera de cribro*, American Mathematical Society Colloquium Publications **57**, American Mathematical Society, Providence, RI, 2010. [MR 2011d:11227](#) [Zbl 1226.11099](#)
- [Granville and Soundararajan 2007] A. Granville and K. Soundararajan, “Large character sums: pretentious characters and the Pólya–Vinogradov theorem”, *J. Amer. Math. Soc.* **20**:2 (2007), 357–384. [MR 2007k:11128](#) [Zbl 1210.11090](#)
- [Green and Tao 2010] B. Green and T. Tao, “Linear equations in primes”, *Ann. of Math. (2)* **171**:3 (2010), 1753–1850. [MR 2011j:11177](#) [Zbl 1242.11071](#)
- [Iwaniec and Kowalski 2004] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications **53**, American Mathematical Society, Providence, RI, 2004. [Zbl 1059.11001](#)
- [Kátai 1986] I. Kátai, “A remark on a theorem of H. Daboussi”, *Acta Math. Hungar.* **47**:1–2 (1986), 223–225. [MR 87j:11072](#) [Zbl 0607.10034](#)
- [Kawada 1993] K. Kawada, “The prime k -tuplets in arithmetic progressions”, *Tsukuba J. Math.* **17**:1 (1993), 43–57. [MR 94i:11068](#) [Zbl 0797.11076](#)
- [Kawada 1995] K. Kawada, “A Montgomery–Hooley type theorem for prime k -tuplets”, *Acta Math. Hungar.* **66**:3 (1995), 177–200. [MR 96h:11096](#) [Zbl 0869.11069](#)
- [Matomäki and Radziwiłł 2015] K. Matomäki and M. Radziwiłł, “Multiplicative functions in short intervals”, 2015. To appear in *Ann. of Math.* [arXiv 1501.04585](#)
- [Mikawa 1992] H. Mikawa, “On prime twins in arithmetic progressions”, *Tsukuba J. Math.* **16**:2 (1992), 377–387. [MR 94e:11101](#) [Zbl 0778.11053](#)
- [Montgomery 1994] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, CBMS Regional Conference Series in Mathematics **84**, American Mathematical Society, Providence, RI, 1994. [MR 96i:11002](#) [Zbl 0814.11001](#)
- [Montgomery and Vaughan 1977] H. L. Montgomery and R. C. Vaughan, “Exponential sums with multiplicative coefficients”, *Invent. Math.* **43**:1 (1977), 69–82. [MR 56 #15579](#) [Zbl 0362.10036](#)
- [Montgomery and Vaughan 2007] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory, I: Classical theory*, Cambridge Studies in Advanced Mathematics **97**, Cambridge Univ. Press, 2007. [MR 2009b:11001](#) [Zbl 1142.11001](#)

Communicated by Andrew Granville

Received 2015-04-17

Revised 2015-08-28

Accepted 2015-10-06

ksmato@utu.fi

*Department of Mathematics and Statistics,
University of Turku, FI-20014 Turku, Finland*

maksym.radziwill@gmail.com

*Department of Mathematics, Rutgers University, Hill Center
for the Mathematical Sciences, 110 Frelinghuysen Rd.,
Piscataway, NJ 08854-8019, United States*

tao@math.ucla.edu

*Department of Mathematics,
University of California Los Angeles, 405 Hilgard Avenue,
Los Angeles, CA 90095-1555, United States*

Algebra & Number Theory

msp.org/ant

EDITORS

MANAGING EDITOR

Bjorn Poonen
Massachusetts Institute of Technology
Cambridge, USA

EDITORIAL BOARD CHAIR

David Eisenbud
University of California
Berkeley, USA

BOARD OF EDITORS

Georgia Benkart	University of Wisconsin, Madison, USA	Susan Montgomery	University of Southern California, USA
Dave Benson	University of Aberdeen, Scotland	Shigefumi Mori	RIMS, Kyoto University, Japan
Richard E. Borcherds	University of California, Berkeley, USA	Raman Parimala	Emory University, USA
John H. Coates	University of Cambridge, UK	Jonathan Pila	University of Oxford, UK
J-L. Colliot-Thélène	CNRS, Université Paris-Sud, France	Anand Pillay	University of Notre Dame, USA
Brian D. Conrad	Stanford University, USA	Victor Reiner	University of Minnesota, USA
Hélène Esnault	Freie Universität Berlin, Germany	Peter Sarnak	Princeton University, USA
Hubert Flenner	Ruhr-Universität, Germany	Joseph H. Silverman	Brown University, USA
Sergey Fomin	University of Michigan, USA	Michael Singer	North Carolina State University, USA
Edward Frenkel	University of California, Berkeley, USA	Vasudevan Srinivas	Tata Inst. of Fund. Research, India
Andrew Granville	Université de Montréal, Canada	J. Toby Stafford	University of Michigan, USA
Joseph Gubeladze	San Francisco State University, USA	Ravi Vakil	Stanford University, USA
Roger Heath-Brown	Oxford University, UK	Michel van den Bergh	Hasselt University, Belgium
Craig Huneke	University of Virginia, USA	Marie-France Vignéras	Université Paris VII, France
Kiran S. Kedlaya	Univ. of California, San Diego, USA	Kei-Ichi Watanabe	Nihon University, Japan
János Kollár	Princeton University, USA	Efim Zelmanov	University of California, San Diego, USA
Yuri Manin	Northwestern University, USA	Shou-Wu Zhang	Princeton University, USA
Philippe Michel	École Polytechnique Fédérale de Lausanne		

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor


See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2015 is US \$255/year for the electronic version, and \$440/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2015 Mathematical Sciences Publishers

Algebra & Number Theory

Volume 9 No. 9 2015

Families of nearly ordinary Eisenstein series on unitary groups XIN WAN	1955
Classifying orders in the Sklyanin algebra DANIEL ROGALSKI, SUSAN J. SIERRA and J. TOBY STAFFORD	2055
Congruence property in conformal field theory CHONGYING DONG, XINGJUN LIN and SIU-HUNG NG	2121
An averaged form of Chowla's conjecture KAISA MATOMÄKI, MAKSYM RADZIWIŁŁ and TERENCE TAO	2167