

# *Algebra & Number Theory*

Volume 10

2016

No. 1

**Stable sets of primes in number fields**

Alexander Ivanov



# Stable sets of primes in number fields

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We define a new class of sets — stable sets — of primes in number fields. For example, Chebotarev sets  $P_{M/K}(\sigma)$ , with  $M/K$  Galois and  $\sigma \in G(M/K)$ , are very often stable. These sets have positive (but arbitrarily small) Dirichlet density and they generalize sets with density one in the sense that arithmetic theorems such as certain Hasse principles, the Grunwald–Wang theorem, and Riemann’s existence theorem hold for them. Geometrically, this allows us to give examples of infinite sets  $S$  with arbitrarily small positive density such that  $\text{Spec } \mathbb{C}_{K,S}$  is a  $K(\pi, 1)$  (simultaneously for all  $p$ ).

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## 1. Introduction

The main goal of this article is to define a new class of sets of primes of positive Dirichlet density in number fields — stable sets. These sets have a positive but arbitrarily small density and they generalize, in many aspects, sets of density one. In particular, most of the arithmetic theorems, such as certain Hasse principles, the Grunwald–Wang theorem, Riemann’s existence theorem,  $K(\pi, 1)$ -property, etc., which hold for sets of density one (see [NSW 2008, Chapters IX and X]), also hold for stable sets. Our goals are on the one hand to prove these arithmetic results, and on the other hand to give many examples of stable sets.

The idea is as follows: let  $\lambda > 1$ . A set  $S$  of primes in a number field  $K$  is  $\lambda$ -stable for the extension  $\mathcal{L}/K$  if there is a subset  $S_0 \subseteq S$ , a finite subextension  $\mathcal{L}/L_0/K$

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The author was supported by the Mathematics Center Heidelberg.

MSC2010: primary 11R34; secondary 11R45.

Keywords: number field, Galois cohomology, restricted ramification, Dirichlet density.

and some  $a > 0$  such that we have  $\delta_L(S_0) \in [a, \lambda a)$  for all finite subextensions  $\mathcal{L}/L/L_0$ , where  $\delta_L$  is the Dirichlet density. We call the field  $L_0$  a  $\lambda$ -stabilizing field for  $S$  for  $\mathcal{L}/K$ . A more restrictive version is the notion of persistent sets:  $S$  is *persistent* if the function  $L \mapsto \delta_L(S_0)$  gets constant in the tower  $\mathcal{L}/K$  beginning from some finite subextension  $L_0/K$  (see [Definition 2.4](#)). In particular, for any  $\lambda > 1$ , a  $\lambda$ -stable set is persistent.

The main result in this article is the following theorem, which links stability to vanishing of certain Shafarevich–Tate groups. Let  $\text{III}^1$  denote the usual Shafarevich–Tate group, consisting of global cohomology classes which vanish locally in a given set of primes. If  $A$  is a module over a finite group  $G$ , then  $H_*^1(G, A)$  denotes the subgroup of  $H^1(G, A)$  consisting of precisely those classes which vanish after restriction to all cyclic subgroups of  $G$ . Moreover, if  $\mathcal{L}/L$  is a Galois extension of fields, then  $G_{\mathcal{L}/L}$  denotes its Galois group, and if  $A$  is a  $G_{\mathcal{L}/L}$ -module, then  $L(A)/L$  denotes the trivializing extension of  $A$ .

**Theorem 4.1.** *Let  $K$  be a number field,  $T$  a set of primes of  $K$  and  $\mathcal{L}/K$  a Galois extension. Let  $A$  be a finite  $G_{\mathcal{L}/K}$ -module. Assume that  $T$  is  $p$ -stable for  $\mathcal{L}/K$ , where  $p$  is the smallest prime divisor of  $|A|$ . Let  $L$  be a  $p$ -stabilizing field for  $T$  for  $\mathcal{L}/K$ . Then*

$$\text{III}^1(\mathcal{L}/L, T; A) \subseteq H_*^1(L(A)/L, A).$$

*In particular, if  $H_*^1(L(A)/L, A) = 0$ , then  $\text{III}^1(\mathcal{L}/L, T; A) = 0$ .*

This theorem has numerous applications to the structure of the Galois group  $G_{K,S} := \text{Gal}(K_S/K)$ , where  $K$  is a number field and  $S$  is stable. To explain our results, we need some notation. If  $S, R$  are two sets of primes of a number field  $K$ , then we denote by  $K_S^R$  the maximal extension of  $K$ , which is unramified outside  $S$  and completely split in  $R$ . Moreover, we denote by  $G_{K,S}^R$  the Galois group of  $K_S^R/K$ . Let  $\mathcal{L}/K$  be any Galois extension. For a prime  $\mathfrak{p}$  of  $K$  we denote by  $\mathcal{L}_{\mathfrak{p}}$  the completion of  $\mathcal{L}$  at a (any) extension of  $\mathfrak{p}$  to  $\mathcal{L}$  (the isomorphism class of the completion  $\mathcal{L}_{\mathfrak{p}}$  does not depend on the particular choice of the extension of  $\mathfrak{p}$  to  $\mathcal{L}$  as  $\mathcal{L}/K$  is Galois, and we suppress this choice in our notation). Furthermore,  $\mathcal{G}_{\mathfrak{p}}$  denotes the absolute Galois group of  $K_{\mathfrak{p}}$ , and  $K_{\mathfrak{p}}(p)$  (resp.  $K_{\mathfrak{p}}^{\text{nr}}(p)$ ) denotes the maximal (resp. maximal unramified) pro- $p$  extension of  $K_{\mathfrak{p}}$ . Moreover, for a profinite group  $G$ , we denote the pro- $p$  completion of  $G$  by  $G(p)$ . For more notation, see also the end of this introduction.

**Theorem** (cf. [Theorems 5.1](#) and [6.4](#)). *Let  $K$  be a number field,  $p$  a rational prime,  $\mathfrak{p}$  a prime of  $K$  and  $T \supseteq S \supseteq R$  sets of primes of  $K$  with  $R$  finite. Assume that  $S$  is  $p$ -stable<sup>1</sup> for  $K_S^R(\mu_p)/K$ . Then:*

<sup>1</sup>In fact a weaker condition would suffice; see [Theorem 5.1](#).

(A) (*Local extensions*)

$$K_{S,p}^R \cong \begin{cases} K_p(p) & \text{if } p \in S \setminus R, \\ K_p^{\text{nr}}(p) & \text{if } p \notin S. \end{cases}$$

(B) (*Riemann's existence theorem*) Let  $I'_p(p)$  denote the Galois group of the maximal pro- $p$  extension of  $K_{S,p}^R$  and let  $K'_T(p)/K_S^R$  denote the maximal pro- $p$  subextension of  $K_T/K_S^R$ . The natural map

$$\phi_{T,S}^R : \underset{p \in R(K_S^R)}{\ast} \mathcal{G}_p(p) \underset{p \in (T \setminus S)(K_S^R)}{\ast} I'_p(p) \xrightarrow{\sim} G_{K'_T(p)/K_S^R}$$

is an isomorphism (where  $\ast$  is to be understood in the sense of [NSW 2008, Chapter IV]).

(C) (*Cohomological dimension*) Assume that either  $p$  is odd or  $K$  is totally imaginary. Then

$$\text{cd}_p G_{K,S}^R = \text{scd}_p G_{K,S}^R = 2.$$

(D) ( $K(\pi, 1)$ -property) Assume additionally that  $R = \emptyset$ ,  $S \supseteq S_\infty$  and that either  $p$  is odd or  $K$  is totally imaginary. Then  $\text{Spec } \mathbb{O}_{K,S}$  is a  $K(\pi, 1)$  for  $p$  (see Definition 6.1).

There are also corresponding results for the maximal pro- $p$  quotient  $G_{K,S}^R(p)$  of  $G_{K,S}^R$ . These results are essentially well-known (see [NSW 2008]) if  $\delta_K(S) = 1$  and respectively if  $S \supseteq S_p \cup S_\infty$ . Also, A. Schmidt showed recently that if  $T_0$  is any fixed set with  $\delta_K(T_0) = 1$  and  $S$  is an arbitrary finite set of primes, then there is a finite subset  $T_1 \subseteq T_0$  (depending on  $S$ ) such that the pro- $p$  versions of the above results essentially (e.g., except the result on  $\text{scd}_p$ ) hold if one replaces  $S$  by  $S \cup T_1$  (see [Schmidt 2007; 2009; 2010]).

A further application of stable sets concerns a generalization of the Neukirch–Uchida theorem, which is a result of anabelian nature. More details on this can be found in [Ivanov 2013, Section 6]. Now we see many examples of stable (even persistent) sets:

**Corollary 3.4.** *Let  $M/K$  be finite Galois and let  $\sigma \in G_{M/K}$ . Let  $S \simeq P_{M/K}(\sigma)$  (i.e., up to a density-zero subset,  $S$  is equal to  $P_{M/K}(\sigma)$ ). Let  $\mathcal{L}/K$  be any extension. Then  $S$  is persistent — or, equivalently, stable (see Corollary 3.6) — for  $\mathcal{L}/K$  if and only if*

$$G_{M/M \cap \mathcal{L}} \cap C(\sigma; G_{M/K}) \neq \emptyset,$$

where  $C(\sigma; G_{M/K})$  denotes the conjugacy class of  $\sigma$  in  $G_{M/K}$ . In particular:

- (i) If  $\sigma = 1$ , then  $S \simeq P_{M/K}(1) = \text{cs}(M/K)$  is persistent for any extension  $\mathcal{L}/K$ .
- (ii) If  $M \cap \mathcal{L} = K$ , then  $S \simeq P_{M/K}(\sigma)$  is persistent for  $\mathcal{L}/K$ .

**Outline.** In [Section 2](#) we introduce stable, sharply  $p$ -stable, strongly  $p$ -stable and persistent sets. [Section 3](#) is devoted to examples: in particular, we introduce *almost Chebotarev sets*, which provide us with a rich supply of persistent sets ([Section 3B](#)), and we show essentially that an almost Chebotarev set is sharply and strongly  $p$ -stable for almost all  $p$  ([Section 3C](#)). In [Section 4A](#) we prove our main result which is a general Hasse principle. In [Sections 4B–4D](#) we discuss some further Hasse principles and uniform bounds on Shafarevich–Tate groups for stable sets. In [Section 5](#) we deduce arithmetic applications such as the Grunwald–Wang theorem, realization of local extensions, Riemann’s existence theorem and cohomological dimension. In [Section 6](#) we use results from [Section 5](#) to deduce the  $K(\pi, 1)$ -property at  $p$  for  $\text{Spec } \mathbb{O}_{K,S}$  with  $S$  being sharply  $p$ -stable.

**Notation.** Our notation essentially coincides with the notation in [\[NSW 2008\]](#). We collect some of the most important notation here. For a profinite group  $G$  we denote by  $G(p)$  its maximal pro- $p$  quotient. For a subgroup  $H \subseteq G$ , we denote by  $N_G(H)$  its normalizer in  $G$ . If  $\sigma \in G$ , then we write  $C(\sigma; G)$  for its conjugacy class. For two finite groups  $H \subseteq G$ , we write  $m_H^G$  (or  $m_H$ , if  $G$  is clear from the context) for the character of the induced representation  $\text{Ind}_H^G \mathbf{1}_H$ .

For a Galois extension  $M/L$  of fields,  $G_{M/L}$  denotes its Galois group and  $L(p)$  denotes the maximal pro- $p$  extension of  $L$  (in a fixed algebraic closure). By  $K$  we always denote an algebraic number field, that is, a finite extension of  $\mathbb{Q}$ . If  $\mathfrak{p}$  is a prime of  $K$  and  $L/K$  is a Galois extension, then  $D_{\mathfrak{p},L/K} \subseteq G_{L/K}$  denotes the decomposition subgroup of  $\mathfrak{p}$ . We write  $\Sigma_K$  for the set of all primes of  $K$  and  $S, T, R, \dots$  will usually denote subsets of  $\Sigma_K$ . If  $L/K$  is an extension and  $S$  a set of primes of  $K$ , then we denote the pull-back of  $S$  to  $L$  by  $S_L, S(L)$  or  $S$  (if no ambiguity can occur). We write  $K_S^R/K$  for the maximal extension of  $K$ , which is unramified outside  $S$  and completely split in  $R$ , and we write  $G_S^R := G_{K,S}^R$  for its Galois group. We use the abbreviations  $K_S := K_S^\emptyset$  and  $G_S := G_S^\emptyset$ . Further, for  $p \leq \infty$  a (archimedean or nonarchimedean) prime of  $\mathbb{Q}$ , we let  $S_p = S_p(K)$  denote the set of all primes of  $K$  lying over  $p$ . Further, if  $S \subseteq \Sigma_K$ , we write  $\mathbb{N}(S) := \mathbb{N} \cap \mathbb{O}_{K,S}^*$ , i.e.,  $p \in \mathbb{N}(S)$  if and only if  $S_p \subseteq S$ .

We write  $\delta_K$  for the Dirichlet density on  $\Sigma_K$ . For  $S, T$  subsets of  $\Sigma_K$ , we use (following [\[NSW 2008, Definition 9.1.2\]](#))

$$S \lesssim T : \Leftrightarrow \delta_K(S \setminus T) = 0, \quad S \simeq T : \Leftrightarrow (S \lesssim T) \text{ and } (T \lesssim S).$$

Thus  $S \lesssim T$  if  $S$  is contained in  $T$  up to a set of primes of density zero. For a finite Galois extension  $M/K$  and  $\sigma \in G_{M/K}$ , we have the Chebotarev set

$$P_{M/K}(\sigma) = \{\mathfrak{p} \in \Sigma_K : \mathfrak{p} \text{ is unramified in } M/K \text{ and } (\mathfrak{p}, M/K) = C(\sigma; G_{M/K})\},$$

where  $(\mathfrak{p}, M/K)$  denotes the conjugacy class of Frobenius elements corresponding to primes of  $M$  lying over  $\mathfrak{p}$ .

## 2. Stable and persistent sets

**2A. Warm-up: preliminaries on Dirichlet density.** Let  $\mathcal{P}_K$  denote the set of all subsets of  $\Sigma_K$ . The Dirichlet density  $\delta_K$  is not defined for all elements in  $\mathcal{P}_K$ . Moreover, there are examples of finite extensions  $L/K$  and  $S \in \mathcal{P}_K$  such that  $S$  has a density but the pull-back  $S_L$  of  $S$  to  $L$  has no density. To avoid dealing with such sets we make the following convention, which holds until the end of this article.

**Convention 2.1.** If  $S \in \mathcal{P}_K$  is a set of primes of  $K$ , then we assume implicitly that, for all finite extensions  $L/K$ , all finite Galois extensions  $M/L$  and all  $\sigma \in \mathbf{G}_{M/L}$ , the set  $S_L \cap P_{M/L}(\sigma)$  has a Dirichlet density.<sup>2</sup>

Convention 2.1 is satisfied for all sets lying in the rather large subset

$$\mathcal{A}_K := \left\{ S \subseteq \Sigma_K : S \simeq \bigcup_i P_{L_i/K_i}(\sigma_i)_K \text{ for some } K/K_i/\mathbb{Q} \right. \\ \left. \text{and } L_i/K_i \text{ finite Galois and } \sigma_i \in \mathbf{G}_{L_i/K_i} \right\}$$

of  $\mathcal{P}_K$ , where the unions are disjoint and countable (or finite or empty). The set  $\mathcal{A}_K$  cannot be closed simultaneously under (arbitrary) unions and complements: otherwise it would be a  $\sigma$ -algebra and hence would be equal to  $\mathcal{P}_K$ .

To compute the density of pull-backs of sets we use the following two lemmas. Let  $L/K$  be a finite extension of degree  $n$  (not necessarily Galois). For  $0 \leq m \leq n$ , define the sets

$$P_m(L/K) := \{\mathfrak{p} \in \Sigma_K : \mathfrak{p} \text{ is unramified and has exactly } m \text{ degree-1 factors in } L\}.$$

In particular,  $P_n(L/K) = \text{cs}(L/K)$ ,  $P_{n-1}(L/K) = \emptyset$ . Recall that if  $H \subseteq G$  are finite groups, then  $m_H$  denotes the character of the  $G$ -representation  $\text{Ind}_H^G \mathbf{1}$ . One has

$$m_H(\sigma) = |\{gH : \langle \sigma \rangle^g \subseteq H\}| = |\{\langle \sigma \rangle gH : \langle \sigma \rangle^g \subseteq H\}|,$$

where  $\langle \sigma \rangle \subseteq G$  denotes the subgroup generated by  $\sigma$  and where  $\langle \sigma \rangle^g := g^{-1} \langle \sigma \rangle g$ . The second equality follows immediately from the fact that if  $\langle \sigma \rangle^g \subseteq H$ , then  $gH = \langle \sigma \rangle gH$ .

**Lemma 2.2.** *Let  $L/K$  be a finite extension and  $N/K$  a finite Galois extension containing  $L$ , with Galois group  $G$ , such that  $L$  corresponds to a subgroup  $H \subseteq G$ .*

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<sup>2</sup>The optimal way to omit sets having no density would be to find an appropriate sub- $\sigma$ -algebra of  $\mathcal{P}_K$  (for any  $K$ ) such that the restriction of  $\delta_K$  to it is a measure (and the pull-back maps  $\mathcal{P}_K \rightarrow \mathcal{P}_L$  attached to finite extensions  $L/K$  restrict to pull-back maps on these sub- $\sigma$ -algebras). Unfortunately, there is no satisfactory way to find such a  $\sigma$ -algebra  $\mathcal{B}_K$ , at least not if one requires that if  $S \in \mathcal{B}_K$ , then  $T \in \mathcal{B}_K$  for any  $T \simeq S$ , or, if one requires the weaker condition that any finite set of primes of  $K$  lies in  $\mathcal{B}_K$ . Indeed, countability of  $\Sigma_K$  would imply  $\mathcal{B}_K = \mathcal{P}_K$  in this case, but not all elements of  $\mathcal{P}_K$  have a Dirichlet density.

Then

$$P_m(L/K) \simeq \{\mathfrak{p} \in P_m(L/K) : \mathfrak{p} \text{ is unramified in } N/K\} = \bigcup_{\substack{C(\sigma; G) \subseteq G \\ m_H(\sigma) = m}} P_{N/K}(\sigma),$$

where the right-hand side is a disjoint union. In particular,  $P_m(L/K) \in \mathcal{A}_K$  and

$$\delta_K(P_m(L/K)) = |G|^{-1} \sum_{\substack{C(\sigma; G) \subseteq G \\ m_H(\sigma) = m}} |C(\sigma; G)|.$$

*Proof.* The proof of the first statement is an elementary exercise in Galois theory: if  $\mathfrak{p}$  is a prime of  $K$  unramified in  $N$ , then the primes of  $L$  lying over  $\mathfrak{p}$  are in one-to-one correspondence with double cosets  $\langle \sigma \rangle gH$ , where  $\sigma$  is arbitrary in the Frobenius class of  $\mathfrak{p}$ ; the residue field extension of a prime belonging to the coset  $\langle \sigma \rangle gH$  over  $\mathfrak{p}$  has the Galois group  $\langle \sigma \rangle^g / \langle \sigma \rangle^g \cap H$ . The second statement follows from the first and the Chebotarev density theorem.  $\square$

**Lemma 2.3.** *Let  $L/K$  be a finite extension of degree  $n$ , let  $S$  be a set of primes of  $K$  and let  $N/K$  be a Galois extension containing  $L$  such that  $G := G_{N/K} \supseteq G_{N/L} =: H$ . Then*

$$\delta_L(S) = \sum_{m=1}^n m \delta_K(S \cap P_m(L/K)) = \sum_{C(\sigma; G) \subseteq G} m_H(\sigma) \delta_K(S \cap P_{N/K}(\sigma)).$$

If, in particular,  $L/K$  is Galois, we get the well-known formula

$$\delta_L(S) = [L : K] \delta_K(S \cap \text{cs}(L/K)).$$

*Proof.* The first equation is an easy computation and the second follows from Lemma 2.2.  $\square$

**2B. Key definitions.** Let  $K$  be a number field and  $S$  a set of primes. If  $\delta_K(S) = 0$  (resp.  $= 1$ ), then  $\delta_L(S) = 0$  (resp.  $= 1$ ) for all finite  $L/K$ . Now, if  $0 < \delta_K(S) < 1$ , then it can happen that there is some finite  $L/K$  with  $\delta_L(S) = 0$  (take a finite Galois extension  $L/K$  and set  $S := \Sigma_K \setminus \text{cs}(L/K)$ ; then  $\delta_K(S) = 1 - [L : K]^{-1}$  and  $\delta_L(S_L) = 0$ ). For stable sets, defined below, this possibility is excluded.

**Definition 2.4.** Let  $S$  be a set of primes of  $K$ , let  $\mathcal{L}/K$  be any extension and let  $\lambda > 1$ . A finite subextension  $\mathcal{L}/L_0/K$  is called  $\lambda$ -stabilizing for  $S$  for  $\mathcal{L}/K$  if there exists a subset  $S_0 \subseteq S$  and some  $a \in (0, 1]$  such that  $\lambda a > \delta_L(S_0) \geq a > 0$  for all finite subextensions  $\mathcal{L}/L/L_0$ . Moreover, we call  $L_0$  persisting for  $S$  for  $\mathcal{L}/K$  if there exists a subset  $S_0 \subseteq S$  such that  $\delta_L(S_0) = \delta_{L_0}(S_0) > 0$  for all finite subextensions  $\mathcal{L}/L/L_0$ . Further:

- (i) We call  $S$   $\lambda$ -stable (resp. persistent) for  $\mathcal{L}/K$  if it has a  $\lambda$ -stabilizing (resp. persisting) extension for  $\mathcal{L}/K$ .

(ii) We call  $S$  *stable* for  $\mathcal{L}/K$  if there is a  $\lambda > 1$  such that  $S$  is  $\lambda$ -stable for  $\mathcal{L}/K$ .

Assume that  $\lambda = p$  is a rational prime.

(iii) We call  $S$  *sharply  $p$ -stable* for  $\mathcal{L}/K$  if  $\mu_p \subseteq \mathcal{L}$  and if  $S$  is  $p$ -stable for  $\mathcal{L}/K$ , or if  $\mu_p \not\subseteq \mathcal{L}$  and if  $S$  is stable for  $\mathcal{L}(\mu_p)/K$ .

In applications we will often use the case  $\mathcal{L} = K_S$ . Therefore, we provide the following definition:

**Definition 2.5.** Let  $S$  be a set of primes of  $K$  and let  $\lambda > 1$ .

(i) We call  $S$   $\lambda$ -*stable* (resp. *stable*, resp. *persistent*) if  $S$  is  $\lambda$ -stable (resp. stable, resp. persistent) for  $K_S/K$ .

Assume that  $\lambda = p$  is a rational prime.

(ii) We call  $S$  *sharply  $p$ -stable* if  $S$  is sharply  $p$ -stable for  $K_S/K$ . Moreover, we define the exceptional set  $E^{\text{sharp}}(S)$  to be the set of all rational primes  $p$  such that  $S$  is not sharply  $p$ -stable.

(iii) We call  $S$  *strongly  $p$ -stable* if  $S$  is  $p$ -stable for  $K_{S \cup S_p \cup S_\infty}/K$  with a  $p$ -stabilizing field contained in  $K_S$ . Further, we call  $S$  *strongly  $\infty$ -stable* if  $S$  is stable for  $K_{S \cup S_\infty}/K$ . Moreover, we define the exceptional set  $E^{\text{strong}}(S)$  to be the set of all rational primes  $p$  or  $p = \infty$  such that  $S$  is not strongly  $p$ -stable.

Clearly, a strongly  $p$ -stable set is also  $p$ -stable and sharply  $p$ -stable. In particular, we have  $E^{\text{strong}}(S) \supseteq E^{\text{sharp}}(S)$ . On the other side, in general, neither one of the properties ‘ $p$ -stable’ or ‘sharply  $p$ -stable’ implies the other.

**Lemma 2.6.** Let  $\mathcal{L}/K$  be an extension and  $S$  a set of primes of  $K$ .

- (i) Let  $\lambda \geq \mu > 1$ . If  $S$  is  $\mu$ -stable with  $\mu$ -stabilizing field  $L_0$ , then  $S$  is  $\lambda$ -stable with  $\lambda$ -stabilizing field  $L_0$ .
- (ii) If  $L_0$  is a  $\lambda$ -stabilizing (resp. persisting) field for  $S$  for  $\mathcal{L}/K$ , then any finite subextension  $\mathcal{L}/L_1/L_0$  has the same property.
- (iii) Let  $S'$  be a further set of primes of  $K$ . If  $S \subsetneq S'$  and if  $S$  is  $\lambda$ -stable (resp. persistent) for  $\mathcal{L}/K$ , then  $S'$  also has this property. Any  $\lambda$ -stabilizing (resp. persisting) field for  $S$  has the same property for  $S'$ .
- (iv) Let  $\mathcal{L}/\mathcal{N}/M/K$  be subextensions. If  $S$  is  $\lambda$ -stable (resp. persistent) for  $\mathcal{L}/K$  with  $\lambda$ -stabilizing (resp. persisting) field  $L_0 \subseteq \mathcal{N}$ , then  $S_M$  is  $\lambda$ -stable (resp. persistent) for  $\mathcal{N}/M$ .

**Lemma 2.7.** Let  $\mathcal{L}/K$  be an extension and  $S$  a set of primes of  $K$ . Assume that  $S$  is sharply  $p$ -stable for  $\mathcal{L}/K$ . There is a finite subextension  $\mathcal{L}/L_0/K$  such that, for any subextensions  $\mathcal{L}/\mathcal{N}/L/L_0$  (with  $L/L_0$  finite),  $S$  is sharply  $p$ -stable for  $\mathcal{N}/L$ .



The proofs of these lemmas are straightforward. The following proposition gives another characterization of stable sets and shows, in particular, that if  $S$  is stable for  $\mathcal{L}/K$ , then any finite subfield  $\mathcal{L}/L/K$  is  $\lambda$ -stabilizing for  $S$  with a certain  $\lambda > 1$  depending on  $L$ .

**Proposition 2.8.** *Let  $S$  be a set of primes of  $K$  and  $\mathcal{L}/K$  an extension. The following are equivalent:*

- (i)  $S$  is stable for  $\mathcal{L}/K$ .
- (ii) There exists some  $\lambda > 1$  such that  $S$  is  $\lambda$ -stable for  $\mathcal{L}/K$  with  $\lambda$ -stabilizing field  $K$ .
- (iii) There exists some  $\epsilon > 0$  such that  $\delta_L(S) > \epsilon$  for all finite  $\mathcal{L}/L/K$ .

*Proof.* The directions (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i) are trivial. We prove (i)  $\Rightarrow$  (iii). Let  $\lambda > 1$  and let  $S$  be  $\lambda$ -stable for  $\mathcal{L}/K$  with  $\lambda$ -stabilizing field  $L_0$ . Then there is some  $a > 0$  and a subset  $S_0 \subseteq S$  such that  $a \leq \delta_L(S_0) < \lambda a$  for all  $\mathcal{L}/L/L_0$ . Suppose there is no  $\epsilon > 0$  such that  $\delta_L(S_0) > \epsilon$  for all  $\mathcal{L}/L/K$ . This implies that there is a family  $(M_i)_{i=1}^\infty$  of finite subextensions of  $\mathcal{L}/K$  with  $\delta_{M_i}(S_0) \rightarrow 0$  as  $i \rightarrow \infty$ . Then  $d_i = [L_0 M_i : M_i] = [L_0 : L_0 \cap M_i]$  is bounded from above by  $[L_0 : K]$  and hence, by [Lemma 2.3](#),

$$\delta_{L_0 M_i}(S_0) = \sum_{m=1}^{d_i} m \delta_{M_i}(S_0 \cap P_m(L_0 M_i / M_i)) \leq [L_0 : K] \delta_{M_i}(S_0) \rightarrow 0$$

for  $i \rightarrow \infty$ . This contradicts the  $\lambda$ -stability of  $S_0$  with respect to the  $\lambda$ -stabilizing field  $L_0$ .  $\square$

Here is a brief overview of the use of these conditions and some examples:

- Most examples of stable sets are given by (almost) Chebotarev sets, i.e., sets of the form  $S \simeq P_{M/K}(\sigma)$ , or sets containing them (see [Section 3B](#)).
- If  $S$  is stable for  $\mathcal{L}/K$ , then  $\delta_L(S) > 0$  for all finite  $\mathcal{L}/L/K$ . The converse is not true in general (see [\[Ivanov 2013, Section 3.5.4\]](#)), but it is true for almost Chebotarev sets (see [Section 3B](#)).
- If an almost Chebotarev set is stable for an extension, then it is also persistent for it (see [Corollary 3.6](#)). It is not clear whether there are examples of stable but not persistent sets (but see [\[Ivanov 2013, Section 3.5.4\]](#)).
- For a stable almost Chebotarev set  $S$ , the set  $E^{\text{sharp}}(S)$  is finite and  $E^{\text{strong}}(S)$  is either  $\Sigma_{\mathbb{Q}}$  or finite (see [Section 3C](#)).
- Roughly speaking,  $p$ -stability (for  $\mathcal{L}/K$ ) is enough to prove Hasse principles in dimension 1 for  $p$ -primary  $(G_{\mathcal{L}/K})$ -modules. See [Section 4](#).

- To prove Hasse principles in dimension 2 and Grunwald–Wang-type results for  $p$ -primary  $G_{K,S}$ -modules, we need strong  $p$ -stability. We will give examples of persistent sets  $S$  together with a finite set  $T$  such that Grunwald–Wang (even stably) fails, i.e.,  $\text{coker}^1(K_{S \cup T}/L, T; \mathbb{Z}/p\mathbb{Z}) \neq 0$  for all finite subextensions  $K_S/L/K$ . But it is not clear whether one can find such an example with the additional requirement that  $T \subseteq S$  (and necessarily  $S$  being not strongly  $p$ -stable). See [Section 5B](#).
- On the other side, for applications of Grunwald–Wang (i.e., to prove Riemann’s existence theorem, to realize local extensions by  $K_S/K$ , to compute (strict) cohomological dimension, etc.), it is enough to require that  $S$  is sharply  $p$ -stable. See [Sections 5A, 5C](#) and [5D](#).

### 3. Examples

In this section we construct examples of stable sets. First, in [Section 3A](#), we see to which extent ‘stable’ is more general than ‘of density 1’. Then, in [Sections 3B](#) and [3C](#), we introduce almost Chebotarev sets and determine when they are stable, strongly  $p$ -stable, and sharply  $p$ -stable. Finally, in [Section 3D](#), we construct a stable almost Chebotarev set  $S$  with  $\mathbb{N}(S) = \{1\}$ .

**3A. Sets of density one.** Stable and persistent sets generalize sets of density one. In particular, every set of primes of  $K$  of density one is persistent for any extension  $\mathcal{L}/K$  with persisting field  $K$  and is strongly  $p$ -stable for each  $p$ . Nevertheless, sets of density one have some properties which stable and persistent sets do not have in general:

- (i) The intersection of two sets of density one again has density one, which is not true for stable and persistent sets: the intersection of two sets persistent for  $\mathcal{L}/K$  can be empty (see [Corollary 3.4](#) and explicit examples below).
- (ii) If  $S \subseteq \Sigma_K$  has density one, then there are infinitely many primes  $p \in \Sigma_{\mathbb{Q}}$  such that  $S_p \subseteq S$  (otherwise, for all primes  $p \in \text{cs}(K/\mathbb{Q})$  one could choose a prime  $\mathfrak{p} \in S_p \setminus S$  of  $K$  and we would have  $\delta_K(S) \leq 1 - [K^n : \mathbb{Q}]^{-1}$ , where  $K^n$  denotes the normal closure of  $K$  over  $\mathbb{Q}$ ). On the other side, it is easy to construct a persistent set  $S \subseteq \Sigma_K$  with  $\mathbb{N}(S) = \{1\}$ , i.e.,  $S_\ell \not\subseteq S$  for all  $\ell \in \Sigma_{\mathbb{Q}}$  (see [Section 3D](#) for an example).

Observe that, for sets  $S$  with  $\mathbb{N}(S) = \{1\}$ , mentioned above, none of the  $\ell$ -adic representations  $\rho_{A,\ell} : G_K \rightarrow \text{GL}_d(\overline{\mathbb{Q}}_\ell)$  which come from an abelian variety  $A/K$  factor through the quotient  $G_K \twoheadrightarrow G_{K,S}$  (indeed, the Tate-pairing on  $A$  shows that the determinant of  $\rho_{A,\ell}$  is the  $\ell$ -part of the cyclotomic character of  $K$  and, in particular,  $\rho_{A,\ell}$  is highly ramified at all primes of  $K$  lying over  $\ell$ . If  $\rho_{A,\ell}$  factored over  $G_{K,S}$ , then we would have  $S_\ell \subseteq S$ ). In particular, this makes it very hard, if

not impossible, to study the group  $G_{K,S}$  via the Langlands program (for example, like in [Chenevier 2007] and [Chenevier and Clozel 2009], where a prime  $\ell \in \mathbb{N}(S)$  is always necessary). If  $S$  is additionally stable, then methods involving stability allow us to study  $G_{K,S}$ .

### 3B. Almost Chebotarev sets.

**Definition 3.1.** Let  $K$  be a number field and  $S$  a set of primes of  $K$ . Then  $S$  is called a *Chebotarev set* (resp. an *almost Chebotarev set*) if  $S = P_{M/K}(\sigma)$  (resp.  $S \simeq P_{M/K}(\sigma)$ ), where  $M/K$  is a finite Galois extension and  $\sigma \in G_{M/K}$ .

**Remark 3.2.**  $M$  and the conjugacy class of  $\sigma$  are not unique, i.e., there are pairs  $(M/K, \sigma)$ ,  $(N/K, \tau)$  such that  $M \neq N$  and  $P_{M/K}(\sigma) \simeq P_{N/K}(\tau)$  (or even  $P_{M/K}(\sigma) = P_{N/K}(\tau)$ ). If one restricts attention to pairs  $(M/K, \sigma)$  such that  $\sigma$  is central in  $G_{M/K}$ , then  $(M/K, \sigma)$  is indeed unique. See [Ivanov 2013, Remark 3.13].

**Proposition 3.3.** Let  $M/K$  be a finite Galois extension and let  $\sigma \in G_{M/K}$ . Let  $L/K$  be any finite extension and set  $L_0 := L \cap M$ . Then

$$\delta_L(P_{M/K}(\sigma)_L) = \frac{|C(\sigma; G_{M/K}) \cap G_{M/L_0}|}{|G_{M/L_0}|}.$$

Thus  $\delta_L(P_{M/K}(\sigma)_L) \neq 0$  if and only if  $C(\sigma; G_{M/K}) \cap G_{M/L_0} \neq \emptyset$ . In particular, this is always the case if  $L_0 = K$  or if  $\sigma = 1$ .

*Proof.* Let  $N/K$  be a finite Galois extension with  $N \supseteq ML$ . Define  $H := G_{N/L}$  and  $\bar{H} := G_{M/L_0}$ . We have a natural surjection  $H \twoheadrightarrow \bar{H}$ . Let  $\mathbf{1}_\sigma$  denote the class function on  $G_{M/K}$ , which has value 1 on  $C(\sigma; G_{M/K})$  and 0 outside. Finally, let  $m_H$  denote the character on  $G := G_{N/K}$  of the induced representation  $\text{Ind}_H^G \mathbf{1}_H$ . Then we have

$$\begin{aligned} \delta_L(P_{M/K}(\sigma)_L) &= \sum_{C(g; G) \mapsto C(\sigma; G_{M/K})} \delta_L(P_{N/K}(g)_L) = \sum_{C(g; G) \mapsto C(\sigma; G_{M/K})} m_H(g) \delta_K(P_{N/K}(g)) \\ &= \sum_{C(g; G) \mapsto C(\sigma; G_{M/K})} m_H(g) \frac{|C(g; G)|}{|G|} = \frac{1}{|G|} \sum_{g \mapsto C(\sigma; G_{M/K})} m_H(g) \\ &= \langle m_H, \text{inf}_{G_{M/K}}^G \mathbf{1}_\sigma \rangle_G = \langle \mathbf{1}_H, \text{inf}_{G_{M/K}}^H \mathbf{1}_\sigma \rangle_H = \langle \mathbf{1}_{\bar{H}}, \mathbf{1}_\sigma|_{\bar{H}} \rangle_{\bar{H}} \\ &= \frac{|C(\sigma; G_{M/K}) \cap \bar{H}|}{|\bar{H}|}. \end{aligned}$$

The first equality follows from [Wingberg 2006, Proposition 2.1] and the second from Lemma 2.3. The third-to-last equality is Frobenius reciprocity, and the second-to-last equality follows from the easy fact that if  $H \twoheadrightarrow \bar{H}$  is a surjection of finite groups and  $\chi, \rho$  are two characters of  $\bar{H}$ , then  $\langle \text{inf}_{\bar{H}}^H \chi, \text{inf}_{\bar{H}}^H \rho \rangle_H = \langle \chi, \rho \rangle_{\bar{H}}$ .  $\square$

**Corollary 3.4.** *Let  $M/K$  be finite Galois and let  $\sigma \in G_{M/K}$ . Let  $\mathcal{L}/K$  be any extension and set  $L_0 := M \cap \mathcal{L}$ . Then a set  $S \simeq P_{M/K}(\sigma)$  is persistent for  $\mathcal{L}/K$  if and only if*

$$C(\sigma; G_{M/K}) \cap G_{M/L_0} \neq \emptyset.$$

*If this is the case,  $L_0$  is a persistent field for  $S$  for  $\mathcal{L}/K$ . In particular:*

- (i) *Any set  $S \simeq \text{cs}(M/K)$  is persistent for any extension  $\mathcal{L}/K$ .*
- (ii) *Any set  $S \simeq P_{M/K}(\sigma)$  is persistent for any extension  $\mathcal{L}/K$  with  $\mathcal{L} \cap M = K$ .*

**Example 3.5** (a persistent set). Let  $K$  be a number field,  $M/K$  a finite Galois extension which is totally ramified in a prime  $\mathfrak{p}$  of  $K$ . Let  $\sigma \in G_{M/K}$  and let  $S$  be a set of primes of  $K$  such that  $S \simeq P_{M/K}(\sigma)$  and  $\mathfrak{p} \notin S$ . Then  $S$  is persistent with persisting field  $K$ . Indeed, we have  $K_S \cap M = K$  by construction, and the claim follows from [Corollary 3.4](#).

**Corollary 3.6.** *Let  $S$  be an almost Chebotarev set and  $\mathcal{L}/K$  an extension. Then the following are equivalent:*

- (i)  *$S$  is stable for  $\mathcal{L}/K$ .*
- (ii)  *$S$  is persistent for  $\mathcal{L}/K$ .*
- (iii)  *$\delta_L(S) > 0$  for all finite  $\mathcal{L}/L/K$ .*

*Proof.* Suppose  $S \simeq P_{M/K}(\sigma)$ , with  $M/K$  a finite Galois extension and  $\sigma \in G_{M/K}$ . By [Proposition 3.3](#), the density of  $S$  is constant and equal to some  $d \geq 0$  in the tower  $\mathcal{L}/L_0$  with  $L_0 = \mathcal{L} \cap M$ . There are two cases: either  $d = 0$  or  $d > 0$ . If  $d = 0$ , then  $S$  is not stable and hence also not persistent for  $\mathcal{L}/K$  by [Proposition 2.8](#), i.e., (i), (ii) and (iii) do not hold in this case. If  $d > 0$ , then  $S$  is obviously persistent for  $\mathcal{L}/K$  with persisting field  $L_0$  and hence also stable, i.e., (i), (ii), (iii) hold.  $\square$

**Remark 3.7.** If  $S$  is any stable set, then (ii)  $\Rightarrow$  (i)  $\Rightarrow$  (iii) still holds. But (iii)  $\Rightarrow$  (i) fails in general (see [[Ivanov 2013](#), Section 3.5.4]) and it is not clear whether (i)  $\Rightarrow$  (ii) holds.

### 3C. Finiteness of $E^{\text{sharp}}(S)$ , $E^{\text{strong}}(S)$ and examples.

**Proposition 3.8.** *Let  $S \simeq P_{M/K}(\sigma)$  with  $\sigma \in G_{M/K}$ .*

- (i) *If  $\infty \in E^{\text{strong}}(S)$ , then  $E^{\text{strong}}(S)$  contains all rational primes. If  $\infty \notin E^{\text{strong}}(S)$ , then  $E^{\text{strong}}(S)$  is finite.*
- (ii) *Assume  $S$  is stable. If  $\mu_p \subset K_S$  or if  $M/K$  is unramified in  $S_p \setminus S$ , then  $S$  is sharply  $p$ -stable. In particular, if  $S$  is stable, then  $E^{\text{sharp}}(S)$  is finite.*

*Proof.* (i) If  $\infty \in E^{\text{strong}}(S)$ , then  $S$  does not have a stabilizing field for  $K_{S \cup S_\infty}/K$  which is contained in  $K_S$ . This is, by [Proposition 2.8](#), equivalent to the fact that  $S$  is not stable for  $K_{S \cup S_\infty}/K$ , which in turn is equivalent, by [Corollary 3.6](#), to the

fact that  $\delta_L(S) = 0$  for all  $K_{S \cup S_\infty}/L/L_0$ , where  $L_0$  is some finite subextension of  $K_{S \cup S_\infty}/K$ . Thus  $p \in E^{\text{strong}}(S)$  for any  $p$ .

Now assume  $\infty \notin E^{\text{strong}}(S)$ . Set  $L_0 := M \cap K_{S \cup S_\infty}$  and  $L_p := M \cap K_{S \cup S_p \cup S_\infty}$ . By [Proposition 3.3](#), the density of  $S$  is constant in the towers  $K_{S \cup S_\infty}/L_0$  and  $K_{S \cup S_p \cup S_\infty}/L_p$  and equal to some real numbers  $d_0$  and  $d_p$ , respectively. Since  $S$  is stable for  $K_{S \cup S_\infty}/K$ , we have  $d_0 > 0$ .

We claim that for almost all  $ps$  we have  $L_p = L_0$ . More precisely, this is true for all  $ps$  such that the set

$$\{\mathfrak{p} \in (S_p \setminus S)_{L_0} : \mathfrak{p} \text{ is ramified in } M/L_0\}$$

is empty. In fact, if this set is empty for  $p$ , then the extension  $L_p/L_0$  is unramified in  $S_p \setminus S(L_0)$ , since it is contained in  $M/L_0$ . But, being contained in  $K_{S \cup S_p \cup S_\infty}$  and unramified in  $S_p \setminus S(L_0)$ , it is contained in  $K_{S \cup S_\infty}$ , and hence also in  $M \cap K_{S \cup S_\infty} = L_0$ , which proves our claim.

Now let  $p$  be such that  $L_p = L_0$ . We claim that  $S$  is  $([L_0 : K]d_0^{-1})$ -stable for  $K_{S \cup S_p \cup S_\infty}/K$  with  $([L_0 : K]d_0^{-1})$ -stabilizing field  $K$ . Indeed, as  $L_p = L_0$ , we have  $d_p = d_0 > 0$ . Let  $K_{S \cup S_p \cup S_\infty}/N/K$  be any finite subextension. We have

$$d_0 = \delta_{L_0 N}(S) = [L_0 N : N] \delta_N(S \cap \text{cs}(L_0 N/N)) \leq [L_0 : K] \delta_N(S),$$

i.e.,  $\delta_N(S) \geq [L_0 : K]^{-1} d_0$  for all  $N$ , and our claim follows.

Finally, almost all primes satisfy  $p > [L_0 : K]d_0^{-1}$  and  $L_p = L_0$ . For such primes,  $S$  is  $p$ -stable for  $K_{S \cup S_p \cup S_\infty}/K$  with stabilizing field  $K$ .

(ii) The second assertion of (ii) follows from the first. If  $\mu_p \subseteq K_S$ , then  $S$  is sharply  $p$ -stable by [Corollary 3.6](#). Assume  $M/K$  is unramified in  $S_p \setminus S$ . Set  $L_0 := M \cap K_S$ ,  $L'_0 := L_0(\mu_p) \cap K_S$  and  $L_p := M \cap K_S(\mu_p)$ . From these definitions and from our assumption on  $M/K$  we have

- (1)  $\mathbf{G}_{K_S(\mu_p)/L'_0} \cong \mathbf{G}_{K_S/L'_0} \times \mathbf{G}_{L_0(\mu_p)/L'_0}$ , and  $L_0(\mu_p)/L'_0$  has no subextension unramified in  $S_p \setminus S$ ,
- (2)  $L_p \cap K_S = L_0$ , and
- (3)  $L_p/L_0$  is unramified in  $S_p \setminus S$ .

By item (3) the extension  $L_p L'_0/L'_0$  is unramified in  $S_p \setminus S$ , and by item (1) we get  $L_p \subseteq L_p L'_0 \subseteq K_S$ . Hence, (2) gives  $L_p = L_0$ . Thus for all  $K_S(\mu_p)/L/L_0$  we have, by [Proposition 3.3](#),  $\delta_L(S) = \delta_{L_p}(S) = \delta_{L_0}(S) > 0$ , since  $S$  is stable.  $\square$

**Remark 3.9.** Suppose  $S \simeq P_{M/K}(\sigma)$ . We have the following equivalences:

$$p \notin E^{\text{sharp}}(S) \Leftrightarrow S \text{ stable for } K_S(\mu_p)/K \Leftrightarrow C(\sigma; \mathbf{G}_{M/K}) \cap \mathbf{G}(M/M \cap K_S(\mu_p)) \neq \emptyset.$$

**Example 3.10** (persistent sets with  $E^{\text{strong}}(S)$  finite but nonempty). Let  $K$  be a totally imaginary number field and let  $M/K$  be a finite Galois extension such that

- $M/K$  is totally ramified in a prime  $\mathfrak{p} \in S_p(K)$ ,
- $d := [M : K] > p$ .

Let  $\sigma \in G_{M/K}$  and let  $S$  be a set of primes of  $K$  such that

- $S \simeq P_{M/K}(\sigma)$ ,
- $\text{Ram}(M/K) \setminus S = \{\mathfrak{p}\}$ .

Then  $S$  is persistent ( $\delta_L(S) = d^{-1}$  for all  $K_S/L/K$ ) with persisting field  $K$ . Further,  $S$  is not strongly  $p$ -stable, i.e.,  $p \in E^{\text{strong}}(S)$  and  $\infty \notin E^{\text{strong}}(S)$ , i.e.,  $E^{\text{strong}}(S)$  is finite by [Proposition 3.8](#). Indeed,  $M \subseteq K_{S \cup S_p \cup S_\infty}$  and there are two cases,  $\sigma = 1$  and  $\sigma \neq 1$ . In the second case, the density of  $S$  in  $K_{S \cup S_p \cup S_\infty}/K$  is zero beginning from  $M$ , hence  $S$  is nonstable for this extension, and  $S$  is not strongly  $p$ -stable. In the first case, we have  $\delta_L(S) = 1$  for all  $K_{S \cup S_p \cup S_\infty}/L/M$ . Assume there is a  $p$ -stabilizing field  $N \subseteq K_S$  for  $S$  for  $K_{S \cup S_p \cup S_\infty}/K$ , i.e., there is some  $S_0 \subseteq S$  and some  $a \in (0, 1]$  with  $a \leq \delta_L(S_0) < pa$  for all  $K_{S \cup S_p \cup S_\infty}/L/N$ . But this leads to a contradiction. Indeed,

$$\delta_{MN}(S_0) = [MN : N]\delta_N(S_0 \cap \text{cs}(MN/N)) = [M : K]\delta_N(S_0) \geq p\delta_N(S_0),$$

since  $N \cap M = K$  and  $S_0 \subseteq S \simeq \text{cs}(M/K)$ .

**Example 3.11** (persistent sets with  $E^{\text{strong}}(S) = \emptyset$ ). Let  $M/K$  be a finite Galois extension of degree  $d$ , with  $K$  totally imaginary, which is totally ramified in at least two primes  $\mathfrak{p}$  and  $\mathfrak{l}$  with different residue characteristics  $\ell_1$  and  $\ell_2$ , respectively. Let  $S \simeq P_{M/K}(\sigma)$  for some  $\sigma \in G_{M/K}$  such that  $\mathfrak{p}, \mathfrak{l} \notin S$ . Then  $M \cap K_S = K$ , hence  $S$  is persistent with persisting field  $K$ . Let  $p$  be a rational prime. Then  $M \cap K_{S \cup S_p \cup S_\infty} = K$ , since  $M/K$  is totally ramified over primes with *different* residue characteristics  $\ell_1$  and  $\ell_2$ . Hence  $S$  is strongly  $p$ -stable for every prime  $p$  and  $K$  is a persisting field for  $S$  for  $K_{S \cup S_p \cup S_\infty}/K$ .

**Example 3.12** (persistent sets with  $E^{\text{strong}}(S) = \emptyset$ ). There is also another possibility, to construct sets  $S$  with  $E^{\text{strong}}(S) = \emptyset$ , using the same idea as in the preceding example. Assume for simplicity that  $K$  is totally imaginary. Let  $M_1, M_2/K$  be two Galois extensions of  $K$ , and let  $\sigma_1 \in G_{M_1/K}$ ,  $\sigma_2 \in G_{M_2/K}$ . Assume  $M_i/K$  is totally ramified in a nonarchimedean prime  $\mathfrak{p}_i$  of  $K$  such that the residue characteristics of  $\mathfrak{p}_1, \mathfrak{p}_2$  are unequal. Then let  $S$  be a set of primes of  $K$  such that

- $S \supseteq P_{M_1/K}(\sigma_1) \cup P_{M_2/K}(\sigma_2)$ ,
- $\{\mathfrak{p}_1, \mathfrak{p}_2\} \notin S$ .

Then, by the same reasoning as in the preceding example,  $S$  is persistent with persisting field  $K$  and  $E^{\text{strong}}(S) = \emptyset$ . Moreover, for each rational prime  $p$ , the field  $K$  is persisting for  $S$  for  $K_{S \cup S_p \cup S_\infty}/K$ .

**3D. Stable sets with  $\mathbb{N}(S) = \{1\}$ .** Let  $M/K/K_0$  be two finite Galois extensions of a number field  $K_0$ . Then the natural map  $G_{M/K_0} \rightarrow \text{Aut}(G_{M/K})$  induces an exterior action

$$G_{K/K_0} \rightarrow \text{Out}(G_{M/K}),$$

thus inducing a natural action of  $G_{K/K_0}$  on the set of all conjugacy classes of  $G_{M/K}$ . For any  $g \in G_{K/K_0}$  and  $\sigma \in G_{M/K}$ , we choose a representative of the conjugacy class  $g \cdot C(\sigma; G_{M/K})$  and denote it by  $g \cdot \sigma$ . Further,  $G_{K/K_0}$  acts naturally on  $\Sigma_K$ , and we have

$$g \cdot P_{M/K}(\sigma) = P_{M/K}(g \cdot \sigma).$$

Let  $K_0 = \mathbb{Q}$  and let  $\sigma \in G_{M/K}$  be an element such that  $C(\sigma; G_{M/K})$  is not fixed by the action of  $G_{K/\mathbb{Q}}$ . Then set

$$S := \text{cs}(K/\mathbb{Q})_K \cap P_{M/K}(\sigma).$$

If  $p \in \Sigma_{\mathbb{Q}, f} \setminus \text{cs}(K/\mathbb{Q})$ , then  $S \cap S_p = \emptyset$ . If  $p \in \text{cs}(K/\mathbb{Q})$  such that  $S_p \cap S \neq \emptyset$ , then the action of  $g \in G_{K/K_0}$ , chosen such that  $C(\sigma; G_{M/K}) \neq C(g \cdot \sigma; G_{M/K})$ , defines an isomorphism between the disjoint sets  $S_p \cap P_{M/K}(\sigma)$  and  $S_p \cap P_{M/K}(g \cdot \sigma)$ , hence the last of these two sets is nonempty. From this we obtain  $S_p \not\subseteq S$ . Thus  $\mathbb{N}(S) = \{1\}$ . Moreover, if we choose  $\sigma$  such that the stabilizer of  $C(\sigma; G_{M/K})$  in  $G_{K/\mathbb{Q}}$  is trivial, then for any  $p$  the intersection  $S_p \cap S$  is either empty or contains exactly one element.

Now we have to choose  $M$  in such a way that  $S$  is stable. This is easy: e.g. take  $M/K$  such that it is totally ramified in a fixed prime which is (by definition of  $S$ ) not contained in  $S$ . Then  $K_S \cap M = K$ , i.e.,  $S$  is stable for  $K_S/K$  with stabilizing field  $K$ , as  $\delta_K(\text{cs}(K/\mathbb{Q})_K) = 1$  and hence  $S \simeq P_{M/K}(\sigma)$ .

#### 4. Shafarevich–Tate groups of stable sets

In this section we generalize many Hasse principles to stable sets and additionally prove finiteness and uniform bounds of certain Shafarevich–Tate groups associated with stable sets. The main result is the Hasse principle in [Theorem 4.1](#). Further, there are two variants of uniform bounds on the size of  $\text{III}^i$ : on the one side we can vary the coefficients, and on the other side the base field. We study both variants, the first in [Section 4C](#) and the second in [Section 4D](#). These results are used in later sections.

**4A. Stable sets and  $\text{III}^1$ : key result.** Let  $K$  be a number field and  $\mathcal{L}/K$  a (possibly infinite) Galois extension. Let  $A$  be a finite  $G_{\mathcal{L}/K}$ -module. Let now  $T$  be a set of primes of  $K$ . Consider the  $i$ -th Shafarevich–Tate group with respect to  $T$ ,

$$\text{III}^i(\mathcal{L}/K, T; A) := \ker(\text{res}^i : H^i(\mathcal{L}/K, A) \rightarrow \prod_{p \in T} H^i(\mathcal{G}_p, A)),$$

where  $\mathcal{G}_{\mathfrak{p}} = G_{K_{\mathfrak{p}}^{\text{sep}}/K_{\mathfrak{p}}}$  is the local absolute Galois group (the map res is essentially independent of the choice of this separable closure and we suppress it in the notation). We also write  $\text{III}^i(K_S/K; A)$  instead of  $\text{III}^i(K_S/K, S; A)$ . We denote by  $K(A)$  the *trivializing extension* for  $A$ , i.e., the smallest field between  $K$  and  $\mathcal{L}$  such that the subgroup  $G_{\mathcal{L}/K(A)}$  of  $G_{\mathcal{L}/K}$  acts trivially on  $A$ . It is a finite Galois extension of  $K$ .

Let  $G$  be a finite group and  $A$  a  $G$ -module. Following Serre [1964, §2] and Jannsen [1982], let  $H_*^i(G, A)$  be defined by exactness of the sequence

$$0 \rightarrow H_*^i(G, A) \rightarrow H^i(G, A) \rightarrow \prod_{\substack{H \subseteq G \\ \text{cyclic}}} H^i(H, A).$$

Our key result is the following theorem. All subsequent results make use of this theorem in a crucial way.

**Theorem 4.1.** *Let  $K$  be a number field,  $T$  a set of primes of  $K$  and  $\mathcal{L}/K$  a Galois extension. Let  $A$  be a finite  $G_{\mathcal{L}/K}$ -module. Assume that  $T$  is  $p$ -stable for  $\mathcal{L}/K$ , where  $p$  is the smallest prime divisor of  $|A|$ . Let  $L$  be a  $p$ -stabilizing field for  $T$  for  $\mathcal{L}/K$ . Then*

$$\text{III}^1(\mathcal{L}/L, T; A) \subseteq H_*^1(L(A)/L, A).$$

*In particular, if  $H_*^1(L(A)/L, A) = 0$ , then  $\text{III}^1(\mathcal{L}/L, T; A) = 0$ .*

**Lemma 4.2.** *Let  $\mathcal{L}/L/K$  be two Galois extensions of  $K$  and  $T$  a set of primes of  $K$ . Let  $A$  be a  $G_{\mathcal{L}/K}$ -module such that for any  $\mathfrak{p} \in T$  one has  $A^{G_{\mathcal{L}/L}} = A^{D_{\mathfrak{p}, \mathcal{L}/L}}$ . Then there is an exact sequence*

$$0 \rightarrow \text{III}^1(L/K, T; A^{G_{\mathcal{L}/L}}) \rightarrow \text{III}^1(\mathcal{L}/K, T; A) \rightarrow \text{III}^1(\mathcal{L}/L, T_L; A).$$

*Proof.* The proof is an easy and straightforward exercise. □

**Lemma 4.3.** *Let  $L/K$  be a finite Galois extension,  $T$  a set of primes of  $K$  and  $A$  a finite  $G_{L/K}$ -module. Let  $i > 0$ . Assume that  $T$  is  $p$ -stable for  $L/K$  with  $p$ -stabilizing field  $K$ , where  $p$  is the smallest prime divisor of  $|A|$ . Then*

$$\text{III}^i(L/K, T; A) \subseteq H_*^i(L/K, A).$$

*Proof.* Since any  $p$ -stable set is  $\ell$ -stable for all  $\ell > p$ , we can assume that  $A$  is  $p$ -primary. We have to show that any cyclic  $p$ -subgroup of  $G_{L/K}$  is a decomposition subgroup of a prime in  $T$ . This is the content of the next lemma. □

**Lemma 4.4.** *Let  $L/K$  be a finite Galois extension,  $T$  a set of primes of  $K$  and  $p$  a rational prime such that  $T$  is  $p$ -stable for  $L/K$  with  $p$ -stabilizing field  $K$ . Then any cyclic  $p$ -subgroup of  $G_{L/K}$  is the decomposition group of a prime in  $T$ .*

**Remark 4.5.** (i) This lemma shows automatically that there are infinitely many primes in  $T$  for which the given cyclic group is a decomposition group.



- (ii) In some sense this lemma ‘generalizes’ Chebotarev’s density theorem, which says, in particular, that if  $S$  has density one and  $L/K$  is finite Galois, then any element of  $G_{L/K}$  is a Frobenius of a prime in  $S$ .

*Proof.* Assume that the cyclic  $p$ -subgroup  $H \subseteq G_{L/K}$  is not a decomposition group of a prime in  $T$ . Let  $pH \subseteq H$  be the subgroup of index  $p$ . Then one computes directly  $m_{pH}(\sigma) = pm_H(\sigma)$  for any  $\sigma \in pH$ . Since  $H$  is not a decomposition subgroup of a prime  $\mathfrak{p} \in T$ , no generator of  $H$  is a Frobenius at  $T$ , i.e.,  $P_{L/K}(\sigma) \cap T = \emptyset$  for any  $\sigma \in H \setminus pH$ . By  $p$ -stability of  $T$ , there is a subset  $T_0 \subseteq T$  and an  $a > 0$  such that  $pa > \delta_{L'}(T_0) \geq a$  for all  $L/L'/K$ . Let  $L_0 = L^H$  and  $L_1 = L^{pH}$ . Then, by [Lemma 2.3](#),

$$\begin{aligned} \delta_{L_0}(T_0) &= \sum_{\sigma \in H} m_H(\sigma) \delta_K(P_{L/K}(\sigma) \cap T_0) = \sum_{\sigma \in pH} m_H(\sigma) \delta_K(P_{L/K}(\sigma) \cap T_0) \\ &= p^{-1} \sum_{\sigma \in pH} m_{pH}(\sigma) \delta_K(P_{L/K}(\sigma) \cap T_0) = p^{-1} \delta_{L_1}(T_0). \end{aligned}$$

This contradicts our assumption on  $T_0$ . □

*Proof of Theorem 4.1.* We can assume that  $L = K$ . By applying [Lemma 4.2](#) to  $\mathcal{L}/K(A)/K$  and using [Lemma 4.3](#), we are reduced to showing that if  $A$  is a trivial  $G$ -module, then  $\text{III}^1(\mathcal{L}/K, T; A) = 0$ . Let  $T_0 \subseteq T$  and  $a > 0$  be such that  $pa > \delta_{L'}(T_0) \geq a$  for all  $\mathcal{L}/L'/K$ . Let  $G_{\mathcal{L}/K}^T$  be the quotient of  $G_{\mathcal{L}/K}$ , corresponding to the maximal subextension of  $\mathcal{L}/K$ , which is completely split in  $T$ . We have then

$$\text{III}^1(\mathcal{L}/K, T; A) = \ker \left( \text{Hom}(G_{\mathcal{L}/K}, A) \rightarrow \prod_{\mathfrak{p} \in T} \text{Hom}(\mathcal{G}_{\mathfrak{p}}, A) \right) = \text{Hom}(G_{\mathcal{L}/K}^T, A).$$

If  $0 \neq \phi \in \text{Hom}(G_{\mathcal{L}/K}^T, A)$ , then  $M := \mathcal{L}^{\ker(\phi)}/K$  is a finite extension inside  $\mathcal{L}/K$  with Galois group  $\text{im}(\phi) \neq 0$  and completely decomposed in  $T$ , and in particular in  $T_0$ . Thus

$$pa > \delta_M(T_0) = [M : K] \delta_K(T_0 \cap \text{cs}(M/K)) = |\text{im}(\phi)| \delta_K(T_0) \geq pa,$$

since  $\delta_K(T_0) \geq a$ . This is a contradiction, and hence we obtain

$$\text{III}^1(\mathcal{L}/K, T; A) = \text{Hom}(G_{\mathcal{L}/K}^T, A) = 0. \quad \square$$

**4B. Hasse principles.** Let  $K, S, T$  be a number field and two sets of primes of  $K$ . Various conditions on  $S, T, A$  which imply the Hasse principles in cohomological dimensions 1 and 2 are considered in [\[NSW 2008, Chapter IX, §1\]](#). We prove analogous results for stable sets. Before stating them, we refer the reader to [\[NSW 2008, Definitions 9.1.5 and 9.1.7\]](#) for definitions of the special cases.

**Corollary 4.6.** *Let  $K$  be a number field, let  $T$  and  $S$  be sets of primes of  $K$ , and let  $A$  be a finite  $G_{K,S}$ -module. Assume that  $T$  is  $p$ -stable for  $K_S/K$ , where  $p$  is the*

smallest prime divisor of  $|A|$ . If  $L$  is a  $p$ -stabilizing field for  $T$  for  $K_S/K$  and if  $H_*^1(L(A)/L, A) = 0$ , then

$$\text{III}^1(K_S/L, T; A) = 0.$$

In particular:

- (i) Let  $L_0$  be a  $p$ -stabilizing field for  $T$  for  $K_S/K$  which trivializes  $A$ . Then  $\text{III}^1(K_S/L, T; A) = 0$  for any finite  $K_S/L/L_0$ .
- (ii) Assume  $S \supseteq S_\infty$  and  $n \in \mathbb{N}(S)$  with the smallest prime divisor equal  $p$ . If  $L_0$  is a  $p$ -stabilizing field for  $T$  for  $K_S/K$ , then  $\text{III}^1(K_S/L, T; \mu_n) = 0$  for any finite  $K_S/L/L_0$  such that we are not in the special case  $(L, n, T)$ . In the special case  $(L, n, T)$  we have  $\text{III}^1(K_S/L, T; \mu_n) = \mathbb{Z}/2\mathbb{Z}$ .

The same also holds if one replaces  $G_{K,S}$  by the quotient  $G_{K,S}(c)$ , where  $c$  is a full class of finite groups in the sense of [NSW 2008, Definition 3.5.2].

*Proof.* (i) The first statement follows directly from Theorem 4.1. Since  $L_0$  is a  $p$ -stabilizing field trivializing  $A$ , any finite subextension  $L$  of  $K_S/L_0$  has the same property. Hence (i) follows.

(ii) To prove (ii), we can assume  $n = p^r$ . If we are not in the special case  $(L, p^r)$ , [NSW 2008, Proposition 9.1.6] implies  $H^1(L(\mu_{p^r})/L, \mu_{p^r}) = 0$ , i.e., we are done by Theorem 4.1. Assume we are in the special case  $(L, p^r)$ . In particular, we have  $p = 2$ . Then  $H^1(L(\mu_{2^r})/L, \mu_{2^r}) = \mathbb{Z}/2\mathbb{Z}$ . Since by Theorem 4.1 we have

$$\text{III}^1(K_S/L(\mu_{2^r}), T; \mu_{2^r}) = 0,$$

we see from Lemma 4.2 that

$$\text{III}^1(K_S/L, T; \mu_{2^r}) = \text{III}^1(L(\mu_{2^r})/L, T; \mu_{2^r}).$$

Now the same argument as in the proof of [NSW 2008, Theorem 9.1.9(ii)] finishes the proof.  $\square$

We turn to  $\text{III}^2$ . For a  $G_{K,S}$ -module  $A$  such that  $|A| \in \mathbb{N}(S)$ , we denote by

$$A' := \text{Hom}(A, \mathbb{C}_{K,S}^*)$$

the dual of  $A$ . As in [NSW 2008, Corollary 9.1.10], we obtain:

**Corollary 4.7.** *Let  $K$  be a number field,  $S \supseteq S_\infty$  a set of primes of  $K$ , and  $A$  a finite  $G_{K,S}$ -module with  $|A| \in \mathbb{N}(S)$ . Assume that  $S$  is  $p$ -stable (i.e.,  $p$ -stable for  $K_S/K$ ), where  $p$  is the smallest prime divisor of  $|A|$ . Let  $L$  be a  $p$ -stabilizing field for  $S$  for  $K_S/K$  such that  $H_*^1(L(A')/L, A') = 0$ . Then*

$$\text{III}^2(K_S/L; A) = 0.$$

*In particular:*

- (i) Let  $L_0$  be a  $p$ -stabilizing field for  $S$  for  $K_S/K$  which trivializes  $A'$ . Then  $\text{III}^2(K_S/L; A) = 0$  for any finite  $K_S/L/L_0$ .
- (ii) Let  $n \in \mathbb{N}(S)$  with smallest prime divisor  $p$ . If  $L$  is a  $p$ -stabilizing field for  $S$  and we are not in the special case  $(L, n, S)$ , then  $\text{III}^2(K_S/L, \mathbb{Z}/n\mathbb{Z}) = 0$ . In the special case, we have  $\text{III}^2(K_S/L; \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ .

**Remark 4.8.** The condition  $|A| \in \mathbb{N}(S)$  is not necessary if  $A$  is trivial: we postpone the proof of this until all necessary ingredients (in particular the Grunwald–Wang theorem, Riemann’s existence theorem and  $\text{cd}_p G_{K,S} = 2$ ) are proven. See [Proposition 5.13](#).

*Proof of Corollary 4.7.* By Poitou–Tate duality [[NSW 2008](#), Theorem 8.6.7] (this is the reason why we need  $S \supseteq S_\infty$  and  $|A| \in \mathbb{N}(S)$ ) we have

$$\text{III}^2(K_S/L, A) \cong \text{III}^1(K_S/L, A')^\vee,$$

where  $X^\vee := \text{Hom}(X, \mathbb{R}/\mathbb{Z})$  is the Pontrjagin dual. An application of [Theorem 4.1](#) to  $K_S/K$ , the sets  $S = T$  and the module  $A'$  gives the desired result. Now (i) and (ii) follow from [Corollary 4.6](#).  $\square$

**4C. Finiteness of the Shafarevich–Tate group with divisible coefficients.** As a version of [Corollary 4.6\(i\)](#), we have the following proposition.

**Proposition 4.9.** *Let  $K$  be a number field,  $\mathcal{L}/K$  a Galois extension,  $p^m$  some rational prime power ( $m \geq 1$ ). Let  $T$  be a set of primes of  $K$  which is  $p^m$ -stable for  $\mathcal{L}/K$ , with  $p^m$ -stabilizing field  $L_0$ . Then*

$$|\text{III}^1(\mathcal{L}/L, T; \mathbb{Z}/p^r\mathbb{Z})| < p^m$$

for any  $r > 0$  and any finite subextension  $\mathcal{L}/L/L_0$ .

*Proof.* Let  $T_0 \subseteq T$  and  $a > 0$  be such that  $a \leq \delta_L(T_0) < p^m a$  for all finite  $\mathcal{L}/L/L_0$ . Let  $\mathcal{L}/L/L_0$  be a finite extension. Assume that  $|\text{III}^1(\mathcal{L}/L, T; \mathbb{Z}/p^r\mathbb{Z})| \geq p^m$ . Then

$$|\text{III}^1(\mathcal{L}/L, T_0; \mathbb{Z}/p^r\mathbb{Z})| \geq p^m$$

and we have

$$\text{III}^1(\mathcal{L}/L, T_0; \mathbb{Z}/p^r\mathbb{Z}) \cong \text{Hom}(G_{\mathcal{L}/L}^{T_0}(p), \mathbb{Z}/p^r\mathbb{Z}) = (G_{\mathcal{L}/L}^{T_0}(p)^{\text{ab}}/p^r)^\vee.$$

Thus,  $|\text{III}^1(\mathcal{L}/L, T_0; \mathbb{Z}/p^r\mathbb{Z})| \geq p^m$  implies  $|G_{\mathcal{L}/L}^{T_0}(p)^{\text{ab}}/p^r| \geq p^m$ . Now, if  $M/L$  is the subextension of  $\mathcal{L}/L$  corresponding to  $G_{\mathcal{L}/L}^{T_0}(p)^{\text{ab}}/p^r$ , then it has a finite subextension  $M_1$  of degree at least  $p^m$  which is completely split in  $T_0$ . Hence, we have  $\delta_{M_1}(T_0) \geq p^m \delta_L(T_0)$ , which is a contradiction to the  $p^m$ -stability of  $T_0$ .  $\square$

**Corollary 4.10.** *Let  $K$  be a number field,  $\mathcal{L}/K$  a Galois extension, and  $T$  a set of primes of  $K$  stable for  $\mathcal{L}/K$ . Then  $\text{III}^1(\mathcal{L}/K, T; \mathbb{Q}_p/\mathbb{Z}_p)$  is finite for any  $p$ . Moreover,  $\text{III}^1(\mathcal{L}/K, T; \mathbb{Q}/\mathbb{Z})$  is finite.*

*Proof.* For the first statement it is enough to show that  $|\text{III}^1(\mathcal{L}/K, T; \mathbb{Z}/p^r\mathbb{Z})|$  is uniformly bounded for  $r > 0$ . By [Proposition 2.8](#), there is some  $m \geq 1$  such that  $K$  is a  $p^m$ -stabilizing field for  $T$  for  $\mathcal{L}/K$ . Then [Proposition 4.9](#) implies  $|\text{III}^1(\mathcal{L}/K, T; \mathbb{Z}/p^r\mathbb{Z})| < p^m$ . For the last statement, we note the decomposition  $\text{III}^1(\mathcal{L}/K, T; \mathbb{Q}/\mathbb{Z}) = \bigoplus_p \text{III}^1(\mathcal{L}/K, T; \mathbb{Q}_p/\mathbb{Z}_p)$ . The proven part shows that each of the summands is finite. Moreover, almost all are zero: there is some  $\lambda > 1$  such that  $K$  is a  $\lambda$ -stabilizing field for  $T$  for  $\mathcal{L}/K$ . Thus, for any  $p \geq \lambda$ , the group  $\text{III}^1(\mathcal{L}/K, T; \mathbb{Q}_p/\mathbb{Z}_p)$  vanishes.  $\square$

**4D. Uniform bound.** For later needs (see [Section 5C](#)) we prove the following uniform bounds. The results of this section are not part of [\[Ivanov 2013\]](#).

**Proposition 4.11.** *Let  $\mathcal{M}/\mathcal{L}/K$  be Galois extensions, let  $A$  be a finite  $\mathbf{G}_{\mathcal{M}/K}$ -module and let  $S$  be stable for  $\mathcal{L}(A)/K$ . Then there is some  $C > 0$  such that*

$$|\text{III}^1(\mathcal{M}/L, S; A)| < C$$

for all finite subextensions  $\mathcal{L}/L/K$ .

*Proof.* For each  $\mathcal{L}/L/K$ , [Lemma 4.2](#) applied to  $\mathcal{M}/L(A)/L$  gives an exact sequence

$$0 \rightarrow \text{III}^1(L(A)/L, S; A) \rightarrow \text{III}^1(\mathcal{M}/L, S; A) \rightarrow \text{III}^1(\mathcal{M}/L(A), S_{L(A)}; A). \quad (4-1)$$

Now  $\text{III}^1(L(A)/L, S; A) \subseteq \text{H}^1(L(A)/L, A)$  and we have that  $\mathbf{G}_{L(A)/L}$  is a subgroup of the finite group  $\mathbf{G}_{K(A)/K}$ , thus for all  $\mathcal{L}/L/K$  we have

$$|\text{III}^1(L(A)/L, S; A)| < m := 1 + \max_{H \subseteq \mathbf{G}_{K(A)/K}} \text{H}^1(H, A).$$

As  $S$  is stable for  $\mathcal{L}(A)/K$ , by [Proposition 2.8](#) there is some  $\epsilon > 0$  such that  $\delta_N(S) > \epsilon$  for all  $\mathcal{L}(A)/N/K(A)$ . Suppose that  $|\text{III}^1(\mathcal{M}/L(A), S, A)| \geq \epsilon^{-1}$  for some  $\mathcal{L}/L/K$ . Then, exactly as in the proof of [Proposition 4.9](#), there is an extension  $M/L(A)$  of degree  $\geq \epsilon^{-1}$  which is completely split in  $S$ . We obtain

$$\delta_M(S) = [M : L(A)]\delta_{L(A)}(S) > \epsilon^{-1}\epsilon = 1,$$

which is a contradiction. Taking into account (4-1), we obtain the statement of the proposition with respect to  $C := m\epsilon^{-1}$ .  $\square$

**Corollary 4.12.** *Let  $K$  be a number field,  $S$  and  $T$  sets of primes of  $K$ , and  $n$  a natural number.*

- (i) Assume that  $K_S/\mathcal{L}/K$  is a subextension such that  $S$  is stable for  $\mathcal{L}/K$  and  $T$  has density 0. Then there is some real  $C > 0$  such that for any  $\mathcal{L}/L/K$  one has

$$|\text{III}^1(K_{S \cup T}/L, S \setminus T, \mathbb{Z}/n\mathbb{Z})| < C.$$

- (ii) Assume that  $T \supseteq (S_\infty \setminus S)$  has density 0 and that  $n \in \mathbb{C}_{K, S \cup T}^*$ . Let  $K_S/\mathcal{L}/K$  be a subextension such that  $S$  is stable for  $\mathcal{L}(\mu_n)/K$ . There is some real  $C > 0$  such that for any  $\mathcal{L}/L/K$  one has

$$|\text{III}^1(K_{S \cup T}/L, S \setminus T, \mu_n)| < C.$$

**Remark 4.13.** The case  $S$  stable for  $\mathcal{L}/K$ , but not stable for  $\mathcal{L}(\mu_p)/K$ , still remains mysterious: one can neither show such a uniform bound by the same methods nor find counterexamples. Moreover, the same kind of arguments do not even show that  $\text{III}^1(K_{S \cup T}/K, S \setminus T, \mu_p)$  must be finite.

## 5. Arithmetic applications

**5A. Overview and results.** In this section we will be interested in the applications of the Hasse principles proven in the preceding section for stable sets. In particular, we will show two versions of the Grunwald–Wang theorem for them, with varying assumptions: we will have a strong Grunwald–Wang result if we assume strong  $p$ -stability (Section 5B) and only a weaker  $\varinjlim$ -version (which is still enough for applications) after weakening the assumption to sharp  $p$ -stability (Section 5C). After this we will consider realization of local extensions, Riemann’s existence theorem and the cohomological dimension of  $G_{K,S}$ . For each of these three results there is a profinite and a pro- $p$  version respectively. We state them below and give proofs in Section 5D. Further, in Section 5E we prove a Hasse principle for  $\text{III}^2$  for constant  $p$ -primary coefficients without the assumption  $p \in \mathbb{C}_{K,S}^*$  (see Corollary 4.7 and Remark 4.8).

**Theorem 5.1.** *Let  $K$  be a number field,  $p$  a rational prime and  $T \supseteq S \supseteq R$  sets of primes of  $K$  with  $R$  finite.*

(A<sub>p</sub>) Assume  $S$  is sharply  $p$ -stable for  $K_S^R(p)/K$ . Then

$$K_S^R(p)_{\mathfrak{p}} = \begin{cases} K_{\mathfrak{p}}(p) & \text{if } \mathfrak{p} \in S \setminus R, \\ K_{\mathfrak{p}}^{\text{nr}}(p) & \text{if } \mathfrak{p} \notin S. \end{cases}$$

(A) Assume  $S$  is sharply  $p$ -stable for  $K_S^R/K$ . Then

$$K_{S,p}^R \supseteq \begin{cases} K_{\mathfrak{p}}(p) & \text{if } \mathfrak{p} \in S \setminus R, \\ K_{\mathfrak{p}}^{\text{nr}}(p) & \text{if } \mathfrak{p} \notin S. \end{cases}$$

(B<sub>p</sub>) Assume  $S$  is sharply  $p$ -stable for  $K_S^R(p)/K$ . Then the natural map

$$\phi_{T,S}^R(p) : \prod_{\mathfrak{p} \in R(K_S^R(p))} \mathcal{G}_{\mathfrak{p}}(p) * \prod_{\mathfrak{p} \in (T \setminus S)(K_S^R(p))} I_{\mathfrak{p}}(p) \xrightarrow{\sim} \mathbf{G}_{K_T(p)/K_S^R(p)}$$

is an isomorphism, where  $I_{\mathfrak{p}}(p) := \mathbf{G}_{K_{\mathfrak{p}}(p)/K_{\mathfrak{p}}^{\text{nr}}(p)} \subseteq \mathcal{G}_{\mathfrak{p}}(p) := \mathbf{G}_{K_{\mathfrak{p}}(p)/K_{\mathfrak{p}}}$ .

Let  $K'_T(p)/K_S^R$  denote the maximal pro- $p$  subextension of  $K_T/K_S^R$ .

(B) Assume  $S$  is sharply  $p$ -stable for  $K_S^R/K$ . Then the natural map

$$\phi_{T,S}^R : \prod_{\mathfrak{p} \in R(K_S^R)} \mathcal{G}_{\mathfrak{p}}(p) * \prod_{\mathfrak{p} \in (T \setminus S)(K_S^R)} I'_{\mathfrak{p}}(p) \xrightarrow{\sim} \mathbf{G}_{K'_T(p)/K_S^R}$$

is an isomorphism, where  $I'_{\mathfrak{p}}(p)$  denotes the Galois group of the maximal pro- $p$  extension of  $K_{S,\mathfrak{p}}^R$ .

Assume  $p$  is odd or  $K$  is totally imaginary.

(C<sub>p</sub>) Assume  $S$  is sharply  $p$ -stable for  $K_S^R(p)/K$ . Then

$$\text{cd } \mathbf{G}_{K,S}^R(p) = \text{scd } \mathbf{G}_{K,S}^R(p) = 2.$$

(C) Assume  $S$  is sharply  $p$ -stable for  $K_S^R/K$ . Then

$$\text{cd}_p \mathbf{G}_{K,S}^R = \text{scd}_p \mathbf{G}_{K,S}^R = 2.$$

**5B. Grunwald–Wang theorem and strong  $p$ -stability.** Consider the cokernel of the global-to-local restriction homomorphism

$$\text{coker}^i(K_S/K, T; A) := \text{coker}(\text{res}^i : H^i(K_S/K, A) \rightarrow \prod'_{\mathfrak{p} \in T} H^i(\mathcal{G}_{\mathfrak{p}}, A)),$$

where  $A$  is a finite  $\mathbf{G}_{K,S}$ -module,  $T$  is a subset of  $S$  and  $\prod'$  means that almost all classes are unramified. If  $A$  is a trivial  $\mathbf{G}_{K,S}$ -module, then the vanishing of this cokernel is equivalent to the existence of global extensions unramified outside  $S$ , which realize given local extensions at primes in  $T$ . If  $S$  has density 1, the set  $T$  is finite,  $A$  is constant and we are not in a special case, this vanishing is essentially the statement of the Grunwald–Wang theorem. Certain conditions on  $S, T, A$ , under which this cokernel vanishes are considered in [NSW 2008, Chapter IX, §2]. All of them require  $S$  to have certain minimal density. We prove analogous results for stable sets.

**Corollary 5.2.** *Let  $K$  be a number field,  $T \subseteq S$  sets of primes of  $K$  with  $S_{\infty} \subseteq S$ . Let  $A$  be a finite  $\mathbf{G}_{K,S}$ -module with  $|A| \in \mathbb{N}(S)$ . Assume that  $T$  is finite and  $S$  is  $p$ -stable, where  $p$  is the smallest prime divisor of  $|A|$ . For any  $p$ -stabilizing field  $L$  for  $S$  for  $K_S/K$  such that  $H_*^1(L(A')/L, A') = 0$ , we have*

$$\text{coker}^1(K_S/L, T; A) = 0.$$

*Proof.* Since  $T$  is finite and  $S$  is  $p$ -stable for  $K_S/K$ , we have that  $S \setminus T$  is also  $p$ -stable for  $K_S/K$ , and the  $p$ -stabilizing fields for  $S$  and  $S \setminus T$  are equal. Let  $L$  be as in the corollary. By [Theorem 4.1](#) applied to  $K_S/L$ ,  $S \setminus T$  and  $A'$ , we obtain  $\text{III}^1(K_S/L, S \setminus T; A') = 0$ . Then [\[NSW 2008, Lemma 9.2.2\]](#) implies that  $\text{coker}^1(K_S/L, T; A) = 0$ .  $\square$

Now we give a generalization of [\[NSW 2008, Theorem 9.2.7\]](#).

**Theorem 5.3.** *Let  $K$  be a number field,  $S$  a set of primes of  $K$ . Let  $T_0, T \subseteq S$  be two disjoint subsets such that  $T_0$  is finite. Let  $p$  be a rational prime and  $r > 0$  an integer. Assume  $S \setminus T$  is  $p$ -stable for  $K_{S \cup S_p \cup S_\infty}/K$  with  $p$ -stabilizing field  $L_0$ , which is contained in  $K_S$ . Then, for any finite  $K_S/L/L_0$  such that we are not in the special case  $(L, p', S \setminus (T_0 \cup T))$ , the canonical map*

$$H^1(K_S/L, \mathbb{Z}/p^r\mathbb{Z}) \rightarrow \bigoplus_{p \in T_0(L)} H^1(\mathcal{G}_p, \mathbb{Z}/p^r\mathbb{Z}) \oplus \bigoplus_{p \in T(L)} H^1(\mathcal{I}_p, \mathbb{Z}/p^r\mathbb{Z})^{\mathcal{G}_p}$$

is surjective, where  $\mathcal{I}_p \subseteq \mathcal{G}_p = \mathbf{G}_{K_p^{\text{sep}}/L_p}$  is the inertia subgroup. If we are in the special case  $(L, p', S \setminus (T_0 \cup T))$ , then  $p = 2$  and the cokernel of this map is of order 1 or 2.

*Proof.* This follows from [Corollary 4.6\(ii\)](#) in exactly the same way as [\[NSW 2008, Theorem 9.2.7\]](#) follows from [\[NSW 2008, Theorem 9.2.3\(ii\)\]](#).  $\square$

**Remarks 5.4.** (i) If  $\delta_K(T) = 0$ , the condition ‘ $S \setminus T$  is  $p$ -stable for  $K_{S \cup S_p \cup S_\infty}/K$  with a  $p$ -stabilizing field contained in  $K_S$ ’ is equivalent to ‘ $S$  is strongly  $p$ -stable’.

(ii) If  $\delta_K(S) = 1$  and  $\delta_K(T) = 0$ , then  $L_0 = K$  is a persisting field for  $S \setminus T$  for any  $\mathcal{L}/K$  and the condition in the theorem is automatically satisfied. Thus our result is a generalization of [\[NSW 2008, Theorem 9.2.7\]](#). To show that it is a proper generalization, we give the following example. Let  $N/M/K$  be finite Galois extensions of  $K$  such that  $N/K$  (and hence also  $M/K$ ) is totally ramified in a nonarchimedean prime  $\mathfrak{l}$  of  $K$ , lying over the rational prime  $\ell$ . Suppose  $\sigma \in \mathbf{G}_{M/K}$  and let  $\tilde{\sigma} \in \mathbf{G}_{N/K}$  be a preimage of  $\sigma$ . Let  $S \supseteq T$  be such that

$$S \simeq P_{M/K}(\sigma), \mathfrak{l} \notin S \text{ and } T \simeq P_{M/K}(\sigma) \setminus P_{N/K}(\tilde{\sigma}).$$

Then  $S \setminus T \simeq P_{N/K}(\tilde{\sigma})$  is persistent for  $K_{S \cup S_p \cup S_\infty}/K$  for any  $p \neq \ell$ , and, moreover,  $K$  is a persisting field (indeed, this follows from  $K_{S \cup S_p \cup S_\infty} \cap N = K$ ). Hence the sets  $S \supseteq T$  satisfy the conditions of the theorem with respect to each  $p \neq \ell$ . Observe that in this example  $T$  is itself persistent for  $K_{S \cup S_p \cup S_\infty}/K$  with persisting field  $K$ . In [\[NSW 2008, Theorem 9.2.7\]](#), the set  $T$  must have density zero.

From this we obtain the following classical form of the Grunwald–Wang theorem. The proof is the same as in [\[NSW 2008, Theorem 9.2.8\]](#).

**Corollary 5.5.** *Let  $T \subseteq S$  be sets of primes of a number field  $K$ . Let  $A$  be a finite abelian group. Assume that  $T$  is finite and that, for any prime divisor  $p$  of  $|A|$ ,  $S$  is  $p$ -stable for  $K_{S \cup S_p \cup S_\infty}/K$  with stabilizing field  $K$ . For all  $\mathfrak{p} \in T$ , let  $L_{\mathfrak{p}}/K_{\mathfrak{p}}$  be a finite abelian extension such that its Galois group can be embedded into  $A$ . Assume that we are not in the special case  $(K, \exp(A), S \setminus T)$ . Then there exists a global abelian extension  $L/K$  with Galois group  $A$ , unramified outside  $S$ , such that  $L$  has completion  $L_{\mathfrak{p}}$  at  $\mathfrak{p} \in T$ .*

**Example 5.6** (a set with persistent subset for which Grunwald–Wang stably fails). Let  $p$  be an odd prime and assume  $\mu_p \subset K$  (in particular,  $K$  is totally imaginary and we can ignore the infinite primes). Let  $S$  be a set of primes of  $K$ . For  $V = S_p \setminus S$ , let  $T \supseteq V$  be a finite set of primes of  $K$ . By [NSW 2008, Theorem 9.2.2] we have for all  $K_S/L/K$  a short exact sequence (recall that  $\mu_p \cong \mathbb{Z}/p\mathbb{Z}$  by assumption)

$$\begin{aligned} 0 \rightarrow \text{III}^1(K_{S \cup T}/L, S \cup T; \mathbb{Z}/p\mathbb{Z}) &\rightarrow \text{III}^1(K_{S \cup T}/L, S \setminus T; \mathbb{Z}/p\mathbb{Z}) \\ &\rightarrow \text{coker}^1(K_{S \cup T}/L, T; \mathbb{Z}/p\mathbb{Z})^\vee \rightarrow 0. \end{aligned}$$

Assume now that  $S$  is  $p$ -stable with  $p$ -stabilizing field  $K$ . Then

$$\text{III}^1(K_{S \cup T}/L, S \cup T; \mathbb{Z}/p\mathbb{Z}) \subseteq \text{III}^1(K_S/L, S; \mathbb{Z}/p\mathbb{Z}) = 0$$

and hence we have

$$\text{coker}^1(K_{S \cup T}/L, T; \mathbb{Z}/p\mathbb{Z}) \cong \text{III}^1(K_{S \cup T}/L, S \setminus T; \mathbb{Z}/p\mathbb{Z})^\vee.$$

We can find such a set  $S$  for which additionally  $\text{III}^1(K_{S \cup T}/L, S \setminus T; \mathbb{Z}/p\mathbb{Z}) \neq 0$  for each  $K_S/L/K$ . For an explicit example, assume  $K = \mathbb{Q}(\mu_p)$  and let  $T \supseteq S_p(K)$  be a finite set of primes of  $K$  ( $S_p(K)$  consists of exactly one prime). Let  $M/K$  be a Galois extension of degree  $p$  with  $\emptyset \neq \text{Ram}(M/K) \subseteq T$  (e.g.,  $M = \mathbb{Q}(\mu_{p^2})$ ). Define  $S := \text{cs}(M/K)$ . Then  $M \cap K_S = K$  and hence  $ML \cap K_S = L$  for each  $K_S/L/K$ . Thus  $S$  is persistent with persisting field  $K$ . Further,  $ML/L$  is a Galois extension of degree  $p$  which is completely split in  $S \setminus T$  and unramified outside  $S \cup T$ , hence the subgroup  $\text{G}_{K_{S \cup T}/ML} \subsetneq \text{G}_{K_{S \cup T}/L}$  is the kernel of a nontrivial homomorphism  $0 \neq \phi_M \in \text{III}^1(K_{S \cup T}/L, S \setminus T; \mathbb{Z}/p\mathbb{Z})$ . Hence this group is nontrivial.

Thus,  $S$  is persistent but not strongly  $p$ -stable — in particular, no  $p$ -stabilizing field for  $S \simeq S \cup T$  for  $K_{S \cup S_p \cup S_\infty}/K$  is contained in  $K_S$  — and Grunwald–Wang does not hold for  $S \cup T \supseteq T$  (i.e., the cokernel in Theorem 5.3 is nonzero). It is still unclear whether there is an example of sets  $\tilde{S} \supseteq \tilde{T}$  such that  $\tilde{S}$  is persistent but not strongly  $p$ -stable and Grunwald–Wang fails for  $\tilde{S} \supseteq \tilde{T}$ .

Finally, we have two corollaries generalizing [NSW 2008, Theorems 9.2.4 and 9.2.9] to stable sets.

**Corollary 5.7.** *Let  $K$  be a number field,  $T \subseteq S$  sets of primes of  $K$  with  $T$  finite. Let  $K_S/L/K$  be a finite Galois subextension with Galois group  $G$ . Let  $p$  be a*



prime and  $A = \mathbb{F}_p[G]^n$  a  $G_{K,S}$ -module. Assume  $S$  is  $p$ -stable for  $K_{S \cup S_p \cup S_\infty}/K$  with  $p$ -stabilizing field  $L$ . Then the restriction map

$$H^1(K_S/K, A) \rightarrow \bigoplus_{p \in T} H^1(\mathcal{G}_p, A)$$

is surjective.

*Proof.* (See [NSW 2008, Corollary 9.2.4]) We have the following commutative diagram, in which the vertical maps are Shapiro-isomorphisms:

$$\begin{array}{ccc} H^1(K_S/K, A) & \longrightarrow & \bigoplus_{p \in T} H^1(\mathcal{G}_p, A) \\ \downarrow \sim & & \downarrow \sim \\ H^1(K_S/L, \mathbb{F}_p^n) & \longrightarrow & \bigoplus_{\mathfrak{p} \in T(L)} H^1(\mathcal{G}_{\mathfrak{p}}, \mathbb{F}_p^n) \end{array}$$

The lower map is surjective by [Theorem 5.3](#), and so is the upper.  $\square$

**Corollary 5.8.** *Let  $K$  be number field,  $S$  a set of primes of  $K$ . Let  $K_S/L/K$  be a finite Galois subextension with Galois group  $G$ . Let  $p$  be a prime and  $A = \mathbb{F}_p[G]^n$  a  $G_{K,S}$ -module. Assume that  $S$  is  $p$ -stable for  $K_{S \cup S_p \cup S_\infty}/L$  with  $p$ -stabilizing field  $L$ . Then the embedding problem*

$$\begin{array}{ccccccc} & & & & G_{K,S} & & \\ & & & & \downarrow & & \\ 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

is properly solvable.

*Proof.* It follows from [Corollary 5.7](#) in the same way as [NSW 2008, Proposition 9.2.9] follows from [NSW 2008, Corollary 9.2.4].  $\square$

**5C. Grunwald–Wang cokernel in the limit and sharp  $p$ -stability.** If one is interested (motivated by [Theorem 5.1](#), we are) in the vanishing of the direct limit over  $K_S/L/K$  of the Grunwald–Wang cokernel, rather than in the vanishing of the cokernel for each  $L$ , one can use sharp  $p$ -stability instead of strong  $p$ -stability, which is considerably weaker.

**Theorem 5.9.** *Let  $K$  be a number field,  $S$  a set of primes of  $K$  and  $\mathcal{L} \subseteq K_S$  a subextension normal over  $K$  such that  $S$  is sharply  $p$ -stable for  $\mathcal{L}/K$ . Let  $T$  be a finite set of primes of  $K$  containing  $(S_p \cup S_\infty) \setminus S$ . If  $p^\infty | [\mathcal{L} : K]$ , then*

$$\varinjlim_{\mathcal{L}/L/K, \text{res}} \text{coker}^1(K_{S \cup T}/L, T, \mathbb{Z}/p\mathbb{Z}) = 0.$$

*Proof.* For any finite subextension  $\mathcal{L}/L/K$  we have the short exact sequence

$$0 \rightarrow \text{III}^1(K_{S \cup T}/L, S \cup T; \mu_p) \rightarrow \text{III}^1(K_{S \cup T}/L, S \setminus T; \mu_p) \\ \rightarrow \text{coker}^1(K_{S \cup T}/L, T; \mathbb{Z}/p\mathbb{Z})^\vee \rightarrow 0.$$

Dualizing, we see that it is enough to show that

$$\varinjlim_{\mathcal{L}/L/K, \text{cor}^\vee} \text{III}^1(K_{S \cup T}/L, S \setminus T; \mu_p)^\vee = 0.$$

For any two finite subextensions  $\mathcal{L}/L'/L/K$  we have the maps

$$\text{res}_L^{L'} : \text{III}^1(K_{S \cup T}/L, S \setminus T; \mu_p) \rightleftharpoons \text{III}^1(K_{S \cup T}/L', S \setminus T; \mu_p) : \text{cor}_L^{L'}. \quad (5-1)$$

**Lemma 5.10.** *There is a finite subextension  $\mathcal{L}/L_1/K$  such that, for all  $\mathcal{L}/L'/L/L_1$ , the map  $\text{res}_L^{L'}$  is an isomorphism.*

*Proof.* First we claim that  $\text{res}_L^{L'}$  is injective if  $L$  is big enough. Assume first that  $\mu_p \subseteq \mathcal{L}$  and that  $S$  is  $p$ -stable for  $\mathcal{L}/K$ . Let  $\mathcal{L}/L_0/K$  be a finite subextension which  $p$ -stabilizes  $S$  and contains  $\mu_p$ . Then any finite subextension  $\mathcal{L}/L/L_0$  satisfies the same. Assume  $\text{res}_L^{L'}$  is not injective, i.e., there is some nonzero  $\phi$  in  $\text{III}^1(K_{S \cup T}/L, S \setminus T; \mathbb{Z}/p\mathbb{Z})$  with  $\text{res}_L^{L'}(\phi) = 0$  (we have chosen some trivialization of  $\mu_p$ ). This  $\phi$  can be seen as a homomorphism  $\phi : G_{K_{S \cup T}/L} \rightarrow \mathbb{Z}/p\mathbb{Z}$  which is trivial on all decomposition subgroups of primes in  $S \setminus T$ . Define  $M := (K_{S \cup T})^{\ker \phi}$ . This is a finite Galois extension of  $L$  with Galois group  $\mathbb{Z}/p\mathbb{Z}$  and  $\text{cs}(M/L) \supseteq S \setminus T$ . But then

$$\delta_M(S) = [M : L]\delta_L(S \cap \text{cs}(M/L)) = p\delta_L(S),$$

since  $T$  is finite. Now  $\text{res}_L^{L'}(\phi) = 0$  implies  $M \subseteq L' \subseteq \mathcal{L}$  and hence we get a contradiction to the  $p$ -stability of  $S$ .

Now assume that  $\mu_p \not\subseteq K_S$ . Then  $\text{res}_L^{L'}$  is always injective. Indeed, suppose there is a nonzero  $x$  in

$$\text{III}^1(K_{S \cup T}/L, S \setminus T; \mu_p) \\ = \{x \in L^*/p : x \in U_{\mathfrak{p}}L_{\mathfrak{p}}^{*,p} \text{ for } \mathfrak{p} \notin S \cup T \text{ and } x \in L_{\mathfrak{p}}^{*,p} \text{ for } \mathfrak{p} \in S \setminus T\}$$

with  $\text{res}_L^{L'}(x) = 0$ . This implies  $x \in L'^{*,p}$ . Let  $y^p = x$  with  $y \in L'$ . Then  $L(y) \subseteq L' \subseteq \mathcal{L}$ . Since the polynomial  $T^p - x$  is irreducible over  $L$  (since  $x \notin L^{*,p}$ ), the conjugates of  $y$  over  $L$  are precisely the roots of this polynomial, which are clearly  $\{\zeta^i y\}_{i=0}^{p-1}$  for  $\zeta \in \mu_p(\bar{K}) \setminus \{1\}$ . Since  $\mathcal{L}$  is normal over  $L$ , these conjugates lie in  $\mathcal{L}$ . In particular, we deduce that  $\zeta \in \mathcal{L}$ , which contradicts  $\mu_p \not\subseteq \mathcal{L}$ . This finishes the proof of the injectivity claim.

By [Corollary 4.12\(ii\)](#), there is a constant  $C > 0$  such that

$$|\text{III}^1(K_{S \cup T}/L, S \setminus T, \mu_p)| < C$$

for all  $\mathcal{L}/L/K$ . Together with the injectivity shown above, this shows that there is a finite subextension  $\mathcal{L}/L_1/K$  such that, for all  $\mathcal{L}/L'/L/L_1$ , the map  $\text{res}_L^{L'}$  is bijective.  $\square$

Now we can finish the proof of [Theorem 5.9](#). Assume  $L_1$  is as in [Lemma 5.10](#). Let  $\mathcal{L}/L/L_1$ . Since  $p^\infty \mid [\mathcal{L} : K]$ , there is a further extension  $\mathcal{L}/L'/L$  such that  $p$  divides  $[L' : L]$ . In the situation of (5-1) we have  $\text{cor} \circ \text{res} = [L' : L] = 0$  since  $\mu_p$  is  $p$ -torsion. Dualizing gives  $\text{res}^\vee \circ \text{cor}^\vee = (\text{cor} \circ \text{res})^\vee = 0$ . But, along with  $\text{res}$ ,  $\text{res}^\vee$  is also an isomorphism, hence we obtain  $\text{cor}^\vee = 0$ . This shows that

$$\varinjlim_{\mathcal{L}/L/K, \text{cor}^\vee} \text{III}^1(K_{S \cup T}/L, S \setminus T; \mu_p)^\vee = 0. \quad \square$$

We have the same arguments for  $\text{III}^2$ .

**Proposition 5.11.** *Let  $K$  be a number field,  $S$  a set of primes of  $K$  and  $\mathcal{L} \subseteq K_S$  a subextension normal over  $K$  such that  $S$  is sharply  $p$ -stable for  $\mathcal{L}/K$ . Let  $T \supseteq S \cup S_p \cup S_\infty$  be a further set of primes. If  $p^\infty \mid [\mathcal{L} : K]$ , then*

$$\varinjlim_{\mathcal{L}/L/K, \text{res}} \text{III}^2(K_T/L, T; \mathbb{Z}/p\mathbb{Z}) = 0.$$

*Proof.* By Poitou–Tate duality this is equivalent to

$$\varinjlim_{\mathcal{L}/L/K, \text{cor}^\vee} \text{III}^1(K_T/L, T; \mu_p)^\vee = 0.$$

This follows in the same way as in the proof of [Theorem 5.9](#).  $\square$

**5D. Consequences.** Here we prove [Theorem 5.1](#).

**Lemma 5.12.** *Let  $S \supseteq R$  be sets of primes of  $K$ . Assume that  $R$  is finite and that  $S \cap \text{cs}(K(\mu_p)/K)$  is infinite. Then  $p^\infty \mid [K_S^R(p) : K]$ .*

*Proof.* By [\[NSW 2008, Corollary 10.7.7\]](#), for any  $C > 0$  there is some finite subset  $S_C \subseteq S \cap \text{cs}(K(\mu_p)/K)$  such that  $R \subseteq S_C$  and

$$\dim_{\mathbb{F}_p} H^1(G_{K, S_C}^R(p), \mathbb{Z}/p\mathbb{Z}) > C.$$

Since each group  $G_{K, S_C}^R(p)$  is a quotient of  $G_{K, S}^R(p)$ , the lemma follows.  $\square$

*Proof of [Theorem 5.1](#).* ( $A_p$ ), (A): Let  $\mathfrak{p}$  be a prime of  $K$  which is not contained in  $R$ . Since the local group  $\mathcal{G}_{\mathfrak{p}}(p)$  is solvable and the assumptions carry over to extensions of  $K$  in  $K_S^R(p)$ , it is enough to show that any class  $\alpha_{\mathfrak{p}} \in H^1(\mathcal{G}_{\mathfrak{p}}(p), \mathbb{Z}/p\mathbb{Z})$  (which has to be unramified if  $\mathfrak{p} \notin S$ ) is realized by a global class after a finite extension. Define  $T := \{\mathfrak{p}\} \cup R \cup S_p \cup S_\infty$  and let  $(\alpha_q) \in \prod_{q \in T} H^1(\mathcal{G}_{\mathfrak{p}}(p), \mathbb{Z}/p\mathbb{Z})$  such that  $\alpha_q$  is unramified if  $q \notin S$  and 0 if  $\mathfrak{p} \in R$ . By [Theorem 5.9](#), there is some finite extension  $K_S^R(p)/L/K$  such that  $(\alpha_q)$  comes from a global class  $\alpha \in H^1(G_{L, S \cup T}^R(p), \mathbb{Z}/p\mathbb{Z})$ .

The  $\mathbb{Z}/p\mathbb{Z}$ -extension of  $L$  corresponding to  $\alpha$  is unramified outside  $S$ , completely split in  $R$  and hence contained in  $K_S^R(p)$ . (A) has analogous proof.

(B<sub>p</sub>): The proof of this part essentially coincides with the proofs of [NSW 2008, Theorem 10.5.8] and [Ivanov 2013, Theorem 4.26]. As done there, we can restrict ourselves to the case  $T \supseteq S_p \cup S_\infty$ . All cohomology groups in the proof have  $\mathbb{Z}/p\mathbb{Z}$ -coefficients and we omit them from the notation. After computing the cohomology on the left side, by [NSW 2008, Proposition 1.6.15] we have to show that the map

$$H^i(\phi_{T,S}^R(p)) : H^i(K_T(p)/K_S^R(p)) \rightarrow \bigoplus'_{p \in R(K_S^R(p))} H^i(\mathcal{G}_p(p)) \oplus \bigoplus'_{p \in (T \setminus S)(K_S^R(p))} H^i(I_p(p))$$

induced by  $\phi_{T,S}^R(p)$  in the cohomology is bijective for  $i = 1$  and injective for  $i = 2$ . (Here  $\bigoplus'$  means the restricted direct sum in the sense of [NSW 2008, Definition 4.3.13].) Now  $H^1(\phi_{T,S}^R(p))$  is injective since  $\phi_{T,S}^R(p)$  is clearly surjective. To show surjectivity for  $i = 1$ , consider, for any finite subset  $T_1 \subseteq T \setminus S$  which contains  $(S_p \cup S_\infty) \setminus S$  and any finite  $K_S^R(p)/L/K$ , the composed maps

$$H^1(K_{S \cup T_1}(p)/L) \rightarrow \bigoplus_{p \in (R \cup T_1)(L)} H^1(\mathcal{G}_p) \twoheadrightarrow \bigoplus_{p \in R(L)} H^1(\mathcal{G}_p) \oplus \bigoplus_{p \in T_1(L)} H^1(\mathcal{J}_p)^{\mathcal{G}_p},$$

where  $\mathcal{J}_p = I_{\overline{K}_p/L_p} \subseteq G_{\overline{K}_p/L_p} = \mathcal{G}_p$  is the inertia subgroup. Passing to the direct limit over  $K_S^R(p)/L/K$ , we obtain by Theorem 5.9 the surjection

$$H^1(K_{S \cup T_1}(p)/K_S^R(p)) \twoheadrightarrow \bigoplus'_{p \in R(K_S^R(p))} H^1(\mathcal{G}_p(p)) \oplus \bigoplus'_{p \in T_1(K_S^R(p))} H^1(I_{\overline{K}_p/K_p})^{G_{\overline{K}_p/K_S^R(p)}},$$

which is, after passing to the direct limit over all finite  $T_1 \subseteq T \setminus S$ , exactly  $H^1(\phi_{T,S}^R(p))$ , since by (A<sub>p</sub>) we have  $K_S^R(p)_p = K_p^{\text{nr}}(p)$  for  $p \in T \setminus S$  and hence

$$H^1(I_{\overline{K}_p/K_p})^{G_{\overline{K}_p/K_S^R(p)}} = H^1(I_p(p))$$

(see the proofs of [NSW 2008, Theorem 10.5.8] and [Ivanov 2013, Theorem 4.26]). Finally, the injectivity of  $H^2(\phi_{T,S}^R(p))$  follows by passing to the limit and using Proposition 5.11.

(B): By Lemma 2.7, there is some  $K_S^R/L_0/K$  such that, for all  $K_S^R/L/L_0$ , the set  $S$  is sharply  $p$ -stable for  $L_S^R(p)/L$ . Thus (B) follows from (B<sub>p</sub>) as we have

$$I'_p(p) = \varprojlim_{K_S^R/L/K} I_{L_p(p)/L_p} \quad \text{and} \quad G_{K'_T(p)/K_S^R} = \varprojlim_{K_S^R/L/K} G_{L_T(p)/L_S^R(p)}.$$

(C<sub>p</sub>), (C): The proof essentially coincides with the proofs of [NSW 2008, Theorem 10.5.10 and Corollary 10.5.11] and [Ivanov 2013, Theorem 4.31, Corollary 4.33]. To avoid many repetitions, we only recall the argument for  $\text{cd } G_{K,S}^R(p) \leq 2$  in the case  $R = \emptyset$  (which differs in one aspect from the cited proofs). Therefore, set

$V = (S_p \cup S_\infty) \setminus S$  and consider the Hochschild–Serre spectral sequence  $(E_n^{ij}, \delta_n^{ij})$  for the Galois groups of the global extensions  $K_{S \cup V}(p)/K_S(p)/K$ . By [NSW 2008, Proposition 8.3.18 and Corollary 10.4.8], we have

$$\text{cd } G_{K, S \cup V}(p) \leq \text{cd}_p G_{K, S \cup V} \leq 2.$$

By Riemann’s existence theorem,  $(B_p)$ , the group  $G_{K_{S \cup V}(p)/K_S(p)}$  is a free pro- $p$  group. Hence  $E_n^{ij}$  degenerates in the second tableau and, in particular, we have (omitting  $\mathbb{Z}/p\mathbb{Z}$ -coefficients from the notation)

$$\text{coker}(\delta_2^{11}) = E_3^{30} = E_\infty^{30} \subseteq H^3(G_{K, S \cup V}(p)) = 0,$$

i.e.,  $\delta_2^{11}$  is surjective. Again by Riemann’s existence theorem we have

$$H^1(K_{S \cup V}(p)/K_S(p)) \cong \bigoplus_{p \in V} \text{Ind}_{D_p, K_S(p)/K}^{G_{K, S}(p)} H^1(I_p(p)).$$

This and Shapiro’s lemma imply

$$E_2^{11} = \bigoplus_{p \in V} H^2(K_p(p)/K_p). \quad (5-2)$$

Further, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} & & \bigoplus_{p \in S} H^2(K_p(p)/K_p) & \twoheadrightarrow & H^0(K_{S \cup V}/K, \mu_p)^\vee \\ & & \downarrow \hookrightarrow & & \downarrow = \\ H^2(K_{S \cup V}(p)/K) & \longrightarrow & \bigoplus_{p \in S \cup V} H^2(K_p(p)/K_p) & \twoheadrightarrow & H^0(K_{S \cup V}/K, \mu_p)^\vee \\ \downarrow & & \downarrow & & \downarrow \\ H^1(K_S(p)/K, H^1(K_{S \cup V}(p)/K_S(p))) & \xrightarrow{\sim} & \bigoplus_{p \in V} H^2(K_p(p)/K_p) & \longrightarrow & 0 \\ \downarrow \delta_2^{11} & & & & \\ H^3(K_S(p)/K) & & & & \end{array}$$

The second row comes from the Poitou–Tate long exact sequence. The first map in the third row is the isomorphism (5-2). The map in the first row is surjective since its dual map  $\mu_p(K) \rightarrow \bigoplus_{p \in S} \mu_p(K_p)$  is injective. Now (in contrast to proofs cited from [NSW 2008] and [Ivanov 2013]) the first map in the second row is not necessarily injective, but one can simply replace the first entry in the second row by  $H^2(K_{S \cup V}(p)/K)/\text{III}^2(K_{S \cup V}/K, S \cup V; \mathbb{Z}/p\mathbb{Z})$ , as both maps in the diagram which start at this entry factor through this quotient. Now apply the snake lemma to the second and third row and obtain  $H^3(K_S(p)/K) = 0$  and hence also  $\text{cd } G_{K, S}(p) \leq 2$  by [NSW 2008, Proposition 3.3.2].  $\square$

**5E. Vanishing of  $\text{III}^2(G_S; \mathbb{Z}/p\mathbb{Z})$  without  $p \in \mathbb{C}_{K,S}^*$ .** We generalize [Corollary 4.7](#) for the constant module. The proof makes use of [Theorem 5.1](#) parts (A), (B), (C) along with the result of Neumann showing the vanishing of certain cohomology groups. Its special case  $\delta_K(S) = 1$  is not contained in [\[NSW 2008\]](#).

**Proposition 5.13.** *Let  $K$  be a number field,  $S$  a set of primes of  $K$ . Let  $p$  be a rational prime,  $r > 0$  an integer. Assume that either  $p$  is odd or  $K_S$  is totally imaginary. Then the following hold:*

- (i) [\[Ivanov 2013, Proposition 4.34\]](#) *Assume  $S$  is strongly  $p$ -stable and  $L_0$  is a  $p$ -stabilizing field for  $S$  for  $K_{S \cup S_p \cup S_\infty}/K$ . Assume  $p$  is odd or  $L_0$  is totally imaginary. Then*

$$\text{III}^2(K_S/L; \mathbb{Z}/p^r\mathbb{Z}) = 0$$

*for any finite  $K_S/L/L_0$  such that we are not in the special case  $(L, p^r, S)$ .*

- (ii) *Let  $K_S/\mathcal{L}/K$  be a normal subextension. Assume that  $S$  is sharply  $p$ -stable for  $\mathcal{L}/K$  and  $p^\infty \nmid [\mathcal{L} : K]$ . Then*

$$\varinjlim_{\mathcal{L}/L/K} \text{III}^2(K_S/L; \mathbb{Z}/p^r\mathbb{Z}) = 0.$$

*Proof.* Define  $V := (S_p \cup S_\infty) \setminus S$ . We write  $H^*(\cdot)$  instead of  $H^*(\cdot, \mathbb{Z}/p^r\mathbb{Z})$  and  $\text{III}^*(\cdot, \cdot)$  instead of  $\text{III}^*(\cdot, \cdot; \mathbb{Z}/p^r\mathbb{Z})$ . Let  $K'_{S \cup V}(p)$  be the maximal pro- $p$  subextension of  $K_{S \cup V}/K_S$ . Let  $K_S/L/K$  be a finite subextension and consider the tower of extensions

$$\begin{array}{c}
 \left. \begin{array}{c} K_{S \cup V} \\ \left| \begin{array}{c} N \\ K'_{S \cup V}(p) \\ \left| \begin{array}{c} H \\ K_S \\ \left| \begin{array}{c} G_{L,S} \\ L \end{array} \right. \end{array} \right. \end{array} \right. \\
 \left. \right\} G'_{L,S \cup V}(p)
 \end{array}$$

with  $N := G_{K_{S \cup V}/K'_{S \cup V}(p)}$ ,  $H := G_{K'_{S \cup V}(p)/K_S}$  and  $G'_{L,S \cup V}(p) := G_{K'_{S \cup V}(p)/L}$ . We claim that for any such  $L$  we have under the assumptions of (i) the natural isomorphisms

$$\begin{aligned}
 \text{III}^2(K'_{S \cup V}(p)/L, S \cup V) &= \text{III}^2(K_{S \cup V}/L, S \cup V) \quad \text{for any } K_S/L/K, \\
 \text{III}^2(K_S/L, S) &= \text{III}^2(K'_{S \cup V}(p)/L, S \cup V) \quad \text{for any } K_S/L/L_0,
 \end{aligned} \tag{5-3}$$

and under (ii) the natural isomorphism

$$\varinjlim_{\mathcal{L}/L/K} \text{III}^2(K_S/L, S) = \varinjlim_{\mathcal{L}/L/K} \text{III}^2(K'_{S \cup V}(p)/L, S \cup V). \tag{5-4}$$

Once this claim is shown, (i) follows immediately from [Corollary 4.7](#) and (ii) follows from [Proposition 5.11](#). Thus it is enough to prove the above claim. The first isomorphism in (5-3) follows immediately from the definition of  $\text{III}^2$ , once we know that the inflation map  $\text{H}^2(G'_{L,S \cup V}(p)) \rightarrow \text{H}^2(G_{L,S \cup V})$  is an isomorphism. To show this last assertion, consider the Hochschild–Serre spectral sequence

$$E_2^{ij} = \text{H}^i(G'_{L,S \cup V}(p), \text{H}^j(N)) \Rightarrow \text{H}^{i+j}(G_{L,S \cup V}).$$

A result of Neumann [[NSW 2008](#), Theorem 10.4.2] applied to  $K_{S \cup V}/K'_{S \cup V}(p)$  (the upper field is  $p$ -( $S \cup V$ )-closed, the lower is  $p$ -( $S_p \cup S_\infty$ )-closed) implies  $E_2^{ij} = 0$  for  $j > 0$ . Hence the sequence degenerates in the second tableau and

$$\text{H}^i(G'_{S \cup V}(p)) = \text{H}^i(G_{S \cup V}),$$

for  $i \geq 0$ , proving our claim. Thus we are reduced to showing that the second map in (5-3) and the map in (5-4) are isomorphisms. For  $\mathfrak{p} \in V$ , let  $K'_\mathfrak{p}(p)$  denote the maximal pro- $p$  extension of  $K_{S,\mathfrak{p}}$ . Define

$$I'_\mathfrak{p}(p) := G_{K'_\mathfrak{p}(p)/K_{S,\mathfrak{p}}}.$$

(Observe that if  $\mathfrak{p} \in S_\infty$ , then  $I'_\mathfrak{p}(p) = 1$ . Indeed, if  $p > 2$ , this is always the case, and if  $p = 2$ , then  $K_{S,\mathfrak{p}} = \mathbb{C}$  using the assumption that  $K_S$  is totally imaginary.) By [[Ivanov 2013](#), Lemma 4.23] (which was only shown there under strong  $p$ -stability assumption on  $S$ , but due to [Theorem 5.1\(A\)](#) it also holds under sharp  $p$ -stability assumption with exactly the same proof), we have  $I'_\mathfrak{p}(p) = D_{\mathfrak{p},K'_{S \cup V}(p)/K_S}$ . Next, by Riemann's existence theorem, [Theorem 5.1\(B\)](#), applied to  $K'_{S \cup V}(p)/K_S/K$ , we have

$$H \cong \bigast_{\mathfrak{p} \in V(K_S)} I'_\mathfrak{p}(p).$$

By [[Ivanov 2013](#), Corollary 4.24], the groups  $I'_\mathfrak{p}(p)$  are free pro- $p$  groups, and hence  $H$  is a free pro- $p$  group. Thus  $\text{cd}_p H \leq 1$ . Consider the exact sequence

$$1 \rightarrow H \rightarrow G'_{L,S \cup V}(p) \rightarrow G_{L,S} \rightarrow 1$$

and the corresponding Hochschild–Serre spectral sequence

$$E_2^{ij} = \text{H}^i(G_{L,S}, \text{H}^j(H)) \Rightarrow \text{H}^{i+j}(G'_{L,S \cup V}(p)).$$

Since by [Theorem 5.1\(C\)](#) we know that  $\text{cd}_p G_{L,S} = 2$ , we have  $E_2^{ij} = 0$  if  $i > 2$  or  $j > 1$ . Moreover, we have

$$\text{H}^1(H) = \bigoplus_{V(K_S)}' \text{H}^1(I'_\mathfrak{p}(p)) = \bigoplus_{V(L)} \text{Ind}_{D_{\mathfrak{p},K_S/L}}^{G_{L,S}} \text{H}^1(I'_\mathfrak{p}(p))$$

as  $G_{L,S}$ -modules, where  $D_{\mathfrak{p},K_S/L} \subseteq G_{L,S}$  is the decomposition group at  $\mathfrak{p}$ , which is in particular procyclic and has an infinite  $p$ -Sylow subgroup (by [Theorem 5.1\(A\)](#)). Using this, an easy computation involving Frobenius reciprocity, Shapiro's lemma

and [Ivanov 2013, Lemma 4.24] allows us to compute the terms  $E_2^{01}$  and  $E_2^{11}$ . We obtain the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^1(\mathrm{G}_{L,S}) \longrightarrow \mathrm{H}^1(\mathrm{G}'_{L,S\cup V}(p)) \longrightarrow \bigoplus_{V(L)} \mathrm{H}^1(I'_p(p))^{D_{p,K_S/L}} \longrightarrow \\ &\xrightarrow{\delta} \mathrm{H}^2(\mathrm{G}_{L,S}) \longrightarrow \mathrm{H}^2(\mathrm{G}'_{L,S\cup V}(p)) \xrightarrow{d} \bigoplus_{V(L)} \mathrm{H}^2(\mathcal{G}_p) \longrightarrow 0, \end{aligned}$$

where  $\delta := \delta_2^{01} : E_2^{01} \rightarrow E_2^{20}$  denotes the differential in the second tableau. Assume first that we are in the situation of (i) and let  $L$  be as introduced there. Then we have the surjections

$$\mathrm{H}^1(\mathrm{G}'_{L,S\cup V}(p)) \twoheadrightarrow \bigoplus_{p \in V(L)} \mathrm{H}^1(\mathcal{G}_p) = \bigoplus_{p \in V(L)} \mathrm{H}^1(D_{p,K'_{S\cup V}(p)/L}) \twoheadrightarrow \bigoplus_{V(L)} \mathrm{H}^1(I'_p(p))^{D_{p,K_S/L}}.$$

The first map is surjective by Grunwald–Wang (Theorem 5.3), and the second and the third maps follow from [Ivanov 2013, Lemma 4.24]. Hence the map preceding  $\delta$  is surjective and hence  $\delta = 0$ . Thus the lower row of the above 6-term exact sequence gives the short exact sequence

$$0 \longrightarrow \mathrm{III}^2(K_S/L, S) \longrightarrow \mathrm{III}^2(K'_{S\cup V}(p)/L, S) \xrightarrow{d} \bigoplus_{V(L)} \mathrm{H}^2(\mathcal{G}_p).$$

On the other side, by definition of  $\mathrm{III}^2$ , we have that the kernel of  $d$  is precisely  $\mathrm{III}^2(K'_{S\cup V}(p)/K, S \cup V)$ , which shows the second equality in (5-3). The equality in (5-4) follows from the assumptions in (ii) by the same arguments after taking  $\varinjlim$  over  $\mathcal{L}/L/K$  (and using Theorem 5.9 instead of Theorem 5.3).  $\square$

## 6. $K(\pi, 1)$ -property

Assume that either  $p$  is odd or  $K$  is totally imaginary, and let  $X = \mathrm{Spec} \mathbb{O}_{K,S}$ . While it is well known that  $X$  is a  $K(\pi, 1)$  for  $p$  if either  $S \supseteq S_p \cup S_\infty$  ('wild case') or  $\delta_K(S) = 1$ , it is a challenging problem to determine whether  $X$  is a  $K(\pi, 1)$  if  $S$  is finite and does not necessarily contain  $S_p \cup S_\infty$ . Until recently there were no nontrivial examples of  $(K, S)$  such that  $X$  is a  $K(\pi, 1)$  for  $p$  or a pro- $p$   $K(\pi, 1)$  and, say,  $S \cap S_p = \emptyset$ . Recent results of Schmidt [2007; 2009; 2010] show that any point of  $\mathrm{Spec} \mathbb{O}_K$  has a basis for Zariski-topology consisting of pro- $p$   $K(\pi, 1)$ -schemes. More precisely, given  $K$ , a finite set  $S$  of primes of  $K$ , a rational prime  $p$  and any set  $T$  of primes of  $K$  of density 1, Schmidt showed that one can find a finite subset  $T_1 \subseteq T$  such that  $X \setminus T_1$  is pro- $p$   $K(\pi, 1)$ . The main ingredient in the proof is the theory of mild pro- $p$  groups, developed by Labute. We conjecture that one can replace the condition  $\delta_K(T) = 1$  in Schmidt's work by the weaker condition that  $T$  is strongly  $p$ -stable (or even that  $T$  is sharply  $p$ -stable for  $K_T(p)/K$ ).



In the present section we enlarge the set of the examples of such pairs  $(K, S)$  for which  $X$  is a  $K(\pi, 1)$  for  $p$  and prove essentially that if  $S$  is sharply  $p$ -stable, then  $X$  is a  $K(\pi, 1)$  for  $p$ . In particular, if  $S$  is a stable almost Chebotarev set with  $S_\infty \subseteq S$ , then  $X$  is a  $K(\pi, 1)$  for almost all primes  $p$  (see [Proposition 3.8](#) and [Example 3.10](#)), and if  $E^{\text{sharp}}(S) = \emptyset$  and  $K$  is totally imaginary, then  $X$  is a  $K(\pi, 1)$ .

**6A. Generalities on the  $K(\pi, 1)$ -property.** There are many equivalent ways to characterize the  $K(\pi, 1)$ -property of schemes (see [\[Stix 2002, Appendix A\]](#), where they are discussed in detail). Without repeating all of it, we want to introduce a small refinement of terminology which is better adapted to formulating our results.

Let  $X$  be a connected scheme,  $X_{\text{ét}}$  the étale site on  $X$ . Fix a geometric point  $\bar{x} \in X$  and let  $\pi := \pi_1(X, \bar{x})$  be the étale fundamental group of  $X$ . Let  $\mathcal{B}\pi$  denote the site of continuous  $\pi$ -sets endowed with the canonical topology. Further, let  $p$  be a rational prime and let  $\mathcal{B}\pi^p$  denote the site of continuous  $\pi^{(p)}$ -sets, where  $\pi^{(p)}$  is the pro- $p$  completion of  $\pi$ . As in [\[Stix 2002, Appendix A.1\]](#), we have natural continuous maps of sites:

$$\begin{array}{ccc} X_{\text{ét}} & \xrightarrow{\gamma} & \mathcal{B}\pi \\ & \searrow \gamma_p & \downarrow \\ & & \mathcal{B}\pi^p \end{array}$$

For a site  $Y$ , let  $\mathcal{S}(Y)$  denote the category of sheaves of abelian groups on  $Y$ , let  $\mathcal{S}(Y)_f$  be the subcategory of locally constant torsion sheaves, and  $\mathcal{S}(Y)_p$  the subcategory of locally constant  $p$ -primary torsion sheaves. Let  $A \in \mathcal{S}(\mathcal{B}\pi)_f$  and  $B \in \mathcal{S}(\mathcal{B}\pi^p)_p$ . Then we have the natural transformations of functors  $\text{id} \rightarrow \mathbb{R}\gamma_*\gamma^*$  and  $\text{id} \rightarrow \mathbb{R}\gamma_{p,*}\gamma_p^*$ , which induce maps in the cohomology:

$$c_A^i : H^i(\pi, A) \longrightarrow H^i(X_{\text{ét}}, \gamma^* A), \quad c_{p,B}^i : H^i(\pi^{(p)}, B) \longrightarrow H^i(X_{\text{ét}}, \gamma_p^* B).$$

Let  $\tilde{X}$  (resp.  $\tilde{X}(p)$ ) denote the universal (resp. universal pro- $p$ ) covering of  $X$ . Since

$$H^1(\tilde{X}_{\text{ét}}, A) = H^1(\tilde{X}(p)_{\text{ét}}, B) = 0$$

for each  $A, B$ , the maps  $c_A^i, c_{p,B}^i$  are isomorphisms for  $i = 0, 1$  and injective for  $i = 2$ .

**Definition 6.1.** Let  $X$  be a connected scheme.

- (i)  $X$  is a  $K(\pi, 1)$  if  $c_A^i$  is an isomorphism for all  $A \in \mathcal{S}(\mathcal{B}\pi)_f$  and  $i \geq 0$ .
- (ii)  $X$  is a  $K(\pi, 1)$  for  $p$  if  $c_A^i$  is an isomorphism for all  $A \in \mathcal{S}(\mathcal{B}\pi)_p$  and  $i \geq 0$ .
- (iii)  $X$  is a pro- $p$   $K(\pi, 1)$  if  $c_{p,B}^i$  is an isomorphism for all  $B \in \mathcal{S}(\mathcal{B}\pi^p)_p$  and  $i \geq 0$ .

Note that we use a shift in definitions compared with [\[Schmidt 2007\]](#) or [\[Wingberg 2007\]](#): what there is called a  $K(\pi, 1)$  for  $p$ , we call here a pro- $p$   $K(\pi, 1)$ . Parts (i) and (iii) of our definition coincide with the definition of a  $K(\pi, 1)$  in [\[Stix 2002, Definition A.1.2\]](#). By decomposing any sheaf into  $p$ -primary components we obtain:

**Lemma 6.2.**  *$X$  is a  $K(\pi, 1)$  if and only if it is a  $K(\pi, 1)$  for all  $p$ .*

Now we have a criterion for being  $K(\pi, 1)$ . For a scheme  $X$ , let  $\text{Fet}_X$  (resp.  $\text{Fet}_X^{(p)}$ ) denote the category of all finite étale coverings (resp. finite étale  $p$ -coverings) of  $X$ . For a number field  $K$ , let

$$\delta_K = \begin{cases} 1 & \text{if } \mu_p \subseteq K, \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 6.3.** *Let  $K$  be a number field,  $S \supseteq S_\infty$  a set of primes of  $K$  such that either  $\delta_K = 0$  or  $S_f \neq \emptyset$ . Assume that either  $p$  is odd or  $K$  is totally imaginary. Let  $X = \text{Spec } \mathbb{O}_{K,S}$ . The following are equivalent:*

- (i)  $X$  is a  $K(\pi, 1)$  for  $p$ .
- (ii)  $\varinjlim_{Y \in \text{Fet}_X} H^2(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) = 0$ .

*The same also holds if one replaces ‘ $K(\pi, 1)$  for  $p$ ’ by ‘pro- $p$   $K(\pi, 1)$ ’ and ‘ $\text{Fet}_X$ ’ by ‘ $\text{Fet}_X^{(p)}$ ’, respectively.*

*Proof.* For the full proof, see [Ivanov 2013, Proposition 5.5]. For convenience, we sketch here the main steps. (i)  $\Rightarrow$  (ii) holds for any connected scheme and follows from [Stix 2002, Proposition A.3.1] and (ii)  $\Rightarrow$  (i) follows from the well-known criterion [Stix 2002, Proposition A.3.1] and the fact that, for every  $q > 0$  and every locally constant  $p$ -primary torsion sheaf  $A$  on  $X_{\text{ét}}$ , we have

$$\varinjlim_{Y \in \text{Fet}_X} H^q(Y_{\text{ét}}, A|_Y) = 0.$$

Since  $A$  is trivialized on some  $Y \in \text{Fet}_X$ , we can assume that  $A$  is constant. By dévissage we are reduced to the case  $A = \mathbb{Z}/p\mathbb{Z}$ . The elements of  $H^1(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z})$  can be interpreted as torsors, which kill themselves, i.e., the case  $q = 1$  follows. Further by [Artin et al. 1973, Exposé X, Proposition 6.1],  $H^q(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) = 0$  for  $q > 3$ . The case  $q = 3$  follows from Artin–Verdier duality. Finally, (ii) implies the case  $q = 2$ . The pro- $p$  case has a similar proof.  $\square$

## 6B. $K(\pi, 1)$ and sharp $p$ -stability.

**Theorem 6.4.** *Let  $K$  be a number field,  $S \supseteq S_\infty$  a set of primes of  $K$  and  $p$  a rational prime. Assume that either  $p$  is odd or  $K$  is totally imaginary. Then:*

- (i) *If  $S$  is sharply  $p$ -stable for  $K_S(p)/K$ , then  $\text{Spec } \mathbb{O}_{K,S}$  is a pro- $p$   $K(\pi, 1)$ .*
- (ii) *If  $S$  is sharply  $p$ -stable, then  $\text{Spec } \mathbb{O}_{K,S}$  is a  $K(\pi, 1)$  for  $p$ .*

**Remark 6.5.** If  $K$  is totally imaginary or in the pro- $p$  case, the assumption  $S_\infty \subseteq S$  is superfluous as  $G_S(p) = G_{S \cup S_\infty}(p)$ : if  $p > 2$ , then this is true in general and if  $p = 2$ , then this is true since we have assumed that  $K$  is totally imaginary.

**Corollary 6.6.** *Let  $K$  be a number field,  $S \supseteq S_\infty$  a stable set of primes of  $K$  such that  $E^{\text{sharp}}(S)$  is finite (in particular,  $S$  can be any stable almost Chebotarev set with  $S \supseteq S_\infty$ ). Then  $\text{Spec } \mathbb{O}_{K,S}$  is a  $K(\pi, 1)$  for almost all primes  $p$ . If  $E^{\text{sharp}}(S) = \emptyset$  and  $K$  is totally imaginary, then  $\text{Spec } \mathbb{O}_{K,S}$  is a  $K(\pi, 1)$ .*

**Example 6.7.** Let  $K$  be totally imaginary. Define  $\tilde{K} := \bigcup_p K(\mu_p)$ . Let  $M/K$  be finite Galois with  $M \cap \tilde{K} = K$  and let  $\sigma \in G_{M/K}$ . Assume that  $S \simeq P_{M/K}(\sigma)$  is stable. Then  $\text{Spec } \mathbb{O}_{K,S}$  is a  $K(\pi, 1)$ .

*Proof of Theorem 6.4.* The proof essentially coincides with that of [Ivanov 2013, Theorem 5.12]. We only prove (ii) (the pro- $p$  case (i) has a similar proof). Define  $X := \text{Spec } \mathbb{O}_{K,S}$ . As  $L$  goes through finite subextensions of  $K_S/K$ , the normalization  $Y$  of  $X$  in  $L$  goes through all finite étale connected coverings of  $X$ . Define  $V := S_p \setminus S$ . For any such  $Y$  we have a decomposition

$$Y \setminus V \hookrightarrow Y \xleftarrow{i} V$$

in an open and a closed part. Now we see that  $Y \setminus V$  is a  $K(\pi, 1)$  for  $p$  and that  $\pi_1(Y \setminus V) = G_{L, S \cup V}$ . Hence

$$c_A^i : H^i(G_{L, S \cup V}) \xrightarrow{\sim} H^i((Y \setminus V)_{\text{ét}}, A) \quad (6-1)$$

is an isomorphism for any  $i \geq 0$  and any  $p$ -primary  $G_{L, S \cup V}$ -module  $A$ . We have the Lerray spectral sequence for  $j$ :

$$E_2^{mn} = H^m(Y, R^n j_* \mathbb{Z}/p\mathbb{Z}) \Rightarrow H^{m+n}(Y \setminus V, \mathbb{Z}/p\mathbb{Z}).$$

Let us compute the terms in this spectral sequence. First of all we have

$$R^n j_* \mathbb{Z}/p\mathbb{Z} = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } n = 0, \\ \bigoplus_{\mathfrak{p} \in V} H^1(\mathcal{I}_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}) & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

where  $\mathcal{I}_{\mathfrak{p}} \subseteq \mathcal{G}_{\mathfrak{p}}$  denotes the inertia subgroup of the full local Galois group at  $\mathfrak{p}$ . Thus

$$E_2^{01} = \bigoplus_{\mathfrak{p} \in V} H^1(\mathcal{I}_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})^{\mathcal{G}_{\mathfrak{p}}^{\text{nr}}},$$

$$E_2^{11} = H^1(Y_{\text{ét}}, \bigoplus_{\mathfrak{p} \in V} H^1(\mathcal{I}_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})) = \bigoplus_{\mathfrak{p} \in V} H^2(\mathcal{G}_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}),$$

and  $E_2^{mn} = 0$  if  $n > 1$  or if  $n = 1$  and  $m > 1$  (as  $\text{cd}_p(\mathcal{G}_{\mathfrak{p}}^{\text{nr}}) = 1$ ). Further,  $E_2^{m0} = 0$  for  $m > 3$ , as  $\text{cd}_p Y \leq 3$  and  $E_2^{30} = H^3(Y, \mathbb{Z}/p\mathbb{Z}) = 0$  by [Ivanov 2013, Lemma 5.9], and

$$E_2^{10} = H^1(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) = H^1(G_{L,S}, \mathbb{Z}/p\mathbb{Z}).$$

Thus we have the following nonzero entries in the second tableau:

$$\begin{array}{ccccccc}
 \bigoplus_{\mathfrak{p} \in V} \mathbf{H}^1(\mathcal{J}_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})^{\text{gnr}_{\mathfrak{p}}} & & \bigoplus_{\mathfrak{p} \in V} \mathbf{H}^2(\mathcal{G}_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}) & & 0 & & 0 \\
 & & \delta_2^{01} & & & & \\
 & & \searrow & & & & \\
 \mathbb{Z}/p\mathbb{Z} & & \mathbf{H}^1(\mathbf{G}_{L,S}, \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \mathbf{H}^2(Y_{\text{ét}}, \mathbb{Z}/p\mathbb{Z}) & & 0
 \end{array}$$

From this and the isomorphism (6-1) we obtain the following exact sequence (from now on, we omit the  $\mathbb{Z}/p\mathbb{Z}$ -coefficients):

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbf{H}^1(\mathbf{G}_{L,S}) & \longrightarrow & \mathbf{H}^1(\mathbf{G}_{L,S \cup V}) & \longrightarrow & \bigoplus_{\mathfrak{p} \in V} \mathbf{H}^1(\mathcal{J}_{\mathfrak{p}})^{\text{gnr}_{\mathfrak{p}}} \xrightarrow{\delta_2^{01}} \\
 & & & & & & \\
 & \longrightarrow & \mathbf{H}^2(Y_{\text{ét}}) & \longrightarrow & \mathbf{H}^2(\mathbf{G}_{L,S \cup V}) & \longrightarrow & \bigoplus_{\mathfrak{p} \in V} \mathbf{H}^2(\mathcal{G}_{\mathfrak{p}}) \longrightarrow 0
 \end{array}$$

By [Proposition 6.3](#) it is enough to show that  $\varinjlim_{Y \in \text{Fet}_X} \mathbf{H}^2(Y_{\text{ét}}) = 0$ . Taking the limit over all  $Y \in \text{Fet}_X$  of this sequence, we see by [Theorem 5.9](#) that the direct limit of the maps preceding  $\delta_2^{01}$  is surjective, hence we obtain

$$\varinjlim_{Y \in \text{Fet}_X} \mathbf{H}^2(Y_{\text{ét}}) \cong \varinjlim_{Y \in \text{Fet}_X} \text{III}^2(K_{S \cup V}/L, V; \mathbb{Z}/p\mathbb{Z}).$$

To finish the proof consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 \mathbf{H}^2(\mathbf{G}_{L,S \cup V}) & \longrightarrow & \bigoplus_{\mathfrak{p} \in S \cup V} \mathbf{H}^2(\mathcal{G}_{\mathfrak{p}}) & \longrightarrow & \mu_p(L)^{\vee} & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \bigoplus_{\mathfrak{p} \in V} \mathbf{H}^2(\mathcal{G}_{\mathfrak{p}}) & \xrightarrow{=} & \bigoplus_{\mathfrak{p} \in V} \mathbf{H}^2(\mathcal{G}_{\mathfrak{p}}) & \longrightarrow & 0 \longrightarrow 0
 \end{array}$$

Here the first map in the upper row becomes injective after taking the limit by [Proposition 5.11](#). The snake lemma shows that

$$\varinjlim_{Y \in \text{Fet}_X} \mathbf{H}^2(Y_{\text{ét}}) \cong \varinjlim_{Y \in \text{Fet}_X} \text{III}^2(K_{S \cup V}/L, V; \mathbb{Z}/p\mathbb{Z}) \subseteq \varinjlim_{Y \in \text{Fet}_X} \bigoplus_{\mathfrak{p} \in S} \mathbf{H}^2(\mathcal{G}_{\mathfrak{p}}),$$

and the last limit vanishes as  $p^{\infty} | [K_{S,\mathfrak{p}} : K_{\mathfrak{p}}]$  for all  $\mathfrak{p} \in S$  by [Theorem 5.1\(A\)](#). This finishes the proof of (ii).  $\square$

### Acknowledgements

Some of the results in this article coincide with the author's Ph.D. thesis [[Ivanov 2013](#)], written under the supervision of Jakob Stix at the University of Heidelberg. The author is grateful to him for the very good supervision, and to Kay Wingberg,

Johannes Schmidt and many other people for helpful remarks and interesting discussions. The work on the author's thesis was partially supported by the Mathematics Center Heidelberg and the Mathematical Institute Heidelberg. The author thanks both institutions for their hospitality and the excellent working conditions.

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Communicated by Bjorn Poonen

Received 2014-06-23

Revised 2015-09-07

Accepted 2015-10-23

[ivanov@ma.tum.de](mailto:ivanov@ma.tum.de)

Zentrum Mathematik, Technischen Universität,  
Boltzmannstraße 3, D-85747 Garching bei München, Germany

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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ANT peer review and production are managed by EditFLOW<sup>®</sup> from MSP.

PUBLISHED BY

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Volume 10    No. 1    2016

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