

Algebra & Number Theory

Volume 10

2016

No. 10

Nonvanishing of Dirichlet L -functions

Rizwanur Khan and Hieu T. Ngo



Nonvanishing of Dirichlet L -functions

Rizwanur Khan and Hieu T. Ngo

We show that for at least $\frac{3}{8}$ of the primitive Dirichlet characters χ of large prime modulus, the central value $L(\frac{1}{2}, \chi)$ does not vanish.

1. Introduction

The zeros of L -functions on the critical line are as important in number theory as they are mysterious. At the real point on the critical line (the central point), an L -function is expected to vanish only for either a good reason or a trivial reason. A good reason is when the central value has some arithmetic significance which explains why it may vanish. For example, the central value of the L -function attached to an elliptic curve over a number field is expected to vanish if and only if the elliptic curve has positive rank (according to the Birch and Swinnerton-Dyer conjecture). A trivial reason is when the functional equation implies that the central value is zero. For instance, the L -function of any odd Hecke–Maass form f has functional equation $L(\frac{1}{2}, f) = -L(\frac{1}{2}, f)$ at the central point. In all other cases, the most extensive success in proving the nonvanishing of L -functions has been achieved through the use of mollifiers. For notable examples of the mollifier method, see [Kowalski et al. 2000a; 2000b; Iwaniec and Sarnak 2000; Soundararajan 2000] as well as the works discussed below.

In this paper, we study the classical nonvanishing problem of primitive Dirichlet L -functions. It is conjectured that $L(\frac{1}{2}, \chi) \neq 0$ for every primitive Dirichlet character χ . Consider for each odd prime p the family of L -functions

$$\{L(s, \chi) : \chi \text{ is primitive modulo } p\};$$

this family has size $p - 2$. Viewing $L(\frac{1}{2}, \chi)$ as a statistical object, we would like to understand its distribution as $p \rightarrow \infty$. One way to get a handle on the distribution is through understanding the moments of $L(\frac{1}{2}, \chi)$, but currently only moments of small order are known. Nevertheless, this is enough to make some progress in proving that a positive proportion of the family is nonvanishing.

MSC2010: 11M20.

Keywords: L -functions, Dirichlet characters, nonvanishing central value, mollifier.

Asymptotic expressions for the first and second moments of $L(\frac{1}{2}, \chi)$ are well known. By a classical result of Paley [1931], we have

$$\begin{aligned} \frac{1}{p} \sum_{\chi \bmod p}^* L(\tfrac{1}{2}, \chi) &\sim 1, \\ \frac{1}{p} \sum_{\chi \bmod p}^* |L(\tfrac{1}{2}, \chi)|^2 &\sim \log p, \end{aligned}$$

where \sum^* restricts the summation to the primitive characters. The discrepancy between the first and second moments indicates fluctuations in the sizes of the central values. Using these moments and the Cauchy–Schwarz inequality, one can only infer that at least 0% of the family is nonvanishing, since

$$\frac{1}{p} \sum_{\substack{\chi \bmod p \\ L(1/2, \chi) \neq 0}}^* 1 \geq \frac{|\frac{1}{p} \sum_{\chi \bmod p}^* L(\frac{1}{2}, \chi)|^2}{\frac{1}{p} \sum_{\chi \bmod p}^* |L(\frac{1}{2}, \chi)|^2} \gg \frac{1}{\log p}.$$

The mollifier method is used to remedy this situation. The origin of the method traces back to the works of Bohr and Landau [1914] and Selberg [1942] on zeros of the Riemann zeta function. The starting idea is to introduce a quantity $M(\chi)$, called the “mollifier”, which, on average, approximates the inverses of the supposedly nonvanishing values $L(\frac{1}{2}, \chi)$. The goal is to choose a mollifier such that the mollified first and second moments are comparable; that is,

$$\begin{aligned} \frac{1}{p} \sum_{\chi \bmod p}^* L(\tfrac{1}{2}, \chi)M(\chi) &\asymp 1, \\ \frac{1}{p} \sum_{\chi \bmod p}^* |L(\tfrac{1}{2}, \chi)M(\chi)|^2 &\asymp 1. \end{aligned}$$

From this, a positive nonvanishing proportion can be inferred:

$$\frac{1}{p} \sum_{\substack{\chi \bmod p \\ L(1/2, \chi) \neq 0}}^* 1 \geq \frac{1}{p} \sum_{\substack{\chi \bmod p \\ L(1/2, \chi)M(\chi) \neq 0}}^* 1 \geq \frac{|\frac{1}{p} \sum_{\chi \bmod p}^* L(\frac{1}{2}, \chi)M(\chi)|^2}{\frac{1}{p} \sum_{\chi \bmod p}^* |L(\frac{1}{2}, \chi)M(\chi)|^2} \gg 1. \quad (1-1)$$

Balasubramanian and Murty [1992] were the first to do this; however, their mollifier was inefficient and they obtained only a very small positive proportion of nonvanishing.

Next came the work of Iwaniec and Sarnak [1999], who introduced a systematic technique that has since served as a model for other families of L -functions. Iwaniec

and Sarnak took the mollifier

$$M(\chi) = \sum_{m \leq M} \frac{y_m \chi(m)}{m^{1/2}}, \tag{1-2}$$

where $M = p^\theta$ is the mollifier length and (y_m) is a sequence of real numbers satisfying $y_m \ll p^\epsilon$. They established the asymptotics of the mollified first and second moments for $\theta < \frac{1}{2}$ and found that the choice of coefficients which maximizes the ratio in (1-1) is essentially

$$y_m = \mu(m) \frac{\log(M/m)}{\log M}, \tag{1-3}$$

yielding a nonvanishing proportion of

$$\frac{1}{p} \sum_{\substack{\chi \pmod p \\ L(1/2, \chi) \neq 0}}^* 1 \geq \frac{\theta}{1+\theta}.$$

This can be taken as close to $\frac{1}{3}$ as possible on letting θ approach $\frac{1}{2}$. Computing the mollified moments for larger values of θ would result in a higher proportion of nonvanishing, but this appears to be very difficult to do. The problem seems to have been attempted by Bettin, Chandee, and Radziwiłł. In [Bettin et al. 2015], these authors solved the parallel problem for the Riemann zeta function, by obtaining the asymptotics as $T \rightarrow \infty$ of

$$\int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left| \sum_{m \leq M} \frac{y_m}{m^{(1/2)+it}} \right|^2 dt,$$

where $M = T^\theta$, for values of θ slightly larger than $\frac{1}{2}$. However with regard to the problem for Dirichlet L -functions, the authors remarked, “Our proof would not extend to give an asymptotic formula in this case, and additional input is needed.”

Shortly after the work of Iwaniec and Sarnak, in their study of the nonvanishing of high derivatives of Dirichlet L -functions, Michel and VanderKam [2000] used the “twisted” mollifier

$$M(\chi) = \sum_{m \leq M} \frac{y_m \chi(m)}{m^{1/2}} + \frac{\bar{\tau}_\chi}{p^{1/2}} \sum_{m \leq M} \frac{y_m \bar{\chi}(m)}{m^{1/2}}, \tag{1-4}$$

where $M = p^\theta$, y_m is as in (1-3), and τ_χ is the Gauss sum as defined in their paper. Heuristically, this is a better mimic of $L(\frac{1}{2}, \chi)^{-1}$ because the approximate functional equation of $L(\frac{1}{2}, \chi)$ essentially consists of a sum of two Dirichlet polynomials, one multiplied by a Gauss sum. A similar two-piece mollifier was first used by Soundararajan [1995] in the context of the Riemann zeta function. Michel and

VanderKam [2000] proved for $\theta < \frac{1}{4}$ a nonvanishing proportion of

$$\frac{1}{p} \sum_{\substack{\chi \pmod{p} \\ L(1/2, \chi) \neq 0}}^* 1 \geq \frac{2\theta}{1+2\theta},$$

recovering the $\frac{1}{3}$ proportion of Iwaniec and Sarnak [1999]. For this method too, computing the mollified moments for larger θ would result in a higher proportion of nonvanishing.

The nonvanishing problem was stuck at the proportion $\frac{1}{3}$ for ten years until Bui [2012] dexterously proved a nonvanishing proportion of 0.3411. His breakthrough was not to increase the length of any existing mollifier but to use an ingenious new two-piece mollifier. Bui [2012, page 1857] commented that “There are two different approaches to improve the results in this and other problems involving mollifiers. One can either extend the length of the Dirichlet polynomial or use some ‘better’ mollifiers. The former is certainly much more difficult.” We take the former, more difficult approach.

Our first idea to attack the nonvanishing problem is to increase the length of the Michel–VanderKam mollifier. This may be a somewhat unexpected avenue because previous attempts at lengthening mollifiers have, as far as we are aware, been directed at the Iwaniec–Sarnak mollifier. Our second idea is to establish an estimate for a trilinear sum of Kloosterman sums with general coefficients (Lemma 3.2). To prove this, we appeal to some work of Fouvry, Ganguly, Kowalski and Michel [Fouvry et al. 2014]. These authors proved best possible estimates for sums of products of Kloosterman sums to prime moduli by using powerful algebro-geometric methods (this work built on [Fouvry et al. 2004] and was later generalized in [Fouvry et al. 2015]). We stress that although the deepest part of our proof comes from [Fouvry et al. 2014], it is not clear how this work is related to the nonvanishing problem. We figure out this relationship.

Before stating our result, it should be said that the works [Iwaniec and Sarnak 1999; Michel and VanderKam 2000; Bui 2012] actually treat general moduli, while we are restricting to prime moduli, which is arguably the most interesting case.

Theorem 1.1. *Let $\epsilon > 0$ be arbitrary. For all primes p large enough in terms of ϵ , there are at least $(\frac{3}{8} - \epsilon)$ of the primitive Dirichlet characters $\chi \pmod{p}$ for which $L(\frac{1}{2}, \chi) \neq 0$.*

The significance of our work is that we show for the first time how to increase the length of a classical mollifier in this context. An interesting open problem that remains is to increase the length of the Iwaniec–Sarnak mollifier. Our nonvanishing proportion $\frac{3}{8}$ improves upon that of Bui for prime moduli. For general moduli, Bui’s nonvanishing proportion 0.3411 is still the best known.

Throughout the paper, we use the standard convention that ϵ denotes an arbitrarily small positive constant which may differ from one occurrence to the next, and that the implied constants in the various estimates depend on ϵ .

2. The work of Michel and VanderKam

We briefly summarize the mollifier method of Michel and VanderKam [2000], setting the ground for our further discussion.

Let the mollifier $M(\chi)$ be given by (1-4), where the mollifier length is $M = p^\theta$ and the real mollifying coefficients y_m are given by (1-3). Michel and VanderKam asymptotically evaluated the mollified first moment

$$\frac{2}{p} \sum_{\chi \bmod p}^+ L\left(\frac{1}{2}, \chi\right) M(\chi)$$

for $\theta < \frac{1}{2}$, where \sum^+ restricts the summation to the even primitive characters, of which there are about $\frac{p}{2}$. The evaluation for the odd primitive characters is entirely similar. They evaluated the mollified second moment

$$\begin{aligned} \frac{2}{p} \sum_{\chi \bmod p}^+ |L\left(\frac{1}{2}, \chi\right) M(\chi)|^2 &= \frac{4}{p} \sum_{\chi \bmod p}^+ |L\left(\frac{1}{2}, \chi\right)|^2 \left| \sum_{m \leq M} \frac{y_m \chi(m)}{m^{1/2}} \right|^2 \\ &+ \frac{4}{p} \sum_{\chi \bmod p}^+ |L\left(\frac{1}{2}, \chi\right)|^2 \frac{\tau_\chi}{p^{1/2}} \left(\sum_{m \leq M} \frac{y_m \chi(m)}{m^{1/2}} \right)^2 \end{aligned} \tag{2-1}$$

for $\theta < \frac{1}{4}$; see [Michel and VanderKam 2000, Equation (10)] for the above identity. An asymptotic for the first sum on the right-hand side of (2-1) is derived for $\theta < \frac{1}{2}$, as was done by Iwaniec and Sarnak [1999], but the second sum is more difficult and could only be handled for $\theta < \frac{1}{4}$. In the end, the main terms of the mollified moments of Michel and VanderKam yield a nonvanishing proportion of $2\theta/(1+2\theta)$, by taking $P_0(t) = t$ in [Michel and VanderKam 2000, Section 7].

Let us concentrate on the second sum on the right-hand side of (2-1). Recall the standard approximate functional equation (see for example [Michel and VanderKam 2000, Equation (3)]):

$$|L\left(\frac{1}{2}, \chi\right)|^2 = 2 \sum_{n_1, n_2 \geq 1} \frac{\chi(n_1) \bar{\chi}(n_2)}{(n_1 n_2)^{1/2}} V\left(\frac{n_1 n_2}{p}\right), \tag{2-2}$$

where

$$V(x) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma\left(\frac{s}{2} + \frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2} (\pi x)^{-s} \frac{ds}{s}.$$

By moving the line of integration, one shows that $V(x) \ll_c x^{-c}$ for any $c > 0$, whence the sum in (2-2) is essentially supported on $n_1 n_2 \leq p^{1+\epsilon}$. Therefore,

$$\begin{aligned} & \frac{4}{p} \sum_{\chi \bmod p}^+ |L(\tfrac{1}{2}, \chi)|^2 \frac{\tau_\chi}{p^{1/2}} \left(\sum_{m \leq M} \frac{y_m \chi(m)}{m^{1/2}} \right)^2 \\ &= \sum_{\substack{n_1, n_2 \geq 1 \\ m_1, m_2 \leq M}} \frac{y_{m_1} y_{m_2}}{(n_1 n_2 m_1 m_2)^{1/2}} V\left(\frac{n_1 n_2}{p}\right) \frac{4}{p} \sum_{\chi \bmod p}^+ \frac{\tau_\chi}{p^{1/2}} \chi(n_1 m_1 m_2) \bar{\chi}(n_2). \end{aligned} \quad (2-3)$$

By [Michel and VanderKam 2000, Equation (17)] or [Iwaniec and Sarnak 1999, Equation (3.4)], for $(n, p) = 1$ we have

$$\sum_{\chi \bmod p}^+ \tau_\chi \chi(n) = p \cos\left(\frac{2\pi \bar{n}}{p}\right) + O(1),$$

so that (2-3) equals

$$\frac{4}{p^{1/2}} \operatorname{Re} \sum_{\substack{n_1, n_2 \geq 1 \\ m_1, m_2 \leq M \\ (n_1 n_2 m_1 m_2, p) = 1}} \frac{y_{m_1} y_{m_2}}{(n_1 n_2 m_1 m_2)^{1/2}} V\left(\frac{n_1 n_2}{p}\right) e\left(\frac{n_2 \bar{n}_1 m_1 m_2}{p}\right) + O\left(\frac{M}{p^{1-\epsilon}}\right) \quad (2-4)$$

for any $\epsilon > 0$, where $e(x) = e^{2\pi i x}$ and \bar{n} denotes the multiplicative inverse of n modulo p for $(n, p) = 1$. The terms with $m_1 m_2 = 1$ contain a main term of (2-3); see [Michel and VanderKam 2000, Section 6]. Consider the rest of the terms in dyadic intervals. Let

$$\begin{aligned} \mathcal{B}(M_1, M_2, N_1, N_2) &= \frac{1}{(p M_1 M_2 N_1 N_2)^{1/2}} \\ &\times \sum_{\substack{n_1, n_2 \geq 1 \\ M_1 \leq m_1 \leq 2M_1 \\ M_2 \leq m_2 \leq 2M_2 \\ (n_1 n_2 m_1 m_2, p) = 1}} y_{m_1} y_{m_2} e\left(\frac{n_2 \bar{n}_1 m_1 m_2}{p}\right) V\left(\frac{n_1 n_2}{p}\right) f_1\left(\frac{n_1}{N_1}\right) f_2\left(\frac{n_2}{N_2}\right) \end{aligned} \quad (2-5)$$

for $1 \leq M_1, M_2 \leq \frac{1}{2} M$, $M_1 M_2 \geq 2$, $1 \leq N_1 N_2 \leq p^{1+\epsilon}$, and any fixed smooth functions f_1, f_2 compactly supported on the positive reals. Michel and VanderKam [2000, Equations (24) and (27)] proved the bounds

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^\epsilon \left(\frac{M^2 N_1}{p N_2}\right)^{1/2} \quad (2-6)$$

and

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^\epsilon \left(\frac{M^2 N_2}{N_1}\right)^{1/2} + \frac{M}{p^{1-\epsilon}}. \quad (2-7)$$

These bounds together yield $\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$, provided that $M \leq p^{(1/4)-\epsilon}$. Thus the contribution to (2-4) of the terms with $m_1 m_2 \geq 2$ is $O(p^{-\epsilon})$ for $\theta < \frac{1}{4}$.

In the next section we will show how to improve the bound (2-7), in the ranges where (2-6) is not useful. This together with (2-6) will imply that

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$$

for larger values of θ , thereby extending the asymptotics of Michel and VanderKam.

3. Proof of Theorem 1.1

To get the bounds (2-6) and (2-7), Michel and VanderKam obtained cancellation in only the (n_1, n_2) -sums of $\mathcal{B}(M_1, M_2, N_1, N_2)$. On the other hand, we use the (m_1, m_2) -sums to our advantage. To set this up, we first prove some estimates for averages of products of Kloosterman sums. Let

$$S(a, b; c) = \sum_{\substack{x \pmod c \\ x\bar{x} \equiv 1 \pmod c}} e\left(\frac{ax + b\bar{x}}{c}\right)$$

denote the Kloosterman sum. The following lemma is a consequence of a result from [Fouvry et al. 2014].

Lemma 3.1. *For $B < p$, we have*

$$\sum_{1 \leq b_1, b_2, b_3, b_4 \leq B} \left| \sum_{h \pmod p} S(h, \bar{b}_1; p) S(h, \bar{b}_2; p) S(h, \bar{b}_3; p) S(h, \bar{b}_4; p) \right| \ll B^4 p^{5/2} + B^2 p^3. \quad (3-1)$$

Proof. Write the left-hand side of (3-1) as

$$\sum_{b_1, b_2, b_3, b_4 \leq B} = \sum_{\substack{b_1, b_2, b_3, b_4 \leq B \\ (b_1, b_2, b_3, b_4) \in \mathfrak{D}}} + \sum_{\substack{b_1, b_2, b_3, b_4 \leq B \\ (b_1, b_2, b_3, b_4) \notin \mathfrak{D}}}$$

where \mathfrak{D} is the set of tuples (b_1, b_2, b_3, b_4) such that no component b_i is distinct from the others. Note that $|\mathfrak{D}| \ll B^2$.

On the one hand, it follows from the Weil bound for Kloosterman sums that

$$\sum_{\substack{b_1, b_2, b_3, b_4 \leq B \\ (b_1, b_2, b_3, b_4) \in \mathfrak{D}}} \left| \sum_{h \pmod p} S(h, \bar{b}_1; p) S(h, \bar{b}_2; p) S(h, \bar{b}_3; p) S(h, \bar{b}_4; p) \right| \ll B^2 p^3.$$

On the other hand, if $(b_1, b_2, b_3, b_4) \notin \mathfrak{D}$, then in the language of [Fouvry et al. 2014, Definition 3.1], $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ is not in “mirror configuration”. Thus [Fouvry

et al. 2014, Proposition 3.2] asserts that

$$\sum_{h \pmod p} S(h, \bar{b}_1; p)S(h, \bar{b}_2; p)S(h, \bar{b}_3; p)S(h, \bar{b}_4; p) \ll p^{5/2},$$

saving a factor of $p^{1/2}$ over Weil’s bound. So

$$\sum_{\substack{b_1, b_2, b_3, b_4 \leq B \\ (b_1, b_2, b_3, b_4) \notin \mathfrak{D}}} \left| \sum_{h \pmod p} S(h, \bar{b}_1; p)S(h, \bar{b}_2; p)S(h, \bar{b}_3; p)S(h, \bar{b}_4; p) \right| \ll B^4 p^{5/2}.$$

The lemma follows. □

Let now

$$S = \sum_{\substack{1 \leq |n| \leq N \\ 1 \leq a \leq A \\ 1 \leq b \leq B}} x_n y_a z_b S(n, \bar{a}\bar{b}; p),$$

where the coefficients satisfy $x_n, y_a, z_b \ll p^\epsilon$, $y_a = 0$ for $p|a$, and $z_b = 0$ for $p|b$.

Lemma 3.2. *For $NA \leq \frac{p}{2}$ and $B < p$, we have*

$$S \ll p^\epsilon N^{3/4} A^{3/4} (Bp^{5/8} + B^{1/2} p^{3/4}).$$

Proof. On applying the Cauchy–Schwarz inequality, we infer

$$|S|^2 \ll p^\epsilon NA \sum_{\substack{|n| \leq N \\ a \leq A}} \left| \sum_{b \leq B} z_b S(n\bar{a}, \bar{b}; p) \right|^2.$$

Hence

$$|S|^2 \ll p^\epsilon NA \sum_{h \pmod p} v(h) \left| \sum_{b \leq B} z_b S(h, \bar{b}; p) \right|^2, \tag{3-2}$$

where

$$v(h) = \sum_{\substack{|n| \leq N \\ a \leq A \\ n\bar{a} \equiv h \pmod p}} 1.$$

On applying Cauchy–Schwarz to (3-2), we find that

$$|S|^4 \ll p^\epsilon N^2 A^2 \left(\sum_{h \pmod p} v(h)^2 \right) \left(\sum_{h \pmod p} \left| \sum_{b \leq B} z_b S(h, \bar{b}; p) \right|^4 \right). \tag{3-3}$$

Observe that

$$\sum_{h \pmod p} v(h)^2 = \sum_{\substack{|n_1|, |n_2| \leq N \\ a_1, a_2 \leq A \\ n_1 \bar{a}_1 \equiv n_2 \bar{a}_2 \pmod p}} 1 = \sum_{\substack{|n_1|, |n_2| \leq N \\ a_1, a_2 \leq A \\ n_1 a_2 \equiv n_2 a_1 \pmod p}} 1.$$

Since $NA \leq \frac{p}{2}$ by assumption, it follows that

$$\sum_{h \pmod p} v(h)^2 = \sum_{\substack{n_1 a_2 = n_2 a_1 \\ |n_1|, |n_2| \leq N \\ a_1, a_2 \leq A}} 1 \ll p^\epsilon NA.$$

Therefore, (3-3) becomes

$$|S|^4 \ll p^\epsilon N^3 A^3 \sum_{b_1, b_2, b_3, b_4 \leq B} \left| \sum_{h \pmod p} S(h, \bar{b}_1; p) S(h, \bar{b}_2; p) S(h, \bar{b}_3; p) S(h, \bar{b}_4; p) \right|.$$

Finally, we apply Lemma 3.1 to conclude that

$$|S|^4 \ll p^\epsilon N^3 A^3 (B^4 p^{5/2} + B^2 p^3),$$

and the lemma is proved. □

We are now in a position to prove a new bound for our nonvanishing problem.

Lemma 3.3. *For $N_1/N_2 > p^\epsilon M$ and $M < p^{1-\epsilon}$, we have*

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^\epsilon \left(\frac{N_2 M^3}{N_1 p^3} \right)^{1/4} \left(p^{5/8} + \frac{p^{3/4}}{M^{1/2}} \right) + \frac{M}{p^{1-\epsilon}}. \tag{3-4}$$

Proof. In (2-5), separate n_1 into residue classes modulo p and apply the Poisson summation formula to get

$$\begin{aligned} & \mathcal{B}(M_1, M_2, N_1, N_2) \\ &= \frac{1}{(pM_1M_2N_1N_2)^{1/2}} \frac{N_1}{p} \sum_{\substack{k \in \mathbb{Z} \\ n_2 \geq 1, (n_2, p) = 1 \\ M_1 \leq m_1 \leq 2M_1 \\ M_2 \leq m_2 \leq 2M_2}} y_{m_1} y_{m_2} S(kn_2, \overline{m_1 m_2}; p) f_2\left(\frac{n_2}{N_2}\right) F(k), \end{aligned} \tag{3-5}$$

where

$$F(k) = \int_{-\infty}^{\infty} f_1(x) V\left(\frac{xN_1n_2}{p}\right) e\left(\frac{-xkN_1}{p}\right) dx.$$

Repeatedly integrating by parts, we find that $F(k) \ll_c (kN_1/p)^{-c}$ for any $c > 0$. Thus, the k -sum may be restricted to $|k| \leq p^{1+\epsilon}/N_1$.

The contribution to (3-5) of the terms with $k = 0$ is

$$\begin{aligned} & \frac{1}{(pM_1M_2N_1N_2)^{1/2}} \frac{N_1}{p} \sum_{\substack{n_2 \geq 1, (n_2, p) = 1 \\ M_1 \leq m_1 \leq 2M_1 \\ M_2 \leq m_2 \leq 2M_2}} y_{m_1} y_{m_2} S(0, \overline{m_1 m_2}; p) f_2\left(\frac{n_2}{N_2}\right) F(0) \\ & \ll \frac{(N_1N_2M_1M_2)^{1/2}}{p^{(3/2)-\epsilon}} \ll \frac{M}{p^{1-\epsilon}}, \end{aligned}$$

on using that the Ramanujan sum $S(0, \overline{m_1 m_2}; p)$ equals -1 . This is the last term in (3-4). The contribution of the terms with $|k| > 0$ is bounded using Lemma 3.2, by putting

$$\begin{aligned} n = kn_2, \quad x_n = f_2(n_2/N_2)F(k) \text{ if } (n_2, p) = 1, \quad x_n = 0 \text{ if } p|n_2, \quad N = N_2 p^{1+\epsilon}/N_1, \\ a = m_1, \quad y_a = y_{m_1}, \quad A = 2M_1, \\ b = m_2, \quad z_b = y_{m_2}, \quad B = 2M_2. \end{aligned}$$

Note that the conditions of Lemma 3.2, namely $B < p$ and $NA \leq \frac{p}{2}$, are satisfied by the assumptions that $M < p^{1-\epsilon}$ and that $N_1/N_2 > p^\epsilon M$. The bound (3-4) follows. \square

Finally, we sum up the work done to arrive at the following power-saving result.

Lemma 3.4. *We have $\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$ for $M < p^{(3/10)-\epsilon}$.*

Proof. Assume first that $M < p^{(1/3)-\epsilon}$. If $N_1/N_2 \leq p^\epsilon M$, it follows from (2-6) that $\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$, whence the lemma follows.

We therefore suppose that $N_1/N_2 > p^\epsilon M$. Now since the conditions of Lemma 3.3 are met, we have the bound (3-4). In this bound, we may suppose $N_2/N_1 < M^2/p^{1-\epsilon}$, since otherwise by (2-6), we have $\mathcal{B}(M_1, M_2, N_1, N_2) \ll p^{-\epsilon}$. Thus, (3-4) becomes

$$\mathcal{B}(M_1, M_2, N_1, N_2) \ll \frac{M^{5/4}}{p^{1-\epsilon}} \left(p^{5/8} + \frac{p^{3/4}}{M^{1/2}} \right) + p^{-(1/6)+\epsilon}.$$

The bound is $O(p^{-\epsilon})$ precisely when $M \ll p^{(3/10)-\epsilon}$. The lemma follows. \square

Proof of Theorem 1.1. By Lemma 3.4, the nonvanishing proportion $2\theta/(1+2\theta)$ of Michel and VanderKam is valid for any $\theta < \frac{3}{10}$. On letting θ approach $\frac{3}{10}$, we infer that the nonvanishing proportion is at least $\frac{3}{8} - \epsilon$ for any $\epsilon > 0$. \square

References

- [Balasubramanian and Murty 1992] R. Balasubramanian and V. K. Murty, “Zeros of Dirichlet L -functions”, *Ann. Sci. École Norm. Sup. (4)* **25**:5 (1992), 567–615. MR 1191737 Zbl 0771.11033
- [Bettin et al. 2015] S. Bettin, V. Chandee, and M. Radziwiłł, “The mean square of the product of the Riemann zeta-function with Dirichlet polynomials”, *J. Reine Angew. Math.* (online publication February 2015).
- [Bohr and Landau 1914] H. Bohr and E. Landau, “Sur les zéros de la fonction $\zeta(s)$ de Riemann”, *C. R. Acad. Sci., Paris* **158** (1914), 106–110. Zbl 45.0716.02
- [Bui 2012] H. M. Bui, “Non-vanishing of Dirichlet L -functions at the central point”, *Int. J. Number Theory* **8**:8 (2012), 1855–1881. MR 2978845 Zbl 1292.11093
- [Fouvry et al. 2004] É. Fouvry, P. Michel, J. Rivat, and A. Sárközy, “On the pseudorandomness of the signs of Kloosterman sums”, *J. Aust. Math. Soc.* **77**:3 (2004), 425–436. MR 2099811 Zbl 1063.11023

- [Fouvry et al. 2014] É. Fouvry, S. Ganguly, E. Kowalski, and P. Michel, “Gaussian distribution for the divisor function and Hecke eigenvalues in arithmetic progressions”, *Comment. Math. Helv.* **89**:4 (2014), 979–1014. MR 3284303 Zbl 1306.11079
- [Fouvry et al. 2015] É. Fouvry, E. Kowalski, and P. Michel, “A study in sums of products”, *Philos. Trans. A* **373**:2040 (2015), 20140309, 26. MR 3338119
- [Iwaniec and Sarnak 1999] H. Iwaniec and P. Sarnak, “Dirichlet L -functions at the central point”, pp. 941–952 in *Number theory in progress (Zakopane-Kościelisko, 1997)*, vol. 2, de Gruyter, Berlin, 1999. MR 1689553 Zbl 0929.11025
- [Iwaniec and Sarnak 2000] H. Iwaniec and P. Sarnak, “The non-vanishing of central values of automorphic L -functions and Landau–Siegel zeros”, *Israel J. Math.* **120**:1 (2000), 155–177. MR 1815374 Zbl 0992.11037
- [Kowalski et al. 2000a] E. Kowalski, P. Michel, and J. VanderKam, “Mollification of the fourth moment of automorphic L -functions and arithmetic applications”, *Invent. Math.* **142**:1 (2000), 95–151. MR 1784797 Zbl 1054.11026
- [Kowalski et al. 2000b] E. Kowalski, P. Michel, and J. VanderKam, “Non-vanishing of high derivatives of automorphic L -functions at the center of the critical strip”, *J. Reine Angew. Math.* **526** (2000), 1–34. MR 1778299 Zbl 1020.11033
- [Michel and VanderKam 2000] P. Michel and J. VanderKam, “Non-vanishing of high derivatives of Dirichlet L -functions at the central point”, *J. Number Theory* **81**:1 (2000), 130–148. MR 1743500 Zbl 1001.11032
- [Paley 1931] R. E. A. C. Paley, “On the k -analogues of some theorems in the theory of the Riemann ζ -function”, *Proc. London Math. Soc.* **S2-32**:1 (1931), 273–311. MR 1575993 Zbl 0002.01601
- [Selberg 1942] A. Selberg, “On the zeros of Riemann’s zeta-function”, *Skr. Norske Vid. Akad. Oslo I.* **1942**:10 (1942), 59. MR 0010712 Zbl 68.0161.01
- [Soundararajan 1995] K. Soundararajan, “Mean-values of the Riemann zeta-function”, *Mathematika* **42**:1 (1995), 158–174. MR 1346680 Zbl 0830.11032
- [Soundararajan 2000] K. Soundararajan, “Nonvanishing of quadratic Dirichlet L -functions at $s = \frac{1}{2}$ ”, *Ann. of Math. (2)* **152**:2 (2000), 447–488. MR 1804529 Zbl 0964.11034

Communicated by Andrew Granville

Received 2015-12-13 Revised 2016-08-18 Accepted 2016-09-17

rizwanur.khan@qatar.tamu.edu *Science Program, Texas A&M University at Qatar,
PO Box 23874, Doha, Qatar*

trunghieu.ay@gmail.com *Science Program, Texas A&M University at Qatar,
PO Box 23874, Doha, Qatar*

Algebra & Number Theory

msp.org/ant

EDITORS

MANAGING EDITOR

Bjorn Poonen
Massachusetts Institute of Technology
Cambridge, USA

EDITORIAL BOARD CHAIR

David Eisenbud
University of California
Berkeley, USA

BOARD OF EDITORS

Dave Benson	University of Aberdeen, Scotland	Susan Montgomery	University of Southern California, USA
Richard E. Borcherds	University of California, Berkeley, USA	Shigefumi Mori	RIMS, Kyoto University, Japan
John H. Coates	University of Cambridge, UK	Raman Parimala	Emory University, USA
J-L. Colliot-Thélène	CNRS, Université Paris-Sud, France	Jonathan Pila	University of Oxford, UK
Brian D. Conrad	Stanford University, USA	Anand Pillay	University of Notre Dame, USA
Hélène Esnault	Freie Universität Berlin, Germany	Victor Reiner	University of Minnesota, USA
Hubert Flenner	Ruhr-Universität, Germany	Peter Sarnak	Princeton University, USA
Sergey Fomin	University of Michigan, USA	Joseph H. Silverman	Brown University, USA
Edward Frenkel	University of California, Berkeley, USA	Michael Singer	North Carolina State University, USA
Andrew Granville	Université de Montréal, Canada	Vasudevan Srinivas	Tata Inst. of Fund. Research, India
Joseph Gubeladze	San Francisco State University, USA	J. Toby Stafford	University of Michigan, USA
Roger Heath-Brown	Oxford University, UK	Ravi Vakil	Stanford University, USA
Craig Huneke	University of Virginia, USA	Michel van den Bergh	Hasselt University, Belgium
Kiran S. Kedlaya	Univ. of California, San Diego, USA	Marie-France Vignéras	Université Paris VII, France
János Kollár	Princeton University, USA	Kei-Ichi Watanabe	Nihon University, Japan
Yuri Manin	Northwestern University, USA	Efim Zelmanov	University of California, San Diego, USA
Philippe Michel	École Polytechnique Fédérale de Lausanne	Shou-Wu Zhang	Princeton University, USA

PRODUCTION

production@msp.org
Silvio Levy, Scientific Editor

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2016 is US \$290/year for the electronic version, and \$485/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

Algebra & Number Theory

Volume 10 No. 10 2016

Weight functions on Berkovich curves	2053
MATTHEW BAKER and JOHANNES NICAISE	
Nonvanishing of Dirichlet L -functions	2081
RIZWANUR KHAN and HIEU T. NGO	
Every integer greater than 454 is the sum of at most seven positive cubes	2093
SAMIR SIKSEK	
Constructible isocrystals	2121
BERNARD LE STUM	
Canonical heights on genus-2 Jacobians	2153
JAN STEFFEN MÜLLER and MICHAEL STOLL	
Combinatorial degenerations of surfaces and Calabi–Yau threefolds	2235
BRUNO CHIARELLOTTO and CHRISTOPHER LAZDA	
The Voronoi formula and double Dirichlet series	2267
EREN MEHMET KIRAL and FAN ZHOU	
Finite dimensional Hopf actions on algebraic quantizations	2287
PAVEL ETINGOF and CHELSEA WALTON	