ERRATUM: CANONICAL HEIGHTS ON GENUS-2 JACOBIANS

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In this note, we correct a mistake in the paper

> J.S. Müller, M. Stoll: *Canonical heights on genus-2 Jacobians,* Algebra & Number Theory **10**:10 (2016), 2153–2234.

We recently found an example of a curve of genus 2 over the rational numbers, for whose Jacobian J `Magma` computed that the rational torsion subgroup is trivial, although an explicit point P of order 7 in $J(\mathbb{Q})$ is known. Closer inspection revealed that the error was caused by a height difference bound that was too small (in particular, invalid for one of the multiples of P). This, in turn, came down to a bug in the code that computes the height difference bound, which was found to be a consequence of a wrong estimate in part (c) of Corollary 9.10 in the paper. (The mistake will be corrected in the next `Magma` release.)

This Corollary gives bounds for the local height difference β in the case that the reduction $\mathcal{C}$ of the given genus 2 curve has three nodes and therefore has two geometric components. The nodes resolve into chains of $m_j - 1$ projective lines in the special fiber of the minimal regular model of the curve, where $j \in \{1,2,3\}$. The geometric component group of the Néron model has order $M = m_1 m_2 + m_2 m_3 + m_3 m_1$.

Part (c) of the corollary is about the case that Frobenius acts by swapping two of the nodes (corresponding to m_1 and m_2) and fixing the third. We say that $\mathcal{C}$ is *split* if Frobenius fixes the geometric components, and *nonsplit* otherwise. The bounds given in the paper are

$$\beta = \begin{cases} \dfrac{m_1}{M} \max\Bigl\{ \Bigl\lfloor \dfrac{m_1^2}{2} \Bigr\rfloor + m_1 m_3, \Bigl\lfloor \dfrac{m_3^2}{2} \Bigr\rfloor + m_1 \Bigl\lfloor \dfrac{m_3}{2} \Bigr\rfloor \Bigr\} & \text{if } \mathcal{C} \text{ is split,} \\ \dfrac{m_1}{2} & \text{if } \mathcal{C} \text{ is nonsplit and } m_1 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Looking at this case again in detail, we noticed that there are two mistakes in the proof of part (c). One is that the claim that "some power of Frobenius acts as negation on the component group" in the nonsplit case is incorrect (this is not true of Frobenius itself, since it swaps two of the nodes, but also not for its square, which is the identity). The other is an incorrect evaluation of μ on elements of the form $[D_i - D_j]$ in the split case. Correcting

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these mistakes results in the following bounds.

$$\beta = \begin{cases} \dfrac{1}{m_1+2m_3} \max\Big\{ \Big\lfloor \dfrac{m_1^2}{2} \Big\rfloor + m_1 m_3, \Big\lfloor \dfrac{(m_1+2m_3)^2}{8} \Big\rfloor \Big\} & \text{if } \mathcal{C} \text{ is split,} \\ \dfrac{1}{m_1} \Big\lfloor \dfrac{m_1^2}{2} \Big\rfloor & \text{otherwise.} \end{cases}$$

Note that $m_1/M = 1/(m_1+2m_3)$ when $m_1 = m_2$.

The second alternative in the split case is larger than the second alternative in the split casse as stated in the paper, and in the nonsplit case the bound is larger when m_1 is odd (this applies to the example mentioned at the beginning).

In the split case, the part of the component group that is fixed by Frobenius is cyclic of order $m_1 + 2m_3$, consisting of elements of the form

$$[B_i - C_{m_1-i}], \quad [C_0 - D_i] \qquad \text{and} \qquad [D_i - B_0]$$

with values of μ given by

$$\frac{2i(m_1-i) + m_1 m_3}{m_1+2m_3} \qquad \text{for } 0 \le i \le m_1$$

in the first case and

$$\frac{i(m_1+2(m_3-i))}{m_1+2m_3} \qquad \text{for } 0 \le i \le m_3$$

in the last two cases. Taking the maximum of these gives the bound as stated above.

In the nonsplit case, the group is cyclic of order m_1, consisting of elements of the form $[B_i - C_i]$ with values of μ given by

$$\frac{2i(m_1-i)}{m_1} \qquad \text{for } 0 \le i \le m_1.$$

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