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of modular forms modulo p**

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Generalizing the recent results of Bellaïche and Khare for the level-1 case, we study the structure of the local components of the shallow Hecke algebras (i.e., Hecke algebras without U_p and U_ℓ for all primes ℓ dividing the level N) acting on the space of modular forms modulo p for $\Gamma_0(N)$ and $\Gamma_1(N)$. We relate them to pseudodeformation rings and prove that in many cases, the local components are regular complete local algebras of dimension 2.

1. Introduction

The p -adic Hecke algebra acting on modular forms of level N of all weights, which is generated by the Hecke operators away from Np , has been well studied in the past due to its connection with p -adic families of modular forms and deformation rings of Galois representations (see [Emerton 2011] for the precise definition and more details). One can similarly define a mod p Hecke algebra acting on modular forms modulo p of level N (in the sense of Serre and Swinnerton-Dyer) of all weights. The main aim of this paper is to study the structure of these Hecke algebras acting on modular forms modulo p and their relation with suitable deformation rings in characteristic p . These objects were previously studied by Jochnowitz [1982], Khare [1998], Nicolas and Serre [2012a; 2012b], and Bellaïche and Khare [2015]. In this article, we generalize, with minor changes, the results Bellaïche and Khare [2015] proved for $p \geq 5$ and $N = 1$.

Before proceeding further, we fix some notation. In all of this paper, we fix a prime number $p > 3$ and a positive integer $N \geq 1$ not divisible by p . We shall denote by K a finite extension of \mathbb{Q}_p , by \mathcal{O} the ring of integers of K , by \mathfrak{p} the maximal ideal of \mathcal{O} , by π the generator of \mathfrak{p} and by \mathbb{F} the finite residue field of \mathcal{O} . We call $G_{\mathbb{Q}, Np}$ the Galois group of a maximal algebraic extension of \mathbb{Q} unramified outside $\{\ell \text{ s.t. } \ell | Np\} \cup \{\infty\}$ over \mathbb{Q} . For a prime q not dividing Np , we denote by $\text{Frob}_q \in G_{\mathbb{Q}, Np}$ a Frobenius element at q . We denote by c a complex conjugation in $G_{\mathbb{Q}, Np}$. We write $G_{\mathbb{Q}_\ell}$ for $\text{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ for every prime

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ℓ dividing Np . There are natural maps $i_\ell : G_{\mathbb{Q}_\ell} \rightarrow G_{\mathbb{Q}, Np}$, well defined up to conjugacy. For every prime ℓ dividing Np , we write I_ℓ for the inertia subgroup of $G_{\mathbb{Q}_\ell}$. For a representation ρ of $G_{\mathbb{Q}, Np}$, we shall denote by $\rho|_{G_{\mathbb{Q}_\ell}}$ the composition of i_ℓ with ρ : this is a representation of $G_{\mathbb{Q}_\ell}$, well defined up to an isomorphism. We denote by $\omega_p : G_{\mathbb{Q}, Np} \rightarrow \mathbb{F}^*$ the cyclotomic character modulo p .

We will now define the Hecke algebra that we want to study and the space of modular forms on which it acts. For the rest of the introduction, Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$ and $\Gamma(M)$ means either $\Gamma_1(M)$ or $\Gamma_0(M)$ accordingly. Following [Bellaïche and Khare 2015] (henceforth abbreviated as [BK]), we shall denote by $S_k^\Gamma(\mathcal{O})$ the module of cuspidal modular forms of weight k for Γ with Fourier coefficients in \mathcal{O} . We see it as a submodule of $\mathcal{O}[[q]]$ by the q -expansion. We denote by $S_{\leq k}^\Gamma(\mathcal{O})$ the submodule $\sum_{i=0}^{i=k} S_i^\Gamma(\mathcal{O})$ of $\mathcal{O}[[q]]$. Note that this sum is direct (see [BK, Section 1.2]). We denote by $S_{\leq k}^\Gamma(\mathbb{F})$ the image of $S_{\leq k}^\Gamma(\mathcal{O})$ under the reduction map $\mathcal{O}[[q]] \rightarrow \mathbb{F}[[q]]$, which reduces each coefficient of a power series in $\mathcal{O}[[q]]$ modulo \mathfrak{p} . Thus, $S_{\leq k}^\Gamma(\mathbb{F}) = \sum_{i=0}^{i=k} S_i^\Gamma(\mathbb{F})$, where $S_i^\Gamma(\mathbb{F})$ is the space of cuspidal modular forms of weight i for Γ over \mathbb{F} in the sense of Serre and Swinnerton-Dyer, i.e., it is the image of $S_i^\Gamma(\mathcal{O})$ under the reduction map considered above. The map $S_{\leq k}^\Gamma(\mathcal{O})/\mathfrak{p}S_{\leq k}^\Gamma(\mathcal{O}) \rightarrow S_{\leq k}^\Gamma(\mathbb{F})$ is surjective but not an isomorphism in general. Let

$$S^\Gamma(\mathcal{O}) = \bigcup_{k=0}^{\infty} S_{\leq k}^\Gamma(\mathcal{O}) \quad \text{and} \quad S^\Gamma(\mathbb{F}) = \bigcup_{k=0}^{\infty} S_{\leq k}^\Gamma(\mathbb{F}).$$

All the modules considered above have a natural action of the Hecke operators T_n for $(n, Np) = 1$. We denote by \mathbb{T}_k^Γ the \mathcal{O} -subalgebra of $\text{End}_{\mathcal{O}}(S_{\leq k}^\Gamma(\mathcal{O}))$ generated by the T_n 's with $(n, Np) = 1$. We denote by A_k^Γ the \mathbb{F} -subalgebra of $\text{End}_{\mathbb{F}}(S_{\leq k}^\Gamma(\mathbb{F}))$ generated by the T_n 's with $(n, Np) = 1$. From the relations between the Hecke operators, we see that \mathbb{T}_k^Γ or A_k^Γ is generated by the Hecke operators T_q and S_q for primes q not dividing Np (see [BK, Section 1.2] for more details). Here, S_q is the operator acting on forms of weight k as the multiplication by $\langle q \rangle q^{k-2}$, where $\langle q \rangle$ is the diamond operator corresponding to q .

We have a natural morphism of \mathbb{F} -algebras $\mathbb{T}_k^\Gamma/\mathfrak{p}\mathbb{T}_k^\Gamma \rightarrow A_k^\Gamma$, which is surjective, but in general not an isomorphism. We set

$$\mathbb{T}^\Gamma = \varprojlim \mathbb{T}_k^\Gamma \quad \text{and} \quad A^\Gamma = \varprojlim A_k^\Gamma.$$

Thus, the Hecke algebras \mathbb{T}^Γ and A^Γ act on $S^\Gamma(\mathcal{O})$ and $S^\Gamma(\mathbb{F})$, respectively. We obtain a surjective map $\mathbb{T}^\Gamma/\mathfrak{p}\mathbb{T}^\Gamma \rightarrow A^\Gamma$ from the surjective maps considered above. We call A^Γ the Hecke algebra modulo p of level Γ and this is the central object of our study.

The rings \mathbb{T}^Γ and A^Γ are complete and semilocal. Actually, if \mathbb{F} is large enough, then, by the existence of Galois representations attached to eigenforms and by the Deligne–Serre lifting lemma, the maximal ideals, and hence the local components of

both \mathbb{T}^Γ and A^Γ , are in bijection with the set of isomorphism classes of Γ -modular Galois representations $\bar{\rho} : G_{\mathbb{Q}, Np} \rightarrow \mathrm{GL}_2(\mathbb{F})$ (see [BK, Section 1.2] for more details). Here and below, Γ -modular means that $\bar{\rho}$ is the semisimplified reduction of a stable lattice for the Galois representation $\rho : G_{\mathbb{Q}, Np} \rightarrow \mathrm{GL}_2(K)$ attached by Deligne to an eigenform in $S^\Gamma(\mathcal{O})$. Observe that, since $\bar{\rho}$ is odd, if $\bar{\rho}$ is irreducible, then it is absolutely irreducible. We define \mathbb{T}_ρ^Γ and A_ρ^Γ to be the local components of \mathbb{T}^Γ and A^Γ corresponding to a Γ -modular representation $\bar{\rho}$. These rings are complete local rings. The surjective map $\mathbb{T}^\Gamma/\mathfrak{p}\mathbb{T}^\Gamma \rightarrow A^\Gamma$ sends $\mathbb{T}_\rho^\Gamma/\mathfrak{p}\mathbb{T}_\rho^\Gamma$ onto A_ρ^Γ . As A^Γ (resp. \mathbb{T}^Γ) is semilocal and an inverse limit of artinian rings, A^Γ (resp. \mathbb{T}^Γ) splits into the product of its local components. Thus, it is enough to study the structure of each local component to understand the structure of A^Γ , once we determine the number of its local components.

The advantage of working with the local components is that one can relate them to suitable deformation rings. By gluing pseudorepresentations attached to modular eigenforms of level Γ lifting the system of eigenvalues corresponding to a Γ -modular representation $\bar{\rho}$, one gets a pseudorepresentation of $G_{\mathbb{Q}, Np}$ taking values in A_ρ^Γ and deforming $(\mathrm{tr} \bar{\rho}, \det \bar{\rho})$. Let \tilde{R}_ρ^0 be the universal deformation ring with constant determinant of the pseudorepresentation $(\mathrm{tr} \bar{\rho}, \det \bar{\rho})$ in the category of local, profinite \mathbb{F} -algebras with residue field \mathbb{F} . The pseudorepresentation obtained above induces a local, surjective morphism $\tilde{R}_\rho^0 \rightarrow A_\rho^\Gamma$ when either $\Gamma = \Gamma_0(N)$ or $p \nmid \phi(N)$ and $\Gamma = \Gamma_1(N)$. Otherwise, it induces a local, surjective map $\tilde{R}_\rho^0 \rightarrow (A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$. See Section 2 for more details.

We define a Γ -modular representation $\bar{\rho}$ to be *unobstructed* if the tangent space of \tilde{R}_ρ^0 has dimension 2. If $\bar{\rho}$ is irreducible and $p \nmid \phi(N)$, then $\bar{\rho}$ is unobstructed in our sense if and only if it is unobstructed in the sense of Mazur [1989, Section 1.6]. But our notion is weaker than Mazur's notion if $p \mid \phi(N)$. See Section 10, where the notion of unobstructedness is studied in detail.

We prove the following results concerning the Hecke algebra A_ρ^Γ and its relation with the deformation ring \tilde{R}_ρ^0 :

Theorem 1. *Let $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$. If $\bar{\rho}$ is a Γ -modular representation, then both \tilde{R}_ρ^0 and A_ρ^Γ have Krull dimension at least 2.*

Theorem 2. *Suppose either $\Gamma = \Gamma_0(N)$ or $p \nmid \phi(N)$ and $\Gamma = \Gamma_1(N)$. If $\bar{\rho}$ is a Γ -modular representation which is unobstructed, then the morphism $\tilde{R}_\rho^0 \rightarrow A_\rho^\Gamma$ is an isomorphism, and A_ρ^Γ is isomorphic to a power series ring in two variables $\mathbb{F}[[x, y]]$. If $p \mid \phi(N)$ and $\bar{\rho}$ is an unobstructed $\Gamma_1(N)$ -modular representation, then the morphism $\tilde{R}_\rho^0 \rightarrow (A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ is an isomorphism, and $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\mathrm{red}}$ is isomorphic to a power series ring in two variables $\mathbb{F}[[x, y]]$.*

In the last section, we shall give some conditions under which a $\Gamma_1(N)$ -modular representation $\bar{\rho}$ is unobstructed and the corresponding local component $A_\rho^{\Gamma_1(N)}$

is not reduced: see [Proposition 27](#) and [Proposition 28](#). We shall also give some examples of nonreduced Hecke algebras: see the remarks and discussion after [Proposition 28](#).

Remark. (1) Our approach to prove [Theorem 1](#) and [Theorem 2](#) does not depend on whether $\bar{\rho}$ is irreducible or not. This is mainly because the results that we use from deformation theory of Galois representations are available for both reducible and irreducible $\bar{\rho}$'s. So, the proof does not become simpler when $\bar{\rho}$ is irreducible.

(2) Note that from [Theorem 2](#), it follows that if $\bar{\rho}$ is a $\Gamma_0(N)$ -modular representation which is unobstructed and if $p \nmid \phi(N)$, then the natural restriction morphism $A_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_0(N)}$ is an isomorphism. In fact, if $p \nmid \phi(N)$, then this happens for any $\Gamma_0(N)$ -modular representation $\bar{\rho}$. Indeed, fixing such a $\bar{\rho}$, we see that the corresponding system of eigenvalues for the diamond operators is trivial. As $p \nmid \phi(N)$, by Hensel's lemma, any system of eigenvalues lifting $\bar{\rho}$ is trivial for the diamond operators. Hence, the diamond operators act trivially on $S^{\Gamma_1(N)}(\mathcal{O})_{\bar{\rho}}$. Therefore, $S^{\Gamma_1(N)}(\mathcal{O})_{\bar{\rho}} = S^{\Gamma_0(N)}(\mathcal{O})_{\bar{\rho}}$. So, $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ and $\mathbb{T}_{\bar{\rho}}^{\Gamma_0(N)}$ are isomorphic. Thus, $A_{\bar{\rho}}^{\Gamma_1(N)}$ and $A_{\bar{\rho}}^{\Gamma_0(N)}$ are isomorphic for all the $\Gamma_0(N)$ -modular representations $\bar{\rho}$ if p does not divide $\phi(N)$. However, this argument breaks down if p divides $\phi(N)$.

Theorem 3. *Assume that $\bar{\rho}$ is a $\Gamma_1(N)$ -modular representation coming from a newform of level N and is absolutely irreducible after restriction to the Galois group of $\mathbb{Q}(\zeta_p)$. If $\ell \mid N$, $p \mid \ell^2 - 1$ and $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is unramified, then assume $\ell^2 \mid N$. If $\ell \mid N$, $p \mid \ell - 1$, and $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible, ramified and not a sum of two ramified characters, then assume that the highest power of ℓ dividing N is greater than the highest power of ℓ dividing the Artin conductor of $\bar{\rho}$. If $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is reducible, assume, in addition, that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is not isomorphic to $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ nor to $\chi \otimes \begin{pmatrix} 1 & * \\ 0 & \omega_p \end{pmatrix}$, where χ is any character $G_{\mathbb{Q}_p} \rightarrow \mathbb{F}^*$. Then $A_{\bar{\rho}}^{\Gamma_1(N)}$ has Krull dimension 2. Moreover, $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}}$ is isomorphic to $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$.*

Similarly, if $\bar{\rho}$ is a $\Gamma_0(N)$ -modular representation satisfying the hypotheses as above, then $A_{\bar{\rho}}^{\Gamma_0(N)}$ has Krull dimension 2. Moreover, if p does not divide $\phi(N)$, then $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}}$ is isomorphic to $(A_{\bar{\rho}}^{\Gamma_0(N)})^{\text{red}}$.

These results are generalizations of [Theorem III](#), [Theorem I](#) and [Theorem II](#) of [\[BK\]](#), respectively. Note that, it is possible to have mod p deformation rings with constant determinant which are nonreduced, but we do not know of any such examples.

Remark. If $p \nmid N$, then we know that the module of mod p modular forms for $\Gamma_1(N)$ is the same as the module of mod p modular forms for $\Gamma_1(Np^e)$ for $e \geq 1$ (see the remark after [Corollary I.3.6](#) on page 23 of [\[Gouvêa 1988\]](#) for more details).

Hence, the corresponding mod p Hecke algebras are also the same. So, even though we are assuming that $p \nmid N$, all the theorems that we prove for mod p Hecke algebras for $\Gamma_1(N)$ above will still be true without this assumption on N .

Note that, [Theorem 2](#) easily follows from [Theorem 1](#) and the definition of unobstructed.

The idea of the proof of [Theorem 1](#) is similar to the idea used in proving [Theorem III](#) in [\[BK\]](#). So, we try to find a relation between the characteristic-0 Hecke algebra $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ with the characteristic- p Hecke algebra $A_{\bar{\rho}}^{\Gamma}$. However, there are some difficulties, such as $S^{\Gamma}(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F}$ not being isomorphic to $S^{\Gamma}(\mathbb{F})$, in comparing them directly. To overcome this problem, like [Bellaïche and Khare \[2015\]](#), we also work with the divided congruence modules of Katz. Using the results of Katz and the methods of [\[BK\]](#), we get the relation between characteristic-0 and characteristic- p full Hecke algebras, i.e., the relation between the characteristic-0 and characteristic- p Hecke algebras generated by the Hecke operators T_q , qS_q for primes q not dividing Np , U_{ℓ} for primes ℓ dividing N and U_p .

Now, we need to analyze how the addition of the U_{ℓ} and U_p operators changes our Hecke algebras in characteristic 0 and p . We can control the change caused by the U_p operator in a similar way to that done in the level-1 case in [\[BK\]](#). This allows us to get a relation between Hecke algebras in characteristic 0 and p generated by Hecke operators away from p . Note that, the proof for the $N = 1$ case finishes at this step. However, for $N > 1$, we still need to study the effect of adding the extra operators U_{ℓ} for primes ℓ dividing N to our original Hecke algebras. This differentiates the case of $N > 1$ from $N = 1$. In this direction, we prove that in characteristic 0, if $\bar{\rho}$ is new, i.e., if $\bar{\rho}$ is not $\Gamma(N')$ -modular for any proper divisor N' of N , then the operator U_{ℓ} acting on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ for every prime ℓ dividing N . This gives us the finiteness of the U_{ℓ} 's over the mod p Hecke algebra automatically for a new $\bar{\rho}$ and, along with the Gouvêa–Mazur infinite fern argument, leads us to [Theorem 1](#) in those cases. Finally, we prove that if $\bar{\rho}$ is also a $\Gamma(M)$ -modular representation for some M dividing N , then the natural map $A_{\bar{\rho}}^{\Gamma} \rightarrow A_{\bar{\rho}}^{\Gamma(M)}$ is surjective using the pseudorepresentation attached to it and the Chebotarev density theorem. [Theorem 1](#) in the case where $\bar{\rho}$ is new, along with the surjectivity established above, leads us to [Theorem 1](#) for all the local components of the mod p Hecke algebra. We would like to point out that, in contrast with [\[BK\]](#), our method does not give the precise kernel of the map $\mathbb{T}_{\bar{\rho}}^{\Gamma} \rightarrow A_{\bar{\rho}}^{\Gamma}$ in all the cases while proving [Theorem 1](#). However, in many cases, we can find the kernel up to some nilpotence.

To prove [Theorem 3](#), we need to use a result of Böckle and flatness of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ over the Iwasawa algebra $\mathcal{O}[[T]]$ which is proved in the same way as in [\[BK\]](#). This, along with [Theorem 1](#), would imply the first part of [Theorem 3](#). We prove, using techniques and results similar to the ones sketched in the previous paragraph, that

the map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/(\mathfrak{p}, T) \rightarrow A_{\bar{\rho}}^{\Gamma_1(N)}$ has nilpotent kernel if $\bar{\rho}$ satisfies the hypotheses of [Theorem 3](#). We use it, along with the results of Böckle, to conclude the second part of [Theorem 3](#) for the $\Gamma_1(N)$ case. The theorem for the $\Gamma_0(N)$ case then follows easily from the theorem for the $\Gamma_1(N)$ case.

Finally, we would like to remark on a possible generalization of these results to the case of Hilbert modular forms. A well behaved theory for mod p Hilbert modular forms with arbitrary weights, which includes a lot of tools and facts for mod p modular forms that we use, is available (see the works of Andreatta and Goren [\[2005\]](#) for more details). Moreover, an analogue of the Gouvêa–Mazur infinite fern argument is also true for certain local components of the p -adic Hecke algebra acting on the space of Hilbert modular forms due to work of Chenevier [\[2011, Theorem 5.9\]](#). However, contrary to the case of modular forms, the theory of divided congruence modules of Katz, and properties of Hecke operators acting on them, is not known for Hilbert modular forms. We expect results similar to what we have proved above to hold for Hilbert modular forms once we know the theory of divided congruence modules of Hilbert modular forms and the infinite fern argument for all the local components. The results will depend on the space of mod p Hilbert modular forms we consider. For instance, we expect the lower bound on the Krull dimension of the mod p Hecke algebra for Hilbert modular forms of parallel weights to be 2, while the corresponding lower bound for the mod p Hecke algebra for Hilbert modular forms of arbitrary weights to be $2n$, where n is the degree of extension of the totally real field over \mathbb{Q} .

2. Deformation rings and Hecke algebras

The goal of this section is to relate mod p Hecke algebras with appropriate deformation rings.

Using [\[Bellaïche 2012b, Step 1 of the proof of Theorem 1\]](#), which is essentially an argument of gluing pseudorepresentations attached to modular eigenforms of a fixed level and all weights, we get the following lemma (see also [\[BK, Proposition 2\]](#)):

Lemma 4. *Let Γ be either $\Gamma_0(N)$ or $\Gamma_1(N)$. For a Γ -modular representation $\bar{\rho}$, there exists a unique continuous pseudorepresentation*

$$(\tau^\Gamma, \delta^\Gamma) : G_{\mathbb{Q}, Np} \rightarrow \mathbb{T}_{\bar{\rho}}^\Gamma$$

such that $\tau^\Gamma(c) = 0$, $\tau^\Gamma(\text{Frob}_q) = T_q$ and $\delta^\Gamma(\text{Frob}_q) = qS_q$ for all the primes q not dividing Np . We have

$$\tau^\Gamma \pmod{m_{\mathbb{T}_{\bar{\rho}}^\Gamma}} = \text{tr } \bar{\rho} \quad \text{and} \quad \delta^\Gamma \pmod{m_{\mathbb{T}_{\bar{\rho}}^\Gamma}} = \det \bar{\rho}.$$

By composing $(\tau^\Gamma, \delta^\Gamma)$ with the natural morphism $\mathbb{T}_{\bar{\rho}}^\Gamma \rightarrow A_{\bar{\rho}}^\Gamma$, we get a pseudorepresentation $(\tilde{\tau}^\Gamma, \tilde{\delta}^\Gamma) : G_{\mathbb{Q}, Np} \rightarrow A_{\bar{\rho}}^\Gamma$ lifting $(\text{tr } \bar{\rho}, \det \bar{\rho})$.

A deformation (t, d) of a pseudorepresentation (\bar{t}, \bar{d}) is called a deformation with constant determinant if $d = \bar{d}$.

Lemma 5. *If $\Gamma = \Gamma_0(N)$ or if $p \nmid \phi(N)$ and $\Gamma = \Gamma_1(N)$, then for a Γ -modular representation $\bar{\rho}$, $(\tilde{\tau}^\Gamma, \tilde{\delta}^\Gamma)$ is a deformation of $(\text{tr } \bar{\rho}, \det \bar{\rho})$ with constant determinant.*

Proof. Let q be a prime not dividing Np . Note that, in the $\Gamma_0(N)$ case, S_q acts as multiplication by q^{k-2} on a weight k modular form with Fourier coefficients in \mathcal{O} . If two modular forms of level N are congruent modulo \mathfrak{p} , then their weights are congruent modulo $p-1$. So, S_q acts like a constant on $S^{\Gamma_0(N)}(\mathbb{F})_{\bar{\rho}}$ for every $\bar{\rho}$. Since $\tilde{\delta}^\Gamma(\text{Frob}_q) = qS_q$ and the set of Frob_q for primes q not dividing Np is dense in $G_{\mathbb{Q}, Np}$, we get that $\tilde{\delta}^\Gamma$ is constant and we are done in the $\Gamma_0(N)$ case.

Applying the same reasoning to the $\Gamma_1(N)$ case, we get that, for every prime q not dividing Np , S_q acts like $c_q \langle q \rangle$ on $S^{\Gamma_1(N)}(\mathbb{F})_{\bar{\rho}}$, where c_q is an invertible constant. Since $S_q \in A_{\bar{\rho}}^{\Gamma_1(N)}$, it follows that $\langle q \rangle \in A_{\bar{\rho}}^{\Gamma_1(N)}$ for every prime q not dividing Np . If p does not divide $\phi(N)$, then the order of every diamond operator $\langle q \rangle$ is coprime to p . We have chosen \mathbb{F} to be large enough so that it contains all the mod p system of eigenvalues. Thus, by Hensel's lemma, $\langle q \rangle$, and hence S_q , will be constant in $A_{\bar{\rho}}^{\Gamma_1(N)}$ for every $\bar{\rho}$ and every prime q not dividing Np . Therefore, $\tilde{\delta}^\Gamma$ is constant in the $\Gamma_1(N)$ case if p does not divide $\phi(N)$, by the same argument as in the $\Gamma_0(N)$ case. \square

If $p \mid \phi(N)$, then the determinant $\tilde{\delta}^\Gamma$ may not be constant. See [Section 10](#) for more details. If the order of $\langle q \rangle$ is p^e , then $(\langle q \rangle - 1)^{p^e} = 0$ in $A_{\bar{\rho}}^{\Gamma_1(N)}$. Therefore, in $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$, we have $\langle q \rangle = 1$ for all such $\langle q \rangle$. Thus, from the proof of the lemma above, it follows that if $p \mid \phi(N)$, then the determinant

$$(\tilde{\delta}^{\Gamma_1(N)})^{\text{red}} : G_{\mathbb{Q}, Np} \rightarrow ((A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}})^*$$

is constant.

Let $R_{\bar{\rho}}$ be the universal deformation ring of the pseudorepresentation $(\text{tr } \bar{\rho}, \det \bar{\rho})$ in the category of local profinite \mathcal{O} -algebras with residue field \mathbb{F} , $\tilde{R}_{\bar{\rho}}$ be the corresponding universal deformation ring mod \mathfrak{p} and $\tilde{R}_{\bar{\rho}}^0$ be the corresponding universal deformation ring mod \mathfrak{p} with constant determinant (see [[Chenevier 2014](#); [BK](#), Section 1.4] for more details regarding the existence and properties of these rings). For a Γ -modular representation $\bar{\rho}$, the pseudorepresentation $(\tau^\Gamma, \delta^\Gamma)$ defines a local morphism $R_{\bar{\rho}} \rightarrow \mathbb{T}_{\bar{\rho}}^\Gamma$ which is the identity, modulo their maximal ideals. Similarly, we get a local morphism $\tilde{R}_{\bar{\rho}} \rightarrow A_{\bar{\rho}}^\Gamma$. From the previous paragraphs, we see that the morphism $\tilde{R}_{\bar{\rho}} \rightarrow A_{\bar{\rho}}^\Gamma$ factors through $\tilde{R}_{\bar{\rho}}^0$ if $\Gamma = \Gamma_0(N)$ or if $p \nmid \phi(N)$ and $\Gamma = \Gamma_1(N)$. However, this is not true in general. But, from the above discussion, we see that for a $\Gamma_1(N)$ -modular representation $\bar{\rho}$, the map $\tilde{R}_{\bar{\rho}} \rightarrow (A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$ factors through $\tilde{R}_{\bar{\rho}}^0$. All the morphisms considered above are surjective. Indeed, for a prime q not dividing Np , the images of the trace and the determinant, coming from the universal pseudorepresentation of Frob_q , under the morphisms above are T_q and qS_q , respectively.

3. Relation between the full Hecke algebras in characteristic 0 and p

The goal of this section is to obtain a relation, similar to [BK, Proposition 16], between the full Hecke algebras, but for both the cases, $\Gamma_1(N)$ and $\Gamma_0(N)$. We follow the approach of [BK]. We briefly recall, without proofs, some important results of the theory of divided congruences of Katz needed for our purpose. Then, we state some results about the comparisons of various Hecke algebras; these are similar to the results given in Sections 3–6 of [BK], and since their proofs are more or less the same, we do not give them in full detail here. Instead, we mostly refer readers to the proofs of corresponding results in [BK] and provide some additional details when required. In this section, we follow the notation of [BK] for the divided congruence modules and the Hecke algebras along with an additional index representing the level. For instance, $D^{\Gamma_1(N)}(\mathcal{O})$ will represent the divided congruence module of cuspidal forms for $\Gamma_1(N)$ over \mathcal{O} . These modules and the Hecke algebras acting on them are defined in exactly the same way as their level-1 counterparts in [BK] after making appropriate level changes. The main references for this section are [Katz 1975; Hida 1986; BK]. Throughout this section, Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$.

3.1. The divided congruence modules of Katz. In this and the following subsection, we quickly list all the results that are needed from the theory of divided congruence modules. These results are the level- N counterparts of the results that appear in [BK].

We now define the divided congruence modules of Katz. Let $S_{\leq k}^{\Gamma}(K)$ be the subspace of $K[[q]]$ given by $\sum_{i=0}^{i=k} S_i^{\Gamma}(K)$, where $S_i^{\Gamma}(K)$ is the space of cusp forms of weight i and level Γ with Fourier coefficients in K , which is identified as a subspace of $K[[q]]$ via q -expansions. Let $D_{\leq k}^{\Gamma}(\mathcal{O})$ be the \mathcal{O} -submodule of $\mathcal{O}[[q]]$ given by the intersection of $S_{\leq k}^{\Gamma}(K)$ with $\mathcal{O}[[q]]$. These modules are called *divided congruence modules* because they capture congruences between cusp forms of different weights (see Remark 3 of Section 2 of [BK] for more details). We define $D_{\leq k}^{\Gamma}(\mathbb{F})$ as the image of $D_{\leq k}^{\Gamma}(\mathcal{O})$ under the reduction map $\mathcal{O}[[q]] \rightarrow \mathbb{F}[[q]]$. Let

$$D^{\Gamma}(\mathcal{O}) = \bigcup_{k=0}^{\infty} D_{\leq k}^{\Gamma}(\mathcal{O}) \quad \text{and} \quad D^{\Gamma}(\mathbb{F}) = \bigcup_{k=0}^{\infty} D_{\leq k}^{\Gamma}(\mathbb{F}).$$

We call $D^{\Gamma}(\mathcal{O})$ the divided congruence module of cuspidal forms of level Γ and $D^{\Gamma}(\mathbb{F})$ the divided congruence module of cuspidal forms modulo p of level Γ . See Section 2 of [BK] for more details.

Lemma 6. *The natural map $D_{\leq k}^{\Gamma}(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \rightarrow D_{\leq k}^{\Gamma}(\mathbb{F})$ is an isomorphism.*

Proof. If $f \in D_{\leq k}^{\Gamma}(\mathcal{O})$ lies in the kernel of the natural surjective map

$$D_{\leq k}^{\Gamma}(\mathcal{O}) \rightarrow D_{\leq k}^{\Gamma}(\mathbb{F}),$$

then f/π lies in both $S_{\leq k}^{\Gamma}(K)$ and $\mathcal{O}[[q]]$. Hence, it lies in $D_{\leq k}^{\Gamma}(\mathcal{O})$ which implies the lemma. See the proof of Lemma 5 of [BK]. \square

We now recall, without proof, two theorems of Katz which relate the divided congruence module of cuspidal forms mod p of level Γ with the space of mod p cuspforms of level Γ :

Proposition 7 [Katz 1975, Corollary 1.7]. (1) *There exists a unique action of $\mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*$ on $D^{\Gamma_1(N)}(\mathcal{O})$, denoted by $((x, y), f) \mapsto (x, y).f$, such that for $(x, y) \in \mathbb{Z}_p^* \times (\mathbb{Z}/N\mathbb{Z})^*$, and $f \in S_k^{\Gamma_1(N)}(\mathcal{O}) \subset D^{\Gamma_1(N)}(\mathcal{O})$,*

$$(x, y).f = x^k \langle y \rangle f.$$

(2) *There exists a unique action of \mathbb{Z}_p^* on $D^{\Gamma_0(N)}(\mathcal{O})$ denoted by $(x, f) \mapsto x.f$ such that for $x \in \mathbb{Z}_p^*$, and $f \in S_k^{\Gamma_0(N)}(\mathcal{O}) \subset D^{\Gamma_0(N)}(\mathcal{O})$,*

$$x.f = x^k f.$$

Theorem 8 [Katz 1975, Section 4]. *The space $S^{\Gamma}(\mathbb{F})$ is the space of invariants of $1 + p\mathbb{Z}_p$ acting on $D^{\Gamma}(\mathbb{F})$.*

See also [Hida 1986, Theorem 1.1].

The two results of Katz recalled in this subsection are proved only for $p > 3$. We do not know whether they also hold for $p = 2, 3$.

3.2. Hecke operators on the divided congruence modules. On $D^{\Gamma}(\mathcal{O})$ we can define the Hecke operators T_q and S_q , for primes q not dividing Np , and U_{ℓ} , for primes ℓ dividing N . Their action on the q -expansions is given in the same way as it is given on the q -expansions of the classical modular forms. See the proof of Corollary-and-Definition 7 of [BK] and [Hida 1986, page 243] for more details.

Now we introduce *partially full* Hecke algebras which are generated by the U_{ℓ} 's for every prime ℓ dividing N along with the T_n 's for $(n, Np) = 1$. We denote these Hecke algebras by supplementing the index with pf. Thus, $\mathbb{T}_k^{\Gamma, \text{pf}}$ is the \mathcal{O} -subalgebra of $\text{End}_{\mathcal{O}}(S_{\leq k}^{\Gamma}(\mathcal{O}))$ generated by the T_n 's with $(n, Np) = 1$ and U_{ℓ} 's with prime ℓ dividing N , while $A_k^{\Gamma, \text{pf}}$ is the \mathbb{F} -subalgebra of $\text{End}_{\mathbb{F}}(S_{\leq k}^{\Gamma}(\mathbb{F}))$ generated by the T_n 's with $(n, Np) = 1$ and U_{ℓ} 's with prime ℓ dividing N . We can consider the projective limits

$$\mathbb{T}^{\Gamma, \text{pf}} = \varprojlim_k \mathbb{T}_k^{\Gamma, \text{pf}}, \quad A^{\Gamma, \text{pf}} = \varprojlim_k A_k^{\Gamma, \text{pf}}.$$

Lemma 9. (1) *The subalgebra of $\text{End}_{\mathcal{O}}(D_{\leq k}^{\Gamma}(\mathcal{O}))$ generated by the Hecke operators T_q, S_q for primes q not dividing Np and U_{ℓ} for primes ℓ dividing N is naturally isomorphic to $\mathbb{T}_k^{\Gamma, \text{pf}}$.*

(2) *The subalgebra of $\text{End}_{\mathcal{O}}(D_{\leq k}^{\Gamma}(\mathcal{O}))$ generated by the Hecke operators T_q, S_q for primes q not dividing Np is naturally isomorphic to \mathbb{T}_k^{Γ} .*

Proof. Both parts of the lemma follow from the observations that $S_{\leq k}^{\Gamma}(\mathcal{O})$ is a cotorsion submodule of $D_{\leq k}^{\Gamma}(\mathcal{O})$ and that the action of Hecke operators on $D_{\leq k}^{\Gamma}(\mathcal{O})$ extends their action on $S_{\leq k}^{\Gamma}(\mathcal{O})$. Indeed, as $S_{\leq k}^{\Gamma}(\mathcal{O})$ is a cotorsion submodule of $D_{\leq k}^{\Gamma}(\mathcal{O})$, if $f \in D_{\leq k}^{\Gamma}(\mathcal{O})$, then $\pi^n f \in S_{\leq k}^{\Gamma}(\mathcal{O})$ for some n which implies that a Hecke operator vanishing on $S_{\leq k}^{\Gamma}(\mathcal{O})$ also vanishes on $D_{\leq k}^{\Gamma}(\mathcal{O})$. See the proof of Lemma 8 of [BK] for more details. \square

Lemma 10. (1) *The homomorphism $\phi : \mathbb{Z}_p^* \rightarrow \text{End}_{\mathcal{O}}(D^{\Gamma_1(N)}(\mathcal{O}))$, defined by $\phi(x)f = (x, 1).f$ for $f \in D^{\Gamma_1(N)}(\mathcal{O})$, takes values in the subalgebra $\mathbb{T}^{\Gamma_1(N)}$ and hence, in $\mathbb{T}^{\Gamma_1(N), \text{pf}}$.*

(2) *The homomorphism $\phi : \mathbb{Z}_p^* \rightarrow \text{End}_{\mathcal{O}}(D^{\Gamma_0(N)}(\mathcal{O}))$, defined by $\phi(x)f = x.f$ for $f \in D^{\Gamma_0(N)}(\mathcal{O})$, takes values in the subalgebra $\mathbb{T}^{\Gamma_0(N)}$ and hence, in $\mathbb{T}^{\Gamma_0(N), \text{pf}}$.*

Proof. The proof of part (1) is almost the same as the proof of Lemma 9 of [BK]. We only need to change the last step of the proof slightly, so we sketch it briefly here. Following the same proof, we see that $\mathbb{T}^{\Gamma_1(N)}$ is a closed subset of $\text{End}_{\mathcal{O}}(D^{\Gamma_1(N)}(\mathcal{O}))$ under the weak topology, and the map $\phi : \mathbb{Z}_p^* \rightarrow \text{End}_{\mathcal{O}}(D^{\Gamma_1(N)}(\mathcal{O}))$ is continuous for the weak topology. For a prime q which does not divide Np and is $1 \pmod{N}$, one has $\phi(q) = q^2 S_q \in \mathbb{T}^{\Gamma_1(N)}$, since $\langle q \rangle$ is the trivial operator as q is $1 \pmod{N}$. If $x \in \mathbb{Z}_p^*$, there exists, by the Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic progressions, a sequence of primes q_n , (different from primes dividing Np) that are $1 \pmod{N}$, that converges to x p -adically. Hence, $\phi(q_n)$ converges to $\phi(x)$ in $\text{End}_{\mathcal{O}}(D^{\Gamma_1(N)}(\mathcal{O}))$. Therefore, $\phi(x) \in \mathbb{T}^{\Gamma_1(N)}$. The proof of part (2) is the same as the proof of Lemma 9 of [BK]. \square

Let Λ be the Iwasawa algebra $\mathcal{O}[[1 + p\mathbb{Z}_p]]$. By choosing a topological generator of $1 + p\mathbb{Z}_p$, say $1 + p$, one gets an isomorphism $\Lambda \simeq \mathcal{O}[[T]]$. Under this isomorphism, the maximal ideal m_{Λ} of Λ gets mapped to (π, T) . We get a morphism $\psi : \Lambda \rightarrow \mathbb{T}^{\Gamma}$ of \mathcal{O} -algebras from the group homomorphism $\phi : 1 + p\mathbb{Z}_p \rightarrow (\mathbb{T}^{\Gamma})^*$. Using the morphism ψ , we can consider \mathbb{T}^{Γ} as a Λ -algebra.

3.3. Divided congruence modules of level $\Gamma_0(Np)$ and $\Gamma_1(Np)$. In this subsection, we consider the divided congruence modules of cuspidal forms for levels $\Gamma_0(Np)$ and $\Gamma_1(Np)$. They are defined in the same way as in the level- N case after just changing the level. For the rest of this section, Γ still means $\Gamma_0(N)$ or $\Gamma_1(N)$ and $\Gamma(p)$ means either $\Gamma_0(Np)$ or $\Gamma_1(Np)$ accordingly.

Proposition 11. *The closures of $D^{\Gamma}(\mathcal{O})$ and $D^{\Gamma(p)}(\mathcal{O})$ in $\mathcal{O}[[q]]$ provided with the topology of uniform convergence, are equal.*

Proof. See [Gouvêa 1988, Proposition I.3.9] and Section 1 of [Hida 1986]. \square

Corollary 12. *There is an isomorphism preserving q -expansions of*

$$D^{\Gamma(p)}(\mathcal{O}) \otimes_{\mathcal{O}} \mathbb{F} \simeq D^{\Gamma}(\mathbb{F}).$$

Corollary 13. *The algebras $\mathbb{T}^{\Gamma(p),\text{pf}}$ and $\mathbb{T}^{\Gamma,\text{pf}}$ are naturally isomorphic, as are also the algebras $\mathbb{T}^{\Gamma(p)}$ and \mathbb{T}^{Γ} .*

Proof. The natural restriction maps between the Hecke algebras of level Np and N are isomorphisms because their action is continuous and the modules on which they act have the same closure. See the proof of Corollary 13 of [BK] for more details. \square

3.4. Full Hecke algebras. In this subsection, we consider *full* Hecke algebras, i.e., the Hecke algebras generated by all the operators T_q, S_q for primes q with $(q, Np) = 1$, U_ℓ for primes ℓ dividing N and U_p . So, $\mathbb{T}_k^{\Gamma(p),\text{full}}, DA_k^{\Gamma,\text{full}}$ and $A_k^{\Gamma,\text{full}}$ are the Hecke algebras generated by the Hecke operators T_q, S_q for primes q with $(q, Np) = 1$, U_ℓ for primes ℓ dividing N and U_p acting on $D_{\leq k}^{\Gamma(p)}(\mathcal{O}), D_{\leq k}^{\Gamma}(\mathbb{F})$ and $S_{\leq k}^{\Gamma}(\mathbb{F})$, respectively. We denote by $\mathbb{T}^{\Gamma(p),\text{full}}, DA^{\Gamma,\text{full}}$ and $A^{\Gamma,\text{full}}$ the full Hecke algebras acting on $D^{\Gamma(p)}(\mathcal{O}), D^{\Gamma}(\mathbb{F})$ and $S^{\Gamma}(\mathbb{F})$, respectively, which are obtained by taking the inverse limits over the weights k of appropriate Hecke algebras, as before.

Proposition 14 (perfect duality). *The pairings*

$$\mathbb{T}_k^{\Gamma(p),\text{full}} \times D_{\leq k}^{\Gamma(p)}(\mathcal{O}) \rightarrow \mathcal{O}, \quad DA_k^{\Gamma,\text{full}} \times D_{\leq k}^{\Gamma}(\mathbb{F}) \rightarrow \mathbb{F}, \quad A_k^{\Gamma,\text{full}} \times S_{\leq k}^{\Gamma}(\mathbb{F}) \rightarrow \mathbb{F},$$

given by $(t, f) \mapsto a_1(tf)$, are perfect.

Proof. This is well known but we recall the proof here. Suppose $f \in D_{\leq k}^{\Gamma(p)}(\mathcal{O})$ is such that $a_1(tf) = 0$ for all $t \in \mathbb{T}_k^{\Gamma(p),\text{full}}$. This means $a_1(T_n f) = a_n(f) = 0$ for all n coprime to Np , $a_1(U_\ell f) = a_\ell(f) = 0$ for all primes ℓ dividing N and $a_1(U_p f) = a_p(f) = 0$. Thus, we have $a_n(f) = 0$ for all n which implies $f = 0$. Now, suppose $t \in \mathbb{T}_k^{\Gamma(p),\text{full}}$ is such that $a_1(tf) = 0$ for all $f \in D_{\leq k}^{\Gamma(p)}(\mathcal{O})$. This means $a_1(s(tf)) = a_1(t(sf)) = 0$ for all $s \in \mathbb{T}_k^{\Gamma(p),\text{full}}$. Thus, from the previous part, we get that $tf = 0$ for all $f \in D_{\leq k}^{\Gamma(p)}(\mathcal{O})$ which means $t = 0$. The proof for other cases goes in the exact same way. \square

Corollary 15. *The map $\mathbb{T}_k^{\Gamma(p),\text{full}} \rightarrow DA_k^{\Gamma,\text{full}}$ induces an isomorphism*

$$\mathbb{T}_k^{\Gamma(p),\text{full}} \otimes_{\mathcal{O}} \mathbb{F} \xrightarrow{\sim} DA_k^{\Gamma,\text{full}}.$$

Hence, we get an isomorphism $\mathbb{T}^{\Gamma(p),\text{full}} \otimes_{\mathcal{O}} \mathbb{F} \simeq DA^{\Gamma,\text{full}}$.

Proof. By the perfect duality above and Lemma 6, we see that the rank of the torsion-free \mathcal{O} -module $\mathbb{T}_k^{\Gamma(p),\text{full}}$ and the dimension of $DA_k^{\Gamma,\text{full}}$ as an \mathbb{F} -vector space are the same, which implies the result. See the proof of Corollary 15 of [BK] for more details. \square

The composition $\Lambda \rightarrow \mathbb{T}^{\Gamma,\text{pf}} \rightarrow \mathbb{T}^{\Gamma(p),\text{full}}$ defines a Λ -algebra structure on $\mathbb{T}^{\Gamma(p),\text{full}}$.

Proposition 16. $\mathbb{T}^{\Gamma(p),\text{full}} / m_\Lambda \mathbb{T}^{\Gamma(p),\text{full}} \simeq A^{\Gamma,\text{full}}$.

Proof. The proposition follows from [Theorem 8](#) (a theorem of Katz that we recalled in [Section 3.1](#)), the perfect duality and its corollary above. See the proof of Proposition 16 of [\[BK\]](#) for more details. \square

4. Relation between the components of Partial and Full Hecke algebras

Throughout this section Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$ and $\Gamma(p)$ means either $\Gamma_1(Np)$ or $\Gamma_0(Np)$ accordingly. Recall that we have a direct product decomposition $\mathbb{T}^\Gamma = \prod \mathbb{T}_{\bar{\rho}}^\Gamma$ and a direct sum decomposition $S^\Gamma(\mathcal{O}) = \bigoplus S^\Gamma(\mathcal{O})_{\bar{\rho}}$, where the product and sum are taken over all the Γ -modular representations. Note that, $S^\Gamma(\mathcal{O})_{\bar{\rho}}$ is the intersection of the subspace of $S^\Gamma(\overline{\mathbb{Q}}_p)$ generated by the eigenforms lifting the system of eigenvalues corresponding to $\bar{\rho}$ with $S^\Gamma(\mathcal{O})$. The decompositions above are such that $\mathbb{T}_{\bar{\rho}}^\Gamma$ is also the largest quotient of \mathbb{T}^Γ which acts faithfully on $S^\Gamma(\mathcal{O})_{\bar{\rho}}$. Let $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ be the largest quotient of $\mathbb{T}^{\Gamma, \text{pf}}$ which acts faithfully on $S^\Gamma(\mathcal{O})_{\bar{\rho}}$. Define $A_{\bar{\rho}}^{\Gamma, \text{pf}}$ in a similar way.

Note that, since $D^\Gamma(\mathcal{O})$ contains $S^\Gamma(\mathcal{O})$ as a cotorsion submodule, we have a direct sum decomposition $D^\Gamma(\mathcal{O}) = \bigoplus D^\Gamma(\mathcal{O})_{\bar{\rho}}$ similar to that of $S^\Gamma(\mathcal{O})$. Moreover, $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ and $\mathbb{T}_{\bar{\rho}}^\Gamma$ are the largest of quotients of $\mathbb{T}^{\Gamma, \text{pf}}$ and \mathbb{T}^Γ respectively, acting faithfully on $D^\Gamma(\mathcal{O})_{\bar{\rho}}$.

By a result of Serre and Tate (see [\[Jochowitz 1982, Lemma 4.4\]](#)), the subalgebras of $\mathbb{T}^{\Gamma(p), \text{full}}$ and $A^{\Gamma, \text{full}}$ generated by the Hecke operators T_q, S_q for primes q not dividing Np and U_p are semilocal and the local components of both of them are in bijection with the set of \mathbb{F} -valued systems of eigenvalues of the Hecke operators T_q, S_q for primes q not dividing Np and U_p appearing in $S^\Gamma(\mathbb{F})$. Hence, they are in bijection with the pairs $(\bar{\rho}, \lambda)$, where $\bar{\rho}: G_{\mathbb{Q}, Np} \rightarrow \text{GL}_2(\mathbb{F})$ is a Γ -modular representation attached to some eigenform $f \in S^\Gamma(\mathbb{F})$ and λ is the eigenvalue of U_p on f (see [\[Jochowitz 1982\]](#)).

So, we get a direct sum decomposition $D^{\Gamma(p)}(\mathcal{O}) = \bigoplus D^{\Gamma(p)}(\mathcal{O})_{\bar{\rho}, \lambda}$ similar to that of $S^\Gamma(\mathcal{O})$ seen above. Now, let us define $\mathbb{T}_{\bar{\rho}, \lambda}^{\Gamma(p), \text{full}}$ to be the largest quotient of $\mathbb{T}^{\Gamma(p), \text{full}}$ acting faithfully on $D^{\Gamma(p)}(\mathcal{O})_{\bar{\rho}, \lambda}$ and $A_{\bar{\rho}, \lambda}^{\Gamma, \text{full}}$ to be the largest quotient of $A^{\Gamma, \text{full}}$ acting faithfully on $S^\Gamma(\mathbb{F})_{\bar{\rho}, \lambda}$.

We get, using the Chinese remainder theorem and the definitions above, the product decompositions $\mathbb{T}^{\Gamma(p), \text{full}} = \prod \mathbb{T}_{\bar{\rho}, \lambda}^{\Gamma(p), \text{full}}$, and $\mathbb{T}^{\Gamma, \text{pf}} = \prod \mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$. Similarly, we get product decompositions of $A^{\Gamma, \text{full}}$ and $A^{\Gamma, \text{pf}}$. Here, the products are finite products as the pairs $(\bar{\rho}, \lambda)$ are finitely many.

Proposition 17. (1) For a Γ -modular representation $\bar{\rho}$, one has a natural isomorphism of $A_{\bar{\rho}}^{\Gamma, \text{pf}}$ -algebras $A_{\bar{\rho}}^{\Gamma, \text{pf}}[[U_p]] \simeq A_{\bar{\rho}, 0}^{\Gamma, \text{full}}$.

(2) For a Γ -modular representation $\bar{\rho}$, one has a natural isomorphism of $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ -algebras $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}[[U_p]] \simeq \mathbb{T}_{\bar{\rho}, 0}^{\Gamma(p), \text{full}}$.

Proof. The first part of this proposition is proved by Jochnowitz [1982, Theorem 6.3]. The proof of the second part of the proposition is the same as that of Proposition 17 of [BK], where they adapt the proof of Jochnowitz in characteristic 0, which relies on the interplay between the operators U_p and V . Here, V is the operator which sends $\sum a_n q^n$ to $\sum a_n q^{pn}$. The same argument works here. \square

5. Finiteness of $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ for new $\bar{\rho}$

Throughout this section, Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$ and $\Gamma(M)$ means either $\Gamma_1(M)$ or $\Gamma_0(M)$ accordingly. Let us call a Γ -modular representation $\bar{\rho}$ *new* if it is not $\Gamma(M)$ -modular for any proper divisor M of N . Thus, the system of eigenvalues of the Hecke operators corresponding to $\bar{\rho}$ does not have a nontrivial eigenspace in $S^{\Gamma(M)}(\mathbb{F})$ for any proper divisor M of N . Let $S_k^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p)$ be the \mathcal{O} -submodule of $S_k^{\Gamma}(\overline{\mathbb{Q}}_p)$ consisting of new modular forms of weight k and level Γ . Let

$$S_{\leq k}^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p) = \sum_{i=0}^k S_i^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p) \quad \text{and} \quad S^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p) = \bigcup_{k=0}^{\infty} S_{\leq k}^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p).$$

The following lemma follows directly from the discussion above and the description of $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ given in the previous section:

Lemma 18. *If $\bar{\rho}$ is a new Γ -modular representation, then $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ is an \mathcal{O} -submodule of $S^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p)$.*

Now we recall a well-known result regarding the Galois representations attached to level- N newforms:

Lemma 19 [Emerton et al. 2006, Lemma 2.6.1]. *Let f be a classical newform of tame level N over $\overline{\mathbb{Q}}_p$. Let ρ_f be the p -adic Galois representation attached to f and let V_f denote its underlying space. Let ℓ be a prime dividing N and let $a_{\ell}(f)$ be the U_{ℓ} eigenvalue of f . Let $(V_f)_{I_{\ell}}$ be the vector space of I_{ℓ} coinvariants of V_f . Then the following are equivalent:*

- (1) $a_{\ell}(f)$ is a nonunit,
- (2) $a_{\ell}(f) = 0$,
- (3) $(V_f)_{I_{\ell}} = 0$.

If these equivalent conditions do not hold, then we have that $(V_f)_{I_{\ell}}$ is one dimensional and $a_{\ell}(f)$ is equal to the eigenvalue of Frob_{ℓ} acting on this line.

Proposition 20. *If $\bar{\rho}$ is a new Γ -modular representation, then $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ is finite over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. More precisely, U_{ℓ} (for every $\ell \mid N$) is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ of degree at most 2. If one of the following condition holds:*

- (1) $\Gamma = \Gamma_0(N)$ and there does not exist a prime ℓ such that $\ell \parallel N$ and $p \mid \ell + 1$.

(2) $\Gamma = \Gamma_1(N)$ and if there exists a prime ℓ such that $p \mid \ell^2 - 1$ and $\ell \parallel N$, then $p \mid \ell + 1$ and $\det \bar{\rho}(I_\ell) \neq 1$.

then $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}} = \mathbb{T}_{\bar{\rho}}^{\Gamma}$.

Proof. Let $\Gamma = \Gamma_0(N)$. Since $\bar{\rho}$ is new, we know, from [Lemma 18](#), that

$$S^{\Gamma}(\mathcal{O})_{\bar{\rho}} \subset S^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p).$$

Now, $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}}$ is the largest quotient of $\mathbb{T}^{\Gamma, \text{Pf}}$ which acts faithfully on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$. Let ℓ be a prime dividing N . By [Theorem 5 of \[Atkin and Lehner 1970\]](#), if ℓ^2 divides N , then U_ℓ acts like 0 on $S_k^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p)$ and if $\ell \parallel N$, then U_ℓ^2 acts like ℓ^{k-2} on $S_k^{\Gamma, \text{new}}(\overline{\mathbb{Q}}_p)$. Hence, if ℓ^2 divides N , then U_ℓ acts like 0 on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ and if $\ell \parallel N$, then U_ℓ^2 acts like $\ell^{-2}\phi(\ell)$ on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$, where $\phi: \mathbb{Z}_p^* \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma}$ is the map considered in [Lemma 10](#). Thus, if $\ell^2 \mid N$, then $U_\ell = 0$, and if $\ell \parallel N$, then $U_\ell^2 - \ell^{-2}\phi(\ell) = 0$ in $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}}$.

Suppose $\ell \parallel N$ and let f be a newform of level $\Gamma_0(N)$ lifting the system of eigenvalues corresponding to $\bar{\rho}$. Let π_ℓ be the ℓ -component of the automorphic representation corresponding to f and ρ_f be the p -adic Galois representation attached to f . As $\ell^2 \nmid N$, π_ℓ is either principal series or special (see [\[Carayol 1989, Section 1.2\]](#)). So, it follows from the local Langlands correspondence, that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is either a direct sum of two characters or a nontrivial extension of a character by its cyclotomic twist (see [Sections 3 and 5 of \[Weston 2004\]](#)). As f is a newform of level Γ , the Artin conductor of ρ_f is N (the level of f). Since $\ell \parallel N$, the exponent of ℓ appearing in the Artin conductor of ρ_f is exactly 1 which means $(\rho_f)^{I_\ell}$, the subspace of ρ_f on which I_ℓ acts trivially, is one dimensional. So, if $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a direct sum of two characters, then one of them is unramified and the other is tamely ramified. Otherwise, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of an unramified character by its cyclotomic twist. As f is a modular form of level $\Gamma_0(N)$, its nebentypus is trivial, which means $\det \rho_f(I_\ell) = 1$. This implies that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is not a direct sum of two characters and hence, $\rho_f|_{G_{\mathbb{Q}_\ell}} \simeq \begin{pmatrix} \epsilon_p \chi & * \\ 0 & \chi \end{pmatrix}$, where $*$ is nonzero and ramified, χ is an unramified character and ϵ_p is the p -adic cyclotomic character of $G_{\mathbb{Q}_\ell}$.

As $a_\ell(f)$, the U_ℓ eigenvalue of f , is nonzero, by [Lemma 19](#) above, it is the eigenvalue of Frob_ℓ acting on $(\rho_f)_{I_\ell}$. Thus, $a_\ell(f) = \chi(\text{Frob}_\ell)$. Let x be a lift of Frob_ℓ in $G_{\mathbb{Q}_\ell}$. Note that

$$\text{tr}(\rho_f \circ i_\ell(x)) = \epsilon_p \chi(\text{Frob}_\ell) + \chi(\text{Frob}_\ell) = (\ell + 1)\chi(\text{Frob}_\ell) = (\ell + 1)a_\ell(f),$$

which means $a_\ell(f) = \text{tr}(\rho_f \circ i_\ell(x))/(\ell + 1)$. Suppose $p \nmid \ell + 1$, which implies that $\ell + 1$ is a unit in $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. Then on every newform f of level Γ lifting $\bar{\rho}$, the action of U_ℓ coincides with the action of $(\tau^\Gamma \circ i_\ell)(x)/(\ell + 1)$, which lies in $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ as $\ell + 1$ is a unit in $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. As $\bar{\rho}$ is new, every eigenform of level Γ lifting $\bar{\rho}$ is a newform. This implies that $U_\ell - (\tau^\Gamma \circ i_\ell)(x)/(\ell + 1)$ acts like 0 on $S^{\Gamma}(\mathcal{O})_{\bar{\rho}}$ and hence, $U_\ell = (\tau^\Gamma \circ i_\ell)(x)/(\ell + 1)$ in $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}}$. Therefore, $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if $\ell \parallel N$ and $p \nmid \ell + 1$.

Note that, $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}}$ is generated by the Hecke operators U_ℓ over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. So, by combining the discussion of the last two paragraphs, we get that $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}} = \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if there does not exist a prime ℓ such that $\ell \parallel N$ and $p \mid \ell + 1$. Otherwise, we have $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if either $\ell^2 \mid N$ or $p \nmid \ell + 1$ and $U_\ell^2 - \ell^{-2}\phi(\ell) = 0$ if $\ell \parallel N$. Therefore, we conclude that $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}}$ is a finite extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma}$.

Let $\Gamma = \Gamma_1(N)$. There is a continuous pseudorepresentation

$$(\tau^\Gamma, \delta^\Gamma) : G_{\mathbb{Q}, Np} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma}$$

such that $\tau^\Gamma(\text{Frob}_q) = T_q$, $\delta^\Gamma(\text{Frob}_q) = qS_q$ for primes q not dividing Np . Hence, we get a continuous pseudorepresentation $(t, d) : G_{\mathbb{Q}, Np} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}}$ since $\mathbb{T}_{\bar{\rho}}^{\Gamma} \subset \mathbb{T}_{\bar{\rho}}^{\Gamma, \text{Pf}}$. For every prime ℓ dividing N , let us choose an element g_ℓ of $G_{\mathbb{Q}_\ell}$ which gets mapped to Frob_ℓ under the quotient map $G_{\mathbb{Q}_\ell} \rightarrow G_{\mathbb{Q}_\ell}/I_\ell$. We have already fixed a natural map $i_\ell : G_{\mathbb{Q}_\ell} \rightarrow G_{\mathbb{Q}, Np}$. Hence, we get a pseudorepresentation

$$(t \circ i_\ell, d \circ i_\ell) : G_{\mathbb{Q}_\ell} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{Pf}}.$$

Now, consider the characteristic polynomial $Q_\ell(x)$ of g_ℓ which is defined by

$$Q_\ell(x) = x^2 - (t \circ i_\ell)(g_\ell)x + (d \circ i_\ell)(g_\ell).$$

Let f be a newform of level N lifting the system of eigenvalues corresponding to $\bar{\rho}$. Denote by ρ_f the p -adic Galois representation attached to f and by $a_\ell(f)$ its U_ℓ eigenvalue. By [Lemma 19](#), if $a_\ell(f) = 0$, then U_ℓ kills f . If $a_\ell(f) \neq 0$, then $a_\ell(f)$ is the root of the characteristic polynomial $P_f(x)$ of $\rho_f \circ i_\ell(g_\ell)$. But $Q_\ell(U_\ell)f = P_f(a_\ell(f))f = 0$. Thus, in this case, $Q_\ell(U_\ell)$ kills f . We will now determine if it is possible to have two newforms f and g of level N lifting $\bar{\rho}$ such that $a_\ell(f) \neq 0$ but $a_\ell(g) = 0$ or equivalently (by [Lemma 19](#)), $(\rho_f)_{I_\ell}$ is one dimensional but $(\rho_g)_{I_\ell} = 0$.

Let f be a newform of level Γ lifting $\bar{\rho}$ as above and π_ℓ be the ℓ -component of the automorphic representation corresponding to f . So, π_ℓ is one of the following: principal series, special and supercuspidal (see [\[Weston 2004, Section 3\]](#)). As f is a newform, the Artin conductor of ρ_f is N (the level of f). Suppose $\ell \mid N$ but $\ell^2 \nmid N$. Then, from the analysis carried out in the $\Gamma_0(N)$ case above, we see that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is either a sum of an unramified character and a tamely ramified character or a nontrivial extension of an unramified character by its cyclotomic twist. Hence, the space $(\rho_f)_{I_\ell}$ of I_ℓ coinvariants is also one dimensional. Therefore, if $\ell \parallel N$, then for every newform f of level Γ lifting $\bar{\rho}$, $(\rho_f)_{I_\ell}$ is one dimensional and hence, $a_\ell(f) \neq 0$.

Now suppose $\ell^2 \mid N$, and moreover, the space of I_ℓ coinvariants of ρ_f is one dimensional. So, the exponent of ℓ appearing in the Artin conductor of ρ_f is at least 2. This means that π_ℓ is not special, as otherwise $(\rho_f)_{I_\ell}$ being one dimensional will imply that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of an unramified character by its

cyclotomic twist, and hence, the exponent of ℓ appearing in the Artin conductor of ρ_f is 1. The nontriviality of $(\rho_f)_{I_\ell}$ also implies that π_ℓ is not extraordinary supercuspidal (see the proof of [Weston 2004, Proposition 3.2] for more details). Suppose π_ℓ is supercuspidal but not extraordinary. Then, by the local Langlands correspondence,

$$\rho_f|_{G_{\mathbb{Q}_\ell}} = \text{Ind}_{G_K}^{G_{\mathbb{Q}_\ell}} \chi,$$

where K is a quadratic extension of \mathbb{Q}_ℓ , G_K is the absolute Galois group of K , χ is a character of G_K taking values in $\overline{\mathbb{Q}_p}$ and moreover, ρ_f is irreducible (see Section 3 of [Weston 2004]). But as $(\rho_f)_{I_\ell}$ is one dimensional, we get that χ is an unramified character of G_K . However, since the maximal unramified extension of K is an abelian extension of \mathbb{Q}_ℓ , this implies that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a sum of two characters, contradicting the hypothesis that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is irreducible. Hence, π_ℓ is not supercuspidal. Therefore, π_ℓ is principal series which means that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a sum of two characters χ_1 and χ_2 . Moreover, $(\rho_f)_{I_\ell} \neq 0$ implies that one of them is unramified, while $\ell^2|N$ implies that the other is wildly ramified. Without loss of generality, suppose χ_1 is wildly ramified and χ_2 is unramified.

Thus, $\bar{\rho}|_{G_{\mathbb{Q}_\ell}} = \bar{\chi}_1 \oplus \bar{\chi}_2$, where $\bar{\chi}_1$ and $\bar{\chi}_2$ are the reductions of χ_1 and χ_2 in characteristic p , respectively. Note that, $\bar{\chi}_2$ is unramified while $\bar{\chi}_1$ is wildly ramified as $\ell \neq p$. Let g be another newform of level Γ lifting $\bar{\rho}$ and π'_ℓ be the ℓ -component of the automorphic representation corresponding to g . If π'_ℓ is special, then $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of a character by the cyclotomic twist of itself. This would imply that both $\bar{\chi}_1$ and $\bar{\chi}_2$ are either unramified or ramified, which is not the case. Hence, π'_ℓ is not special. If π'_ℓ is extraordinary supercuspidal, then $\bar{\rho}|_{I_\ell}$ is irreducible as $p \geq 5$ (see proof of [Weston 2004, Proposition 3.2]). So, π'_ℓ is not extraordinary supercuspidal. If π'_ℓ is supercuspidal but not extraordinary, then $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is induced from a character of the absolute Galois group of a quadratic extension of \mathbb{Q}_ℓ . Moreover, the subspace of ρ_g fixed by I_ℓ is trivial. But the subspace of $\bar{\rho}$ fixed by I_ℓ is one dimensional. Thus, the exponent of ℓ in the Artin conductor of ρ_g is greater than the exponent of ℓ in the Artin conductor of $\bar{\rho}$ (see [Carayol 1989, Section 1.1]). But [Carayol 1989, Proposition 2], along with the assumption that π'_ℓ is supercuspidal, implies that $\bar{\rho}$ is unramified at ℓ which gives us a contradiction. Therefore, π'_ℓ is not supercuspidal.

This means that π'_ℓ is principal series and hence, $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is a sum of two characters, say χ'_1 and χ'_2 . Without loss of generality, suppose χ'_1 is a lift of $\bar{\chi}_1$ and χ'_2 is a lift of $\bar{\chi}_2$. As $\bar{\chi}_1$ is wildly ramified, the Artin conductor of $\bar{\chi}_1$ is same as the Artin conductor of χ'_1 and the Artin conductor of χ_1 (see [Carayol 1989, Section 1.2]). As the exponent of ℓ in the Artin conductor of ρ_f is the sum of the exponents of ℓ in the Artin conductors of χ_1 and χ_2 and its exponent in the Artin conductor of ρ_g is the sum of its exponents in the Artin conductors of χ'_1 and χ'_2 . As both f

and g are newforms of level N , the exponents of ℓ in the Artin conductors of ρ_f and ρ_g are same. This implies that the Artin conductors of χ_2 and χ'_2 are same, which means that χ'_2 is also unramified. In particular, we see that $(\rho_g)_{I_\ell}$ is one dimensional. As a consequence, we see that if f is a newform of level Γ lifting $\bar{\rho}$ and $(\rho_f)_{I_\ell}$ is one dimensional, then, for every newform g of level Γ lifting $\bar{\rho}$, $(\rho_g)_{I_\ell}$ is one dimensional.

Let us continue with the assumption that there is a newform f_0 of level Γ lifting $\bar{\rho}$ such that $(\rho_{f_0})_{I_\ell}$ is one dimensional. Observe that in this case, from the discussion so far, we get that for every newform g of level Γ lifting $\bar{\rho}$, $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is reducible and moreover, at least one character appearing in the semisimplification of $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is unramified. Thus, $(\bar{\rho}|_{G_{\mathbb{Q}_\ell}})^{\text{ss}} = \alpha \oplus \beta$, where $(\bar{\rho}|_{G_{\mathbb{Q}_\ell}})^{\text{ss}}$ is the semisimplification of $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ and α and β are characters of $G_{\mathbb{Q}_\ell}$ such that at least one of them is unramified. Note that, both α and β are defined over \mathbb{F} . Indeed, both $\alpha|_{I_\ell}$ and $\beta|_{I_\ell}$ take values in \mathbb{F} as one of them is unramified and $\det(\bar{\rho})$ is defined over \mathbb{F} . So it follows from the previous discussion that the image of I_ℓ under $\bar{\rho}$ in $\text{GL}_2(\mathbb{F})$ is abelian and hence, is upper-triangular under a suitable basis. Therefore, since I_ℓ is normal in $G_{\mathbb{Q}_\ell}$, it follows that if $\bar{\rho}$ is ramified at ℓ , then under the same basis, the image of $G_{\mathbb{Q}_\ell}$ under $\bar{\rho}$ in $\text{GL}_2(\mathbb{F})$ is also upper-triangular, which implies that both α and β are defined over \mathbb{F} . If $\bar{\rho}$ is unramified at ℓ , then $\bar{\rho}$ is reducible as it is new. As $\bar{\rho}$ is semisimple, it follows that both α and β are defined over \mathbb{F} .

Suppose $\alpha \neq \beta$. Then, by [Bellaïche and Chenevier 2009, Theorem 1.4.4, Chapter 1], $\mathbb{T}_{\bar{\rho}}^\Gamma[G_{\mathbb{Q}_\ell}]/(\ker(\tau^\Gamma \circ i_\ell(G_{\mathbb{Q}_\ell}))$) is a generalized matrix algebra (GMA) of the form

$$\begin{pmatrix} \mathbb{T}_{\bar{\rho}}^\Gamma & B \\ C & \mathbb{T}_{\bar{\rho}}^\Gamma \end{pmatrix}, \quad (1)$$

where B and C are finitely generated $\mathbb{T}_{\bar{\rho}}^\Gamma$ -modules contained in the total fraction ring of $\mathbb{T}_{\bar{\rho}}^\Gamma$, and the diagonal entries reduce to α and β modulo the maximal ideal of $\mathbb{T}_{\bar{\rho}}^\Gamma$. Moreover, $BC \subset \mathbb{T}_{\bar{\rho}}^\Gamma$ and it is an ideal of $\mathbb{T}_{\bar{\rho}}^\Gamma$. Let us call it I .

For a newform f of level Γ lifting $\bar{\rho}$, let $\phi_f : \mathbb{T}_{\bar{\rho}}^\Gamma \rightarrow \overline{\mathbb{Q}}_p$ be the map which sends a Hecke operator to its f -eigenvalue and denote its kernel by P_f . From the previous paragraph, we see that the 2 dimensional pseudocharacter $\tau^\Gamma \circ i_\ell$ of $G_{\mathbb{Q}_\ell}$ is a sum of two characters modulo P_f and hence, is reducible modulo P_f for every newform f of level Γ lifting $\bar{\rho}$. Therefore, by [Bellaïche and Chenevier 2009, Proposition 1.5.1, Chapter 1], it follows that $I \subset P_f$ for every newform f of level Γ lifting $\bar{\rho}$. This means that if $x \in I$, then $x(f) = 0$ for every newform f of level Γ lifting $\bar{\rho}$. As every eigenform of level Γ lifting $\bar{\rho}$ is a newform, we see that $I = 0$. Thus, the projection on the diagonal entries of the GMA (1) above gives two characters $\tilde{\alpha}$ and $\tilde{\beta}$ of $G_{\mathbb{Q}_\ell}$ taking values in $\mathbb{T}_{\bar{\rho}}^\Gamma$ such that, $\tilde{\alpha}$ is a deformation of α , while $\tilde{\beta}$ is a deformation of β .

As at least one of α and β is unramified; without loss of generality, assume that β is unramified. Suppose α is ramified. Then, it follows, from the analysis above, that for a newform f of level Γ lifting $\bar{\rho}$, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a direct sum of an unramified character and a ramified character. As α is ramified, the unramified character appearing in $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a lift of β and hence, it is the image of $\tilde{\beta}$ modulo P_f . This means that the reduction of $\tilde{\beta}$ modulo P_f gives an unramified character for every newform f of level Γ lifting $\bar{\rho}$. Hence, by the reasoning used in the previous paragraph, we see that $\tilde{\beta}$ is an unramified character of $G_{\mathbb{Q}_\ell}$. By [Lemma 19](#), it follows that for every newform f of level Γ lifting $\bar{\rho}$, the U_ℓ eigenvalue of f is the reduction of $\tilde{\beta}(\text{Frob}_\ell)$ modulo P_f . Thus, $U_\ell - \tilde{\beta}(\text{Frob}_\ell)$ annihilates every newform f of level Γ lifting $\bar{\rho}$. Therefore, $U_\ell = \tilde{\beta}(\text{Frob}_\ell)$ in $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ and hence, $U_\ell \in \mathbb{T}_{\bar{\rho}}^\Gamma$ as $\tilde{\beta}(\text{Frob}_\ell) \in \mathbb{T}_{\bar{\rho}}^\Gamma$.

Now, suppose α is also unramified. This means that for a newform f of level Γ lifting $\bar{\rho}$, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is either a nontrivial extension of an unramified character by its cyclotomic twist or a direct sum of an unramified character and a tamely ramified character and in both cases, $\ell \parallel N$ (see [\[Carayol 1989, Proposition 2\]](#)). Moreover, in the second case, $p \mid \ell - 1$. Suppose $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a direct sum of an unramified character and a tamely ramified character. Let ϵ be the nebentypus of f . Note that,

$$\det(\rho_f(I_\ell)) = \epsilon((\mathbb{Z}/\ell\mathbb{Z})^*) \neq 1,$$

but its reduction is 1 in characteristic p . So, by [\[Carayol 1989, Proposition 3\]](#), there exists a newform g of level Γ lifting $\bar{\rho}$ such that $\epsilon'((\mathbb{Z}/\ell\mathbb{Z})^*) = 1$, where ϵ' is the nebentypus of g . Thus, $\rho_g|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of an unramified character by its cyclotomic twist. This means that either α/β or β/α is the cyclotomic character ω_p . But as $p \mid \ell - 1$, $\omega_p(\text{Frob}_\ell) = 1$ and hence, ω_p is the trivial character. However, this means that $\alpha = \beta$, which contradicts our assumption that $\alpha \neq \beta$. Therefore, we get that, if α is unramified, then $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of an unramified character by the cyclotomic twist of itself for every newform f of level Γ lifting $\bar{\rho}$. By [\[Carayol 1989, Proposition 3\]](#) and the discussion above, it follows that $p \nmid \ell - 1$, for otherwise there would exist a newform g lifting $\bar{\rho}$ such that $\det(\rho_g(I_\ell)) \neq 1$, which gives a contradiction.

As α is also unramified, we saw above that either α/β or β/α is ω_p . If both of them are ω_p , then $\omega_p^2 = 1$, which means $p \mid \ell^2 - 1$. As $p \nmid \ell - 1$, we get that $p \mid \ell + 1$ if both of them are ω_p . Suppose $p \nmid \ell + 1$, which implies that exactly one of them is ω_p . Without loss of generality, assume α/β is ω_p . Now in this case, the image of $\tilde{\alpha}$ is the cyclotomic twist of the image of $\tilde{\beta}$ modulo P_f for every newform f of level Γ lifting $\bar{\rho}$. Thus, for every such newform f , $G_{\mathbb{Q}_\ell}$ acts by the image of $\tilde{\beta}$ modulo P_f on $(\rho_f)_{I_\ell}$. Hence, the U_ℓ eigenvalue of f is the reduction of $\tilde{\beta}(\text{Frob}_\ell)$ modulo P_f for every newform f of level Γ lifting $\bar{\rho}$. Therefore, by the reasoning used in the previous case, we see that $U_\ell = \tilde{\beta}(\text{Frob}_\ell)$ in $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ which means that $U_\ell \in \mathbb{T}_{\bar{\rho}}^\Gamma$.

Now suppose that there exists a newform f_0 of level Γ lifting $\bar{\rho}$ such that $(\rho_{f_0})_{I_\ell} = 0$, which means $U_\ell f_0 = 0$. Then, by our analysis above, it follows that $\ell^2 | N$ and $U_\ell f = 0$ for all newforms f of level Γ lifting $\bar{\rho}$. As $\bar{\rho}$ is new of level Γ , we get, by the reasoning used above, that $U_\ell = 0$ in $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$.

From the discussion so far, we see that for a prime ℓ dividing N , $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if one of the following conditions hold:

- (1) $\ell^2 | N$,
- (2) $\ell \parallel N$ and $p \nmid \ell^2 - 1$,
- (3) $\ell \parallel N$, $p | \ell + 1$ and $\det \bar{\rho}(I_\ell) \neq 1$.

Otherwise, $Q_\ell(U_\ell)$ kills every newform f of level N lifting $\bar{\rho}$. Since $\bar{\rho}$ is new, every eigenform of level N lifting $\bar{\rho}$ is a newform. This implies that either $U_\ell \in \mathbb{T}_{\bar{\rho}}^{\Gamma}$ or $Q_\ell(U_\ell) = 0$ for every prime ℓ dividing N . Since the pseudorepresentation (t, d) takes values in $\mathbb{T}_{\bar{\rho}}^{\Gamma}$, it follows that for every prime ℓ dividing N , $Q_\ell(x)$ is a monic polynomial with coefficients in $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. Hence, U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ for every prime ℓ dividing N of degree at most 2. Therefore, $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ is finite over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ and moreover, $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}} = \mathbb{T}_{\bar{\rho}}^{\Gamma}$ if $\ell \parallel N$ and $p | \ell^2 - 1$ implies that $p | \ell + 1$ and $\det \bar{\rho}(I_\ell) \neq 1$ as $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ is generated by these U_ℓ 's over $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. \square

Remark. (1) Note that, the proof given above for the $\Gamma_1(N)$ case works for the $\Gamma_0(N)$ case as well. But we give a different proof because it gives us a more precise result in the $\Gamma_0(N)$ case and it is also simpler than the proof in the $\Gamma_1(N)$ case.

- (2) Even though we do a detailed analysis in the $\Gamma_1(N)$ case above, it is not really necessary just to prove that U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. Indeed, by the reasoning used in the last paragraph of the proof above, we can easily prove that $U_\ell Q_\ell(U_\ell) = 0$, which proves that U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. For most of our purposes, we only need the result that U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. But we give this detailed analysis to obtain a more precise result which is helpful in getting a more precise version of [Theorem 3](#) in some cases (see the remark after the proof of [Theorem 3](#)).

6. The case of $\bar{\rho}$ which is not new

Throughout this section Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$ and $\Gamma(M)$ means either $\Gamma_1(M)$ or $\Gamma_0(M)$ accordingly. Let $\bar{\rho}$ be a Γ -modular representation which is not new. Thus, the system of \mathbb{F} -valued eigenvalues corresponding to $\bar{\rho}$ has a nontrivial eigenspace in $S^{\Gamma(M)}(\mathbb{F})$ for some proper divisor M of N . Let us denote by $M_{\bar{\rho}}$ the smallest divisor of N such that the system of \mathbb{F} -valued eigenvalues corresponding to $\bar{\rho}$ has a nontrivial eigenspace in $S^{\Gamma(M_{\bar{\rho}})}(\mathbb{F})$. Note that, $\bar{\rho}$ is then a new $\Gamma(M_{\bar{\rho}})$ -modular representation.

Lemma 21. *If $\bar{\rho}$ is a Γ -modular representation which is not new, then the natural map of local algebras $r : A_{\bar{\rho}}^{\Gamma} \rightarrow A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$ obtained by restriction is surjective.*

Proof. First note that $S^{\Gamma(M_{\bar{\rho}})}(\mathbb{F})_{\bar{\rho}} \subset S^{\Gamma}(\mathbb{F})_{\bar{\rho}}$. Hence, using this inclusion, we obtain a natural map $r : A_{\bar{\rho}}^{\Gamma} \rightarrow A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$ by restriction. Its image $\text{Im}(r)$ is complete and hence, closed. It contains all the Hecke operators T_q and qS_q (considered as operators on $S^{\Gamma(M_{\bar{\rho}})}(\mathbb{F})_{\bar{\rho}}$) for all the primes q not dividing Np . There is a continuous pseudorepresentation $(\tilde{\tau}^{\Gamma(M_{\bar{\rho}})}, \tilde{\delta}^{\Gamma(M_{\bar{\rho}})}) : G_{\mathbb{Q}, M_{\bar{\rho}}p} \rightarrow A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$ and $\text{Im}(r)$ contains the Hecke operators $T_q = \tilde{\tau}^{\Gamma(M_{\bar{\rho}})}(\text{Frob}_q)$ for primes q not dividing Np . By the Chebotarev density theorem, the set of Frob_q for primes q not dividing Np is dense in $G_{\mathbb{Q}, M_{\bar{\rho}}p}$. Since $\tilde{\tau}^{\Gamma(M_{\bar{\rho}})}$ is continuous and $\text{Im}(r)$ is closed, it contains $\tilde{\tau}^{\Gamma(M_{\bar{\rho}})}(g)$ for every $g \in G_{\mathbb{Q}, M_{\bar{\rho}}p}$. It also contains $qS_q = \tilde{\delta}^{\Gamma(M_{\bar{\rho}})}(\text{Frob}_q)$ for all the primes q not dividing Np . As $\tilde{\delta}^{\Gamma(M_{\bar{\rho}})}$ is continuous and $\text{Im}(r)$ is closed, by the Chebotarev density theorem, we see that it contains $\tilde{\delta}^{\Gamma(M_{\bar{\rho}})}(g)$ for every $g \in G_{\mathbb{Q}, M_{\bar{\rho}}p}$. For every prime q not dividing $M_{\bar{\rho}}$, we have $\tilde{\tau}^{\Gamma(M_{\bar{\rho}})}(\text{Frob}_q) = T_q$ and $\tilde{\delta}^{\Gamma(M_{\bar{\rho}})}(\text{Frob}_q) = qS_q$. Hence, we see that $\text{Im}(r)$ contains T_q, qS_q for all the primes q not dividing $M_{\bar{\rho}}p$. Thus, $\text{Im}(r)$ is a closed subalgebra of $A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$ containing all of its generators. Hence, $\text{Im}(r) = A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$. \square

7. Proof of Theorem 1

Throughout this section Γ means either $\Gamma_1(N)$ or $\Gamma_0(N)$, $\Gamma(p)$ means either $\Gamma_1(Np)$ or $\Gamma_0(Np)$ accordingly and $\Gamma(M)$ means $\Gamma_1(M)$ or $\Gamma_0(M)$ accordingly. As we noted before, the natural map $\mathbb{T}^{\Gamma, \text{pf}} \rightarrow \prod \mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ is an isomorphism and it lifts the natural map $\mathbb{T}^{\Gamma} \rightarrow \prod \mathbb{T}_{\bar{\rho}}^{\Gamma}$. The corresponding statement for $A^{\Gamma, \text{pf}}$ and A^{Γ} is also true. The natural surjective map $\mathbb{T}^{\Gamma, \text{pf}} \rightarrow A^{\Gamma, \text{pf}}$ sends $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ onto $A_{\bar{\rho}}^{\Gamma, \text{pf}}$. The natural surjective map $\mathbb{T}^{\Gamma(p), \text{full}} \rightarrow A^{\Gamma, \text{full}}$ takes $\mathbb{T}_{\bar{\rho}, \lambda}^{\Gamma(p), \text{full}}$ onto $A_{\bar{\rho}, \lambda}^{\Gamma, \text{full}}$.

Thus, from [Proposition 16](#) and the discussion above, it follows that $A_{\bar{\rho}, 0}^{\Gamma, \text{full}}$ is isomorphic to $\mathbb{T}_{\bar{\rho}, 0}^{\Gamma(p), \text{full}} / m_{\Lambda} \mathbb{T}_{\bar{\rho}, 0}^{\Gamma(p), \text{full}}$. By [Proposition 17](#), $\mathbb{T}_{\bar{\rho}, 0}^{\Gamma(p), \text{full}} = \mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}} \llbracket U_p \rrbracket$ and $A_{\bar{\rho}, 0}^{\Gamma, \text{full}} = A_{\bar{\rho}}^{\Gamma, \text{pf}} \llbracket U_p \rrbracket$. Thus, $A_{\bar{\rho}}^{\Gamma, \text{pf}}$ is isomorphic to $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}} / m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$. As the ideal m_{Λ} is generated by two elements, the *Hauptidealsatz* implies that

$$\dim A_{\bar{\rho}}^{\Gamma, \text{pf}} \geq \dim \mathbb{T}_{\bar{\rho}}^{\Gamma(p), \text{pf}} - 2.$$

Suppose $\bar{\rho}$ is a new Γ -modular representation. Then, we have proved so far that $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}}$ is a finite extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma}$. Observe that $\mathbb{T}_{\bar{\rho}}^{\Gamma}$ is mapped onto $A_{\bar{\rho}}^{\Gamma}$ under the surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}} \rightarrow A_{\bar{\rho}}^{\Gamma, \text{pf}}$. This follows from the fact that the map between the partially full Hecke algebras is obtained by first reducing the partially full Hecke algebra in characteristic 0 modulo \mathfrak{p} to get an action on $D^{\Gamma}(\mathbb{F})_{\bar{\rho}}$ and then restricting it to the subspace $S^{\Gamma}(\mathbb{F})_{\bar{\rho}}$. Hence, we see that $A_{\bar{\rho}}^{\Gamma, \text{pf}}$ is finite over $A_{\bar{\rho}}^{\Gamma}$.

Therefore, $\dim A_{\bar{\rho}}^{\Gamma, \text{pf}} = \dim A_{\bar{\rho}}^{\Gamma}$ and $\dim \mathbb{T}_{\bar{\rho}}^{\Gamma, \text{pf}} = \dim \mathbb{T}_{\bar{\rho}}^{\Gamma}$. By the Gouvêa–Mazur infinite fern argument, we see that $\dim \mathbb{T}_{\bar{\rho}}^{\Gamma} \geq 4$ (see [[Emerton 2011](#), Corollary 2.28; [Gouvêa and Mazur 1998](#), Theorem 1]). Hence, $\dim A_{\bar{\rho}}^{\Gamma} \geq \dim \mathbb{T}_{\bar{\rho}}^{\Gamma} - 2 \geq 2$.

Now suppose that $\bar{\rho}$ is not a new Γ -modular representation. Thus, as remarked before, there exists a proper divisor $M_{\bar{\rho}}$ of N such that $\bar{\rho}$ is a new $\Gamma(M_{\bar{\rho}})$ -modular representation (we can take it to be the smallest divisor of N such that the eigenspace corresponding to the system of eigenvalues corresponding to $\bar{\rho}$ is nonzero in $S^{\Gamma_1(M_{\bar{\rho}})}(\mathbb{F})$). If $M_{\bar{\rho}} = 1$, then we know that $\dim A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})} = 2$ by Theorem III of [BK]. If $M_{\bar{\rho}} > 1$, then since $\bar{\rho}$ is a new $\Gamma(M_{\bar{\rho}})$ -modular representation, repeating the argument given in the previous paragraph for $A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$ and $\mathbb{T}_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$ gives $\dim A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})} \geq 2$. (We can do this since, all the results till the previous section are valid for any level not divisible by p . This condition is satisfied by $M_{\bar{\rho}}$ as it divides N which is coprime to p .) Thus, in any case, we see that $\dim A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})} \geq 2$. By Lemma 21, we have a surjective map $A_{\bar{\rho}}^{\Gamma} \rightarrow A_{\bar{\rho}}^{\Gamma(M_{\bar{\rho}})}$. Hence, $\dim A_{\bar{\rho}}^{\Gamma} \geq 2$ when $\bar{\rho}$ is not a new Γ -modular representation. Thus, Theorem 1 is proved for all the Γ -modular representations $\bar{\rho}$.

8. Proof of Theorem 3

Theorem 22 (Böckle, Diamond–Flach–Guo, Gouvêa–Mazur, Kisin). *Under the hypotheses of Theorem 3, the natural map $R_{\bar{\rho}} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism between local rings of dimension 4.*

Proof. If $\bar{\rho}$ is a $\Gamma_1(N)$ -modular representation which also satisfies the hypotheses of Theorem 3, then mimicking the proof of Theorem 18 of [BK] gives the result of the theorem. We can mimic the argument since, under all these assumptions, the infinite fern argument of Gouvêa–Mazur and [Böckle 2001, Theorems 2.8, 3.1 and 3.9; Diamond 1996, Theorem 1.1; Diamond et al. 2004, Theorem 3.6; Kisin 2004, Main Theorem], which are the key ingredients of the proof of Theorem 18 of [BK], hold. Note that the Hecke algebra appearing in [Böckle 2001, Theorem 3.9] is of a higher level N' such that $N \mid N'$ and N and N' have the same prime factors (see the discussion after [Böckle 2001, Theorem 2.7] for more details). But the hypotheses of Theorem 3, along with [Carayol 1989, Proposition 2], ensure that all eigenforms of level N' lifting $\bar{\rho}$ arise from newforms of level dividing N , which means that the natural restriction map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N')} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. \square

Lemma 23. *Under the hypotheses of Theorem 3, the algebra $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is flat over Λ for the structure of Λ -algebra on $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ defined at the end of Section 3.2.*

Proof. The proof is similar to that of Lemma 19 of [BK], but we need to make a few changes. Let $\bar{\rho}$ be a $\Gamma_1(N)$ -modular representation. If the p -primary part of $(\mathbb{Z}/N\mathbb{Z})^*$ is $\prod_{i=1}^{i=n} \mathbb{Z}/p^{e_i}\mathbb{Z}$, then the universal deformation ring of $\det \bar{\rho}$ is

$$R_{\det \bar{\rho}} = \Lambda[Y_1, \dots, Y_n]/(Y_1^{p^{e_1}} - 1, \dots, Y_n^{p^{e_n}} - 1). \quad (2)$$

Thus, if p does not divide $\phi(N)$, then $R_{\det \bar{\rho}} = \Lambda$. Using the same arguments as used in the proof of [BK, Lemma 19], we get that $R_{\bar{\rho}}$ is flat over $R_{\det \bar{\rho}}$. Since $R_{\det \bar{\rho}}$

is finite and free over Λ , it is also flat over Λ . So, we get that $R_{\bar{\rho}}$ is flat over Λ . To verify that the map $\Lambda \rightarrow R_{\bar{\rho}} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is the same as the map $\Lambda \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ given at the end of [Section 3.2](#), we first show that it is true for all the primes $q \in 1 + p\mathbb{Z}_p$ which are 1 (mod N). This is shown in exactly the same way as it is showed for all the primes $q \in 1 + p\mathbb{Z}_p$ in the proof of [\[BK, Lemma 19\]](#). Since we are choosing the primes which are 1 (mod N), the nebentypus is trivial at those primes and hence, we can use the argument used in the proof of [\[BK, Lemma 19\]](#) without modifications. By Dirichlet's theorem on primes in arithmetic progressions, we know that the set of all such primes is dense in $1 + p\mathbb{Z}_p$. Thus, by continuity, we see that the two maps considered above are the same. Under the hypotheses of [Theorem 3](#), the map $R_{\bar{\rho}} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. Hence, we conclude that $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is flat over Λ for the Λ -algebra structure defined on $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ at the end of [Section 3.2](#). \square

There is a surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_1(N)}$ whose kernel contains $m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ by [Proposition 16](#). Hence, we get a surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} / m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_1(N)}$. Since $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is flat over Λ by [Lemma 23](#), it follows, from [\[Eisenbud 1995, Theorem 10.10\]](#), that the Krull dimension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} / m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is equal to $\dim \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} - 2 = 4 - 2 = 2$. Thus, the Krull dimension of $A_{\bar{\rho}}^{\Gamma_1(N)}$ is at most 2. But, by [Theorem 1](#), the dimension of $A_{\bar{\rho}}^{\Gamma_1(N)}$ is at least 2. Hence, the dimension of $A_{\bar{\rho}}^{\Gamma_1(N)}$ is exactly 2. This concludes the proof of first part of [Theorem 3](#) for the $\Gamma_1(N)$ case.

Let us assume that $\bar{\rho}$ satisfies the hypothesis of [Theorem 3](#) and is also new. Then we have proved that $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$ is an integral extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ and $A_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$ is an integral extension of $A_{\bar{\rho}}^{\Gamma_1(N)}$. By [Proposition 16](#), we see that the kernel of the natural map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{pf}} \rightarrow A_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$ is $m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$. Thus, the kernel of the natural map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_1(N)}$ is contained in every prime ideal of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$ containing $m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$. Hence, by the going-up theorem, the kernel is contained in every prime ideal of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ containing $m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. Therefore, the natural surjective map $(\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} / m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}} \rightarrow (A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$ is an isomorphism. Under the hypothesis of [Theorem 3](#), the surjective map $R_{\bar{\rho}} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. Thus, from the proof of [Lemma 23](#), it follows that $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}}$ is isomorphic to $(R_{\bar{\rho}} / m_{\Lambda} R_{\bar{\rho}})^{\text{red}}$ (this follows from the fact that the universal deformation ring of $\det \bar{\rho}$ is given by [Equation \(2\)](#) and hence, its maximal ideal is $\text{Rad}(m_{\Lambda})$) and the map $R_{\bar{\rho}} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ considered above induces an isomorphism $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}} \rightarrow (\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} / m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$. As seen before, we have a surjective map $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}} \rightarrow (A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$ and this map factors through $(\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} / m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$. Since both the maps $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}} \rightarrow (\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} / m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$ and $(\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} / m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}} \rightarrow (A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$ are isomorphisms, we see that $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}}$ is isomorphic to $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$ under the map mentioned above.

Now suppose $\bar{\rho}$ satisfies the hypotheses of [Theorem 3](#) but is not new. Note that, while proving the theorem for a new $\bar{\rho}$ of level N , we use the result that $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$ is an integral extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ to prove that the kernel of the surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_1(N)}$ is nilpotent modulo $m_{\Lambda} \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$, which in turn implies the

isomorphism between $(\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}/m_{\Lambda}\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$ and $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$. This, along with the isomorphism between $R_{\bar{\rho}}$ and $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$, gives us the isomorphism between $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}}$ and $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$. In the present situation, we know that $R_{\bar{\rho}}$ and $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ are isomorphic. So, to prove the theorem in the current case, it suffices to prove that $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$ is an integral extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. To prove this, we will first compare the Artin conductor of $\bar{\rho}$ with the level N and then do a case by case analysis using the technique used in the proof of [Proposition 20](#). Let C be the Artin conductor of $\bar{\rho}$. Recall that we assume $\bar{\rho}$ to be semisimple. As there exists a newform of level N giving rise to $\bar{\rho}$, we see that $N = C \prod r^{v(r)}$, where the product is over finitely many primes r and $v(r) \leq 2$ for every r (see [[Carayol 1989](#), Proposition 2] and the discussion before it). Note that the product may include some divisors of C .

Let ℓ be a prime dividing N . So, if f is an eigenform of level N lifting the system of eigenvalues associated to $\bar{\rho}$, then f comes from a newform g of level M such that the highest power of ℓ dividing N/M is at most 2. Thus, $f(z) = \sum_{i=1}^{i=r} g(d_i z)$, where the d_i 's are some divisors of N/M . Now we will analyze the action of U_{ℓ} on f case by case to find a monic polynomial with coefficients in $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ satisfied by U_{ℓ} .

Case 1. $\ell \mid M$ but $\ell \nmid N/M$: In this case, $\ell \nmid d_i$ for all i and hence, f is new at ℓ . Therefore, from the proof of [Proposition 20](#), we see that $U_{\ell} Q_{\ell}(U_{\ell})(f) = 0$, where $Q_{\ell}(x)$ is the characteristic polynomial of a lift of Frob_{ℓ} in $G_{\mathbb{Q}, Np}$, as considered in the proof of [Proposition 20](#).

Case 2. $\ell \mid M$ and $\ell \parallel N/M$: In this case, for every i either $\ell \parallel d_i$ or $\ell \nmid d_i$ and g is an eigenform for U_{ℓ} . So, $U_{\ell} f$ is an eigenform of level N/ℓ , which is also new at ℓ if it is nonzero. So, by the same logic as in the previous case, $U_{\ell} Q_{\ell}(U_{\ell})(U_{\ell} f) = 0$.

Case 3. $\ell \mid M$ and $\ell^2 \mid N/M$: In this case, for every i either $\ell \nmid d_i$ or ℓ divides d_i with multiplicity at most 2 and g is an eigenform for U_{ℓ} . So, $U_{\ell}^2 f$ is an eigenform of level N/ℓ^2 and it is new at ℓ when it is nonzero. Thus, we get $U_{\ell} Q_{\ell}(U_{\ell})(U_{\ell}^2 f) = 0$.

Case 4. $\ell \nmid M$ and $\ell \nmid d_i$ for all i : If $\ell \nmid M$ and $\ell \nmid d$, then it follows directly from the description of the action of T_{ℓ} and U_{ℓ} on the q -expansions of modular forms, along with the assumption that g is an eigenform of level M , that U_{ℓ} stabilizes the subspace of modular forms of level $Md\ell$ generated by $g(dz)$ and $g(\ell dz)$ (see [[Bellaïche 2010](#), Lemma III.7.2] and the discussion around it). Moreover, the characteristic polynomial of U_{ℓ} over this subspace is $(U_{\ell}^2 - \text{tr } \rho_g(\text{Frob}_{\ell})U_{\ell} + \det \rho_g(\text{Frob}_{\ell}))$, where ρ_g is the p -adic Galois representation of $G_{\mathbb{Q}, Mp}$ attached to g . Thus, in this case, we get that $Q_{\ell}(U_{\ell})(f) = 0$.

Case 5. $\ell \nmid M$ and ℓ divides at least one of the d_i 's: Note that, $U_{\ell} g(d_i z)$ is an eigenform of level N/ℓ if $\ell \parallel d_i$ and $U_{\ell}^2 g(d_i z)$ is an eigenform of level N/ℓ^2 if $\ell^2 \mid d_i$. This means that for all i , one of $g(d_i z)$, $U_{\ell} g(d_i z)$ or $U_{\ell}^2 g(d_i z)$ is an eigenform of level coprime to ℓ . So, by case 4, we have $Q_{\ell}(U_{\ell})(U_{\ell}^2 g(d_i z)) = 0$ for all i as

both $Q_\ell(U_\ell)(U_\ell g(d_i z)) = 0$ and $Q_\ell(U_\ell)(g(d_i z)) = 0$ imply $Q_\ell(U_\ell)(U_\ell^2 g(d_i z)) = 0$. Hence, we have $Q_\ell(U_\ell)(U_\ell^2 f) = 0$.

From the previous paragraphs, we conclude that, if f is an eigenform of level N lifting $\bar{\rho}$, then $(U_\ell^3 Q_\ell(U_\ell))f = 0$, which means $U_\ell^3 Q_\ell(U_\ell) = 0$. As $Q_\ell(x)$ is a monic polynomial with coefficients in $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$, U_ℓ is integral over $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ for all primes ℓ dividing N . Therefore, $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$ is an integral extension of $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ and $A_{\bar{\rho}}^{\Gamma_1(N), \text{pf}}$ is an integral extension of $A_{\bar{\rho}}^{\Gamma_1(N)}$. Now using the argument used for the new case above, we get that $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}}$ is isomorphic to $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$. This concludes the proof of the second part of [Theorem 3](#) for the $\Gamma_1(N)$ case.

Let $\bar{\rho}$ be a $\Gamma_0(N)$ -modular representation which satisfies the hypothesis of [Theorem 3](#). Then, by the arguments above, it follows that $A_{\bar{\rho}}^{\Gamma_1(N)}$ has Krull dimension 2 and $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$ is isomorphic to $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}}$. We have a natural surjective map $A_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_0(N)}$. By [Theorem 1](#), the Krull dimension of $A_{\bar{\rho}}^{\Gamma_0(N)}$ is at least 2. Hence, we conclude that the Krull dimension of $A_{\bar{\rho}}^{\Gamma_0(N)}$ is exactly 2. If p does not divide $\phi(N)$, then as observed in the introduction, the natural surjective map $A_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_0(N)}$ is an isomorphism. Hence, we get that $(A_{\bar{\rho}}^{\Gamma_0(N)})^{\text{red}}$ is isomorphic to $(\tilde{R}_{\bar{\rho}}^0)^{\text{red}}$. Thus, [Theorem 3](#) is proved in the $\Gamma_0(N)$ case.

Remark. Let $\bar{\rho}$ be a $\Gamma_1(N)$ -modular representation satisfying the hypothesis of [Theorem 3](#) and assume $p \nmid \phi(N)$. Moreover, assume that if $\ell \parallel N$ and $p \mid \ell + 1$, then $\det \bar{\rho}(I_\ell) \neq 1$. Hence, from the proof of [Proposition 20](#) and the proof of [Theorem 1](#), we see that the kernel of the map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_1(N)}$ is $m_\Lambda \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. As $p \nmid \phi(N)$, the kernel of the surjective map $R_{\bar{\rho}} \rightarrow \tilde{R}_{\bar{\rho}}^0$ is $m_\Lambda R_{\bar{\rho}}$. As $\bar{\rho}$ satisfies the hypotheses of [Theorem 3](#), we get that $R_{\bar{\rho}} \simeq \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. Hence, by combining all the observations above, we see that the natural surjective map $\tilde{R}_{\bar{\rho}}^0 \rightarrow A_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. Thus, for this case, we get a stronger statement than [Theorem 3](#).

9. Proof of [Theorem 2](#)

Throughout this section Γ means either $\Gamma_0(N)$ or $\Gamma_1(N)$ with $p \nmid \phi(N)$. Assume that $\bar{\rho}$ is an unobstructed Γ -modular representation. So, by assumption, the cotangent space of $\tilde{R}_{\bar{\rho}}^0$ has dimension 2 and this implies that its Krull dimension is at most 2. We have a surjective morphism $\tilde{R}_{\bar{\rho}}^0 \rightarrow A_{\bar{\rho}}^\Gamma$. The Krull dimension of $A_{\bar{\rho}}^\Gamma$ is at least 2 by [Theorem 1](#). Hence, the Krull dimension of $\tilde{R}_{\bar{\rho}}^0$ is exactly 2 and it is a regular local ring of dimension 2. Therefore, by [[Eisenbud 1995](#), Proposition 10.16], it is isomorphic to $\mathbb{F}[[x, y]]$. Hence, the surjective map $\tilde{R}_{\bar{\rho}}^0 \rightarrow A_{\bar{\rho}}^\Gamma$ is an isomorphism. This proves [Theorem 2](#) for the cases considered above.

If $\bar{\rho}$ is an unobstructed $\Gamma_1(N)$ -modular representation and $p \mid \phi(N)$, then we have a surjective morphism $\tilde{R}_{\bar{\rho}}^0 \rightarrow (A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$. By [Theorem 1](#), the Krull dimension of $A_{\bar{\rho}}^{\Gamma_1(N)}$, and hence also of $(A_{\bar{\rho}}^{\Gamma_1(N)})^{\text{red}}$, is at least 2. Therefore, the argument used

in the previous paragraph proves [Theorem 2](#) in this case as well. This completes the proof of [Theorem 2](#) in all the cases.

10. Unobstructed modular representations

In this section, we study $\Gamma_1(N)$ -modular representations to determine if and when they can be unobstructed. If $\bar{\rho}$ is irreducible, then $\bar{\rho}$ is unobstructed in our sense if and only if $H^1(G_{\mathbb{Q}, Np}, \text{ad}^0 \bar{\rho})$ has dimension 2 (see [[BK](#), Section 1.4]). By Tate's global Euler characteristic formula, we see that $\dim H^1(G_{\mathbb{Q}, Np}, \text{ad}^0 \bar{\rho}) \geq 2$ with equality if and only if $H^2(G_{\mathbb{Q}, Np}, \text{ad}^0 \bar{\rho}) = 0$. If p does not divide $\phi(N)$, then this is equivalent to $H^2(G_{\mathbb{Q}, Np}, \text{ad} \bar{\rho}) = 0$, as $\text{ad} \bar{\rho} = 1 \oplus \text{ad}^0 \bar{\rho}$ and, by the global Euler characteristic formula, $H^2(G_{\mathbb{Q}, Np}, 1) = 0$. Thus, in this case, $\bar{\rho}$ is unobstructed in our sense if and only if it is unobstructed in the sense of Mazur [[1989](#), Section 1.6]. The study of such unobstructed representations is carried out in [[Weston 2004; 2005](#)] when p does not divide $\phi(N)$.

However, if $\bar{\rho}$ is irreducible and p divides $\phi(N)$, then the global Euler characteristic formula tells us that $H^2(G_{\mathbb{Q}, Np}, 1) \neq 0$ and hence, $H^2(G_{\mathbb{Q}, Np}, \text{ad} \bar{\rho}) \neq 0$. Thus, we see that $\bar{\rho}$ is always obstructed in the sense of Mazur if $p \mid \phi(N)$ (see [[Weston 2004](#), Theorem 4.5]). However, it is not clear a priori if $\bar{\rho}$ can be unobstructed in our sense when $p \mid \phi(N)$. We devote most of this section to study the unobstructedness of reducible and irreducible $\bar{\rho}$'s when $p \mid \phi(N)$.

Throughout this section, we assume that $p \mid \phi(N)$ unless otherwise stated. Note that, as $p \nmid N$ and $p \mid \phi(N)$, there exists at least one prime divisor ℓ of N such that $p \mid \ell - 1$, i.e., $\ell \equiv 1 \pmod{p}$. Let \mathcal{C} be the category of local profinite \mathbb{F} -algebras. Let $D_{\bar{\rho}}$ be the functor from \mathcal{C} to *Set* of deformations of the pseudorepresentation $(\text{tr } \bar{\rho}, \det \bar{\rho})$ and $D_{\bar{\rho}}^0$ be its subfunctor of deformations with constant determinant.

Proposition 24. *If $p \mid \phi(N)$, then every reducible $\Gamma_1(N)$ -modular representation is obstructed.*

Proof. Let $\bar{\rho}$ be a reducible $\Gamma_1(N)$ -modular representation. Up to a twist, $\bar{\rho}$ is of the form $1 \oplus \chi$ where χ is an odd character of $G_{\mathbb{Q}, Np}$ taking values in \mathbb{F} . The character χ is odd because $\bar{\rho}$ is odd. By the main theorem of [[Bellaïche 2012a](#)], we have the following exact sequence involving the tangent space of $D_{\bar{\rho}}$:

$$0 \rightarrow \text{Tan}(D_1 \oplus D_{\chi}) \xrightarrow{i} \text{Tan}(D_{\bar{\rho}}) \rightarrow H^1(G_{\mathbb{Q}, Np}, \chi) \otimes H^1(G_{\mathbb{Q}, Np}, \chi^{-1}) \rightarrow H^2(G_{\mathbb{Q}, Np}, 1)^2.$$

Here, D_{χ} and D_1 are the deformation functors of χ and 1 as characters of $G_{\mathbb{Q}, Np}$, and i is the map which sends a pair of deformations (α, β) of $(\chi, 1)$ on $\mathbb{F}[\epsilon]/(\epsilon^2)$ to the deformation $\alpha + \beta$ of $\text{tr } \bar{\rho} = 1 + \chi$.

Let γ be an element of $H^1(G_{\mathbb{Q}, Np}, \chi) \otimes H^1(G_{\mathbb{Q}, Np}, \chi^{-1})$ which is the image of some deformation (t, d) of $(\text{tr } \bar{\rho}, \det \bar{\rho})$ to $\mathbb{F}[\epsilon]/(\epsilon^2)$. Thus, d is a deformation of $\det \bar{\rho} = \chi$ and hence, an element of $\text{Tan}(D_{\chi}) \subset \text{Tan}(D_1 \oplus D_{\chi})$. Thus, subtracting the

image of this element under the map i from (t, d) gives us an element of $\text{Tan}(D_{\bar{\rho}}^0)$ whose image in $H^1(G_{\mathbb{Q}, Np}, \chi) \otimes H^1(G_{\mathbb{Q}, Np}, \chi^{-1})$ is still γ . Therefore, we have an exact sequence:

$$0 \rightarrow \text{Tan}((D_1 \oplus D_\chi)^0) \xrightarrow{i} \text{Tan}(D_{\bar{\rho}}^0) \rightarrow H^1(G_{\mathbb{Q}, Np}, \chi) \otimes H^1(G_{\mathbb{Q}, Np}, \chi^{-1}) \rightarrow H^2(G_{\mathbb{Q}, Np}, 1)^2.$$

Here, $(D_1 \oplus D_\chi)^0$ is the subfunctor of $D_1 \oplus D_\chi$ parameterizing the deformations (α, β) of $(1, \chi)$ such that $\alpha\beta$ is constant.

Observe that, $\dim H^1(G_{\mathbb{Q}, Np}, 1) = n + 1$, where n is the number of prime divisors of N which are $1 \pmod p$. Hence, $\dim \text{Tan}((D_1 \oplus D_\chi)^0) = n + 1$. Since, i is an injective map, we get $\dim \text{Tan}(D_{\bar{\rho}}^0) \geq n + 1$. Thus, if $n \geq 2$, then $\dim \text{Tan}(D_{\bar{\rho}}^0) \geq 3$. Hence, $\bar{\rho}$ is obstructed when $n \geq 2$, i.e., when there are at least two primes dividing N which are $1 \pmod p$.

Let us now assume that $n = 1$. Thus, there is a unique prime ℓ which divides N and which is $1 \pmod p$. In this case, $\dim H^1(G_{\mathbb{Q}, Np}, 1) = 2$ and by Tate's global Euler characteristic formula, $\dim H^2(G_{\mathbb{Q}, Np}, 1) = 1$. From the exact sequence above, we get that

$$\dim \text{Tan}(D_{\bar{\rho}}^0) \geq 2 + \dim H^1(G_{\mathbb{Q}, Np}, \chi) \dim H^1(G_{\mathbb{Q}, Np}, \chi^{-1}) - 2 \dim H^2(G_{\mathbb{Q}, Np}, 1).$$

We shall distinguish between two cases:

First case. $\chi|_{G_{\mathbb{Q}_\ell}} = 1$: Note that $\omega_p|_{G_{\mathbb{Q}_\ell}} = 1$, since $\ell \equiv 1 \pmod p$.

If $\chi \neq \omega_p$, then, by the Greenberg–Wiles version of Poitou–Tate duality [Washington 1997, Theorem 2], we get that

$$\begin{aligned} \dim H^1(G_{\mathbb{Q}, Np}, \chi) &\geq \dim H^1(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) - \dim H^0(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) \\ &\quad + \dim H^1(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}) - \dim H^0(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}). \end{aligned}$$

As $\ell \equiv 1 \pmod p$, $\dim H^1(G_{\mathbb{Q}_\ell}, 1) = 2$. By the local Euler characteristic formula,

$$\dim H^1(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}) - \dim H^0(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}) = 1 + \dim H^2(G_{\mathbb{Q}_p}, \chi|_{G_{\mathbb{Q}_p}}) \geq 1.$$

Since, $\chi|_{G_{\mathbb{Q}_\ell}} = 1$, we see that $\dim H^1(G_{\mathbb{Q}, Np}, \chi) \geq 2 - 1 + 1 = 2$.

If $\chi = \omega_p$, then, using the Kummer exact sequence, we get that a class in $H^1(G_{\mathbb{Q}, Np}, \omega_p)$ is represented by a cocycle of the form $g \mapsto g(\alpha)/\alpha$, with $\alpha \in \overline{\mathbb{Q}}$, $\alpha^p \in \mathbb{Q}$ and $v_q(\alpha^p) = 0$ for all primes $q \nmid Np$ (See [Washington 1997, Section 1] for more details). Hence, $\dim H^1(G_{\mathbb{Q}, Np}, \omega_p) = n'$, where n' = number of distinct prime factors of Np . Hence, $\dim H^1(G_{\mathbb{Q}, Np}, \omega_p) \geq 2$. Since, $\chi^{-1}|_{G_{\mathbb{Q}_\ell}} = 1$ as well, the same argument gives us $\dim H^1(G_{\mathbb{Q}, Np}, \chi^{-1}) \geq 2$.

Combining all the above calculations yields $\dim \text{Tan}(D_{\bar{\rho}}^0) \geq 2 + (2)(2) - 2(1) = 4$. Hence, if $n = 1$ and $\chi|_{G_{\mathbb{Q}_\ell}} = 1$, then $\bar{\rho}$ is obstructed.

Second case. $\chi|_{G_{\mathbb{Q}_\ell}} \neq 1$: As $\chi|_{G_{\mathbb{Q}_\ell}} \neq 1$, $\omega_p \otimes \chi \neq 1$. Thus, $H^0(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = 0$, $H^2(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = H^0(G_{\mathbb{Q}_\ell}, \omega_p \otimes \chi|_{G_{\mathbb{Q}_\ell}}) = 0$ and hence, $H^1(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = 0$.

Similarly, we get that $H^1(G_{\mathbb{Q}_\ell}, \chi^{-1}|_{G_{\mathbb{Q}_\ell}}) = 0$. Thus, the calculations done in the first case do not imply that $\dim \text{Tan}(D_{\bar{\rho}}^0) > 2$ which is required to prove that $\bar{\rho}$ is obstructed.

Assume that $\bar{\rho}$ is unobstructed. So, the dimension of the tangent space of $\tilde{R}_{\bar{\rho}}^0$ is 2. By [Theorem 1](#), its Krull dimension is at least 2. Hence, $\tilde{R}_{\bar{\rho}}^0$ is isomorphic to $\mathbb{F}[[x, y]]$, the power series ring in two variables. Let $(t^{\text{univ}}, d^{\text{univ}})$ be the universal pseudorepresentation with constant determinant in characteristic p deforming $(\text{tr } \bar{\rho}, \det \bar{\rho})$.

By [\[Bellaïche and Chenevier 2009, Theorem 1.4.4, Chapter 1\]](#),

$$\tilde{R}_{\bar{\rho}}^0[G_{\mathbb{Q}_\ell}]/(\ker(t^{\text{univ}} \circ i_\ell(G_{\mathbb{Q}_\ell}))) \text{ is a GMA of the form } \begin{pmatrix} \tilde{R}_{\bar{\rho}}^0 & B \\ C & \tilde{R}_{\bar{\rho}}^0 \end{pmatrix},$$

where B and C are finitely generated $\tilde{R}_{\bar{\rho}}^0$ -modules and the diagonal entries reduce to 1 and χ modulo the maximal ideal of $\tilde{R}_{\bar{\rho}}^0$. By [\[Bellaïche and Chenevier 2009, Theorem 1.5.5, Chapter 1\]](#), we get injective maps $(B/m_{\tilde{R}_{\bar{\rho}}^0} B)^* \hookrightarrow \text{Ext}_{G_{\mathbb{Q}_\ell}}^1(\chi, 1)$ and $(C/m_{\tilde{R}_{\bar{\rho}}^0} C)^* \hookrightarrow \text{Ext}_{G_{\mathbb{Q}_\ell}}^1(1, \chi)$. Since we have

$$H^1(G_{\mathbb{Q}_\ell}, \chi|_{G_{\mathbb{Q}_\ell}}) = H^1(G_{\mathbb{Q}_\ell}, \chi^{-1}|_{G_{\mathbb{Q}_\ell}}) = 0,$$

it implies that $\text{Ext}_{G_{\mathbb{Q}_\ell}}^1(\chi, 1) = \text{Ext}_{G_{\mathbb{Q}_\ell}}^1(1, \chi) = 0$ and hence, $B = C = 0$. Thus, $t^{\text{univ}} \circ i_\ell(G_{\mathbb{Q}_\ell}) = \kappa_1 + \kappa_2$, where κ_1 and κ_2 are characters of $G_{\mathbb{Q}_\ell}$ taking values in $(\tilde{R}_{\bar{\rho}}^0)^*$ and are deformations of 1 and χ , respectively. Therefore, $t^{\text{univ}} \circ i_\ell$ factors through $G_{\mathbb{Q}_\ell}^{\text{ab}}$, the abelianization of $G_{\mathbb{Q}_\ell}$.

By local class field theory, $G_{\mathbb{Q}_\ell}^{\text{ab}} = \mathbb{Z}_\ell^* \times \hat{\mathbb{Z}}$. Let p^e be the highest power of p dividing $\ell - 1$. Let $a \in G_{\mathbb{Q}_\ell}^{\text{ab}}$ be the unique element of order p^e . Since, a has order p^e and $\mathbb{F}[[x, y]]$ does not have any nontrivial element of order p , it follows that $\kappa_1(a) = \kappa_2(a) = 1$. Let b be a lift of a in $G_{\mathbb{Q}_\ell}$. Hence, $t^{\text{univ}} \circ i_\ell(b) = 2$, i.e., it is constant. Hence, $t \circ i_\ell(b) = 2$ for any deformation t of $\text{tr } \bar{\rho}$ as a pseudocharacter of $G_{\mathbb{Q}, Np}$ with constant determinant in characteristic p . But it is easy to construct an explicit such deformation t with $t \circ i_\ell(b) \neq 2$.

Indeed, to construct such a deformation, first observe that the maximal pro- p subgroup $G_{\mathbb{Q}, Np}^{\text{ab}, p}$ of $G_{\mathbb{Q}, Np}^{\text{ab}}$ is $\mathbb{Z}/p^e\mathbb{Z} \times \mathbb{Z}_p$. Consider a deformation $\tilde{1}$ of 1 as a character of $G_{\mathbb{Q}, Np}$ to $R = \mathbb{F}[[x]][[y]]/(y^{p^e} - 1)$ which maps $G_{\mathbb{Q}, Np}^{\text{ab}, p} \rightarrow 1 + m_R$ in the following way: the topological generator of \mathbb{Z}_p is mapped to $1 + x$ and the generator of $\mathbb{Z}/p^e\mathbb{Z}$ is mapped to y . The character $\chi/\tilde{1}$, which we will denote by $\tilde{\chi}$, is a deformation of χ . Let $\tilde{\rho} = \tilde{1} \oplus \tilde{\chi}$. Thus, $(\text{tr } \tilde{\rho}, \det \tilde{\rho})$ is a deformation of $(\text{tr } \bar{\rho}, \det \bar{\rho})$ to R with constant determinant. We claim that $\text{tr}(\tilde{\rho} \circ i_\ell(b)) \neq 2$.

To prove the claim, first consider the map $i_\ell^{\text{ab}} : G_{\mathbb{Q}_\ell}^{\text{ab}} \rightarrow G_{\mathbb{Q}, Np}^{\text{ab}}$ induced from i_ℓ by passing to the abelianizations of both the groups. By class field theory, $G_{\mathbb{Q}, Np}^{\text{ab}} = \prod \mathbb{Z}_q^*$, where the product is taken over primes q which divide Np . By the local-global compatibility of class field theory, the \mathbb{Z}_ℓ^* component of $G_{\mathbb{Q}, Np}^{\text{ab}}$ lies in the image of $\mathbb{Z}_\ell^* \subset G_{\mathbb{Q}_\ell}^{\text{ab}}$ under the map i_ℓ^{ab} . Thus, the unique element of $G_{\mathbb{Q}, Np}^{\text{ab}}$

of order p^e is $i_\ell^{\text{ab}}(a)$. Therefore,

$$\text{tr}(\tilde{\rho} \circ i_\ell(b)) = \tilde{\Gamma}(i_\ell^{\text{ab}}(a)) + \tilde{\chi}(i_\ell^{\text{ab}}(a)) = y + y^{-1}.$$

If $y + y^{-1} = 2$, then it would imply $(y - 1)^2 = 0$. But this relation does not hold in R . Hence, $\text{tr}(\tilde{\rho} \circ i_\ell(b)) \neq 2$ and our claim is proved.

Thus, we get a contradiction to our hypothesis that $\tilde{\rho}$ is unobstructed. Therefore, in this case as well, $\tilde{\rho}$ is obstructed.

Therefore, combining all the results above, we get that if $\tilde{\rho}$ is a reducible $\Gamma_1(N)$ -modular representation and if $p \mid \phi(N)$, then $\tilde{\rho}$ is obstructed. \square

Remark. Now suppose that $\tilde{\rho}$ is a reducible $\Gamma_1(N)$ -modular representation and p does not divide $\phi(N)$. Up to a twist, $\tilde{\rho} = 1 \oplus \chi$. If $p \nmid \phi(N)$, then

$$\dim H^1(G_{\mathbb{Q}, Np}, 1) = 1,$$

and Tate's global Euler characteristic formula implies that $H^2(G_{\mathbb{Q}, Np}, 1) = 0$. The exact sequence of [Bellaïche 2012a] considered on page 25 implies

$$\dim \text{Tan}(D_{\tilde{\rho}}^0) = 1 + \dim H^1(G_{\mathbb{Q}, Np}, \chi) \dim H^1(G_{\mathbb{Q}, Np}, \chi^{-1}).$$

If $\chi = \omega_p$, then by the Kummer theory argument above, it follows that the number of distinct prime divisors of Np is $\dim H^1(G_{\mathbb{Q}, Np}, \omega_p)$ and hence, greater than 1. Hence, if $\chi = \omega_p, \omega_p^{-1}$, then $\dim \text{Tan}(D_{\tilde{\rho}}^0) > 2$, so in this case $\tilde{\rho}$ is obstructed. If $\chi \neq \omega_p, \omega_p^{-1}$, then by the Greenberg–Wiles version of Poitou–Tate duality, we get that

$$k + 1 \leq \dim H^1(G_{\mathbb{Q}, Np}, \chi) \leq k + 1 + \dim A(\chi^{-1}\omega_p)$$

and

$$k' + 1 \leq \dim H^1(G_{\mathbb{Q}, Np}, \chi^{-1}) \leq k' + 1 + \dim A'(\chi\omega_p),$$

where k is the number of prime divisors ℓ of N such that $\chi|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$, k' is the number of prime divisors ℓ of N such that $\chi^{-1}|_{G_{\mathbb{Q}_\ell}} = \omega_p|_{G_{\mathbb{Q}_\ell}}$, $A(\chi^{-1}\omega_p)$ is the part of the p -torsion subgroup of the class group of the totally real abelian extension F of \mathbb{Q} fixed by $\text{Ker}(\chi^{-1}\omega_p)$ on which $\text{Gal}(F/\mathbb{Q})$ acts by $\chi^{-1}\omega_p$ and $A'(\chi\omega_p)$ is the part of the p -torsion subgroup of the class group of the totally real abelian extension F' of \mathbb{Q} fixed by $\text{Ker}(\chi\omega_p)$ on which $\text{Gal}(F'/\mathbb{Q})$ acts by $\chi\omega_p$.

Proposition 25. *Suppose $p \mid \phi(N)$. Let $\tilde{\rho}$ be an absolutely irreducible $\Gamma_1(N)$ -modular representation such that $\tilde{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible for at least one prime ℓ dividing N which is $1 \pmod p$. Then, $\tilde{\rho}$ is obstructed.*

Proof. Let $\tilde{\rho}$ be an absolutely irreducible $\Gamma_1(N)$ -modular representation. By [Weston 2005, Lemma 2.5], we have

$$\dim_{\mathbb{F}} H^2(G_{\mathbb{Q}, Np}, \text{ad}\tilde{\rho}) = \dim_{\mathbb{F}} \text{III}^1(G_{\mathbb{Q}, Np}, \omega_p \otimes \text{ad}\tilde{\rho}) + \sum_{q \mid Np} \dim_{\mathbb{F}} H^0(G_{\mathbb{Q}_q}, \omega_p \otimes \text{ad}\tilde{\rho}).$$

Note that $\text{ad}\bar{\rho} = 1 \oplus \text{ad}^0\bar{\rho}$. By Tate's global Euler characteristic formula, we get that $\dim_{\mathbb{F}} H^2(G_{\mathbb{Q}, Np}, 1) = n$, where $n =$ number of prime divisors of N which are $1 \pmod p$. For a prime q , $\dim_{\mathbb{F}} H^0(G_{\mathbb{Q}_q}, \omega_p)$ is 1 if q is $1 \pmod p$ and 0, otherwise. By removing the contributions of the trivial representation from both sides of the formula above, we get

$$\dim_{\mathbb{F}} H^2(G_{\mathbb{Q}, Np}, \text{ad}^0\bar{\rho}) = \dim_{\mathbb{F}} \text{III}^1(G_{\mathbb{Q}, Np}, \omega_p \otimes \text{ad}^0\bar{\rho}) + \sum_{q|Np} \dim_{\mathbb{F}} H^0(G_{\mathbb{Q}_q}, \omega_p \otimes \text{ad}^0\bar{\rho}).$$

Now, let ℓ be a prime dividing N such that $\ell - 1$ is divisible by p . Suppose that $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible. Thus, $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is an extension of a character χ_1 by a character χ_2 . If $\chi_1 \neq \chi_2$, then, by Tate's local Euler characteristic formula, $\text{Ext}_{G_{\mathbb{Q}_\ell}}^1(\chi_1, \chi_2) = 0$. So, $\bar{\rho}|_{G_{\mathbb{Q}_\ell}} = \chi_1 \oplus \chi_2$ and $\text{ad}^0\bar{\rho}|_{G_{\mathbb{Q}_\ell}} = 1 \oplus \chi_1\chi_2^{-1} \oplus \chi_1^{-1}\chi_2$. As $p|\ell - 1$, $\omega_p(G_{\mathbb{Q}_\ell}) = 1$ and $H^0(G_{\mathbb{Q}_\ell}, \omega_p \otimes \text{ad}^0\bar{\rho}) = H^0(G_{\mathbb{Q}_\ell}, \text{ad}^0\bar{\rho})$ and in this case, both of them are nonzero. Hence, if $\chi_1 \neq \chi_2$, then, by the formula above, $H^2(G_{\mathbb{Q}, Np}, \text{ad}^0\bar{\rho}) \neq 0$ and $\bar{\rho}$ is obstructed.

If $\chi_1 = \chi_2 = \chi$, then $\rho|_{G_{\mathbb{Q}_\ell}}$ is either $\chi \oplus \chi$ or a nontrivial extension of χ by itself. If $\rho|_{G_{\mathbb{Q}_\ell}} = \chi \oplus \chi$, then clearly $1 \subset \text{ad}^0\bar{\rho}$. If $\rho|_{G_{\mathbb{Q}_\ell}}$ is a nontrivial extension of χ by itself, then choose a basis of $\bar{\rho}$ such that $\bar{\rho}(G_{\mathbb{Q}_\ell})$ is upper triangular and identify $\text{ad}^0\bar{\rho}$ with the subspace of trace 0 matrices of $M_2(\mathbb{F})$. Then, $G_{\mathbb{Q}_\ell}$ acts trivially on the subspace of $\text{ad}^0\bar{\rho}$ generated by the element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, $1 \subset \text{ad}^0\bar{\rho}$ in this case also. Therefore, if $\chi_1 = \chi_2$, then

$$H^0(G_{\mathbb{Q}_\ell}, \omega_p \otimes \text{ad}^0\bar{\rho}) = H^0(G_{\mathbb{Q}_\ell}, \text{ad}^0\bar{\rho}) \neq 0$$

and hence, by the formula above, $H^2(G_{\mathbb{Q}, Np}, \text{ad}^0\bar{\rho}) \neq 0$, which implies that $\bar{\rho}$ is obstructed. This concludes the proof of [Proposition 25](#). \square

We would like to determine the cases when the situation considered as above would arise, i.e., when $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ would be reducible. Note that, if f is an eigenform of level N and ρ_f is the Galois representation attached to it, then we know all the possible descriptions of $\rho_f|_{G_{\mathbb{Q}_\ell}}$ for a prime ℓ dividing N . So, we will now analyze all the possible descriptions of $\rho_f|_{G_{\mathbb{Q}_\ell}}$ for an eigenform f lifting $\bar{\rho}$ to determine which of them will make $\bar{\rho}$ obstructed and when can they arise. This will give us some conditions on $\bar{\rho}$ which will force it to be obstructed.

Let f be an eigenform of level N lifting $\bar{\rho}$, ℓ be a prime dividing N which is $1 \pmod p$, ρ_f be the p -adic Galois representation attached to f and π_ℓ be the ℓ -component of the automorphic representation associated to f . If π_ℓ is either principal series or special, then by the local Langlands correspondence, we see that $\rho_f|_{G_{\mathbb{Q}_\ell}}$, and hence $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$, is reducible (see Sections 3 and 5 of [\[Weston 2004\]](#) for a similar analysis). This implies that $\bar{\rho}$ is obstructed. In particular, if $\ell|N$ but $\ell^2 \nmid N$, then we know that π_ℓ is either special or principal series (see [\[Carayol 1989, Section 1.2\]](#)). If $(\bar{\rho})^{\ell^2}$, the subspace of $\bar{\rho}$ on which I_ℓ acts trivially, is nonzero, then

clearly $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible. If there exists an eigenform f of level M which lifts $\bar{\rho}$ such that there exists at least one prime divisor ℓ of N which is $1 \pmod p$ and which does not divide M , then $\bar{\rho}$ is unramified at ℓ and hence, $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible. Therefore, in these cases, $\bar{\rho}$ is obstructed.

On the other hand, if π_ℓ is supercuspidal, then, as $\ell \neq 2$, it follows, from the local Langlands correspondence, that $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is induced from a character χ of $G_K = \text{Gal}(\overline{\mathbb{Q}_\ell}/K)$, where K is a quadratic extension of \mathbb{Q}_ℓ and moreover, $\rho_f|_{G_{\mathbb{Q}_\ell}}$ is irreducible (see the proof of Proposition 3.2 of [Weston 2004]). Let χ^σ be the $\text{Gal}(K/\mathbb{Q}_\ell)$ -conjugate character of χ . So, $\rho_f|_{G_K} \simeq \chi \oplus \chi^\sigma$. If $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is reducible, then clearly $\bar{\chi} = \overline{\chi^\sigma}$, where $\bar{\chi}$ and $\overline{\chi^\sigma}$ are the reductions of χ and χ^σ in characteristic p , respectively. This means that χ/χ^σ factors through the maximal abelian pro- p quotient of G_K . But it follows, from local class field theory and the assumption that $p|\ell - 1$, that the maximal abelian pro- p extension of K is also abelian over \mathbb{Q}_ℓ . So, if χ/χ^σ factors through the maximal abelian pro- p quotient of G_K , then it implies that $\rho_f|_{G_{\mathbb{Q}_\ell}} = \text{Ind}_{G_K}^{G_{\mathbb{Q}_\ell}} \chi$ is reducible, contradicting the hypothesis that it is irreducible. Thus, we see that if π_ℓ is supercuspidal and $p|\ell - 1$, then $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is not reducible. In summary we have:

Corollary 26. *Suppose $p|\phi(N)$. Let $\bar{\rho}$ be an absolutely irreducible $\Gamma_1(N)$ -modular representation satisfying one of the following conditions:*

- (1) *There exists at least one prime ℓ which is $1 \pmod p$ such that $\ell|N$ and $\ell^2 \nmid N$.*
- (2) *There exists an eigenform f of level M which lifts $\bar{\rho}$ such that there is at least one prime divisor ℓ of N which is $1 \pmod p$ and which either does not divide M or divides M with multiplicity 1.*
- (3) *The subspace $(\bar{\rho})^{\ell\ell}$ is nontrivial.*

Then, $\bar{\rho}$ is obstructed.

The only case which remains to be considered is when $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is irreducible for all the prime divisors ℓ of N which are $1 \pmod p$. Let $\bar{\rho}$ be such a $\Gamma_1(N)$ -modular representation. In this case, $H^0(G_{\mathbb{Q}_\ell}, \omega_p \otimes \text{ad}^0 \bar{\rho}) = H^0(G_{\mathbb{Q}_\ell}, \text{ad}^0 \bar{\rho}) = 0$ for all such primes ℓ . Note that, if f is any eigenform which lifts $\bar{\rho}$, then the local component π_ℓ of the automorphic representation associated to f is supercuspidal at all such primes. The analysis of contributions coming from p and other prime divisors of N in the formula above, which are not $1 \pmod p$, is done in [Weston 2004, Sections 3–5; 2005, Section 3] to give conditions for vanishing of corresponding H^0 's. We can assume that those conditions hold as well so that we can ignore those primes. Note that these conditions neither depend on each other nor put any restrictions on each other. So assuming them together does not yield any immediate contradiction. Now, moreover assume that $\bar{\rho}$ does not come from a weight 1 modular form. We need this hypothesis to use the results of Section 3 of [Diamond et al. 2004]. We will

analyze the only remaining group $\text{III}^1(G_{\mathbb{Q}, Np}, \omega_p \otimes \text{ad}^0 \bar{\rho})$ in this case following the approach of Weston [2004; 2005].

Let g be an eigenform such that ρ_g is a minimally ramified lift of $\bar{\rho}$ (such a lift does exist; see [Diamond et al. 2004, Section 3.2]). Hence, it is an eigenform of level N and its weight k lies between 2 and $p - 1$. Let K be a finite extension of \mathbb{Q}_p generated by the eigenvalues of g , \mathcal{O} be its ring of integers and \mathfrak{m} be its maximal ideal. Denote by A_{ρ_g} the module $(K/\mathcal{O})^3$ on which $G_{\mathbb{Q}, Np}$ acts via $\text{ad}^0 \rho_g$. We define $H_{\mathcal{O}}^1(G_{\mathbb{Q}}, A_{\rho_g})$ as in [Diamond et al. 2004, Section 2.1] (see also [Weston 2005, Section 2.2]).

By [Diamond et al. 2004, Theorem 2.15], we get that

$$H_f^1(G_{\mathbb{Q}}, \rho_g) = H_f^1(G_{\mathbb{Q}}, \rho_g(1)) = 0,$$

where $\rho_g(1)$ is the Tate twist of ρ_g and $H_f^1(G_{\mathbb{Q}}, \rho_g)$, $H_f^1(G_{\mathbb{Q}}, \rho_g(1))$ are the Bloch–Kato Selmer groups defined as in Section 2.1 of [Diamond et al. 2004]. It follows, from [Weston 2005, Lemma 2.6], that

$$\dim_{\mathbb{F}} \text{III}^1(G_{\mathbb{Q}, Np}, \omega_p \otimes \text{ad}^0 \bar{\rho}) \leq \dim_{\mathbb{F}} H_{\mathcal{O}}^1(G_{\mathbb{Q}}, A_{\rho_g})[\mathfrak{m}],$$

where $H_{\mathcal{O}}^1(G_{\mathbb{Q}}, A_{\rho_g})[\mathfrak{m}]$ is the \mathfrak{m} -torsion of $H_{\mathcal{O}}^1(G_{\mathbb{Q}}, A_{\rho_g})$. Now, [Diamond et al. 2004, Theorem 3.7] implies that the length of $H_{\mathcal{O}}^1(G_{\mathbb{Q}}, A_{\rho_g})$ is $v_{\mathfrak{m}}(\eta_g^{\mathcal{O}})$. Here, $v_{\mathfrak{m}}$ is the \mathfrak{m} -adic valuation and $\eta_g^{\mathcal{O}}$ is the congruence ideal of g defined in [Diamond et al. 2004, Section 1.7]. From the proof of [Weston 2005, Proposition 4.2], we see that $v_{\mathfrak{m}}(\eta_g^{\mathcal{O}}) > 0$ if and only if \mathfrak{m} is a congruence prime for g , i.e., if there exists an eigenform h which lifts $\bar{\rho}$ and which is not a Galois conjugate of g .

Let ϵ be the nebentypus of g . Let ψ be a character of $(\mathbb{Z}/N\mathbb{Z})^*$ whose order is a power of p (there exists such a character as $p \mid \phi(N)$). Thus, ϵ and $\psi\epsilon$ have the same reduction modulo \mathfrak{m} . Hence, from [Carayol 1989, Proposition 3], it follows that there exists an eigenform h of level N and nebentypus $\psi\epsilon$ which lifts $\bar{\rho}$. If h is a Galois conjugate of g , then $\epsilon((\mathbb{Z}/N\mathbb{Z})^*)$ and $\psi\epsilon((\mathbb{Z}/N\mathbb{Z})^*)$ will be Galois conjugates of each other. Since, g is a minimal lift of $\bar{\rho}$, the p -part of $\epsilon((\mathbb{Z}/N\mathbb{Z})^*)$ is trivial. But the p -part of $\psi\epsilon((\mathbb{Z}/N\mathbb{Z})^*)$ is nontrivial as ψ is a character of p -power order. So, they can't be Galois conjugates of each other. Therefore, g and h are not Galois conjugates of each other. Thus, \mathfrak{m} is a congruence prime for g and $H_{\mathcal{O}}^1(G_{\mathbb{Q}}, A_{\rho_g}) \neq 0$. However, this does not ensure that $\text{III}^1(G_{\mathbb{Q}, Np}, \omega_p \otimes \text{ad}^0 \bar{\rho})$ is nonzero because we do not know whether the injection $\text{III}^1(G_{\mathbb{Q}, Np}, \omega_p \otimes \text{ad}^0 \bar{\rho}) \rightarrow H_{\mathcal{O}}^1(G_{\mathbb{Q}}, A_{\rho_g})[\mathfrak{m}]$ given in the proof of Lemma 2.6 of [Weston 2005], which we used in the last paragraph, is an isomorphism, and in general it is very difficult to prove such a result (see [Weston 2005, Lemma 2.6, Remark 2.7] and the discussion preceding them for more details). But, based on the calculations above, we do expect that $\text{III}^1(G_{\mathbb{Q}, Np}, \omega_p \otimes \text{ad}^0 \bar{\rho})$ is nonzero.

Let us consider the modular representations coming from a weight-1 modular form. In the proposition below, we do not put the condition $p \mid \phi(N)$. For a number field L with ring of integers \mathcal{O}_L , we denote by $A(L)[p]$ the p -torsion subgroup of the class group of L and we let

$$U(L) = \ker \left(\frac{\mathcal{O}_L^*}{(\mathcal{O}_L^*)^p} \rightarrow \left(\prod_{v \mid p} \frac{\mathcal{O}_{L_v}^*}{(\mathcal{O}_{L_v}^*)^p} \right) \right).$$

Proposition 27. *Let $\bar{\rho}$ be a $\Gamma_1(N)$ -modular representation coming from a regular modular form f of weight 1 which has either RM or CM by F (see [Bellaïche and Dimitrov 2016] for the definition of regular). Let H be the extension of \mathbb{Q} which is fixed by $\text{Ker}(\text{ad}^0 \bar{\rho})$. Moreover, assume the following conditions:*

- (1) *If $\ell \mid N$ and $p \mid \ell + 1$, then ℓ does not stay inert in F .*
- (2) *If ℓ is a prime divisor of N , then $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is irreducible.*
- (3) *$A(F)[p] = 0$ and $\text{Hom}_G(A(H)[p], \text{ad}^0 \bar{\rho}) = 0$.*
- (4) *$\text{Hom}_G(U(F), \text{ad}^0 \bar{\rho}) = 0$ and if F is imaginary, then $\text{Hom}_G(U(H), \text{ad}^0 \bar{\rho}) = 0$.*

Then, $\bar{\rho}$ is unobstructed.

Proof. Note that, $\bar{\rho} = \text{Ind}_{G_F}^{G_{\mathbb{Q}}} \delta$, where F is a quadratic extension of \mathbb{Q} which is either real or imaginary and $\delta : G_F \rightarrow \mathbb{F}^*$ is a character, as it comes from a weight-1 form which has RM or CM. Let $\text{Gal}(F/\mathbb{Q}) = G' = \{1, \sigma\}$. As f is a regular weight-1 eigenform, $\rho_f|_{G_{\mathbb{Q}_p}} = \chi_1 \oplus \chi_2$, where χ_1 and χ_2 are distinct, unramified characters from $G_{\mathbb{Q}_p} \rightarrow (\mathbb{Q}_p^*)^*$. So, F and the fixed field H of $\text{Ker}(\text{ad}^0 \bar{\rho})$ are unramified at p . From the hypothesis (2) above, we see that all prime divisors ℓ of N are either inert or ramified in F .

Thus, from above, it follows that $\text{ad}^0 \bar{\rho} = \epsilon \oplus \text{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi$, where ϵ is the character of $G_{\mathbb{Q}}$ of order 2 corresponding to F and $\chi = \delta/\delta^\sigma$. So

$$H^1(G_{\mathbb{Q}, Np}, \text{ad}^0 \bar{\rho}) = H^1(G_{\mathbb{Q}, Np}, \epsilon) \oplus H^1(G_{\mathbb{Q}, Np}, \text{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi).$$

We will analyze each part of this sum separately. We mostly follow the notation of [Bellaïche and Dimitrov 2016].

First, we will look at the group $H^1(G_{\mathbb{Q}, Np}, \epsilon)$ appearing above. From the inflation-restriction sequence, we see that $H^1(G_{\mathbb{Q}, Np}, \epsilon) = (\text{Hom}(G_{F,S}, \mathbb{F}) \otimes \epsilon)^{G'}$, where S is the set of the places of F dividing Np . From class field theory and the hypothesis (3) above, we have the following exact sequence of G' -modules:

$$0 \rightarrow \text{Hom}(G_{F,S}^{\text{ab}}, \mathbb{F}) \rightarrow \text{Hom} \left(\prod_{v \mid Np} \mathcal{O}_{F_v}^*, \mathbb{F} \right) \rightarrow \text{Hom}(\mathcal{O}_F^*, \mathbb{F}).$$

Now, by hypothesis (1), we have:

$$\mathrm{Hom}\left(\prod_{v|Np} \mathcal{O}_{F_v}^*, \mathbb{F}\right) = \mathrm{Hom}\left(\prod_{v|p} \mathcal{O}_{F_v} \prod_{\substack{\ell|N \\ p|\ell-1}} \prod_{v|\ell} \mathbb{Z}/(\ell-1)\mathbb{Z}, \mathbb{F}\right)$$

as F is unramified at p . The factors in the second product come from places of F dividing such primes and, since ℓ is either inert or ramified in F , each factor appears only once. Observe that G' acts trivially on each of them. Thus,

$$\left(\mathrm{Hom}\left(\prod_{\substack{\ell|N \\ p|\ell-1}} \prod_{v|\ell} \mathbb{Z}/(\ell-1)\mathbb{Z}, \mathbb{F}\right) \otimes \epsilon\right)^{G'} = 0.$$

If F is imaginary, then \mathcal{O}_F^* is a finite group and its order is not divisible by p , as p is unramified in F . Thus, $U(F) = \mathcal{O}_F^*/(\mathcal{O}_F^*)^p = 0$. Hence, in this case, it follows that $(\mathrm{Hom}(\prod_{v|p} \mathcal{O}_{F_v}, \mathbb{F}) \otimes \epsilon)^{G'}$ has dimension 1. Therefore, $H^1(G_{\mathbb{Q}, Np}, \epsilon)$ has dimension 1. If F is real, then the free part of \mathcal{O}_F^* has rank 1 and the torsion part is $\{1, -1\}$. Suppose p is not split in F . Then, it is inert in F and the fundamental unit of F generates the residue field of \mathcal{O}_{F_p} over \mathbb{F}_p . Suppose p is split in F . Then $\prod_{v|p} \mathcal{O}_{F_v} = \mathbb{Z}_p \times \mathbb{Z}_p$ and the action of G' switches them. Moreover, under the G' -equivariant diagonal embedding $\mathcal{O}_F^* \rightarrow \prod_{v|p} \mathcal{O}_{F_v}^* = \mathbb{Z}_p^* \times \mathbb{Z}_p^*$, the fundamental unit of F gets mapped to an element of the form $(a, -a)$ as the nontrivial element of G' sends the unit to its inverse. So, it follows, from the discussion above and hypothesis (4), that in both the cases, G' acts trivially on the subspace of elements of $\mathrm{Hom}(\prod_{v|p} \mathcal{O}_{F_v}, \mathbb{F})$ which vanish on \mathcal{O}_F^* . An element of $\mathrm{Hom}(G_{F,S}^{\mathrm{ab}}, \mathbb{F})$ vanishes on the fundamental unit. Thus, combining this and the previous paragraph, we see that G' acts trivially on $\mathrm{Hom}(G_{F,S}^{\mathrm{ab}}, \mathbb{F})$, when F is totally real. It follows that, in this case, $H^1(G_{\mathbb{Q}, Np}, \epsilon) = (\mathrm{Hom}(G_{F,S}^{\mathrm{ab}}, \mathbb{F}) \otimes \epsilon)^{G'} = 0$.

We shall now analyze the second factor appearing in the sum above. Let H be the extension of \mathbb{Q} which is fixed by $\mathrm{Ker}(\mathrm{ad}^0 \bar{\rho})$. Thus, $G = \mathrm{Gal}(H/\mathbb{Q})$ is a dihedral group D_{2n} which is nonabelian and $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi$ is an irreducible representation of G . By the arguments used above, we get that

$$H^1(G_{\mathbb{Q}, Np}, \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi) = (H^1(G_{H,S}, \mathbb{F}) \otimes \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \chi)^G.$$

By hypothesis (3), $H^1(G_{H,S}, \mathbb{F})$ is the subspace of $\mathrm{Hom}(\prod_{v|Np} \mathcal{O}_{H_v}^*, \mathbb{F})$ vanishing on \mathcal{O}_H^* . Since H is unramified at p , we get

$$\mathrm{Hom}\left(\prod_{v|Np} \mathcal{O}_{H_v}^*, \mathbb{F}\right) = \mathrm{Hom}\left(\prod_{v|p} \mathcal{O}_{H_v} \prod \mathbb{Z}/p\mathbb{Z}, \mathbb{F}\right),$$

where the last product is taken over all places v of H which divide N and whose residue fields have p -th roots of unity. If ℓ is a such a prime, then the G -submodule

$\text{Hom}(\prod_{v|\ell} \mathbb{Z}/p\mathbb{Z}, \mathbb{F})$ of the module above is isomorphic to $\text{Ind}_{D_\ell}^G \alpha$, where D_ℓ is the image of $G_{\mathbb{Q}_\ell}$ under $\text{ad}^0 \bar{\rho}$ and α is the character by which it acts on the p -th roots of unity. Note that, by hypothesis (2) above, $\text{Ind}_{G_F}^{G_\mathbb{Q}} \chi$ is an irreducible representation of D_ℓ for every $\ell | N$. So, $(\text{Hom}(\prod \mathbb{Z}/p\mathbb{Z}, \mathbb{F}) \otimes \text{Ind}_{G_F}^{G_\mathbb{Q}} \chi)^G = 0$.

Assume F is imaginary. As H is unramified at p and $p \nmid |G|$, we see, using arguments from [Bellaïche and Dimitrov 2016, Section 3.2], that every irreducible representation of G occurs in $\text{Hom}(G_{H,S}^{\text{ab}}, \mathbb{F})$ with multiplicity at least 1. The additional multiplicities would only arise from $U(H)$. But, from hypothesis (4) above, we have $\text{Hom}_G(U(H), \text{Ind}_{G_F}^{G_\mathbb{Q}} \chi) = 0$. Therefore, $\text{Ind}_{G_F}^{G_\mathbb{Q}} \chi$ occurs in $\text{Hom}(G_{H,S}^{\text{ab}}, \mathbb{F})$ with multiplicity 1. Hence, $(\text{Hom}(G_{H,S}, \mathbb{F}) \otimes \text{Ind}_{G_F}^{G_\mathbb{Q}} \chi)^G = H^1(G_{\mathbb{Q}, Np}, \text{Ind}_{G_F}^{G_\mathbb{Q}} \chi)$ has dimension 1. If F is real, then $\text{Ind}_{G_F}^{G_\mathbb{Q}} \chi$ is totally odd. Thus, from the arguments in [Bellaïche and Dimitrov 2016, Section 3.2], we see that $\text{Ind}_{G_F}^{G_\mathbb{Q}} \chi$ occurs in the subspace of $\text{Hom}(\prod_{v|p} \mathcal{O}_{H_v}^*, \mathbb{F})$ vanishing on \mathcal{O}_H^* with multiplicity 2. Note that, we do not need to consider contribution from $U(H)$ as $\text{Ind}_{G_F}^{G_\mathbb{Q}} \chi$ does not occur in $\mathcal{O}_H^*/(\mathcal{O}_H^*)^p$. Therefore, combining this with the previous paragraph, we see that $\text{Ind}_{G_F}^{G_\mathbb{Q}} \chi$ occurs in $\text{Hom}(G_{H,S}^{\text{ab}}, \mathbb{F})$ with multiplicity 2. Hence, $H^1(G_{\mathbb{Q}, Np}, \text{Ind}_{G_F}^{G_\mathbb{Q}} \chi)$ has dimension 2.

Combining these results we see that if $\bar{\rho}$ satisfies the conditions of the proposition, then $H^1(G_{\mathbb{Q}, Np}, \text{ad}^0 \bar{\rho})$ has dimension 2. Hence, $\bar{\rho}$ is unobstructed in our sense. \square

- Remark.** (1) The hypotheses (3) and (4) are similar to those used by Mazur [1989, Sections 1.12 and 1.13], where he studies unobstructed representations unramified outside a single prime. Note that, for a number field L , $A(L(\zeta_p))[p] = 0$ implies that $U(L) = 0$ (see the remark after Proposition 1 of [Boston 1990]).
- (2) We know that $G' = \text{Gal}(F/\mathbb{Q})$ acts on $A(F)[p]$ by the character ϵ . So, if $A(F)[p] \neq 0$, then it would contribute to $H^1(G_{\mathbb{Q}, Np}, \epsilon)$ in addition to what we have calculated above. Hence, in this case, we have $\dim H^1(G_{\mathbb{Q}, Np}, \epsilon) > 1$. Thus, considering the above calculations, we see that $\dim H^1(G_{\mathbb{Q}, Np}, \text{ad}^0 \bar{\rho}) > 2$. Hence, if $A(F)[p] \neq 0$, then $\bar{\rho}$ will be obstructed.
- (3) Suppose f is a weight-1 form which has RM or CM by F , and there exists a prime ℓ such that $\ell | N$ and $p | \ell + 1$. Moreover, assume that ℓ stays inert in F . Thus, using the notations of the proof above, we see that

$$\text{Hom}\left(\prod_{v|Np} \mathcal{O}_{F_v}^*, \mathbb{F}\right) = \text{Hom}\left(\prod_{v|p} \mathcal{O}_{F_v} \prod_{\substack{\ell | N \\ p | \ell - 1}} \prod_{v|\ell} \mathbb{Z}/(\ell - 1)\mathbb{Z} \prod \mathbb{Z}/p\mathbb{Z}, \mathbb{F}\right),$$

where the last product is taken over all the prime divisors ℓ of N which stay inert in F and which are $-1 \pmod{p}$. Observe that, they are isomorphic to ϵ as G' representations. The projection of the image of \mathcal{O}_F^* in $\prod_{v|Np} \mathcal{O}_{F_v}^*$ onto this product is also trivial as F is unramified at p . Hence, this product also

contributes to $H^1(G_{\mathbb{Q}, Np}, \epsilon)$, making it bigger. All the other calculations at the places above p in the proof above still remain valid. As a result, we see that $\dim H^1(G_{\mathbb{Q}, Np}, \text{ad}^0 \bar{\rho}) > 2$. Hence, in this case, $\bar{\rho}$ is obstructed.

Note that, the assumptions in the proposition above do not yield any immediate contradiction. Thus, we see that our notion of unobstructedness is weaker than Mazur's notion of unobstructedness.

Proposition 28. *Let $\bar{\rho}$ be a modular representation satisfying the hypotheses of Proposition 27 and suppose $p \mid \phi(N)$. Let the p -primary part of $(\mathbb{Z}/N\mathbb{Z})^*$ be $\mathbb{Z}/p^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{e_n}\mathbb{Z}$. Assume moreover that:*

- (1) $\bar{\rho}$ is a new $\Gamma_1(N)$ -modular representation.
- (2) For all primes ℓ dividing N such that $p \mid \ell^2 - 1$, U_ℓ acts like 0 on all newforms of level N which lift $\bar{\rho}$.

Then, the corresponding local component $A_{\bar{\rho}}$ of the mod p Hecke algebra is isomorphic to $\mathbb{F}[[x, y]][y_1, \dots, y_n]/(y_1^{p^{e_1}}, \dots, y_n^{p^{e_n}})$ and thus, is not reduced.

Proof. As $\bar{\rho}$ satisfies the hypotheses of Proposition 27, it satisfies the hypotheses of Theorem 3 as well. Indeed, $\bar{\rho}$ comes from a regular, weight-1 form and is unramified at p . So, $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is of the required form. If ℓ is a prime divisor of N which is 1 mod p , then it splits completely in $\mathbb{Q}(\zeta_p)$. As $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ is irreducible, $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is also irreducible. Therefore, from the results of Böckle [2001], Diamond [1996], Diamond et al. [2004], and Kisin [2004], it follows that the surjective map $R_{\bar{\rho}} \rightarrow \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$ is an isomorphism. The hypothesis (2) along with the proof of Proposition 20 and the proof of Theorem 1 implies that the kernel of the surjective map $\mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)} \rightarrow A_{\bar{\rho}}^{\Gamma_1(N)}$ is $m_\Delta \mathbb{T}_{\bar{\rho}}^{\Gamma_1(N)}$. By Proposition 27, we see that $R_{\bar{\rho}} \simeq \mathcal{O}[[x, y, T]][z_1, \dots, z_n]/(z_1^{p^{e_1}} - 1, \dots, z_n^{p^{e_n}} - 1)$. Combining all of the above gives us that $A_{\bar{\rho}} \simeq \mathbb{F}[[x, y]][y_1, \dots, y_n]/(y_1^{p^{e_1}}, \dots, y_n^{p^{e_n}})$. \square

Remark. (1) If the first condition of Proposition 28 is satisfied, then to check the second condition, it is sufficient, by Lemma 19, to check that $\bar{\rho}|_{I_\ell}$ is irreducible.

- (2) Suppose $\bar{\rho}$ is a $\Gamma_1(N)$ -modular representation which satisfies the two assumptions of Proposition 28 and the assumptions of Theorem 3. Moreover, assume $p \mid \phi(N)$ and let the p -primary part of $(\mathbb{Z}/N\mathbb{Z})^*$ be $\mathbb{Z}/p^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{e_n}\mathbb{Z}$. Then, from the proof above, we see that $A_{\bar{\rho}}^{\Gamma_1(N)} \simeq R_{\bar{\rho}}/(\pi, T)$. Now, as $p \neq 2$,

$$R_{\bar{\rho}} \simeq R_{\bar{\rho}}^0[T, y_1, \dots, y_n]/(y_1^{p^{e_1}} - 1, \dots, y_n^{p^{e_n}} - 1),$$

where $R_{\bar{\rho}}^0$ is the universal deformation ring of $\bar{\rho}$ with constant determinant (see the proof of [BK, Lemma 19] for more details). So, $A_{\bar{\rho}}^{\Gamma_1(N)}$ is isomorphic to $\tilde{R}_{\bar{\rho}}^0[z_1, \dots, z_n]/(z_1^{p^{e_1}}, \dots, z_n^{p^{e_n}})$ and hence, is not reduced. So, the assumptions of Proposition 27 are not necessary to get nonreduced Hecke algebras but are necessary to find the precise structure of the Hecke algebra.

We now give some examples of nonreduced Hecke algebras following the previous remark. Let ℓ be a prime such that $p \mid \ell - 1$ and ℓ is 3 modulo 4. Let $K = \mathbb{Q}(\sqrt{-\ell})$ and let h_K be its class number. If p splits in K , then define n to be the smallest integer such that p is not split in the anticyclotomic extension of degree ℓ^n of K and $\ell^n \nmid h_K$. Otherwise, define it to be the smallest integer such that $\ell^n \nmid h_K$. Let $\chi : G_K \rightarrow \mathbb{Z}_\ell \rightarrow \mathbb{Z}/\ell^n \mathbb{Z} \rightarrow \overline{\mathbb{F}}^*$, where the first map is given by the anticyclotomic \mathbb{Z}_ℓ extension of K , and the last map is the inclusion of the ℓ^n -th roots of unity into $\overline{\mathbb{F}}^*$. Now, we see, from a classical theorem of Hecke, that $\bar{\rho} = \text{Ind}_{G_K}^{G_{\mathbb{Q}}}\chi$ on $G_{\mathbb{Q}, p\ell}$ is an odd, irreducible representation coming from a weight-1 newform f of level ℓ^{2n+3} .

As $\chi^\sigma = \chi^{-1}$ and χ has odd order, they are distinct characters of G_K and they only coincide on $\ker(\chi)$. Now, ℓ ramifies in K and the prime v' of K lying above ℓ ramifies in the anticyclotomic extension of K of degree ℓ^n as $\ell^n \nmid h_K$. As a consequence, we see that $\bar{\rho}|_{I_\ell}$ and $\bar{\rho}|_{G_{\mathbb{Q}_\ell}}$ are irreducible. As ℓ splits completely in $\mathbb{Q}(\zeta_p)$, it follows that $\bar{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is also irreducible. If p is inert in K , then clearly $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is a sum of distinct characters as p is unramified in K . If p is split in K , then for a place v of K above p , χ and χ^σ are distinct characters of G_{K_v} , as χ has odd order and p does not split completely in the anticyclotomic extension of K of degree ℓ^n . As the anticyclotomic extension of K is unramified at p , we see that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is not a direct sum of characters which are cyclotomic twists of each other. Therefore, $\bar{\rho}$ is a new $\Gamma_1(\ell^{2n+3})$ -modular representation satisfying the two conditions of [Proposition 28](#) and the conditions of [Theorem 3](#). Hence, from the remark above, it follows that $A_{\bar{\rho}}^{\Gamma_1(\ell^{2n+3})}$ is not reduced.

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