Existence of compatible systems of
lisse sheaves on arithmetic schemes

Koji Shimizu

Existence of compatible systems of lisse sheaves on arithmetic schemes

Koji Shimizu

Deligne conjectured that a single ℓ-adic lisse sheaf on a normal variety over a finite field can be embedded into a compatible system of ℓ'-adic lisse sheaves with various ℓ'. Drinfeld used Lafforgue's result as an input and proved this conjecture when the variety is smooth. We consider an analogous existence problem for a regular flat scheme over $\mathbb{Z}$ and prove some cases using Lafforgue's result and the work of Barnet-Lamb, Gee, Geraghty, and Taylor.

1. Introduction

Deligne [1980] conjectured that all the $\overline{\mathbb{Q}}_\ell$-sheaves on a variety over a finite field are mixed. A standard argument reduces this conjecture to the following one.

Conjecture 1.1 (Deligne). *Let p and ℓ be distinct primes. Let X be a connected normal scheme of finite type over $\mathbb{F}_p$ and $\mathscr{E}$ an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf whose determinant has finite order. Then the following properties hold*:

 (i) *$\mathscr{E}$ is pure of weight 0.*

 (ii) *There exists a number field $E \subset \overline{\mathbb{Q}}_\ell$ such that the polynomial $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ has coefficients in E for every $x \in |X|$.*

 (iii) *The roots of $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ are λ-adic units for any nonarchimedean place λ of E prime to p.*

 (iv) *For a sufficiently large E and for every nonarchimedean place λ of E prime to p, there exists an E_λ-sheaf $\mathscr{E}_\lambda$ compatible with $\mathscr{E}$, that is, $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}}) = \det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\lambda, \bar{x}})$ for every $x \in |X|$.*

Here $|X|$ denotes the set of closed points of X and $\bar{x}$ is a geometric point above x.

The conjecture for curves is proved by L. Lafforgue [2002]. He also deals with parts (i) and (iii) in general by reducing them to the case of curves (see also [Deligne 2012]). Deligne [2012] proves part (ii), and Drinfeld [2012] proves part (iv) for smooth varieties. They both use Lafforgue's results.

We can consider similar questions for arbitrary schemes of finite type over $\mathbb{Z}[\ell^{-1}]$. This paper focuses on part (iv), namely the problem of embedding a single lisse $\overline{\mathbb{Q}}_\ell$-sheaf into a compatible system of lisse sheaves. We have the following folklore conjecture in this direction.

Conjecture 1.2. *Let ℓ be a rational prime. Let X be an irreducible regular scheme that is flat and of finite type over $\mathbb{Z}[\ell^{-1}]$. Let E be a finite extension of $\mathbb{Q}$ and λ a prime of E above ℓ. Let $\mathscr{E}$ be an irreducible lisse E_λ-sheaf on X and ρ the corresponding representation of $\pi_1(X)$. Assume the following conditions*:

(i) *The polynomial $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ has coefficients in E for every $x \in |X|$.*

(ii) *$\mathscr{E}$ is de Rham at ℓ (see below for the definition).*

Then for each rational prime ℓ' and each prime λ' of E above ℓ' there exists a lisse $\overline{E}_{\lambda'}$-sheaf on $X[\ell'^{-1}]$ which is compatible with $\mathscr{E}|_{X[\ell'^{-1}]}$.

The conjecture when $\dim X = 1$ is usually rephrased in terms of Galois representations of a number field (see Conjecture 1.3 of [Taylor 2002], for example).

When $\dim X > 1$, a lisse sheaf $\mathscr{E}$ is called *de Rham at ℓ* if for every closed point $y \in X \otimes \mathbb{Q}_\ell$, the representation $i_y^* \rho$ of $\mathrm{Gal}\big(\overline{k(y)}/k(y)\big)$ is de Rham, where i_y is the morphism $\mathrm{Spec}\, k(y) \to X$. Ruochuan Liu and Xinwen Zhu [2017] have shown that this is equivalent to the condition that the lisse E_λ-sheaf $\mathscr{E}|_{X\otimes\mathbb{Q}_\ell}$ on $X \otimes \mathbb{Q}_\ell$ is a de Rham sheaf in the sense of relative p-adic Hodge theory.

Now we discuss our main results. They concern Conjecture 1.2 for schemes over the ring of integers of a totally real or CM field.

Theorem 1.3. *Let ℓ be a rational prime and K a totally real field which is unramified at ℓ. Let X be an irreducible smooth $\mathbb{O}_K[\ell^{-1}]$-scheme such that*

- *the generic fiber is geometrically irreducible,*
- *$X_K(\mathbb{R}) \neq \varnothing$ for every real place $K \hookrightarrow \mathbb{R}$, and*
- *X extends to an irreducible smooth $\mathbb{O}_K$-scheme with nonempty fiber over each place of K above ℓ.*

Let E be a finite extension of $\mathbb{Q}$ and λ a prime of E above ℓ. Let $\mathscr{E}$ be a lisse E_λ-sheaf on X and ρ the corresponding representation of $\pi_1(X)$. Suppose that $\mathscr{E}$ satisfies the following assumptions:

(i) *The polynomial $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ has coefficients in E for every $x \in |X|$.*

(ii) *For every totally real field L which is unramified at ℓ and every morphism $\alpha \colon \mathrm{Spec}\, L \to X$, the E_λ-representation $\alpha^* \rho$ of $\mathrm{Gal}(\overline{L}/L)$ is crystalline at each prime v of L above ℓ, and for each $\tau \colon L \hookrightarrow \overline{E}_\lambda$ it has distinct τ-Hodge–Tate numbers in the range $[0, \ell - 2]$.*

(iii) ρ *can be equipped with a symplectic* (*resp. orthogonal*) *structure with multiplier* $\mu\colon \pi_1(X) \to E_\lambda^\times$ *such that* $\mu|_{\pi_1(X_K)}$ *admits a factorization*

$$\mu|_{\pi_1(X_K)}\colon \pi_1(X_K) \to \mathrm{Gal}(\overline{K}/K) \xrightarrow{\mu_K} E_\lambda^\times$$

with a totally odd (*resp. totally even*) *character* μ_K (*see below for the definitions*).

(iv) *The residual representation* $\bar{\rho}|_{\pi_1(X[\zeta_\ell])}$ *is absolutely irreducible. Here* ζ_ℓ *is a primitive* ℓ-*th root of unity and* $X[\zeta_\ell] = X \otimes_{\mathbb{O}_K} \mathbb{O}_K[\zeta_\ell]$.

(v) $\ell \geq 2(\mathrm{rank}\,\mathscr{E} + 1)$.

Then for each rational prime ℓ' *and each prime* λ' *of* E *above* ℓ' *there exists a lisse* $\overline{E}_{\lambda'}$-*sheaf on* $X[\ell'^{-1}]$ *which is compatible with* $\mathscr{E}|_{X[\ell'^{-1}]}$.

For an E_λ-representation $\rho\colon \pi_1(X) \to \mathrm{GL}(V_\rho)$, a symplectic (resp. orthogonal) structure with multiplier is a pair $((\cdot,\cdot),\mu)$ consisting of a symplectic (resp. orthogonal) pairing $\langle\cdot,\cdot\rangle\colon V_\rho \times V_\rho \to E_\lambda$ and a continuous homomorphism $\mu\colon \pi_1(X) \to E_\lambda^\times$ satisfying $\langle\rho(g)v, \rho(g)v'\rangle = \mu(g)\langle v, v'\rangle$ for any $g \in \pi_1(X)$ and $v, v' \in V_\rho$.

We show a similar theorem without assuming that K is unramified at ℓ using the potential diagonalizability assumption. See Theorem 4.1 for this statement and Theorem 4.2 for the corresponding statement when K is CM.

The proof of Theorem 1.3 uses Lafforgue's work and the work of Barnet-Lamb, Gee, Geraghty, and Taylor [Barnet-Lamb et al. 2014, Theorem C]. The latter work concerns Galois representations of a totally real field, and it can be regarded as a special case of Conjecture 1.2 when $\dim X = 1$. We remark that their theorem has several assumptions on Galois representations since they use potential automorphy. Hence Theorem 1.3 needs assumptions (ii)–(v) on lisse sheaves.

The main part of this paper is devoted to constructing a compatible system of lisse sheaves on a scheme from those on curves. Our method is modeled after Drinfeld's [2012] result, which we explain now.

For a given lisse sheaf on a scheme, one can obtain a lisse sheaf on each curve on the scheme by restriction. Conversely, Drinfeld [2012, Theorem 2.5] proves that a collection of lisse sheaves on curves on a regular scheme defines a lisse sheaf on the scheme if it satisfies some compatibility and tameness conditions. See also a remark after Theorem 1.4. This method originates from the work of Wiesend [2006] on higher dimensional class field theory [Kerz and Schmidt 2009].

Drinfeld uses this method to reduce part (iv) of Conjecture 1.1 for smooth varieties to the case when $\dim X = 1$, where he can use Lafforgue's result. Similarly, one can use his result to reduce Conjecture 1.2 to the case when $\dim X = 1$.

However, Drinfeld's result cannot be used to reduce Theorem 1.3 to the results of Lafforgue and Barnet-Lamb, Gee, Geraghty, and Taylor since his theorem needs a lisse sheaf on every curve on the scheme as an input. On the other hand, the

results of [Lafforgue 2002] and [Barnet-Lamb et al. 2014] only provide compatible systems of lisse sheaves on special types of curves on the scheme: curves over finite fields and *totally real curves*, that is, open subschemes of the spectrum of the ring of integers of a totally real field. Thus the goal of this paper is to deduce Theorem 1.3 using the existence of compatible systems of lisse sheaves on these types of curves.

We now explain our method. Fix a prime ℓ and a finite extension E_λ of $\mathbb{Q}_\ell$. Fix a positive integer r. On a normal scheme X of finite type over $\operatorname{Spec} \mathbb{Z}[\ell^{-1}]$, each lisse E_λ-sheaf $\mathscr{E}$ of rank r defines a polynomial-valued map $f_\mathscr{E} \colon |X| \to E_\lambda[t]$ of degree r given by $f_{\mathscr{E},x}(t) = \det(1 - \operatorname{Frob}_x t, \mathscr{E}_{\bar{x}})$. Here we say that a polynomial-valued map is of degree r if its values are polynomials of degree r. Moreover, $f_\mathscr{E}$ determines $\mathscr{E}$ up to semisimplifications by the Chebotarev density theorem. Conversely, we can ask whether a polynomial-valued map $f \colon |X| \to E_\lambda[t]$ of degree r arises from a lisse sheaf of rank r on X in this way.

Let K be a totally real field. Let X be an irreducible smooth $\mathbb{O}_K$-scheme which has geometrically irreducible generic fiber and satisfies $X_K(\mathbb{R}) \neq \varnothing$ for every real place $K \hookrightarrow \mathbb{R}$. In this situation, we show the following theorem.

Theorem 1.4. *A polynomial-valued map f of degree r on $|X|$ arises from a lisse sheaf on X if and only if it satisfies the following conditions*:

(i) *The restriction of f to each totally real curve arises from a lisse sheaf.*

(ii) *The restriction of f to each separated smooth curve over a finite field arises from a lisse sheaf.*

We prove a similar theorem when K is CM (Theorem 3.14).

Drinfeld's theorem involves a similar equivalence, which holds for arbitrary regular schemes of finite type, although his condition (i) is required to hold for arbitrary regular curves and there is an additional tameness assumption in his condition (ii).[1]

If K and X satisfy the assumptions in Theorem 1.3, then we prove a variant of Theorem 1.4, where we require condition (i) to hold only for totally real curves which are unramified over ℓ (Remark 3.13). This variant, combined with the results by Lafforgue and Barnet-Lamb, Gee, Geraghty, and Taylor, implies Theorem 1.3.

One of the main ingredients for the proof of these types of theorems is an approximation theorem: one needs to find a curve passing through given points in given tangent directions and satisfying a technical condition coming from a given étale covering. To prove this Drinfeld uses the Hilbert irreducibility theorem. In our case, we need to further require that such a curve be totally real or CM. For this we use a theorem of Moret-Bailly.

---

[1]We do not need tameness assumption in condition (ii) in Theorem 1.4. This was pointed out by Drinfeld.

We briefly mention a topic related to Conjecture 1.2. As conjectures on Galois representations suggest, the following stronger statement should hold.

Conjecture 1.5. (With notation as in Conjecture 1.2). *Condition* (ii) *implies condition* (i) *after replacing E by a bigger number field inside E_λ.*

This is an analogue of Conjecture 1.1(ii) and (iii). Even if we assume the conjectures for curves, that is, Galois representations of a number field, no method is known to prove Conjecture 1.5 in full generality. However in [Shimizu 2015] we show the conjecture for smooth schemes assuming conjectures on Galois representations of a number field and the Generalized Riemann Hypothesis. Note that Deligne's [2012] proof of Conjecture 1.1(iii) uses the Riemann Hypothesis for varieties over finite fields, or more precisely, the purity theorem of [Deligne 1980].

We now explain the organization of this paper. In Section 2, we review the theorem of Moret-Bailly and prove an approximation theorem for "schemes with enough totally real curves." We show a similar theorem in the CM case. In Section 3, we prove Theorem 1.4 and its variants using the approximation theorems. Most arguments in Section 3 originate from [Drinfeld 2012]. Finally, we prove the main theorems in Section 4.

Notation. For a number field E and a place λ of E, we denote by $\bar{E}_\lambda$ a fixed algebraic closure of E_λ.

For a scheme X, we denote by $|X|$ the set of closed points of X. We equip finite subsets of $|X|$ with the reduced scheme structure. We denote the residue field of a point x of X by $k(x)$. An étale covering over X means a scheme which is finite and étale over X.

For a number field K and an $\mathbb{O}_K$-scheme X, X_K denotes the generic fiber of X regarded as a K-scheme. In particular, for a K-algebra R, $X_K(R)$ means $\mathrm{Hom}_K(\mathrm{Spec}\,R, X_K)$, not $\mathrm{Hom}_\mathbb{Z}(\mathrm{Spec}\,R, X_K)$. We also write $X(R)$ instead of $X_K(R)$.

For simplicity, we omit base points of fundamental groups and we often change base points implicitly in the paper.

2. Existence of totally real and CM curves via the theorem of Moret-Bailly

Theorem 2.1 [Moret-Bailly 1989]. *Let K be a number field. We consider a quadruple $(X_K, \Sigma, \{M_v\}_{v\in\Sigma}, \{\Omega_v\}_{v\in\Sigma})$ consisting of*

(i) *a geometrically irreducible, smooth and separated K-scheme X_K,*

(ii) *a finite set Σ of places of K,*

(iii) *a finite Galois extension M_v of K_v for every $v \in \Sigma$, and*

(iv) *a nonempty $\mathrm{Gal}(M_v/K_v)$-stable open subset Ω_v of $X_K(M_v)$ with respect to the M_v-topology.*

Then there exist a finite extension L of K and an L-rational point $x \in X_K(L)$ satisfying the following two conditions:

- *For every $v \in \Sigma$, L is M_v-split, that is, $L \otimes_K M_v \cong M_v^{[L:K]}$.*
- *The images of x in $X_K(M_v)$ induced from embeddings $L \hookrightarrow M_v$ lie in Ω_v.*

Remark 2.2. Our formulation is slightly different from that of Moret-Bailly, but Theorem 2.1 is a simple consequence of [Moret-Bailly 1989, théorème 1.3]: We can always find an integral model $f \colon X \to B$ of $X_K \to \operatorname{Spec} K$ over a sufficiently small open subscheme B of $\operatorname{Spec} \mathcal{O}_K$ such that $(X \to B, \Sigma, \{M_v\}_{v \in \Sigma}, \{\Omega_v\}_{v \in \Sigma})$ is an incomplete Skolem datum (see [Moret-Bailly 1989, définition 1.2]). Then Theorem 2.1 follows from théorème 1.3 of [Moret-Bailly 1989] applied to this incomplete Skolem datum.

Since the set Σ can contain infinite places, the above theorem implies the existence of totally real or CM valued points.

Lemma 2.3.   (i) *Let K be a totally real field and X_K a geometrically irreducible smooth K-scheme such that $X_K(\mathbb{R}) \neq \varnothing$ for every real place $K \hookrightarrow \mathbb{R}$. For any dense open subscheme U_K of X_K, there exists a totally real extension L of K such that $U_K(L) \neq \varnothing$.*

(ii) *Let F be a CM field and Z_F a geometrically irreducible smooth F-scheme. For any dense open subscheme V_F of Z_F, there exists a CM extension L of F such that $V_F(L) \neq \varnothing$.*

Proof. In either setting, we may assume that the scheme is separated over the base field by replacing it by an open dense subscheme.

(i) For every real place $v \colon K \hookrightarrow \mathbb{R}$, let $U_v = U_K \cap (X_K \otimes_{K,v} \mathbb{R})(\mathbb{R})$. It follows from the assumptions and the implicit function theorem that U_v is a nonempty open subset of $(X_K \otimes_{K,v} \mathbb{R})(\mathbb{R})$ with respect to real topology.

We apply the theorem of Moret-Bailly to the datum $(X_K, \{v\}, \{\mathbb{R}\}_v, \{U_v\}_v)$ to find a finite extension L of K and a point $x \in X_K(L)$ such that $L \otimes_K \mathbb{R} \cong \mathbb{R}^{[L:K]}$ and the images of x induced from real embeddings $L \hookrightarrow \mathbb{R}$ above v lie in U_v. Then L is totally real and $x \in U_K(L)$. Hence $U_K(L) \neq \varnothing$.

(ii) Let F^+ be the maximal totally real subfield of F. Define $Z_{F^+}^+$ (resp. $V_{F^+}^+$) to be the Weil restriction $\operatorname{Res}_{F/F^+} Z_F$ (resp. $\operatorname{Res}_{F/F^+} V_F$). Denote the nontrivial element of $\operatorname{Gal}(F/F^+)$ by c and $Z_F \otimes_{F,c} F$ by $c^* Z_F$. Then we have $Z_{F^+}^+ \otimes_{F^+} F \cong Z_F \times_F c^* Z_F$, and this scheme is geometrically irreducible over F. Thus $Z_{F^+}^+$ is a geometrically irreducible smooth F^+-scheme, and $V_{F^+}^+$ is dense and open in $Z_{F^+}^+$. Moreover, for every real place $F^+ \hookrightarrow \mathbb{R}$, we can extend it to a complex place $F \hookrightarrow \mathbb{C}$ and get $F \otimes_{F^+} \mathbb{R} \cong \mathbb{C}$. Hence we have $Z_{F^+}^+(\mathbb{R}) = Z_F(\mathbb{C}) \neq \varnothing$. Therefore we can apply (i) to the triple $(F^+, Z_{F^+}^+, V_{F^+}^+)$ and find a totally real extension L^+ of

F^+ such that $V_F(L^+ \otimes_{F^+} F) = V_{F^+}^+(L^+) \neq \varnothing$. Since $L^+ \otimes_{F^+} F$ is a CM extension of F, this completes the proof. $\qquad\square$

This lemma leads to the following definitions.

Definition 2.4. A *totally real curve* is an open subscheme of the spectrum of the ring of integers of a totally real field. A *CM curve* is an open subscheme of the spectrum of the ring of integers of a CM field.

Definition 2.5. Let K be a totally real field and X an irreducible regular $\mathbb{O}_K$-scheme. We say that X is an $\mathbb{O}_K$-scheme *with enough totally real curves* if X is flat and of finite type over $\mathbb{O}_K$ with geometrically irreducible generic fiber and $X_K(\mathbb{R}) \neq \varnothing$ for every real place $K \hookrightarrow \mathbb{R}$.

Now we introduce some notation and state our approximation theorems.

Definition 2.6. Let $g: X \to Y$ be a morphism of schemes. For $x \in X$, consider the tangent space $T_x X = \mathrm{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x))$ at x, where $\mathfrak{m}_x$ denotes the maximal ideal of the local ring at x. This contains $T_x(X_{g(x)})$, where $X_{g(x)} = X \otimes_Y k(g(x))$. A one-dimensional subspace l of $T_x X$ is said to be *horizontal* (with respect to g) if l does not lie in the subspace $T_x(X_{g(x)})$.

Definition 2.7. Let X be a connected scheme and Y a generically étale X-scheme. A point $x \in X(L)$ with some field L is said to *be inert in $Y \to X$* if for each irreducible component Y_α of Y, $(\mathrm{Spec}\, L) \times_{x,X} Y_\alpha$ is nonempty and connected.

Theorem 2.8. *Let K be a totally real field and X an irreducible smooth separated $\mathbb{O}_K$-scheme with enough totally real curves. Consider the following data*:

(i) *a flat $\mathbb{O}_K$-scheme Y which is generically étale over X*;

(ii) *a finite subset $S \subset |X|$ such that $S \to \mathrm{Spec}\,\mathbb{O}_K$ is injective*;

(iii) *a one-dimensional subspace l_s of $T_s X$ for every $s \in S$.*

Then there exist a totally real curve C with fraction field L, a morphism $\varphi: C \to X$ and a section $\sigma: S \to C$ of φ over S such that $\varphi(\mathrm{Spec}\, L)$ is inert in $Y \to X$ and $\mathrm{Im}(T_{\sigma(s)}C \to T_s X) = l_s$ for every $s \in S$.

Proof. We will use the theorem of Moret-Bailly to find the desired curve. Note that we can replace Y by any dominant étale Y-scheme.

Let v denote the structure morphism $X \to \mathrm{Spec}\,\mathbb{O}_K$. Take an open subscheme $U \subset X$ such that $v(U) \cap v(S) = \varnothing$ and the morphism $Y \times_X U \to U$ is finite and étale. Replacing each connected component of $Y \times_X U$ by its Galois closure, we may assume that each connected component of $Y \times_X U$ is Galois over U. Write $Y \times_X U = \coprod_{1 \leq i \leq k} W_i$ as the disjoint union of connected components and denote by G_i the Galois group of the covering $W_i \to U$. Let $H_{i1}, \ldots, H_{ir_i}$ be all the proper subgroups of G_i.

We will choose a quadruple of the form

$$\big(X_K, \{v_{ij}\}_{i,j} \cup \{v(s)\}_{s \in S} \cup \{v_{\infty,i}\}_i, \{K_{v_{ij}}\} \cup \{M_s\} \cup \{\mathbb{R}\}, \{V_{ij}\} \cup \{V_s\} \cup \{V_{\infty,i}\}\big).$$

First we choose v_{ij} and V_{ij} ($1 \le i \le k$, $1 \le j \le r_i$) which control the inertness property.

Claim. *There exist a finite set $\{v_{ij}\}_{i,j}$ of finite places of K and a nonempty open subset V_{ij} of $X(K_{v_{ij}})$ with respect to the $K_{v_{ij}}$-topology for each (i,j) such that*:

(i) *For any finite extension L of K and any L-rational point $x \in X(L)$, x is inert in $Y \to X$ if L is $K_{v_{ij}}$-split and if all the images of x under the induced maps $X(L) \to X(K_{v_{ij}})$ lie in V_{ij}.*

(ii) *The v_{ij} are different from any element of $v(S)$ regarded as a finite place of K.*

Proof of Claim. For each $i = 1, \ldots, k$ and $j = 1, \ldots, r_i$, let $\pi_{H_{ij}}$ denote the induced morphism $W_i/H_{ij} \to X$ and let M_{ij} be the algebraic closure of K in the field of rational functions of W_i/H_{ij}. Then we have a canonical factorization $W_i/H_{ij} \to \mathrm{Spec}\,\mathbb{O}_{M_{ij}} \to \mathrm{Spec}\,\mathbb{O}_K$.

If $M_{ij} = K$, then the generic fiber $(W_i/H_{ij})_K$ is geometrically integral over K. It follows from Proposition 3.5.2 of [Serre 2008] that there are infinitely many finite places v_0 of K such that $U(K_{v_0}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_0}))$ is a nonempty open subset of $U(K_{v_0})$. Thus choose such a finite place v_{ij} and put

$$V_{ij} = U(K_{v_{ij}}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_{ij}})).$$

Next consider the case where $M_{ij} \neq K$. The Lang–Weil theorem and the Chebotarev density theorem show that there are infinitely many finite places v_0 of K such that $U(K_{v_0}) \neq \varnothing$ and v_0 does not split completely in M_{ij}, that is, $M_{ij} \otimes_K K_{v_0} \not\cong K_{v_0}^{[M_{ij}:K]}$ (see [Serre 2008, Propositions 3.5.1 and 3.6.1], for example). In this case, choose such a finite place v_{ij} and put

$$V_{ij} = U(K_{v_{ij}}).$$

It is easy to see that we can choose v_{ij} satisfying condition (ii). We now show that these v_{ij} and V_{ij} satisfy condition (i). Take L and $x \in X(L)$ as in condition (i). By Lemma 2.9 below, it suffices to prove that $x \notin \pi_{H_{ij}}(W_i/H_{ij}(L))$ for any H_{ij}.

When $M_{ij} = K$, this is obvious because the images of x under the maps $X(L) \to X(K_{v_{ij}})$ lie in $V_{ij} = U(K_{v_{ij}}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_{ij}}))$. When $M_{ij} \neq K$, we know that $M_{ij} \otimes_K K_{v_{ij}}$ is not $K_{v_{ij}}$-split. Since L is assumed to be $K_{v_{ij}}$-split, M_{ij} cannot be embedded into L. On the other hand, we have a canonical factorization $W_i/H_{ij} \to \mathrm{Spec}\,\mathbb{O}_{M_{ij}}$. Therefore $W_i/H_{ij}(L) = \varnothing$. Thus x is inert in $Y \to X$ in both cases. $\square$

Next we choose a finite Galois extension M_s of $K_{v(s)}$ and a $\mathrm{Gal}(M_s/K_{v(s)})$-stable nonempty open subset V_s of $X(M_s)$ with respect to the M_s-topology to make a

totally real curve pass through s in the tangent direction l_s. Here $K_{v(s)}$ denotes the completion of K with respect to the finite place $v(s)$ of K. Let $\hat{\mathbb{O}}_{X,s}$ denote the completed local ring of X at $s \in S$. Since $\hat{\mathbb{O}}_{X,s}$ is regular, we can find a regular one-dimensional closed subscheme $\operatorname{Spec} R_s \subset \operatorname{Spec} \hat{\mathbb{O}}_{X,s}$ which is tangent to l_s and satisfies $R_s \otimes_{\mathbb{O}_K} K \neq \varnothing$ (see [Drinfeld 2012, Lemma A.6]).

It follows from the construction that R_s is a complete discrete valuation ring which is finite and flat over $\mathbb{O}_{K_{v(s)}}$ and has residue field $k(s)$. Let M'_s be the fraction field of R_s. For each $s \in S$ we first choose M_s and a local homomorphism $\mathbb{O}_{X,s} \to \mathbb{O}_{M_s}$. There are two cases.

If l_s is horizontal, then R_s is unramified over $\mathbb{O}_{K_{v(s)}}$ and hence M'_s is Galois over $K_{v(s)}$. Put $M_s := M'_s$ in this case. Then we have a natural local homomorphism $\mathbb{O}_{X,s} \to \hat{\mathbb{O}}_{X,s} \to \mathbb{O}_{M_s}$.

If l_s is not horizontal, then R_s is ramified over $\mathbb{O}_{K_{v(s)}}$. Let $K'_{v(s)}$ be the maximal unramified extension of $K_{v(s)}$ in M'_s and M_s the Galois closure of M'_s over $K_{v(s)}$. Then both $K'_{v(s)}$ and M_s have the same residue field $k(s)$.

We construct a local homomorphism $\hat{\mathbb{O}}_{X,s} \to \mathbb{O}_{M_s}$ in this setting. Since X is smooth over $\mathbb{O}_K$, the ring $\hat{\mathbb{O}}_{X,s}$ is isomorphic to the ring of formal power series $\mathbb{O}_{K'_{v(s)}}[\![t_1, \ldots, t_m]\!]$ for some m and we identify these rings.

Let $u_i \in \mathbb{O}_{M'_s}$ denote the image of t_i under the homomorphism

$$\mathbb{O}_{K'_{v(s)}}[\![t_1, \ldots, t_m]\!] = \hat{\mathbb{O}}_{X,s} \to R_s = \mathbb{O}_{M'_s}.$$

Let π (resp. ϖ) be a uniformizer of $\mathbb{O}_{M'_s}$ (resp. $\mathbb{O}_{M_s}$) and consider π-adic expansion $u_i = \sum_{j=0}^{\infty} a_{ij} \pi^j$. Since $\hat{\mathbb{O}}_{X,s} \to \mathbb{O}_{M'_s}$ is a local homomorphism, we have $a_{i0} = 0$ for each i.

Consider the differential of $\operatorname{Spec} \mathbb{O}_{M'_s} = \operatorname{Spec} R_s \to \operatorname{Spec} \hat{\mathbb{O}}_{X,s}$ at the closed point. The tangent vector $\partial/\partial\pi$ is sent to $\sum_{i=1}^{m} a_{i1} \partial/\partial t_i$ under this map, and the latter spans the tangent line l_s.

Define a local homomorphism $\hat{\mathbb{O}}_{X,s} \to \mathbb{O}_{M_s}$ by sending t_i to $\sum_{j=1}^{\infty} a_{ij} \varpi^j$. Then the image of the differential of the corresponding morphism $\operatorname{Spec} \mathbb{O}_{M_s} \to X$ at the closed point is l_s by the same computation as above.

In either case, we have chosen M_s and a homomorphism $\mathbb{O}_{X,s} \to \mathbb{O}_{M_s}$. Let $\hat{s} \in X(\mathbb{O}_{M_s})$ be the point induced by the homomorphism. Note that $X(\mathbb{O}_{M_s})$ is an open subset of $X(M_s)$ by separatedness. Let $\alpha \colon X(\mathbb{O}_{M_s}) \to X(\mathbb{O}_{M_s}/\mathfrak{m}_{M_s}^2)$ be the reduction map, where $\mathfrak{m}_{M_s}$ denotes the maximal ideal of $\mathbb{O}_{M_s}$. Define $V'_s = \alpha^{-1}(\alpha(\hat{s}))$, which is a nonempty open subset of $X(M_s)$, and put

$$V_s = \bigcup_{\sigma} \sigma(V'_s),$$

where σ runs over all the elements of $\operatorname{Gal}(M_s/K_{v(s)})$. Since $\operatorname{Gal}(M_s/K_{v(s)})$ acts continuously on $X(M_s)$, V_s is a nonempty $\operatorname{Gal}(M_s/K_{v(s)})$-stable open subset of $X(M_s)$.

Finally, let $v_{\infty,1}, \ldots, v_{\infty,n}$ be the real places of K and put

$$V_{\infty,i} = X(\mathbb{R})$$

for each $i = 1, \ldots, n$. This is nonempty by our assumption.

It follows from the theorem of Moret-Bailly (Theorem 2.1) that there exist a finite extension L of K and an L-rational point $x \in X(L)$ satisfying the following properties:

(i) $L \otimes_K K_{v_{ij}}$ is $K_{v_{ij}}$-split and x goes into V_{ij} under any embedding $L \hookrightarrow K_{v_{ij}}$.

(ii) $L \otimes_K M_s$ is M_s-split and x goes into V_s under any embedding $L \hookrightarrow M_s$.

(iii) L is totally real.

We can spread out the L-rational point $x \colon \operatorname{Spec} L \to X$ to a morphism $\varphi \colon C \to X$, where C is a totally real curve with fraction field L. By property (ii), we can choose C and φ so that all the points of $\operatorname{Spec} \mathbb{O}_L$ above $v(S) \subset \operatorname{Spec} \mathbb{O}_K$ are contained in C. The claim on page 188 shows that x is inert in $Y \to X$. Thus it remains to prove that there exists a section σ of φ over S such that $\operatorname{Im}(T_{\sigma(s)}C \to T_s X) = l_s$ for every $s \in S$.

It follows from property (ii) and the definition of V_s that there exists an embedding $L \hookrightarrow M_s$ such that the image of x under the associated map $X(L) \to X(M_s)$ lies in V_s'. Let $s' \in \operatorname{Spec} \mathbb{O}_L$ be the closed point corresponding to this embedding. Then we have $s' \in C$, $k(s') = k(s)$, and $\operatorname{Im}(T_{s'}C \to T_s X) = l_s$. Hence we can define a desired section of φ over S. $\qquad\square$

Lemma 2.9. *Let L be a field, U a locally noetherian connected scheme and $\pi \colon W \to U$ a Galois covering with Galois group G. For any subgroup $H \subset G$, let π_H denote the induced morphism $W/H \to U$. An L-valued point of X is inert in π if and only if it lies in $U(L) \setminus \bigcup_{H \subsetneq G} \pi_H(W/H(L))$.*

Proof. Let x denote the L-valued point. Choose a point of W above x and fix a geometric point above it. This also defines a geometric point above x and we have a homomorphism $\pi_1(x) \to G$, where $\pi_1(x)$ is the absolute Galois group of L. Let H_0 denote the image of this homomorphism. Then x is inert in π if and only if the homomorphism is surjective, that is, $H_0 = G$. On the other hand, for a subgroup $H \subset G$, the point x lies in $\pi_H(W/H(L))$ if and only if $\operatorname{Spec} L \times_{x,U} W/H \to \operatorname{Spec} L$ has a section, which is equivalent to the condition that some conjugate of H contains H_0. The lemma follows from these two observations. $\qquad\square$

For our applications, we need a stronger variant of the theorem.

Corollary 2.10. *Let K be a totally real field and X an irreducible smooth separated $\mathbb{O}_K$-scheme with enough totally real curves. Let U be a nonempty open subscheme of X. Suppose that we are given the following data:*

(i) *a flat $\mathbb{O}_K$-scheme Y which is generically étale over X;*

(ii) *a closed normal subgroup $H \subset \pi_1(U)$ such that $\pi_1(U)/H$ contains an open pro-ℓ subgroup;*

(iii) *a finite subset $S \subset |X|$ such that $S \to \mathrm{Spec}\, \mathbb{O}_K$ is injective;*

(iv) *a one-dimensional subspace l_s of $T_s X$ for every $s \in S$.*

Then there exist a totally real curve C with fraction field L, a morphism $\varphi \colon C \to X$ with $\varphi^{-1}(U) \neq \varnothing$ and a section $\sigma \colon S \to C$ of φ over S such that

- *$\varphi(\mathrm{Spec}\, L)$ is inert in $Y \to X$,*
- *$\pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective, and*
- *$\mathrm{Im}(T_{\sigma(s)} C \to T_s X) = l_s$ for every $s \in S$.*

Proof. As is shown in the proof of Proposition 2.17 of [Drinfeld 2012], we can find an open normal subgroup $G_0 \subset \pi_1(U)/H$ satisfying the following property: Every closed subgroup $G \subset \pi_1(U)/H$ such that the map $G \to (\pi_1(U)/H)/G_0$ is surjective equals $\pi_1(U)/H$.

Let Y' be the Galois covering of U corresponding to G_0. Then we can apply Theorem 2.8 to $(Y \sqcup Y', S, (l_s)_{s \in S})$ and get the desired triple (C, φ, σ).   □

We have a similar approximation theorem in the CM case. The proof uses the Weil restriction and is essentially similar to the totally real case, although one has to check that the conditions are preserved under the Weil restriction.

Theorem 2.11. *Let F be a CM field and Z an irreducible smooth separated $\mathbb{O}_F$-scheme with geometrically irreducible generic fiber. Let U be a nonempty open subscheme of Z. Suppose that we are given the following data*:

(i) *a flat $\mathbb{O}_F$-scheme W which is generically étale over Z;*

(ii) *a closed normal subgroup $H \subset \pi_1(U)$ such that $\pi_1(U)/H$ contains an open pro-ℓ subgroup;*

(iii) *a finite subset $S \subset |Z|$ such that $S \to \mathrm{Spec}\, \mathbb{O}_{F^+}$ is injective;*

(iv) *a one-dimensional subspace l_s of $T_s Z$ for every $s \in S$.*

Then there exist a CM curve C with fraction field L, a morphism $\varphi \colon C \to Z$ with $\varphi^{-1}(U) \neq \varnothing$ and a section $\sigma \colon S \to C$ of φ over S such that

- *$\varphi(\mathrm{Spec}\, L)$ is inert in $W \to Z$,*
- *$\pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective, and*
- *$\mathrm{Im}(T_{\sigma(s)} C \to T_s Z) = l_s$ for every $s \in S$.*

Proof. Let F^+ be the maximally totally real subfield of F. Let w (resp. v) denote the structure morphism $Z \to \mathrm{Spec}\, \mathbb{O}_F$ (resp. $Z \to \mathrm{Spec}\, \mathbb{O}_{F^+}$). As in the proof of

Corollary 2.10, we may omit the datum (ii) by replacing W by another flat, generically étale Z-scheme and prove the first and third properties of the triple (C, φ, σ).

Define Z^+ to be the Weil restriction $\mathrm{Res}_{\mathbb{O}_F/\mathbb{O}_{F^+}} Z$. Then we have $Z^+ \otimes_{\mathbb{O}_{F^+}} \mathbb{O}_F \cong Z \times_{\mathbb{O}_F} c^*Z$, where c denotes the nontrivial element of $\mathrm{Gal}(F/F^+)$ and c^*Z denotes $Z \otimes_{\mathbb{O}_F, c} \mathbb{O}_F$. It follows from the assumptions that Z^+ is an irreducible smooth $\mathbb{O}_{F^+}$-scheme with enough totally real curves. We will apply the theorem of Moret-Bailly to Z^+ with appropriate data.

We may assume that each connected component of W is a Galois cover over its image in Z by replacing W if necessary. Put $Y = W \times_{\mathbb{O}_F} c^*W$ and regard it as an $\mathbb{O}_{F^+}$-scheme. Then $Y \to Z \times_{\mathbb{O}_F} c^*Z \to Z^+$ is flat and generically étale, and therefore satisfies the same assumptions as $Y \to X$ in Theorem 2.8 and the second paragraph of its proof. Hence, as in the claim on page 188, there exist a finite set $\{v_{ij}\}_{1 \le i \le k, 1 \le j \le r_i}$ of finite places of F^+ and a nonempty open subset V_{ij} of $Z^+(F^+_{v_{ij}})$ for each (i, j) satisfying the following properties:

(i) For any finite extension L^+ of F^+ and any L^+-rational point $z^+ \in Z^+(L^+)$, z^+ is inert in $Y \to Z^+$ if L^+ is $F^+_{v_{ij}}$-split and if z^+ lands in V_{ij} under any embedding $L^+ \hookrightarrow F^+_{v_{ij}}$.

(ii) The v_{ij} are different from any element of $v(S)$.

Next take any $s \in S$. We will choose a finite Galois extension M_s of $F_{w(s)}$ and a $\mathrm{Gal}(M_s/F^+_{v(s)})$-stable nonempty subset of $Z^+(M_s)$ with respect to the M_s-topology. Here we regard $w(s)$ (resp. $v(s)$) as a finite place of F (resp. F^+). Then M_s is Galois over $F^+_{v(s)}$ since $w(s)$ lies above $v(s)$ and $[F_{w(s)} : F^+_{v(s)}]$ is either 1 or 2.

As in the proof of Theorem 2.8, we can find a finite Galois extension M_s of $F_{w(s)}$ with residue field $k(s)$ and a homomorphism $\mathbb{O}_{Z,s} \to \mathbb{O}_{M_s}$ such that the image of the differential of the corresponding morphism $\mathrm{Spec}\, \mathbb{O}_{M_s} \to Z$ at the closed point is l_s. Denote by $\hat{s} \in Z(\mathbb{O}_{M_s}) \subset Z(M_s)$ the point corresponding to this morphism.

Let $\alpha \colon Z(\mathbb{O}_{M_s}) \to Z(\mathbb{O}_{M_s}/\mathfrak{m}_{M_s}^2)$ be the reduction map, where $\mathfrak{m}_{M_s}$ denotes the maximal ideal of $\mathbb{O}_{M_s}$. Define $V'_s = \alpha^{-1}(\alpha(\hat{s}))$, which is a nonempty open subset of $Z(M_s)$, and put $V''_s = \bigcup_\sigma \sigma(V'_s)$, where σ runs over all the elements of $\mathrm{Gal}(M_s/F_{w(s)})$.

Denote by ι a natural embedding $F \hookrightarrow F_{w(s)} \hookrightarrow M_s$. We have $\mathrm{Hom}_{F^+}(F, M_s) = \{\iota, \iota \circ c\}$ and the F^+-homomorphism $(\iota, \iota \circ c) \colon F \to M_s \times M_s$ induces an isomorphism $M_s \otimes_{F^+} F \cong M_s \times M_s$ which sends $a \otimes b$ to $\big(a\iota(b), a\iota(c(b))\big)$. Hence we get identifications

$$Z^+(M_s) = Z(M_s \otimes_{F^+} F) = Z(M_s) \times c^*Z(M_s).$$

Here $Z(M_s)$ denotes $\mathrm{Hom}_{\mathbb{O}_F}(\mathrm{Spec}\, M_s, Z)$ by regarding $\mathrm{Spec}\, M_s$ as an F-scheme via ι.

Define

$$V_s := V_s'' \times c^* V_s'' \subset Z(M_s) \times c^* Z(M_s) = Z^+(M_s).$$

This is a nonempty open subset of $Z^+(M_s)$. Since V_s'' is $\mathrm{Gal}(M_s/F_{w(s)})$-stable and $\mathrm{Gal}(F/F^+) = \{\mathrm{id}, c\}$, V_s is $\mathrm{Gal}(M_s/F_{v(s)}^+)$-stable.

Let $v_{\infty,1}, \ldots, v_{\infty,n}$ be the real places of F^+. Then for each $1 \le i \le n$ put

$$V_{\infty,i} = Z^+(\mathbb{R}) = Z(\mathbb{C})$$

via an isomorphism $F \otimes_{F^+} \mathbb{R} \cong \mathbb{C}$. This is a nonempty open set.

We apply Theorem 2.1 to the quadruple

$$\left(Z_{F^+}^+, \{v_{ij}\}_{i,j} \cup \{v(s)\}_{s \in S} \cup \{v_{\infty,i}\}_i, \{F_{v_{ij}}^+\} \cup \{M_s\} \cup \{\mathbb{R}\}, \{V_{ij}\} \cup \{V_s\} \cup \{V_{\infty,i}\} \right)$$

and find a totally real finite extension L^+ of F^+ and an L^+-rational point $z^+ \in Z^+(L^+)$ satisfying the following properties:

(i) z^+ is inert in $Y \to Z^+$.

(ii) L^+ is M_s-split and z^+ goes into V_s via any embedding $L^+ \hookrightarrow M_s$.

Let L be the CM field $L^+ \otimes_{F^+} F$ and $z \in Z(L)$ be the L-rational point corresponding to $z^+ \in Z^+(L^+)$. Then the morphism z is equal to the composite

$$\mathrm{pr}_Z \circ (z^+ \otimes_{F^+} F) \colon \ \mathrm{Spec}\, L \to Z^+ \otimes_{\mathbb{O}_{F^+}} \mathbb{O}_F = Z \times_{\mathbb{O}_F} c^* Z \to Z.$$

We can spread out $z \colon \mathrm{Spec}\, L \to Z$ to a morphism $\varphi \colon C \to Z$ for some CM curve C with fraction field L. We may assume that C contains all the points of $\mathrm{Spec}\, \mathbb{O}_L$ above $w(S) \subset \mathrm{Spec}\, \mathbb{O}_F$. It follows from property (ii) and the definition of V_s that φ has a section σ over S such that $\mathrm{Im}(T_{\sigma(s)} C \to T_s X) = l_s$ for every $s \in S$.

It remains to prove that $z = \varphi(\mathrm{Spec}\, L)$ is inert in $W \to Z$. Without loss of generality, we may assume that W_F is connected, and thus it suffices to show that $\mathrm{Spec}\, L \times_{z,Z} W = \mathrm{Spec}\, L \times_{z,Z_F} W_F$ is connected. Define the schemes P and Q such that the squares in the following diagram are Cartesian:

$$
\begin{array}{ccccccc}
P & \longrightarrow & Q & \longrightarrow & \mathrm{Spec}\, L & \longrightarrow & \mathrm{Spec}\, L^+ \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle z^+ \otimes_{F^+} F} & & \downarrow {\scriptstyle z^+} \\
W_F \times_F c^* W_F & \longrightarrow & W_F \times_F c^* Z_F & \longrightarrow & Z_F \times_F c^* Z_F & \longrightarrow & Z_{F^+}^+ \\
& & \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_{Z_F}} & & \\
& & W_F & \longrightarrow & Z_F & &
\end{array}
$$

Since $W_F \times_F c^* W_F = Y_{F^+}$, we have $P \cong \mathrm{Spec}\, L^+ \times_{z^+, Z_{F^+}^+} Y_{F^+}$. As $Q \cong \mathrm{Spec}\, L \times_{z,Z_F} W_F$, we need to show that Q is connected.

Note that $W_F \times_F c^* Z_F$ is connected; this follows from the fact that $c^* Z_F$ is geometrically connected over F and W_F is connected. Now take any connected component T of $Y_{F^+} = W_F \times_F c^* W_F$. Since $W_F \times_F c^* W_F \to W_F \times_F c^* Z_F$ is an étale covering with connected base, T surjects onto $W_F \times_F c^* Z_F$. At the same time, the subscheme $\operatorname{Spec} L^+ \times_{z^+, Z_{F^+}^+} T \subset P$ is connected because z^+ is inert in $Y \to Z^+$. Since the connected scheme $\operatorname{Spec} L^+ \times_{z^+, Z_{F^+}^+} T$ surjects onto Q, the latter is also connected. $\qquad\square$

Remark 2.12. In Theorem 2.8, Corollary 2.10, and Theorem 2.11, we assume that the scheme in question is smooth and separated. If $S = \varnothing$, then we can replace these two assumptions by regularity. In fact, if $S = \varnothing$, we can replace the scheme by an open subscheme, and thus reduce to the separated case. Moreover, the regularity implies that the generic fiber of the scheme is smooth. So we can apply the theorem of Moret-Bailly to our scheme. Note that the smoothness assumption was used only when $S \neq \varnothing$ and l_s is not horizontal for some $s \in S$.

3. Proofs of Theorem 1.4 and its variants

In this section, we prove Theorem 1.4 and its variants following [Drinfeld 2012]. First we set up our notation. Fix a prime ℓ and a finite extension E_λ of $\mathbb{Q}_\ell$. Let $\mathbb{O}$ be the ring of integers of E_λ and $\mathfrak{m}$ its maximal ideal.

Fix a positive integer r. For a normal scheme X of finite type over $\operatorname{Spec} \mathbb{Z}[\ell^{-1}]$, $\operatorname{LS}_r^{E_\lambda}(X)$ denotes the set of equivalence classes of lisse E_λ-sheaves on X of rank r, and $\widetilde{\operatorname{LS}}_r^{E_\lambda}(X)$ denotes the set of maps from the set of closed points of X to the set of polynomials of the form $1 + c_1 t + \cdots + c_r t^r$ with $c_i \in \mathbb{O}$ and $c_r \in \mathbb{O}^\times$. Here we say that two lisse E_λ-sheaves on X are *equivalent* if they have isomorphic semisimplifications. Since the coefficient field E_λ is fixed throughout this section, we simply write $\operatorname{LS}_r(X)$ or $\widetilde{\operatorname{LS}}_r(X)$.

For an element $f \in \widetilde{\operatorname{LS}}_r(X)$, we denote by $f_x(t)$ or $f(x)(t)$ the value of f at $x \in X$; this is a polynomial in t. By the Chebotarev density theorem, we can regard $\operatorname{LS}_r(X)$ as a subset of $\widetilde{\operatorname{LS}}_r(X)$ by attaching to each equivalence class its Frobenius characteristic polynomials. For another scheme Y and a morphism $\alpha \colon Y \to X$, we have a canonical map $\alpha^* \colon \widetilde{\operatorname{LS}}_r(X) \to \widetilde{\operatorname{LS}}_r(Y)$ whose restriction to $\operatorname{LS}_r(X)$ coincides with the pullback map of sheaves $\operatorname{LS}_r(X) \to \operatorname{LS}_r(Y)$. We also denote $\alpha^*(f)$ by $f|_Y$.

Let C be a separated smooth curve over a finite field and $\bar{C}$ the smooth compactification of C. We define $\operatorname{LS}_r^{\text{tame}}(C)$ to be the subset of $\operatorname{LS}_r(C)$ consisting of equivalence classes of lisse E_λ-sheaves on C which are tamely ramified at each point of $\bar{C} \setminus C$. This condition does not depend on the choice of a lisse sheaf in the equivalence class. Let φ be a morphism $C \to X$ and $f \in \widetilde{\operatorname{LS}}_r(X)$. When $\varphi^*(f) \in \operatorname{LS}_r(C)$ (resp. $\varphi^*(f) \in \operatorname{LS}_r^{\text{tame}}(C)$), we simply say that f *arises from* a lisse sheaf (resp. a tame lisse sheaf) over the curve C.

To show Theorem 2.5 of [Drinfeld 2012], which is a prototype of Theorem 1.4, Drinfeld considers a subset $\mathrm{LS}'_r(X)$ of $\widetilde{\mathrm{LS}}_r(X)$ which contains $\mathrm{LS}_r(X)$ and is characterized by a group-theoretic property. He then proves the following three statements for $f \in \widetilde{\mathrm{LS}}_r(X)$, which imply Theorem 2.5 of [Drinfeld 2012].

- If the map f satisfies two conditions[2] similar to those in Theorem 1.4, then $f|_U \in \mathrm{LS}'_r(U)$ for some dense open subscheme $U \subset X$.

- If U is regular, then $\mathrm{LS}'_r(U) = \mathrm{LS}_r(U)$. In particular, the restriction $f|_U \in \mathrm{LS}'_r(U)$ arises from a lisse sheaf.

- If $f|_U$ arises from a lisse sheaf, then so does f under the assumptions that X is regular and that $f|_C$ arises from a lisse sheaf for every regular curve C.

Following Drinfeld, we will introduce the group-theoretic notion of "having a kernel" and prove similar statements: Propositions 3.4, 3.10, and 3.11. Then Theorem 1.4 and its variants will be deduced from them at the end of the section.

Definition 3.1. Let X be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$ and $f \in \widetilde{\mathrm{LS}}_r(X)$. For a nonzero ideal $I \subset \mathcal{O}$, the map f is said to be *trivial modulo I* if it has the value congruent to $(1-t)^r$ modulo I at every closed point of X.

When X is connected, the map f is said to *have a kernel* if there exists a closed normal subgroup $H \subset \pi_1(X)$ satisfying the following conditions:

(i) $\pi_1(X)/H$ contains an open pro-ℓ subgroup.

(ii) For every $n \in \mathbb{N}$, there exists an open subgroup $H_n \subset \pi_1(X)$ containing H such that the pullback of f to X_n is trivial modulo $\mathfrak{m}^n$. Here X_n denotes the covering of X corresponding to H_n.

When X is disconnected, the map f is said to have a kernel if the restriction of f to each connected component of X has a kernel.

Remark 3.2. If f arises from a lisse sheaf on X, it has a kernel. To see this, we may assume that X is connected. Then the kernel of the E_λ-representation of $\pi_1(X)$ corresponding to the lisse sheaf satisfies the conditions.

Remark 3.3. The set $\mathrm{LS}'_r(X)$ defined by Drinfeld [2012, Definition 2.11] consists of the maps f which have a kernel and arise from a lisse sheaf over every regular curve.

Proposition 3.4. *Let K be a totally real field. Let X be an irreducible regular $\mathcal{O}_K[\ell^{-1}]$-scheme with enough totally real curves and $f \in \widetilde{\mathrm{LS}}_r(X)$. Assume that*

(i) *f arises from a lisse sheaf over every totally real curve, and*

---

[2]One needs a tameness assumption in the second condition (it is identical to condition (ii) of Proposition 3.4).

(ii) *there exists a dominant étale morphism $X' \to X$ such that the pullback $f|_{X'}$ arises from a tame lisse sheaf over every separated smooth curve over a finite field.*

Then there exists a dense open subscheme $U \subset X$ such that $f|_U$ has a kernel.

We will first show two lemmas and then prove Proposition 3.4 by induction on the dimension of X. For this we use elementary fibrations, which we recall now.

Definition 3.5. A morphism of schemes $\pi : X \to S$ is called an *elementary fibration* if there exist an S-scheme $\bar{\pi} : \bar{X} \to S$ and a factorization $X \to \bar{X} \xrightarrow{\bar{\pi}} S$ of π such that

(i) the morphism $X \to \bar{X}$ is an open immersion and X is fiberwise dense in $\bar{\pi} : \bar{X} \to S$,

(ii) $\bar{\pi}$ is a smooth and projective morphism whose geometric fibers are nonempty irreducible curves, and

(iii) the reduced closed subscheme $\bar{X} \setminus X$ is finite and étale over S.

The next lemma, which is due to Drinfeld and Wiesend, is a key to our induction argument in the proof of Proposition 3.4.

Lemma 3.6. *Let X be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$ and $f \in \widetilde{\mathrm{LS}}_r(X)$. Suppose that X admits an elementary fibration $X \to S$ with a section $\sigma : S \to X$. Assume that*

(i) *f arises from a tame lisse sheaf over every fiber of $X \to S$, and*

(ii) *there exists a dense open subscheme $V \subset S$ such that $\sigma^*(f)|_V$ has a kernel.*

Then there exists a dense open subscheme $U \subset X$ such that $f|_U$ has a kernel.

Proof. This is shown in the latter part of the proof of Lemma 3.1 of [Drinfeld 2012]. For the reader's convenience, we summarize the proof below.

We may assume that X is connected and normal, and that $V = S$. For every $n \in \mathbb{N}$, consider the functor which attaches to an S-scheme S' the set of isomorphism classes of $\mathrm{GL}_r(\mathbb{O}/\mathfrak{m}^n)$-torsors on $X \times_S S'$ tamely ramified along $(\bar{X} \setminus X) \times_S S'$ relative to S' with trivialization over the section $S' \hookrightarrow X \times_S S'$. Then this functor is representable by an étale scheme T_n of finite type over S and the morphism $T_{n+1} \to T_n$ is finite for each n. By shrinking S, we may assume that the morphism $T_n \to S$ is finite for each n. We will prove that f has a kernel in this situation.

Since $\sigma^*(f)$ has a kernel by assumption (ii), there exist connected étale coverings S_n of S such that

• the pullback of $\sigma^*(f)$ to S_n is trivial modulo $\mathfrak{m}^n$, and

- for some (or any) geometric point $\bar{s}$ of S, the quotient of the group $\pi_1(S, \bar{s})$ by the intersection of the kernels of its actions on the fibers $(S_n)_{\bar{s}}$, where n runs in $\mathbb{N}$, contains an open pro-ℓ subgroup.

Let $\mathfrak{T}_n$ be the universal tame $\mathrm{GL}_r(\mathbb{O}/\mathfrak{m}^n)$-torsor over $X \times_S T_n$. Define the X-scheme Y_n to be the Weil restriction $\mathrm{Res}_{X \times_S T_n / X} \mathfrak{T}_n$ and let X_n denote $Y_n \times_S S_n$. We thus have a diagram whose squares are Cartesian

$$\begin{array}{ccccc} X_n & \longrightarrow & X \times_S S_n & \longrightarrow & S_n \\ \downarrow & & \downarrow & \swarrow & \downarrow \\ Y_n & \longrightarrow & X & \longrightarrow & S \end{array}$$

and regard X_n as an étale covering of X. Here the morphism $S_n \to X$ is the composite of $S_n \to S$ and the section $\sigma \colon S \to X$.

It suffices to prove the following two assertions:

(a) The pullback of f to X_n is trivial modulo $\mathfrak{m}^n$.

(b) For some (or any) geometric point $\bar{x}$ of X, the quotient of the group $\pi_1(X, \bar{x})$ by the intersection of the kernels of its actions on the fibers $(X_n)_{\bar{x}}$, where n runs in $\mathbb{N}$, contains an open pro-ℓ subgroup.

In fact, if we take a Galois covering X'_n of X splitting the (possibly disconnected) covering X_n, the corresponding subgroup $H_n := \pi(X'_n, \bar{x}) \subset \pi_1(X, \bar{x})$ and the intersection $H := \bigcap_n H_n$ satisfy the conditions for the map f to have a kernel.

First we prove assertion (a). Take an arbitrary closed point $x \in X_n$. Let $s \in S$ denote the image of x and choose a geometric point $\bar{s}$ above $s \in S$. By assumption (i), the restriction $f|_{X_s}$ arises from a lisse E_λ-sheaf of rank r. Let $\mathscr{F}$ be a locally constant constructible sheaf of free $(\mathbb{O}/\mathfrak{m}^n)$-modules of rank r obtaining from the above lisse sheaf modulo $\mathfrak{m}^n$.

Consider the $X_{\bar{s}}$-scheme $(Y_n)_{\bar{s}}$. The scheme $(X \times_S T_n)_{\bar{s}}$ is the disjoint union of copies of $X_{\bar{s}}$, and $(\mathfrak{T}_n)_{\bar{s}}$ is a disjoint union of the $\mathrm{GL}_r(\mathbb{O}/\mathfrak{m}^n)$-torsors, each of which lies above a copy of $X_{\bar{s}}$ in $(X \times_S T_n)_{\bar{s}}$. Since the Weil restriction and the base change commute, $(Y_n)_{\bar{s}}$ is the fiber product of the tame $\mathrm{GL}_r(\mathbb{O}/\mathfrak{m}^n)$-torsors over $X_{\bar{s}}$. Hence $\mathscr{F}|_{(Y_n)_{\bar{s}}}$ is constant, and so is $\mathscr{F}|_{(X_n)_{\bar{s}}}$.

Now let $s' \in S_n$ be the image of x. By the choice of S_n, we have

$$(\sigma^*(f))|_{S_n}(s')(t) \equiv (1 - t)^r \mod \mathfrak{m}^n.$$

Since we have shown that $\mathscr{F}|_{(X_n)_{\bar{s}}}$ is constant, it follows that

$$f|_{X_n}(x)(t) \equiv (1 - t)^r \mod \mathfrak{m}^n.$$

Finally, we prove assertion (b). Let η be the generic point of S. Choose a geometric point $\bar{\eta}$ above $\eta \in S$ and let $\bar{x}$ denote the geometric point above $\sigma(\eta)$

induced from $\bar{\eta}$. Let H be the intersection of the kernels of actions of $\pi_1(X, \bar{x})$ on the fibers $(X_n)_{\bar{x}}$, where n runs in $\mathbb{N}$. We need to show that $\pi_1(X, \bar{x})/H$ contains an open pro-ℓ subgroup.

Using the fact that the tame fundamental group $\pi_1^{\text{tame}}(X_{\bar{\eta}}, \bar{x})$ is topologically finitely generated, one can prove that the quotient of the group $\pi_1(X, \bar{x})$ by the intersection H_Y of the kernels of its actions on the fibers $(Y_n)_{\bar{x}}, n \in \mathbb{N}$ contains an open pro-ℓ subgroup (see the last part of the proof of Lemma 3.1 in [Drinfeld 2012]).

Let H_S' be the intersection of the kernels of actions of $\pi_1(S, \bar{\eta})$ on the fibers $(S_n)_{\bar{\eta}}$, where n runs in $\mathbb{N}$, and let H_S be the inverse image of H_S' with respect to the homomorphism $\pi_1(X, \bar{x}) \to \pi_1(S, \bar{\eta})$. By the choice of S_n, the group $\pi_1(S, \bar{\eta})/H_S'$ contains an open pro-ℓ subgroup. Since we have a surjection

$$\pi_1(X, \bar{x})/(H_Y \cap H_S) \to \pi_1(X, \bar{x})/H$$

and an injection

$$\pi_1(X, \bar{x})/(H_Y \cap H_S) \to \pi_1(X, \bar{x})/H_Y \times \pi_1(S, \bar{\eta})/H_S',$$

the group $\pi_1(X, \bar{x})/H$ also contains an open pro-ℓ subgroup. $\square$

To use the above lemma, we show that there exists a chain of split fibrations ending with a totally real curve.

Definition 3.7. A sequence of schemes $X_n \to X_{n-1} \to \cdots \to X_1$ is called *a chain of split fibrations* if the morphism $X_{i+1} \to X_i$ is an elementary fibration which admits a section $X_i \to X_{i+1}$ for each $i = 1, \ldots, n-1$.

Lemma 3.8. *Let K be a totally real field and X an n-dimensional irreducible regular $\mathbb{O}_K$-scheme with enough totally real curves. Then there exist an étale X-scheme X_n, a totally real curve X_1 and a chain of split fibrations $X_n \to \cdots \to X_1$.*

Proof. We prove the lemma by induction on $n = \dim X$. When $\dim X = 1$, the lemma holds by assumption. Thus we assume $\dim X \geq 2$.

By induction on $\dim X$, it suffices to prove that after replacing K by a totally real field extension and X by a nonempty étale X-scheme, there exist an irreducible regular $\mathbb{O}_K$-scheme S with enough totally real curves and an elementary fibration $X \to S$ with a section $S \to X$.

It follows from Lemma 2.3(i) that there exists a totally real extension L of K such that $X_K(L) \neq \varnothing$. Replacing K by L and X by a nonempty open subscheme of $X \otimes_{\mathbb{O}_K} \mathbb{O}_L$ that is étale over X, we may further assume that the generic fiber $X_K \to \operatorname{Spec} K$ has a section $x \colon \operatorname{Spec} K \to X_K$. We also denote the image of x in X_K by x.

If $\dim X = 2$, then X_K is a smooth and geometrically connected curve over K. Take the smooth compactification $\bar{X}_K$ of X_K over K. Then the structure morphism

$X_K \to \operatorname{Spec} K$ has the factorization $X_K \subset \bar{X}_K \to \operatorname{Spec} K$ and thus it is an elementary fibration with a section x. After shrinking X, we can spread it out into an elementary fibration $X \to S$ over an open subscheme S of $\operatorname{Spec} \mathbb{O}_K$ such that it admits a section $S \to X$.

Now assume $\dim X \geq 3$. We apply Artin's theorem on elementary fibration [SGA 4_3 1973, Exposé XI, proposition 3.3] to the pair (X_K, x), and by shrinking X if necessary we get an elementary fibration $\pi \colon X_K \to S_K$ over K with a geometrically irreducible smooth K-scheme S_K. Note that this theorem holds if the base field is perfect and infinite.

Since X_K is smooth over S_K, there exist an open neighborhood V_K of x in X_K and an étale morphism $\alpha \colon V_K \to \mathbb{A}^1_{S_K}$ such that $\pi|_{V_K} \colon V_K \to S_K$ admits a factorization

$$V_K \xrightarrow{\alpha} \mathbb{A}^1_{S_K} \to S_K.$$

Take a section $\tau \colon S_K \to \mathbb{A}^1_{S_K}$ of the projection such that $\alpha(x)$ lies in $\tau(S_K)$.

Consider the connected component S'_K of $S_K \times_{\tau, \mathbb{A}^1_{S_K}} V_K$ that contains the K-rational point $(\pi(x), x)$. This is étale over S_K and satisfies $S'_K(K) \neq \varnothing$. Moreover, S'_K is geometrically integral over K since it is a connected regular K-scheme containing a K-rational point.

We replace S_K by S'_K and X_K by $X_K \times_{S_K} S'_K$. By this replacement, the elementary fibration $\pi \colon X_K \to S_K$ admits a section and $S_K(K) \neq \varnothing$. After shrinking X, we can spread it out into an elementary fibration $X \to S$ with a section $S \to X$, where S is an irreducible regular scheme which is flat and of finite type over $\mathbb{O}_K$ with geometrically irreducible generic fiber and contains a K-rational point. The existence of a K-rational point implies that S has enough totally real curves.   □

Proof of Proposition 3.4. First note that if $\alpha^*(f)|_{U''}$ has a kernel for a nonempty étale X-scheme $\alpha \colon X'' \to X$ and a dense open subscheme $U'' \subset X''$, then so does $f|_{\alpha(U'')}$.

Let n denote the dimension of X. Replacing X by the image of $X' \to X$, we may assume that $X' \to X$ is surjective.

Take a chain of split fibrations $X_n \to X_{n-1} \to \cdots \to X_1$ with a totally real curve X_1 as in Lemma 3.8. We regard X_1 as an X-scheme via $X_n \to X$ and the sections $X_i \to X_{i+1}$. Put $X'_1 = X' \times_X X_1$. This is a nonempty scheme. For $i = 2, \ldots, n$ we put $X'_i = X_i \times_{X_1} X'_1$ via the morphism $X_i \to X_{i-1} \to \cdots \to X_1$. Then $X'_n \to X'_{n-1} \to \cdots \to X'_1$ is a chain of split fibrations.

Since $f|_{X_1}$ lies in $\operatorname{LS}_r(X_1)$ by assumption (i), we have $f|_{X'_1} = (f_{X_1})|_{X'_1} \in \operatorname{LS}_r(X'_1)$. Then we get the result for $(X'_2, f|_{X'_2})$ by Lemma 3.6. Repeating this argument for the chain of split fibrations $X'_n \to \cdots \to X'_2$ we get the result for $(X'_n, f|_{X'_n})$. Applying the remark at the beginning to the morphism $X'_n \to X$, we get the result for (X, f).   □

For the later use, we prove variants of Lemma 3.8. The proof given below is similar to that of Lemma 3.8, but instead of Lemma 2.3 we will use Corollary 2.10 and Theorem 2.11.

Lemma 3.9.   (i) (With notation as in Lemma 3.8). *Suppose that we are given a connected étale covering $Y \to X$. Then there exist an étale X-scheme X_n, a totally real curve X_1, and a chain of split fibrations $X_n \to \cdots \to X_1$ such that $X_1 \times_X Y$ is connected. Here $X_1 \to X$ is the composite of sections $X_{i+1} \to X_i$ and $X_n \to X$.*

 (ii) *Let F be a CM field and Z an n-dimensional irreducible regular $\mathbb{O}_F$-scheme with geometrically irreducible generic fiber. Let $Y \to Z$ be a connected étale covering. Then there exist an étale Z-scheme Z_n, a CM curve Z_1, and a chain of split fibrations $Z_n \to \cdots \to Z_1$ such that $Z_1 \times_Z Y$ is connected. Here $Z_1 \to Z$ is the composite of sections $Z_{i+1} \to Z_i$ and $Z_n \to Z$.*

Proof. First we prove (i) by induction on $n = \dim X$. Since the claim is obvious when $\dim X = 1$, we assume $\dim X \geq 2$.

By induction on $\dim X$, it suffices to prove that after replacing K by a totally real field extension, X by a nonempty étale X-scheme, and the covering $Y \to X$ by its pullback, there exist an irreducible regular $\mathbb{O}_K$-scheme S with enough totally real curves and an elementary fibration $X \to S$ with a section $S \to X$ such that $S \times_X Y$ is connected. The construction of such an S will be the same as that of Lemma 3.8.

It follows from Corollary 2.10 and Remark 2.12 that there exist a totally real extension L of K and an L-rational point $x \in X(L)$ such that $\operatorname{Spec} L \times_{x,X} Y$ is connected. Note that $Y \otimes_{\mathbb{O}_K} \mathbb{O}_L$ is connected because

$$Y \otimes_{\mathbb{O}_K} \mathbb{O}_L \to X \otimes_{\mathbb{O}_K} \mathbb{O}_L$$

is an étale covering with connected base and

$$\operatorname{Spec} L \times_{x,(X \otimes_{\mathbb{O}_K} \mathbb{O}_L)} (Y \otimes_{\mathbb{O}_K} \mathbb{O}_L) = \operatorname{Spec} L \times_{x,X} Y$$

is connected. Thus replacing K by L, X by a nonempty open subscheme of $X \otimes_{\mathbb{O}_K} \mathbb{O}_L$ that is étale over X, and Y by its pullback, we may further assume that the generic fiber $X_K \to \operatorname{Spec} K$ has a section $x \colon \operatorname{Spec} K \to X_K$ such that $\operatorname{Spec} K \times_{x,X} Y$ is connected. We also denote the image of x in X_K by x.

If $\dim X = 2$, the morphism $X_K \to \operatorname{Spec} K$ is an elementary fibration with a section x. After shrinking X, we can spread it out into an elementary fibration $X \to S$ over an open subscheme S of $\operatorname{Spec} \mathbb{O}_K$ such that it admits a section $S \to X$. By construction, $S \times_X Y$ is connected.

Now assume $\dim X \geq 3$. We apply Artin's theorem on elementary fibration to the pair (X_K, x), and by shrinking X if necessary we get an elementary fibration $\pi \colon X_K \to S_K$ over K with a geometrically irreducible smooth K-scheme S_K.

By smoothness, there exist an open neighborhood V_K of x in X_K and an étale morphism $\alpha \colon V_K \to \mathbb{A}^1_{S_K}$ such that $\pi|_{V_K} \colon V_K \to S_K$ admits a factorization

$$V_K \xrightarrow{\alpha} \mathbb{A}^1_{S_K} \to S_K.$$

Take a section $\tau \colon S_K \to \mathbb{A}^1_{S_K}$ of the projection such that $\alpha(x)$ lies in $\tau(S_K)$.

Consider the connected component S'_K of $S_K \times_{\tau, \mathbb{A}^1_{S_K}} V_K$ that contains the K-rational point $(\pi(x), x)$. As was shown in Lemma 3.8, S'_K is étale over S_K and geometrically integral over K, and $S'_K(K) \neq \varnothing$.

The section τ defines the morphism $S'_K \to X_K \times_{S_K} S'_K \to X_K$. The composite of this morphism and $(\pi(x), x) \colon \operatorname{Spec} K \to S'_K$ coincides with $x \colon \operatorname{Spec} K \to X_K$. Since $S'_K \times_X Y \to S'_K$ is an étale covering with connected base and

$$\operatorname{Spec} K \times_{(\pi(x), x), S'_K} (S'_K \times_X Y) = \operatorname{Spec} K \times_{x, X} Y$$

is connected, it follows that $S'_K \times_X Y$ is connected.

We replace S_K by S'_K, X_K by $X_K \times_{S_K} S'_K$ and Y_K by $Y_K \times_{S_K} S'_K$. By this replacement, the elementary fibration $\pi \colon X_K \to S_K$ admits a section such that $S_K(K) \neq \varnothing$ and $S_K \times_{X_K} Y_K$ is connected. As is discussed in the last paragraph of the proof of Lemma 3.8, after shrinking X, we can spread out $X_K \to S_K$ and $Y_K \to S_K$ into an elementary fibration $X \to S$ with a section $S \to X$ and a covering $Y \to X$, where S is an irreducible regular $\mathbb{O}_K$-scheme with enough totally real curves. Since $S \times_X Y$ is connected by construction, this S works.

For part (ii), it is easy to verify that the same argument works if we apply Theorem 2.11 instead of Corollary 2.10. □

Next we show that if f has a kernel and arises from a lisse sheaf over every totally real curve then it actually arises from a lisse sheaf.

Proposition 3.10. *Let K be a totally real field and X an irreducible smooth separated $\mathbb{O}_K[\ell^{-1}]$-scheme with enough totally real curves. Suppose that $f \in \widetilde{\mathrm{LS}}_r(X)$ satisfies the following conditions:*

  (i) *f arises from a lisse sheaf over every totally real curve.*

  (ii) *f has a kernel.*

Then $f \in \mathrm{LS}_r(X)$.

Proof. We follow Section 4 of [Drinfeld 2012]. Since f has a kernel, we take a closed subgroup H of $\pi_1(X)$ as in the definition of having a kernel. In particular, $\pi_1(X)/H$ contains an open pro-ℓ subgroup.

By Corollary 2.10, there exists a totally real curve C with a morphism $\varphi \colon C \to X$ such that $\varphi_* \colon \pi_1(C) \to \pi_1(X)/H$ is surjective. By assumption (i), for any such pair

(C, φ), the pullback $\varphi^*(f)$ arises from a semisimple representation $\rho_C \colon \pi_1(C) \to$ $\mathrm{GL}_r(E_\lambda)$. Define

$$H_C := \mathrm{Ker}\big(\varphi_* \colon \pi_1(C) \to \pi_1(X)/H\big).$$

Then condition (ii) in the definition of having a kernel (Definition 3.1), together with the Chebotarev density theorem, shows that $\mathrm{Ker}\,\rho_C$ contains H_C. See [Drinfeld 2012, Lemma 4.1] for details. Thus we regard ρ_C as a representation

$$\rho_C \colon \pi_1(X) \to \pi_1(X)/H \to \mathrm{GL}_r(E_\lambda)$$

of $\pi_1(X)$. Note that $\rho_C|_{\pi_1(C)}$ gives the original representation of $\pi_1(C)$.

We will show that the lisse sheaf on X corresponding to this representation gives f. For this, we need to show that

$$\det\big(1 - t\rho_C(\mathrm{Frob}_x)\big) = f_x(t)$$

for all closed points $x \in X$. We know that this equality holds for each closed point $x \in \varphi(C)$ such that $\varphi^{-1}(x)$ contains a point whose residue field is equal to $k(x)$.

Take any closed point $x \in X$. We will first construct a curve C' passing through x and some finitely many points on C specified below. We will then construct a lisse sheaf on C' whose Frobenius polynomial at x is $f_x(t)$, and prove that the lisse sheaf on C' extends over X and the corresponding representation of $\pi_1(X)$ coincides with ρ_C.

We use a lemma by Faltings; define T_0 to be the set of closed points of C which have the same image in $\mathrm{Spec}\,\mathbb{O}_K$ as that of x. By the theorem of Hermite, the Chebotarev density theorem, and the Brauer–Nesbitt theorem, there exists a finite set $T \subset |C| \setminus T_0$ satisfying the following properties:

(i) $T \to \mathrm{Spec}\,\mathbb{O}_K$ is injective.

(ii) For any semisimple representations $\rho_1, \rho_2 \colon \pi_1(C) \to \mathrm{GL}_r(E_\lambda)$, the equality $\mathrm{tr}\,\rho_1(\mathrm{Frob}_y) = \mathrm{tr}\,\rho_2(\mathrm{Frob}_y)$ for all $y \in T$ implies $\rho_1 \cong \rho_2$.

See [Faltings 1983, Satz 5] or [Deligne 1985, théorème 3.1] for details.

By Corollary 2.10 applied to $S = \varphi(T) \cup \{x\}$, there exists a totally real curve C' with a morphism $\varphi' \colon C' \to X$ such that the map $\varphi'_* \colon \pi_1(C') \to \pi_1(X)/H$ is surjective and for each $y \in \varphi(T) \cup \{x\}$ there exists a point in $\varphi'^{-1}(y)$ whose residue field is equal to $k(y)$. As discussed before, this pair (C', φ') also defines a semisimple representation

$$\rho_{C'} \colon \pi_1(X) \to \pi_1(X)/H \to \mathrm{GL}_r(E_\lambda)$$

such that $\det(1 - t\rho_{C'}(\mathrm{Frob}_y)) = f_y(t)$ for each $y \in \varphi(T) \cup \{x\}$. Note that the surjectivity of φ_* implies that $\rho_{C'}|_{\pi_1(C)}$ is semisimple.

It follows from property (ii) of T that $\rho_C|_{\pi_1(C)}$ and $\rho_{C'}|_{\pi_1(C)}$ are isomorphic as representations of $\pi_1(C)$. Since the map $\varphi_*\colon \pi_1(C) \to \pi_1(X)/H$ is surjective, we have $\rho_C \cong \rho_{C'}$ as representations of $\pi_1(X)/H$ and thus they are also isomorphic as representations of $\pi_1(X)$. In particular, we have

$$\det\big(1 - t\rho_C(\mathrm{Frob}_x)\big) = \det\big(1 - t\rho_{C'}(\mathrm{Frob}_x)\big) = f_x(t).$$

Hence f comes from the lisse sheaf on X corresponding to ρ_C. $\square$

We now prove the last proposition of our three key ingredients for Theorem 1.4. This concerns extendability of a lisse sheaf on a dense open subset to the whole scheme. In the proof we use the Zariski–Nagata purity theorem; thus the regularity assumption for X is crucial. We further need to assume that X is smooth as we use Corollary 2.10 to find a totally real curve passing through a given point in a given tangent direction.

Proposition 3.11. *Let K be a totally real field and X an irreducible smooth separated $\mathcal{O}_K[\ell^{-1}]$-scheme with enough totally real curves. Suppose that $f \in \widetilde{\mathrm{LS}}_r(X)$ satisfies the following conditions*:

 (i) *f arises from a lisse sheaf over every totally real curve.*

 (ii) *There exists a dense open subscheme $U \subset X$ such that $f|_U \in \mathrm{LS}_r(U)$.*

Then $f \in \mathrm{LS}_r(X)$.

Proof. We follow Section 5.2 of [Drinfeld 2012]. Let $\mathcal{E}_U$ be the semisimple lisse E_λ-sheaf on U corresponding to $f|_U$. First we show that $\mathcal{E}_U$ extends to a lisse E_λ-sheaf on X.

Suppose the contrary. Since X is regular, the Zariski–Nagata purity theorem implies that there exists an irreducible divisor D of X contained in $X \setminus U$ such that $\mathcal{E}_U$ is ramified along D. Then by a specialization argument [Drinfeld 2012, Corollary 5.2], we can find a closed point $x \in X \setminus U$ and a one-dimensional subspace $l \subset T_x X$ satisfying the following property:

 ($*$) Consider a triple (C, c, φ) consisting of a regular curve C, a closed point $c \in C$, and a morphism $\varphi\colon C \to X$ such that $\varphi(c) = x$, $\varphi^{-1}(U) \neq \varnothing$, and $\mathrm{Im}\big(T_c C \to T_x X \otimes_{k(x)} k(c)\big) = l \otimes_{k(x)} k(c)$. For any such triple, the pullback of $\mathcal{E}_U$ to $\varphi^{-1}(U)$ is ramified at c.

Let H be the kernel of the representation $\rho_U\colon \pi_1(U) \to \mathrm{GL}_r(E_\lambda)$ corresponding to $\mathcal{E}_U$. The group $\pi_1(U)/H \cong \mathrm{Im}\, \rho_U$ contains an open pro-ℓ subgroup because $\mathrm{Im}\, \rho_U$ is a compact open subgroup of $\mathrm{GL}_r(E_\lambda)$. Therefore by Corollary 2.10 we can find a totally real curve C, a closed point $c \in C$, and a morphism $\varphi\colon C \to X$ such that

  • $\varphi(c) = x$ and $k(c) \cong k(x)$,

- $\varphi^{-1}(U) \neq \varnothing$ and $\varphi_* \colon \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective, and
- $\mathrm{Im}(T_c C \to T_x X) = l$.

Since $\varphi_* \colon \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective, the pullback of $\mathscr{E}_U$ to $\varphi^{-1}(U)$ is semisimple. Thus this lisse E_λ-sheaf has no ramification at c by assumption (i), which contradicts property (∗). Hence $\mathscr{E}_U$ extends to a lisse E_λ-sheaf $\mathscr{E}$ on X.

Let f' be the element of $\mathrm{LS}_r(X)$ corresponding to $\mathscr{E}$. We know $f|_U = f'|_U$. Take any closed point $x \in X$. It suffices to show that $f(x) = f'(x)$. We can find a totally real curve C', a closed point $c' \in C'$, and a morphism $\varphi' \colon C' \to X$ such that $\varphi'(c') = x$, $k(c') = k(x)$, and $\varphi'^{-1}(U) \neq \varnothing$. Then

$$\varphi'^*(f)|_{\varphi'^{-1}(U)} = (f|_U)|_{\varphi'^{-1}(U)} = (f'|_U)|_{\varphi'^{-1}(U)} = \varphi'^*(f')|_{\varphi'^{-1}(U)}.$$

Since $\varphi'^{-1}(U) \neq \varnothing$, the homomorphism $\pi_1(\varphi'^{-1}(U)) \to \pi_1(C')$ is surjective and thus $\varphi'^*(f) = \varphi'^*(f')$. In particular, $f(x) = \varphi'^*(f)(c') = \varphi'^*(f')(c') = f'(x)$. □

Proof of Theorem 1.4. First note that a polynomial-valued map f of degree r in the theorem lies in $\widetilde{\mathrm{LS}}_r(X)$. One direction of the equivalence is obvious, and thus it suffices to prove that if f satisfies conditions (i) and (ii), then f lies in $\mathrm{LS}_r(X)$.

First assume that X is separated. Let k_λ be the residue field of E_λ and N be the cardinality of $\mathrm{GL}_r(k_\lambda)$. Put $X' := X \otimes_{\mathbb{Z}} \mathbb{Z}[N^{-1}]$. Then $X' \to X$ is a dominant étale morphism and satisfies the following property:

> The pullback $f|_{X'}$ arises from a tame lisse sheaf over every separated smooth curve over a finite field.

Thus by Proposition 3.4, there exists an open dense subscheme U of X such that $f|_U$ has a kernel. Therefore $f|_U$ lies in $\mathrm{LS}_r(U)$ by Proposition 3.10 and we have $f \in \mathrm{LS}_r(X)$ by Proposition 3.11.

In the general case, we consider a covering $X = \bigcup_i U_i$ by open separated subschemes. Then we can apply the above discussion to each $f|_{U_i}$ and obtain a lisse E_λ-sheaf $\mathscr{E}_i$ on U_i that represents $f|_{U_i}$. Since U_i is normal, we can replace $\mathscr{E}_i$ by its semisimplification and assume that each $\mathscr{E}_i$ is semisimple.

Put $U = \bigcap_i U_i$. This is nonempty, and the restrictions $\mathscr{E}_i|_U$ are isomorphic to each other. Thus $\{\mathscr{E}_i\}_i$ glues to a lisse E_λ-sheaf on X and this sheaf represents f. □

We end this section with variants of Theorem 1.4. Condition (i) in Theorem 3.12 or Remark 3.13 is weaker than that of Theorem 1.4 since they concern only totally real curves with additional properties. This weaker condition is essential to use the result of [Barnet-Lamb et al. 2014] in the proof of our main theorems in the next section. Theorem 3.14 is a variant in the CM case.

Theorem 3.12. *Let K be a totally real field. Let X be an irreducible smooth $\mathbb{O}_K[\ell^{-1}]$-scheme with enough totally real curves. An element $f \in \widetilde{\mathrm{LS}}_r(X)$ belongs to $\mathrm{LS}_r(X)$ if and only if it satisfies the following conditions*:

(i) *There exists a connected étale covering $Y \to X$ such that f arises from a lisse sheaf over every totally real curve C with the property that $C \times_X Y$ is connected.*

(ii) *The restriction of f to each separated smooth curve over a finite field arises from a lisse sheaf.*

Proof. Recall that Theorem 1.4 is deduced from Propositions 3.4, 3.10, and 3.11 and that these propositions have the same condition (i) that f arises from a lisse sheaf over every totally real curve. Consider the variant statements of Propositions 3.4, 3.10, and 3.11 where we replace condition (i) by

(i′) f arises from a lisse sheaf over every totally real curve C such that $C \times_X Y$ is connected.

It suffices to prove that these variants also hold; then the theorem is deduced from them in the same way as Theorem 1.4.

   The variant of Proposition 3.4 is proved in the same way as Proposition 3.4 if one uses Lemma 3.9(i) instead of Lemma 3.8. For the variants of Propositions 3.10 and 3.11, the same proof works; observe that whenever one uses Corollary 2.10 in the proof to find a totally real curve C, one can impose the additional condition that $C \times_X Y$ is connected by adding the covering $Y \to X$ to the input of Corollary 2.10. $\qquad\square$

Remark 3.13. We need another variant of Theorem 1.4 to prove Theorem 1.3: With the notation as in Theorem 3.12, suppose further that

- K is unramified at ℓ, and

- X extends to an irreducible smooth $\mathcal{O}_K$-scheme X' with nonempty fiber over each place of K above ℓ.

   Then condition (i) in Theorem 3.12 can be replaced by

(i′) There exists a connected étale covering $Y \to X$ such that f arises from a lisse sheaf over every totally real curve C with the properties that

- $C \times_X Y$ is connected and that
- the fraction field of C is unramified at ℓ.

   This statement is proved in the same way as Theorem 3.12; it suffices to prove variants of Propositions 3.4, 3.10, and 3.11 where condition (i) in these propositions is replaced by the following condition:

   The map f arises from a lisse sheaf over every totally real curve C such that $C \times_X Y$ is connected and the fraction field of C is unramified at ℓ.

For the proof of the variant of Proposition 3.4, we also need to consider the variant of Lemma 3.9(i) where we further require that the fraction field of X_1 is unramified at ℓ.

We now explain how to prove the variants of Lemma 3.9(i) and Propositions 3.4, 3.10, and 3.11. By the additional condition on X, for each place v of K above ℓ, there exist a finite unramified extension L of K_v and a morphism $\operatorname{Spec} \mathbb{O}_L \to X'$. We denote the image of the closed point of $\operatorname{Spec} \mathbb{O}_L$ by s_v. Since L is unramified over K_v, we can find a horizontal one-dimensional subspace l_v of $T_{s_v} X'$ with respect to $X' \to \operatorname{Spec} \mathbb{O}_K$.

If we add $\{s_v\}_{v|\ell}$ and l_v to the input when we use Corollary 2.10, the fraction field of the resulting totally real curve is unramified over K at each v, hence unramified at ℓ. Thus we can prove the variant of Lemma 3.9(i) in the same way as Lemma 3.9(i), and the arguments given in Theorem 3.12 work for the current variants of Propositions 3.4, 3.10, and 3.11. Hence the statement of this remark follows.

Theorem 3.14. *Let F be a CM field. Let Z be an irreducible smooth $\mathbb{O}_F[\ell^{-1}]$-scheme with geometrically irreducible generic fiber. An element $f \in \widetilde{\operatorname{LS}}_r(Z)$ belongs to $\operatorname{LS}_r(Z)$ if and only if it satisfies the following conditions*:

 (i) *There exists a connected étale covering $Y \to Z$ such that f arises from a lisse sheaf over every CM curve C with the property that $C \times_Z Y$ is connected.*

 (ii) *The restriction of f to each separated smooth curve over a finite field arises from a lisse sheaf.*

Proof. We can prove variants of Propositions 3.4, 3.10, and 3.11 for the CM case using Theorem 2.11 and Lemma 3.9(ii). Then the theorem is deduced from them in the same way as Theorems 1.4 and 3.12. $\qquad\square$

4. Proofs of the main theorems

In this section, we prove theorems on the existence of the compatible system of a lisse sheaf. Theorem 4.1 concerns the totally real case and Theorem 4.2 concerns the CM case. Theorem 1.3 in the introduction is proved after Theorem 4.1. Following the discussion in [Drinfeld 2012, Section 2.3], we deduce these main theorems from Theorems 3.12, 3.14, and theorems in [Lafforgue 2002; Barnet-Lamb et al. 2014].

As we mentioned in the introduction, some of the assumptions in the main theorems come from the potential diagonalizability condition, which is introduced in [Barnet-Lamb et al. 2014, Section 1.4]. We first review this notion; see [loc. cit.] for details.

Let L be a finite extension of $\mathbb{Q}_\ell$. Let E_λ be a finite extension of $\mathbb{Q}_\ell$. We say that an $\bar{E}_\lambda$-representation ρ of $\operatorname{Gal}(\bar{L}/L)$ is *potentially diagonalizable* if it is potentially

crystalline and there is a finite extension L' of L such that $\rho|_{\mathrm{Gal}(\bar{L}/L')}$ lies on the same irreducible component of the universal crystalline lifting ring of the residual representation $\bar{\rho}|_{\mathrm{Gal}(\bar{L}/L')}$ with fixed Hodge–Tate numbers as a sum of characters lifting $\bar{\rho}|_{\mathrm{Gal}(\bar{L}/L')}$.

There are two important examples of this notion (see [Barnet-Lamb et al. 2014, Lemma 1.4.3]): Ordinary representations are potentially diagonalizable. When L is unramified over $\mathbb{Q}_\ell$, a crystalline representation is potentially diagonalizable if for each $\tau \colon L \hookrightarrow \bar{E}_\lambda$ the τ-Hodge–Tate numbers lie in the range $[a_\tau, a_\tau + \ell - 2]$ for some integer a_τ.

We first prove our main theorem for the totally real case.

Theorem 4.1. *Let ℓ be a rational prime. Let K be a totally real field and X an irreducible smooth $\mathbb{O}_K[\ell^{-1}]$-scheme with enough totally real curves. Let E be a finite extension of $\mathbb{Q}$ and λ a prime of E above ℓ. Let $\mathscr{E}$ be a lisse E_λ-sheaf on X and ρ the corresponding representation of $\pi_1(X)$. Suppose that $\mathscr{E}$ satisfies the following assumptions*:

(i) *The polynomial $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ has coefficients in E for every $x \in |X|$.*

(ii) *For every totally real field L and every morphism $\alpha \colon \mathrm{Spec}\, L \to X$, the E_λ-representation $\alpha^* \rho$ of $\mathrm{Gal}(\bar{L}/L)$ is potentially diagonalizable at each prime v of L above ℓ and for each $\tau \colon L \hookrightarrow \bar{E}_\lambda$ it has distinct τ-Hodge–Tate numbers.*

(iii) *ρ can be equipped with a symplectic (resp. orthogonal) structure with multiplier $\mu \colon \pi_1(X) \to E_\lambda^\times$ such that $\mu|_{\pi_1(X_K)}$ admits a factorization*

$$\mu|_{\pi_1(X_K)} \colon \pi_1(X_K) \to \mathrm{Gal}(\bar{K}/K) \xrightarrow{\mu_K} E_\lambda^\times$$

with a totally odd (resp. totally even) character μ_K.

(iv) *The residual representation $\bar{\rho}|_{\pi_1(X[\zeta_\ell])}$ is absolutely irreducible.*

(v) *$\ell \geq 2(\mathrm{rank}\,\mathscr{E} + 1)$.*

Then for each rational prime ℓ' and each prime λ' of E above ℓ' there exists a lisse $\bar{E}_{\lambda'}$-sheaf on $X[\ell'^{-1}]$ which is compatible with $\mathscr{E}|_{X[\ell'^{-1}]}$.

Proof. Replacing X by $X[\ell'^{-1}]$, we may assume that ℓ' is invertible in $\mathbb{O}_X$. Let r be the rank of $\mathscr{E}$. Take an arbitrary extension M of $E_{\lambda'}$ of degree $r!$. By assumption (i), we regard the map $f \colon x \mapsto \det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ as an element of $\widetilde{\mathrm{LS}}_r^M(X)$ via the embedding $E \hookrightarrow E_{\lambda'} \hookrightarrow M$.

We will apply Theorem 3.12 to $f \in \widetilde{\mathrm{LS}}_r^M(X)$. Here we use the prime ℓ' and the field M (we used ℓ and E_λ in Section 3).

First we show that the map f satisfies condition (i) in Theorem 3.12. Let Y be the connected étale covering $Y \to X[\zeta_\ell]$ that corresponds to $\mathrm{Ker}\,\bar{\rho}|_{\pi_1(X[\zeta_\ell])}$. We regard Y as a connected étale covering over X via $Y \to X[\zeta_\ell] \to X$. We will prove

that this $Y \to X$ satisfies condition (i). Take any totally real curve $\varphi \colon C \to X$ such that $C \times_X Y$ is connected.

To show that $\varphi^*(f)$ arises from a lisse M-sheaf on C, it suffices to prove that there exists a lisse $\bar{E}_{\lambda'}$-sheaf on C which is compatible with $\varphi^*\mathscr{E}$; this follows from [Drinfeld 2012, Lemma 2.7]. Namely, let $\rho'_C \colon \pi_1(C) \to \mathrm{GL}_r(\bar{E}_{\lambda'})$ denote the semisimplification of the corresponding $\bar{E}_{\lambda'}$-representation. Since

$$\det\big(1 - t\rho'_C(\mathrm{Frob}_x)\big) = f_x(t) \in E_{\lambda'}[t]$$

for every closed point $x \in C$, the character of ρ'_C is defined over $E_{\lambda'}$ by the Chebotarev density theorem. It follows from $[M : E_{\lambda'}] = r!$ that the Brauer obstruction of ρ'_C in the Brauer group $\mathrm{Br}(E_{\lambda'})$ vanishes in $\mathrm{Br}(M)$ and ρ'_C can be defined over M. This means $\varphi^*(f) \in \mathrm{LS}_r^M(C)$.

We will construct a lisse $\bar{E}_{\lambda'}$-sheaf on C which is compatible with $\varphi^*\mathscr{E}$. For this we will apply Theorem C from [Barnet-Lamb et al. 2014] to the $\bar{E}_\lambda$-representation $\varphi_L^*\rho$ of $\mathrm{Gal}(\bar{L}/L)$, where L denotes the fraction field of C and $\varphi_L \colon \mathrm{Spec}\, L \to X$ denotes $\varphi|_{\mathrm{Spec}\, L}$.

We need to see that the Galois representation $\varphi_L^*\rho$ satisfies the assumptions in Theorem C. By assumptions (ii) and (v) it remains to check that

  (a) $\varphi_L^*\rho$ can be equipped with a symplectic (resp. orthogonal) structure with totally odd (resp. totally even) multiplier, and

  (b) the residual representation $(\varphi_L^*\bar{\rho})|_{\mathrm{Gal}(\bar{L}/L(\zeta_\ell))}$ is absolutely irreducible.

Assumption (a) follows from assumption (iii). To see (b), recall that $C \times_X Y$ is connected. Hence $C \times_X X[\zeta_\ell]$ is connected with fraction field $L(\zeta_\ell)$, and $C \times_X Y \to C \times_X X[\zeta_\ell]$ is a connected étale covering. It follows from the definition of Y that $\mathrm{Im}(\varphi_L^*\bar{\rho})|_{\mathrm{Gal}(\bar{L}/L(\zeta_\ell))}$ coincides with $\mathrm{Im}\,\bar{\rho}|_{\pi_1(X[\zeta_\ell])}$, and thus $(\varphi_L^*\bar{\rho})|_{\mathrm{Gal}(\bar{L}/L(\zeta_\ell))}$ is absolutely irreducible by assumption (iv).

Hence by [Barnet-Lamb et al. 2014, Theorem C] we obtain an $\bar{E}_{\lambda'}$-representation of $\mathrm{Gal}(\bar{L}/L)$. The proof of the theorem, which uses potential automorphy and Brauer's theorem, shows that this representation is unramified at each closed point of C, and thus it gives rise to a lisse $\bar{E}_{\lambda'}$-sheaf on C which is compatible with $\varphi^*\mathscr{E}$. Hence f satisfies condition (i) in Theorem 3.12.

Next we show that f satisfies condition (ii) in Theorem 3.12. Let C be a separated smooth curve over $\mathbb{F}_p$ for some prime p and denote the structure morphism $C \to \mathrm{Spec}\,\mathbb{F}_p$ by α. Let $\varphi \colon C \to X$ be a morphism. Note that p is different from ℓ and ℓ'.

We write the semisimplification of $\varphi^*\mathscr{E}$ as $\bigoplus_i \mathscr{E}_i^{\oplus r_i}$, where $\mathscr{E}_i$ are distinct irreducible lisse $\bar{E}_\lambda$-sheaves on C. Then there exist an irreducible lisse $\bar{E}_\lambda$-sheaf $\mathscr{F}_i$ on C and a lisse $\bar{E}_\lambda$-sheaf $\mathscr{G}_i$ of rank 1 on $\mathrm{Spec}\,\mathbb{F}_p$ such that $\mathscr{F}_i$ has determinant of finite order and $\mathscr{E}_i \cong \mathscr{F}_i \otimes \alpha^*\mathscr{G}_i$ (see [Deligne 1980, Section I.3] or [Deligne 2012, Section 0.4], for example).

By théorèm VII.6 of [Lafforgue 2002], for each closed point $x \in C$, the roots of $\det(1 - \mathrm{Frob}_x\, t, \mathscr{F}_{i,\bar{x}})$ are algebraic numbers that are λ'-adic units. Moreover, there exists an irreducible lisse $\bar{E}_{\lambda'}$-sheaf $\mathscr{F}_i'$ on C which is compatible with $\mathscr{F}_i$.

We will show that there exists a lisse $\bar{E}_{\lambda'}$-sheaf $\mathscr{G}_i'$ on $\mathrm{Spec}\,\mathbb{F}_p$ which is compatible with $\mathscr{G}_i$. Note that the lisse $\bar{E}_\lambda$-sheaf $\mathscr{G}_i$ is determined by the value of the corresponding character of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ at the geometric Frobenius. Denote this value by $\beta_i \in \bar{E}_\lambda^\times$. It suffices to prove that β_i is an algebraic number that is a λ'-adic unit. Since the roots of $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ and $\det(1 - \mathrm{Frob}_x\, t, \mathscr{F}_{i,\bar{x}})$ are all algebraic numbers, so is β_i.

We prove that β_i is a λ'-adic unit. To see this, take a closed point x of C. Then by Corollary 2.10 we can find a totally real curve C' and a morphism $\varphi' \colon C' \to X$ such that $\varphi(x) \in \varphi'(C')$ and $C' \times_X Y$ is connected. As discussed before, Theorem C of [Barnet-Lamb et al. 2014] implies that there exists a lisse $\bar{E}_{\lambda'}$-sheaf on C' whose Frobenius characteristic polynomial map is $\varphi'^*(f)$. Thus for each closed point $y \in C'$ the roots of $\varphi'^*(f)(y)$ are algebraic numbers that are λ'-adic units. Considering a point $y \in \varphi'^{-1}(\varphi(x))$, we conclude that some power of β_i is a λ'-adic unit and thus so is β_i. Hence there exists a lisse $\bar{E}_{\lambda'}$-sheaf $\mathscr{G}_i'$ on $\mathrm{Spec}\,\mathbb{F}_p$ which is compatible with $\mathscr{G}_i$.

The Frobenius characteristic polynomial map associated with the semisimple lisse $\bar{E}_{\lambda'}$-sheaf $\bigoplus_i (\mathscr{F}_i' \otimes \alpha^* \mathscr{G}_i')^{\oplus r_i}$ is $\varphi^*(f)$. As discussed before, this sheaf can be defined over M. Thus f satisfies condition (ii) in Theorem 3.12.

Therefore by Theorem 3.12 there exists a lisse M-sheaf on X which is compatible with $\mathscr{E}$. $\square$

Proof of Theorem 1.3. All the discussions in the proof of Theorem 4.1 also work in this setting by using Remark 3.13 instead of Theorem 3.12. $\square$

We also have a theorem for the CM case.

Theorem 4.2. *Let ℓ be a rational prime, E a finite extension of $\mathbb{Q}$, and λ a prime of E above ℓ. Let F be a CM field with $\zeta_\ell \notin F$ and Z an irreducible smooth $\mathbb{O}_F[\ell^{-1}]$-scheme with geometrically irreducible generic fiber. Let $\mathscr{E}$ be a lisse E_λ-sheaf on X and ρ the corresponding representation of $\pi_1(Z)$. Suppose that $\mathscr{E}$ satisfies the following assumptions*:

  (i) *The polynomial $\det(1 - \mathrm{Frob}_x\, t, \mathscr{E}_{\bar{x}})$ has coefficients in E for every $x \in |X|$.*

 (ii) *For any CM field L with $\zeta_\ell \notin L$ and any morphism $\alpha \colon \mathrm{Spec}\, L \to Z$, the E_λ-representation $\alpha^* \rho$ of $\mathrm{Gal}(\bar{L}/L)$ satisfies the following two conditions*:

   (a) *$\alpha^* \rho$ is potentially diagonalizable at each prime v of L above ℓ and for each $\tau \colon L \hookrightarrow \bar{E}_\lambda$ it has distinct τ-Hodge–Tate numbers.*

   (b) *$\alpha^* \rho$ is totally odd and polarizable (in the sense of [Barnet-Lamb et al. 2014, Section 2.1]).*

(iii) *The residual representation $\bar{\rho}|_{\pi_1(Z[\zeta_\ell])}$ is absolutely irreducible.*

(iv) $\ell \geq 2(\text{rank}\,\mathscr{E} + 1)$.

Then for each rational prime ℓ' and each prime λ' of E above ℓ' there exists a lisse $\bar{E}_{\lambda'}$-sheaf on $Z[\ell'^{-1}]$ which is compatible with $\mathscr{E}|_{Z[\ell'^{-1}]}$.

Proof. In the same way as Theorem 4.1, the theorem is deduced from Theorem 3.14, Theorem 5.5.1 of [Barnet-Lamb et al. 2014], théorèm VII.6 of [Lafforgue 2002], and the following remark: If $Y \to Z[\zeta_\ell]$ denotes the connected étale covering defined by $\text{Ker}\,\bar{\rho}_{\pi_1(Z[\zeta_\ell])}$ and C is a CM curve with a morphism to Z such that $C \times_Z Y$ is connected, then $C \times_Z Z[\zeta_\ell] = C \otimes_{\mathcal{O}_F} \mathcal{O}_F(\zeta_\ell)$ is connected. In particular, the fraction field of C does not contain ζ_ℓ. $\qquad\square$

Acknowledgments

References

[Barnet-Lamb et al. 2014] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, "Potential automorphy and change of weight", *Ann. of Math.* (2) **179**:2 (2014), 501–609. MR Zbl

[Deligne 1980] P. Deligne, "La conjecture de Weil, II", *Inst. Hautes Études Sci. Publ. Math.* **52** (1980), 137–252. MR Zbl

[Deligne 1985] P. Deligne, "Preuve des conjectures de Tate et de Shafarevitch (d'après G. Faltings)", exposé no. 616, 25–41 in *Séminar Bourbaki* 1983/1984, Astérisque **121-122**, 1985. MR Zbl

[Deligne 2012] P. Deligne, "Finitude de l'extension de $\mathbb{Q}$ engendrée par des traces de Frobenius, en caractéristique finie", *Mosc. Math. J.* **12**:3 (2012), 497–514, 668. MR

[Drinfeld 2012] V. Drinfeld, "On a conjecture of Deligne", *Mosc. Math. J.* **12**:3 (2012), 515–542, 668. MR Zbl

[Faltings 1983] G. Faltings, "Endlichkeitssätze für abelsche Varietäten über Zahlkörpern", *Invent. Math.* **73**:3 (1983), 349–366. MR

[Kerz and Schmidt 2009] M. Kerz and A. Schmidt, "Covering data and higher dimensional global class field theory", *J. Number Theory* **129**:10 (2009), 2569–2599. MR

[Lafforgue 2002] L. Lafforgue, "Chtoucas de Drinfeld et correspondance de Langlands", *Invent. Math.* **147**:1 (2002), 1–241. MR Zbl

[Liu and Zhu 2017] R. Liu and X. Zhu, "Rigidity and a Riemann–Hilbert correspondence for p-adic local systems", *Invent. Math.* **207**:1 (2017), 291–343.

[Moret-Bailly 1989] L. Moret-Bailly, "Groupes de Picard et problèmes de Skolem, II", *Ann. Sci. École Norm. Sup.* (4) **22**:2 (1989), 181–194. MR Zbl

[Serre 2008] J.-P. Serre, *Topics in Galois theory*, 2nd ed., Research Notes in Mathematics **1**, A. K. Peters, Wellesley, MA, 2008. MR

[SGA 4₃ 1973] M. Artin, A. Grothendieck, and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas, Tome 3: Exposés IX–XIX* (Séminaire de Géométrie Algébrique du Bois Marie 1963–1964), Lecture Notes in Math. **305**, Springer, Berlin, 1973. MR Zbl

[Shimizu 2015] K. Shimizu, "Finiteness of Frobenius traces of a sheaf on a flat arithmetic scheme", preprint, 2015, Available at http://www.math.harvard.edu/~shimizu/fin_Frob_traces.pdf.

[Taylor 2002] R. Taylor, "Galois representations", pp. 449–474 in *Proceedings of the International Congress of Mathematicians* (Beijing, 2002), vol. I, Higher Ed. Press, Beijing, 2002. MR Zbl

[Wiesend 2006] G. Wiesend, "A construction of covers of arithmetic schemes", *J. Number Theory* **121**:1 (2006), 118–131. MR Zbl

shimizu@math.harvard.edu   *Department of Mathematics, Harvard University, Cambridge, MA 02138, United States*

msp

Algebra & Number Theory

msp.org/ant

Algebra & Number Theory