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Test vectors and central L-values for GL(2)

Daniel File, Kimball Martin and Ameya Pitale

We determine local test vectors for Waldspurger functionals for GL_2, in the case where both the representation of GL_2 and the character of the degree two extension are ramified, with certain restrictions. We use this to obtain an explicit version of Waldspurger's formula relating twisted central L-values of automorphic representations on GL_2 with certain toric period integrals. As a consequence, we generalize an average value formula of Feigon and Whitehouse, and obtain some nonvanishing results.

1. Introduction

1A. ***Global results.*** Let F be a number field and π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let L/F be a quadratic extension and Ω an idèle class character of $L^\times$ such that $\Omega|_{\mathbb{A}_F^\times} = \omega_\pi$, the central character of π. We are interested in the central value of the L-function

$$L(s, \pi_L \otimes \Omega) = L(s, \pi \times \theta_\Omega),$$

where π_L denotes the base change of π to $\mathrm{GL}_2(\mathbb{A}_L)$ and θ_Ω denotes the theta series on $\mathrm{GL}_2(\mathbb{A}_F)$ associated to Ω. Note this contains the following interesting special case: when Ω is trivial, then $L(s, \pi_L \otimes \Omega) = L(s, \pi)L(s, \pi \otimes \eta)$, where $\eta = \eta_{L/F}$ denotes the quadratic character of $\mathbb{A}_F^\times$ associated to L via class field theory. Assume

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that ω_π is trivial or η. Then $\epsilon\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = \pm 1$, even though $\pi_L \otimes \Omega$ need not be self-dual (cf. [Jacquet and Chen 2001]). In the case where $\epsilon\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = -1$, the central value $L\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = 0$. Henceforth, assume $\epsilon\left(\frac{1}{2}, \pi_L \otimes \Omega\right) = +1$.

Let D be a quaternion algebra over F containing L such that π has a Jacquet–Langlands transfer to an automorphic representation π' of $D^\times(\mathbb{A}_F)$. We allow for the possibility that $D = M_2(F)$ and $\pi' = \pi$, so there is always at least one such π'. Embed $L^\times$ as a torus T inside $D^\times$. The period integrals we are interested in are

$$P_D(\phi) = \int_{Z(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)} \phi(t)\Omega^{-1}(t)\, dt, \tag{1-1}$$

where $\phi \in \pi'$ and Z denotes the center of $D^\times$ (with dt as in Section 7). If $F = \mathbb{Q}$ and L is imaginary quadratic, then this period simplifies to a finite sum over certain "CM points".

When ω_π is trivial, a beautiful theorem of Waldspurger [1985] states that

$$\frac{|P_D(\phi)|^2}{(\phi, \phi)} = \zeta(2) \prod_v \alpha_v(L, \Omega, \phi) \frac{L\left(\frac{1}{2}, \pi_L \otimes \Omega\right)}{L(1, \pi, \mathrm{Ad})} \tag{1-2}$$

for any $\phi \in \pi'$. Here $(\cdot, \cdot)$ is a certain inner product on π' and the factors $\alpha_v(L, \Omega, \phi)$ are certain local integrals which equal 1 at almost all places. For all but one D, $P_D \equiv 0$ for local reasons. Namely, the linear functional P_D factors into a product of local linear functionals $P_{D,v}$. There is a unique $D \supset L$ for which all $P_{D,v} \neq 0$, and this D is determined by local epsilon factors in work of Tunnell [1983] and Saito [1993]. Fixing this D, one now gets the nonvanishing criterion: $L\left(\frac{1}{2}, \pi_L \otimes \Omega\right) \neq 0$ if and only if $P_D \neq 0$.

It is useful to have a more explicit version of this formula for certain applications like equidistribution, nonvanishing, subconvexity, p-adic L-functions, etc.; see, e.g., [Popa 2006; Martin and Whitehouse 2009; Feigon and Whitehouse 2009; Hsieh 2014]. In particular, it is not even obvious from (1-2) that $L\left(\frac{1}{2}, \pi_L \otimes \Omega\right) \geq 0$, as predicted by the grand Riemann hypothesis. This positivity result was subsequently shown by Jacquet and Chen [2001] using a trace formula identity.

Explicit versions of (1-2) have been considered by many authors under various assumptions; see, e.g., [Gross 1987; Zhang 2001; Xue 2006; Popa 2006; Martin and Whitehouse 2009; Murase 2010; Hida 2010; Hsieh 2014]. These explicit formulas rely on picking out a suitable *test vector* ϕ in (1-2). All of these works rely on the theta correspondence (as did [Waldspurger 1985]), except for [Martin and Whitehouse 2009], which uses the trace formula identity from [Jacquet and Chen 2001]. The only assumption in [Martin and Whitehouse 2009] is that π and Ω have disjoint ramification, i.e., for any finite place v of F, π and Ω are not both ramified at v. In this case one has a natural choice for the test vector ϕ from the work of Gross and Prasad [1991] on *local* test vectors. In [Martin and Whitehouse

2009], it was noted that this restriction of disjoint ramification is not essential to the method and could be removed if one had a reasonable way to define the test vector ϕ in a more general setting.

The main local results of this paper (see Theorems 1.6 and 1.7 below) are the existence and characterization of suitable local test vectors in the case of joint ramification under certain conditions. This allows us to extend the formula of [Martin and Whitehouse 2009] to these cases. To be precise, for a finite place v of F, let $c(\pi_v)$ be the (exponent of) the conductor of π_v and $c(\Omega_v)$ be the (exponent of) the "F-conductor" of Ω (see (2-19)). Then we make the following assumption:

$$\textit{If } v<\infty \textit{ is inert in } L \textit{ and } c(\pi_v),\, c(\Omega_v)>0, \textit{ then we have } c(\Omega_v)\ge c(\pi_v). \tag{1-3}$$

In particular, if the level $N=\prod_{v<\infty}\varpi_v^{c(\pi_v)}$ of π is squarefree, there is no condition on Ω. We note that a consequence of our determination of test vectors is that assumption (1-3) implies that D and Ω do not have joint ramification at any finite place.

Theorems 1.6 and 1.7 below give suitable local test vectors ϕ_v under assumption (1-3), which yields the desired global test vector ϕ. Here suitable essentially means that the local test vectors can be described purely in terms of ramification data, and do not require more refined information about local representations. This is crucial for global applications. Note that it is not even a priori clear if suitable test vectors should exist in general.

Let us now describe the L-value formula more precisely. Denote the absolute value of the discriminants of F and L by Δ and Δ_L. Let $e(L_v/F_v)$ be the ramification degree of L_v/F_v. Let S_{inert} be the set of places of F inert in L. Let $S(\pi)$ be the set of finite places of F where π is ramified, $S(\Omega)$ the set of finite places of F where Ω is ramified, $S_1(\pi)$ the set of places in $S(\pi)$ where $c(\pi_v)=1$ and $S_2(\pi)$ the set of places in $S(\pi)$ where $c(\pi_v)\ge 2$. Finally, let $S_0(\pi)=S_2(\pi)\cup\{v\in S_1(\pi): L_v/F_v \text{ is ramified and } \Omega_v \text{ is unramified}\}$, and denote by $c(\Omega)$ the absolute norm of the conductor of Ω.

Theorem 1.1. *Let π be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_F)$ *with trivial central character and Ω a character of $\mathbb{A}_L^\times/L^\times\mathbb{A}_F^\times$. Assume $\epsilon\big(\tfrac12,\pi_L\otimes\Omega\big)=1$ and that π and Ω satisfy* (1-3). *Then, with the test vector $\phi\in\pi'$ defined in Section 7A and archimedean factors $C_v(L,\pi,\Omega)$ defined in Section 7B, we have*

$$\begin{aligned}\frac{|P_D(\phi)|^2}{(\phi,\phi)} &= \frac12\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}}\,L_{S(\Omega)}(1,\eta)L_{S(\pi)\cup S(\Omega)}(1,\eta)L_{S(\pi)\cap S(\Omega)}(1,1_F)L^{S(\pi)}(2,1_F)\\ &\qquad\times\prod_{v\in S(\pi)\cap S(\Omega)^c} e(L_v/F_v)\prod_{v\mid\infty}C_v(L,\pi,\Omega)\cdot\frac{L^{S_0(\pi)}\big(\tfrac12,\pi_L\otimes\Omega\big)}{L^{S_0(\pi)}(1,\pi,\mathrm{Ad})}.\end{aligned}$$

Here $(\,\cdot\,,\cdot\,)$ is the standard inner product on π' with respect to the measure on $D^\times(\mathbb{A}_F)$ which is the product of local Tamagawa measures.

After our paper was originally completed, the paper [Cai et al. 2014] appeared, which gives a similar formula using a less explicit choice of test vector.

Note ϕ is specified up to a scalar, and the left-hand side is invariant under scaling. As in [Martin and Whitehouse 2009, Theorem 4.2], one can rewrite this formula using the Petersson norm of a normalized newform in π instead of $L(1,\pi,\mathrm{Ad})$. See (8-19) for when π corresponds to a holomorphic Hilbert modular form. If $F=\mathbb{Q}$ and π corresponds to a holomorphic new form of squarefree level N with $N \mid c(\Omega)$, then the above formula simplifies considerably:

Corollary 1.2. *Let f be a normalized holomorphic modular eigenform of weight k and squarefree level N. Let S be the set of primes $p \mid N$ which split in L. Let Ω be any ideal class character of L such that $N \mid c(\Omega)$ and $\epsilon\big(\frac{1}{2}, f\times\Omega\big)=1$. Then*

$$\frac{|P_D(\phi)|^2}{(\phi,\phi)}=\frac{C_\infty(L,f,\Omega)}{2^{k+1}\sqrt{c(\Omega)\Delta_L}}\,L_{S(\Omega)}(1,\eta)^2\prod_{p\mid N}(1+p^{-1})^{\epsilon_p}\times\frac{L^S\big(\frac{1}{2},f\times\Omega\big)}{\langle f,f\rangle},$$

where ϵ_p is $+1$ if p splits in L and -1 otherwise, and $\langle\,\cdot\,,\cdot\,\rangle$ is the Petersson inner product.

In the setting of the corollary, $C_\infty(L,f,\Omega)$ is also easier to describe. If L is real quadratic, then $C_\infty(L,f,\Omega)=2^k$. If L is imaginary quadratic, it is described by beta functions, and if we also assume Ω_∞ is trivial, then

$$C_\infty(L,f,\Omega)=\frac{\big(\frac{1}{2}k-1\big)!^2}{\pi(k-1)!}.$$

We prove Theorem 1.1 by computing local spectral distributions appearing in the trace formula identity of [Jacquet and Chen 2001], just as in [Martin and Whitehouse 2009]. For simplicity, we only do this when $\omega_\pi=1$, though the case of $\omega_\pi=\eta$ should be similar. (One needs either $\omega_\pi=1$ or $\omega_\pi=\eta$ to use the identity from [Jacquet and Chen 2001].) Note this formula is considerably more general than the one in [Martin and Whitehouse 2009] (for trivial central character) and one expects that it should generalize the applications of the previously mentioned formulas. For instance, we obtain the following generalization of an average value result of Feigon and Whitehouse [2009, Theorem 1.1] by computing the geometric side of a certain trace formula.

Theorem 1.3. *Let F be a totally real number field with $d=[F:\mathbb{Q}]$. Let $\mathcal{F}(\mathfrak{N},2\boldsymbol{k})$ be the set of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to the holomorphic Hilbert modular eigen newforms of weight $2\boldsymbol{k}$ and level $\mathfrak{N}$, with $\boldsymbol{k}=(k_1,\dots,k_d)\neq(1,\dots,1)$ and $\mathfrak{N}$ squarefree. Let L be a totally imaginary* *quadratic extension of F, which is inert and unramified above each place $\mathfrak{p} \mid \mathfrak{N}$. Fix a unitary character Ω of $\mathbb{A}_L^\times / L^\times \mathbb{A}_F^\times$, and let $\mathfrak{C}$ be the norm of its conductor in F. Suppose $\mathfrak{N} = \mathfrak{N}_0 \mathfrak{N}_1$ and $\mathfrak{C} = \mathfrak{C}_0 \mathfrak{N}_1$ with $\mathfrak{N}_0$, $\mathfrak{N}_1$ and $\mathfrak{C}_0$ all coprime. Assume $\mathfrak{N}_1$ is odd, and that the number of primes dividing $\mathfrak{N}_0$ has the same parity as d. Further assume that for each infinite place v of F, we have $k_v > |m_v|$, where $\Omega_v(z) = (z/\bar{z})^{m_v}$.*

Then, if

$$|\mathfrak{N}_0| > d_{L/F}(|\mathfrak{C}_0|/|\mathfrak{N}_1|)^{h_F},$$

where h_F is the class number of F, we have

$$\prod_{v \mid \infty} \binom{2k_v - 2}{k_v - m_v - 1} \sum_{\mathfrak{N}'} \sum_{\pi \in \mathcal{F}(\mathfrak{N}', 2\boldsymbol{k})} \frac{L\left(\frac{1}{2}, \pi_L \otimes \Omega\right)}{L^{S(\mathfrak{N})}(1, \pi, \mathrm{Ad})} = 2^{2-d} \Delta^{3/2} |\mathfrak{N}| L_{S(\mathfrak{N}_0)}(2, 1_F) L_{S(\mathfrak{N}_1)}(1, 1_F) L^{S(\mathfrak{C}_0)}(1, \eta),$$

where $\mathfrak{N}'$ runs over ideals dividing $\mathfrak{N}$ which are divisible by $\mathfrak{N}_0$, and $S(\mathfrak{J})$ denotes the set of all primes dividing $\mathfrak{J}$.

The parity condition guarantees the sign $\epsilon\left(\frac{1}{2}, \pi_L \otimes \Omega\right)$ of the relevant functional equation is $+1$ for $\pi \in \mathcal{F}(\mathfrak{N}, \boldsymbol{k})$. Without a condition to the effect that $\mathfrak{N}$ (or $\mathfrak{N}_0$) is large, one does not expect a nice explicit formula, but rather just an asymptotic in $\mathfrak{N}$, which miraculously stabilizes for $\mathfrak{N}$ large (cf. [Michel and Ramakrishnan 2012; Feigon and Whitehouse 2009]). Hence the condition above on the size of $\mathfrak{N}_0$ means we are in the *stable range*. The other assumptions in the theorem allow for simplifications of the trace formula we will use, but are not necessary to express such averages as the geometric side of an appropriate trace formula.

Theorem 1.3 specializes to [Feigon and Whitehouse 2009, Theorem 1.1] in the case that $\mathfrak{N}$ and $\mathfrak{C}$ are coprime, i.e., $\mathfrak{N} = \mathfrak{N}_0$. This case $\mathfrak{N} = \mathfrak{N}_0$ is particularly nice as one can transfer the problem to a trace formula computation on a quaternion algebra that only picks up forms of exact level $\mathfrak{N}$. Additionally, one can rewrite the formula in terms of the complete adjoint L-value at 1, as in [Feigon and Whitehouse 2009]. However, this is impossible to manage in general, and the primary difficulty in going from Theorem 1.1 to Theorem 1.3 is to determine the contribution to the spectral side of the relevant trace formula coming from the oldforms. (In general, it is not easy to isolate the newforms in such formulas — see, e.g., [Knightly and Li 2010] or [Nelson 2013] — and the issue for us is that the contribution from the oldforms is now weighted by local adjoint L-factors.)

Still, one can use the above formula together with formulas for smaller levels to get both explicit bounds and asymptotics for average values over just the forms of exact level $\mathfrak{N}$. We do this in the case $\mathfrak{N}_1$ is prime. This immediately implies $L\left(\frac{1}{2}, \pi_L \otimes \Omega\right) \neq 0$ for some $\pi_L \in \mathcal{F}(\mathfrak{N}, 2\boldsymbol{k})$.

Theorem 1.4. *With assumptions as in Theorem 1.3, and further assuming that* $\mathfrak{N}_1 = \mathfrak{p}$ *is an odd prime and* $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{C}|^{h_F}$, *we have*

$$|\mathfrak{p}| - \frac{1}{1 - 2|\mathfrak{p}|^{-1} + |\mathfrak{p}|^{-2}} \leq \Sigma(\mathfrak{N}) \leq |\mathfrak{p}| - \frac{1}{1 + 2|\mathfrak{p}|^{-1} + |\mathfrak{p}|^{-2}},$$

where $\Sigma(\mathfrak{N})$ *is equal to*

$$\frac{2^{d-2}}{\Delta^{3/2}|\mathfrak{N}_0| L(1, 1_{F_\mathfrak{p}}) L_{S(\mathfrak{N}_0)}(2, 1_F) L^{S(\mathfrak{C}_0)}(1, \eta)} \times \prod_{v|\infty} \binom{2k_v - 2}{k_v - m_v - 1} \sum_{\pi \in \mathcal{F}(\mathfrak{N}, 2\boldsymbol{k})} \frac{L\big(\frac{1}{2}, \pi_L \otimes \Omega\big)}{L(1, \pi, \mathrm{Ad})}.$$

In particular, $\Sigma(\mathfrak{N}) \sim |\mathfrak{p}| - 1 + O(|\mathfrak{p}|^{-1})$ *as* $|\mathfrak{N}_0\mathfrak{p}| \to \infty$ *such that* $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{C}|^{h_F}$. *Furthermore, with* $\mathfrak{p}$ *fixed, we have*

$$\lim_{|\mathfrak{N}_0| \to \infty} \Sigma(\mathfrak{N}) = |\mathfrak{p}| - 1.$$

In both of these asymptotics, $\mathfrak{N}_0$ *travels along squarefree ideals coprime to* $\mathfrak{C}$ *which are products of unramified primes and satisfy our previous parity assumption.*

Note the above theorem implies the nonvanishing of $\Sigma(\mathfrak{N})$, and therefore at least one of these central values, provided $|\mathfrak{p}| > \frac{1}{2}(3 + \sqrt{5})$ and $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{C}|^{h_F}$, or $\mathfrak{p}$ is arbitrary and $|\mathfrak{N}_0|$ is sufficiently large.

We remark that the bounds come from having to estimate the p-th Hecke eigenvalues $\{a_p, a_p^{-1}\}$ of the oldforms of level $\mathfrak{N}_0$. The latter asymptotic comes from an asymptotic for a weighted analogue of Theorem 1.3 in the case of disjoint ramification (see [Feigon and Whitehouse 2009, Theorem 1.2]) to pick off the contribution from the oldforms. One should be able to prove a version of Theorem 1.3 involving weighting by Hecke eigenvalues (namely, extend [Feigon and Whitehouse 2009, Theorem 6.1] to the case of joint ramification) whereby one could inductively obtain asymptotics for the average values $\Sigma(\mathfrak{N})$ in the case where $\gcd(\mathfrak{N}, \mathfrak{C})$ has an arbitrary number of prime factors. (We remark Sugiyama and Tsuzuki [2016] have recently obtained asymptotics for weighted averages using a different relative trace formula approach when Ω is trivial, but $\mathfrak{N}$ need not be squarefree.)

Note that in previous studies of such averages, $\mathfrak{N}$ is typically required to be prime (e.g., [Ramakrishnan and Rogawski 2005]) or have an odd number of prime factors (e.g., [Feigon and Whitehouse 2009]) to force the sign of the functional equation to be $+1$ if, say, d is odd. However, allowing for joint ramification we can treat levels $\mathfrak{N}$ with an arbitrary number of prime divisors, though we do not always get an exact formula in this situation.

Lastly, we include another application of Theorem 1.3 when $\mathfrak{N} = \mathfrak{N}_0$ (i.e., [Feigon and Whitehouse 2009, Theorem 1.1]). Here, having an exact formula for the average value over newforms allows us to deduce the nonvanishing mod p of the algebraic part $L^{\mathrm{alg}}\bigl(\frac{1}{2}, \pi_L \otimes \Omega\bigr)$ (see (8-18)) of the central value for p suitably large.

Theorem 1.5. *With notation and assumptions as in Theorem 1.3, suppose that* $|\mathfrak{N}| > d_{L/F}|\mathfrak{C}|^{h_F}$, *that* $\mathfrak{N}$ *is coprime to* $\mathfrak{C}$, *and that* m_v *is even for each* $v \mid \infty$. *Let* p *be an odd rational prime satisfying* $p > q+1$ *for all primes* $q \in S(\Omega)$, *and* $\mathcal{P}$ *a prime of* $\overline{\mathbb{Q}}$ *above* p. *Then there exists* $\pi \in \mathcal{F}(\mathfrak{N}, 2\boldsymbol{k})$ *such that*

$$L^{\mathrm{alg}}\bigl(\tfrac{1}{2}, \pi_L \otimes \Omega\bigr) \not\equiv 0 \mod \mathcal{P}.$$

This generalizes a theorem of Michel and Ramakrishnan [2012] on the case $F = \mathbb{Q}$ and $\mathfrak{N} = N$ is prime. The parity condition on m_v ensures that Ω is algebraic and that the above central value is critical.

As in [Feigon and Whitehouse 2009], one should be able to use Theorem 1.1 to get estimates on more general averages of L-values, and apply this to subconvexity and equidistribution problems, but we do not address this here. Theorem 1.1 has also been used in very recent works of Hamieh [2014] on valuations of Rankin–Selberg L-values in anticyclotomic towers and Van Order [2014] on constructing p-adic L-functions.

We remark that similar L-value formulas have been recently proven in certain cases of joint ramification with L totally imaginary, namely in Hida [2010] for $F = \mathbb{Q}$ and in Hsieh [2014] for Hilbert modular forms of squarefree level (these works have some additional conditions, but they do not assume trivial central character). In general, when the joint ramification does not satisfy (1-3), this problem appears considerably more complicated.

1B. ***Local results.*** Now, we pass to the local situation and discuss the local test vectors in some detail.

Let F be a p-adic field and L a quadratic separable extension of F (either a field or $F \oplus F$). We may then embed $L^\times$ as a torus $T(F)$ of $\mathrm{GL}_2(F)$. All such embeddings are conjugate in $\mathrm{GL}_2(F)$, so the choice of embedding will be merely one of convenience. Consider an (infinite-dimensional) irreducible admissible representation π of $\mathrm{GL}_2(F)$. We do not assume that the central character ω_π is trivial. A basic question to ask is the following: which characters of $T(F)$ appear as quotients in $\pi|_{T(F)}$? Let Ω be a character of $T(F)$. If Ω is an irreducible constituent of $\pi|_{T(F)}$, i.e., if

$$\mathrm{Hom}_{T(F)}(\pi, \Omega) \neq 0,$$

then we must have $\Omega|_{Z(F)} = \omega_\pi$, where Z denotes the center of GL_2. Hence we will assume $\Omega|_{Z(F)} = \omega_\pi$.

Let D be the unique quaternion division algebra over F, and let π' be the Jacquet–Langlands transfer to $D^\times(F)$ when it exists. If π' exists and $T(F)$ embeds into $D^\times(F)$, put $A(\pi) = \{\pi, \pi'\}$. Otherwise, put $A(\pi) = \{\pi\}$. From [Waldspurger 1985], one knows that

$$\sum_{\tau \in A(\pi)} \dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\tau, \Omega) = 1.$$

In other words, Ω is a constituent of $\pi|_{T(F)}$ if and only if it does not occur in that of $\pi'|_{T(F)}$ (when this makes sense), and it occurs with multiplicity at most one. Further, Tunnell [1983] and Saito [1993] gave a local ϵ-factor criterion:

$$\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = \tfrac{1}{2}\bigl(1 + \varepsilon\bigl(\tfrac{1}{2}, \pi_L \otimes \Omega\bigr)\omega_\pi(-1)\bigr).$$

Applications to a global L-value formula (discussed in Section 1A) require finer information than this. Namely, suppose $\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = 1$ and let $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$ be nonzero. Then one would like to have a *test vector* for ℓ, i.e., an element $\phi \in \pi$ such that $\ell(\phi) \neq 0$. For the applications, we will need ϕ to satisfy two further conditions:

(i) $\phi \in V_\pi^K$ for a compact subgroup K of $\mathrm{GL}_2(F)$ with $\dim(V_\pi^K) = 1$.

(ii) The compact subgroup K above depends only on the ramification data attached to π and Ω.

Let us note that, if $\ell \neq 0$, then some translate of the new vector of π is always a test vector for ℓ. Hence, we can always find a test vector satisfying the first condition above. Under some restriction on the conductors of π and Ω, we will obtain a test vector satisfying the second condition as well.

Specifically, let $\mathfrak{o}$ be the ring of integers of F, $\mathfrak{p}$ its maximal ideal and ϖ a uniformizer. Let $c(\pi)$ be the exponent of the conductor of π as defined in Section 2A, and let

$$K_1(\mathfrak{p}^{c(\pi)}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o}) : c \in \mathfrak{p}^{c(\pi)},\ d \in 1 + \mathfrak{p}^{c(\pi)} \right\}.$$

Let $c(\Omega)$ be the conductor of Ω as defined in (2-19). Gross and Prasad [1991] determine a test vector when $c(\pi) = 0$ (π is unramified) or $c(\Omega) = 0$ (Ω is unramified). In particular, when $c(\pi) = 0$ so $A(\pi) = \{\pi\}$, the vector they obtain can be described as a translate of the new vector.

We will now describe test vectors when π and Ω are both ramified. We will distinguish the split and field case.

1B1. *The split case.* In the following, $w = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

Theorem 1.6. *Suppose $L = F \oplus F$ and let $T(F) \cong L^\times$ be the diagonal torus in $\mathrm{GL}_2(F)$. Let π be any infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}_2(F)$ with central character ω_π and conductor $\mathfrak{p}^{c(\pi)}$, $c(\pi) \geq 0$. Let*

$\Omega(\mathrm{diag}(x, y)) = \Omega_1(x)\Omega_2(y)$ *be a character of* $T(F)$ *such that* $\Omega_1\Omega_2 = \omega_\pi$. *Without loss of generality, assume that* $c(\Omega_1) \geq c(\Omega_2)$. *Write* $\Omega_1 = |\cdot|^{1/2-s_0}\mu$ *for some* $s_0 \in \mathbb{C}$ *and some unitary character* μ *of* $F^\times$ *such that* $\mu(\varpi) = 1$. *Then* $\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = 1$, *and for nonzero* $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$, *the subgroup* $hK_1(\mathfrak{p}^{c(\pi)})h^{-1}$ *fixes a* 1*-dimensional space of* π *consisting of test vectors for* ℓ, *where*

$$h = \begin{cases} \begin{bmatrix} 1 & \varpi^{-c(\Omega)} \\ 0 & 1 \end{bmatrix} & \text{if } c(\mu) = 0 \\ & \quad \text{or } L(s, \pi \otimes \mu^{-1}) \text{ does not have a pole at } s = s_0; \\ w\begin{bmatrix} 1 & \varpi^{-c(\Omega)} \\ 0 & 1 \end{bmatrix} & \text{if } c(\mu) > 0 \text{ and } L(s, \pi \otimes \mu^{-1}) \text{ has a pole at } s = s_0, \\ & \quad \text{but } L(1-s, \tilde{\pi} \otimes \mu) \text{ does not have a pole at } s = s_0. \end{cases}$$

In particular, if both Ω *and* π *are unitary, then we are always in the first case above.*

The proof of the above theorem uses the theory of zeta integrals for GL_2 representations given by their Whittaker models. The zeta integral $Z(s_0, *, \mu^{-1})$ (defined in (3-1)) divided by the L-value $L(s_0, \pi \otimes \mu^{-1})$ gives a concrete realization of a nonzero $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. One checks that the newform in the Whittaker model translated by the matrix h in the statement of the above theorem is a test vector for ℓ.

Note that we do not give a compact subgroup that fixes a 1-dimensional space of π consisting of test vectors for ℓ when both $L(s, \pi \otimes \mu^{-1})$ and $L(1-s, \tilde{\pi} \otimes \mu)$ have a pole at $s = s_0$.

1B2. *The field case.*

Theorem 1.7. *Suppose* L *is a field. Let* π *be any infinite-dimensional, irreducible, admissible representation of* $\mathrm{GL}_2(F)$ *with central character* ω_π *and conductor* $\mathfrak{p}^{c(\pi)}$. *Let* Ω *be a character on* $L^\times$ *such that* $\Omega|_{F^\times} = \omega_\pi$. *Assume that* $c(\Omega) \geq c(\pi) > 0$. *Embed* $L^\times$ *as a torus* $T(F)$ *in* $\mathrm{GL}_2(F)$ *as in Section 2C. Then* $\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = 1$, *and for a nonzero* $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$, *the subgroup*

$$\begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{c(\Omega)} \\ \mathfrak{p}^{c(\pi)-c(\Omega)} & 1+\mathfrak{p}^{c(\pi)} \end{bmatrix} \cap \mathrm{GL}_2(F) = hK_1(\mathfrak{p}^{c(\pi)})h^{-1}, \quad h = \begin{bmatrix} \varpi^{c(\Omega)-c(\pi)} & 0 \\ 0 & 1 \end{bmatrix} w,$$

fixes a 1*-dimensional space of* π *consisting of test vectors for* ℓ.

If π has trivial central character, then we can replace the compact subgroup in the statement of the above theorem by

$$\begin{bmatrix} \varpi^{c(\Omega)} & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^{c(\pi)} & \mathfrak{o}^\times \end{bmatrix} \begin{bmatrix} \varpi^{-c(\Omega)} & \\ & 1 \end{bmatrix},$$

since the Atkin–Lehner element normalizes the group $\left[\begin{smallmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^{c(\pi)} & \mathfrak{o}^\times \end{smallmatrix}\right]$.

The proof of the theorem breaks up into several cases depending on the type of the representation π. Although the proofs are quite different in all cases, it turns out that one of the key ingredients of the proof is that a function roughly of the form $x \mapsto \Omega(1 + x\beta)$ (see Section 2B for details on notation) is an additive character of $\mathfrak{o}$ of a specific conductor. The condition $c(\Omega) \geq c(\pi)$ is required to make this key ingredient work. Also, in certain cases we obtain test vectors for more general situations than the one mentioned above.

Principal series. If π is a principal series representation, then we realize it in its induced model and explicitly define a linear functional

$$\ell(f) = \int\limits_{Z(F)\backslash T(F)} f(t)\Omega^{-1}(t)\,dt.$$

It is easy to see that $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. We are able to show, for any $c(\pi), c(\Omega) \geq 0$, that $\ell \neq 0$. See (4-4) and (4-5) for details. It is not clear if the explicit test vector for ℓ obtained in (4-4) belongs to a 1-dimensional subspace of π of vectors right-invariant under a compact subgroup. It is also not clear how to obtain a component that is right invariant under a conjugate of $K_1(\mathfrak{p}^{c(\pi)})$. To obtain a test vector with the right invariance mentioned in the statement of the theorem, we evaluate ℓ at a translate of the newform of π by h and show that that is nonzero. For this, we need $c(\Omega) \geq c(\pi) > 0$. If we replace the h in the statement of the theorem by $h = \left[\begin{smallmatrix} \varpi^s & \\ & 1 \end{smallmatrix}\right]$, $s = c(\pi) - c(\Omega) - v(\boldsymbol{a})$, where $\boldsymbol{a}$ depends on a particular embedding of $T(F)$ in $\mathrm{GL}_2(F)$, then we can extend the result to the case $c(\Omega) \geq 2c(\chi_1)$ (see Proposition 4.2). Here, $\pi = \chi_1 \times \chi_2$ and $c(\chi_1) \leq c(\chi_2)$.

Twists of the Steinberg representation. If π is a twist of the Steinberg representation by a ramified character χ, then realizing it as a subrepresentation of the reducible induced representation $\chi|\cdot|^{1/2} \times \chi|\cdot|^{-1/2}$, we see that we get the same linear functional and the same nonvanishing of the translate of newform as in the irreducible principal series case.

If π is a twist of the Steinberg representation by an unramified character χ, then we use the fact that such representations are characterized by the existence of a unique (up to constant) vector that is right invariant under the Iwahori subgroup I and is an eigenvector of the Atkin–Lehner operator with eigenvalue $-\chi(\varpi)$. If we assume that $c(\Omega) \geq c(\pi)$, then [Waldspurger 1985] implies the existence of a nonzero $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. As in Section 2D, we can then realize π as a subrepresentation of the space of smooth functions $B : \mathrm{GL}_2(F) \to \mathbb{C}$ satisfying $B(tg) = \Omega(t)B(g)$. In this latter space, we look for a vector B with three properties: one that is right invariant under I, is zero when averaged over $\mathrm{GL}_2(\mathfrak{o})/I$, and is an eigenvector for the Atkin–Lehner operator with eigenvalue $-\chi(\varpi)$. Using a double coset decomposition for $T(F)\backslash\mathrm{GL}_2(F)/I$, we obtain in Lemma 4.4 the explicit values of such a B for all $g \in \mathrm{GL}_2(F)$. This gives us $B(h) \neq 0$, for h defined in the statement of the theorem. The advantage of the above method is twofold. It gives us the explicit values of the newform in the Waldspurger model and it also gives another proof of the uniqueness of the Waldspurger model. One can also obtain an independent proof of existence using the methods of [Pitale 2011], but we do not do that here.

Supercuspidal representations. In the case that π is an irreducible supercuspidal representation we may appeal to Mackey theory. We begin with the explicit construction of supercuspidal representations of $\mathrm{GL}_2(F)$ by induction from an open subgroup that is compact modulo the center. Suppose that J is such a subgroup and $\pi = c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)} \rho$. We first describe the situation when π is minimal, i.e., when the conductor of π cannot be lowered upon twisting by a character.

We say that ρ and Ω intertwine on $T(F)gJ$ if $\mathrm{Hom}_{J \cap g^{-1}T(F)g}(\rho, \Omega^g) \neq 0$. Understanding $\mathrm{Hom}_{T(F)}(\pi, \Omega)$ then reduces to understanding the double cosets $T(F)\backslash \mathrm{GL}_2(F)/J$ on which ρ and Ω intertwine. We do this in two steps. The first step is to consider a larger subgroup $K_{\mathfrak{A}} \supseteq J$ where $K_{\mathfrak{A}}$ is one of two subgroups depending on J. There is a unique double coset $T(F)h_0K_{\mathfrak{A}}$ that depends only on $c(\pi)$ and $c(\Omega)$ containing a $T(F)\backslash \mathrm{GL}_2(F)/J$ double coset on which ρ and Ω can possibly intertwine. This double coset decomposes as the disjoint union of finitely many $T(F)\backslash \mathrm{GL}_2(F)/J$ double cosets

$$T(F)h_0K_{\mathfrak{A}} = \bigsqcup_i T(F)h_iJ.$$

When $c(\Omega) > \left[\frac{1}{2}c(\pi)\right]$, we describe this decomposition explicitly, show that one may choose the representatives h_i to be diagonal matrices, and show for each i that

$$(J \cap h_i^{-1}T(F)h_i) \ker\rho / Z(F) \ker\rho \cong (J \cap \overline{N})/(\ker\rho \cap \overline{N}),$$

where $\overline{N}$ is the subgroup of lower triangular unipotent matrices. It suffices to examine $\rho|_{J\cap\overline{N}}$, which decomposes as a direct sum of characters. We show that there is a unique i_0 such that ρ and Ω intertwine on $T(F)h_{i_0}J$. We conclude that there exists a nonzero linear functional $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. We describe the translate of the newvector in the induced model explicitly, and show that this translate is a test vector.

Finally, we deal with the case of an irreducible supercuspidal representation τ that is not minimal. In this case $\tau \cong \pi \otimes \chi$, where π is a minimal supercuspidal representation and χ is a character of $F^\times$. We construct a vector $\varphi_\chi \in \pi$ so that $\varphi_\chi \otimes \chi$ is a translate of the newvector in τ. Using the results of the minimal case, we show that φ_χ is a test vector for $\Omega \otimes \chi^{-1}$.

Similarly to the irreducible principal series case, if we replace h in the statement of the theorem by $h = \left[\begin{smallmatrix} \varpi^s & \\ & 1 \end{smallmatrix}\right]$, $s = c(\pi) - c(\Omega) - v(\boldsymbol{a})$, then in the *minimal* supercuspidal case, we can extend the result to the case $c(\Omega) \geq \left\lfloor \frac{3}{4}c(\pi) \right\rfloor + 1$.

1B3. *Relation to test vectors of Gross–Prasad.* We recall some results of Gross and Prasad [1991]. For simplicity assume that $\omega_\pi = 1$, that L/F is unramified and that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{T(F)}(\pi, \Omega) = 1.$$

For an order R of $M_2(F)$, let $d(R)$ be the exponent of its reduced discriminant and $c(R)$ be the smallest $c \geq 0$ such that $\mathfrak{o} + \varpi^c \mathfrak{o}_L \subset R$. It is clear that $R^\times$ can only fix a test vector if $c(R) \geq c(\Omega)$. Moreover, if we want $R^\times$ to fix a line in π, it is reasonable to try R with $d(R) = c(\pi)$. Thus one might consider orders with $c(R) = c(\Omega)$ and $d(R) = c(\pi)$. If either $c(\Omega) = 0$ or $c(\pi) = 0$, then there is a unique-up-to-$L^\times$-conjugacy order R with $c(R) = c(\Omega)$ and $d(R) = c(\pi)$, and [Gross and Prasad 1991] shows that $R^\times$ fixes a line consisting of test vectors. If $c(\pi) = 0$ then R is a maximal order, but in general R is not an Eichler order.

When $c(\Omega) > 0$ and $c(\pi) > 0$, the invariants $c(R)$ and $d(R)$ no longer specify R uniquely up to conjugacy by $L^\times$. However, with the above assumptions, Theorem 1.7 can be interpreted as follows: when $c(\Omega) \geq c(\pi)$, there is an Eichler order R with $c(R) = c(\Omega)$ and $d(R) = c(\pi)$ such that $R^\times$ fixes a line in π which consists of test vectors. Moreover, this R can be described uniquely up to $L^\times$-conjugacy as the intersection of two maximal ideals R_1 and R_2, with $c(R_1) = c(\Omega)$ and $c(R_2) = c(\Omega) - c(\pi)$, which are the maximal possible distance apart in the Bruhat–Tits tree, i.e., $d(R_1, R_2) = c(\pi)$. This provides an intrinsic description of our test vectors, i.e., one without reference to a specific embedding of $L^\times$ in $\mathrm{GL}_2(F)$. It would be interesting to know whether other Eichler orders R satisfying $c(R) = c(\Omega)$ and $d(R) = c(\pi)$ also pick out test vectors.

Note that if $c(\pi) > 2c(\Omega)$, there is no Eichler order with $c(R) = c(\Omega)$ and $d(R) = c(\pi)$, which suggests that the case when π is highly ramified, in comparison with Ω, is more complicated than the reverse situation.

1C. ***Outline.*** Our paper consists of two parts, one local and one global.

In the first (local) part of the paper we prove our results on local test vectors, which we treat in three separate cases. Section 2 contains our local notation and embedding of $L^\times$ into $\mathrm{GL}_2(F)$. Then in Section 3 we treat the case where L/F is split, using zeta integrals. This proves Theorem 1.6. Now assume L/F is inert. In Section 4, we treat the case of principal series and Steinberg representations. In Section 5, we treat the case of supercuspidal representations. These two sections complete Theorem 1.7. Finally, in Section 6 we compute certain local spectral distributions associated to our local test vectors.

The global part of the paper consists of two sections. In Section 7, we use the local spectral calculations of Section 6 to prove our L-value formula (Theorem 1.1). In Section 8, we deduce our results on average values and nonvanishing (Theorems 1.3, 1.4 and 1.5).

2. Local setup

Let F be a nonarchimedean local field of characteristic zero, $\mathfrak{o}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$ and ϖ a generator of $\mathfrak{p}$. Denote by q the size of the residue field and by v the normalized valuation map on F.

For a character χ of $F^\times$, let $c(\chi)$ be the exponent of its conductor, i.e., $c(\chi) \geq 0$ is minimal such that χ is trivial on $(1+\mathfrak{p}^{c(\chi)}) \cap \mathfrak{o}^\times$.

2A. ***Subgroups and representations of* GL₂.** We use the following compact subgroups of $\mathrm{GL}_2(F)$. Put $K_1(\mathfrak{o}) = K_2(\mathfrak{o}) = \mathrm{GL}_2(\mathfrak{o})$. For $n > 0$, put

$$K_1(\mathfrak{p}^n) = \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{bmatrix}, \tag{2-1}$$

$$K_2(\mathfrak{p}^n) = \begin{bmatrix} 1+\mathfrak{p}^n & \mathfrak{o} \\ \mathfrak{p}^n & 1+\mathfrak{p}^n \end{bmatrix}. \tag{2-2}$$

For $s \in \mathbb{Z}$, $n \geq 0$, let

$$K_1^{(s)}(\mathfrak{p}^n) = \begin{bmatrix} \varpi^s & \\ & 1 \end{bmatrix} K_1(\mathfrak{p}^n) \begin{bmatrix} \varpi^{-s} & \\ & 1 \end{bmatrix}. \tag{2-3}$$

We also have the Iwahori subgroup

$$I = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{bmatrix} \cap \mathrm{GL}_2(\mathfrak{o}). \tag{2-4}$$

Let (π, V) be an infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}_2(F)$. For $n \geq 0$, denote by V^n the subspace of $K_1(\mathfrak{p}^n)$-fixed vectors. By [Jacquet et al. 1981], one knows $V^n \neq 0$ for some n. Further, if $c(\pi)$ is the minimal n such that $V^n \neq 0$, then $\dim(V^{c(\pi)}) = 1$. Call the ideal $\mathfrak{p}^{c(\pi)}$ the *conductor* of π. If $c(\pi) = 0$, then π is unramified.

Such a π is a principal series, twist of Steinberg (special), or supercuspidal representation. Let χ_1, χ_2 be two characters of $F^\times$. The representation $\pi = \chi_1 \times \chi_2$ is the standard induced representation of $\mathrm{GL}_2(F)$ consisting of locally constant functions $f : \mathrm{GL}_2(F) \to \mathbb{C}$ such that

$$f\Big(\begin{bmatrix} a & b \\ & d \end{bmatrix} g\Big) = \chi_1(a)\chi_2(d)\big|ad^{-1}\big|^{1/2} f(g),$$
$$\text{for all } g \in \mathrm{GL}_2(F),\, a, d \in F^\times,\, b \in F. \tag{2-5}$$

This is irreducible if and only if $\chi_1\chi_2 \neq |\cdot|^{\pm 1}$, in which case we say $\chi_1 \times \chi_2$ is a principal series representation. For a character χ of $F^\times$, the twist of the Steinberg representation by χ, which we denote by $\chi\,\mathrm{St}_{\mathrm{GL}_2}$, is the unique irreducible subrepresentation of the induced representation $\chi|\cdot|^{1/2} \times \chi|\cdot|^{-1/2}$. The supercuspidal representations are described in Section 5.

2B. ***The degree-two extension.*** As in [Furusawa 1993], we fix three elements $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in F$ such that $\boldsymbol{d} = \boldsymbol{b}^2 - 4\boldsymbol{a}\boldsymbol{c} \neq 0$. We let $L = F(\sqrt{\boldsymbol{d}})$ if $\boldsymbol{d} \notin F^{\times 2}$, and $L = F \oplus F$ otherwise. In the latter case we consider F diagonally embedded in L. Let $z \mapsto \bar{z}$ be the obvious involution on L whose fixed point set is F. We define the Legendre symbol as

$$\left(\frac{L}{\mathfrak{p}}\right) = \begin{cases} -1 & \text{if } L/F \text{ is an unramified field extension,} \\ 0 & \text{if } L/F \text{ is a ramified field extension,} \\ 1 & \text{if } L = F \oplus F. \end{cases} \tag{2-6}$$

We make the following assumptions:

- $\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{o}$ and $\boldsymbol{c} \in \mathfrak{o}^\times$.
- If $\boldsymbol{d} \notin F^{\times 2}$, then $\boldsymbol{d}$ is a generator of the discriminant of L/F.
- If $\boldsymbol{d} \in F^{\times 2}$, then $\boldsymbol{d} \in \mathfrak{o}^\times$.

We define elements β and ξ_0 of L by

$$\beta = \begin{cases} \dfrac{\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}} & \text{if } L \text{ is a field,} \\ \left(\dfrac{\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}}, \dfrac{\boldsymbol{b}-\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}}\right) & \text{if } L = F \oplus F, \end{cases} \tag{2-7}$$

$$\xi_0 = \begin{cases} \dfrac{-\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2} & \text{if } L \text{ is a field,} \\ \left(\dfrac{-\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2}, \dfrac{-\boldsymbol{b}-\sqrt{\boldsymbol{d}}}{2}\right) & \text{if } L = F \oplus F. \end{cases} \tag{2-8}$$

If L is a field, let $\mathfrak{o}_L$ be its ring of integers, ϖ_L a uniformizer, and v_L the normalized valuation. If $L = F \oplus F$, put $\mathfrak{o}_L = \mathfrak{o} \oplus \mathfrak{o}$ and $\varpi_L = (\varpi, 1)$. By [Pitale and Schmidt 2009, Lemma 3.1.1], in either case,

$$\mathfrak{o}_L = \mathfrak{o} + \mathfrak{o}\beta = \mathfrak{o} + \mathfrak{o}\xi_0. \tag{2-9}$$

Lemma 2.1. *Suppose L is a field. The possible valuations of β and $\boldsymbol{a}$ are*

$$v_L(\beta) = v(\boldsymbol{a}) = 0 \qquad \textit{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \tag{2-10}$$

$$v_L(\beta) = v(\boldsymbol{a}) \in \{0, 1\} \quad \textit{if } \left(\frac{L}{\mathfrak{p}}\right) = 0. \tag{2-11}$$

Proof. Consider the identity

$$\frac{\boldsymbol{b}+\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}} \cdot \frac{\boldsymbol{b}-\sqrt{\boldsymbol{d}}}{2\boldsymbol{c}} = \frac{\boldsymbol{a}}{\boldsymbol{c}}. \tag{2-12}$$

If $\left(\frac{L}{\mathfrak{p}}\right) = -1$, we get the result by observing that d is a nonsquare unit. If $\left(\frac{L}{\mathfrak{p}}\right) = 0$, we get the result since $1, \beta$ is an integral basis. $\square$

Fix the ideal in $\mathfrak{o}_L$ given by

$$\mathfrak{P}_L := \mathfrak{p}\mathfrak{o}_L = \begin{cases} \mathfrak{p}_L & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = -1, \\ \mathfrak{p}_L^2 & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 0, \\ \mathfrak{p} \oplus \mathfrak{p} & \text{if } \left(\frac{L}{\mathfrak{p}}\right) = 1. \end{cases} \tag{2-13}$$

Here $\mathfrak{p}_L$ is the maximal ideal of $\mathfrak{o}_L$ when L is a field. We have $\mathfrak{P}_L^n \cap \mathfrak{o} = \mathfrak{p}^n$ for all $n \geq 0$.

Under our stated assumptions, it makes sense to consider the quadratic equation $\boldsymbol{c}u^2 + \boldsymbol{b}u + \boldsymbol{a} = 0$ over the residue class field $\mathfrak{o}/\mathfrak{p}$. The number of solutions of this equation is $\left(\frac{L}{\mathfrak{p}}\right) + 1$. In the ramified case we will fix an element $u_0 \in \mathfrak{o}$ such that

$$\boldsymbol{c}u_0^2 + \boldsymbol{b}u_0 + \boldsymbol{a} \in \mathfrak{p}; \tag{2-14}$$

see [Pitale and Schmidt 2009, Lemma 3.1.1]. Further, note that in the ramified case we have

$$\boldsymbol{b} + 2\boldsymbol{c}u_0 \in \mathfrak{p}. \tag{2-15}$$

This follows from the fact that u_0 is a double root of $\boldsymbol{c}u^2 + \boldsymbol{b}u + \boldsymbol{a}$ over $\mathfrak{o}/\mathfrak{p}$.

2C. ***The torus.*** We now specify an embedding of $L^\times$ as a torus in GL_2 for convenience of calculations. With $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ as above, let

$$S = \begin{bmatrix} \boldsymbol{a} & \frac{1}{2}\boldsymbol{b} \\ \frac{1}{2}\boldsymbol{b} & \boldsymbol{c} \end{bmatrix}, \qquad \xi = \begin{bmatrix} \frac{1}{2}\boldsymbol{b} & \boldsymbol{c} \\ -\boldsymbol{a} & -\frac{1}{2}\boldsymbol{b} \end{bmatrix}.$$

Then $F(\xi) = F \cdot I_2 + F \cdot \xi$ is a 2-dimensional F-algebra isomorphic to L. If L is a field, then an isomorphism is given by $x + y\xi \mapsto x + y\sqrt{\boldsymbol{d}}/2$. If $L = F \oplus F$, then an isomorphism is given by $x + y\xi \mapsto \left(x + y\sqrt{\boldsymbol{d}}/2, x - y\sqrt{\boldsymbol{d}}/2\right)$. The determinant map on $F(\xi)$ corresponds to the norm map on L. Let

$$T(F) = \{g \in \mathrm{GL}_2(F) : {}^t g S g = \det(g) S\}. \tag{2-16}$$

One can check that $T(F) = F(\xi)^\times$. Note that $T(F) \cong L^\times$ via the isomorphism $F(\xi) \cong L$. Under the same isomorphism the group $T(\mathfrak{o}) := T(F) \cap \mathrm{GL}_2(\mathfrak{o})$ is isomorphic to $\mathfrak{o}_L^\times$. Note that $T(F)$ consists of all matrices

$$t(x, y) = \begin{bmatrix} x + \frac{1}{2}y\boldsymbol{b} & \boldsymbol{c}y \\ -\boldsymbol{a}y & x - \frac{1}{2}y\boldsymbol{b} \end{bmatrix},$$
$$\text{for all } x, y \in F,\ \det(g) = x^2 - \tfrac{1}{4}y^2(\boldsymbol{b}^2 - 4\boldsymbol{a}\boldsymbol{c}) \neq 0. \tag{2-17}$$

We give a useful structural lemma here.

Lemma 2.2. *Let L/F be a field extension. For any $m, n \geq 0$, we have*

$$T(F)\begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix}K_1(\mathfrak{p}^n) = T(F)\begin{bmatrix} \varpi^{m-v(\boldsymbol{a})} & \\ & 1 \end{bmatrix}wK_1(\mathfrak{p}^n). \tag{2-18}$$

Proof. Set $y = \varpi^{-m}$ and $x = \frac{1}{2}y\boldsymbol{b}$. Then

$$\begin{bmatrix} \varpi^{-m} & \\ & 1 \end{bmatrix} = \frac{-1}{\boldsymbol{a}\varpi^{-v(\boldsymbol{a})}}t(x, y)\begin{bmatrix} \varpi^{m-v(\boldsymbol{a})} & \\ & 1 \end{bmatrix}wk,$$

with

$$k = \begin{bmatrix} \dfrac{\boldsymbol{a}\varpi^{-v(\boldsymbol{a})}}{\boldsymbol{c}} & \dfrac{\boldsymbol{b}\varpi^{m-v(\boldsymbol{a})}}{\boldsymbol{c}} \\ & 1 \end{bmatrix} \in K_1(\mathfrak{p}^n) \quad \text{for all } n \geq 0,$$

since $v(\boldsymbol{b}) \geq 1$ whenever $v(\boldsymbol{a}) = 1$. □

2D. ***The Waldspurger model.*** Let Ω be any character of $L^\times$, which we may view as a character of the torus $T(F)$. Define

$$c(\Omega) := \min\{m \geq 0 : \Omega|_{(1+\mathfrak{P}_L^m)\cap\mathfrak{o}_L^\times} \equiv 1\}. \tag{2-19}$$

Note that this is the (exponent of the) conductor of Ω only in the case L/F is an unramified field extension. Let $\mathcal{B}(\Omega)$ be the space of all locally constant functions $B : \mathrm{GL}_2(F) \to \mathbb{C}$ satisfying

$$B(tg) = \Omega(t)B(g) \quad \text{for all } t \in T(F),\ g \in \mathrm{GL}_2(F). \tag{2-20}$$

Let (π, V) be any infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}_2(F)$. We say that π has an Ω-*Waldspurger model* if π is isomorphic to a subrepresentation of $\mathcal{B}(\Omega)$. We call a linear functional ℓ on π an Ω-*Waldspurger functional* if it satisfies

$$\ell(\pi(t)v) = \Omega(t)\ell(v) \quad \text{for all } t \in T(F),\ v \in V. \tag{2-21}$$

If π has an Ω-Waldspurger model then we obtain an Ω-Waldspurger functional ℓ by $\ell(B) = B(1)$. On the other hand, if π has an Ω-Waldspurger functional ℓ, we obtain an Ω-Waldspurger model for π by the map $v \mapsto B_v$, where $B_v(g) = \ell(\pi(g)v)$. Observe that a necessary condition for an Ω-Waldspurger model or functional to exist is that $\Omega|_{F^\times} = \omega_\pi$, the central character of π.

If π has an Ω-Waldspurger functional ℓ, we say that $v \in V$ is a *test vector* for ℓ if $\ell(v) \neq 0$. From the discussion above, this is equivalent to $B_v(1) \neq 0$. Suppose B_0 is the newform in an Ω-Waldspurger model of π. Lemma 2.2 states that, in the field case, for $m \geq 0$, the vector

$$\pi\left(\begin{bmatrix} \varpi^{m-v(\boldsymbol{a})} & \\ & 1 \end{bmatrix}w\right)B_0$$

is a test vector if and only if $\pi\left(\begin{bmatrix}\varpi^{-m} & \\ & 1\end{bmatrix}\right)B_0$ is also a test vector. This will be used in the proof of Theorem 1.7 below.

Criteria for existence of Waldspurger functionals, which must be unique up to scalars, are given in Section 1B.

3. Zeta integrals and test vectors for split Waldspurger models

In this section we show that any irreducible admissible representation π of $\mathrm{GL}_2(F)$ has a split Ω-Waldspurger model for every character Ω of $L^\times = F^\times \oplus F^\times$. Under certain restrictions on the poles of the L-function of π, we also determine test vectors for the Waldspurger functional that are right invariant under certain conjugates of the compact group $K_1(\mathfrak{p}^{c(\pi)})$. The conjugating elements depend only on $c(\pi)$ and $c(\Omega)$.

Let π be any irreducible admissible representation of $\mathrm{GL}_2(F)$ with central character ω_π (not assumed to be trivial). Let π be given by its Whittaker model $\mathcal{W}(\pi,\psi)$, where ψ is a nontrivial character of F with conductor $\mathfrak{o}$. For any $W \in \mathcal{W}(\pi,\psi)$ and a unitary character μ of $F^\times$, define the zeta integral

$$Z(s, W, \mu^{-1}) := \int_{F^\times} W\left(\begin{bmatrix} x & \\ & 1\end{bmatrix}\right)|x|^{s-1/2}\mu^{-1}(x)\, d^\times x, \tag{3-1}$$

where $d^\times x$ is the Haar measure on $F^\times$ giving $\mathfrak{o}^\times$ volume $1-q^{-1}$. Since μ is unitary, there is an $r \in \mathbb{R}$ not depending on μ such that $Z(s, W, \mu^{-1})$ converges absolutely for $\Re(s) > r$. By the theory of L-functions, we have

$$\frac{Z(s, W, \mu^{-1})}{L(s, \mu^{-1}\otimes\pi)} \in \mathbb{C}[q^{-s}, q^s] \tag{3-2}$$

and the functional equation

$$\frac{Z(1-s, \pi(w)W, \mu\omega_\pi^{-1})}{L(1-s, \mu\otimes\tilde{\pi})} = \varepsilon(s, \mu^{-1}\otimes\pi, \psi)\frac{Z(s, W, \mu^{-1})}{L(s, \mu^{-1}\otimes\pi)} \tag{3-3}$$

for any $W \in \mathcal{W}(\pi,\psi)$. Here $w = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$. Please refer to Theorem 6.12 of [Gelbart 1975] for details.

Let W_0 be the unique $K_1(\mathfrak{p}^{c(\pi)})$-right invariant vector in $\mathcal{W}(\pi,\psi)$ such that $W_0(1) = 1$. The formula for $W_0\left(\begin{bmatrix} x & \\ & 1\end{bmatrix}\right)$ in various cases is given in Table 1 (see, e.g., [Schmidt 2002]).

Proposition 3.1. *Let π be any irreducible, admissible representation of $\mathrm{GL}_2(F)$ with central character ω_π and conductor $\mathfrak{p}^{c(\pi)}$. Let W_0 be the newform in the Whittaker model $\mathcal{W}(\pi,\psi)$ of π such that $W_0(1)=1$. Let μ be a unitary character of $F^\times$.*

(i) *If* $c(\mu) = 0$ *then, for any* π, *we have*

$$Z\left(s, \pi\left(\begin{bmatrix} 1 & \varpi^{-c(\mu)} \\ & 1 \end{bmatrix}\right) W_0, \mu^{-1}\right) = \left(1 - \frac{1}{q}\right) L(s, \mu^{-1} \otimes \pi).$$

(ii) *If* $c(\mu) > 0$ *then*

$$Z\left(s, \pi\left(\begin{bmatrix} 1 & \varpi^{-c(\mu)} \\ & 1 \end{bmatrix}\right) W_0, \mu^{-1}\right) = q^{-c(\mu)/2} \mu(\varpi^{-c(\mu)}) \varepsilon\left(\frac{1}{2}, \mu, \psi\right).$$

Proof. If $c(\mu) = 0$, then the values of the newform W_0 from Table 1 and the normalization of the measure give us (i). We have, for any $k \in \mathbb{Z}$ and any π,

$$\begin{aligned} Z\left(s, \pi\left(\begin{bmatrix} 1 & \varpi^{k} \\ & 1 \end{bmatrix}\right) W_0, \mu^{-1}\right) & \\ &= \int_{F^\times} \psi(a\varpi^k) W_0\left(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\right) |a|^{s-1/2} \mu^{-1}(a)\, d^\times a \\ &= \sum_{j \in \mathbb{Z}} \int_{\mathfrak{o}^\times} \psi(a\varpi^{j+k}) W_0\left(\begin{bmatrix} a\varpi^j & \\ & 1 \end{bmatrix}\right) |a\varpi^j|^{s-1/2} \mu^{-1}(a\varpi^j)\, d^\times a \\ &= \sum_{j \in \mathbb{Z}} q^{-j(s-1/2)} \mu^{-1}(\varpi^j) W_0\left(\begin{bmatrix} \varpi^j & \\ & 1 \end{bmatrix}\right) \int_{\mathfrak{o}^\times} \psi(a\varpi^{j+k}) \mu^{-1}(a)\, d^\times a. \end{aligned}$$

If $c(\mu) > 0$, then, by the definition of the epsilon factor for μ (see [Schmidt 2002, equation (5)]), we have

$$\begin{aligned} \int_{\mathfrak{o}^\times} \psi(a\varpi^{j+k}) \mu^{-1}(a)\, d^\times a & \\ &= \begin{cases} q^{-c(\mu)/2} \mu(\varpi^{j+k}) \varepsilon\left(\frac{1}{2}, \mu, \psi\right) & \text{if } j + k = -c(\mu), \\ 0 & \text{if } j + k \neq -c(\mu). \end{cases} \end{aligned} \tag{3-4}$$

Now the proposition follows since $W_0(1) = 1$. □

π	$W_0\left(\begin{bmatrix} x & \\ & 1 \end{bmatrix}\right)$
$\chi_1 \times \chi_2$, with χ_1, χ_2 unramified, $\chi_1 \chi_2^{-1} \neq \lvert\cdot\rvert^{\pm 1}$	$\lvert x\rvert^{1/2} \left(\sum_{k+l=v(x)} \chi_1(\varpi^k) \chi_2(\varpi^l) \right) 1_{\mathfrak{o}}(x)$
$\chi_1 \times \chi_2$, with χ_1 unramified, χ_2 ramified	$\lvert x\rvert^{1/2} \chi_1(x) 1_{\mathfrak{o}}(x)$
$\chi\, \mathrm{St}_{\mathrm{GL}_2}$, with χ unramified	$\lvert x\rvert \chi(x) 1_{\mathfrak{o}}(x)$
$L(s, \pi) = 1$	$1_{\mathfrak{o}^\times}(x)$

Table 1. Whittaker newform values.

Proof of Theorem 1.6. For any $W \in \mathcal{W}(\pi, \psi)$, define

$$\ell(W) := \frac{Z(s_0, W, \mu^{-1})}{L(s_0, \mu^{-1} \otimes \pi)}. \tag{3-5}$$

The well-definedness of ℓ for all s_0 and μ follows from (3-2). By [Gelbart 1975, Theorem 6.12], ℓ is nonzero. The definition of the zeta integral and $\Omega_1\Omega_2 = \omega_\pi$ gives us the transformation property

$$\ell\Big(\pi\Big(\begin{bmatrix} x & \\ & y \end{bmatrix}\Big)W\Big) = \Omega_1(x)\Omega_2(y)\ell(W), \quad x, y \in F^\times.$$

Hence, we get $\mathrm{Hom}_{T(F)}(\pi, \Omega) \neq 0$. The 1-dimensionality follows from [Waldspurger 1985]. Note that, if $c(\mu) = 0$ or if $L(s, \mu^{-1} \otimes \pi)$ does not have a pole at $s = s_0$, then we have

$$\ell\Big(\pi\Big(\begin{bmatrix} 1 & \varpi^{-c(\Omega)} \\ & 1 \end{bmatrix}\Big)W_0\Big) \neq 0$$

by Proposition 3.1. If $c(\mu) > 0$ and $L(s, \mu^{-1} \otimes \pi)$ has a pole at $s = s_0$, then

$$\ell\Big(\pi\Big(\begin{bmatrix} 1 & \varpi^{-c(\Omega)} \\ & 1 \end{bmatrix}\Big)W_0\Big) = 0$$

by Proposition 3.1. In this case, if we assume that $L(1-s, \mu \otimes \tilde{\pi})$ does not have a pole at $s = s_0$, then we can use the local functional equation (3-3), which gives us the test vectors for ℓ. The uniqueness statement follows from the uniqueness of W_0. If Ω and π are unitary, then $s_0 = \frac{1}{2}$ and one can check that $L(s, \mu^{-1} \otimes \pi)$ does not have a pole at $s = \frac{1}{2}$. □

4. Nonsupercuspidal representations

Here we assume that L is a field and prove Theorem 1.7 when π is not supercuspidal.

Let us define Haar measures dg on $\mathrm{GL}_2(F)$ such that $\mathrm{GL}_2(\mathfrak{o})$ has volume 1; $d^\times x$ on $F^\times = Z(F)$, the center of $\mathrm{GL}_2(F)$, such that $\mathfrak{o}^\times$ has volume 1 (note this is different from Section 3); and dt on $T(F) = L^\times$ such that the volume of $\mathfrak{o}_L^\times$ is 1.

4A. ***Irreducible principal series representation.*** Let π be a *ramified* irreducible principal series representation of $\mathrm{GL}_2(F)$ given by

$$\begin{gathered} \pi = \chi_1 \times \chi_2, \qquad \chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}, \qquad c(\chi_2) \geq c(\chi_1), \\ c(\pi) = c(\chi_1) + c(\chi_2) > 0, \qquad \omega_\pi = \chi_1\chi_2. \end{gathered} \tag{4-1}$$

Recall that π consists of locally constant functions f on $\mathrm{GL}_2(F)$ satisfying (2-5). The unique, up to scalars, right $K_1(\mathfrak{p}^{c(\pi)})$-invariant vector f_0 in π is given by the formula

$$f_0(g)=\begin{cases}|a/d|^{1/2}\chi_1(a)\chi_2(d) & \text{if } g\in\begin{bmatrix}a&*\\&d\end{bmatrix}\gamma_{c(\chi_2)}K_1(\mathfrak{p}^{c(\pi)}),\\ 0 & \text{if } g\notin B(F)\gamma_{c(\chi_2)}K_1(\mathfrak{p}^{c(\pi)}),\end{cases}\tag{4-2}$$

where $\gamma_{c(\chi_2)}=\left[\begin{smallmatrix}1&\\ \varpi^{c(\chi_2)}&1\end{smallmatrix}\right]$ and $B(F)$ is the Borel subgroup of $\mathrm{GL}_2(F)$ consisting of upper triangular matrices. See [Schmidt 2002] for details.

Let Ω be a character of $L^\times$ such that $\Omega|_{F^\times}=\omega_\pi$. Let $\mathcal{B}(\Omega)$ be the space of all locally constant functions $B:\mathrm{GL}_2(F)\to\mathbb{C}$ satisfying (2-20) defined in Section 2D. Define an intertwining operator $\mathcal{A}:\pi\to\mathcal{B}(\Omega)$ by the formula

$$(\mathcal{A}(f))(g):=\int\limits_{Z(F)\backslash T(F)} f(tg)\Omega^{-1}(t)\,dt,\quad f\in\pi,\ g\in\mathrm{GL}_2(F).\tag{4-3}$$

Since $Z(F)\backslash T(F)$ is compact and $\Omega|_{F^\times}=\omega_\pi$, this integral is well defined and convergent. Note $Z(F)\backslash T(F)$ is isomorphic to $Z(\mathfrak{o})\backslash T(\mathfrak{o})$ if $\left(\frac{L}{\mathfrak{p}}\right)=-1$, and to $Z(\mathfrak{o})\backslash T(\mathfrak{o})\sqcup\varpi_L(Z(\mathfrak{o})\backslash T(\mathfrak{o}))$ if $\left(\frac{L}{\mathfrak{p}}\right)=0$.

Next we show that $\mathcal{A}$ is nonzero for all Ω and, assuming $c(\Omega)\geq 2c(\chi_1)$, obtain $g\in\mathrm{GL}_2(F)$ such that $(\mathcal{A}(f_0))(g)\neq 0$. First observe that we can write $\mathrm{GL}_2(F)=M_2(F)T(F)$, where $M_2(F)=\left\{\left[\begin{smallmatrix}a&b\\0&1\end{smallmatrix}\right]:a,b\in F\right\}\cap\mathrm{GL}_2(F)$ is the mirabolic subgroup of $\mathrm{GL}_2(F)$ and $M_2(F)\cap T(F)=\{1\}$. Hence, the function $\hat{f}$ defined by

$$\hat{f}\left(\begin{bmatrix}a&b\\0&1\end{bmatrix}t\right)=|a|^{1/2}\chi_1(a)\Omega(t),\quad \begin{bmatrix}a&b\\0&1\end{bmatrix}\in M_2(F),\ t\in T(F),\tag{4-4}$$

is a well-defined element of π and, for $t\in T(F)$, satisfies $\pi(t)\hat{f}=\Omega(t)\hat{f}$, which implies

$$\mathcal{A}(\hat{f})\neq 0.\tag{4-5}$$

For the computation of $\mathcal{A}$ applied to the newvector f_0, we need to know when the argument tg of f_0 is in the support of f_0 for certain elements $g\in\mathrm{GL}_2(F)$. We obtain that information in the following lemma.

Lemma 4.1. *Let $t=t(x,y)\in T(F)$. For $s\in\mathbb{Z}$, we have the following decomposition of $t\left[\begin{smallmatrix}\varpi^s&\\&1\end{smallmatrix}\right]$ as bk with $b\in B(F)$ and $k\in\mathrm{GL}_2(\mathfrak{o})$.*

(i) *If $x-\frac{1}{2}\boldsymbol{b}y\in\varpi^{-l}\mathfrak{o}^\times$, $l\geq 0$, $\boldsymbol{a}\varpi^{s+l}y\in\mathfrak{o}$, then*

$$t\begin{bmatrix}\varpi^s&\\&1\end{bmatrix}=\begin{bmatrix}\det(t)\varpi^s/\left(x-\frac{1}{2}\boldsymbol{b}y\right)&\varpi^{-l}\boldsymbol{c}y/\left(x-\frac{1}{2}\boldsymbol{b}y\right)\\0&\varpi^{-l}\end{bmatrix}\times\begin{bmatrix}1&0\\-\boldsymbol{a}\varpi^{s+l}y&\varpi^l\left(x-\frac{1}{2}\boldsymbol{b}y\right)\end{bmatrix}.$$

(ii) *If* $x-\frac{1}{2}\boldsymbol{b}y\in\varpi^{-l}\mathfrak{o}^\times$, $l\geq 0$, $\boldsymbol{a}\varpi^{s+l}y\notin\mathfrak{o}$, *then*

$$t\begin{bmatrix}\varpi^s & \\ & 1\end{bmatrix}=\begin{bmatrix}\det(t)/(\boldsymbol{a}y) & -\varpi^s\big(x+\frac{1}{2}\boldsymbol{b}y\big)\\ 0 & \boldsymbol{a}\varpi^s y\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & \big(x-\frac{1}{2}\boldsymbol{b}y\big)/(\boldsymbol{a}\varpi^s y)\end{bmatrix}.$$

(iii) *If* $x-\frac{1}{2}\boldsymbol{b}y\in\mathfrak{p}$, $\big(x-\frac{1}{2}\boldsymbol{b}y\big)/(\varpi^s\boldsymbol{a}y)\in\mathfrak{o}$, *then*

$$t\begin{bmatrix}\varpi^s & \\ & 1\end{bmatrix}=\begin{bmatrix}\det(t)/(\boldsymbol{a}y) & -\varpi^s\big(x+\frac{1}{2}\boldsymbol{b}y\big)\\ 0 & \varpi^s\boldsymbol{a}y\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & \big(x-\frac{1}{2}\boldsymbol{b}y\big)/(\varpi^s\boldsymbol{a}y)\end{bmatrix}.$$

(iv) *If* $x-\frac{1}{2}\boldsymbol{b}y\in\mathfrak{p}$, $\big(x-\frac{1}{2}\boldsymbol{b}y\big)/(\varpi^s\boldsymbol{a}y)\notin\mathfrak{o}$, *then*

$$t\begin{bmatrix}\varpi^s & \\ & 1\end{bmatrix}=\begin{bmatrix}\det(t)\varpi^s/\big(x-\frac{1}{2}\boldsymbol{b}y\big) & \boldsymbol{c}y\\ 0 & x-\frac{1}{2}\boldsymbol{b}y\end{bmatrix}\begin{bmatrix}1 & 0\\ -\boldsymbol{a}\varpi^s y/\big(x-\frac{1}{2}\boldsymbol{b}y\big) & 1\end{bmatrix}.$$

Proof. The lemma is obtained by direct computation. □

Proposition 4.2. *Let* $c(\Omega)\geq 2c(\chi_1)$ *and set* $s=c(\pi)-c(\Omega)-v(\boldsymbol{a})$. *Then*

$$(\mathcal{A}(f_0))\left(\begin{bmatrix}\varpi^s & \\ & 1\end{bmatrix}\right)\neq 0.$$

Proof. Since $\Omega|_{F^\times}=\omega_\pi$ and $c(\Omega)\geq 2c(\chi_1)$, we have $c(\Omega)>0$. Let us first compute the part of the integral $(\mathcal{A}(f_0))\big(\big[\begin{smallmatrix}\varpi^s & \\ & 1\end{smallmatrix}\big]\big)$ over $Z(\mathfrak{o})\backslash T(\mathfrak{o})$. The argument of f_0 is given by $t\big[\begin{smallmatrix}\varpi^s & \\ & 1\end{smallmatrix}\big]$, where

$$t=\begin{bmatrix}x+\frac{1}{2}\boldsymbol{b}y & \boldsymbol{c}y\\ -\boldsymbol{a}y & x-\frac{1}{2}\boldsymbol{b}y\end{bmatrix}\in T(\mathfrak{o}),$$

i.e., $y, x-\frac{1}{2}\boldsymbol{b}y\in\mathfrak{o}$ and $x^2-\frac{1}{4}y^2\boldsymbol{d}\in\mathfrak{o}^\times$. We write $t\big[\begin{smallmatrix}\varpi^s & \\ & 1\end{smallmatrix}\big]$ as bk with $b\in B(F)$ and $k\in\mathrm{GL}_2(\mathfrak{o})$ according to Lemma 4.1. Since $t\in T(\mathfrak{o})$, we must have $l=0$ in parts (i) and (ii) of Lemma 4.1, and $\boldsymbol{a}$, $y\in\mathfrak{o}^\times$ in parts (iii) and (iv) of Lemma 4.1. The support of f_0 is $B(F)\gamma_{c(\chi_2)}K_1(\mathfrak{p}^{c(\pi)})$ and an element $k\in\mathrm{GL}_2(\mathfrak{o})$ lies in the support if and only if the (2, 1) entry of k has (strictly positive) valuation $c(\chi_2)$ if $c(\chi_1)>0$ and $\geq c(\chi_2)$ if $c(\chi_1)=0$. Hence, the k obtained in parts (ii) and (iii) of Lemma 4.1 is never in the support of f_0. Since $s<c(\chi_2)-v(\boldsymbol{a})$, the k obtained in part (iv) of Lemma 4.1 is not in the support of f_0 as well. Hence, the only possibility is part (i) of Lemma 4.1. First suppose that $c(\chi_1)>0$. By successive change of variable

$x \to x + \frac{1}{2}\boldsymbol{b}y$ and $y \to xy$, we have

$$\int_{Z(\mathfrak{o})\backslash T(\mathfrak{o})} f_0\Big(t\begin{bmatrix} \varpi^s & \\ & 1\end{bmatrix}\Big)\Omega^{-1}(t)\,dt$$
$$= \int_{y\in\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}\mathfrak{o}^\times} f_0\Big(\begin{bmatrix} \varpi^s(1+\boldsymbol{b}y+\boldsymbol{ac}y^2) & \boldsymbol{c}y \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}y\varpi^s & 1\end{bmatrix}\Big) \times \Omega^{-1}(1+\boldsymbol{c}y\beta)\,dy$$
$$= q^{-s/2}\chi_1(\varpi^s)\int_{y\in\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}\mathfrak{o}^\times} \chi_1(1+\boldsymbol{b}y+\boldsymbol{ac}y^2)\,f_0\Big(\begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}y\varpi^s & 1\end{bmatrix}\Big) \times \Omega^{-1}(1+\boldsymbol{c}y\beta)\,dy$$
$$= (1-q^{-1})q^{s/2-c(\chi_2)+v(\boldsymbol{a})}\chi_1(\varpi^s) \times \int_{\mathfrak{o}^\times} \chi_1(1+\boldsymbol{b}\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}y+\boldsymbol{ac}\varpi^{2(c(\chi_2)-s-v(\boldsymbol{a}))}y^2)$$
$$\times f_0\Big(\begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}y\varpi^{c(\chi_2)-v(\boldsymbol{a})} & 1\end{bmatrix}\Big)\Omega^{-1}(1+\boldsymbol{c}\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}y\beta)\,d^\times y. \quad (4\text{-}6)$$

We get the factor $(1-q^{-1})$ above by the normalization of measures. Now, we have

$$\begin{bmatrix} 1 & 0 \\ -\boldsymbol{a}y\varpi^{c(\chi_2)-v(\boldsymbol{a})} & 1\end{bmatrix} = \begin{bmatrix} -\varpi^{v(\boldsymbol{a})}/(\boldsymbol{a}y) & 0 \\ 0 & 1\end{bmatrix}\gamma_{c(\chi_2)}\begin{bmatrix} -\boldsymbol{a}y\varpi^{-v(\boldsymbol{a})} & \\ & 1\end{bmatrix}.$$

Hence the integral (4-6) is equal to

$$(1-q^{-1})q^{s/2-c(\chi_2)+v(\boldsymbol{a})}\chi_1(\varpi^s) \times \int_{\mathfrak{o}^\times} \chi_1(1+\boldsymbol{b}\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}y+\boldsymbol{ac}\varpi^{2(c(\chi_2)-s-v(\boldsymbol{a}))}y^2)$$
$$\times \chi_1(-\varpi^{v(\boldsymbol{a})}/(\boldsymbol{a}y))\Omega^{-1}(1+\boldsymbol{c}\varpi^{c(\chi_2)-s-v(\boldsymbol{a})}y\beta)\,d^\times y \quad (4\text{-}7)$$

Using $c(\chi_2)-s-v(\boldsymbol{a}) = c(\Omega)-c(\chi_1) \geq c(\chi_1)$, we get

$$(1-q^{-1})q^{(c(\chi_1)-c(\chi_2)-c(\Omega)+v(\boldsymbol{a}))/2}\chi_1(-\varpi^{c(\pi)-c(\Omega)}/\boldsymbol{a}) \times \int_{\mathfrak{o}^\times} \chi_1^{-1}(y)\Omega^{-1}(1+\boldsymbol{c}\varpi^{c(\Omega)-c(\chi_1)}y\beta)\,d^\times y.$$

Since $c(\Omega) \geq 2c(\chi_1)$, the map $y \mapsto \Omega^{-1}(1+\boldsymbol{c}\varpi^{c(\Omega)-c(\chi_1)}y\beta)$ is an additive character of $\mathfrak{o}$ of conductor $c(\chi_1)$. This character extends to a character $\widehat{\psi}$ of F with conductor $c(\chi_1)$.

Hence, using (3-4), we get

$$\int_{Z(\mathfrak{o})\backslash T(\mathfrak{o})} f_0\left(t\begin{bmatrix} \varpi^{c(\pi)-c(\Omega)-v(\boldsymbol{a})} & \\ & 1 \end{bmatrix}\right)\Omega^{-1}(t)\,dt$$
$$= (1-q^{-1})q^{(-c(\chi_2)-c(\Omega)+v(\boldsymbol{a}))/2}\chi_1(-\varpi^{c(\chi_2)-c(\Omega)}/\boldsymbol{a})\varepsilon\left(\tfrac{1}{2},\chi_1,\widehat{\psi}\right). \tag{4-8}$$

If $c(\chi_1)=0$, the integral is much simpler. We get

$$\int_{Z(\mathfrak{o})\backslash T(\mathfrak{o})} f_0\left(t\begin{bmatrix} \varpi^{s} & \\ & 1 \end{bmatrix}\right)\Omega^{-1}(t)\,dt$$
$$= \int_{y\in\mathfrak{p}^{c(\Omega)}} f_0\left(\begin{bmatrix} \varpi^s(1+\boldsymbol{b}y+\boldsymbol{a}\boldsymbol{c}y^2) & \boldsymbol{c}y \\ 0 & 1 \end{bmatrix}\right)\Omega^{-1}(1+\boldsymbol{c}y\beta)\,dy$$
$$= \chi_1(\varpi^s)q^{-s/2-c(\Omega)}. \tag{4-9}$$

If L/F is a ramified field extension, then it is also necessary to integrate over $\varpi_L(Z(\mathfrak{o})\backslash T(\mathfrak{o}))$. Let

$$t=\begin{bmatrix} x+\frac{1}{2}\boldsymbol{b}y & \boldsymbol{c}y \\ -\boldsymbol{a}y & x-\frac{1}{2}\boldsymbol{b}y \end{bmatrix}\in\varpi_L T(\mathfrak{o}).$$

Hence, we have

$$x^2-\tfrac{1}{4}y^2\boldsymbol{d}=\left(x+\tfrac{1}{2}\boldsymbol{b}y\right)\left(x-\tfrac{1}{2}\boldsymbol{b}y\right)+\boldsymbol{a}\boldsymbol{c}y^2\in\varpi\mathfrak{o}^\times.$$

We claim that, for $r<c(\chi_2)-v(\boldsymbol{a})$, the element $t\left[\begin{smallmatrix}\varpi^r & \\ & 1\end{smallmatrix}\right]$ is never in the support of f_0. We look at the four possibilities from Lemma 4.1. We know that the values of x, y satisfying the conditions of parts (ii) and (iii) never give elements in the support of f_0.

- Suppose $x-\frac{1}{2}\boldsymbol{b}y\in\varpi^{-l}\mathfrak{o}^\times$ with $l\geq 0$ and $\boldsymbol{a}y\varpi^{r+l}\in\mathfrak{o}$. To prove the claim, it is enough to show that $v(y)\leq -l$. Suppose $y\in\mathfrak{p}^{-l+1}$. Then we have $\varpi^l y\in\mathfrak{p}$. By assumption, we have $\varpi^l\left(x-\frac{1}{2}\boldsymbol{b}y\right)\in\mathfrak{o}^\times$. Hence $\varpi^l\left(x+\frac{1}{2}\boldsymbol{b}y\right)\in\mathfrak{o}^\times$. But then we get $\varpi^{2l}\left(x^2-\frac{1}{4}y^2\boldsymbol{d}\right)\in\mathfrak{o}^\times$, which is a contradiction.
- Suppose $x-\frac{1}{2}\boldsymbol{b}y\in\mathfrak{p}$, $\left(x-\frac{1}{2}\boldsymbol{b}y\right)/(\varpi^r\boldsymbol{a}y)\notin\mathfrak{o}$. To prove the claim, it is enough to show that $v(y)\leq 0$. Suppose $y\in\mathfrak{p}$. Then $x+\frac{1}{2}\boldsymbol{b}y\in\mathfrak{p}$. But then we get $\left(x^2-\frac{1}{4}y^2\boldsymbol{d}\right)\in\mathfrak{p}^2$, a contradiction.

Hence, for $r<c(\chi_2)-v(\boldsymbol{a})$, we have

$$\int_{\varpi_L(Z(\mathfrak{o})\backslash T(\mathfrak{o}))} f_0\left(t\begin{bmatrix} \varpi^{r} & \\ & 1 \end{bmatrix}\right)\Omega^{-1}(t)\,dt=0. \tag{4-10}$$

This completes the proof of the proposition by observing that $s<c(\chi_2)-v(\boldsymbol{a})$. $\square$

Observe that in the above proof, we have used $c(\Omega) \geq 2c(\chi_1)$ at two crucial steps to simplify the integral. In the case $c(\Omega) < 2c(\chi_1)$, it is not clear if the statement of the proposition still remains valid.

Proof of Theorem 1.7 for principal series representations. By the definition (4-3) of $\mathcal{A}$ and (4-5), the linear functional on π given by $\ell(f) = (\mathcal{A}(f))(1)$ is a nonzero functional satisfying $\ell(\pi(t)f) = \Omega(t)\ell(f)$ for all $t \in T(F)$ and $f \in \pi$. Hence, $\mathrm{Hom}_{T(F)}(\pi, \Omega) \neq 0$. The 1-dimensionality follows from [Waldspurger 1985]. Since $c(\Omega) \geq c(\pi)$, we can apply Lemma 2.2 together with Proposition 4.2 to obtain the existence of the required test vector. The uniqueness follows from the uniqueness of f_0. □

4B. ***Steinberg representation.*** Let $\pi = \chi|\cdot|^{1/2} \times \chi|\cdot|^{-1/2}$. Let V_0 be the unique invariant (infinite-dimensional) subspace of π, so $\pi|_{V_0}$ is the twisted Steinberg representation $\chi\,\mathrm{St}_{\mathrm{GL}_2}$. If we set $\chi_1 = \chi|\cdot|^{1/2}$ and $\chi_2 = \chi|\cdot|^{-1/2}$, then V_0 is characterized as the kernel of the intertwining operator $M : \chi_1 \times \chi_2 \to \chi_2 \times \chi_1$, given by

$$(M(f))(g) = \int_F f\left(\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} g\right) dx.$$

Case 1: $\boldsymbol{\chi}$ ***ramified.*** If χ is a ramified character, then f_0, defined as in (4-2), is in V_0 and is, in fact, the unique (up to a constant) newform in $\chi\,\mathrm{St}_{\mathrm{GL}_2}$ (see [Schmidt 2002]). Hence the proof of Proposition 4.2 is valid in this case without any modification.

Case 2: $\boldsymbol{\chi}$ ***unramified.*** If χ is unramified, then the vector f_0, defined as in (4-2), is a spherical vector in $\chi_1 \times \chi_2$, hence clearly not the newform of $\chi\,\mathrm{St}_{\mathrm{GL}_2}$, which has conductor $\mathfrak{p}$. Any vector in $\chi\,\mathrm{St}_{\mathrm{GL}_2}$ which is right $K_1(\mathfrak{p})$-invariant is also right I-invariant, where I is the Iwahori subgroup defined in (2-4). It is known (see [Schmidt 2002]) that the newform in the induced model is given by

$$f_0(g) = \begin{cases} |a/d|\chi(ad)q & \text{if } g \in \begin{bmatrix} a & * \\ & d \end{bmatrix} I, \\[2ex] -|a/d|\chi(ad) & \text{if } g \in \begin{bmatrix} a & * \\ & d \end{bmatrix} wI. \end{cases} \tag{4-11}$$

We can try to compute $(\mathcal{A}(f_0))(g)$ (defined in (4-3)) in this case for various values of g. But instead, we use a double coset decomposition and properties of the Steinberg representation to obtain the test vector. This has the added advantage of obtaining a new proof of the uniqueness (up to a constant) of the Waldspurger functional, and also gives us the explicit formula for $B(g)$, where B is the newform in the corresponding Waldspurger model and g is any element of $\mathrm{GL}_2(F)$. By

[Sugano 1985, Lemma 2-4], there is the disjoint double coset decomposition

$$\mathrm{GL}_2(F)=\bigsqcup_{r=0}^{\infty} T(F)\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}\mathrm{GL}_2(\mathfrak{o}). \tag{4-12}$$

Note that, by the Iwasawa decomposition of $\mathrm{SL}_2(\mathfrak{o}/\mathfrak{p})$, we have

$$\mathrm{GL}_2(\mathfrak{o})=wI\sqcup\bigsqcup_{u\in\mathfrak{o}/\mathfrak{p}}\begin{bmatrix}1 & \\ u & 1\end{bmatrix}I,\qquad w=\begin{bmatrix} & 1\\ -1 & \end{bmatrix}. \tag{4-13}$$

For $u\in\mathfrak{o}$ and $r\ge 0$ set $\beta_{u,r}:=\boldsymbol{a}\varpi^{2r}+\boldsymbol{b}\varpi^r u+\boldsymbol{c}u^2$. Arguing as in Lemma 3.1 of [Pitale 2011], we have

$$T(F)\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}wI=T(F)\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}\begin{bmatrix}1 & \\ u & 1\end{bmatrix}I\iff\beta_{u,r}\in\mathfrak{o}^\times. \tag{4-14}$$

Lemma 3.2 of [Pitale 2011] tells us exactly when $\beta_{u,r}\in\mathfrak{o}^\times$. Putting everything together, we get the following proposition.

Proposition 4.3. *For $r>0$, we have*

$$T(F)\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}\mathrm{GL}_2(\mathfrak{o})=T(F)\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}I\sqcup T(F)\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}wI.$$

For $r=0$, $\left(\frac{L}{\mathfrak{p}}\right)=-1$, we have

$$T(F)\mathrm{GL}_2(\mathfrak{o})=T(F)I=T(F)wI.$$

For $r=0$, $\left(\frac{L}{\mathfrak{p}}\right)=0$, we have

$$T(F)\mathrm{GL}_2(\mathfrak{o})=T(F)wI\sqcup T(F)\begin{bmatrix}1 & \\ u_0 & 1\end{bmatrix}I,$$

where u_0 is chosen as in (2-14).

The twisted Steinberg representation is characterized as the representation π with a newform v_0 which is invariant under I and satisfies the two conditions

$$\sum_{\gamma\in\mathrm{GL}_2(\mathfrak{o})/I}\pi(\gamma)v_0=0,\qquad \pi\left(\begin{bmatrix} & 1\\ \varpi & \end{bmatrix}\right)v_0=-\chi(\varpi)v_0.$$

These conditions follow from the action of the Atkin–Lehner element and the fact that π does not have a vector invariant under $\mathrm{GL}_2(\mathfrak{o})$. See Proposition 3.1.2 of [Schmidt 2002]. Let Ω be a character of $L^\times$ with $\Omega|_{F^\times}=\omega_\pi$.

Let $B : \mathrm{GL}_2(F) \to \mathbb{C}$ be a function that satisfies $B(tgk) = \Omega(t)B(g)$ for all $t \in T(F)$, $g \in \mathrm{GL}_2(F)$, $k \in I$ and

$$\sum_{u \in \mathfrak{o}/\mathfrak{p}} B\Big(g\begin{bmatrix} 1 & \\ u & 1 \end{bmatrix}\Big) = -B(gw), \tag{4-15}$$

$$B\Big(g\begin{bmatrix} & 1 \\ \varpi & \end{bmatrix}\Big) = -\chi(\varpi)B(g) \tag{4-16}$$

for all $g \in \mathrm{GL}_2(F)$. If π has a Ω-Waldspurger model, then B will precisely be the unique (up to scalars) newform of π in the Ω-Waldspurger model; otherwise B will be 0.

Lemma 4.4. (i) *If $c(\Omega) \geq 2$, then*

$$B\Big(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}w\Big) = 0 \quad \text{if } r \leq c(\Omega) - 2. \tag{4-17}$$

(ii) *For $r > 0$, we have*

$$B\Big(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}\Big) = \begin{cases} -qB\Big(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}w\Big) & \text{if } r \geq c(\Omega), \\ 0 & \text{if } r < c(\Omega). \end{cases} \tag{4-18}$$

(iii) *For $r \geq \max\{c(\Omega) - 1, 0\}$, we have*

$$B\Big(\begin{bmatrix} \varpi^{r+1} & \\ & 1 \end{bmatrix}w\Big) = \frac{\chi(\varpi)}{q}B\Big(\begin{bmatrix} \varpi^r & \\ & 1 \end{bmatrix}w\Big). \tag{4-19}$$

(iv) *If $\left(\frac{L}{\mathfrak{p}}\right) = 0$, then*

$$B\Big(\begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix}\Big) = \begin{cases} -qB(w) & \text{if } c(\Omega) = 0, \\ 0 & \text{if } c(\Omega) > 0. \end{cases} \tag{4-20}$$

(v) *If $c(\Omega) = 0$ and $\Omega = \chi \circ N_{L/F}$, then*

$$B(w) = 0. \tag{4-21}$$

Proof. We illustrate the proof of (i) and (ii) in detail here. Let $u, v \in \mathfrak{p}^{c(\Omega)-1}$ be such that $\Omega(1 + u + v\beta) \neq 1$. Take $y = v/\boldsymbol{c}$, $x = 1 + u + \frac{1}{2}\boldsymbol{b}y$ and, for $r \leq c(\Omega) - 2$, let

$$k = \begin{bmatrix} 1+u & \boldsymbol{a}/\boldsymbol{c}v\varpi^r \\ -\varpi^{-r}v & 1+u+\boldsymbol{b}/\boldsymbol{c}v \end{bmatrix} \in I.$$

Then

$$B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}w\right)=B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}wk\right)=B\left(t(x,y)\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}w\right)$$
$$=\Omega(1+u+v\beta)B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}w\right).$$

This gives us (4-17) and completes the proof of (i).

Next, we give the proof of (ii). Substitute $g=\left[\begin{smallmatrix}\varpi^r & \\ & 1\end{smallmatrix}\right]$ in (4-15) to get

$$\sum_{u\in\mathfrak{o}/\mathfrak{p}} B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}\begin{bmatrix}1 & \\ u & 1\end{bmatrix}\right)=-B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}w\right).$$

For $u\neq 0$, setting $x=\boldsymbol{b}/2\varpi^r+\boldsymbol{c}u$, $y=\varpi^r$, we get

$$\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}\begin{bmatrix}1 & \\ u & 1\end{bmatrix}\begin{bmatrix}-\boldsymbol{c} & \boldsymbol{b}\varpi^r+\boldsymbol{c}u \\ & -\beta_{u,r}\end{bmatrix}=t(x,y)\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}w.$$

Since $r>0$ by assumption, $\beta_{u,r}\in\mathfrak{o}^\times$. Hence, for $u\neq 0$ we have

$$B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}\begin{bmatrix}1 & \\ u & 1\end{bmatrix}\right)=\Omega(u+\varpi^r\beta)B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}w\right).$$

This gives us

$$B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}\right)=-\left(\sum_{u\in(\mathfrak{o}/\mathfrak{p})^\times}\Omega(u+\varpi^r\beta)+\Omega(1)\right)B\left(\begin{bmatrix}\varpi^r & \\ & 1\end{bmatrix}w\right).$$

Using (4-17) and the definition of $c(\Omega)$ we get the result for $r\geq c(\Omega)$ or $r\leq c(\Omega)-2$. For $r=c(\Omega)-1$, using Lemma 3.4 of [Pitale 2011], we see that the expression in the parentheses on the right-hand side above is 0. This completes the proof of (ii).

Using (4-15), (4-16) and similar calculations as above, we get the remaining results. □

Proof of Theorem 1.7 for twists of Steinberg representations. Let D be the quaternion division algebra over F and $N_{D/F}$ be the reduced norm. Since $\pi=\chi\,\mathrm{St}$ corresponds to the 1-dimensional representation $\pi'=\chi\circ N_{D/F}$ of $D^\times(F)$, one knows by [Waldspurger 1985] that π has an Ω-Waldspurger model if and only if $\Omega\neq\chi\circ N_{L/F}$. Since $c(\Omega)\geq c(\pi)$, this must be the case, i.e., $\dim_{\mathbb{C}}\mathrm{Hom}_{T(F)}(\pi,\Omega)=1$. The case of ramified χ follows exactly as in the principal series case. For χ unramified, the result follows from Lemma 4.4. □

5. Supercuspidal representations

Throughout this section we continue to assume that L/F is a field.

5A. ***Chain orders and strata.*** This section contains a summary of the facts about chain orders and fundamental strata that we will use to construct test vectors for the supercuspidal representations π of $\mathrm{GL}_2(F)$, all of which can be found in [Bushnell and Henniart 2006, Chapter 4].

Let $\mathfrak{A}$ be a chain order in $\mathrm{M}_2(F)$. Up to $\mathrm{GL}_2(F)$-conjugacy, $\mathfrak{A}$ must be either $\mathfrak{M} = \mathrm{M}_2(\mathfrak{o})$ or $\mathfrak{J} = \left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{smallmatrix}\right]$, so we always take $\mathfrak{A}$ to be $\mathfrak{M}$ or $\mathfrak{J}$.

Write $e_{\mathfrak{A}} = 1$ if $\mathfrak{A} = \mathfrak{M}$ and $e_{\mathfrak{A}} = 2$ if $\mathfrak{A} = \mathfrak{J}$. For more intrinsic definitions, see [Bushnell and Henniart 2006]. Let $\mathfrak{P} = \operatorname{rad}\mathfrak{A}$, the Jacobson radical of $\mathfrak{A}$. There is an element $\Pi \in \mathrm{GL}_2(F)$ such that $\mathfrak{P} = \Pi\mathfrak{A}$, and one has

$$\operatorname{rad}\mathfrak{M} = \varpi\mathfrak{M}, \qquad \operatorname{rad}\mathfrak{J} = \begin{bmatrix} & 1 \\ \varpi & \end{bmatrix}\mathfrak{J}.$$

Let $\mathfrak{P}^n = \Pi^n\mathfrak{A}$ for $n \in \mathbb{Z}$. Let $U_{\mathfrak{A}}^0 = U_{\mathfrak{A}} := \mathfrak{A}^\times$, $U_{\mathfrak{A}}^n := 1 + \mathfrak{P}^n$ for $n \geq 1$, and $K_{\mathfrak{A}} = \{g \in \mathrm{GL}_2(F) : g\mathfrak{A}g^{-1} = \mathfrak{A}\}$. Then

$$K_{\mathfrak{A}} = \begin{cases} Z(F)\mathrm{GL}_2(\mathfrak{o}) & \text{if } \mathfrak{A} = \mathfrak{M}, \\ \left\langle \begin{bmatrix} & 1 \\ \varpi & \end{bmatrix} \right\rangle \ltimes \mathfrak{J}^\times & \text{if } \mathfrak{A} = \mathfrak{J}. \end{cases}$$

We fix a character $\psi_1 : F \to \mathbb{C}^\times$ so that the conductor of ψ_1 is $\mathfrak{p}$. For $\alpha \in \mathrm{M}_2(F)$, define a function of $U_{\mathfrak{A}}$ by $\psi_\alpha(x) = \psi_1(\operatorname{Tr}\alpha(x-1))$. Then for $1 \leq m \leq n \leq 2m$, there is an isomorphism

$$\begin{aligned} \mathfrak{P}^{-n}/\mathfrak{P}^{-m} &\to \left(U_{\mathfrak{A}}^{m+1}/U_{\mathfrak{A}}^{n+1}\right)^\wedge, \\ \alpha + \mathfrak{P}^{-m} &\mapsto \psi_\alpha. \end{aligned}$$

The normalized level of π is defined to be

$$\ell(\pi) = \min\{n/e_{\mathfrak{A}} : \pi|_{U_{\mathfrak{A}}^{n+1}} \text{ contains the trivial character}\}.$$

A stratum in $\mathrm{M}_2(F)$ is a triple $(\mathfrak{A}, n, \alpha)$ where $\mathfrak{A}$ is a chain order in $\mathrm{M}_2(F)$ with radical $\mathfrak{P}$, n is an integer and $\alpha \in \mathfrak{P}^{-n}$. For $n \geq 1$ one associates to a stratum the character ψ_α of $U_{\mathfrak{A}}^n$ which is trivial on $U_{\mathfrak{A}}^{n+1}$.

We say that a smooth representation π contains the stratum $(\mathfrak{A}, n, \alpha)$ if $\pi|_{U_{\mathfrak{A}}^n}$ contains ψ_α. A fundamental stratum is one such that $\alpha + \mathfrak{P}^{1-n}$ contains no nilpotent elements. If an irreducible smooth representation π of $\mathrm{GL}_2(F)$ contains a stratum $(\mathfrak{A}, n, \alpha)$, then $(\mathfrak{A}, n, \alpha)$ is fundamental if and only if $\ell(\pi) = n/e_{\mathfrak{A}}$ [Bushnell and Henniart 2006, 12.9 Theorem].

Suppose that $(\mathfrak{A}, n, \alpha)$ is a fundamental stratum with $e_{\mathfrak{A}} = 1$. Write $\alpha = \varpi^{-n}\alpha_0$ for $\alpha_0 \in \mathfrak{A}$. Let $f_\alpha(t) \in \mathfrak{o}[t]$ be the characteristic polynomial of α_0, and let $\tilde{f}_\alpha \in \boldsymbol{k}[t]$ be its reduction modulo $\mathfrak{p}$. Here $\boldsymbol{k}$ is the residue class field. If $\tilde{f}_\alpha$ has two solutions in $\boldsymbol{k}$, then $(\mathfrak{A}, n, \alpha)$ is said to be a split fundamental stratum. If $\tilde{f}_\alpha$ is irreducible, then the stratum $(\mathfrak{A}, n, \alpha)$ is said to be unramified simple. On the other hand, if $(\mathfrak{A}, n, \alpha)$ is a fundamental stratum with $e_{\mathfrak{A}} = 2$, and n odd, then $(\mathfrak{A}, n, \alpha)$ is said to be ramified simple. A simple stratum is either a simple unramified stratum or a simple ramified stratum. Suppose that $(\mathfrak{A}, n, \alpha)$ is a simple stratum with α_0 as above and let $E = F[\alpha_0]$. Bushnell and Henniart define what it means for α to be minimal [Bushnell and Henniart 2006, 13.4 Definition], and when this is the case $\mathfrak{o}_E = \mathfrak{o}[\alpha_0]$ [Bushnell and Henniart 2006, 13.4 Lemma]. If $(\mathfrak{A}, n, \alpha)$ is a simple stratum with $\mathfrak{A} = \mathfrak{M}$, then $\alpha_0 \in \mathfrak{M}$ but $\alpha_0 \notin \mathfrak{P}$.

Define π to be *minimal* if, for all characters χ of $F^\times$, $\ell(\pi \otimes \chi) \geq \ell(\pi)$. Every irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ is either minimal, or isomorphic to the twist of a minimal irreducible supercuspidal representation. Every minimal irreducible smooth representation of $\mathrm{GL}_2(F)$ contains exactly one of the following: a ramified simple stratum, an unramified simple stratum, or a split fundamental stratum [Bushnell and Henniart 2006, 13.3 Corollary]. If π contains a split fundamental stratum, then π is not supercuspidal.

5B. ***Construction of minimal supercuspidals.*** In this section we review the construction of minimal irreducible supercuspidal representations. See [Bushnell and Henniart 2006, Section 19] for more details. In each case we describe a distinguished vector v_0 in the inducing representation. This vector v_0 will be used to construct a test vector for π.

We remark that if a representation π contains a simple stratum $(\mathfrak{A}, n, \alpha)$, then it contains all $\mathrm{GL}_2(F)$-conjugates of $(\mathfrak{A}, n, \alpha)$. Therefore, we may always take $\mathfrak{A}$ to be either $\mathfrak{M}$ or $\mathfrak{J}$. Since $K_{\mathfrak{A}}$ normalizes $U_{\mathfrak{A}}$, we may also consider α up to $K_{\mathfrak{A}}$-conjugacy.

For the rest of Section 5 we assume that all supercuspidal representations are irreducible.

5B1. $\mathfrak{A} = \mathfrak{M}$, $\ell(\pi) = 2r+1$. Suppose that π is a minimal supercuspidal representation containing the simple stratum given by $(\mathfrak{M}, 2r+1, \alpha)$. Then $E = F[\alpha]$ is an unramified quadratic extension of F, and $\pi \cong c\text{-}\mathrm{Ind}_{J_\alpha}^{\mathrm{GL}_2(F)} \lambda$, where $J_\alpha = E^\times U_{\mathfrak{M}}^{r+1}$ and λ is a character.

We have that $\lambda|_{U_{\mathfrak{M}}^{r+1}} = \psi_\alpha$ with $\alpha \in \mathfrak{P}_{\mathfrak{M}}^{-2r-1}$ and α is minimal. One may take $\alpha_0 = \varpi^{2r+1}\alpha$ to be in rational canonical form, i.e.,

$$\alpha_0 = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix}, \tag{5-1}$$

for $a_i \in \mathfrak{o}$, $i = 0, 1$. Then

$$1 + \begin{bmatrix} \mathfrak{p}^{r+1} & \mathfrak{p}^{2r+2} \\ \mathfrak{p}^{2r+2} & \mathfrak{p}^{2r+2} \end{bmatrix} \subseteq \ker \psi_\alpha.$$

5B2. $\mathfrak{A} = \mathfrak{J}$, $\ell(\pi) = \frac{1}{2}(2r+1)$. Suppose that π is a minimal supercuspidal representation containing the simple stratum $(\mathfrak{J}, 2r+1, \alpha)$, and let $E = F[\alpha]$. In this case E/F is a ramified extension, and $\ell(\pi)e(E/F) = 2r+1$. Then $\pi = c\text{-}\mathrm{Ind}_{J_\alpha}^{\mathrm{GL}_2(F)}\lambda$, where $J_\alpha = E^\times U_{\mathfrak{J}}^{r+1}$ and λ is a character. Observe

$$U_{\mathfrak{J}}^{r+1} = 1 + \mathfrak{P}^{r+1} = \begin{cases} 1 + \begin{bmatrix} \mathfrak{p}^{r/2+1} & \mathfrak{p}^{r/2} \\ \mathfrak{p}^{r/2+1} & \mathfrak{p}^{r/2+1} \end{bmatrix} & \text{if } r \text{ is even,} \\ 1 + \begin{bmatrix} \mathfrak{p}^{(r+1)/2} & \mathfrak{p}^{(r+1)/2} \\ \mathfrak{p}^{(r+3)/2} & \mathfrak{p}^{(r+1)/2} \end{bmatrix} & \text{if } r \text{ is odd.} \end{cases}$$

Note that

$$U_{\mathfrak{J}}^{2r+2} = 1 + \begin{bmatrix} \mathfrak{p}^{r+1} & \mathfrak{p}^{r+1} \\ \mathfrak{p}^{r+2} & \mathfrak{p}^{r+1} \end{bmatrix} \subseteq \ker\lambda.$$

Let $\alpha_0 = \varpi^{r+1}\alpha$ be of the form (5-1), where now $a_0 \in \varpi\mathfrak{o}^\times$ and $a_1 \in \mathfrak{p}$, and $k := \left[\frac{1}{2}r\right] + 1$. Then

$$1 + \begin{bmatrix} \mathfrak{p}^{k} & \mathfrak{p}^{r+1} \\ \mathfrak{p}^{r+2} & \mathfrak{p}^{r+1} \end{bmatrix} \subseteq \ker\lambda.$$

5B3. $\mathfrak{A} = \mathfrak{M}$, $\ell(\pi) = 2r > 0$. Now suppose π contains an unramified simple stratum $(\mathfrak{M}, 2r, \alpha)$ for some $\alpha \in \mathfrak{P}^{-2r}$ so that $\ell(\pi) = 2r > 0$ and $e(E/F) = 1$, where as before $E = F[\alpha]$. Continue to assume that $\alpha_0 = \varpi^{2r}\alpha$ has the form (5-1). In this case, π is not induced from a character, and E is an unramified quadratic extension of F. We describe a representation ρ of $J_\alpha = E^\times U_{\mathfrak{M}}^r$ so that π is compactly induced from ρ, and we follow Kutzko [1978, §1] since his construction is more convenient for our applications.

Write $U_E^1 = U_{\mathfrak{M}}^1 \cap E^\times$. Since $\alpha_0 \in \mathfrak{M} \setminus \mathfrak{P}$ and $\mathfrak{o}_E = \mathfrak{o}[\alpha_0]$ (see Section 5A), a simple argument shows $U_E^1 \subset 1 + \mathfrak{p}_E$. The opposite inclusion is obvious; therefore, $U_E^1 = 1 + \mathfrak{p}_E$. We similarly note that $E^\times \cap U_{\mathfrak{M}} =: U_E \cong \mathfrak{o}_E^\times$. There is a character χ of $E^\times$ such that $\chi(1+x) = \psi_1 \circ \mathrm{Tr}_{E/F}(\alpha x)$ for all $x \in \mathfrak{p}_E^{r+1}$. Define a character

$$\lambda : H_\alpha^1 := U_E^1 U_{\mathfrak{M}}^{r+1} \to \mathbb{C}^\times$$

by $\lambda(ux) = \chi(u)\psi_\alpha(x)$ for $u \in U_E^1$ and $x \in U_{\mathfrak{M}}^{r+1}$.

Let $A = \left\{\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right] : x \in F^\times\right\}$, and set $A^n = A \cap U_{\mathfrak{M}}^n$ for $n \geq 0$. The character λ can be extended to a character $\tilde{\lambda}$ of $A^r H_\alpha^1$ by $\tilde{\lambda}(yx) = \lambda(x)$ for $y \in A^r U_{\mathfrak{M}}^{2r+1}$ and $x \in H_\alpha^1$ [Kutzko 1978, Definition 1.8].

Let $J_\alpha^1 = U_E^1 U_{\mathfrak{M}}^r$, and define $\eta = \mathrm{Ind}_{A^r H_\alpha^1}^{J_\alpha^1}\tilde{\lambda}$. Then η is an irreducible representation of J_α^1 of dimension q. There is an irreducible representation ρ of J_α such that $\pi \cong \mathrm{Ind}_{J_\alpha}^{\mathrm{GL}_2(F)}\rho$, and $\rho|_{J_\alpha^1} \cong \eta$ [Kutzko 1978, Lemma 1.10 and Proposition 1.15]. Note that $U_{\mathfrak{M}}^{2r+1} \subset \ker\rho$. We must compute $\rho|_{A^r} \cong \eta|_{A^r}$. We have $[J_\alpha^1 : A^r H_\alpha^1] = q$, and an irredundant set of coset representatives is given by $\left\{\left[\begin{smallmatrix} 1 & \\ a & 1 \end{smallmatrix}\right] : a \in \mathfrak{p}^r/\mathfrak{p}^{r+1}\right\}$.

It is a simple computation to show that $\eta|_{A^r} = \operatorname{Ind}_{A^r H_\alpha^1}^{J_\alpha^1} \tilde{\lambda}|_{A^r}$ is isomorphic to the regular representation of A^r/A^{r+1}. In particular it contains the trivial character with multiplicity one, so there is a vector $v_0 \in \rho$ that is unique up to scalars which is fixed by $1 + \left[\begin{smallmatrix} \mathfrak{p}^r & \mathfrak{p}^{2r+1} \\ \mathfrak{p}^{2r+1} & \mathfrak{p}^{2r+1} \end{smallmatrix}\right]$.

Sometimes it will be convenient to consider the corresponding vector $f_0 \in \eta$ given by

$$f_0(k) = \begin{cases} \tilde{\lambda}(k) & \text{if } k \in A^r H_\alpha^1, \\ 0 & \text{otherwise.} \end{cases} \tag{5-2}$$

5B4. *Depth zero supercuspidals.* Now, consider a depth zero supercuspidal representation, i.e., $\ell(\pi) = 0$. Then π is induced from a representation ρ of $K_{\mathfrak{M}}$ that is inflated from a cuspidal representation $\tilde{\rho}$ of $\mathrm{GL}_2(\mathfrak{o}/\mathfrak{p})$, i.e., ρ is trivial on $U_{\mathfrak{M}}^1 = 1 + \mathfrak{p}\mathrm{M}_2(\mathfrak{o})$ and it factors through $\tilde{\rho}$. The cuspidal representations $\tilde{\rho}$ are parameterized by Galois conjugacy classes of regular characters $\theta : \mathbb{F}_{q^2}^\times \to \mathbb{C}^1$. Such a character θ can also be regarded as a character of $\mathfrak{o}_E^\times$ that is trivial on $1 + \mathfrak{p}_E$, where E/F is the unique unramified quadratic extension. Embed $\mathfrak{o}_E^\times$ in $\mathrm{GL}_2(\mathfrak{o})$, and identify $\mathbb{F}_{q^2}^\times$ with the image of $\mathfrak{o}_E^\times$ under the reduction map modulo $\mathfrak{p}$. The following proposition gives the character table for $\tilde{\rho}$, which is a well-known result (see, e.g., [Bushnell and Henniart 2006, 6.4.1]).

Proposition 5.1. *The character table of $\tilde{\rho}$ is given by*

$$\begin{aligned} \operatorname{Tr} \tilde{\rho}(z) &= (q-1)\theta(z), && z \in Z; \\ \operatorname{Tr} \tilde{\rho}(zu) &= -\theta(z), && z \in Z,\ u \in N,\ u \neq 1; \\ \operatorname{Tr} \tilde{\rho}(y) &= -(\theta(y) + \theta^q(y)), && y \in \mathbb{F}_{q^2}^\times \smallsetminus Z. \end{aligned}$$

If g is not conjugate to an element of $\mathbb{F}_{q^2}^\times \cup ZN$, then $\operatorname{Tr} \tilde{\rho}(g) = 0$.

From the character table one sees that the restriction of ρ to A^0 is isomorphic to the regular representation of A^0/A^1. In particular there is a vector $v_0 \in \rho$ such that $\rho(a)v_0 = v_0$ for $a \in A^0$.

5C. ***Remarks on minimal supercuspidals.*** We consider a minimal supercuspidal representation $\pi = c\text{-}\operatorname{Ind}_{J_\alpha}^{\mathrm{GL}_2(F)} \rho$, where $E = F[\alpha]$ for $\alpha \in \mathfrak{P}^{-n}$ such that π contains the simple stratum $(\mathfrak{A}, n, \alpha)$. When $e(E/F)\ell(\pi)$ is odd, then $\rho = \lambda$ is a character which restricts to ψ_α. When $e(E/F)\ell(\pi)$ is even, then ρ is not a character, and if additionally $\ell(\pi) > 0$, then we sometimes identify $\rho|_{J_\alpha^1}$ with η as described above.

From the discussion in the previous section, we may always take $J = J_\alpha$ of the form $E^\times\big(1 + \mathfrak{P}^{[(e_{\mathfrak{A}}\ell(\pi)+1)/2]}\big)^\times$, where we take $E = F$ if $\ell(\pi) = 0$. In all cases, we have $E^\times \subset K_{\mathfrak{A}}$, so $J \subset K_{\mathfrak{A}}$.

Definition 5.2. Suppose $\pi = c\text{-}\operatorname{Ind}_J^{\mathrm{GL}_2(F)} \rho$ is an irreducible minimal supercuspidal representation.

(i) If $\rho=\lambda$ is a character, define v_0 to be the unique vector up to scalar multiples in ρ. That is, $\rho(k)v_0=\lambda(k)$ for $k\in J$. Then according to the constructions in Sections 5B1 and 5B2, $\rho(a)v_0=\lambda(a)=1$ for $a\in A\cap J$.

(ii) Suppose $\dim_{\mathbb{C}}\rho>1$. Define $v_0\in\rho$ to be the vector described in Sections 5B3 and 5B4 such that $\rho(a)v_0=v_0$ for $a\in A\cap J$.

Let $N=\left\{\left[\begin{smallmatrix}1&x\\&1\end{smallmatrix}\right]\right\}\subset\mathrm{GL}_2(F)$, $\bar{N}=\left\{\left[\begin{smallmatrix}1&\\x&1\end{smallmatrix}\right]\right\}\subset\mathrm{GL}_2(F)$ and $\bar{N}^r=\bar{N}\cap U^r_{\mathfrak{M}}$ for $r\geq 0$.

Lemma 5.3. *Suppose that π is a minimal supercuspidal representation and that $\ell(\pi)=2r$. Write $\pi=c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$, where ρ is not a character.*

(i) *If $\ell(\pi)=0$, then $\rho|_{\bar{N}^0}=\bigoplus_{i=1}^{q-1}\psi_i$, where ψ_i runs over all the nontrivial characters of $\bar{N}^0/\bar{N}^1$.*

(ii) *If $\ell(\pi)>0$ and $J=J_\alpha$, then $\rho|_{\bar{N}\cap J_\alpha}=\bigoplus_j\psi_j$, where the sum runs over all characters ψ_j of $\bar{N}\cap J_\alpha$ such that $\psi_j|_{\bar{N}\cap H_\alpha}=\psi_\alpha|_{\bar{N}\cap H_\alpha}$, and $H_\alpha:=E^\times U^{r+1}_{\mathfrak{M}}$.*

Proof. The first part may be deduced from Proposition 5.1. Now, suppose that $\ell(\pi)>0$. Since $J_\alpha=E^\times U^r_{\mathfrak{M}}$, we have $\bar{N}\cap J_\alpha=\bar{N}^r$, and similarly $\bar{N}\cap H_\alpha=\bar{N}^{r+1}$. Also, recall that $\rho|_{J^1_\alpha}\cong\eta=\mathrm{Ind}_{A^rH^1_\alpha}^{J^1_\alpha}\tilde{\lambda}$, where $\tilde{\lambda}$ is obtained from λ by extending it trivially to A^r.

A set of irredundant coset representatives for $A^rH^1_\alpha\backslash J^1_\alpha$ is given by

$$\left\{\begin{bmatrix}1&\\a&1\end{bmatrix}:a\in\mathfrak{p}^r/\mathfrak{p}^{r+1}\right\}.$$

Let ψ' be one of the characters ψ_j of $\bar{N}^r$ as in (ii). For $a\in\mathfrak{p}^r/\mathfrak{p}^{r+1}$, define

$$f_a(y)=\begin{cases}\psi'\left(\begin{bmatrix}1&\\a&1\end{bmatrix}\right)\tilde{\lambda}(x) & \text{if } y=x\begin{bmatrix}1&\\a&1\end{bmatrix},\ x\in A^rH^1_\alpha,\\ 0 & \text{otherwise.}\end{cases}\tag{5-3}$$

This is well defined since ψ' and $\tilde{\lambda}$ agree on $\bar{N}^{r+1}$. Note the f_a span η, and when $a=0$, (5-3) agrees with (5-2). Define $f_{\psi'}=\sum_{a\in\mathfrak{p}^r/\mathfrak{p}^{r+1}}f_a$. Then we have

$$\eta\left(\begin{bmatrix}1&\\x&1\end{bmatrix}\right)f_{\psi'}=\psi'(x)f_{\psi'}.$$

From the explicit basis we computed for η, one sees that each of these characters appear with multiplicity one. This proves the lemma. $\square$

Later, it will be useful to have a case-by-case description of the kernel of ρ. We summarize what we know about the kernel from the previous section in Table 2. The quantities in the latter two columns will be denoted i and i' in Propositions 5.5 and 5.9, and are included here for the later convenience of the reader.

$\ell(\pi)$	J	$\subset \ker\rho$	$\left[\frac{\ell(\pi)+3/2}{2}\right]$	$\left[\ell(\pi)+\frac{3}{2}\right]-\left[\frac{\ell(\pi)}{2}\right]-1$
$2r+1$	$E^\times(1+\mathfrak{P}^{r+1})$	$1+\begin{bmatrix}\mathfrak{p}^{r+1} & \mathfrak{p}^{2r+2}\\ \mathfrak{p}^{2r+2} & \mathfrak{p}^{2r+2}\end{bmatrix}$	$r+1$	$r+1$
$2r>0$	$E^\times(1+\mathfrak{P}^{r})$	$1+\begin{bmatrix}\mathfrak{p}^{r+1} & \mathfrak{p}^{2r+1}\\ \mathfrak{p}^{2r+1} & \mathfrak{p}^{2r+1}\end{bmatrix}$	r	r
0	$Z(F)\mathrm{GL}_2(\mathfrak{o})$	$1+\mathfrak{P}$	0	0
$\frac{2r+1}{2}$	$E^\times(1+\mathfrak{P}^{r+1})$	$1+\begin{bmatrix}\mathfrak{p}^{[r/2]+1} & \mathfrak{p}^{r+1}\\ \mathfrak{p}^{r+2} & \mathfrak{p}^{r+1}\end{bmatrix}$	$\left[\frac{r}{2}\right]+1$	$r-\left[\frac{r}{2}\right]+1$

Table 2. Data for minimal supercuspidal representations.

5D. ***Mackey theory.*** In this section we describe the strategy to obtain the desired test vector for π. Consider a minimal supercuspidal representation π of $\mathrm{GL}_2(F)$. There is an open subgroup J of $\mathrm{GL}_2(F)$ that contains the center $Z(F)$, is compact modulo $Z(F)$ and has an irreducible representation ρ of J with $\pi \cong c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$. As before, let $\Omega : T(F) \to \mathbb{C}^\times$ be a character such that $\Omega|_{Z(F)} = \omega_\pi$.

Consider the space

$$\mathrm{Hom}_{T(F)}(\pi, \Omega) \cong \mathrm{Hom}_{\mathrm{GL}_2(F)}\big(c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho,\ \mathcal{B}(\Omega)\big). \tag{5-4}$$

See Section 2D for the definition of $\mathcal{B}(\Omega)$ and details of the above isomorphism. Following the proof of Proposition 1.6 of [Bushnell and Henniart 1998] and [Kutzko 1977], define $\mathscr{H}(\mathrm{GL}_2(F),\ \rho,\ \Omega)$ to be the space of functions

$$f : \mathrm{GL}_2(F) \to \mathrm{Hom}_{\mathbb{C}}(\rho, \mathbb{C})$$

satisfying

$$f(tgk) = \Omega(t) f(g) \circ \rho(k), \quad t \in T(F),\ g \in \mathrm{GL}_2(F),\ k \in J.$$

Then for $\varphi \in c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$ and $f \in \mathscr{H}(\mathrm{GL}_2(F),\ \rho,\ \Omega)$, the convolution $f * \varphi$ defined as

$$f*\varphi(g) = \int_{\mathrm{GL}_2(F)/Z(F)} f(x)\varphi(x^{-1}g)\,d\bar{x}, \quad g \in \mathrm{GL}_2(F),$$

gives a function in the space $\mathcal{B}(\Omega)$. Furthermore, $\mathrm{GL}_2(F)$ acts on $\mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$ through the convolution by $(g \cdot f) * \varphi = f * (g \cdot \varphi)$, and there is a $\mathrm{GL}_2(F)$ homomorphism

$$\begin{gathered}\mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega) \to \mathrm{Hom}_{\mathrm{GL}_2(F)}\big(c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho,\ \mathcal{B}(\Omega)\big),\\ f \mapsto (\varphi \mapsto f*\varphi).\end{gathered}$$

This is in fact an isomorphism. By [Waldspurger 1985],

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_2(F)}\big(c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho,\ \mathcal{B}(\Omega)\big) \le 1.$$

Hence, there is at most one double coset $T(F)h_0J$ which has nontrivial intersection with the support of any $f \in \mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$, and each f is uniquely determined by its value at h_0 (see 1.8 of [Bushnell and Henniart 1998]). Suppose that $f \in \mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$ has support in a double coset $T(F)h_0J$, and that $f(h_0) = \ell_0 \in \mathrm{Hom}(\rho, \mathbb{C})$. For $k \in J \cap h_0^{-1}T(F)h_0$, define $\Omega^{h_0}(k) = \Omega(h_0kh_0^{-1})$. Then ℓ_0 has the property that for $k \in J \cap h_0^{-1}T(F)h_0$,

$$\ell_0(\rho(k)v) = \Omega^{h_0}(k)\ell_0(v).$$

Therefore,

$$\ell_0 \in \mathrm{Hom}_{J\cap h_0^{-1}T(F)h_0}(\rho,\ \Omega^{h_0}). \tag{5-5}$$

When the Hom space in (5-5) is not 0, we say that π and Ω *intertwine* on h_0. If this is the case, then the double coset $T(F)h_0J$ supports a nonzero function in $\mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$, and the Hom space in (5-4) is not zero.

5E. ***Test vectors for minimal supercuspidal representations.*** By [Henniart 2002, Section A.3], if π is a minimal representation with level $\ell(\pi)$, then $c(\pi) = 2\ell(\pi)+2$. Let $\pi = c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$ as described above. Let $v_0 \in \rho$ be the vector described in Definition 5.2. Assume that $c(\Omega) \ge c(\pi)$. Set

$$m_0 = \big[\ell(\pi) + \tfrac{3}{2}\big] - c(\Omega) - v(\boldsymbol{a}). \tag{5-6}$$

In the next proposition, we determine a double coset representative h_0 of $T(F)\backslash\mathrm{GL}_2(F)/J$ such that $\mathrm{Hom}_{J\cap h_0^{-1}T(F)h_0}(\rho,\ \Omega^{h_0}) \ne 0$. We remark that this result depends on our choice of inducing subgroup J, and in particular the quadratic extension $E = F[\alpha] = F[\alpha_0]$, where α_0 is always assumed to be of the form (5-1). For $m \in \mathbb{Z}$ and $z \in \mathfrak{o}^\times$, we define $g(m, z) := \left[\begin{smallmatrix} z\varpi^m & \\ & 1\end{smallmatrix}\right]$.

Lemma 5.4. *For $z \in \mathfrak{o}^\times$, $T(F)\cap g(m_0, z)Jg(m_0, z)^{-1} = F^\times\big(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}\big)$.*

Proof. We give the details when $\mathfrak{A} = \mathfrak{M}$. First, suppose $\ell(\pi) = 0$. In this case $J = Z(F)\mathrm{GL}_2(\mathfrak{o})$ and $m_0 = 1 - c(\Omega) - v(\boldsymbol{a})$. Furthermore, for $z \in \mathfrak{o}^\times$, $g(m_0, z)Jg(m_0, z)^{-1} = g(m_0, 1)Jg(m_0, 1)^{-1}$. Let $t' \in T(F)\cap g(m_0, 1)Jg(m_0, 1)^{-1}$. Since $J = Z(F)\mathrm{GL}_2(\mathfrak{o})$, there is a unique integer k such that

$$\varpi^k t' \in T(F)\cap g(m_0, 1)\mathrm{GL}_2(\mathfrak{o})g(m_0, 1)^{-1}.$$

Let $t = \varpi^k t' = \left[\begin{smallmatrix} x+\boldsymbol{b}y & \boldsymbol{c}y \\ \boldsymbol{a}y & x\end{smallmatrix}\right]$. Then

$$g(m_0, 1)^{-1}tg(m_0, 1) = \begin{bmatrix} x+\boldsymbol{b}y & \boldsymbol{c}y\varpi^{-m_0} \\ \boldsymbol{a}y\varpi^{m_0} & x\end{bmatrix} \in \mathrm{GL}_2(\mathfrak{o}).$$

Therefore, $y \in \mathfrak{p}^{-m_0-v(\boldsymbol{a})} = \mathfrak{p}^{c(\Omega)-1} \subset \mathfrak{p}$. Since $g(m_0,1)^{-1}tg(m_0,1) \in \mathrm{GL}_2(\mathfrak{o})$, we have $x \in \mathfrak{o}^\times$. This proves that $T(F) \cap g(m_0,z)Jg(m_0,z)^{-1} \subseteq F^\times\bigl(1+\mathfrak{P}_L^{c(\Omega)-1}\bigr)$. The other inclusion is straightforward. This completes the proof for $\ell(\pi)=0$.

Now, assume $\ell(\pi)>0$. Note that $t \in T(F)\cap g(m_0,z)Jg(m_0,z)^{-1}$ if and only if $wt \in T(F)\cap g(m_0,z)Jg(m_0,z)^{-1}$ for all $w \in Z(F)$.

Suppose that $t' \in T(F)\cap g(m_0,z)Jg(m_0,z)^{-1}$, where $t' = \left[\begin{smallmatrix} x'+\boldsymbol{b}y' & \boldsymbol{c}y' \\ -\boldsymbol{a}y' & x' \end{smallmatrix}\right]$. Let $k = \max\{-v(x'), -v(y')-m_0-v(\boldsymbol{a})\}$, $x = x'\varpi^k$, $y = y'\varpi^k$ and $t = \varpi^k t'$. Then

$$g(m_0,z)^{-1}tg(m_0,z) = \begin{bmatrix} x+\boldsymbol{b}y & z^{-1}\boldsymbol{c}y\varpi^{-m_0} \\ -z\boldsymbol{a}y\varpi^{m_0} & x \end{bmatrix}.$$

Let $i = \bigl[\frac{1}{2}(\ell(\pi)+1)\bigr]$, so $J = E^\times U^i_{\mathfrak{M}}$. There is a $u \in U^i_{\mathfrak{M}}$ such that

$$g(m_0,z)^{-1}tg(m_0,z)u \in E^\times.$$

Since $g(m_0,z)^{-1}tg(m_0,z) \in \mathrm{M}_2(\mathfrak{o})$ and $u \in U^i_{\mathfrak{M}}$, this implies that

$$a_0 z^{-1}\boldsymbol{c}y\varpi^{-m_0} \equiv -z\boldsymbol{a}y\varpi^{m_0} \mod \mathfrak{p}^i$$

(see (5-1) for a_0). As $y\varpi^{-m_0} \in \mathfrak{p}^{-2m_0-v(\boldsymbol{a})}$, we have $y \in \mathfrak{p}^{i-m_0-v(\boldsymbol{a})} = \mathfrak{p}^{c(\Omega)-[\ell(\pi)/2]-1}$. But this means that $v(\boldsymbol{a}y\varpi^{m_0})>0$, and hence $v(x)=0$ by our choice of k. Therefore, we have $t \in \mathfrak{o}^\times(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1})$. The discussion above shows that $t' \in F^\times(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1})$. The inclusion

$$T(F)\cap g(m_0,z)Jg(m_0,z)^{-1} \supseteq F^\times\bigl(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}\bigr)$$

is straightforward. □

Proposition 5.5. *Let $i = \bigl[\frac{1}{2}\bigl(\ell(\pi)+\frac{3}{2}\bigr)\bigr]$. There is a unique $z_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that we have*

$$\mathrm{Hom}_{J\cap g(m_0,z_0)^{-1}T(F)g(m_0,z_0)}(\rho, \Omega^{g(m_0,z_0)}) \neq 0 \quad \textit{for } g(m_0,z_0) := \begin{bmatrix} z_0\varpi^{m_0} & \\ & 1 \end{bmatrix}.$$

Proof. We give the details when $\ell(\pi)>0$ and $\mathfrak{A}=\mathfrak{M}$. In this case, we have $m_0 = \ell(\pi)+1-c(\Omega)-v(\boldsymbol{a})$ and $i = \bigl[\frac{1}{2}(\ell(\pi)+1)\bigr]$. By Lemma 5.4,

$$T(F)\cap g(m_0,z)Jg(m_0,z)^{-1} = F^\times\bigl(1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}\bigr).$$

Since $\Omega|_{F^\times} = \omega_\pi$, intertwining only depends on $\Omega|_{1+\mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}}$. Recall the definition of ξ from Section 2. The function given by $y \mapsto \Omega\bigl(1+y\bigl(\xi-\frac{1}{2}\boldsymbol{b}\bigr)\bigr)$ is an additive character of $\mathfrak{p}^{c(\Omega)-[\ell(\pi)/2]-1}/\mathfrak{p}^{c(\Omega)}$.

On the other hand, we have an isomorphism

$$\mathfrak{p}^{c(\Omega)-[\ell(\pi)/2]-1}/\mathfrak{p}^{c(\Omega)} \to \bar{N}^i U_{\mathfrak{M}}^{\ell(\pi)+1}/U_{\mathfrak{M}}^{\ell(\pi)+1},$$
$$y \mapsto g(m_0,z)^{-1}\bigl(1+y\bigl(\xi-\tfrac{1}{2}\boldsymbol{b}\bigr)\bigr)g(m_0,z).$$

Therefore, $\Omega^{g(m_0,z)}$ determines a character of $\overline{N}^i U_{\mathfrak{M}}^{\ell(\pi)+1}/U_{\mathfrak{M}}^{\ell(\pi)+1} \cong \overline{N}^i/\overline{N}^{\ell(\pi)+1}$. As z runs over $\mathfrak{o}^\times/(1+\mathfrak{p}^{\ell(\pi)+1-i})$, the character determined by $\Omega^{g(m_0,z)}$ runs over all characters of $\overline{N}^i/\overline{N}^{\ell(\pi)+1}$ that are nontrivial on $\overline{N}^{\ell(\pi)}$. In an abuse of notation we also refer to the character of $\overline{N}^i/\overline{N}^{\ell(\pi)+1}$ by $\Omega^{g(m_0,z)}$.

If $\ell(\pi)$ is odd, then ρ is a character of J. Then $\rho|_{\overline{N}^i} = \psi'$, i.e., ρ restricts to a single character, and $\ell(\pi)+1-i = i$, giving the proposition in this case.

Otherwise $\ell(\pi) > 0$ is even and, according to Lemma 5.3, $\rho|_{\overline{N}^i}$ is the direct sum of characters all of which restrict to the same character ψ' on $\overline{N}^{i+1}$. In this case there is a unique $z_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that

$$\rho|_{\overline{N}^i} \cong \bigoplus_{\substack{z\in\mathfrak{o}^\times/(1+\mathfrak{p}^{i+1}) \\ z\equiv z_0 \pmod{\mathfrak{p}^i}}} \Omega^{g(m_0,z)},$$

proving the proposition. The other cases follow similarly. □

Remark 5.6. Let us comment on the choice of the m_0 in (5-6). Put $g_m = g(m,1)$. One can exhibit the following double coset decomposition:

$$\mathrm{GL}_2(F) = \begin{cases} \bigsqcup_{m\geq v(\boldsymbol{a})} T(F) g_{-m} K_{\mathfrak{M}} & \text{if } \mathfrak{A}=\mathfrak{M}, \\ \bigsqcup_{m\geq 0} T(F) g_{-m} K_{\mathfrak{J}} & \text{if } \mathfrak{A}=\mathfrak{J},\ \left(\frac{L}{\mathfrak{p}}\right)=-1, \\ \bigsqcup_{m\geq v(\boldsymbol{a})} T(F) g_{-m} K_{\mathfrak{J}} \sqcup T(F)\begin{bmatrix} 1 & \\ u_0 & 1 \end{bmatrix} K_{\mathfrak{J}} & \text{if } \mathfrak{A}=\mathfrak{J},\ \left(\frac{L}{\mathfrak{p}}\right)=0. \end{cases}$$

Then, still assuming $c(\Omega) \geq c(\pi)$, one can prove that if $f \in \mathscr{H}(\mathrm{GL}_2(F),\rho,\Omega)$ is nonzero and is supported on the double coset $T(F)g_{-m}K_{\mathfrak{A}}$, one must have $-m = m_0$. Thus it makes sense to look for intertwining on an element of $T(F)g_{-m}K_{\mathfrak{A}}$. The decomposition above involves negative powers of the uniformizer in the double coset representatives, whereas (4-12) uses positive exponents in the representatives. The difference in the indices occurs because for $m \geq v(\boldsymbol{a})$, g_{-m} and $g_{m-v(\boldsymbol{a})}$ represent the same double coset.

Next, we define a vector in π which will be a test vector for a Ω-Waldspurger functional and have the desired right-invariance. Recall v_0 from Definition 5.2. Define $\varphi_0 \in \pi$ by

$$\varphi_0(g) = \begin{cases} \rho(k_1)v_0 & \text{if } g = k_1 g_{m_0}^{-1} k_2,\ k_1 \in J,\ k_2 \in K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{2\ell(\pi)+2}), \\ 0 & \text{otherwise}. \end{cases} \tag{5-7}$$

See (2-3) for the definition of $K_1^{(s)}(\mathfrak{p}^n)$. The vector φ_0 is well defined because of the inclusion

$$J \cap g_{m_0}^{-1} K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{2\ell(\pi)+2}) g_{m_0} \subseteq \mathrm{Stab}(v_0).$$

Since φ_0 is a translate of the newform in π, we see that φ_0 is the unique (up to scalar) $K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{c(\pi)})$ fixed vector in π.

For $z \in \mathfrak{o}^\times$, define

$$\varphi_z(g) = \begin{cases} \rho(k)v_0 & \text{if } g = kg(m_0, z)^{-1},\ k \in J, \\ 0 & \text{otherwise.} \end{cases} \tag{5-8}$$

Proposition 5.7. *Suppose π is a minimal supercuspidal representation. Let i and z_0 be as in Proposition 5.5. Then*

(i) $\varphi_0 = \sum_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)} \varphi_z$, *and*

(ii) $\ell(\varphi_0) = \ell(\varphi_{z_0})$ *for* $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$.

Proof. The space $\big(g_{m_0} J g_{m_0}^{-1} \cap K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{2\ell(\pi)+2})\big)\backslash K_1^{(m_0+[\ell(\pi)+1])}(\mathfrak{p}^{2\ell(\pi)+2})$ has an irredundant set of coset representatives given by $\{g(0, z) : z \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)\}$. This shows (i). It is a straightforward computation to show that the double cosets $T(F)g(m_0, z)J$ for $z \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ are disjoint. Hence, z_0 is the unique element in $\mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that the double coset $T(F)g(m_0, z_0)J$ is in the support of a nonzero $f \in \mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega)$. By the discussion in Section 5D, this gives (ii). $\square$

Proposition 5.8. *Let π be a minimal supercuspidal representation. There is a nonzero $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$ satisfying $\ell(\varphi_0) \neq 0$.*

Proof. Let $z_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ be as in Proposition 5.5 and ℓ_0 be a nonzero element of the space $\mathrm{Hom}_{J\cap g(m_0,z_0)^{-1}T(F)g(m_0,z_0)}(\rho, \Omega^{g(m_0,z_0)})$. Define

$$\xi = 1_{T(F)g(m_0,z_0)J} \otimes \ell_0 \in \mathscr{H}(\mathrm{GL}_2(F), \rho, \Omega).$$

As in Section 5D, define $\ell(\varphi) = \xi * \varphi(1) \in \mathrm{Hom}_{T(F)}(\pi, \Omega)$. After appropriate normalization of measures, $\ell(\varphi_0) = \ell(\varphi_{z_0}) = \ell_0(v_0)$.

When ρ is a character, it follows immediately that $\ell(\varphi_0) \neq 0$. However, when ρ is not a character, it must be shown that $v_0 \notin \ker \ell_0$.

Suppose that $\ell(\pi) = 2r > 0$ and write $J = J_\alpha$. Recall that under the identification

$$\rho|_{J_\alpha^1} \cong \eta = \mathrm{Ind}_{A^r H_\alpha^1}^{J_\alpha^1} \tilde{\lambda},$$

the vector v_0 is identified with f_0 defined by (5-2). Consider any character ψ' which is a summand of $\rho|_{\overline{N}^r}$, and the vectors $f_a \in \eta$ defined by (5-3) with respect to ψ'. We may take ℓ_0 to be given by

$$\ell_0\bigg(\sum_{a \in \mathfrak{p}^r/\mathfrak{p}^{r+1}} c_a f_a\bigg) := \sum_{a \in \mathfrak{p}^r/\mathfrak{p}^{r+1}} c_a.$$

Indeed, with this definition, note that for $f \in \eta$ and $x \in \mathfrak{p}^r$,

$$\ell_0\Big(\eta\Big(\Big[\begin{smallmatrix}1 & \\ x & 1\end{smallmatrix}\Big]\Big)f\Big) = \psi'\Big(\Big[\begin{smallmatrix}1 & \\ x & 1\end{smallmatrix}\Big]\Big)\ell_0(f)$$
$$= \Omega^{g(m_0,z)}\Big(\Big[\begin{smallmatrix}1 & \\ x & 1\end{smallmatrix}\Big]\Big)\ell_0(f),$$

viewing $\Omega^{g(m_0,z)}$ as a character of $\overline{N}^r$ for some choice of $z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{i+1})$ corresponding to ψ' as in the proof of Proposition 5.5. Hence $\ell_0(v_0) = \ell_0(f_0) = 1 \neq 0$.

Finally, suppose that $\ell(\pi) = 0$, so $c(\pi) = 2$ and $s = 1 - c(\Omega) - v(\boldsymbol{a}) < 0$. Let $h = g_s$. The linear functional ℓ_0 is the projection onto one of the irreducible summands of $\rho|_{\overline{N}^0}$. Let $a \in \mathfrak{o}^\times$, and denote by ψ_a the character of $\overline{N}^0$ given by $\psi_a\big(\big[\begin{smallmatrix}1 & \\ u & 1\end{smallmatrix}\big]\big) = \psi_1(au)$. Denote by v_a the vector in ρ such that $\rho\big(\big[\begin{smallmatrix}1 & \\ u & 1\end{smallmatrix}\big]\big)v_a = \psi_a(u)v_a$.

Now, write $v_0 = \sum c_a v_a$, where $c_a \in \mathbb{C}$ and a runs over $\mathfrak{o}^\times/(1+\mathfrak{p})$. For $b \in \mathfrak{o}^\times$, we have $\rho(g(0,b))v_0 = v_0$. However, $\rho(g(0,b))v_a = v_{ba}$. Therefore, $c_a = c_{ba}$ for all $b \in \mathfrak{o}^\times$. Therefore, v_0 has a nonzero component in each summand of $\rho|_{\overline{N}^0}$. □

Proof of Theorem 1.7 for minimal supercuspidal representations. Since $c(\Omega) \geq c(\pi)$, we can apply Lemma 2.2 together with Proposition 5.8 and the definition (5-7) to see that $\mathrm{Hom}_{T(F)}(\pi, \Omega) \neq 0$ and that φ_0 is a test vector with the required properties. □

5F. ***Nonminimal representations.*** In this section we consider a nonminimal supercuspidal representation τ and let Ω be a character of $T(F)$ such that $\Omega|_{Z(F)} = \omega_\tau$ and $c(\Omega) \geq c(\tau)$. There exists a minimal supercuspidal representation π and a (ramified) character χ of $F^\times$ such that $\tau \cong \pi \otimes \chi$. Identify τ with $\pi \otimes \chi$. Since π is minimal and τ is not, Proposition 3.4 of [Tunnell 1978] tells us

$$c(\tau) = 2c(\chi) > c(\pi).$$

Then $c(\Omega \otimes \chi^{-1}) \geq c(\Omega) > c(\pi)$.

Observe that the considerations of the previous section guarantee the existence of a vector in π that is a test vector for an $(\Omega \otimes \chi^{-1})$-Waldspurger functional and is the unique vector (up to scalars) in π that is right invariant under the corresponding conjugate of $K_1(\mathfrak{p}^{c(\pi)})$. To get the desired test vector for τ, we actually need a vector in π with respect to $\Omega \otimes \chi^{-1}$, but which transforms according to $\chi^{-1} \circ \det$ under right translation by a conjugate of $K_1(\mathfrak{p}^{c(\tau)})$. In the next proposition, we obtain a vector φ_χ with the correct right-transformation property, and then show that it is a test vector for the appropriate linear functional.

Proposition 5.9. *Suppose that* $\pi = c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$. *Let* $s = 2c(\chi) - c(\Omega) - v(\boldsymbol{a})$, $b = \big[\ell(\pi) + \frac{3}{2}\big] - c(\chi)$, $m_0 = \big[\ell(\pi) + \frac{3}{2}\big] - c(\Omega) - v(\boldsymbol{a})$, $i = \big[\frac{1}{2}\big(\ell(\pi) + \frac{3}{2}\big)\big]$ *and* $i' = \big[\ell(\pi) + \frac{3}{2}\big] - \big[\frac{1}{2}\ell(\pi)\big] - 1$.

(i) *There is a unique* $u \in \mathfrak{o}^\times/((1+\mathfrak{p}^i)\cap \mathfrak{o}^\times)$, *and* $v_\chi \in \rho$ *depending on* u, *which is unique up to scaling*, *such that for all* $x \in \mathfrak{p}^{i'}$,

$$\rho\left(\begin{bmatrix} 1+x\varpi^{-b} & 0 \\ xu^{-1} & 1 \end{bmatrix}\right)v_\chi = \chi^{-1}(1+x\varpi^{-b})v_\chi. \tag{5-9}$$

(ii) *Suppose* u *and* v_χ *satisfy* (5-9). *Let*

$$\varphi_\chi(g) = \begin{cases} (\chi^{-1}\circ\det)(k_1)\rho(k_2)v_\chi & \text{if } g = k_2 g_\chi k_1,\ k_2 \in J,\ k_1 \in K_1^{(s)}(\mathfrak{p}^{2c(\chi)}), \\ 0 & \text{otherwise}, \end{cases}$$

where $g_\chi = \left[\begin{smallmatrix} \varpi^{-m_0} & 0 \\ u^{-1}\varpi^{c(\chi)-s} & 1 \end{smallmatrix}\right]$. *Then* φ_χ *is well defined*, *and is the unique vector* (*up to scalars*) *in* π *such that*, *for* $k \in K_1^{(s)}(\mathfrak{p}^{2c(\chi)})$, $\pi(k)\varphi_\chi = (\chi^{-1}\circ\det)(k)\varphi_\chi$.

Note that $i = i'$ when $\ell(\pi)\in\mathbb{Z}$ or $\ell(\pi) = \frac{1}{2}(2r+1)$ with r even; otherwise they are off by 1 (see Table 2).

Proof. Observe that for $\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right] \in K_1^{(s)}(\mathfrak{p}^{2c(\chi)})$, we have

$$g_\chi \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} g_\chi^{-1} \in \begin{bmatrix} a_{11} & 0 \\ (a_{11}-1)u^{-1}\varpi^b & 1 \end{bmatrix} + \mathfrak{P}^{\ell(\pi)e_{\mathfrak{A}}+1}. \tag{5-10}$$

We remark that $b \le 0$. If $g_\chi\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]g_\chi^{-1} \in J$, then $a_{11} \equiv 1 \mod \mathfrak{p}^{i'-b}$. To show that φ_χ is well defined, we must check that ρ^{g_χ} and χ agree on $g_\chi^{-1}Jg_\chi \cap K_1^{(s)}(\mathfrak{p}^{2c(\chi)})$. This is precisely the condition (5-9). Once this is established, uniqueness then follows since $\varphi_\chi \otimes \chi$ is a translate of the newform for τ. Therefore, part (ii) of the proposition will follow from part (i).

First, suppose $\rho = \lambda$ is a character. As in Section 5B we have

$$\rho\left(\begin{bmatrix} 1+x\varpi^{-b} & 0 \\ xu^{-1} & 1 \end{bmatrix}\right)v_\chi = \lambda\left(\begin{bmatrix} 1 & 0 \\ xu^{-1} & 1 \end{bmatrix}\right).$$

As a function of x, both sides of (5-9) are nontrivial characters of $\mathfrak{p}^{i'}/\mathfrak{p}^{[\ell(\pi)+3/2]}$ (see Table 2). Therefore, there is a unique $u \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that (5-9) holds. This proves part (i) when ρ is a character.

If ρ is not a character, then $b < 0$. By Section 5B,

$$\rho\left(\begin{bmatrix} 1+x\varpi^{-b} & 0 \\ xu^{-1} & 1 \end{bmatrix}\right)v_\chi = \rho\left(\begin{bmatrix} 1 & 0 \\ xu^{-1} & 1 \end{bmatrix}\right)v_\chi.$$

Suppose that π is a depth zero supercuspidal representation. Let $u = 1$. By Lemma 5.3 there is a unique up to scalar $v_\chi \in \rho$ such that, for $x \in \mathfrak{o}$,

$$\rho\left(\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\right)v_\chi = \chi^{-1}(1+x\varpi^{c(\chi)-1})v_\chi.$$

Finally, suppose that $\ell(\pi) = 2r > 0$, and $\pi = c\text{-}\mathrm{Ind}_J^{\mathrm{GL}_2(F)}\rho$, where ρ is not a character of J. By Lemma 5.3, $\rho|_{\overline{N}\cap H_\alpha}$ is a multiple of $\psi_\alpha|_{\overline{N}\cap H_\alpha}$. There exists a unique $u \in \mathfrak{o}^\times/(1+\mathfrak{p}^i)$ such that (5-9) holds for $x \in \mathfrak{p}^{i'}$. By Lemma 5.3, with this choice of u there is a unique up to scalar multiple $v_\chi \in \rho$ such that (5-9) holds. This completes the proof of (i) of the proposition. □

The next lemma gives a double coset decomposition of the support of φ_χ.

Lemma 5.10. *Let* $g_\chi = \begin{bmatrix} \varpi^{-m_0} & 0 \\ u^{-1}\varpi^{c(\chi)-s} & 1 \end{bmatrix}$ *as in Proposition 5.9. Then*

$$Jg_\chi K_1^{(s)}(\mathfrak{p}^{2c(\chi)}) = \bigsqcup_{z\in\mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-[\ell(\pi)/2]-1})} Jg_\chi \begin{bmatrix} z & \\ & 1 \end{bmatrix}.$$

Proof. Recall

$$K_1^{(s)}(\mathfrak{p}^{2c(\chi)}) = \begin{bmatrix} 1 & \mathfrak{p}^s \\ \mathfrak{p}^{c(\Omega)+v(\boldsymbol{a})} & 1+\mathfrak{p}^{2c(\chi)} \end{bmatrix} \begin{bmatrix} \mathfrak{o}^\times & \\ & 1 \end{bmatrix}.$$

We have, for $z \in \mathfrak{o}^\times$,

$$g_\chi \begin{bmatrix} 1 & \mathfrak{p}^s \\ \mathfrak{p}^{c(\Omega)+v(\boldsymbol{a})} & 1+\mathfrak{p}^{2c(\chi)} \end{bmatrix} g_\chi^{-1} = \begin{bmatrix} 1+\mathfrak{p}^{c(\chi)} & \mathfrak{p}^{2c(\chi)-[\ell(\pi)+3/2]} \\ \mathfrak{p}^{[\ell(\pi)+3/2]} & 1+\mathfrak{p}^{c(\chi)} \end{bmatrix}, \tag{5-11}$$

$$g_\chi \begin{bmatrix} z & \\ & 1 \end{bmatrix} g_\chi^{-1} = \begin{bmatrix} z & \\ u^{-1}\varpi^{c(\chi)-s+m_0}(z-1) & 1 \end{bmatrix}. \tag{5-12}$$

Using the description of J in Table 2, we see that the right-hand side of (5-11) lies in J. Also, the right-hand side of (5-12) lies in J if and only if $z \in 1+\mathfrak{p}^{c(\chi)-[\ell(\pi)/2]-1}$. This completes the proof of the lemma. □

The next two lemmas give a useful decomposition of g_χ. Fix g_χ and u to be as in Proposition 5.9, and set $y_0 = -u^{-1}\boldsymbol{a}^{-1}\varpi^{c(\Omega)+v(\boldsymbol{a})-c(\chi)}$. For $y \in F$, define $t_y = t\big(1+\frac{1}{2}\boldsymbol{b}y, y\big) = \left[\begin{smallmatrix} 1+\boldsymbol{b}y & \boldsymbol{c}y \\ -\boldsymbol{a}y & 1 \end{smallmatrix}\right]$.

Lemma 5.11. *We have* $g_\chi = k_0 g_{m_0}^{-1} t_{y_0}$, *where* $k_0 \in U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$.

Proof. Write $g_\chi = g_{m_0}^{-1} g$, where $g = \left[\begin{smallmatrix} 1 & \\ -\boldsymbol{a}y_0 & 1 \end{smallmatrix}\right]$. Let $k_0^{-1} = g_{m_0}^{-1} t_{y_0} g^{-1} g_{m_0}$. We see that

$$k_0^{-1} = \begin{bmatrix} 1+\boldsymbol{b}y_0+\boldsymbol{ac}y_0^2 & \boldsymbol{c}y_0\varpi^{-m_0} \\ 0 & 1 \end{bmatrix}.$$

So $k_0 \in U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$ and $g_\chi = k_0 g_{m_0}^{-1} t_{y_0}$. □

Lemma 5.12. *For each* $z \in \mathfrak{o}^\times$, *there exists* $k_z \in U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$ *such that*

$$g_{m_0}^{-1} t_{y_0} \begin{bmatrix} z & \\ & 1 \end{bmatrix} = k_z g_{m_0}^{-1} \begin{bmatrix} z & \\ & 1 \end{bmatrix} t_{zy_0}. \tag{5-13}$$

Proof. Write $g_\chi = k_0 g_{m_0}^{-1} t_{y_0}$ as in Lemma 5.11. Then

$$\begin{aligned} g_{m_0}^{-1} t_{y_0} \begin{bmatrix} z & \\ & 1 \end{bmatrix} &= k_0^{-1} g_\chi \begin{bmatrix} z & \\ & 1 \end{bmatrix} = k_0^{-1} g_{m_0}^{-1} \begin{bmatrix} 1 & \\ -\boldsymbol{a} y_0 & 1 \end{bmatrix} \begin{bmatrix} z & \\ & 1 \end{bmatrix} \\ &= k_0^{-1} \begin{bmatrix} z & \\ & 1 \end{bmatrix} g_{m_0}^{-1} \begin{bmatrix} 1 & \\ -\boldsymbol{a} z y_0 & 1 \end{bmatrix} = k_0^{-1} \begin{bmatrix} z & \\ & 1 \end{bmatrix} k_0' g_{m_0}^{-1} t_{z y_0} \\ &= k_z \begin{bmatrix} z & \\ & 1 \end{bmatrix} g_{m_0}^{-1} t_{z y_0}. \end{aligned}$$

For the second to last equality we have used a decomposition similar to Lemma 5.11, and k_0' is the corresponding element of $U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$. For the last equality we use the fact that the subgroup $U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$ is normalized by A^0. □

Let us remark here that $U_{\mathfrak{A}}^{\ell(\pi)e_{\mathfrak{A}}+1}$ lies in the kernel of ρ (see Table 2 for details). For any $g \in \mathrm{GL}_2(F)$ and $v \in \rho$, define

$$\varphi_{g,v}(h) = \begin{cases} \rho(k)v & \text{if } h = kg,\ k \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for any $z \in \mathfrak{o}^\times$, the support of $\pi\left(\begin{bmatrix} z^{-1} & \\ & 1 \end{bmatrix}\right)\varphi_{g_\chi, v_\chi}$ is exactly $J g_\chi \begin{bmatrix} z & \\ & 1 \end{bmatrix}$. Then

$$\begin{aligned} \varphi_\chi &= \sum_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-[\ell(\pi)/2]-1})} \chi^{-1}(z)\, \pi\left(\begin{bmatrix} z^{-1} & \\ & 1 \end{bmatrix}\right)\varphi_{g_\chi, v_\chi} \\ &= q^{-[\ell(\pi)/2]-1} \sum_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)})} \chi(z)\, \pi\left(\begin{bmatrix} z & \\ & 1 \end{bmatrix}\right)\varphi_{g_\chi, v_\chi} \\ &= q^{-[\ell(\pi)/2]-1} \sum_{z \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)})} \chi(z)\, \pi\left(t_{z^{-1}y_0}^{-1}\right)\varphi_{g(m_0,z)^{-1}, v_\chi}. \end{aligned} \tag{5-14}$$

The first equality follows from the double coset decomposition in Lemma 5.10. To get the second equality, note that for $z \in 1 + \mathfrak{p}^{c(\chi)-[\ell(\pi)/2]-1}$, we can apply (5-9) to the right-hand side of (5-12). Finally, the third equality follows from Lemmas 5.11 and 5.12.

Write $c_0 = \left[\frac{1}{2}(c(\chi)+1)\right]$. For $x \in \mathfrak{p}^{c_0}$, $x \mapsto \chi(1+x)$ is an additive character of $\mathfrak{p}^{c_0}/\mathfrak{p}^{c(\chi)}$.

Proposition 5.13. *Suppose $\ell \in \mathrm{Hom}_{T(F)}(\pi, \Omega \otimes \chi^{-1})$. Then there is a unique $w_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-c_0})$ satisfying the following conditions*:

(i) *If*
$$\sum_{z \in (1+\mathfrak{p}^{c_0})/(1+\mathfrak{p}^{c(\chi)})} \chi(zw)\, \ell\left(\pi\left(t_{(zw)^{-1}y_0}^{-1}\right)\varphi_{g(m_0,zw)^{-1}, v_\chi}\right) \neq 0,$$
then $w \equiv w_0 \mod \mathfrak{p}^{c(\chi)-c_0}$.

(ii) $\ell(\varphi_{g(m_0,zw_0)^{-1}, v_\chi}) \neq 0.$

Proof. Recall $y_0 = -u^{-1}\boldsymbol{a}^{-1}\varpi^{c(\Omega)+v(\boldsymbol{a})-c(\chi)}$. First, note $\chi \circ \det$ is trivial on elements t_{zy_0} for $z \in \mathfrak{o}^\times$. We can define an additive character of $\mathfrak{o}$ by $\psi_\Omega(x) := \Omega(t_{xy_0})$. By (5-14), we have

$$\ell(\varphi_\chi) = q^{-[\ell(\pi)/2]-1} \times \sum_{w \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c_0})} \sum_{z \in (1+\mathfrak{p}^{c_0})/(1+\mathfrak{p}^{c(\chi)})} \chi^{-1}(wz)\,\psi_\Omega^{-1}(wz)\,\ell(\varphi_{g(-m_0,zw),v_\chi}). \tag{5-15}$$

If $z \in 1+\mathfrak{p}^{c_0}$, then $\rho(g(0, z^{-1}))v_\chi = v_\chi$, and $\varphi_{g(-m_0,zw),v_\chi} = \varphi_{g(-m_0,w),v_\chi}$. Consider the inner sum of (5-15):

$$\sum_{z \in (1+\mathfrak{p}^{c_0})/(1+\mathfrak{p}^{c(\chi)})} \chi^{-1}(wz)\,\psi_\Omega^{-1}(wz) = \chi^{-1}(w)\,\psi_\Omega^{-1}(w) \sum_{x \in \mathfrak{p}^{c_0}/\mathfrak{p}^{c(\chi)}} \chi^{-1}(1+x)\,\psi_\Omega^{-1}(wx). \tag{5-16}$$

This sum does not equal zero if and only if $\chi^{-1}(1+x) = \psi_\Omega(xw)$ for all $x \in \mathfrak{p}^{c_0}$, and this occurs for exactly one element $w = w_0 \in \mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-c_0})$. This proves the first part.

Consider an element $t \in T(F) \cap g_{m_0} J g_{m_0}^{-1} = F^\times\big(1 + \mathfrak{P}_L^{c(\Omega)-[\ell(\pi)/2]-1}\big)$ (see Lemma 5.4). Since $2c(\chi) > c(\pi) = 2\ell(\pi)+2$, we see

$$\big(c(\Omega) - \big[\tfrac{1}{2}\ell(\pi)\big] - 1\big) - (c(\Omega) - c(\chi)) = c(\chi) - \big[\tfrac{1}{2}\ell(\pi)\big] - 1 > c_0.$$

Therefore, there exist $x \in \mathfrak{p}^{c_0}$ and $z \in F^\times$ such that $t = z t_{xy_0}$, and, by the remarks after (5-16),

$$\chi^{-1}(1 + w_0^{-1}x) = \psi_\Omega(x) = \Omega(t_{xy_0}) = \Omega(z)^{-1}\Omega(t).$$

By (5-9) we have

$$\rho\Big(\begin{bmatrix} 1 & \\ u^{-1}x & 1 \end{bmatrix}\Big)v_\chi = \chi^{-1}(1 + x\varpi^{-b})v_\chi.$$

Therefore, for all $t \in T(F) \cap g_{m_0} J g_{m_0}^{-1} = T(F) \cap g_{m_0,w_0} J g_{m_0,w_0}^{-1}$,

$$\rho(g_{m_0,w_0}^{-1}\, t\, g_{m_0,w_0})v_\chi = \Omega(t)v_\chi = \Omega(t)(\chi^{-1} \circ \det)(t)v_\chi.$$

This implies that there is $\ell_0 \in \mathrm{Hom}_{g(m_0,w_0)^{-1}T(F)g(m_0,w_0)\cap J}(\rho, (\Omega \otimes \chi^{-1})^{g(m_0,w_0)})$ such that $\ell_0(v_\chi) \neq 0$, and from the discussion in Section 5D, after a normalization, $\ell(\varphi_{g(m_0,w_0)^{-1},v_\chi}) = \ell_0(v_\chi)$. □

Proof of Theorem 1.7 for nonminimal supercuspidal representations. We compute, by Proposition 5.13,

$$\begin{aligned}\ell(\varphi_\chi) &= q^{-[\ell(\pi)/2]-1}\ell(\varphi_{g(m_0,w_0)^{-1},v_\chi}) \sum \chi^{-1}(ww_0^{-1})\psi_\Omega^{-1}(ww_0^{-1}) \\ &= q^{-[\ell(\pi)/2]-1-[c(\chi)/2]}\psi_\Omega^{-1}(w_0^{-1})G(\chi, \psi_\Omega^{-1})\ell(\varphi_{g(m_0,w_0)^{-1},v_\chi}) \neq 0,\end{aligned}$$

where the sum is over $w \in (1+\mathfrak{p}^{c(\chi)-c_0})/(1+\mathfrak{p}^{c_0})$, w_0 is the unique element of $\mathfrak{o}^\times/(1+\mathfrak{p}^{c(\chi)-c_0})$ such that $\chi^{-1}(1+z) = \psi_\Omega(zw_0^{-1})$ for all $z \in \mathfrak{p}^{c_0}$, and $G(\chi, \psi_\Omega^{-1})$ is the Gauss sum for the pair χ, ψ_Ω^{-1}. For the last equality see [Bushnell and Henniart 2006, 23.6 Proposition]. This shows that $\mathrm{Hom}_{T(F)}(\tau, \Omega) \neq 0$. The 1-dimensionality follows from [Waldspurger 1985]. Since $c(\Omega) \geq c(\tau)$, we can apply Lemma 2.2 to obtain the test vector with the required properties. The uniqueness of the test vector follows from the uniqueness of the newform in π. □

6. Local spectral distributions

Now we return to the setting where F is a p-adic field and L is a quadratic separable extension as in Section 2B. Let π be an infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}_2(F)$, and Ω a character of $L^\times$ such that $\Omega|_{F^\times} = \omega_\pi$. In this section, we calculate certain local spectral distributions $\tilde{J}_\pi(f)$ defined by Jacquet and Chen [2001] for certain test functions $f \in C_c^\infty(\mathrm{GL}_2(F))$. These are used in Section 7 to generalize the global L-value formula previously obtained in [Martin and Whitehouse 2009]. For simplicity, we prove this global L-value formula when the central character of our automorphic representation is trivial, so we may as well assume $\omega_\pi = 1$ in this section also. We also assume that π and Ω are unitary, since the global objects in the following sections are unitary as well.

The calculation of $\tilde{J}_\pi(f)$ is contained in [Martin and Whitehouse 2009] in the cases where F is archimedean, L/F is split, or π and Ω have disjoint ramification. Hence, we assume L/F is a quadratic extension of nonarchimedean fields and $c(\pi), c(\Omega) > 0$. In particular, either $L(s, \pi) = 1$ or $\pi = \chi\, \mathrm{St}_{\mathrm{GL}_2}$, where, for the rest of this section, χ denotes an unramified quadratic character. Further, we assume $c(\Omega) \geq c(\pi)$ to use our determination of test vectors.

Write $L^\times = F(\xi)^\times$, where $\xi = \frac{1}{2}\sqrt{\boldsymbol{d}}$. Let $T = T(F)$ be the torus in $\mathrm{GL}_2(F)$ isomorphic to $L^\times$ defined in (2-16). Here it is convenient to take a slightly different parameterization for T than the one given by $t(x, y)$ in (2-17). Namely, we map

$$x + y\xi_0 \mapsto \begin{bmatrix} x & \boldsymbol{c}y \\ -\boldsymbol{a}y & x - \boldsymbol{b}y \end{bmatrix}, \tag{6-1}$$

where

$$\xi_0 = \xi - \tfrac{1}{2}\boldsymbol{b} = \tfrac{1}{2}\big(\sqrt{\boldsymbol{d}} - \boldsymbol{b}\big).$$

By [Tunnell 1983; Saito 1993] or Theorem 1.7, the assumption $c(\Omega) \geq c(\pi)$ implies $\dim_{\mathbb{C}} \mathrm{Hom}_T(\pi, \Omega) = 1$. Fix a nonzero linear functional $\ell \in \mathrm{Hom}_T(\pi, \Omega)$.

Consider the Kirillov model for π and the inner product on π given by

$$(\phi_1, \phi_2) = \int_{F^\times} \phi_1(a)\overline{\phi_2(a)}\, d^\times a,$$

where $d^\times a$ is the Haar measure giving $\mathrm{vol}(\mathfrak{o}^\times) = 1$. This inner product is $\mathrm{GL}_2(F)$-invariant. Let e be the unique (up to scalars) vector in π such that $\pi(t)e = \Omega(t)e$ for $t \in T$, which we normalize so that $(e, e) = 1$. Let dg denote the local Tamagawa measure on $\mathrm{GL}_2(F)$. Then the local distribution we are interested in is defined in [Jacquet and Chen 2001] by

$$\tilde{J}_\pi(f) = (\pi(f)e, e) = \int_{\mathrm{GL}_2(F)} f(g)(\pi(g)e, e)\, dg, \quad f \in C_c^\infty(\mathrm{GL}_2(F)). \tag{6-2}$$

Put $s = c(\Omega) - c(\pi)$, $h = \begin{bmatrix} \varpi^s & \\ & 1 \end{bmatrix} w$, and

$$K' = hK_1(\mathfrak{p}^{c(\pi)})h^{-1} = \begin{bmatrix} 1+\mathfrak{p}^{c(\pi)} & \mathfrak{p}^{c(\Omega)} \\ \mathfrak{p}^{c(\pi)-c(\Omega)} & \mathfrak{o}^\times \end{bmatrix}. \tag{6-3}$$

Then Theorem 1.7 says there is a unique (up to scalars) test vector $\phi \in \pi$ which is right invariant by K' such that $\ell(\phi) \neq 0$. Let ϕ_0 be the newvector in π normalized so that $\phi_0(1) = 1$. Then we may take $\phi = \pi(h)\phi_0$.

Observe that Ω is trivial on $T \cap ZK'$, where $Z = Z(T)$, since ϕ is fixed by ZK'. Consider the vector $e' \in \pi$ given by

$$e' = \sum_{t \in T/(T \cap ZK')} \Omega^{-1}(t)\pi(t)\phi. \tag{6-4}$$

Note the index set for the sum is finite, so e' is well defined. Then for any $t \in T$, we have $\pi(t)e' = \Omega(t)e'$, and $\ell(e') \neq 0$. In other words, we may assume

$$e = \frac{e'}{(e', e')^{1/2}}.$$

We take for our test function $f = 1_{K'}/\mathrm{vol}(K')$, so our calculations do not in fact depend on the normalization of dg in (6-2). Then

$$\tilde{J}_\pi(f) = \mathrm{vol}(K')^{-1} \int_{K'} (\pi(k)e, e)\, dk = \mathrm{vol}(K')^{-1} \int_{K'} \frac{(\pi(k)e', e')}{(e', e')}\, dk.$$

Note, using the $\mathrm{GL}_2(F)$-invariance of the inner product, we get

$$\begin{aligned} (e', e') &= \sum_{t \in T/(T \cap ZK')} \Omega^{-1}(t)(\phi, \pi(t^{-1})e') \\ &= |T/(T \cap ZK')|\,(\phi, e'). \end{aligned}$$

Since $\pi(f)$ is simply orthogonal projection onto $\langle \phi \rangle = \pi^{K'}$,

$$\mathrm{vol}(K')^{-1} \int_{K'} (\pi(k)e', e')\, dk = (\pi(f)e', e') = \frac{(e', \phi)(\phi, e')}{(\phi, \phi)}.$$

Hence

$$\tilde{J}_\pi(f) = \frac{1}{|T/(T \cap ZK')|} \frac{(e', \phi)}{(\phi, \phi)}. \tag{6-5}$$

Note that

$$(\phi, \phi) = (\phi_0, \phi_0) = \begin{cases} L(2, 1_F) & \text{if } \pi = \chi \operatorname{St}_{\mathrm{GL}_2}, \\ 1 & \text{if } L(s, \pi) = 1, \end{cases} \tag{6-6}$$

so it remains to compute $|T/(T \cap ZK')|$ and (e', ϕ). (Recall χ denotes an unramified character.) Only the latter computation is involved. This requires knowing some facts about values of the Whittaker newform and determining a set of representatives for $T/(T \cap ZK')$. We first tackle these two tasks, and then compute (e', ϕ), and hence $\tilde{J}_\pi(f)$, under our above assumptions.

Whittaker values. Assume π has trivial central character and let ψ be a nontrivial additive character of F of conductor $\mathfrak{o}$. Let $\mathcal{W}(\pi, \psi)$ be the Whittaker model for π with respect to ψ. Let W_0 be the newform normalized so that $W_0(1) = 1$, and therefore $\phi_0(a) = W_0\left(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right]\right)$.

We are interested in certain values of the Whittaker newform when the local L-factor of π has degree 1 or 0. For this, we recall (see Table 1 in Section 3) that $\phi_0(a) = \chi(a)|a| 1_{\mathfrak{o}}(a)$ when $\pi = \chi \operatorname{St}_{\mathrm{GL}_2}$ and $\phi_0(a) = 1_{\mathfrak{o}^\times}(a)$ when $L(s, \pi) = 1$. From this, one obtains the following result on Whittaker newform values.

Lemma 6.1. (i) *If $u, v \in \mathfrak{o}^\times$, then*

$$W_0\left(g \begin{bmatrix} u & \\ & v \end{bmatrix} w\right) = W_0(gw).$$

(ii) *If $\pi = \chi \operatorname{St}_{\mathrm{GL}_2}$ with χ unramified, then for $j \in \mathbb{Z}$,*

$$W_0\left(\begin{bmatrix} \varpi^j & \\ & 1 \end{bmatrix} w\right) = \begin{cases} -\chi(\varpi)^j q^{-j-1} & \text{if } j \geq -1, \\ 0 & \text{else}. \end{cases}$$

If $L(s, \pi) = 1$, then for any $j \in \mathbb{Z}$,

$$W_0\left(\begin{bmatrix} \varpi^j & \\ & 1 \end{bmatrix} w\right) = \begin{cases} \epsilon\left(\frac{1}{2}, \pi\right) & \text{if } j = -c(\pi), \\ 0 & \text{else}. \end{cases}$$

(iii) *If $\pi = \chi \operatorname{St}_{\mathrm{GL}_2}$ with χ unramified, for $j \geq 0 \geq k$, we have*

$$\int_{\mathfrak{o}^\times} W_0\left(\begin{bmatrix} \varpi^j u & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi^k & 1 \end{bmatrix}\right) d^\times u = -q^{-1}(\chi(\varpi) q^{-1})^{j-2k}.$$

If $L(s, \pi) = 1$, then for all $j, k \in \mathbb{Z}$,

$$\int_{\mathfrak{o}^\times} W_0\left(\begin{bmatrix} \varpi^j u & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ \varpi^k & 1 \end{bmatrix}\right) d^\times u = \begin{cases} 1 & \text{if } j = 0 \text{ and } k \geq c(\pi), \\ (1-q)^{-1} & \text{if } j = 0 \text{ and } k = c(\pi) - 1, \\ 0 & \text{else}. \end{cases}$$

Proof. Part (i) follows simply from the facts that W_0 is right invariant by $K_1(\mathfrak{p}^{c(\pi)})$ and $\omega_\pi = 1$. The proof of parts (ii) and (iii) follows from the functional equation (3-3) with $\mu = 1$ by comparing coefficients of q^s. □

The toric quotient. We identify $t = x + y\xi_0 \in L^\times$ with its image in T via (6-1). Since we have assumed that $c(\Omega) \geq c(\pi)$, we have $t = x + y\xi_0 \in K'$ if and only if $x \in 1 + \mathfrak{p}^{c(\pi)}$ and $y \in \mathfrak{p}^{c(\Omega)}$.

Lemma 6.2. *We have*

$$|T/(T \cap ZK')| = \begin{cases} q^{c(\Omega)}(1+q^{-1}) & \text{if } L/F \text{ is unramified,} \\ 2q^{c(\Omega)} & \text{if } L/F \text{ is ramified.} \end{cases} \tag{6-7}$$

Furthermore, if L/F is unramified or $v(\boldsymbol{a}) = 1$, then a complete set of representatives of $T/(T \cap ZK')$ is given by

$$\{1 + y\xi_0 : y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)}\} \cup \{x + \xi_0 : x \in \mathfrak{p}/\mathfrak{p}^{c(\Omega)+v(\boldsymbol{a})}\}, \tag{6-8}$$

while if L/F is ramified and $v(\boldsymbol{a}) = 0$, then a complete set of representatives of $T/(T \cap ZK')$ is given by

$$\{1 + y\xi_0 : y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)},\ y \not\equiv u_0' \mod \mathfrak{p}\} \\ \cup \{1 + (u_0' + y)\xi_0 : y \in \mathfrak{p}/\mathfrak{p}^{c(\Omega)+1}\} \cup \{x + \xi_0 : x \in \mathfrak{p}/\mathfrak{p}^{c(\Omega)}\}, \tag{6-9}$$

where $u_0' = -u_0/\boldsymbol{a} \in \mathfrak{o}^\times$ with u_0 as in (2-14).

Proof. We obtain the set of representatives of $T/(T \cap ZK')$, from which (6-7) follows. Given an arbitrary $t = x + y\xi_0 \in T$, we may multiply t by an element of Z to assume that $x, y \in \mathfrak{o}$ and either $x = 1$ or $y = 1$. Further, if x and y are both units, we may assume $x = 1$. So we may consider a set of representatives of the form $x + \xi_0$ and $1 + y\xi_0$, where $x \in \mathfrak{p}$ and $y \in \mathfrak{o}$. Observe $\xi_0^2 = -\boldsymbol{ac} - \boldsymbol{b}\xi_0$. For $t, t' \in T$, write $t \sim t'$ if $t = t_0 t'$ for some $t_0 \in T \cap ZK'$.

First we observe that $x + \xi_0 \sim 1 + y\xi_0$, where $x \in \mathfrak{p}$ and $y \in \mathfrak{o}$, is not possible. If it were, there would exist $u \in 1 + \mathfrak{p}^{c(\pi)}$, $r \in \mathfrak{p}^{c(\Omega)}$ and $z \in F^\times$ such that

$$zx + z\xi_0 = (u + r\xi_0)(1 + y\xi_0) = u - \boldsymbol{ac}ry + (uy + r - \boldsymbol{b}ry)\xi_0.$$

Since $u - \boldsymbol{ac}ry \in \mathfrak{o}^\times$, we see $v(z) < 0$, but $z = uy + r - \boldsymbol{b}ry \in \mathfrak{o}$, a contradiction.

Now consider $x_1 + \xi_0 \sim x_2 + \xi_0$ for $x_1, x_2 \in \mathfrak{p}$. Then, for some $u \in 1 + \mathfrak{p}^{c(\pi)}$, $r \in \mathfrak{p}^{c(\Omega)}$ and $z \in F^\times$, we have

$$zx_1 + z\xi_0 = (u + r\xi_0)(x_2 + \xi_0) = ux_2 - \boldsymbol{ac}r + (u + rx_2 - \boldsymbol{b}r)\xi_0.$$

Hence $z = u + rx_2 - \boldsymbol{b}r \in 1 + \mathfrak{p}^{c(\pi)}$ and

$$zx_1 = ux_1 + rx_1x_2 - \boldsymbol{b}rx_1 = ux_2 - \boldsymbol{ac}r,$$

which implies

$$u(x_2 - x_1) = r(\boldsymbol{ac} - \boldsymbol{b}x_1 + x_1x_2).$$

In particular, we must have $x_1 \equiv x_2 \mod \mathfrak{p}^{c(\Omega)+v(\boldsymbol{a})}$.

In fact, if $v(\boldsymbol{a}) = 0$, we have $x_1 + \xi_0 \sim x_2 + \xi_0$ if and only if $x_1 \equiv x_2 \mod \mathfrak{p}^{c(\Omega)}$. Similarly, if $v(\boldsymbol{a}) = 1$, then $v(\boldsymbol{b}) > 0$ and $x_1 + \xi_0 \sim x_2 + \xi_0$ if and only if $x_1 \equiv x_2 \mod \mathfrak{p}^{c(\Omega)+1}$. The rest of the cases are computed similarly. □

Let us remark that the coset representatives in the previous lemma depend only on $c(\Omega)$ since we are in the case $c(\Omega) \geq c(\pi)$.

Projection onto the test vector. Put $e(L/F) = 1$ if L/F is unramified, $e(L/F) = 2$ if L/F is ramified. Denote by η the quadratic character of $F^\times$ associated to L/F.

Proposition 6.3. *If $c(\pi) \geq 2$, then*

$$\tilde{J}_\pi(f) = q^{-c(\Omega)} \frac{L(1, 1_F)L(1, \eta)}{e(L/F)}. \tag{6-10}$$

If $c(\pi) = 1$, then

$$\tilde{J}_\pi(f) = q^{-c(\Omega)} \frac{L(1, 1_F)L(1, \eta)}{e(L/F)L(2, 1_F)}. \tag{6-11}$$

Proof. By (6-5), (6-6) and (6-7), this proposition is equivalent to the statement that

$$(e', \phi) = L(1, 1_F).$$

To show this, first observe

$$(e', \phi) = \sum_{t \in T/(T \cap ZK')} \Omega^{-1}(t)(\pi(t)\phi, \phi) = \sum_{t \in T/(T \cap ZK')} \Omega^{-1}(t)(\pi(h^{-1}th)\phi_0, \phi_0).$$

Recall that $h = \begin{bmatrix} \varpi^s & \\ & 1 \end{bmatrix} w$ with $s = c(\Omega) - c(\pi)$. We give the details of the case $c(\pi) \geq 2$ here. The other case is computed similarly. Hence, assume that $c(\pi) \geq 2$, so $L(s, \pi) = 1$. Then, for $g \in \mathrm{GL}(2)$,

$$(\pi(g)\phi_0, \phi_0) = \int_{\mathfrak{o}^\times} W_0\left(\begin{bmatrix} u & \\ & 1 \end{bmatrix} g\right) d^\times u.$$

First suppose $t = x + \xi_0$, where $x \in \mathfrak{p}$ and $v(x) \leq c(\Omega) + v(\boldsymbol{a})$. Note

$$\begin{aligned} h^{-1}th &= \begin{bmatrix} x - \boldsymbol{b} & \varpi^s \boldsymbol{a} \\ -\varpi^{-s}\boldsymbol{c} & x \end{bmatrix} \\ &= \varpi^{-s}\boldsymbol{c} \begin{bmatrix} 1 & (\boldsymbol{b} - x)\varpi^s/\boldsymbol{c} \\ & 1 \end{bmatrix} \begin{bmatrix} \det(t)\varpi^{2s}/\boldsymbol{c}^2 & \\ & 1 \end{bmatrix} w \begin{bmatrix} 1 & -\varpi^s x/\boldsymbol{c} \\ & 1 \end{bmatrix}. \end{aligned}$$

Since the rightmost matrix lies in $K_1(\mathfrak{p}^{c(\pi)})$, we have

$$(\pi(h^{-1}th)\phi_0, \phi_0) = W_0\left(\begin{bmatrix} \det(t)\varpi^{2s}/\boldsymbol{c}^2 & \\ & 1 \end{bmatrix} w\right) = 0,$$

where the last equality follows from Lemma 6.1(ii). Now suppose $t = 1 + y\xi_0$, where $y \in \mathfrak{o}$. If $y = 0$, then

$$(\pi(h^{-1}th)\phi_0, \phi_0) = (\phi_0, \phi_0) = 1.$$

Otherwise, assume $v(y) < c(\Omega)$ and write

$$h^{-1}th = \begin{bmatrix} 1 & \varpi^s \boldsymbol{a} y \\ & 1 \end{bmatrix}\begin{bmatrix} \det(t) & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -\varpi^{-s}\boldsymbol{c}y & 1 \end{bmatrix}.$$

Then, by Lemma 6.1(iii),

$$\begin{aligned}(\pi(h^{-1}th)\phi_0, \phi_0) &= \int_{\mathfrak{o}^\times} W_0\left(\begin{bmatrix} \det(t)u & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -\varpi^{-s}\boldsymbol{c}y & 1 \end{bmatrix}\right) d^\times u \\ &= \begin{cases} (1-q)^{-1} & \text{if } v(y) = c(\Omega) - 1, \\ 0 & \text{else.} \end{cases}\end{aligned}$$

Observe that $\int_{1+\varpi^k \mathfrak{o}_L} \Omega^{-1}(u)\, d^\times u = 0$ for $0 < k < c(\Omega)$, together with $\Omega^{-1}|_{\mathfrak{o}^\times} = 1$, implies

$$\sum_{y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)} : v(y) \geq k} \Omega^{-1}(1 + y\xi_0) = 0.$$

Hence, for $0 < k \leq c(\Omega)$, we have

$$\sum_{y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)} : v(y) = k} \Omega^{-1}(1 + y\xi_0) = \begin{cases} 0 & \text{if } 0 < k < c(\Omega) - 1, \\ -1 & \text{if } k = c(\Omega) - 1 \text{ and } c(\Omega) > 1, \\ 1 & \text{if } k = c(\Omega). \end{cases}$$

Summing up gives the desired calculation

$$(e', \phi) = 1 + (1-q)^{-1} \sum_{y \in \mathfrak{o}/\mathfrak{p}^{c(\Omega)} : v(y) = c(\Omega) - 1} \Omega(1 + y\xi_0)^{-1} = \frac{1}{1 - q^{-1}},$$

since here $c(\Omega) \geq 2$. □

7. A central-value formula

In this section we work globally. Specifically, let L/F be a quadratic extension of number fields, $\mathbb{A}$ the adèles of F and $\mathbb{A}_L$ the adèles of L. Let Δ and Δ_L be the absolute values of the discriminants of F and L, and let $\eta = \eta_{L/F}$ be the quadratic idèle class character associated to L/F via class field theory.

Set $G = \mathrm{GL}(2)/F$. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character, and Ω a unitary character of $\mathbb{A}_L^\times/L^\times\mathbb{A}^\times$. Assume the sign of the functional equation $\epsilon\big(\frac{1}{2}, \pi_L \otimes \Omega\big) = 1$, where π_L is the base change of π to L. Then by [Waldspurger 1985; Tunnell 1983; Saito 1993], one knows that there is a unique quaternion algebra (possibly the split matrix algebra) D/F in which L embeds, such that π has a Jacquet–Langlands transfer to a representation π' of $D^\times(\mathbb{A})$ and the local Hom spaces $\mathrm{Hom}_{L_v^\times}(\pi'_v, \Omega_v) \neq 0$ for all places v, and in fact have dimension 1. Fix this D and π', and write G' for $D^\times$, regarded as an algebraic group over F. Let T be a torus in G' whose F-points are isomorphic to $L^\times$, and view Ω as a character of $T(\mathbb{A})/Z(\mathbb{A})$, where Z is the center of G'.

Let ψ be the standard additive character on $\mathbb{A}/F$, i.e., the composition of the trace map with the standard additive character on $\mathbb{A}_{\mathbb{Q}}$. Let S be a finite set of places of F containing all archimedean places, such that, for all $v \notin S$, ψ, π and Ω are unramified and L is not ramified at or above v.

Put on $G'(\mathbb{A})$ the product of the local Tamagawa measures times $L^S(2, 1_F)$, i.e., take the local Tamagawa measure dg_v for $v \in S$ and dg_v normalized so that $G(\mathfrak{o}_v) \cong G'(\mathfrak{o}_v)$ has volume 1 if $v \notin S$ (see, e.g., [Jacquet and Chen 2001, Section 2] for the definition of local Tamagawa measures). Note we will renormalize our measure on $G'(\mathbb{A})$ later in Section 7C.

Jacquet and Chen [2001] prove a formula for a distribution appearing on the spectral side of the relative trace formula,

$$J_{\pi'}(f) = \sum_{\phi} \int\limits_{T(\mathbb{A})/Z(\mathbb{A})T(F)} \pi'(f)\phi(t)\Omega(t)^{-1}\,dt \int\limits_{T(\mathbb{A})/Z(\mathbb{A})T(F)} \overline{\phi(t)\Omega(t)^{-1}}\,dt, \quad (7\text{-}1)$$

where ϕ runs over an orthonormal basis for the space of π'. Here $T(\mathbb{A})$ and $Z(\mathbb{A})$ are given the product of local Tamagawa measures, $T(F)$ has the counting measure, and dt is the quotient measure.

Let S_{inert} be the set of finite places v in S such that L_v/F_v is inert (ramified or unramified). For $v \in S_{\mathrm{inert}}$, as in (6-2), define

$$\tilde{J}_{\pi'_v}(f_v) = \int\limits_{G'(F_v)} f_v(g)(\pi'_v(g)e'_v, e'_v)\,dg_v,$$

where e'_v is a norm 1 vector such that $\pi'_v(t)e'_v = \Omega_v(t)e'_v$ for all $t \in T(F_v)$. For $v \in S - S_{\mathrm{inert}}$, set

$$\tilde{J}_{\pi'_v}(f_v) = \sum_W \bigg(\int\limits_{F_v^\times} \pi'_v(f_v) W\Big(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\Big)\Omega\Big(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\Big)^{-1} d^\times a \\ \times \overline{\int\limits_{F_v^\times} W\Big(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\Big)\Omega\Big(\begin{bmatrix} a & \\ & 1 \end{bmatrix}\Big)^{-1} d^\times a}\bigg),$$

where $d^\times a$ is the local Tamagawa measure and W runs over an orthonormal basis for the local Whittaker model $\mathcal{W}(\pi_v, \psi_v)$.

With the above normalizations, the formula of Jacquet and Chen is as follows.

Theorem 7.1 [Jacquet and Chen 2001]. *Let S be a set of places containing all infinite places and all places at which L, π or Ω is ramified. Let*

$$f = \prod f_v \in C_c^\infty(G'(\mathbb{A}_F))$$

with f_v the unit element of the Hecke algebra for $v \notin S$. Then

$$J_{\pi'}(f) = \frac{1}{2} \prod_S \tilde{J}_{\pi'_v}(f_v) \prod_{v \in S_{\text{inert}}} 2\epsilon(1, \eta_v, \psi_v) L(0, \eta_v) \times \frac{L_S(1, \eta) L^S\big(\frac{1}{2}, \pi_L \otimes \Omega\big)}{L^S(1, \pi, \mathrm{Ad})}.$$

Note that if $\pi'(f)$ is an orthogonal projection onto a 1-dimensional subspace $\langle \phi \rangle$, then

$$J_{\pi'}(f) = \frac{\left|\int_{T(\mathbb{A})/Z(\mathbb{A})T(F)} \phi(t)\Omega(t)^{-1}\, dt\right|^2}{(\phi, \phi)}. \tag{7-2}$$

This expression is written to be invariant under replacing ϕ by a scalar multiple.

7A. *Choice of test vector.* To obtain an explicit L-value formula, we choose $f = \prod f_v$ so that it picks out a global test vector $\phi = \bigotimes \phi_v$ as follows.

First suppose v is a finite place of F. We denote by $\mathfrak{o}_v$, $\mathfrak{o}_{L_v}$, $\mathfrak{p}_v$ and ϖ_v what was denoted in previous sections by these symbols without the subscript v for the local field F_v. Since we have assumed that the central character is trivial, we may work with the congruence subgroups

$$K_{0,v}(\mathfrak{p}_v^n) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G(\mathfrak{o}_v) : c \in \mathfrak{p}_v^n \right\}.$$

We assume that at any finite $v \in S_{\text{inert}}$ such that $c(\Omega_v) > 0$, we have $c(\Omega_v) \geq c(\pi_v)$. Recall that, if L_v/F_v is split or $0 \leq c(\pi_v) \leq c(\Omega_v)$, then we can identify $G'(F_v)$ with $G(F_v)$.

For $v \notin S$, let f_v be the characteristic function of $G(\mathfrak{o}_v)$. Then $\pi_v \cong \pi'_v$, and $\pi'_v(f_v)$ is orthogonal projection onto the local newvector ϕ_v.

Let $v \in S - S_{\text{inert}}$. Take $g_v \in G(F_v)$ such that $g_v^{-1} T(F_v) g_v$ is the diagonal subgroup of $G(F_v)$. Let f_v be the characteristic function of the subgroup of $G(F_v)$ given by

$$g_v^{-1} \begin{bmatrix} 1 & -\varpi_v^{-c(\Omega_v)} \\ & 1 \end{bmatrix} K_{0,v}(\mathfrak{p}_v^{c(\pi_v)}) \begin{bmatrix} 1 & \varpi_v^{-c(\Omega_v)} \\ & 1 \end{bmatrix} g_v$$

divided by its volume. Then ϕ_v is the unique (up to scalar multiples) vector in π_v fixed by this subgroup.

Consider $v \in S_{\text{inert}}$.

Suppose $c(\pi_v) = 0$ or $c(\Omega_v) = 0$. Let $R(\pi'_v)$ be an order in $D(F_v)$ of reduced discriminant $\mathfrak{p}_v^{c(\pi_v)}$ such that $R(\pi'_v) \cap L_v = \mathfrak{o}_v + \varpi_v^{c(\Omega_v)} \mathfrak{o}_{L_v}$ (see [Gross 1988, Proposition 3.4]). Note $R(\pi'_v)$ is unique up to $T(F_v)$-conjugacy. In this case, we take f_v to be the characteristic function of $R(\pi'_v)^\times$ divided by its volume. Then $\pi'_v(f_v)$ acts as orthogonal projection onto the local Gross–Prasad test vector ϕ_v [Gross and Prasad 1991], except in the case that $c(\pi_v) \geq 2$ and L_v/F_v is ramified. (Note [Gross and Prasad 1991] also assumes F_v has odd residual characteristic if π_v is supercuspidal because of this restriction in [Tunnell 1983], but this hypothesis is no longer needed due to [Saito 1993].) When $c(\Omega_v) = 0$, $c(\pi_v) \geq 2$ and L_v/F_v is ramified, $\pi'_v(f_v)$ acts as orthogonal projection onto a 2-dimensional space containing a vector ϕ_v which satisfies $\pi'_v(t_v)\phi_v = \Omega_v(t_v)\phi_v$ for all $t_v \in T(F_v)$ [Gross and Prasad 1991, Remark 2.7]; hence on this space any linear form in $\mathrm{Hom}(\pi_v, \Omega_v)$ is simply a multiple of the map $\phi'_v \mapsto (\phi'_v, \phi_v)$.

If $0 < c(\pi_v) \leq c(\Omega_v)$, take g_v so that $g_v^{-1} T(F_v) g_v$ is of the form (2-16), and let K_v be such that $g_v K_v g_v^{-1}$ is the subgroup in (6-3). Let f_v be the characteristic function of K_v divided by its volume, so $\pi_v(f_v)$ acts as orthogonal projection onto the line generated by ϕ_v, the unique (up to scalar multiples) vector in π_v fixed by K_v.

Lastly, suppose v is an infinite place of F. Let K_v be a maximal compact subgroup of $G'(F_v)$ whose restriction to $T(F_v)$ remains maximal compact. Let ϕ_v be a vector of minimal weight such that $\pi'_v(t_v)\phi_v = \Omega_v(t_v)\phi_v$ for $t_v \in K_v \cap T(F_v)$. Choose f_v so that $\pi'_v(f_v)$ is orthogonal projection onto $\langle \phi_v \rangle$.

Take $f = \prod f_v$ and $\phi = \otimes \phi_v$, so $\pi(f)$ acts as orthogonal projection onto a finite-dimensional space V containing ϕ. Local considerations show the toric period integral vanishes on the orthogonal complement of $\langle \phi \rangle$ in V, and hence one has (7-2).

7B. ***Archimedean factors.*** Here we recall from [Martin and Whitehouse 2009] certain archimedean constants $C_v(L, \pi, \Omega)$. Let v be an infinite place of F. By assumption, Ω_v is a unitary character of L_v.

First suppose $F_v = \mathbb{R}$ and $L_v = \mathbb{R} \oplus \mathbb{R}$. Write

$$\Omega_v(x_1, x_2) = \left| \frac{x_1}{x_2} \right|^{it} \mathrm{sgn}^{m_v}\left(\frac{x_1}{x_2} \right),$$

where $t \in \mathbb{R}$ and m_v is 0 or 1. If $\pi_v = \mu_v \times \mu_v^{-1}$ is a principal series with Laplacian eigenvalue λ_v, let $\epsilon_v \in \{0, 1\}$ such that $\mu_v \Omega_v = |\cdot|^r \mathrm{sgn}^{\epsilon_v}$ for some r. Then we put

$$C_v(L, \pi, \Omega) = \left(\frac{8\pi^2}{\lambda_v} \right)^{\epsilon_v}.$$

If π is a discrete series of weight k_v, put

$$C_v(L, \pi, \Omega) = 2^{k_v}.$$

Now suppose $F_v = \mathbb{R}$ and $L_v = \mathbb{C}$. Write $\Omega_v(z) = (z/\bar{z})^{\pm m_v}$, where $m_v \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. If $\pi_v = \mu_v \times \mu_v^{-1}$ is a principal series where μ_v is of the form $|\cdot|^{r_v}\operatorname{sgn}^{\epsilon_v}$, then

$$C_v(L, \pi, \Omega) = (2\pi)^{2m_v} \prod_{j=0}^{m_v-1} (\lambda_v + j(j+1))^{-1},$$

where $\lambda_v = \frac{1}{4} - r_v^2$. If π_v is a discrete series of weight k_v, then

$$C_v(L, \pi, \Omega) = \frac{1}{\pi B(k_v/2 + m_v, k_v/2 - m_v)}$$

if $m_v < \frac{1}{2}(k_v - 1)$ and

$$C_v(L, \pi, \Omega) = \frac{(2\pi)^{2m_v - k_v} k_v!}{m_v! B(k_v/2 + m_v, 1 - k_v/2 + m_v)}$$

if $m_v \geq \frac{1}{2}(k_v - 1)$. Here $B(x, y)$ denotes the beta function.

Lastly suppose $F_v = \mathbb{C}$, so $L_v = \mathbb{C} \oplus \mathbb{C}$. Write Ω_v in the form

$$\Omega_v(z_1, z_2) = (z_1\bar{z}_1)^{it} \left(\frac{z_1}{\bar{z}_1}\right)^{m_v} (z_2\bar{z}_2)^{-it} \left(\frac{z_2}{\bar{z}_2}\right)^{-m_v},$$

where $t \in \mathbb{R}$ and $m_v \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then π_v is a principal series. Let k_v be its weight, λ_v the Laplacian eigenvalue and $\ell_v = \max(k_v, m_v)$. Then

$$C_v(L, \pi, \Omega) = \left(\frac{1}{2} + \ell_v\right) \binom{2\ell_v}{|k_v - m_v|} \prod_{j=k_v+1}^{\ell_v} \frac{4\pi^2}{4\lambda_v + j^2 - 1}.$$

7C. *Proof of Theorem 1.1.* We consider a measure on $G'(\mathbb{A})$ which is the product of local Tamagawa measures. Write $\Delta = \Delta_{\text{inert}}\Delta_{\text{split}}$, where Δ_{inert} is the part of Δ coprime to every place over which L/F splits. Then note that

$$\begin{aligned}\prod_{v \in S_{\text{inert}}} 2\epsilon(1, \eta_v, \psi_v) L(0, \eta_v) &= \frac{1}{\sqrt{c(\eta)c(\psi)}} \prod_{v \in S_{\text{inert}}} e(L_v/F_v) \\ &= \sqrt{\frac{\Delta_{\text{inert}}}{\Delta_L}} \prod_{v \in S_{\text{inert}}} e(L_v/F_v).\end{aligned}$$

Let $v \in S$ be finite. The calculations of $\tilde{J}_{\pi'_v}(f_v)$ below for when L_v/F_v is split, v is infinite, or at most one of π_v and Ω_v is ramified are taken from [Martin and Whitehouse 2009].

Suppose $L_v = F_v \oplus F_v$. Then

$$\tilde{J}_{\pi'_v}(f_v) = \begin{cases} q_v^{-c(\Omega_v)} \dfrac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)}{(W_{\pi_v}, W_{\pi_v})} & \text{if } \Omega_v \text{ is unramified,} \\ q_v^{-c(\Omega_v)} \dfrac{L(1, 1_{F_v})^2}{(W_{\pi_v}, W_{\pi_v})} & \text{if } \Omega_v \text{ is ramified,} \end{cases}$$

where W_{π_v} is the normalized Whittaker newvector. Furthermore,

$$\mathrm{vol}(\mathfrak{o}_v^\times)(W_{\pi_v}, W_{\pi_v}) = \begin{cases} L(1, \pi_v, \mathrm{Ad})L(1, 1_{F_v})/L(2, 1_{F_v}) & \text{if } \pi_v \text{ is unramified,} \\ L(1, \pi_v, \mathrm{Ad}) = L(2, 1_{F_v}) & \text{if } c(\pi_v) = 1, \\ 1 & \text{if } c(\pi_v) > 1. \end{cases}$$

Since we are using local Tamagawa measures, the product over all such v of $\mathrm{vol}(\mathfrak{o}_v^\times)$ is $\sqrt{\Delta_{\mathrm{split}}}$.

Suppose now L_v/F_v is inert. If π'_v is unramified, then $\tilde{J}_{\pi'_v}(f_v)$ is

$$\frac{q_v^{-c(\Omega_v)}}{e(L_v/F_v)} \frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})} L(1, \eta_v)^{\delta_v},$$

where $\delta_v = -1$ if Ω_v is unramified and $\delta_v = 1$ if Ω_v is ramified. If π_v is ramified and Ω_v is unramified, then $\tilde{J}_{\pi'_v}(f_v) = 1$. When both π_v and Ω_v are ramified, $\tilde{J}_{\pi'_v}(f_v)$ is calculated in Proposition 6.3.

Summing up, if π_v is unramified, then, up to factors of the form $\mathrm{vol}(\mathfrak{o}_v^\times)$ and $e(L_v/F_v)$, $\tilde{J}_{\pi'_v}(f_v)$ is

$$q^{-c(\Omega_v)} \frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})} L(1, \eta_v)^{\delta_v}.$$

If $c(\pi_v) = 1$, then, up to factors of the form $\mathrm{vol}(\mathfrak{o}_v^\times)$ and $e(L_v/F_v)$, $\tilde{J}_{\pi'_v}(f_v)$ is

$$\frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)}{L(1, \pi_v, \mathrm{Ad})}$$

if Ω_v is unramified and L_v/F_v is split or unramified; 1 if Ω_v is unramified and L_v/F_v is ramified; and

$$q^{-c(\Omega_v)} \frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)}{L(1, \pi_v, \mathrm{Ad})} L(1, 1_{F_v})L(1, \eta_v)$$

if Ω_v is ramified.

If $c(\pi_v) \geq 2$, then, up to factors of the form $\mathrm{vol}(\mathfrak{o}_v^\times)$ and $e(L_v/F_v)$, $\tilde{J}_{\pi'_v}(f_v)$ is 1 if Ω_v is unramified and $q^{-c(\Omega_v)}L(1, 1_{F_v})L(1, \eta_v)$ if Ω_v is ramified.

Now suppose $v \mid \infty$. Then from [Martin and Whitehouse 2009] one has

$$\tilde{J}_{\pi'_v}(f_v) = \frac{C_v(L, \pi, \Omega)}{e(L_v/F_v)} \frac{L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})L(1, \eta_v)}.$$

Combining the above calculations completes the proof of Theorem 1.1.

Remark 7.2. When $S(\pi)\cap S(\Omega)=\varnothing$, Theorem 1.1 is exactly the main theorem of [Martin and Whitehouse 2009], though their choice of measure on $G'(\mathbb{A})$ is slightly different. Our set $S_0(\pi)$ is denoted by $S'(\pi)$ in that paper.

As in [Martin and Whitehouse 2009], one can rewrite this formula using the Petersson norm (ϕ_π, ϕ_π) of the new vector $\phi_\pi \in \pi$ instead of $L(1, \pi, \mathrm{Ad})$. The formula in [Martin and Whitehouse 2009] is also valid when $\omega_\pi = \eta$, and one could treat that case here similarly. The restriction that $\omega_\pi \in \{1, \eta\}$ is not inherent in the method, but is due to this assumption in [Jacquet and Chen 2001].

Remark 7.3. For many applications, one would like a formula for the *complete* ratio of L-values $L\big(\frac{1}{2}, \pi_L \otimes \Omega\big)/L(1, \pi, \mathrm{Ad})$. Theorem 1.1 of course gives this when $S_0=\varnothing$ (e.g., if the conductor $c(\pi)$ of π is squarefree and π and L/F have disjoint ramification). In general, one can of course multiply both sides by the appropriate local factors, but then the rest of the formula will depend on more than just the ramification of π and Ω together with their infinity types. Specifically, for $v \in S_1(\pi)$ and $\pi_v = \chi_v \,\mathrm{St}_v$, the local factor $L\big(\frac{1}{2}, \pi_{L_v} \otimes \Omega_v\big)$ depends on the sign of χ_v when L_v/F_v is ramified. Similarly, for $v \in S_2(\pi)$, the local factor $L(1, \pi_v, \mathrm{Ad})$ depends on more than just the ramification of π_v.

8. An average-value formula

In this section, we prove Theorems 1.3, 1.4 and 1.5. Fix notation as in the first paragraph of Theorem 1.3.

8A. ***The trace formula.*** Let D/F be the quaternion algebra which is ramified precisely at the infinite primes and the primes dividing $\mathfrak{N}_0$. Set $G' = D^\times$ and $G=\mathrm{GL}(2)/F$. Let Z denote the center of either of these. Let ϵ be an element of the normalizer of $T(F)$ inside $G'(F)$ which does not lie in $T(F)$, so $\epsilon^2 \in Z(F)$ and $D(F)=L\oplus\epsilon L$. Then we may write an element of $G'(F)$ in the form

$$\begin{bmatrix} \alpha & \beta\epsilon \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, \quad \alpha, \beta \in L.$$

With this representation,

$$T=\left\{\begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}\right\}.$$

As in Section 7, let ψ be the standard additive character of $\mathbb{A}/F$, and take the product of the local Tamagawa measures on $T(\mathbb{A})$, $G'(\mathbb{A})$, $G(\mathbb{A})$ and $Z(\mathbb{A})$. For a cuspidal automorphic representation π' of $G'(\mathbb{A})$, let $\mathrm{JL}(\pi')$ denote its Jacquet–Langlands transfer to $G(\mathbb{A})$. Denote by $\mathcal{F}'(\mathfrak{N}, 2\boldsymbol{k})$ the set of cuspidal automorphic representations π' of $G'(\mathbb{A})$ such that $\mathrm{JL}(\pi') \in \mathcal{F}(\mathfrak{N}, 2\boldsymbol{k})$. We call $\mathfrak{N}$ the conductor of π' and write $c(\pi')=\mathfrak{N}$. Subject to assumption (1-3), we note that our choice of D guarantees $\mathrm{Hom}_T(\pi',\Omega)\neq 0$ for all $\pi'\in\mathcal{F}'(\mathfrak{N},2\boldsymbol{k})$.

We now recall Jacquet's relative trace formula for G' from [Jacquet 1987]. This is an identity of the form

$$I(f)=J(f), \tag{8-1}$$

where $I(f)$ is a certain geometric distribution, and $J(f)$ is a certain spectral distribution. Specifically, let $f=\prod f_v\in C_c^\infty(G'(\mathbb{A}))$. The geometric (relative) orbital integrals of f are defined by

$$I(0,f)=\int_{T(\mathbb{A})} f(t)\Omega(t)\,dt,$$

$$I(\infty,f)=\int_{T(\mathbb{A})} f\Bigl(t\begin{bmatrix}0&\epsilon\\1&0\end{bmatrix}\Bigr)\Omega(t)\,dt$$

and

$$I(b,f)=\int_{T(\mathbb{A})/Z(\mathbb{A})}\int_{T(\mathbb{A})} f\biggl(s\begin{bmatrix}1&\epsilon\beta\\\bar{\beta}&1\end{bmatrix}t\biggr)\Omega(st)\,ds\,dt,$$

where $b=\epsilon N(\beta)$ for $\beta\in L^\times$. Note this latter integral only depends on b and not the choice of a specific β. Then the left-hand (geometric) side of (8-1) is

$$I(f)=\mathrm{vol}(T(\mathbb{A})/Z(\mathbb{A})T(F))(I(0,f)+\delta(\Omega^2)I(\infty,f))+\sum_{b\in\epsilon N(L^\times)} I(b,f), \tag{8-2}$$

where $\delta(\chi)=1$ if χ is trivial and $\delta(\chi)=0$ otherwise.

We now describe $J(f)$, but for simplicity only in the situation that is relevant for us. Namely, for each $v\mid\infty$, fix an embedding $\iota_v: G'(F_v)\hookrightarrow \mathrm{GL}_2(\mathbb{C})$ and let π'_{2k_v} be the irreducible $(2k_v-1)$-dimensional representation of $G'(F_v)$ given by $\pi'_{2k_v}=(\mathrm{Sym}^{2k_v-2}\otimes\det^{1-k_v})\circ\iota_v$. Hence $\mathrm{JL}(\pi'_{2k_v})$ is the holomorphic discrete series of weight $2k_v$ on $G(F_v)$. The assumption that $|m_v|<k_v$ implies that there is a 1-dimensional subspace of π'_{2k_v} consisting of vectors w_v such that $\pi'_{2k_v}(t)w_v=\Omega_v(t)w_v$ for all $t\in T(F_v)$. Fix such a vector $w_v\in\pi'_{2k_v}$ which satisfies $(w_v,w_v)=1$. For all $v\mid\infty$, we may take $f_v\in C_c^\infty(G'(\mathbb{R}))$ as in Section 7A, so that

$$\int_{Z(F_v)} f_v(zg)\,dz=\frac{2k_v-1}{\mathrm{vol}(G'(F_v)/Z(F_v))}\overline{(\pi'_{2k_v}(g)w_v,w_v)}$$

(cf. [Feigon and Whitehouse 2009, Lemma 3.4]).

For a cuspidal automorphic representation π' of $G'(\mathbb{A})/Z(\mathbb{A})$, we consider the spectral distribution

$$J_{\pi'}(f)=\sum_\phi P_D(\pi'(f)\phi)\overline{P_D(\phi)},$$

where ϕ runs over an orthonormal basis for π' and P_D is defined as in (1-1). In general, the spectral side $J(f)$ of (8-1) is a sum over all π' of $J_{\pi'}(f)$ plus a noncuspidal contribution. However, things simplify greatly for our choice of f.

We already specified f_v for $v \mid \infty$. Now let $v < \infty$ and put $m_v = c(\Omega_v)$. For such a v, as in Section 7A, we take f_v to be the characteristic function of $R_v^\times$ divided by its volume, for an order R_v of $G'(F_v)$ chosen as follows. If $v \nmid \mathfrak{N}$, then $G'(F_v) \cong G(F_v)$ and we take R_v to be a maximal order optimally containing $\mathfrak{o}_v + \varpi_v^{m_v}\mathfrak{o}_{L_v}$. If $v \mid \mathfrak{N}_0$, then $G'(F_v)$ is not split and we take R_v to be a maximal order containing $\mathfrak{o}_{L_v}$. If $v \mid \mathfrak{N}_1$, then $G'(F_v) \cong G(F_v)$ and, at least when v is odd, we can take

$$R_v = \left\{ \begin{bmatrix} \alpha & \beta\epsilon_v \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} : \operatorname{Tr}(\alpha), \operatorname{Tr}(\beta) \in \mathfrak{o}_v,\ \alpha, \beta \in \mathfrak{p}_v^{1-m_v}\mathfrak{o}_{L_v}, \right.$$
$$\left. \text{and } \alpha - \beta \in \mathfrak{o}_v + \mathfrak{p}_v^{m_v}\mathfrak{o}_{L_v} \right\}. \quad (8\text{-}3)$$

Note that for each $v \nmid \mathfrak{N}_0$, this agrees with our choice of test functions in Section 7A. The difference of the present choice of f_v for $v \mid \mathfrak{N}_0$ is simply out of convenience so we can directly apply local calculations from [Feigon and Whitehouse 2009]. What is important is that one still has $\pi_v(f_v)$ being orthogonal projection onto our local test vector for $v \mid \mathfrak{N}_0$ (cf. [Feigon and Whitehouse 2009, Lemma 3.3]).

Consequently, for this f, assuming $k_v > 1$ for some $v \mid \infty$, the spectral side of (8-1) is given by

$$J(f) = \sum_{\mathfrak{N}'} \sum_{\pi' \in \mathcal{F}'(\mathfrak{N}', 2\boldsymbol{k})} J_{\pi'}(f), \quad (8\text{-}4)$$

where $\mathfrak{N}'$ runs over ideals which divide $\mathfrak{N}$ and are divisible by $\mathfrak{N}_0$. This is because, for our choice of f', $\pi'(f')$ is zero unless π' is of weight $2\boldsymbol{k}$ and has conductor dividing $\mathfrak{N}$. Furthermore, by our choice of D, $J_{\pi'}(f)$ vanishes for local reasons if the conductor of π' is not divisible by $\mathfrak{N}_0$ (cf. [Feigon and Whitehouse 2009, Lemmas 3.6 and 3.7]). (The avoidance of the case $k_v = 1$ for all $v \mid \infty$ is purely for simplicity, for in this case there is also contribution from the residual spectrum, which one would treat as in [Feigon and Whitehouse 2009].)

8B. ***Spectral calculations.*** Here we compute the spectral expansion (8-4). For $\pi' \in \mathcal{F}'(\mathfrak{N}, 2\boldsymbol{k})$, we see that $J_{\pi'}(f) = |P_D(\phi)|^2/(\phi, \phi)$. Hence Theorem 1.1 implies

$$J_{\pi'}(f) = \frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}}\, L^{S(\mathfrak{N}_0)}(2, 1_F) L_{S(\mathfrak{N})}(1, \eta) L_{S(\mathfrak{C}_0)}(1, \eta)^2$$
$$\times \prod_{v \mid \infty} \frac{2k_v - 1}{\pi} \binom{2k_v - 2}{k_v - m_v - 1} \frac{L\left(\frac{1}{2}, \pi_L \otimes \Omega\right)}{L(1, \pi, \mathrm{Ad})}. \quad (8\text{-}5)$$

We now need to extend this equality to general $\pi' \in \mathcal{F}'(\mathfrak{N}', 2\boldsymbol{k})$, where $\mathfrak{N}'$ divides $\mathfrak{N}$ and is divisible by $\mathfrak{N}_0$. For $v \mid (\mathfrak{N}')^{-1}\mathfrak{N}$, let R'_v be the maximal order of

$G'(F_v) \cong G(F_v)$ which contains R_v given by (8-3). Let $f' = \prod f'_v$, where $f'_v = f_v$ if $v \nmid (\mathfrak{N}')^{-1}\mathfrak{N}$, and f'_v is the characteristic function of $(R'_v)^\times$ divided by its volume if $v \mid (\mathfrak{N}')^{-1}\mathfrak{N}$. Now, f' agrees with our choice of test function for π' in Section 7A, and Theorem 1.1 gives

$$J_{\pi'}(f') = \frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}}\, L^{S(\mathfrak{N}_0)}(2, 1_F) L_{S(\mathfrak{N})}(1,\eta) L_{S(\mathfrak{C}_0)}(1,\eta)^2 \\ \times \prod_{v \mid (\mathfrak{N}')^{-1}\mathfrak{N}} L(1,\eta_v) \prod_{v\mid\infty} \frac{2k_v - 1}{\pi}\binom{2k_v-2}{k_v - m_v - 1} \frac{L\left(\frac{1}{2}, \pi_L\otimes\Omega\right)}{L(1,\pi,\mathrm{Ad})}. \tag{8-6}$$

From Theorem 7.1, we see that

$$J_{\pi'}(f) = J_{\pi'}(f') \prod_{v\mid(\mathfrak{N}')^{-1}\mathfrak{N}} \frac{\tilde{J}_{\pi'_v}(f_v)}{\tilde{J}_{\pi'_v}(f'_v)}.$$

From [Martin and Whitehouse 2009, Section 2.2.4], we know

$$\tilde{J}_{\pi'_v}(f'_v) = q_v^{-m_v} L(2, 1_{F_v}) L(1,\eta_v) \frac{1}{L(1,\pi_v,\mathrm{Ad})},$$

so it remains to compute $\tilde{J}_{\pi'_v}(f_v)$. Here $v \mid (\mathfrak{N}')^{-1}\mathfrak{N} \supset \mathfrak{N}_1$, so $\pi'_v = \pi_v$ is unramified and $m_v = 1$. We may write $\pi_v = \chi \times \chi^{-1}$, where $\chi = \chi_v$ is an unramified (unitary) character of $F_v^\times$.

Note $\pi_v(f_v)$ is orthogonal projection onto $\pi_v^{R_v^\times}$. Embedding L_v in $M_2(F_v)$ as in (2-16), we may write

$$R_v^\times = K_v := \begin{bmatrix} \mathfrak{o}_v^\times & \mathfrak{p}_v^{m_v} \\ \mathfrak{p}_v^{1-m_v} & \mathfrak{o}_v^\times \end{bmatrix} = \begin{bmatrix} \mathfrak{o}_v^\times & \mathfrak{p}_v \\ \mathfrak{o}_v & \mathfrak{o}_v^\times \end{bmatrix}.$$

Note

$$K_v = h_v \mathrm{GL}_2(\mathfrak{o}_v) h_v^{-1} \cap h_v \begin{bmatrix} \varpi_v^{-1} & \\ & 1 \end{bmatrix} \mathrm{GL}_2(\mathfrak{o}_v) \begin{bmatrix} \varpi_v & \\ & 1 \end{bmatrix} h_v^{-1},$$

where $h_v = \left[\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right]$. So if we put ϕ_0 to be a newvector in π_v and $\phi'_0 = \pi_v(h_v^{-1})\phi_0$, then

$$\pi_v^{K_v} = \langle \pi_v(h_v)\phi_0,\ \pi_v(h_v)\phi'_0 \rangle.$$

Normalize ϕ_0 so that $(\phi_0, \phi_0) = 1$.

Lemma 8.1. *We have*

$$(\phi_0, \phi'_0) = (\phi'_0, \phi_0) = \frac{q_v^{-1/2}}{1 + q_v^{-1}} (\chi(\varpi_v) + \chi(\varpi_v)^{-1}). \tag{8-7}$$

Proof. In the induced model for π_v, we have

$$(\phi'_0, \phi_0) = (\pi(h_v)\phi_0, \phi_0) = \int_{\mathrm{GL}_2(\mathfrak{o}_v)} \phi_0(k h_v)\, dk.$$

We may then use the fact that the subgroup K_v of $\mathrm{GL}_2(\mathfrak{o}_v)$ is normalized by h_v to get the lemma. □

Lemma 8.2. *For $v \mid (\mathfrak{N}')^{-1}\mathfrak{N}$, so that π_v is unramified and $m_v = 1$, we have $\tilde{J}_{\pi_v}(f_v) = q_v^{-1}$.*

Proof. Write $\phi_1 = \pi_v(h_v)\phi_0$ and

$$\phi_2 = \frac{\phi_0 - (\phi_0, \phi_1)\phi_1}{(1 - (\phi_0, \phi_0')^2)^{1/2}} = \left(\frac{L(1, \pi_v, \mathrm{Ad})(1 + q_v^{-1})}{L(2, 1_{F_v})}\right)^{1/2} (\phi_0 - (\phi_0, \phi_1)\phi_1),$$

so that $\{\phi_1, \phi_2\}$ forms an orthonormal basis for $\pi_v^{K_v}$. As in Section 6, put

$$e' = \sum_{t \in T_v/(T_v \cap Z_v K_v)} \Omega_v^{-1}(t)\pi_v(t)\phi_1,$$

so

$$\tilde{J}_{\pi_v}(f_v) = \mathrm{vol}(K_v)^{-1} \int_{K_v} \frac{(\pi_v(k)e', e')}{(e', e')}\, dk = \frac{1}{|T_v/(T_v \cap Z_v K_v)|(\phi_1, e')} (\pi_v(f_v)e', e').$$

Since $\pi_v(f_v)e' = (e', \phi_1)\phi_1 + (e', \phi_2)\phi_2$, we have

$$\tilde{J}_{\pi_v}(f_v) = \frac{1}{|T_v/(T_v \cap Z_v K_v)|(\phi_1, e')} \big((e', \phi_1)(\phi_1, e') + (e', \phi_2)(\phi_2, e')\big).$$

From [Martin and Whitehouse 2009, Section 2.2.4], where (ϕ_1, e') is denoted $\langle v_0, e_T''\rangle / \langle v_0, v_0\rangle$, we know $(\phi_1, e') = L(2, 1_{F_v})/L(1, \pi_v, \mathrm{Ad})$. Thus

$$\tilde{J}_{\pi_v}(f_v) = \frac{1}{|T_v/(T_v \cap Z_v K_v)|}\left(\frac{L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})} + \frac{L(1, \pi_v, \mathrm{Ad})}{L(2, 1_{F_v})}|(\phi_2, e')|^2\right). \tag{8-8}$$

Note

$$(\phi_2, e') = \left(\frac{L(1, \pi_v, \mathrm{Ad})(1 + q_v^{-1})}{L(2, 1_{F_v})}\right)^{1/2} \left((\phi_0, e') - (\phi_0', \phi_0)\frac{L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})}\right). \tag{8-9}$$

Hence it suffices to compute

$$\begin{aligned}(\phi_0, e') &= \sum_{t \in T_v/(T_v \cap Z_v K_v)} \Omega_v(t)(\phi_0, \pi_v(t)\phi_1) \\ &= \sum_{t \in T_v/(T_v \cap Z_v K_v)} \Omega_v^{-1}(t)(\pi_v(h_v^{-1} t h_v)\phi_0', \phi_0).\end{aligned}$$

Using the set of representatives for $T_v/(T_v \cap Z_v K_v)$ given in Lemma 6.2, we see

$$\begin{aligned}(\phi_0, e') = \sum_{x \in \mathfrak{p}_v/\mathfrak{p}_v} \Omega_v^{-1}(x + \xi_{0,v})\left(\pi_v\left(\begin{bmatrix} \varpi_v^{-1}x & \boldsymbol{c}\varpi_v^{-1} \\ -\boldsymbol{a} & x - \boldsymbol{b}\end{bmatrix}\right)\phi_0, \phi_0\right) \\ + \sum_{y \in \mathfrak{o}_v/\mathfrak{p}_v} \Omega_v^{-1}(1 + y\xi_{0,v})\left(\pi_v\left(\begin{bmatrix} \varpi_v^{-1} & \boldsymbol{c}\varpi_v^{-1}y \\ -\boldsymbol{a}y & 1 - \boldsymbol{b}y\end{bmatrix}\right)\phi_0, \phi_0\right).\end{aligned} \tag{8-10}$$

Let ϕ and ϕ' be the realizations of the unit newvectors ϕ_0 and ϕ_0', respectively, in the Kirillov model for π_v with respect to an unramified ψ_v. Recall $\phi(z) = 0$ unless $z \in \mathfrak{o}_v$. Recall also the action of the standard Borel on the Kirillov model is given by

$$\left(\pi_v\begin{bmatrix} a & x \\ & d \end{bmatrix}\phi\right)(z) = \psi_v(xz/d)\phi(az/d).$$

Since v is odd and unramified, we may assume $\boldsymbol{b} = 0$ and $\boldsymbol{a}$ is a unit. For the $x = 0$ term in (8-10), note

$$\pi_v\begin{bmatrix} & \boldsymbol{c}\varpi_v^{-1} \\ -\boldsymbol{a} & \end{bmatrix}\phi(z) = \pi_v\begin{bmatrix} \boldsymbol{c}\varpi_v^{-1} & \\ & \boldsymbol{a} \end{bmatrix}\phi(z) = \phi(\varpi_v^{-1}z) = \phi'(z).$$

For $y \in \mathfrak{o}_v$, we have

$$\begin{aligned}\pi_v\left(\begin{bmatrix} \varpi_v^{-1} & \boldsymbol{c}\varpi_v^{-1}y \\ -\boldsymbol{a}y & 1 \end{bmatrix}\right)\phi(z) &= \pi_v\left(\begin{bmatrix} \varpi^{-1}(1+\boldsymbol{ac}y^2) & \boldsymbol{c}\varpi_v^{-1}y \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -\boldsymbol{a}y & 1 \end{bmatrix}\right)\phi(z) \\ &= \psi_v(\boldsymbol{c}\varpi_v^{-1}yz)\phi(\varpi_v^{-1}z) = \phi(\varpi_v^{-1}z) = \phi'(z).\end{aligned}$$

In the last line, we used the facts that $1+\boldsymbol{ac}y^2 \in \mathfrak{o}_v^\times$, ψ is unramified and ϕ vanishes outside of $\mathfrak{o}_v$. Hence (8-10) becomes

$$(\phi_0, e') = \left(\Omega_v^{-1}(\xi_{0,v}) + \sum_{y\in\mathfrak{o}_v/\mathfrak{p}_v} \Omega_v^{-1}(1+y\xi_{0,v})\right)(\phi_0', \phi_0) = 0, \tag{8-11}$$

as this character sum is zero. Combining (8-7), (8-8) and (8-9) gives

$$\begin{aligned}\tilde{J}_{\pi_v}(f_v) &= \frac{1}{|T_v/(T_v\cap Z_vK_v)|}\left(\frac{L(2, 1_{F_v})}{L(1, \pi_v, \mathrm{Ad})} + \frac{q_v^{-1}}{1+q_v^{-1}}\big(\chi(\varpi_v)+\chi(\varpi_v)^{-1}\big)^2\right) \\ &= \frac{1+q_v^{-1}}{|T_v/(T_v\cap Z_vK_v)|}.\end{aligned}$$

This, with Lemma 6.2, gives the result. □

Hence,

$$\frac{\tilde{J}_{\pi_v'}(f_v)}{\tilde{J}_{\pi_v'}(f_v')} = \frac{L(1, \pi_v, \mathrm{Ad})}{L(2, 1_{F_v})L(1, \eta_v)} \tag{8-12}$$

for $v \mid (\mathfrak{N}')^{-1}\mathfrak{N}$, which yields

$$\begin{aligned}J(f) = {}& \\ &\frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}}\, L^{S(\mathfrak{N}_0)}(2, 1_F)L_{S(\mathfrak{N})}(1, \eta)L_{S(\mathfrak{C}_0)}(1, \eta)^2 \prod_{v|\infty}\frac{2k_v-1}{\pi}\binom{2k_v-2}{k_v-m_v-1} \\ &\times\sum_{\mathfrak{N}'}\sum_{\pi\in\mathcal{F}(\mathfrak{N}',2\boldsymbol{k})}\left(\prod_{v|(\mathfrak{N}')^{-1}\mathfrak{N}}\frac{L(1, \pi_v, \mathrm{Ad})}{L(2, 1_{F_v})}\right)\frac{L\big(\frac{1}{2}, \pi_L\otimes\Omega\big)}{L(1, \pi, \mathrm{Ad})}.\end{aligned}$$

Here $\mathfrak{N}'$ runs over all divisors of $\mathfrak{N}$ which are divisible by $\mathfrak{N}_0$. Writing

$$\prod_{v \mid (\mathfrak{N}')^{-1}\mathfrak{N}} \frac{L(1,\pi_v,\mathrm{Ad})}{L(2,1_{F_v})} \cdot \frac{1}{L(1,\pi,\mathrm{Ad})} = \prod_{v \mid \mathfrak{N}'} \frac{L(2,1_{F_v})}{L(1,\pi_v,\mathrm{Ad})} \cdot \frac{1}{L_{S(\mathfrak{N})}(2,1_F)L^{S(\mathfrak{N})}(1,\pi,\mathrm{Ad})}$$

and observing $L(1,\pi_v,\mathrm{Ad}) = L(2,1_{F_v})$ for $v \mid \mathfrak{N}'$ and $\pi \in \mathcal{F}(\mathfrak{N}',\boldsymbol{k})$ gives

$$J(f) = \frac{1}{2}\sqrt{\frac{\Delta}{c(\Omega)\Delta_L}} \frac{L^{S(\mathfrak{N}_0)}(2,1_F)}{L_{S(\mathfrak{N})}(1,1_F)} L_{S(\mathfrak{C}_0)}(1,\eta)^2 \times \prod_{v\mid\infty} \frac{2k_v-1}{\pi}\binom{2k_v-2}{k_v-m_v-1} \sum_{\mathfrak{N}'} \sum_{\pi\in\mathcal{F}(\mathfrak{N}',2\boldsymbol{k})} \frac{L\left(\frac{1}{2},\pi_L\otimes\Omega\right)}{L^{S(\mathfrak{N})}(1,\pi,\mathrm{Ad})}. \tag{8-13}$$

8C. *Geometric calculations.* We now obtain our average value formula from the trace formula (8-1) and spectral calculation (8-5) by computing the geometric side $I(f)$. Most of the calculations we need are done in [Feigon and Whitehouse 2009], with the proviso that our choice of test functions f_v (for $v \nmid \mathfrak{N}_1$) are essentially constant multiples of those therein (the test functions in [Feigon and Whitehouse 2009] also come "preintegrated over the center").

Lemma 8.3. *Let $b \in \epsilon N(L^\times)$. We have the following vanishing of local orbital integrals.*

(i) *If $v \mid \mathfrak{N}_0$, then $I(\infty, f_v) = 0$.*

(ii) *If $v \mid \mathfrak{N}_0$ and $b \notin \mathfrak{p}_v$, then $I(b, f_v) = 0$.*

(iii) *If $v \nmid \mathfrak{N}$ is finite and $v(1-b) > v(\mathfrak{d}_{L/F}c(\Omega))$, then $I(b, f_v) = 0$.*

(iv) *If $v \mid \mathfrak{N}_1$ and $v(1-b) > v(c(\Omega)) - 2$, then $I(b, f_v) = 0$.*

Proof. The first three results are directly from [Feigon and Whitehouse 2009, Lemmas 4.2, 4.10 and 4.11]. So suppose $v \mid \mathfrak{N}_1$ is odd and write $b = \epsilon N(\beta)$ for some $\beta \in L^\times$. For $I(b, f_v)$ to be nonzero we need that, for some $\alpha \in L_v^\times$ and $u \in L_v^1$,

$$\begin{bmatrix} \alpha & \\ & \bar{\alpha} \end{bmatrix} \begin{bmatrix} 1 & \epsilon\beta u \\ \beta u & 1 \end{bmatrix} \in R_v^\times,$$

i.e.,

$$N(\alpha)(1-b) \in \mathfrak{o}_v^\times, \quad \mathrm{Tr}(\alpha) \in \mathfrak{o}_v^\times, \quad \alpha \in \mathfrak{p}_v^{1-m_v}\mathfrak{o}_{L_v}, \quad \alpha(1-\beta u) \in \mathfrak{o}_v + \mathfrak{p}_v^{m_v}\mathfrak{o}_{L_v}.$$

Note this implies $v(1-b) = -v(N(\alpha)) \le 2m_v - 2$. Hence if $v(1-b) \ge 2m_v - 1$, then $I(b, f_v) = 0$. ☐

Proposition 8.4. *If $|\mathfrak{N}_0| > d_{L/F}(|\mathfrak{C}|/|\mathfrak{N}_1|)^{h_F}$, then $I(f) = 2L(1,\eta)I(0,f)$ and*

$$I(0,f) = \frac{\Delta^2|\mathfrak{N}|}{\sqrt{c(\Omega)\Delta_L}} \frac{L_{S(\mathfrak{C}_0)}(1,\eta)}{L_{S(\mathfrak{N}_0)}(1,1_F)} L(2,1_F) \prod_{v|\infty} \frac{2k_v-1}{2\pi}.$$

Proof. This argument is adapted from the proof of [Feigon and Whitehouse 2009, Lemma 4.21]. By the first part of the previous lemma, we know the global orbital integral $I(\infty, f) = 0$. Arguing as in Feigon and Whitehouse's proof, we see that, if $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{N}_1^{-2}\mathfrak{C}|^{h_F}$, then $I(b,f) = 0$ for all b.

Next we compute $I(0,f)$. For $v \nmid \mathfrak{N}_1$, we recall the following calculations from [Feigon and Whitehouse 2009, Section 4.1]; see [Jacquet and Chen 2001, Section 2; Feigon and Whitehouse 2009, Section 2.1 and proof of Proposition 4.20] for necessary facts about local Tamagawa measures. Due to the difference in our definition of test functions from those in [Feigon and Whitehouse 2009], our local orbital integrals $I(0,f_v)$ (for $v \nmid \mathfrak{N}_1$) will be $\mathrm{vol}(Z_v \cap R_v^\times)/\mathrm{vol}(R_v^\times)$ times theirs for finite v, and $(2k_v-1)/\mathrm{vol}(G'(F_v)/Z(F_v))$ times theirs for infinite v.

For $v \mid \mathfrak{N}_0$,

$$\begin{aligned} I(0,f_v) &= \mathrm{vol}(\mathfrak{o}_{L_v}^\times/\mathfrak{o}_v^\times)\mathrm{vol}(Z_v \cap R_v^\times)/\mathrm{vol}(R_v^\times) \\ &= (q_v-1)L(2,1_{F_v})\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(\mathfrak{o}_v^\times)^4, \end{aligned}$$

since $\mathrm{vol}(R_v^\times) = L(2,1_{F_v})^{-1}(q_v-1)^{-1}\mathrm{vol}(\mathfrak{o}_v^\times)^4$ and $R_v^\times \cap Z_v = \mathfrak{o}_v^\times$.

For a finite $v \nmid \mathfrak{N}$, we have $\mathrm{vol}(R_v^\times) = L(2,1_{F_v})^{-1}\mathrm{vol}(\mathfrak{o}_v^\times)^4$ and

$$I(0,f_v) = \begin{cases} \mathrm{vol}(\mathfrak{o}_{L_v}^\times/\mathfrak{o}_v^\times)\mathrm{vol}(Z_v \cap R_v^\times)/\mathrm{vol}(R_v^\times) & \\ \quad = L(2,1_{F_v})\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(\mathfrak{o}_v^\times)^4 & \text{for } m_v = 0, \\ q^{-m_v}L(1,\eta_v)\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(R_v^\times) & \\ \quad = q^{-m_v}L(1,\eta_v)L(2,1_{F_v})\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(\mathfrak{o}_v^\times)^4 & \text{for } m_v > 0. \end{cases}$$

For $v \mid \infty$,

$$I(0,f_v) = \mathrm{vol}(F_v^\times \backslash L_v^\times)\frac{2k_v-1}{\mathrm{vol}(G'(F_v)/Z(F_v))} = \frac{2k_v-1}{2\pi^2}.$$

Now, for $v \mid \mathfrak{N}_1$, our description of R_v readily implies

$$I(0,f_v) = \mathrm{vol}(\mathfrak{o}_v^\times(1+\mathfrak{p}_v^{m_v}\mathfrak{o}_{L_v}))/\mathrm{vol}(R_v^\times) = q^{-m_v}L(1,\eta_v)\mathrm{vol}(\mathfrak{o}_{L_v}^\times)/\mathrm{vol}(R_v)^\times.$$

A simple calculation gives $\mathrm{vol}(R_v^\times) = q_v^{-1}\mathrm{vol}(\mathfrak{o}^\times)^4/L(1,1_{F_v})$. Hence when $v \mid \mathfrak{N}_1$, we have

$$I(0,f_v) = q_v^{1-m_v}L(2,1_{F_v})\mathrm{vol}(\mathfrak{o}_L^\times)/\mathrm{vol}(\mathfrak{o}^\times)^4.$$

Putting together the nonarchimedean calculations gives

$$\prod_{v<\infty} I(0,f_v) = \frac{|\mathfrak{N}|}{\sqrt{c(\Omega)}} L_{\mathrm{fin}}(2,1_F) \prod_{v|\mathfrak{N}_0} L(1,1_{F_v})^{-1} \prod_{v|\mathfrak{C}_0} L(1,\eta_v) \prod_{v<\infty} \frac{\mathrm{vol}(\mathfrak{o}_{L_v}^\times)}{\mathrm{vol}(\mathfrak{o}^\times)^4}.$$

Noting that $\prod_{v<\infty} \mathrm{vol}(\mathfrak{o}^\times) = \Delta^{-1/2}$ (and similarly over L), we have

$$\prod_{v<\infty} I(0, f_v) = \frac{\Delta^2|\mathfrak{N}|}{\sqrt{c(\Omega)\Delta_L}} L_{\mathrm{fin}}(2, 1_F) \prod_{v|\mathfrak{N}_0} L(1, 1_{F_v})^{-1} \prod_{v|\mathfrak{C}_0} L(1, \eta_v).$$

Recalling that $L(2, 1_F) = L_{\mathrm{fin}}(2, 1_F)/\pi^d$, we see

$$I(0, f) = \frac{\Delta^2|\mathfrak{N}|}{\sqrt{c(\Omega)\Delta_L}} L(2, 1_F) \prod_{v|\mathfrak{N}_0} L(1, 1_{F_v})^{-1} \prod_{v|\mathfrak{C}_0} L(1, \eta_v) \prod_{v|\infty} \frac{2k_v - 1}{2\pi}. \quad \square$$

8D. *Proofs.*

Proof of Theorem 1.3. The result immediately follows from our above calculations of both sides of the equality $J(f) = I(f) = 2L(1, \eta)I(0, f)$. $\square$

Proof of Theorem 1.4. Suppose $\mathfrak{N}_1$ contains exactly one prime $\mathfrak{p}$. Put

$$\Sigma_{\mathfrak{N}}(\mathfrak{N}') = \prod_{v|\infty} \binom{2k_v - 2}{k_v - m_v - 1} \sum_{\pi \in \mathcal{F}(\mathfrak{N}', 2\boldsymbol{k})} \frac{L\big(\frac{1}{2}, \pi_L \otimes \Omega\big)}{L^{S(\mathfrak{N})}(1, \pi, \mathrm{Ad})}.$$

Then Theorem 1.3 reads

$$\Sigma_{\mathfrak{N}}(\mathfrak{N}_0) + \Sigma_{\mathfrak{N}}(\mathfrak{N}) = 2^{2-d}\Delta^{3/2}|\mathfrak{N}| L(1, 1_{F_\mathfrak{p}}) L_{S(\mathfrak{N}_0)}(2, 1_F) L^{S(\mathfrak{C}_0)}(1, \eta). \tag{8-14}$$

Applying our average value formula when $\mathfrak{N} = \mathfrak{N}_0$, we also see

$$\Sigma_{\mathfrak{N}_0}(\mathfrak{N}_0) = 2^{2-d}\Delta^{3/2}|\mathfrak{N}_0| L_{S(\mathfrak{N}_0)}(2, 1_F) L^{S(\mathfrak{C}_0)}(1, \eta) \tag{8-15}$$

if $|\mathfrak{N}_0| > d_{L/F}|\mathfrak{C}|^{h_F}$. (This is precisely [Feigon and Whitehouse 2009, Theorem 1.1].) For $\pi_\mathfrak{p}$ unramified, we have

$$L(1, 1_{F_\mathfrak{p}}) \frac{1}{1 + 2q_\mathfrak{p}^{-1} + q_\mathfrak{p}^{-2}} \le L(1, \pi_\mathfrak{p}, \mathrm{Ad}) \le L(1, 1_{F_\mathfrak{p}}) \frac{1}{1 - 2q_\mathfrak{p}^{-1} + q_\mathfrak{p}^{-2}},$$

which implies

$$\begin{aligned} L(1, 1_{F_\mathfrak{p}}) \frac{1}{(1 + q_\mathfrak{p}^{-1})^2} \Sigma_{\mathfrak{N}_0}(\mathfrak{N}_0) &\le \Sigma_{\mathfrak{N}}(\mathfrak{N}_0) \\ &\le L(1, 1_{F_\mathfrak{p}}) \frac{1}{(1 - q_\mathfrak{p}^{-1})^2} \Sigma_{\mathfrak{N}_0}(\mathfrak{N}_0). \end{aligned} \tag{8-16}$$

Combining the (in)equalities above gives

$$\begin{aligned} &\Sigma_{\mathfrak{N}}(\mathfrak{N}) \\ &\le 2^{2-d}\Delta^{3/2}|\mathfrak{N}_0| L(1, 1_{F_\mathfrak{p}}) L_{S(\mathfrak{N}_0)}(2, 1_F) L^{S(\mathfrak{C}_0)}(1, \eta) \left(|\mathfrak{p}| - \frac{1}{1 + 2|\mathfrak{p}|^{-1} + |\mathfrak{p}|^{-2}} \right), \end{aligned}$$

and a similar lower bound, which are precisely the bounds asserted in Theorem 1.4.

To get an asymptotic, we use a special case of [Feigon and Whitehouse 2009, Theorem 1.2], which is an asymptotic for

$$\sum_{\pi\in\mathcal{F}(\mathfrak{N}_0,2\boldsymbol{k})} \frac{L^{\mathfrak{p}}\big(\tfrac{1}{2},\pi_L\otimes\Omega\big)}{L^{\mathfrak{p}}(1,\pi,\mathrm{Ad})} = \prod_{v\mid\infty}\binom{2k_v-2}{k_v-m_v-1}^{-1}\frac{\Sigma_{\mathfrak{N}}(\mathfrak{N}_0)}{L_{S(\mathfrak{N}_0)}(2,1_F)}$$

as $|\mathfrak{N}_0|\to\infty$. (Note $L^{\mathfrak{p}}\big(\tfrac{1}{2},\pi_L\otimes\Omega\big)=L\big(\tfrac{1}{2},\pi_L\otimes\Omega\big)$ since Ω is ramified at $\mathfrak{p}$.) Specifically, [Feigon and Whitehouse 2009, Theorem 1.2] tells us

$$\Sigma_{\mathfrak{N}}(\mathfrak{N}_0)\sim 2^{2-d}\Delta^{3/2}|\mathfrak{N}_0|L(1,1_{F_\mathfrak{p}})L_{S(\mathfrak{N}_0)}(2,1_F)L^{S(\mathfrak{C}_0)}(1,\eta) \tag{8-17}$$

as $|\mathfrak{N}_0|\to\infty$ along a sequence of squarefree ideals $\mathfrak{N}_0$ coprime to $\mathfrak{C}$ satisfying our parity and ramification assumptions. Consequently, we have

$$\Sigma_{\mathfrak{N}}(\mathfrak{N})\sim 2^{2-d}\Delta^{3/2}|\mathfrak{N}_0|L(1,1_{F_\mathfrak{p}})L_{S(\mathfrak{N}_0)}(1,\eta)L^{S(\mathfrak{C}_0)}(1,\eta)(|\mathfrak{p}|-1).$$

This gives the asymptotic asserted in the theorem. □

8E. ***Nonvanishing mod p.*** Let $\pi\in\mathcal{F}(\mathfrak{N},2\boldsymbol{k})$, and let $\boldsymbol{f}$ be the corresponding normalized Hilbert modular newform of weight $2\boldsymbol{k}$ and level $\mathfrak{N}$ over F. As before, Ω is a unitary character of $\mathbb{A}_L^\times/L^\times\mathbb{A}_F^\times$ such that, for all $v\mid\infty$, $\Omega_v(z)=(z/\bar{z})^{\pm m_v}$ with $0\le m_v<k_v$. Put $\boldsymbol{m}=(m_1,\dots,m_d)$. Then Ω gives rise to a Hilbert modular form $\boldsymbol{g}$ over F of weight $\boldsymbol{m}+1=(m_1+1,\dots,m_d+1)$; see [Shimura 1978, Section 5]. Assume $m_1\equiv m_2\equiv\cdots\equiv m_d \mod 2$. This implies Ω is algebraic, so that the field of rationality $\mathbb{Q}(\boldsymbol{g})\subset\overline{\mathbb{Q}}$ [Shimura 1978, Proposition 2.8].

Put $k_0=\max_{v\mid\infty}k_v$ and $m_0=\max_{v\mid\infty}m_v$. Then Shimura [1978, Theorem 4.1] proved

$$\frac{D(s_0,\boldsymbol{f},\boldsymbol{g})}{\sqrt{\Delta}\,\pi^{2|\boldsymbol{k}|}(\boldsymbol{f},\boldsymbol{f})}\in\overline{\mathbb{Q}}(\boldsymbol{g})=\overline{\mathbb{Q}}$$

for any $s_0\in\mathbb{Z}$ such that $\tfrac{1}{2}(2k_0+m_0-1)<s_0<\tfrac{1}{2}(2k_0+m_0+2k_v-m_v)$ for all $v\mid\infty$. Here $D(s,\boldsymbol{f},\boldsymbol{g})$ is the Dirichlet series defined in [Shimura 1978], $(\boldsymbol{f},\boldsymbol{f})$ is the Petersson norm defined as in [Hida 1991], and $|\boldsymbol{k}|=\sum_{v\mid\infty}k_v$. Assume that $m_0\equiv 0 \mod 2$. Then, for $s_0=\tfrac{1}{2}(2k_0+m_0)$, this means

$$L^{\mathrm{alg}}\big(\tfrac{1}{2},\pi_L\otimes\Omega\big):=\frac{1}{L(1,\eta)}\frac{L_{\mathrm{fin}}\big(\tfrac{1}{2},\pi_L\otimes\Omega\big)}{\sqrt{\Delta}\,\pi^{2|\boldsymbol{k}|}(\boldsymbol{f},\boldsymbol{f})}\in\overline{\mathbb{Q}}. \tag{8-18}$$

(Note that we normalize the algebraic part of the L-value in a different way than other authors.) Recall the archimedean L-factors are given by

$$L_v\big(\tfrac{1}{2},\pi_L\otimes\Omega\big)=(2\pi)^{-2k_v}4\Gamma(k_v+m_v)\Gamma(k_v-m_v),\quad v\mid\infty.$$

From [Hida and Tilouine 1993, Theorem 7.1; Hida 1991, (7.2c)] (cf. [Getz and Goresky 2012, Theorem 5.16]), we have

$$L(1,\pi,\mathrm{Ad}) = \frac{2^{2|\boldsymbol{k}|-1}}{\Delta^2 h_F |\mathfrak{N}|}(\boldsymbol{f},\boldsymbol{f}). \tag{8-19}$$

Thus,

$$\frac{L\big(\tfrac12,\pi_L\otimes\Omega\big)}{L(1,\pi,\mathrm{Ad})} = 2^{2d+1-4|\boldsymbol{k}|}\Delta^{5/2} h_F |\mathfrak{N}| L(1,\eta) L^{\mathrm{alg}}\big(\tfrac12,\pi_L\otimes\Omega\big)\prod_{v|\infty}\Gamma(k_v+m_v)\Gamma(k_v-m_v).$$

Hence we can rewrite the average value formula from Theorem 1.3 as

$$2^{3d-4|\boldsymbol{k}|-1}\Delta h_F \prod_v (2k_v-2)! \sum_{\pi\in\mathcal{F}(\mathfrak{N},2\boldsymbol{k})} L^{\mathrm{alg}}\big(\tfrac12,\pi_L\otimes\Omega\big) = \frac{1}{L_{S(\Omega)}(1,\eta)}. \tag{8-20}$$

This immediately implies Theorem 1.5.

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