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First covering of the Drinfel'd upper half-plane and Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

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For an odd prime p, we construct some admissible Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that conjecturally should correspond to some 2-dimensional tamely ramified, potentially Barsotti–Tate representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ via the p-adic local Langlands correspondence. To achieve this, we generalize Breuil's work in the semistable case and work on the first covering of the Drinfel'd upper half-plane. Our main tool is an explicit semistable model of the first covering.

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1. Introduction

Breuil [2004] constructed some Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, which conjecturally should be nonzero and admissible and correspond to 2-dimensional semistable, noncrystalline representations of $G_{\mathbb{Q}_p}$ under the p-adic local Langlands correspondence. Here $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, where $\overline{\mathbb{Q}_p}$ is some fixed algebraic closure of $\mathbb{Q}_p$. Later on Colmez [2004] found the relationship between these Banach representations and (ϕ, Γ)-modules and proved their admissibility. Breuil and Mézard [2010] also proved the admissibility in some cases by explicitly computing the mod p reductions of these Banach representations. The aim of this paper is to generalize Breuil's work to some 2-dimensional tamely ramified, potentially Barsotti–Tate representations of $G_{\mathbb{Q}_p}$.

First we recall some of Breuil's [2004] construction. Let E be a finite extension of $\mathbb{Q}_p$ and k an integer greater than 2. Up to a twist by some character, all 2-dimensional semistable, noncrystalline E-representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, k-1)$ are classified by the "$\mathcal{L}$-invariant" [Breuil 2004, exemple 1.3.5]. We use $V(k, \mathcal{L})$ to denote this Galois representation. Here $\mathcal{L}$ is an element in E and basically tells you the position of the Hodge filtration on the Weil–Deligne representation associated to $V(k, \mathcal{L})$. Notice that this Weil–Deligne representation does not depend on $\mathcal{L}$. So via the classical local Langlands correspondence, all $V(k, \mathcal{L})$ correspond to the same smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, which is a twist of St, the usual Steinberg representation.

Breuil's idea is that for each $\mathcal{L}$, there should exist a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant norm on $\underline{\mathrm{Sym}}^{k-2}E^2 \otimes \mathrm{St}$; here $\underline{\mathrm{Sym}}^{k-2}E^2$ is a twist of the algebraic representation $\mathrm{Sym}^{k-2}E^2$. Different $\mathcal{L}$ should give different noncommensurable unit balls of $\underline{\mathrm{Sym}}^{k-2}E^2 \otimes \mathrm{St}$. If we take the completion, we get a Banach representation $B(k, \mathcal{L})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ for each $\mathcal{L}$. Moreover, we hope this representation is admissible in the sense of [Schneider and Teitelbaum 2002] and the correspondence between $V(k, \mathcal{L})$ and $B(k, \mathcal{L})$ is compatible with the mod p correspondence defined by Breuil [2003].

So how to construct these $B(k, \mathcal{L})$? For simplicity, I assume $E = \mathbb{Q}_p$ and k is even. The strategy of Breuil is to realize the unit ball $O(k, \mathcal{L})^U$ of the dual representation of $B(k, \mathcal{L})$ in $O(k) = \Gamma(\Omega, \mathcal{O}(k))$, where $\mathcal{O}(k)$ is a coherent sheaf on the Drinfel'd upper half-plane Ω over $\mathbb{Q}_p$. Concretely, $\mathcal{O}(2)$ is the sheaf of rigid differential forms and $\mathcal{O}(2n) = \mathcal{O}(2)^{\otimes n}$. Here Ω is considered as a rigid analytic space and $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on everything. We note that the de Rham cohomology of Ω is nothing but $\mathrm{St}^\vee$, the algebraic dual representation of St [Schneider and Stuhler 1991, Theorem 1]. The construction of $O(k, \mathcal{L})^U$, as far as I understand, has the following two important properties:

(1) $O(k, \mathcal{L})^U$ is "globally bounded" and hence compact. In other words, it is contained in $\Gamma(\widehat{\Omega}, \omega^{\otimes k/2})$, where $\widehat{\Omega}$ is a semistable model of Ω and ω is an

integral structure of $\mathcal{O}(2)$. This guarantees that the dual of $O(k, \mathcal{L})^U$ is indeed a Banach representation (after inverting p).

(2) If $f \in O(k)$ comes from a modular form of weight k (see [Breuil 2004, section 5] for the precise meaning), then $f \in O(k, \mathcal{L}_0)^U$ if and only the $\mathcal{L}$-invariant of f is $\mathcal{L}_0$.

Now consider the case where the Galois representation is tamely ramified. We will see later that the situation is very similar. Fix E a finite extension of $\mathbb{Q}_p$ large enough and let O_E be its ring of integers. This time we need to work on the first covering of Drinfel'd upper half-plane. According to Drinfel'd, there is a universal p-divisible group X over $\widehat{\Omega}\hat{\otimes}\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ and O_D acts on it, where O_D is the ring of integers inside the quaternion algebra D over $\mathbb{Q}_p$. Fix a uniformizer $\Pi \in O_D$ and define $\mathcal{X}_n$ as the generic fiber of $X[\Pi^n]$. The first covering $\Sigma_1 = \mathcal{X}_1 - \mathcal{X}_0$, also carries the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $O_D^\times$. It was shown by Drinfel'd [1976] that the action of $O_D^\times$ can be extended to $D^\times$. This is a left action and we will keep this convention in this paper unless explicitly inverting it. One remark is that the actions of $\mathbb{Q}_p^\times$ inside $D^\times$ and $\mathrm{GL}_2(\mathbb{Q}_p)$ become the same once we invert the action of $D^\times$.

First we note that the (E-coefficient) de Rham cohomology $H^1_{\mathrm{dR}}(\Sigma_1, E) \overset{\mathrm{def}}{=} H^1_{\mathrm{dR}}(\Sigma_1) \otimes_{\mathbb{Q}_p} E$ of Σ_1 has the following decomposition. Let $\psi : \mathbb{Q}_p^\times \to O_E^\times$ be a unitary character of level 0 in the sense that $1 + p\mathbb{Z}_p$ is contained in the kernel of ψ. We will view it as a character of $\mathbb{Q}_p^\times \subset D^\times$. In the following theorem, we invert the action of $D^\times$ so that it acts on the cohomology on the left. We denote the ψ-isotypic component of $H^1_{\mathrm{dR}}(\Sigma_1, E)$ by $H^1_{\mathrm{dR}}(\Sigma_1, E)^\psi$.

Theorem 1.1. *As a representation of $D^\times \times \mathrm{GL}_2(\mathbb{Q}_p)$,*

$$H^1_{\mathrm{dR}}(\Sigma_1, E)^\psi \simeq \bigoplus_{\pi \in \mathcal{A}^0(D^\times)(\psi^\vee)_0} (\pi \otimes \mathrm{JL}(\pi))^\vee \otimes_E D_\pi,$$

where $\cdot^\vee$ denotes the algebraic dual representation, $\mathcal{A}^0(D^\times)(\psi^\vee)_0$ is the space of admissible irreducible representations of $D^\times$ of level 0 *over E that are not characters and with central character $\psi^\vee$* (see [Bushnell and Henniart 2006, Chapter 13]), $\mathrm{JL}(\pi)$ *is the representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *associated to π by the Jacquet–Langlands correspondence*, *and D_π is a two-dimensional vector space over E.*

Remark 1.2. In fact, we can define more structures on D_π. Roughly speaking, we may find a finite extension F of $\mathbb{Q}_p$ such that

$$F \otimes_{\mathbb{Q}_p} D_\pi \simeq F \otimes_{F_0} D_{\mathrm{crys},\pi},$$

where F_0 is the maximal unramified extension of $\mathbb{Q}_p$ inside F and $D_{\mathrm{crys},\pi}$ is a $(\varphi, N, F/\mathbb{Q}_p, E)$-module (see Section 13 for the notation here). Then up to some unramified character, the Weil–Deligne representation associated to $D_{\mathrm{crys},\pi}$

corresponds to $\mathrm{JL}(\pi)$ under the classical local Langlands correspondence. See Theorem 1.10 below.

Explicitly, any $\pi \in \mathcal{A}^0(D^\times)(\psi^\vee)_0$ is an induced representation

$$\pi \simeq \mathrm{Ind}_{O_D^\times \mathbb{Q}_p^\times}^{D^\times} \Xi,$$

where $\Xi : O_D^\times \mathbb{Q}_p^\times \to O_E^\times$ is a character which extends $\psi^\vee$ and is trivial on $1+\Pi O_D$. It is clear that π has an integral structure π_0 over O_E.

As we noted before, we need to construct a $\mathrm{GL}_2(\mathbb{Q}_p)\times D^\times$-equivariant formal model $\widehat{\Sigma_1^{\mathrm{nr}}}$ of Σ_1. This will be done by using Raynaud's theory of $\mathbb{F}$-vector space schemes. As Breuil did in the case of the Drinfel'd upper half-plane, we can define a $\mathrm{GL}_2(\mathbb{Q}_p)\times D^\times$-equivariant integral model ω^1 of $\Omega^1_{\Sigma_1}$ on this formal model, where $\Omega^1_{\Sigma_1}$ is the sheaf of differential forms (see Remark 14.2). Consider the composition of the following maps:

$$H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1) \to H^0(\Sigma_1, \Omega^1_{\Sigma_1}) \to H^1_{\mathrm{dR}}(\Sigma_1).$$

We will show that this map is injective (Proposition 14.6), so that $H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)$ can be viewed as a subspace in the de Rham cohomology. Rewrite Theorem 1.1 as

$$H^1_{\mathrm{dR}}(\Sigma_1, E)(\pi^\vee) = H^1_{\mathrm{dR}}(\Sigma_1, E)^{\psi}(\pi^\vee) \simeq \mathrm{JL}(\pi)^\vee \otimes D_\pi,$$

where $(\cdot)(\pi^\vee) = \mathrm{Hom}_{E[D^\times]}(\pi^\vee, \cdot)$. For any line $\mathcal{L}$ inside D_π (the $\mathcal{L}$-invariant in our case), we may view $\mathrm{JL}(\pi)^\vee \otimes \mathcal{L}$ as a subspace inside $H^1_{\mathrm{dR}}(\Sigma_1, E)(\pi^\vee)$ by the above isomorphism. We can now define the (dual) of our Banach space representations:

Definition 1.3. $M(\pi, \mathcal{L}) \stackrel{\text{def}}{=} \big(H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E\big)(\pi_0^\vee) \cap \big(\mathrm{JL}(\pi)^\vee \otimes \mathcal{L}\big).$

Recall that π_0 is some integral structure of π. Notice that $M(\pi, \mathcal{L})$ is contained in $\big(H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E\big)(\pi^\vee)$, a natural subspace of $\big(H^0(\Sigma_1, \Omega^1_{\Sigma_1}) \otimes_{\mathbb{Q}_p} E\big)(\pi^\vee)$. This last space has a natural Fréchet space structure over E. The induced topology on $M(\pi, \mathcal{L})$ makes it into a compact topological space, and thus allows us to introduce:

Definition 1.4. $B(\pi, \mathcal{L}) = \mathrm{Hom}^{\mathrm{cont}}_{O_E}(M(\pi, \mathcal{L}), E).$

This is a unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Remark 1.5. The argument of [Breuil 2004, lemme 4.1.1] shows that $H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)$ and $H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega'^1)$ are commensurable, where ω'^1 is any other $\mathrm{GL}_2(\mathbb{Q}_p)\times D^\times$-equivariant integral model of $\Omega^1_{\Sigma_1}$. Hence $B(\pi, \mathcal{L})$ is independent of the choice of ω^1.

Now we can state the main result of this paper. Assume p is an odd prime.

Theorem 1.6. (1) *$B(\pi, \mathcal{L})$ is nonzero and admissible as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. In fact, its mod p reduction can be computed explicitly.*

(2) *$B(\pi,\mathcal{L})$ is a unitary completion of* $\mathrm{JL}(\pi)$.

The computation will give us an interesting $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant short exact sequence (Corollaries 16.28 and 17.5):

Corollary 1.7. *The sequence*

$$0 \to \widehat{\mathrm{JL}(\pi)} \to H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)^d_E(\pi) \to B(\pi,\mathcal{L}) \to 0,$$

is exact, where $\widehat{\mathrm{JL}(\pi)}$ *is the universal unitary completion of* $\mathrm{JL}(\pi)$ (see [Emerton 2005]), *and*

$$H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1)^d_E = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_p}\big(H^0(\widehat{\Sigma_1^{\mathrm{nr}}}, \omega^1), E\big).$$

Note that the kernel and the middle term are independent of $\mathcal{L}$ while the map between them depends on $\mathcal{L}$.

Remark 1.8. Unfortunately, we have to assume $p \geq 3$ in the proof of Theorem 1.6 (for example in the proof of Lemma 16.4). However Theorem 1.1 is also true for $p = 2$.

Now we explain the strategy of proving Theorem 1.1. By twisting with some unramified unitary characters, it suffices to deal with the case where the central character ψ satisfies $\psi(p) = 1$. This suggests we descend Σ_1 from $\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$ to $\mathbb{Q}_{p^2}$, the unramified quadratic extension of $\mathbb{Q}_p$, by taking the "p-invariant" of Σ_1 (see Section 7). We use Σ_1^p to denote this rigid analytic space. One warning here: even though Σ_1^p has a structure map to $\mathbb{Q}_{p^2}$, I will view it as a rigid space over $\mathbb{Q}_p$. A semistable model of Σ_1^p is very helpful (see Theorem 8.4):

Theorem 1.9. $\Sigma_1^p \times_{\mathbb{Q}_p} F$ *has an explicit* $D^\times \times \mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant semistable model* $\widehat{\Sigma^{(0)}_{1,O_F}}$ *over* O_F, *where* $F \simeq \mathbb{Q}_{p^2}[(-p)^{1/(p^2-1)}]$.

Similar results have been obtained before by Teitelbaum [1990].

Denote the generic fiber of this semistable model by $\Sigma^{(0)}_{1,F} = \Sigma_1^p \times_{\mathbb{Q}_p} F$. With the help of the semistable model, we can compute its de Rham cohomology. Let $\chi(E)$ be the character group of $O_D^\times/(1+\Pi O_D)$ with values in $E^\times$. Recall that $O_D^\times$ acts on $\Sigma^{(0)}_{1,F}$. We have the following result (see Section 12, especially Corollary 12.10 and Remark 12.11):

Theorem 1.10. *For any* $\chi \in \chi(E)$ *such that* $\chi \neq \chi^p$, *we have a* $\mathrm{Gal}(F/\mathbb{Q}_p) \times O_D^\times \times \mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant isomorphism*:

$$F \otimes_{F_0} D_{\mathrm{crys},\chi} \otimes_E \big(\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_{\chi^{-1}}\big)^\vee \xrightarrow{\sim} \big(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E\big)^\chi,$$

where $F_0 \simeq \mathbb{Q}_{p^2}$, c-Ind *is the induction with compact support*, $\cdot^\vee$ *means the algebraic dual*, $\rho_{\chi^{-1}}$ *is a cuspidal representation of* $\mathrm{GL}_2(\mathbb{F}_p)$ *over* E *defined via Deligne–Lusztig theory*, *and* $D_{\mathrm{crys},\chi}$ *is a free* $F_0 \otimes E$*-module of rank* 2 *with an action of* $\mathrm{Gal}(F/\mathbb{Q}_p)$. *In addition, we can define a Frobenius operator* φ *acting on it. It is explicitly described in Proposition 12.8.*

Take $\pi = \mathrm{Ind}_{O_D^\times \mathbb{Q}_p^\times}^{D^\times} \chi$, where χ is viewed as a character of $O_D^\times \mathbb{Q}_p^\times$ that is trivial on p. Then $\mathrm{JL}(\pi) \simeq \text{c-}\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$. Theorem 1.1 follows from the above theorem by taking the $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariants. There is some inverse involved since we invert the action of $D^\times$ in Theorem 1.1.

It is clear from the theorem that $D_{\mathrm{crys},\chi}$ is a $(\varphi, N, F/\mathbb{Q}_p, E)$-module. A line $\mathcal{L}$ inside D_π, or equivalently, a $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariant "line" inside $F \otimes_{F_0} D_{\mathrm{crys},\chi}$, essentially gives a filtration and makes $D_{\mathrm{crys},\chi}$ into a filtered $(\varphi, N, F/\mathbb{Q}_p, E)$-module. See Section 13 for more details.

After fixing some basis for $D_{\mathrm{crys},\chi}$ (see Proposition 12.8), any line $\mathcal{L}$ can be identified with an element b inside E or ∞. Assume $b \in O_E$ for the moment. We will write

$$M(\chi, [1, b]) = M(\mathrm{Ind}_{O_D^\times \mathbb{Q}_p^\times}^{D^\times} \chi, \mathcal{L}),$$

and similarly $B(\chi, [1, b]) = B(\mathrm{Ind}_{O_D^\times \mathbb{Q}_p^\times}^{D^\times} \chi, \mathcal{L})$.

Some notation here: Fix a $\mathbb{Z}_p$-linear embedding of $W(\mathbb{F}_{p^2})$, the Witt vector of $\mathbb{F}_{p^2}$ into O_D. Then any $\chi \in \chi(E)$ can be viewed as a character of $\mathbb{F}_{p^2}^\times$ by composing this embedding with the Teichmüller character. Also fix an embedding τ of $W(\mathbb{F}_{p^2})$ into E. Similarly the Teichmüller character gives us a character $\chi_\tau : \mathbb{F}_{p^2}^\times \to E^\times$.

Definition 1.11. We define m as the unique integer in $\{0, \dots, p^2 - 2\}$ such that

$$\chi = \chi_\tau^{-m} : \mathbb{F}_{p^2}^\times \to O_E^\times.$$

We will write $m = i + (p+1)j$, where $i \in \{0, \dots, p\}$, $j \in \{0, \dots, p-2\}$ and $[-mp]$ as the unique integer in $\{0, \dots, p^2 - 2\}$ congruent to $-mp$ modulo $p^2 - 1$.

Let $\sigma_i(j)$ be the following representation of $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\sigma_i(j) = \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} (\mathrm{Sym}^i \mathbb{F}_p^2) \otimes O_E/p \otimes \det{}^j,$$

where $\mathrm{Sym}^i \mathbb{F}_p^2$ is the i-th symmetric power of the natural representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on the canonical basis of $\mathbb{F}_p^2$, viewed as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ trivial on $p^{\mathbb{Z}}$.

Using our explicit semistable model, we can compute the mod p reduction of $M(\chi, [1, b])$ (Corollary 16.29, Remark 16.30, Corollary 17.6).

Theorem 1.12. *Let T be the usual Hecke operator* (defined in [Breuil 2007]) *and let $c(\chi, b) = (-1)^{j+1}\tau(w_1^{-i})b \in O_E/p$, where $\tau(w_1)$ satisfies $\tau(w_1)^{p+1} = -1$ and is independent of χ, $\mathcal{L}$.*

(1) *Assume $p^2 - 1 - m \geq [-mp], i \in \{2, \dots, p-1\}$. As a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$,*

$$0 \to \{X \in \sigma_{i-2}(j+1) \mid c(\chi, b)X = T(X)\} \to M(\chi, [1, b])/p \\ \to \{X \in \sigma_{p-1-i}(i+j) \mid X = c(\chi, b)T(X)\} \to 0.$$

(2) *Assume* $p^2 - 1 - m \leq [-mp]$, $i \in \{2, \dots, p-1\}$.

$$0 \to \left\{X \in \sigma_{p-1-i}(i+j) \mid X = c(\chi, b)T(X)\right\} \to M(\chi, [1,b])/p \\ \to \left\{X \in \sigma_{i-2}(j+1) \mid c(\chi, b)X = T(X)\right\} \to 0.$$

(3) *Assume* $i = p$. *Then*

$$M(\chi, [1,b])/p \simeq \left\{X \in \sigma_{p-2}(j+1) \mid -c(\chi, b)X + T(X) - c(\chi, b)T^2(X) = 0\right\}.$$

(4) *Assume* $i = 1$. *Then*

$$M(\chi, [1,b])/p \simeq \{X \in \sigma_{p-2}(j+1),\, X + c(\chi, b)T(X) + T^2(X) = 0\}.$$

Thus in any case, $B(\chi, [1,b])$ *is nonzero and admissible.*

Remark 1.13. In a recent paper Gabriel Dospinescu and Arthur-César Le Bras [Dospinescu and Le Bras 2015] independently use a very similar method to construct some locally analytic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and verify the compatibilities with the p-adic local Langlands correspondence, and thus generalize Breuil's [2004] work in this direction. Their method works for all the coverings of the Drinfel'd upper half-plane and relies on the previous work of Colmez on the relationship between Banach space representations and (ϕ, Γ)-modules. Combining their results with some known results of p-adic local Langlands correspondence, they can also prove Theorem 1.1 and Theorem 1.6. However, it seems that Corollary 1.7 does not follow directly from their work.

We give a brief outline of this paper. The goal of the next eight sections (Sections 2–9) is to explicitly write down a semistable model of Σ_1. Our strategy is to apply Raynaud's [1974] theory of $\mathbb{F}$-vector spaces schemes to X_1. We will collect some basic facts about the Drinfel'd upper half-plane in Section 2 and review Raynaud's theory in Section 3. To compute the data in Raynaud's theory, we need the existence of some "polarization" of X_1 (Proposition 5.1), which comes from a formal polarization of X (Section 4). Using this, a formal model is obtained in Section 5. By comparing the invariant differential forms of X_1 computed in two different ways, we write down the local equation of this formal model in Section 6. From this, it's not too hard to work out a semistable model in Section 8 and make clear how $\mathrm{GL}_2(\mathbb{Z}_p)$, $O_D^\times$, and $\mathrm{Gal}(F/\mathbb{Q}_p)$ act on it in Section 9.

In Sections 10–12 we compute the de Rham cohomology of $\Sigma_{1,F}^{(0)}$. Using our semistable models, this can be expressed by the crystalline cohomology of the irreducible components of the special fiber, which is well-understood via Deligne–Lusztig theory. The main result is Corollary 12.10, which describes the structure of the de Rham cohomology.

In Section 13 we classify all possible filtrations on $D_{\mathrm{crys},\chi}$ with Hodge–Tate weights (0, 1). We use this result to define $M(\chi, [1,b])$ in Section 14.

Sections 15, 16, and 17 contain the computation of $M(\chi,[1,b])/p$. In Section 15, we compute $H^0(\widehat{\Sigma_1^{\mathrm{nr}}},\omega^1)/p$ (not exactly this space, see the precise statement there). Roughly speaking, the method is by carefully studying the shape of differential forms on each irreducible component of the special fiber. The main result is Proposition 15.13 which says that this space is an extension of two inductions. Sections 16 and 17 treat different cases of computations of $M(\chi,[1,b])/p$ according to the value of i, but their strategies are the same: First we interpret $M(\chi,[1,b])$ as the kernel of a map θ_b from $(H^0(\widehat{\Sigma_1^{\mathrm{nr}}},\omega^1)\otimes O_E/p)^\chi$ to a space J_2. Then we compute the mod p reduction $\bar{\theta}_b$ of this map explicitly and show that $\bar{\theta}_b$ is in fact surjective. Hence θ_b has to be surjective as well since both $H^0(\widehat{\Sigma_1^{\mathrm{nr}}},\omega^1)$ and J_2 are p-adically complete. Therefore $M(\chi,[1,b])$ is just the kernel of $\bar{\theta}_b$.

Notation. Throughout this paper, fix an odd prime number p.

Let $\mathbb{Q}_p^{\mathrm{nr}}$ be the maximal unramified extension of $\mathbb{Q}_p$ and $\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$ be its p-adic completion. We will write $\mathbb{Z}_{p^2}=W(\mathbb{F}_{p^2})$, the ring of Witt vectors of $\mathbb{F}_{p^2}$ and fix an embedding of it into $\mathbb{Q}_p^{\mathrm{nr}}$. Denote the fractional field of $\mathbb{Z}_{p^2}$ by $\mathbb{Q}_{p^2}$. We use F_0 to denote the unique unramified quadratic extension of $\mathbb{Q}_p$. Hence the fixed embedding of $\mathbb{Q}_{p^2}$ into $\mathbb{Q}_p^{\mathrm{nr}}$ gives us an isomorphism between F_0 and $\mathbb{Q}_{p^2}$. Later on, F_0 will appear as some intermediate field extension when we try to compute a semistable model. Let O_{F_0} be the ring of integers inside F_0. Frequently we will identify F_0 with $\mathbb{Q}_{p^2}$ by this fixed isomorphism.

We denote by D the quaternion algebra of $\mathbb{Q}_p$ and fix a uniformizer $\Pi\in D$ such that $\Pi^2=p$. We will also fix a $\mathbb{Z}_p$-linear embedding of $\mathbb{Z}_{p^2}$ into O_D, hence an isomorphism:

$$O_D/\Pi O_D\simeq\mathbb{F}_{p^2}.$$

Let E be a finite extension of $\mathbb{Q}_p$ such that $\mathrm{Hom}_{\mathbb{Q}_p}(F_0,E)\neq 0$. We use $\tau,\bar{\tau}$ to denote the embeddings of F_0 into E and O_E to denote its ring of integers. For any O_{F_0}-module A, we denote $A\otimes_{O_{F_0},\tau}O_E$ by A_τ and $A\otimes_{O_{F_0},\bar{\tau}}O_E$ by $A_{\bar{\tau}}$.

For $K=E,F_0$, we use $\chi(K)$ to denote the character group of $O_D^\times/(1+\Pi O_D)=(O_D/\Pi)^\times$ with values in $K^\times$.

For any integer n, we will use $[n]$ to denote the unique integer in $\{0,1,\dots,p^2-2\}$ congruent to n modulo p^2-1.

For any ring A and integer n, we use $\mu_n(A)$ to denote $\{a\in A\mid a^n=1\}$.

For any abelian group M, we denote the p-adic completion of M by $M^\wedge$.

We use $\mathrm{Sym}^i\,\mathbb{F}_p^2$ to denote the i-th symmetric power of the natural representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on the canonical basis of $\mathbb{F}_p^2$ for i nonnegative. Explicitly, we can identify $\mathrm{Sym}^i\,\mathbb{F}_p^2$ with $\bigoplus_{r=0}^i\mathbb{F}_p x^r y^{i-r}$, where the action is given by

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}x^iy^{i-r}=(ax+cy)^r(bx+dy)^{i-r}.$$

Sometimes we will also view it as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ by abuse of notation.

Also, we define an induced representation of $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\sigma_i = \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} (\mathrm{Sym}^i \mathbb{F}_p^2) \otimes O_E/p,$$

where the induction has no restriction on the support and we view $\mathrm{Sym}^i \mathbb{F}_p^2$ as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ trivial on $p^{\mathbb{Z}}$. We define σ_{-1} as 0 and

$$\sigma_i(j) = \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} (\mathrm{Sym}^i \mathbb{F}_p^2) \otimes O_E/p \otimes \det{}^j,$$

where det is the determinant map.

We recall the definition of Hecke operator T here. See Section 3.2 of [Breuil 2007] for more details. Let $\sigma = \mathrm{Sym}^r \mathbb{F}_p^2 \otimes \det^m$, $0 \le r \le p-2$ be an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_p)$ over $\mathbb{F}_p$. I would like to view it as a representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ with p acting trivially. We use V_σ to denote the underlying representation space. Hence,

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma = \bigl\{ f : \mathrm{GL}_2(\mathbb{Q}_p) \to V_\sigma \mid f(hg) = \sigma(h)(f(g)),\ h \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \bigr\}.$$

Note that we put no restriction on the support. Following [Breuil 2007], denote by

$$[g, v] : \mathrm{GL}_2(\mathbb{Q}_p) \to V_\sigma$$

the following element of $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma$:

$$[g, v](g') = \begin{cases} \sigma(g'g)v & \text{if } g' \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times g^{-1}, \\ 0 & \text{if } g' \notin \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times g^{-1}. \end{cases}$$

We have $g([g', v]) = [gg', v]$ and $[gh, v] = [g, \sigma(h)v]$ if $h \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$. It is clear that every element in $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma$ can be written uniquely as an infinite sum of $[g_i, v_i]$ such that no two g_i are within the same coset $\mathrm{GL}_2(\mathbb{Q}_p)/\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$. Identify V_σ with $\bigoplus_{k=0}^r \mathbb{F}_p x^k y^{r-k}$. We define $\varphi_r : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{End}_{\mathbb{F}_p}(V_\sigma, V_\sigma)$ as follows:

$$\varphi_r(g) = 0 \quad \text{if } g \notin \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p),$$

$$\varphi_r\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)(x^k y^{r-k}) = 0 \quad \text{if } k \neq 0,$$

$$\varphi_r\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)(y^r) = y^r,$$

$$\varphi_r(h_1 g h_2) = \sigma(h_1) \circ \varphi_r(g) \circ \sigma(h_2), \qquad h_1, h_2 \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times.$$

The Hecke operator T_{φ_r} (or T for simplicity) is defined as:

$$T([g, v]) = \sum_{g' \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \in \mathrm{GL}_2(\mathbb{Q}_p)/(\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times)} [gg', \varphi_r(g'^{-1})(v)].$$

2. Some facts about the Drinfel'd upper half-plane

Let Ω be the p-adic upper half-plane (or Drinfel'd upper half plane) over $\mathbb{Q}_p$. It is a rigid analytic space over $\mathbb{Q}_p$ and its $\mathbb{C}_p$-points are $\mathbb{C}_p - \mathbb{Q}_p$, where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. There is a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on Ω. On the set of $\mathbb{C}_p$-points, it is given by

$$z \mapsto z|_g = \frac{az+c}{bz+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p).$$

Ω has a $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant formal model $\widehat{\Omega}$ over $\mathbb{Z}_p$, which is described in detail in [Boutot and Carayol 1991]. One warning here: in this paper, the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on Ω is a right action rather than a left action used in Drinfel'd's original paper [1976] and in [Boutot and Carayol 1991]. Our action is the inverse of their action. I apologize here if this causes any confusion.

Let me recall some facts we need to use later. There exists an open covering $\{\widehat{\Omega}_e\}_e$ on $\widehat{\Omega}$ indexed by the set of edges of the Bruhat–Tits tree I of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Two different $\widehat{\Omega}_e$ and $\widehat{\Omega}_{e'}$ have nonempty intersection if and only e and e' share a vertex s. When this happens, $\widehat{\Omega}_e \cap \widehat{\Omega}_{e'}$ only depends on the vertex s. We call it $\widehat{\Omega}_s$. For two adjacent vertices s, s', we denote the unique edge connecting them by $[s, s']$. Explicitly, ($\widehat{\ }$ is for p-adic completion)

$$\widehat{\Omega}_{s'} \simeq \operatorname{Spf} O_\eta \overset{\text{def}}{=} \operatorname{Spf} \mathbb{Z}_p\Big[\eta, \frac{1}{\eta-\eta^p}\Big]^{\widehat{}} \tag{1}$$

$$\widehat{\Omega}_{s} \simeq \operatorname{Spf} O_\zeta \overset{\text{def}}{=} \operatorname{Spf} \mathbb{Z}_p\Big[\zeta, \frac{1}{\zeta-\zeta^p}\Big]^{\widehat{}} \tag{2}$$

$$\widehat{\Omega}_{[s,s']} \simeq \operatorname{Spf} O_{\zeta,\eta} \overset{\text{def}}{=} \operatorname{Spf} \frac{\mathbb{Z}_p[\zeta,\eta]}{\zeta\eta-p}\Big[\frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}\Big]^{\widehat{}}. \tag{3}$$

The inclusion from $\widehat{\Omega}_s$ (resp. $\widehat{\Omega}_{s'}$) to $\widehat{\Omega}_{[s,s']}$ under these isomorphisms is just ζ (resp. η) goes to p/η (resp. p/ζ).

The set of vertices of the tree is in bijection with $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\backslash\mathrm{GL}_2(\mathbb{Q}_p)$. Clearly there is a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on this set and it extends to an action on the set of edges. In fact, this action can be identified with the action on the open covering $\{\widehat{\Omega}_e\}_e$. When s is the vertex that corresponds to the coset $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$, which I call the central vertex s_0', we can choose the isomorphism (1) such that the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on it is given by

$$\eta \mapsto \frac{a\eta+c}{b\eta+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

From the explicit description of $\widehat{\Omega}_{[s,s']}$ and $\widehat{\Omega}_s$ above, it is clear the special fiber of $\widehat{\Omega}$ is a tree of rational curves over $\mathbb{F}_p$ intersecting at all $\mathbb{F}_p$-rational points. The set of irreducible components (singular points) is nothing but the set of vertices (edges) of the tree. The dual graph of the special fiber of $\widehat{\Omega}$ is just the Bruhat–Tits tree.

In [Drinfel'd 1976], it was shown that there exists a universal family of formal groups X of height 4 over $\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$, where $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is the p-adic completion of the ring of integers inside the maximal unramified extension $\mathbb{Q}_p^{\mathrm{nr}}$ of $\mathbb{Q}_p$. We denote by D the "unique" quaternion algebra over $\mathbb{Q}_p$, and O_D the ring of integers inside D. Then from Drinfel'd's construction, we know that O_D acts on the universal formal group on the left.

Fix a uniformizer Π inside O_D such that $\Pi^2 = p$. Define $X_n = X[\Pi^n]$. They are finite formal group schemes over $X_0 = \widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. Let $\mathcal{X}_n$ be the rigid space associated to X_n, or equivalently, $\mathcal{X}_n$ is the generic fiber of X_n. These $\mathcal{X}_n$ are étale coverings of $\mathcal{X}_0 = \Omega\,\hat{\otimes}\,\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$. Then $O_D/(\Pi^n)$ acts on it and we have equivariant inclusions $\mathcal{X}_{n-1} \hookrightarrow \mathcal{X}_n$. Now define

$$\Sigma_n = \mathcal{X}_n - \mathcal{X}_{n-1}.$$

It can be shown that Σ_n is a finite étale Galois covering over $\mathcal{X}_0$ with Galois group $(O_D/(\Pi^n))^\times$.

It is important that all the spaces (X_n, $\mathcal{X}_n$, Σ_n) we construct here have a natural $\mathrm{GL}_2(\mathbb{Q}_p)$ action and all the maps here are $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. On $X_0 = \Omega\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$, $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\widehat{\Omega}$ as we described before and acts on $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ via $\widetilde{\mathrm{Fr}}^{v_p(\det(g))}$, where $\widetilde{\mathrm{Fr}}$ is the (lift of the) arithmetic Frobenius and v_p is the usual p-adic valuation on $\mathbb{Q}_p$. One can show that the action of $\mathbb{Z}_p^\times$ in $\mathrm{GL}_2(\mathbb{Q}_p)$ on Σ_n is inverse to the action of $\mathbb{Z}_p^\times$ in $O_D^\times$.

Now we want to describe the action of Π on the tangent space T of X. It is easy to see from the construction that T is a rank 2 vector bundle on $\widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. Moreover, T splits canonically into a direct sum of two line bundles T_0, T_1 by considering the action of $\mathbb{Z}_{p^2}$ inside O_D (recall that we fix such an embedding in the previous section). Each eigenspace of this action is a line bundle because X is "special" in the sense of Drinfel'd. Π interchanges T_0, T_1 and under the isomorphisms (1)–(3), we can write it down explicitly. But before doing that, I must introduce the notion of odd and even vertices.

Definition 2.1. A class $[g]$ in $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \backslash \mathrm{GL}_2(\mathbb{Q}_p)$ is called odd (resp. even) if $v_p(\det(g))$ is odd (resp. even).

Notice that this is well defined. And we will call a vertex in the tree (or an irreducible component of the special fiber of $\widehat{\Omega}$) even or odd according to the corresponding class.

Back to the discussion of the tangent space. I should mention that all T_0, T_1, Π descend naturally to $\widehat{\Omega}$, and I still call them T_0, T_1, Π by abuse of notation. Suppose s is an odd vertex and s' is adjacent to s, and so must be even. On $\widehat{\Omega}_{[s,s']}$, both

T_0, and T_1 are trivial. If we choose appropriate bases e_0, e_1 of them, then under the isomorphisms (1)–(3), Π becomes

$$\Pi_0 : T_0 \to T_1, \quad e_0 \mapsto \zeta e_1, \tag{4}$$

$$\Pi_1 : T_1 \to T_0, \quad e_1 \mapsto \eta e_0. \tag{5}$$

Identify Π_0, Π_1 with global sections of $T_0^* \otimes T_1$ and $T_1^* \otimes T_0$, where T_i^* denotes the dual of $T_i,\ i = 0, 1$ (the cotangent space). Then the explicit description of Π tells us that on an odd component of the special fiber, Π_0 has a simple zero at each intersection point with other irreducible components. Since each irreducible component is a rational curve over $\mathbb{F}_p$ and intersects other components at $\mathbb{F}_p$-rational points, Π_0 corresponds to the divisor that is the sum of all points of $\mathbb{P}^1(\mathbb{F}_p)$. On the other hand, Π_1 is zero on an odd component because $\eta = p/\zeta = 0$ (we are working over the special fiber, so already modulo p). On an even component, a similar argument shows that Π_0 is zero and Π_1 is the sum of all points of $\mathbb{P}^1(\mathbb{F}_p)$ as a divisor.

Restricting everything to the central vertex s_0', we have an isomorphism $\widehat{\Omega}_{s_0'} \simeq \operatorname{Spf} \widehat{\mathbb{Z}_p}[\eta, 1/(\eta - \eta^p)]^{\widehat{}}$, and $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on it via

$$\eta \mapsto \frac{a\eta + c}{b\eta + d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

The action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on T_0^* is easier to describe than the action on T_0. Using the same basis as in the last paragraph and denoting the dual basis element of e_0 by e_0^*, we have

$$g : T_0^* \to T_0^*, \quad f(\eta) e_0^* \mapsto \frac{1}{b\eta + d} f\Big(\frac{a\eta + c}{b\eta + d}\Big) e_0^* \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p). \tag{6}$$

Most details here can be found in [Boutot and Carayol 1991], especially the first chapter about Deligne's functor (and notice the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ here is the inverse of the action there).

3. Raynaud's theory of $\mathbb{F}$-vector space schemes

We want to write down the equation defining X_1. Recall that there exists an action of $O_D/(\Pi)$ on X_1. But $\mathbb{F} \overset{\text{def}}{=} O_D/(\Pi)$ is a finite field which is isomorphic to $\mathbb{F}_{p^2}$. So X_1 is an "$\mathbb{F}$-vector space scheme" in the sense of Raynaud. Let's recall Raynaud's theory of $\mathbb{F}$-vector space schemes in our situation. The reference is the first section of [Raynaud 1974].

Definition 3.1. Let S be a scheme and $\mathbb{F}$ a finite field. An $\mathbb{F}$-vector space scheme is a group scheme G over S with an embedding of $\mathbb{F}$ into the endomorphism ring of G (over S).

Although the definition here is different from Raynaud's original definition, it's clear that they are equivalent. Now let G be an $\mathbb{F}$-vector space scheme; we also use G to denote the group scheme in the definition by abuse of notation. The action of $\lambda \in \mathbb{F}$ is denoted by $[\lambda]$. Following Raynaud, we assume G is finite, flat and of finite presentation over S.

Let $\mathcal{A}$ be the bialgebra of G and $\mathcal{I}$ be the augmentation ideal. Then $\mathbb{F}^\times$ acts on $\mathcal{A}$ and $\mathcal{I}$. In Raynaud's paper, he defined a ring "D". Since we already use D for the quaternion algebra, I will use D_R for Raynaud's "D". In our case, we can think of D_R as $\mathbb{Z}_{p^2}$, the quadratic extension of $\mathbb{Z}_p$ in $\mathbb{Z}_p^{\mathrm{nr}}$. Although this ring is much bigger than D_R, both of them give the same result here. Under the hypothesis $(*)$ in Raynaud's paper and fixing a map $S \to \operatorname{Spec} D_R$, we have a canonical decomposition of $\mathcal{I}$:

$$\mathcal{I} = \bigoplus_{\chi \in M} \mathcal{I}_\chi,$$

where M is the set of characters of $\mathbb{F}^\times$ with value in $D_R^\times$, and $\mathcal{I}_\chi$ is defined as the "χ-isotypic component". More precisely, for every open set V on S, $H^0(V, \mathcal{I}_\chi)$ is the set of elements $a \in H^0(V, \mathcal{I})$, such that $[\lambda]a = \chi(\lambda)a$ for any $\lambda \in \mathbb{F}^\times$.

Definition 3.2. Let χ_1, χ_2 be the characters of $\mathbb{F}^\times = (O_D/\Pi)^\times$ with values in $D_R^\times = \mathbb{Z}_{p^2}^\times$ such that the composition

$$\mathbb{F}_{p^2}^\times \simeq (O_D/\Pi)^\times \xrightarrow{\chi_i} \mathbb{Z}_{p^2}^\times$$

is the Teichmüller character if $i = 1$ and its Galois twist if $i = 2$. Here, the first isomorphism is the one we fixed in the beginning. They are the fundamental characters defined in Raynaud's paper.

It is clear that $\chi_1^p = \chi_2$ and $\chi_2^p = \chi_1$. Every character χ in M can be written uniquely as

$$\chi = \chi_1^{n_1} \chi_2^{n_2}, \quad 0 \le n_1, n_2 \le p-1, \ (n_1, n_2) \ne (0,0).$$

Now, it is easy see to that given two characters χ, χ' in M, we have two $\mathcal{O}_S$-linear maps

$$\begin{cases} c_{\chi,\chi'} : \mathcal{I}_{\chi\chi'} \to \mathcal{I}_\chi \otimes \mathcal{I}_{\chi'}, \\ d_{\chi,\chi'} : \mathcal{I}_\chi \otimes \mathcal{I}_{\chi'} \to \mathcal{I}_{\chi\chi'} \end{cases}$$

coming from the comultiplication and multiplication structure of $\mathcal{A}$. A slight generalization of this (or equivalently iterate this $p-1$ times) gives us

$$\begin{cases} c_i : \mathcal{I}_{\chi_{i+1}} \to \mathcal{I}_{\chi_i}^{\otimes p}, \\ d_i : \mathcal{I}_{\chi_i}^{\otimes p} \to \mathcal{I}_{\chi_{i+1}} \end{cases}$$

for $i = 1, 2$, and we identify χ_3 as χ_1.

Under the hypothesis ($**$) in Raynaud's paper, which says that each $\mathcal{I}_\chi$ is an invertible sheaf on S, we have the following classification theorem of $\mathbb{F}$-vector space schemes.

Theorem 3.3 [Raynaud 1974]. *Let S be a D_R-scheme. The map*

$$G \mapsto \left(\mathcal{I}_{\chi_i},\ c_i : \mathcal{I}_{\chi_{i+1}} \to \mathcal{I}_{\chi_i}^{\otimes p},\ d_i : \mathcal{I}_{\chi_i}^{\otimes p} \to \mathcal{I}_{\chi_{i+1}}\right)_{i=1,2}$$

defines a bijection between the isomorphism classes of $\mathbb{F}$-vector space schemes over S satisfying ($**$) and the isomorphism classes of $(\mathcal{L}_1, \mathcal{L}_2, c_1, c_2, d_1, d_2)$, where:

(1) *$\mathcal{L}_i$ is an invertible sheaf on S for any $i = 1, 2$.*

(2) *The c_i and d_i are $\mathcal{O}_S$-linear maps*

$$\begin{cases} c_i : \mathcal{L}_{\chi_{i+1}} \to \mathcal{L}_{\chi_i}^{\otimes p} \\ d_i : \mathcal{L}_{\chi_i}^{\otimes p} \to \mathcal{L}_{\chi_{i+1}} \end{cases}$$

such that $d_i \circ c_i = w\,\mathrm{Id}_{\mathcal{L}_{i+1}}$. Here w is a constant in D_R that only depends on $\mathbb{F}$ and can be expressed using Gauss sums. More precisely, if we identify D_R with $\mathbb{Z}_{p^2}$, then $w \in \mathbb{Z}_p \subset \mathbb{Z}_{p^2}$ with p-adic valuation 1. And if we write $w = pu$, then $u \equiv -1 \pmod{p}$.

The inverse map in the theorem is as follows: we define

$$\mathcal{A} = \bigoplus_{0 \le a_i \le p-1} (\mathcal{L}_1^{\otimes a_1} \otimes \mathcal{L}_2^{\otimes a_2})$$

and equip it with the multiplication and comultiplication structure using d_i, c_i. $\mathcal{A}$ is now a bialgebra and thus defines a group scheme over S. The action of $\mathbb{F}^\times$ is defined as the character χ_i on $\mathcal{L}_i$ and more generally as the character $\chi_1^{a_1}\chi_2^{a_2}$ on $\mathcal{L}_1^{\otimes a_1} \otimes \mathcal{L}_2^{\otimes a_2}$. We now define the action of 0 in $\mathbb{F}$ to be trivial on $\mathcal{A}$. The properties of c_i and d_i guarantee that we indeed get a $\mathbb{F}$-vector space scheme. As a corollary, we have a description of the invariant differential forms of G:

Corollary 3.4. $\omega_{G/S} \simeq \mathcal{I}/\mathcal{I}^2 = (\mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p})) \oplus (\mathcal{L}_2/d_1(\mathcal{L}_1^{\otimes p}))$.

Remark 3.5. When S is an affine scheme, say $\mathrm{Spec}(A)$, and $\mathcal{L}_i$ is free over S for all i, we have an explicit description of $\mathcal{A}$. Suppose x_i is a basis of $\mathcal{L}_i$. Under such basis, d_i becomes an element v_i inside A, namely $d_i(x_i^{\otimes p}) = v_i x_{i+1}$. Then the bialgebra $\mathcal{A}$ is isomorphic to $A[x_1, x_2]/(x_1^p - v_1 x_2, x_2^p - v_2 x_1)$ as an A-algebra.

Remark 3.6. The Cartier dual of an $\mathbb{F}$-vector space scheme is also an $\mathbb{F}$-vector space scheme by the dual action of $\mathbb{F}$. On the level of bialgebra, the action of $\mathbb{F}$ is given by its transpose. If G corresponds to $(\mathcal{L}_i, c_i, d_i)$, the Cartier dual G^* corresponds to $(\mathcal{L}_i^*, d_i^*, c_i^*)$, where $\mathcal{L}_i^* = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{L}_i, \mathcal{O}_S)$ and d_i^* (resp. c_i^*) is the transpose of d_i (resp. c_i).

4. Some results about the formal polarization

We want to apply Raynaud's theory to our situation. Although our base scheme is a formal scheme, the argument of Raynaud can be extended naturally to this situation. As we remarked in the beginning of the previous section, $X_1 = X[\Pi]$ is a $\mathbb{F}$-vector space scheme over $X_0 = \widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Q}_p^{\mathrm{nr}}}$, where $\mathbb{F} = O_D/(\Pi)$. Using that its generic fiber $\mathcal{X}_1$ is étale over $\Omega\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ and applying Proposition 1.2.2. in Raynaud's paper, we know that X_1 satisfies hypothesis $(**)$. So the classification theorem tells us there exist 2 invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$, and maps

$$c_1 : \mathcal{L}_2 \mapsto \mathcal{L}_1^{\otimes p}, \quad c_2 : \mathcal{L}_1 \mapsto \mathcal{L}_2^{\otimes p}, \tag{7}$$

$$d_1 : \mathcal{L}_1^{\otimes p} \mapsto \mathcal{L}_2, \quad d_2 : \mathcal{L}_2^{\otimes p} \mapsto \mathcal{L}_1, \tag{8}$$

such that $d_1 \circ c_1 = w\,\mathrm{Id}_{\mathcal{L}_2}$, and $d_2 \circ c_2 = w\,\mathrm{Id}_{\mathcal{L}_1}$.

In order to determine c_i, d_i, we need the existence of "formal $*$-polarization" of the universal formal group X, which is a lemma in the proof of Proposition 4.3. of [Drinfel'd 1976], and proved in detail in [Boutot and Carayol 1991, chapitre III lemma 4.2.]. I would like to recall it here without proof.

Lemma 4.1. *Suppose $t \in D$ such that $t^2 \in pO_D^{\times}$. There exists a symmetric isomorphism $\lambda : X \to X^*$, where X^* is the Cartier dual of X, such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & X^* \\ {\scriptstyle t^{-1}\bar{d}t}\downarrow & & \downarrow{\scriptstyle d^*} \\ X & \xrightarrow{\lambda} & X^* \end{array}$$

commutes for any $d \in O_D$, where $\bar{d}$ is the canonical involution of d in D, and d^* is the dual morphism of the endomorphism d. Here symmetric means $\lambda = \lambda^*$ under the canonical identification between X and X^{**}.

Remark 4.2. This isomorphism is not unique, but is unique up to $\mathbb{Z}_p^{\times}$-action. From now on, we will fix one such isomorphism λ that is defined in [Drinfel'd 1976] and [Boutot and Carayol 1991]. So we also fix such a t.

How does this isomorphism behave under the action $\mathrm{GL}_2(\mathbb{Q}_p)$? Recall that $X_0 = \widehat{\Omega}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$.

Lemma 4.3. *Suppose $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $\det(g) \in p^{\mathbb{Z}}$; then g "commutes" with λ. More precisely, by abuse of notation, let $g : X_0 \to X_0$ be the automorphism of X_0 induced by g. Then there exists a natural isomorphism $\mu_g : X \to g^*X$ over X_0, where g^*X is the pull-back of X under $g : X_0 \to X_0$ by the equivariance of the $\mathrm{GL}_2(\mathbb{Q}_p)$ action. Denote by μ_g^* the dual morphism of μ_g. We have the following* *commutative diagram*:

$$\begin{tikzcd} X^* & (g^*X)^* \simeq g^*X^* \arrow[l, "\mu_g^*"'] \arrow[r] & X^* \\ X \arrow[u, "\lambda"] \arrow[r, "\mu_g"] \arrow[d] & g^*X \arrow[u, "g^*\lambda"'] \arrow[r] \arrow[d] & X \arrow[u, "\lambda"'] \arrow[d] \\ X_0 \arrow[r, equal] & X_0 \arrow[r, "g"] & X_0 \end{tikzcd}$$

In general, for any $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, *we have the same diagram but replace the upper left square by*

$$\begin{tikzcd} X^* & g^*X^* \arrow[l, "\mu_g^*"'] \\ X \arrow[u, "\lambda"] \arrow[r, "\mu_{g\cdot p^n/\det(g)}"'] & g^*X \arrow[u, "g^*\lambda"'] \end{tikzcd}$$

where $n = v_p(\det(g))$. *Notice that this makes sense since* $\mathbb{Z}_p^\times$ *has trivial action on* X_0, *so* $g^*X = (g \cdot p^n/\det(g))^*X$.

Proof. Since I will use some formulas in [Drinfel'd 1976] and [Boutot and Carayol 1991], I think it's better not to translate their left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ to right action here. Hence I will follow their convention in this proof.

It's clear that we only need to prove the general case. Thanks to Drinfel'd's lemma (the lemma on strictness for p-divisible groups in the appendix of [Drinfel'd 1976]), it suffices to verify this commutative diagram after we reduce modulo p. But by Drinfel'd's construction of the universal p-divisible group, $X \times \mathbb{F}_p$ is quasi-isogenous of degree 0 to a constant p-divisible group $\Phi_{X_0 \times \mathbb{F}_p}$ over $X_0 \times \mathbb{F}_p$. Here, recall that Φ is a p-divisible group defined over $\overline{\mathbb{F}}_p$, and $\Phi_{X_0\times\mathbb{F}_p} \overset{\text{def}}{=} \Phi \times_{\overline{\mathbb{F}}_p} X_0$. $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on Φ as quasi-isogenies. A detailed description of Φ can be found in [Boutot and Carayol 1991, chapitre III 4.3] or the proof of Proposition 4.3. of [Drinfel'd 1976]. Besides, the construction of the "formal polarization" λ tells us that λ actually comes from a "formal polarization" λ_0 of Φ that makes the following diagram commutative:

$$\begin{tikzcd} X \times \mathbb{F}_p \arrow[r, "\bar{\lambda}"] \arrow[d, "\rho"'] & X^* \times \mathbb{F}_p \\ \Phi_{X_0\times\mathbb{F}_p} \arrow[r, "\lambda_{0,X_0\times\mathbb{F}_p}"] & \Phi^*_{X_0\times\mathbb{F}_p} \arrow[u, "\rho^*"'] \end{tikzcd}$$

where $\bar{\lambda} \overset{\text{def}}{=} \lambda (\mathrm{mod}\ p)$, ρ is the quasi-isogeny and ρ^* is its dual. From the definition of the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, we know how ρ changes under this action (basically the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on Φ with some twist of Frobenius, see [Drinfel'd 1976, Section 2] or [Boutot and Carayol 1991, chapitre II section 9]). Thus we can translate the diagram of X into a diagram of Φ. It turns out that it suffices to verify that the following diagram is commutative:

$$\begin{tikzcd}
\Phi^* & (\mathrm{Fr}^{-n})^*\Phi^* \arrow[l, "(\mathrm{Frob}_\Phi^{-n}\circ g)^*"'] \\
\Phi \arrow[u, "\lambda_0"] \arrow[r, "\mathrm{Frob}_\Phi^{-n}\circ(g\cdot p^n/\det(g))"] & (\mathrm{Fr}^{-n})^*\Phi \arrow[u, "(\mathrm{Fr}^{-n})^*\lambda_0"']
\end{tikzcd}$$

Here $\mathrm{Fr}:\mathrm{Spec}(\overline{\mathbb{F}}_p)\to\mathrm{Spec}(\overline{\mathbb{F}}_p)$ is the arithmetic Frobenius and $\mathrm{Frob}_\Phi:(\mathrm{Fr}^{-1})^*\Phi\to\Phi$ is the Frobenius morphism over $\mathrm{Spec}(\overline{\mathbb{F}}_p)$. I would like to decompose the diagram as the following diagram (and invert the arrow on the bottom line):

$$\begin{tikzcd}
\Phi^* & \Phi^* \arrow[l, "g^*(\det(g))^{-1}"'] & (\mathrm{Fr}^{-n})^*\Phi^* \arrow[l, "(\det(g)\,\mathrm{Frob}_\Phi^{-n})^*"'] \\
\Phi \arrow[u, "\lambda_0"] & \Phi \arrow[u, "\lambda_0"] \arrow[l, "g^{-1}"'] & (\mathrm{Fr}^{-n})^*\Phi \arrow[u, "(\mathrm{Fr}^{-n})^*\lambda_0"'] \arrow[l, "(\det(g)/p^n)\,\mathrm{Frob}_\Phi^n"']
\end{tikzcd}$$

First we look at the right square:

$$(\det(g)\,\mathrm{Frob}_\Phi^{-n})^*=\Big(\frac{\det(g)}{p^n}\Big)(p^n\,\mathrm{Frob}_\Phi^{-n})^*=\Big(\frac{\det(g)}{p^n}\Big)(\mathrm{Ver}_\Phi^n)^*=\Big(\frac{\det(g)}{p^n}\Big)\mathrm{Frob}_{\Phi^*}^n,$$

where Ver_{Φ^*} is the Verschiebung morphism. Now it is easy to see the diagram commutes from the basic property of the Frobenius morphism.

As for the left square, the commutativity in fact comes from our explicit choice of Φ, λ_0 and the action of $\mathrm{GL}_2(\mathbb{Q}_p)$. See the remarque in [Boutot and Carayol 1991, chapitre III 4.3] which says the Rosati involution associated to λ_0 is nothing but the canonical involution on $\mathrm{M}_2(\mathbb{Q}_p)$. □

Remark 4.4. When $g\in\mathrm{SL}_2(\mathbb{Q}_p)$, the calculation above is essentially given in [Boutot and Carayol 1991, chapitre III 4.5].

5. Structure of $\mathcal{X}_1$ and a formal model of Σ_1

Now let's see how the discussion above helps us study c_i, d_i in (7), (8). The main result is the following:

Proposition 5.1. *There exists an isomorphism λ_1 from $X_1=X[\Pi]$ to $X[\Pi]^*$, the Cartier dual of $X[\Pi]$, such that*:

(1) *The following diagram commutes for any $d \in O_D^\times$:*

$$\begin{array}{ccc} X[\Pi] & \xrightarrow{\lambda_1} & X[\Pi]^* \\ \bar{d}\downarrow & & \downarrow d^* \\ X[\Pi] & \xrightarrow{\lambda_1} & X[\Pi]^* \end{array}$$

Recall that $\bar{d}$ is the canonical involution of d in D.

(2) $\lambda_1^* = \lambda_1 \circ [-1] = [-1]^* \circ \lambda_1$, *where* $[-1]$ *denotes the action of* $-1 \in O_D$.

Proof. We can take $t = \Pi$ in Lemma 4.1. Then if we restrict to the p torsion points of X, we certainly get an isomophism:

$$\lambda_p : X[p] = X[\Pi^{-1}\bar{p}\Pi] \to X^*[p^*] = X^*[p].$$

Notice that $X^*[p]$ is canonically isomorphic to $(X[p])^*$, the Cartier dual of $X[p]$. The inclusion of $X[\Pi]$ into $X[p]$ induces a canonical isomorphism

$$j : X^*[p]/X^*[\Pi^*] = X^*[p]/((X^*[p])[\Pi^*]) \xrightarrow{\sim} (X[p])^*/((X[p])^*[\Pi^*]) \xrightarrow{\sim} X[\Pi]^*.$$

Since $\Pi^2 = p$, the map $\Pi^* : X^*[p] \to X^*[p]$ gives us an isomorphism

$$h : X^*[p]/X^*[\Pi^*] \xrightarrow{\sim} X^*[\Pi^*].$$

Finally, we restrict λ to the Π torsion points of X and get an isomorphism

$$\lambda_\Pi : X[\Pi] = X[\Pi^{-1}\bar{\Pi}\Pi] \to X^*[\Pi^*].$$

Now, we define $\lambda_1 = j \circ h^{-1} \circ \lambda_\Pi : X[\Pi] \to X[\Pi]^*$.

What is the Rosati involution associated to λ_1? I claim the following diagram commutes:

$$\begin{array}{ccccccc} X[\Pi] & \xrightarrow{\lambda_\Pi} & X^*[\Pi^*] & \xleftarrow{h} & X^*[p]/X^*[\Pi^*] & \xrightarrow{j} & X[\Pi]^* \\ \Pi^{-1}\bar{d}\Pi \downarrow & & d^* \downarrow & & (\Pi d \Pi^{-1})^* \downarrow & & \downarrow (\Pi d \Pi^{-1})^* \\ X[\Pi] & \xrightarrow{\lambda_\Pi} & X^*[\Pi^*] & \xleftarrow{h} & X^*[p]/X^*[\Pi^*] & \xrightarrow{j} & X[\Pi]^* \end{array}$$

The left-most square is commutative because we have a similar diagram for λ and λ_Π is a restriction of λ. The right-most diagram is commutative because j comes from the canonical quotient map $X^*[p] \simeq (X[p])^* \to X[\Pi]^*$ and this certainly commutes with the dual endomorphism of O_D. As for the middle square, notice that h is induced by the map $\Pi^* : X^*[p] \to X^*[p]$ and everything is clear.

Since $\Pi^{-1}\bar{d}\Pi \equiv d \pmod{\Pi O_D}$ and everything in the diagram above is killed by Π or Π^*, we can replace $\Pi^{-1}\bar{d}\Pi$ by d and $(\Pi d \Pi^{-1})^*$ by $\bar{d}^*$, and hence get the desired commutative diagram in part (1).

As for part (2), we use G, H to denote $X[p]$, $X[\Pi]$ respectively. Then $G^*=X^*[p]$. We can decompose $-\Pi : G \to G$ as

$$G \xrightarrow{q} G/H \xrightarrow{h_{-\Pi}} H \xrightarrow{i} G,$$

where i (resp. q) is the canonical inclusion of H to G (resp. canonical quotient map of G to G/H). The induced isomorphism is $h_{-\Pi}$.

Notice that $\Pi^{-1}\bar{\Pi}\Pi = -\Pi$ and G is killed by p. We have the following diagram, which is a restriction of the diagram of Lemma 4.1 to G with $d = \Pi$:

$$\begin{tikzcd}
G \arrow[r, "-\Pi"] \arrow[d, "\lambda_p"'] & G \arrow[d, "\lambda_p"] \\
G^* \arrow[r, "\Pi^*"] & G^*
\end{tikzcd}$$

Similarly we can decompose Π^* as we did for $-\Pi$ and have the commutative diagram

$$\begin{tikzcd}
G \arrow[r, "q"] \arrow[d, "\lambda_p"'] & G/H \arrow[r, "h_{-\Pi}"] \arrow[d, "\lambda_{G/H}"'] & H \arrow[r, "i"] \arrow[d, "\lambda_H"'] & G \arrow[d, "\lambda_p"'] \\
G^* \arrow[r, "i^*"] & H^* \arrow[r, "h_{\Pi^*}"] & (G/H)^* \arrow[r, "q^*"] & G^*
\end{tikzcd}$$

such that the composition of all three maps in the bottom line is Π^*. The map h_{Π^*} is induced from Π^*. Thus it's easy to see $([-1]\circ h_{-\Pi})^* = h_{\Pi^*}$ and its dual $h_{\Pi^*}^* = [-1]\circ h_{-\Pi}$.

Since λ is symmetric, so is λ_p and we certainly have $\lambda_{G/H}^* = \lambda_H$. Now it's not hard to see that our λ_1 is nothing but $h_{\Pi^*}^{-1}\circ\lambda_H$. So,

$$\begin{aligned}
\lambda_1^* &= (h_{\Pi^*}^{-1}\circ\lambda_H)^* = \lambda_H^*\circ(h_{\Pi^*}^{-1})^* = \lambda_{G/H}\circ(h_{\Pi^*}^*)^{-1} \\
&= \lambda_{G/H}\circ([-1]\circ h_{-\Pi})^{-1} = \lambda_{G/H}\circ h_{-\Pi}^{-1}\circ[-1]^{-1} = \lambda_1\circ[-1].
\end{aligned}$$

The last identity comes from the middle square of the diagram above. □

Corollary 5.2. *The isomorphism λ_1 induces isomorphisms*

$$\lambda_{\mathcal{L}_1} : \mathcal{L}_2^* \xrightarrow{\sim} \mathcal{L}_1, \quad \lambda_{\mathcal{L}_2} : \mathcal{L}_1^* \xrightarrow{\sim} \mathcal{L}_2.$$

Moreover, $\lambda_{\mathcal{L}_1} = -\lambda_{\mathcal{L}_2}^*$ if p is odd and $\lambda_{\mathcal{L}_1} = \lambda_{\mathcal{L}_2}^*$ if $p = 2$.

Proof. Using Theorem 3.3, we can identify $X_1 = X[\Pi]$ with $(\mathcal{L}_1, \mathcal{L}_2, c_1, c_2, d_1, d_2)$, and the final remark there tells us we can identify $X[\Pi]^*$ with $(\mathcal{L}_1^*, \mathcal{L}_2^*, d_1^*, d_2^*, c_1^*, c_2^*)$.

Now λ_1 gives us an isomorphism from $X[\Pi]$ to $X[\Pi]^*$ but this is not $\mathbb{F} = O_D/(\Pi)$-equivariant. For a character χ of $\mathbb{F}^\times$, considered as a character of $O_D^\times$, we have

$$\chi(\bar{d}) = \chi(d^p) = \chi^p(d)$$

for any $d \in O_D^\times$. This is because when we restrict the canonical involution to a quadratic unramified extension of $\mathbb{Z}_p$ inside O_D, it is nothing but the nontrivial Galois action. So modulo the uniformizer, it becomes the Frobenius automorphism.

Take $\chi = \chi_1$, one of the fundamental characters; then $\chi_1^p = \chi_2$, so $\chi_1(\bar{d}) = \chi_2(d)$. Similarly, we have $\chi_2(\bar{d}) = \chi_1(d)$. From these identities and the commutative diagram in Proposition 5.1, it is easy to see λ_1 really induces isomorphisms

$$\lambda_{\mathcal{L}_1} : \mathcal{L}_2^* \xrightarrow{\sim} \mathcal{L}_1, \quad \lambda_{\mathcal{L}_2} : \mathcal{L}_1^* \xrightarrow{\sim} \mathcal{L}_2.$$

The last identity comes from the consideration that the difference between λ_1 and λ_1^* is the action of -1. And we know $\chi_1(-1) = \chi_2(-1) = -1$ if p is odd and 1 otherwise. □

From now on, I will assume p is odd.

Corollary 5.3. *Under the isomorphism $\lambda_{\mathcal{L}_1}$, we have $-d_1 = c_2^*$. More precisely, we have the following commutative diagram:*

$$\begin{tikzcd} \mathcal{L}_1^{\otimes p} \arrow[r, "-d_1"] & \mathcal{L}_2 \\ (\mathcal{L}_2^*)^{\otimes p} \arrow[u, "\lambda_{\mathcal{L}_1}^{\otimes p}"] \arrow[r, "c_2^*"'] & \mathcal{L}_1^* \arrow[u, "\lambda_{\mathcal{L}_1}^*"'] \end{tikzcd}$$

Proof. It is easy to see λ_1 induces a similar diagram by replacing $-d_1$ with d_1 and $\lambda_{\mathcal{L}_1}^*$ with $\lambda_{\mathcal{L}_2}$. Now the corollary follows from $\lambda_{\mathcal{L}_1} = -\lambda_{\mathcal{L}_2}^*$. □

Corollary 5.4. *Under the isomorphism $\lambda_{\mathcal{L}_1}$, we can identify $d_1 : \mathcal{L}_1^{\otimes p} \to \mathcal{L}_2$ with a global section of $\mathcal{L}_1^{\otimes -p-1}$. Similarly, we can identify d_2 with a global section of $\mathcal{L}_1^{\otimes p+1}$. The canonical pairing*

$$H^0(X_0, \mathcal{L}_1^{\otimes -p-1}) \times H^0(X_0, \mathcal{L}_1^{\otimes p+1}) \to H^0(X_0, \mathcal{O}_{X_0})$$

sends (d_1, d_2) to the constant $-w = -pu$, where w is the constant in Theorem 3.3, and u is w/p.

Proof. Recall that $d_2 \circ c_2 = w\,\mathrm{Id}_{\lambda_{\mathcal{L}_1}}$. Then everything follows from Corollary 5.3. □

Corollary 5.5. *Recall that the bialgebra of X_1 is isomorphic as an $\mathcal{O}_{X_0}$-module to $\bigoplus_{0 \le i,j \le p-1} \mathcal{L}_1^{\otimes i} \otimes \mathcal{L}_2^{\otimes j}$. The isomorphism $\lambda_{\mathcal{L}_1}$ gives a global section $\widetilde{\lambda_{\mathcal{L}_1}}$ of $\mathcal{L}_1 \otimes \mathcal{L}_2$. Then as a global section of X_1, we have*

$$\widetilde{\lambda_{\mathcal{L}_1}}^p = -w\widetilde{\lambda_{\mathcal{L}_1}},$$

where everything is computed inside the bialgebra of X_1.

Proof. We only need to verify this locally. Suppose $\mathcal{L}_1, \mathcal{L}_2$ are free over an open set U and generated by x_1, x_2 such that $x_1 \otimes x_2 = \widetilde{\lambda_{\mathcal{L}_1}}$, or equivalently they are dual to each other under $\lambda_{\mathcal{L}_1}$. Now d_1, d_2 are given by two elements $v_1, v_2 \in H^0(U, \mathcal{O}_{X_0})$. So $x_1^p = v_1 x_2$, and $x_2^p = v_2 x_1$ (see Remark 3.5). But from the last corollary, we have $v_1 v_2 = -w$. Thus the product of these two equations is just what we want. □

Remark 5.6. Perhaps it is better to remark here that $\mathcal{L}_1, \mathcal{L}_2$ are nontrivial on the formal model but we'll see later that they become trivial on the generic fiber (Lemma 10.1).

Now we can describe a formal model of Σ_1. Recall that $\Sigma_1 = \mathcal{X}_1 - \mathcal{X}_0$, where $\mathcal{X}_1, \mathcal{X}_0$ are the rigid analytic spaces associated to X_1, X_0.

Proposition 5.7. *Let $\mathcal{A} = \bigoplus_{0 \le i,j \le p-1} \mathcal{L}_1^{\otimes i} \otimes \mathcal{L}_2^{\otimes j}$ be the bialgebra of X_1. Then $\mathcal{A}/(\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w)$ (the closed subscheme defined by the ideal sheaf $(\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w)$) is a formal model of Σ_1. We will use $\widehat{\Sigma_1^{\mathrm{nr}}}$ to denote this formal model.*

Proof. It suffices to check this locally on X_0, so we can assume $\mathcal{L}_1, \mathcal{L}_2$ are free. A point x on Σ_1 gives a morphism $x : \mathcal{A} \to \mathbb{C}_p$. If it does not factor through $\mathcal{A}/(\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w)$, $x(\widetilde{\lambda_{\mathcal{L}_1}})$ has to be 0 because last corollary tells us $(\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w)\widetilde{\lambda_{\mathcal{L}_1}} = 0$. But

$$x_1^{p+1} = x_1^p x_1 = v_1 x_2 x_1 = v_1 \widetilde{\lambda_{\mathcal{L}_1}},$$

so $x(x_1) = 0$ and $x(x_2) = 0$ by the same argument. Therefore x factors through $\mathcal{A}$ modulo the ideal sheaf generated by x_1, x_2 which is the augmentation ideal. Therefore x is in $\mathcal{X}_0$. The converse is trivial. □

It's easy to see its underlying algebra of $\widehat{\Sigma_1^{\mathrm{nr}}}$ is just

$$\bigoplus_{\substack{0 \le i,j \le p-1, \\ (i,j) \ne (p-1,p-1)}} \mathcal{L}_1^{\otimes i} \otimes \mathcal{L}_2^{\otimes j}.$$

Remark 5.8. There exist natural actions of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $O_D^\times$ on $\widehat{\Sigma_1^{\mathrm{nr}}}$. The action of $O_D^\times$ is clear. To see the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, notice that $\widetilde{\lambda_{\mathcal{L}_1}}$ is a global section of a trivial line bundle on X_0, but $H^0(X_0, \mathcal{O}_{X_0})$ is canonically isomorphic to $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ (I will prove this later; see Lemma 14.7). So $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\widetilde{\lambda_{\mathcal{L}_1}}$ as a scalar. Recall that $\widetilde{\lambda_{\mathcal{L}_1}}^p + w\widetilde{\lambda_{\mathcal{L}_1}} = 0$. This implies $\widetilde{\lambda_{\mathcal{L}_1}}^{p-1} + w$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant. The same argument shows that the action of $O_D^\times$ can be extended to $D^\times$.

But how does $\mathrm{GL}_2(\mathbb{Q}_p)$ act on $\widetilde{\lambda_{\mathcal{L}_1}}$? Here is a direct consequence of Lemma 4.3:

Proposition 5.9. *With $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $n = v_p(\det(g))$,*

$$g(\widetilde{\lambda_{\mathcal{L}_1}}) = \chi_1(\det(g)/p^n)^{-1} \widetilde{\lambda_{\mathcal{L}_1}}.$$

6. Local equation of X_1 and $\widehat{\Sigma_1^{\mathrm{nr}}}$

In order to get a semistable model of $\widehat{\Sigma_1^{\mathrm{nr}}}$, we need to know the local equation defining it. Recall that in Section 2, we described an open covering $\{\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}\}_e$ of X_0 such that

$$\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \operatorname{Spf} \frac{\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\zeta, \eta]}{\zeta\eta - p}\Big[\frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}\Big]^{\widehat{}}.$$

We try to write down the equation of $\widehat{\Sigma_1^{\mathrm{nr}}}$ above each $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. Our first observation is:

Lemma 6.1. *Any line bundle $\mathcal{L}$ over*

$$\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \operatorname{Spf} \frac{\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\zeta, \eta]}{\zeta\eta - p}\Big[\frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}\Big]^{\widehat{}}$$

is trivial.

Proof. Recall (see (3))

$$O_{\zeta,\eta} = \frac{\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\zeta, \eta]}{\zeta\eta - p}\Big[\frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}\Big]^{\widehat{}}.$$

The special fiber of $\operatorname{Spf} O_{\zeta,\eta}$ is $\operatorname{Spec} \overline{\mathbb{F}}_p[\zeta, \eta, 1/(1-\zeta^{p-1}), 1/(1-\eta^{p-1})]/(\zeta\eta)$. I claim every line bundle $\bar{\mathcal{L}}$ over it is trivial. Let $\bar{L}$ be $H^0(\operatorname{Spec} O_{\zeta,\eta}/p, \bar{\mathcal{L}})$. Then we have the exact sequence

$$0 \to \bar{L} \to \bar{L}/(\zeta\bar{L}) \oplus \bar{L}/(\eta\bar{L}) \xrightarrow{-} \bar{L}/(\zeta\bar{L} + \eta\bar{L}) \to 0,$$

where the inclusion is the canonical morphism and $-$ is defined by taking their difference. This sequence is exact because $\bar{L}$ is locally free and thus flat over $O_{\zeta,\eta}/p$. Notice that $\bar{L}/(\zeta\bar{L})$ defines a line bundle on

$$\operatorname{Spec} O_{\zeta,\eta}/(p, \zeta) = \operatorname{Spec} \overline{\mathbb{F}}_p\Big[\eta, \frac{1}{1-\eta^{p-1}}\Big],$$

and hence has to be trivial. The same result holds for $\bar{L}/(\eta\bar{L})$. Also $\bar{L}/(\zeta\bar{L} + \eta\bar{L})$ is nothing but $\overline{\mathbb{F}}_p$. Using these, it's not hard to find an element that generates $\bar{L}$. So $\bar{\mathcal{L}}$ is trivial.

Now we can find an element in $H^0(\operatorname{Spf} O_{\zeta,\eta}, \mathcal{L})$ that generates $\mathcal{L}/p$. But $H^0(\operatorname{Spf} O_{\zeta,\eta}, \mathcal{L})$ is p-adically complete, so this element actually generates the whole $H^0(\operatorname{Spf} O_{\zeta,\eta}, \mathcal{L})$. Therefore $\mathcal{L}$ is trivial. (Here we use the fact that a surjective map between two line bundles has to be an isomorphism.) $\square$

Thanks to this lemma, the restriction of $\mathcal{L}_1$ on $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is trivial. We fix an isomorphism between $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ and $\operatorname{Spf} O_{\zeta,\eta}$. Suppose x_1 is a generator of

$H^0(\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}, \mathcal{L}_1)$, and $x_2 \in H^0(\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}, \mathcal{L}_2)$ is the dual basis under the isomorphism $\lambda_{\mathcal{L}_1}$ defined in the previous section. Let v_1, v_2 be the elements given by d_1, d_2 under the basis x_1, x_2. Then we know that locally X_1 is defined by $x_1^p = v_1 x_2$, $x_2^p = v_2 x_1$.

How to determine v_1, v_2? Our strategy is to compare the invariant differential forms of X_1 computed in two different ways. First recall that the tangent space T of the universal formal group over X_0 is a rank 2 vector bundle over X_0 that naturally splits into a direct sum of two line bundles T_0, T_1. So the sheaf of invariant differential forms is its dual, namely $T_0^* \oplus T_1^*$. The action of Π on T_0 sends T_0 (resp. T_1) into T_1 (resp. T_0), which we denoted by Π_0 (resp. Π_1) in Section 2. Thus Π_0^* (resp. Π_1^*) sends T_1^* (resp. T_0^*) to T_0^* (resp. T_1^*) and the sheaf of invariant forms ω_{X_1/X_0} of $X_1 = X[\Pi]$ is

$$T_0^*/\Pi_0^* T_1^* \oplus T_1^*/\Pi_1^* T_0^*.$$

On the other hand, using Corollary 3.4, we know that this is also

$$\mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p}) \oplus \mathcal{L}_2/d_1(\mathcal{L}_1^{\otimes p}).$$

It is natural to guess:

Lemma 6.2. $\quad T_0^*/\Pi_0^* T_1^* \simeq \mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p}), \quad T_1^*/\Pi_1^* T_0^* \simeq \mathcal{L}_2/d_1(\mathcal{L}_1^{\otimes p}).$

Proof. If we restrict the action of O_D to $\mathbb{Z}_{p^2}$, it acts by identity on T_0 and by conjugation on T_1. Recall that we fix an embedding of $\mathbb{Z}_{p^2}$ into O_D in the beginning. This is just the definition of X being "special". Now our desired identification follows from a simple comparison of the action of $\mathbb{Z}_{p^2}^\times$ in both ways. $\square$

Recall that all irreducible components of the special fiber of X_0 are isomorphic to $\mathbb{P}^1_{\mathbb{F}_p}$ such that the singular points are exactly $\mathbb{P}^1(\mathbb{F}_p)$. From the explicit description (4), (5) of Π_0, Π_1 and the discussion in Section 2, we know that on an odd component of the special fiber s, $T_0^*/\Pi_0^* T_1^*$ is isomorphic to $\bigoplus_{P \in s_{\mathrm{sing}}} i_{P*}\overline{\mathbb{F}}_p$, where s_{sing} is the set of singular points of the special fiber on s, and $i_P : P \to s$ is the embedding.

Restrict $\mathcal{L}_1, \mathcal{L}_2, d_2 : \mathcal{L}_2^{\otimes p} \to \mathcal{L}_1$ to s. From

$$\mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p}) \simeq T_0^*/\Pi_0^* T_1^* \simeq \bigoplus_{P \in s_{\mathrm{sing}}} i_{P*}\overline{\mathbb{F}}_p \qquad (\text{on } s), \tag{9}$$

it's easy to see $\deg(\mathcal{L}_1|_s) - \deg(\mathcal{L}_2^{\otimes p}|_s) = p+1$. But $\mathcal{L}_2 \simeq \mathcal{L}_1^*$, so

$$\deg(\mathcal{L}_2|_s) = -\deg(\mathcal{L}_1|_s). \tag{10}$$

This implies:

Lemma 6.3. *For any odd component s*, $\deg(\mathcal{L}_1|_s) = 1$. *Similarly*, $\deg(\mathcal{L}_2|_{s'}) = 1$ *for any even component s'.*

Now we would like to choose some good basis of $\mathcal{L}_1$ so that v_1, v_2 have a good form. Using the isomorphism $\widehat{\Omega}_e \simeq \mathrm{Spf}\, O_{\zeta,\eta}$, we can identify two irreducible components of its special fiber with $\mathrm{Spec}\, O_{\zeta,\eta}/(\zeta)$ and $\mathrm{Spec}\, O_{\zeta,\eta}/(\eta)$. Assume the second one is odd and we use s to denote the corresponding component in the special fiber of X_0 and use s' for the other component. Moreover $\mathrm{Spec}\, O_{\zeta,\eta}/(\eta) = \mathrm{Spec}\, \overline{\mathbb{F}}_p[\zeta, 1/(1-\zeta^{p-1})]$ hence has an obvious embedding into $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$ which can be identified as the embedding into s.

Choose a global section $\widetilde{x_1}$ of $\mathcal{L}_1|_s$ such that it has a simple zero at infinity under the identification above. It is a basis of $H^0(\mathrm{Spec}\, O_{\zeta,\eta}/(\eta), \mathcal{L}_1)$. Then under this basis,

$$d_2 : \mathcal{L}_2^{\otimes p} \simeq \mathcal{L}_1^{*\otimes p} \to \mathcal{L}_1, \quad \widetilde{x_1}^{*\,\otimes p} \mapsto c(\zeta^p - \zeta)\widetilde{x_1}$$

for some constant $c \in \overline{\mathbb{F}}_p^{\times}$, where $\widetilde{x_1}^*$ is the dual basis of $\widetilde{x_1}^*$.

Notice that $\widetilde{x_1}$ is only defined up to a constant. If we replace $\widetilde{x_1}$ by $d\widetilde{x_1}$, then the constant c is replaced by $d^{-p-1}c$. We can choose $d = c^{1/(p+1)}$ to eliminate c. More precisely, we can choose a section, which I still call $\widetilde{x_1}$ by abuse of notation, such that under this basis, d_2 is just multiplication by $\zeta^p - \zeta$.

We can do a similar thing for s', which means we can choose a basis $\widetilde{x_2}$ of $\mathcal{L}_2|_{\mathrm{Spec}\, O_{\zeta,\eta}/(\zeta)}$ such that under this basis, d_1 is multiplication by $c'(\eta^p - \eta)$. Here we choose $\widetilde{x_2}$ so that $\widetilde{x_1}, \widetilde{x_2}^*$ can glue to a global basis $\overline{x_1}$ of $\mathcal{L}_1|_{\mathrm{Spec}\, O_{\zeta,\eta}/(p)}$ (see the proof of Lemma 6.1). A priori we know nothing about the constant c'.

Now we can lift $\overline{x_1}$ to a global basis x_1 of $\mathcal{L}_1|_{\mathrm{Spec}\, O_{\zeta,\eta}}$, so it determines a basis x_2 of $\mathcal{L}_2|_{\mathrm{Spec}\, O_{\zeta,\eta}}$ under the isomorphism $\lambda_{\mathcal{L}_1}$. And d_1, d_2 are given by two numbers v_1, v_2. The explicit description (4), (5) and Lemma 6.2 imply that

$$v_2 = \zeta u_2, \quad v_1 = \eta u_1 \tag{11}$$

for some units $u_1, u_2 \in O_{\zeta,\eta}^{\times}$. Note that $u_1u_2 = -u$ because $v_1v_2 = -w = -pu$, by Corollary 5.4, and $\eta\zeta = p$. From our choice of x_1, x_2, we have

$$v_2 \equiv \zeta^p - \zeta (\mathrm{mod}\ \eta), \quad v_1 \equiv \eta^p - \eta (\mathrm{mod}\ \zeta), \tag{12}$$

so

$$u_2 \equiv \zeta^{p-1} - 1 \ (\mathrm{mod}\ \eta), \tag{13}$$

$$u_1 \equiv c'(\eta^{p-1} - 1) \ (\mathrm{mod}\ \zeta). \tag{14}$$

This is because (ζ, η) is a regular sequence in $O_{\zeta,\eta}$. In fact $O_{\zeta,\eta}$ is normal. When we take the product of the identities above considered in $O_{\zeta,\eta}/(\zeta,\eta) \simeq \overline{\mathbb{F}}_p$, the left-hand side is $u_1u_2 = -u$, which is 1 modulo p (see Theorem 3.3), while the right-hand side is just c'. Therefore:

Lemma 6.4. $$c' = 1.$$

Notice that $u_2 \equiv u_1^{-1} (\mathrm{mod}\ p)$, and $(\zeta) \cap (\eta) = (p)$ in $O_{\zeta,\eta}$. It's not hard to see that:

Lemma 6.5. $$u_1 \equiv -\frac{\eta^{p-1}-1}{\zeta^{p-1}-1} \ (\mathrm{mod}\ p), \quad u_2 \equiv -\frac{\zeta^{p-1}-1}{\eta^{p-1}-1} \ (\mathrm{mod}\ p).$$

Now if we replace our x_1 by rx_1 for some unit $r \in O_{\zeta,\eta}^{\times}$, then x_2 is replaced by $r^{-1}x_2$ and u_1 (resp. u_2) is replaced by $r^{p+1}u_1$ (resp. $r^{-p-1}u_2$). Write

$$u_1 = -\frac{\eta^{p-1}-1}{\zeta^{p-1}-1} r_1;$$

then $r_1 \equiv 1 (\mathrm{mod}\ p)$. Thus $r_1^{1/(p+1)}$ exists in $O_{\zeta,\eta}$. Hence we can modify our x_1 to make $u_1 = -(\eta^{p-1}-1)/(\zeta^{p-1}-1)$. In summary:

Proposition 6.6. *We can choose appropriate bases x_1, x_2 of $\mathcal{L}_1, \mathcal{L}_2$ over*

$$\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \mathrm{Spf}\, O_{\zeta,\eta}$$

such that they are dual to each other under $\lambda_{\mathcal{L}_1}$, and under these bases,

$$d_1 : \mathcal{L}_1^{\otimes p} \to \mathcal{L}_2, \quad x_1^{\otimes p} \mapsto -\frac{\eta^p - \eta}{\zeta^{p-1}-1} x_2, \tag{15}$$

$$d_2 : \mathcal{L}_2^{\otimes p} \to \mathcal{L}_1, \quad x_2^{\otimes p} \mapsto u\frac{\zeta^p - \zeta}{\eta^{p-1}-1} x_1. \tag{16}$$

Corollary 6.7. *The restriction of X_1 to $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \mathrm{Spf}\, O_{\zeta,\eta}$ is defined by*

$$\mathrm{Spf}\, O_{\zeta,\eta}[x_1, x_2]\Big/\Big(x_1^p + \frac{\eta^p-\eta}{\zeta^{p-1}-1}x_2, x_2^p - u\frac{\zeta^p-\zeta}{\eta^{p-1}-1}x_1\Big). \tag{17}$$

Similarly, the restriction of $\widehat{\Sigma_1^{\mathrm{nr}}}$ to $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is defined by

$$\mathrm{Spf}\, O_{\zeta,\eta}[x_1, x_2]\Big/\Big(x_1^p + \frac{\eta^p-\eta}{\zeta^{p-1}-1}x_2, x_2^p - u\frac{\zeta^p-\zeta}{\eta^{p-1}-1}x_1, (x_1x_2)^{p-1} + pu\Big). \tag{18}$$

Proof. The first statement follows from the above discussion. As for $\widehat{\Sigma_1^{\mathrm{nr}}}$, notice that x_1x_2 is just $\widetilde{\lambda_{\mathcal{L}_1}}$ defined in Corollary 5.5. So this is the definition of $\widehat{\Sigma_1^{\mathrm{nr}}}$. □

Fix a $\tilde{u}_1 = (-u)^{1/(p^2-1)}$ in $\mathbb{Z}_p$. If we replace x_1 by $\tilde{u}_1 x_1$, and x_2 by $\tilde{u}_1^p x_2$, then our new x_1, x_2 are dual to each other under $\tilde{u}_1^{-p-1}\lambda_{\mathcal{L}_1}$. Under this basis, $x_1x_2 = \tilde{u}_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}}$.

Corollary 6.8. *The restriction of X_1 to $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \mathrm{Spf}\, O_{\zeta,\eta}$ is defined by*

$$\mathrm{Spf}\, O_{\zeta,\eta}[x_1, x_2]\Big/\Big(x_1^p + \frac{\eta^p-\eta}{\zeta^{p-1}-1}x_2, x_2^p + \frac{\zeta^p-\zeta}{\eta^{p-1}-1}x_1\Big).$$

Similarly, the restriction of $\widehat{\Sigma_1^{\mathrm{nr}}}$ to $\widehat{\Omega}_e \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is defined by

$$\mathrm{Spf}\, O_{\zeta,\eta}[x_1, x_2]\Big/\Big(x_1^p + \frac{\eta^p-\eta}{\zeta^{p-1}-1}x_2, x_2^p + \frac{\zeta^p-\zeta}{\eta^{p-1}-1}x_1, (x_1x_2)^{p-1} - p\Big).$$

Suppose $e=[s,s']$ and we have (1), (2), and (3). Then $\widehat{\Omega}_{s'}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is obtained by inverting η in $O_{\zeta,\eta}$ and taking the p-adic completion. Therefore, we have:

Corollary 6.9. *The restriction of X_1 to $\widehat{\Omega}_{s'}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}\simeq \operatorname{Spf}\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\eta,1/(\eta^p-\eta)]^\wedge$ is defined by*

$$\operatorname{Spf}\widehat{\mathbb{Z}_p^{\mathrm{nr}}}\Big[\eta,\frac{1}{\eta^p-\eta}\Big]^\wedge[x_1,x_2]\Big/\Big(x_1^p+\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}x_2,\,x_2^p+\frac{(p/\eta)^p-(p/\eta)}{\eta^{p-1}-1}x_1\Big).$$

Similarly, the restriction of $\widehat{\Sigma_1^{\mathrm{nr}}}$ to $\widehat{\Omega}_{s'}\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ is defined by

$$\operatorname{Spf}\widehat{\mathbb{Z}_p^{\mathrm{nr}}}\Big[\eta,\frac{1}{\eta^p-\eta}\Big]^\wedge[x_1,x_2]\Big/\Big(x_1^p+\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}x_2,\,x_2^p+\frac{(p/\eta)^p-(p/\eta)}{\eta^{p-1}-1}x_1,\,(x_1x_2)^{p-1}-p\Big).$$

7. The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{\Sigma_1^{\mathrm{nr}}}$ and a descent $\widehat{\Sigma}_1$ to $\mathbb{Z}_{p^2}$

Recall that we fix an embedding $\mathbb{Z}_{p^2}\hookrightarrow\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. In this section, I want to describe the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{\Sigma_1^{\mathrm{nr}}}$. As a corollary, we can descend the formal model from $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ to $\mathbb{Z}_{p^2}$ by taking the "p-invariants", where p is considered as an element in $\mathrm{GL}_2(\mathbb{Q}_p)$. This descent is not quite canonical. On the other hand, as we explained in the introduction, it suffices to prove Theorem 1.1 when the central character is trivial on p, and this is exactly the descent we are considering here.

Denote the canonical morphism $\widehat{\Sigma_1^{\mathrm{nr}}}\to X_0$ by π and $\pi^{-1}(\widehat{\Omega}_e\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}})$ by $\widehat{\Sigma_{1,e}^{\mathrm{nr}}}$, $\pi^{-1}(\widehat{\Omega}_s\,\hat{\otimes}\,\widehat{\mathbb{Z}_p^{\mathrm{nr}}})$ by $\widehat{\Sigma_{1,s}^{\mathrm{nr}}}$, for edge e and vertex s. Then $\{\widehat{\Sigma_{1,e}^{\mathrm{nr}}}\}_e$ is an open covering of $\widehat{\Sigma_1^{\mathrm{nr}}}$, such that each open set has a nice description as in the previous section. Then the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on this covering can be identified with the action on the Bruhat–Tits tree.

Now let s_0' be the central vertex defined in Section 2. Then, $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on $\widehat{\Sigma_{1,s_0'}^{\mathrm{nr}}}$. I want to write down explicitly this action under the identification in Corollary 6.9. Since π is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant, we only need to describe the action on x_1,x_2. However it's clear from the equation in Corollary 6.9 that x_2 can be expressed using x_1 because $\eta^p-\eta$ is invertible. So it suffices to describe the action on x_1.

We first observe that $T_0^*/(\Pi_0^*T_1^*)\simeq\mathcal{L}_1/d_2(\mathcal{L}_2^{\otimes p})$ is a free O_η/p-module of rank one with a basis x_1. Recall $O_\eta=\widehat{\mathbb{Z}_p^{\mathrm{nr}}}[\eta,1/(\eta^p-\eta)]^\wedge$. In Section 2, we gave an explicit description (6) of the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on T_0^*, which is given by

$$f(\eta)e_0^*\mapsto\frac{1}{b\eta+d}f\Big(\frac{a\eta+c}{b\eta+d}\Big)e_0^*,\quad g=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

for some basis e_o^*. So if we write $x_1 = f(\eta)e_0^*$, for some $f(\eta) \in (O_\eta/p)^\times$, then the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on x_1 in O_η/p is

$$g(x_1) = \frac{1}{b\eta+d} f\Big(\frac{a\eta+c}{b\eta+d}\Big) f(\eta)^{-1} x_1. \tag{19}$$

Notice that on $\widehat{\Sigma^{\mathrm{nr}}_{1,s}}$,

$$x_1^{p+1} \equiv (\eta^p - \eta)x_1x_2 = (\eta^p - \eta)(-u)^{-1/(p-1)}\widetilde{\lambda_{\mathcal{L}_1}} \pmod p.$$

Thanks to Proposition 5.9, we know how $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on the right-hand side:

$$\begin{aligned} &g\big((\eta^p - \eta)(-u)^{-1/(p+1)}\widetilde{\lambda_{\mathcal{L}_1}}\big) \\ &\qquad = \Big(\Big(\frac{a\eta+c}{b\eta+d}\Big)^p - \frac{a\eta+c}{b\eta+d}\Big)(-u)^{-1/(p-1)}(ad-bc)^{-1}\widetilde{\lambda_{\mathcal{L}_1}}. \end{aligned}$$

Here we use the fact $\chi_1(\det(g)) \equiv \det(g) \pmod p$. An easy computation shows this is just $(1/(b\eta+d))^{p+1}(\eta^p - \eta)(-u)^{-1/(p-1)}\widetilde{\lambda_{\mathcal{L}_1}}$.

But from (19),

$$g(x_1)^{p+1} = \Big(\frac{1}{b\eta+d}\Big)^{p+1}\Big(f\Big(\frac{a\eta+c}{b\eta+d}\Big)f(\eta)^{-1}\Big)^{p+1} x_1^{p+1}.$$

Comparing both expressions, we have

$$f\Big(\frac{a\eta+c}{b\eta+d}\Big)^{p+1} = f(\eta)^{p+1} \quad \text{for any } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

Since $f(\eta) \in (O_\eta/p)^\times = \overline{\mathbb{F}}_p[\eta, 1/(\eta^p - \eta)]^\times$, it can only have poles and zeros at $\mathbb{F}_p$-rational points. Now $\mathrm{GL}_2(\mathbb{Z}_p)$ acts transitively on these points, so $f(\eta)$ has to be a constant. In other words:

Proposition 7.1. *The action of* $\mathrm{GL}_2(\mathbb{Z}_p)$ *on the special fiber of* $\widehat{\Sigma^{\mathrm{nr}}_{1,s_0'}}$ *is given by*

$$g(x_1) \equiv \frac{1}{b\eta+d}x_1 \pmod p, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p).$$

Corollary 7.2. *This action factors through* $\mathrm{GL}_2(\mathbb{F}_p)$.

What's the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on $\widehat{\Sigma^{\mathrm{nr}}_{1,s_0'}}$? Using Proposition 7.1 we can write

$$g(x_1)^{p+1} = \Big(\frac{1}{b\eta+d}\Big)^{p+1} x_1^{p+1}(1 + ph(\eta))$$

for some $h(\eta) \in O_\eta$ which only depends on g. Then:

Proposition 7.3. $$g(x_1) = \frac{1}{b\eta+d}x_1(1+ph(\eta))^{1/(p+1)},$$

where $(1+ph(\eta))^{1/(p+1)} = 1 + \frac{1}{p+1}ph(\eta) + \cdots$ *is the binomial expansion.*

Now let e_0 be the edge that connects the central vertex s_0' and the vertex s_0 that corresponds to $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \cdot w$, where $w = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. Then w acts on $\widehat{\Sigma^{\mathrm{nr}}_{1,e_0}}$. What is it?

We fix an isomorphism of $\widehat{\Sigma^{\mathrm{nr}}_{1,e_0}}$ with the explicit formal scheme described above. On $\widehat{\Omega}_{e_0} \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}} \simeq \operatorname{Spf} O_{\zeta,\eta}$, the action of w is given by

$$\eta \mapsto \frac{p}{-\eta} = -\zeta, \quad \zeta \mapsto \frac{p}{-\zeta} = -\eta,$$

and acts as the (lift) of arithmetic Frobenius on $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$. Notice that w interchanges $\mathcal{L}_1$ and $\mathcal{L}_2$ because it acts semilinearly (over $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$). Using this, it's not hard to see w has the form

$$x_1 \mapsto w_1 x_2, \quad x_2 \mapsto w_2 x_1,$$

where $w_1, w_2 \in O_{\zeta,\eta}^\times$.

An easy computation shows that w_1, w_2 must satisfy the following relation:

$$w_1^p = -w_2. \tag{20}$$

Since $w \in \{g \in \mathrm{GL}_2(\mathbb{Q}_p) \mid \det(g) \in p^{\mathbb{Z}}\}$, we can apply Proposition 5.9, which tells us $x_1 x_2 = \widetilde{\lambda_{\mathcal{L}_1}}$ is invariant by w. So,

$$w_1 w_2 = 1. \tag{21}$$

Combining these together, we get:

Lemma 7.4. *The action of $w = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$ on $\widehat{\Sigma^{\mathrm{nr}}_{1,e_0}}$ is given by*

$$x_1 \mapsto w_1 x_2, \quad x_2 \mapsto w_1^{-1} x_1,$$

where $w_1 \in \mathbb{Z}_{p^2}^\times$ satisfies $w_1^{p+1} = -1$.

Now we are ready to prove the main result of this section:

Proposition 7.5. *$\widehat{\Sigma_1^{\mathrm{nr}}}$ can be descended to a formal scheme $\widehat{\Sigma}_1$ over $\mathbb{Z}_{p^2}$. In fact, $\widehat{\Sigma}_1 = \widehat{\Sigma_1^{\mathrm{nr}}}^p$, the formal scheme defined by the $p \in \mathrm{GL}_2(\mathbb{Q}_p)$-invariant sections of $\widehat{\Sigma_1^{\mathrm{nr}}}$.*

Proof. It suffices to prove this locally, so we only need to work on $\widehat{\Sigma^{\mathrm{nr}}_{1,e}}$. Since $\mathrm{GL}_2(\mathbb{Q}_p)$ acts transitively on this covering, and p is in the center of $\mathrm{GL}_2(\mathbb{Q}_p)$, we can just work with $\widehat{\Sigma^{\mathrm{nr}}_{1,e_0}}$. $\widehat{\Omega}_{e_0} \hat{\otimes} \widehat{\mathbb{Z}_p^{\mathrm{nr}}}$ certainly descends to $\mathbb{Z}_{p^2}$. The question is whether the descents of $\mathcal{L}_1, \mathcal{L}_2, d_1, d_2$ are effective. We show this by explicit computations.

Choose $c \in \mathbb{Z}_p^{\mathrm{nr}}$ such that $c^{p+1} = v_1 w_1^{-1}$, where v_1 is a choice of $(p-1)$-th root of -1, then c is a root of unity, and $\widetilde{\mathrm{Fr}}(c) = c^p$. Define $e = c x_1$, $e' = c^{-1} x_2$. We have

$$w^2(e) = w(\widetilde{\mathrm{Fr}}(c) w_1 x_2) = w(c^p w_1 x_2) = c^{p^2} w_1^p w_1^{-1} x_1 = c^{p^2-1} w_1^{p-1} e = -e.$$

Similarly, $w^2(e') = -e'$. Notice that $p = -w^2$ and -1 acts on x_1 as $\chi_1(-1)^{-1} = -1$ (the action of $\mathbb{Z}_p^\times$ in $\mathrm{GL}_2(\mathbb{Q}_p)$ is the inverse of the action of it in $O_D^\times$). So e and e' are invariant by p, and $\mathcal{L}_1$, $\mathcal{L}_2$ can be descended to $\mathbb{Z}_{p^2}$.

What about d_1, d_2? Now

$$d_1 : e^{\otimes p} \mapsto -c^{p+1} \frac{\eta^p - \eta}{\zeta^{p-1} - 1} e'.$$

Since $c^{p+1} = (-1)^{1/(p-1)} w_1^{-1} \in \mathbb{Z}_{p^2}$, d_1 is defined over $\mathbb{Z}_{p^2}$. A similar argument works for d_2. □

Remark 7.6. Sometimes e also denotes an edge of a graph. I hope that it is clear from the context whether e refers to an edge or a section of $\mathcal{L}_1$ (locally).

Corollary 7.7. (1) *The action of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *can also be defined over* $\widehat{\Sigma}_1$.

(2) $\widehat{\Sigma}_1$ *has an open covering* $\{\widehat{\Sigma_{1,e}}\}_e$ *indexed by the edges of the Bruhat–Tits tree, such that this identification is* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant.*

(3) $\widehat{\Sigma_{1,e}}$ *is isomorphic to*

$$\operatorname{Spf} O_{e,e'} = \operatorname{Spf} \frac{\mathbb{Z}_{p^2}\Big[\zeta, \eta, \frac{1}{1-\zeta^{p-1}}, \frac{1}{1-\eta^{p-1}}, e, e'\Big]^{\widehat{}}}{\Big(e^p + v_1 w_1^{-1} \frac{\eta^p - \eta}{\zeta^{p-1} - 1} e',\ e'^p + v_1^{-1} w_1 \frac{\zeta^p - \zeta}{\eta^{p-1} - 1} e,\ (ee')^{p-1} - p,\ \eta\zeta - p\Big)},$$

where $w_1 = (-1)^{1/(p+1)}$ *is a* (p^2-1)*-th root of unity, and* v_1 *is a choice of* $(p-1)$*-th root of* -1.

(4) *The action of* w *on* $\widehat{\Sigma_{1,e_0}}$ *is given by*

$$e \mapsto v_1 e', \quad e' \mapsto v_1^{-1} e.$$

Remark 7.8. The reason that everything can be defined over $\mathbb{Z}_{p^2}$, I believe, is that the universal formal group can be defined over $\mathbb{Z}_{p^2}$. This is because when we formulate the moduli functor it represents, the "unique" 2-dimensional special formal group of height 4 and all endomorphisms can be defined over $\mathbb{F}_{p^2}$.

8. A semistable model of $\widehat{\Sigma}_1$

In this section, our goal is to work out a semistable model of $\widehat{\Sigma}_1$ as a formal scheme over $\mathbb{Z}_p$ (not $\mathbb{Z}_{p^2}$!). Notice that $\widehat{\Sigma}_1$ has a structural map to $\operatorname{Spec} \mathbb{Z}_{p^2}$. Hence if we change our base from $\mathbb{Z}_p$ to O_{F_0}, then

$$\widehat{\Sigma}_1 \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} O_{F_0} \simeq \widehat{\Sigma}_1 \sqcup \widehat{\Sigma}_1'. \tag{22}$$

Here $\widehat{\Sigma}_1'$ is the same scheme $\widehat{\Sigma}_1$ but with twisted map to O_{F_0}. Recall that F_0 is the unique unramified quadratic extension of $\mathbb{Q}_p$, and we fix an isomorphism between it and $\mathbb{Q}_{p^2}$ in the beginning. Hence we may identify $\widehat{\Sigma}_1$ as a formal scheme over O_{F_0}.

Therefore we only need to work over the scheme $\widehat{\Sigma_1}$ as a scheme over Spec $\mathbb{Z}_{p^2}$, and use the equation above to translate everything into the $\mathbb{Z}_p$-scheme $\widehat{\Sigma_1}$. I hope this won't cause too much confusion.

I say a formal scheme X is a semistable curve over Spec R, where R is a complete discrete valuation ring, if:

(1) The generic fiber of X is smooth over the generic fiber of Spec R.

(2) The special fiber of X is reduced.

(3) Each irreducible component of the special fiber of X is a divisor on X.

(4) Each singular point has an étale neighborhood that is étale over

$$\operatorname{Spec} R[x, y]/(xy - \pi_R),$$

where π_R is a uniformizer of R.

Back to our situation; we first work locally on $\widehat{\Sigma_1}$, so we just work with $\widehat{\Sigma_{1,e}}$. Moreover we can assume $e = e_0$ defined in the previous section and use the results there.

First notice that in $O_{e,e'}$ (see the notation in Corollary 7.7), $ee' = \tilde{u}_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}}$ (see the equation before Corollary 6.8 and recall in the proof of Proposition 7.5, $x_1x_2 = ee'$), so it is a globally defined section on $\widehat{\Sigma_1}$, and satisfies $(ee')^{p-1} - p$. Now if we do base change from $\mathbb{Z}_{p^2}$ to $\mathbb{Z}_{p^2}[p^{1/(p-1)}]$, the generic fiber of Spf $O_{e,e'}[p^{1/(p-1)}]$ will split into $p-1$ connected components. Each connected component corresponds to a choice of $(p-1)$-th root of p. Adjoining $ee'/p^{1(p-1)} = \tilde{u}_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}}/p^{1(p-1)}$ into $O_{e,e'}[p^{1/(p-1)}]$, which I would like to call $O^1_{e,e'}$, the formal scheme also splits into $p-1$ connected components, namely,

$$O^1_{e,e'} = \prod_{\varpi_1^{p-1}=p} O^1_{e,e',\varpi_1}.$$

Explicitly, O^1_{e,e',ϖ_1} is

$$\frac{\mathbb{Z}_{p^2}[p^{1/(p-1)}]\Big[\eta, \zeta, e, e', \dfrac{1}{\eta^{p-1}-1}, \dfrac{1}{\zeta^{p-1}-1}\Big]^\wedge}{\Big(e^p + v_1 w_1^{-1}\dfrac{\eta^p-\eta}{\zeta^{p-1}-1}e',\ e'^p + v_1^{-1}w_1\dfrac{\zeta^p-\zeta}{\eta^{p-1}-1}e,\ ee' - \varpi_1\Big)}.$$

Now, we have (write ϖ_1 as $p^{1/(p-1)}$)

$$\begin{aligned} e^{p+1} &= e^p \cdot e = -v_1 w_1^{-1}\frac{\eta^p - \eta}{\zeta^{p-1}-1}e'e = -w_1^{-1}\frac{\eta^p-\eta}{\zeta^{p-1}-1}v_1 p^{1/(p-1)} \\ &= -w_1^{-1}\frac{\eta^p-\eta}{\zeta^{p-1}-1}(-p)^{1/(p-1)}. \end{aligned}$$

Recall v_1 is a $(p-1)$-th root of -1. This clearly shows that if we adjoin a (p^2-1)-th root of $-p$, then the normalization of this ring contains $e/((-p)^{1/(p-1)})^{1/(p+1)} = e/(-p)^{1/(p^2-1)}$. Similarly, $e'/(-p)^{1/(p^2-1)}$ is also contained in the normalization.

Definition 8.1. Let ϖ be a fixed choice of $(-p)^{1/(p^2-1)}$. Define $F = F_0[\varpi]$, and O_F as the ring of integers inside F.

We change our base from $\operatorname{Spec} \mathbb{Z}_{p^2}$ to $\operatorname{Spec} O_F$ via the fixed identification between $\mathbb{Q}_{p^2}$ and F_0, and take the normalization of $O_{e,e',\varpi_1}[\varpi]$ (it's not hard to verify it's integral). Denote the normalization by $\widetilde{O_{e,e',\varpi_1}}[\varpi]$. I claim basically this is just adjoining e/ϖ, e'/ϖ.

Lemma 8.2. $\widetilde{O_{e,e',\varpi_1}}[\varpi] =$

$$\frac{O_F\Big[\eta, \zeta, \dfrac{1}{\eta^{p-1}-1}, \dfrac{1}{\zeta^{p-1}-1}, \dfrac{e}{\varpi}, \dfrac{e'}{\varpi}\Big]^{\widehat{}}}{\left(\left(\dfrac{e}{\varpi}\right)^{p+1} + v_1 w_1^{-1}\xi\dfrac{\eta^p-\eta}{\zeta^{p-1}-1},\ \left(\dfrac{e'}{\varpi}\right)^{p+1} + v_1^{-1}w_1\xi\dfrac{\zeta^p-\zeta}{\eta^{p-1}-1},\ \dfrac{e}{\varpi}\dfrac{e'}{\varpi} - \xi\varpi^{p-1}\right)},$$

where $\xi = \dfrac{\varpi_1}{\varpi^{p+1}}$ *is a* $(p-1)$*-th root of* -1.

Proof. It's clear both sides become the same after inverting p and certainly the right-hand side is contained in the left-hand side. Thus it suffices to prove the right-hand side is normal. First, since the generic fiber is smooth, there is no singular point on the generic fiber. Now if we modulo ϖ, the uniformizer, it's easy to see the only singular point is the maximal ideal $(e/\varpi, e'/\varpi, \varpi)$. We only need to show $(e/\varpi, e'/\varpi)$ is a regular sequence. Simple calculations indicate that the right-hand side is p-torsion free, so e/ϖ is not a zero divisor. In fact this already proves that the right-hand side is integral. Modulo e/ϖ, the right-hand side becomes $\mathbb{Z}_{p^2}[\varpi]/(\varpi^{p-1})[\zeta, e'/\varpi]/((e'/\varpi)^{p+1} + a(\zeta^p - \zeta))$ for some unit a. The element e'/ϖ is clearly neither a zero divisor, nor a unit. So we're done. $\square$

Remark 8.3. The special fiber of $\widetilde{O_{e,e',\varpi_1}}[\varpi]$ has two irreducible components, defined by $e/\varpi = 0$ and $e'/\varpi = 0$. Each one maps to an irreducible component of the special fiber of $\widehat{\Omega}_{e_0} \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} O_F$, and has the form

$$\mathbb{F}_{p^2}\Big[x, y, \frac{1}{y^{p-1}-1}\Big]/(y^p - y - cx^{p+1}),$$

where c is some root of unity. So each irreducible component is smooth and is an open set of an Artin–Schreier curve. In fact, if we do not split these connected components, then the special fiber is isomorphic to

$$\mathbb{F}_{p^2}\Big[x, y, \frac{1}{y^{p-1}-1}\Big]/\big((y^p - y)^{p-1} + w_1^{-2}x^{p^2-1}\big),$$

which is an open set of a twist of Deligne–Lusztig variety of $\mathrm{GL}_2(\mathbb{F}_p)$ (see [Deligne and Lusztig 1976, Section 2]). More precisely, if we invert x and define $X = 1/x$, $Y = y/x$, this curve now has the form $(XY^p - YX^p)^{p-1} = -w_1^{-2}$.

Notice that $\widetilde{O_{e,e',\varpi_1}}[\varpi]$ is not semistable, because locally the singular point is defined by $(e/\varpi)(e'/\varpi) - \varpi^{p-1}\xi$, where ξ is some unit. To get a semistable model, keep blowing up the singular points until our scheme becomes regular. In fact, we need to blow up $[(p-1)/2]$ times. On the level of special fiber, this singular point will be replaced by $p-2$ rational curves in this process. After this, we finally get our desired semistable model of $\widehat{\Sigma_{1,e_0}} \times_{\operatorname{Spec}\mathbb{Z}_{p^2}} \operatorname{Spec} O_F$.

So far we have been working locally on $\widehat{\Sigma}_1$, but our construction above can be done globally. First, we change the base to $\operatorname{Spec} O_F$ and adjoin $u_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}}/\varpi^{p-1}$ (equivalently, $\widetilde{\lambda_{\mathcal{L}_1}}/\varpi^{p-1}$). Here, since the difference between ϖ^{p-1} and a $(p-1)$-th root of p is a $(p-1)$-th root of -1, it doesn't matter which one we use. Then our formal scheme will split into $p-1$ connected components, indexed by $(p-1)$-th roots of -1. Now take the normalization of each connected component. Call the total space $\widetilde{\Sigma_{1,O_F}}$. For each component, it is clear from the above explicit local description that the dual graph of its special fiber is the same as $\widehat{\Omega}$'s, which is nothing but the Bruhat–Tits tree. Finally, blow up each singular point to get rid of singularities and we end up with a semistable model of $\widehat{\Sigma}_1 \times_{\operatorname{Spec}\mathbb{Z}_{p^2}} \operatorname{Spec} O_F$.

Theorem 8.4. *$\widehat{\Sigma}_1$ (over $\operatorname{Spec}\mathbb{Z}_{p^2}$) has a semistable model $\widehat{\Sigma_{1,O_F}}$ over O_F, such that*:

(1) *$\widehat{\Sigma_{1,O_F}}$ has $(p-1)$ connected components, indexed by $(p-1)$-th roots of -1.*

(2) *The dual graph of the special fiber of each connected component is the graph adding $p-2$ vertices to each edge of the Bruhat–Tits tree.*

(3) *Vertices that come from the Bruhat–Tits tree correspond to some Artin–Schreier curves ($y^{p+1} = c(x^p - x)$ in $\mathbb{P}^2$, where $c \in \mathbb{F}_{p^2}^\times$). Singular points are points with $y = 0$. If we put the $p-1$ connected components together, then a dense open set of it is isomorphic to the Deligne–Lusztig variety of $\mathrm{GL}_2(\mathbb{F}_p)$ over any algebraically closed field.*

(4) *Other vertices correspond to rational curves. Singular points are zero and infinity.*

Proof. We only need to prove our assertion for the special fiber. In the previous discussion, we already know the dual graph of the special fiber of each connected component of $\widetilde{\Sigma_{1,O_F}}$ is the Bruhat–Tits tree. Since blow-ups replace each singular point by $p-2$ rational curves, everything is clear. □

Let $\hat{\pi}$ and $\tilde{\pi}$ be the canonical maps from $\widehat{\Sigma_{1,O_F}}$ and $\widetilde{\Sigma_{1,O_F}}$ to $\widehat{\Omega} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F$. For each edge e of the Bruhat–Tits tree, we can define $\widehat{\Sigma_{1,O_F,e}}$ and $\widetilde{\Sigma_{1,O_F,e}}$ as

$\hat{\pi}^{-1}(\widehat{\Omega}_e \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F)$ and $\tilde{\pi}^{-1}(\widehat{\Omega}_e \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F)$, respectively. Similarly we can define $\widehat{\Sigma_{1,O_F,s}} = \widetilde{\Sigma_{1,O_F,s}}$ for each vertex s. Define $\widehat{\Sigma_{1,O_F,e,\xi}}$, $\widehat{\Sigma_{1,O_F,s,\xi}}$, $\widetilde{\Sigma_{1,O_F,e,\xi}}$, $\widetilde{\Sigma_{1,O_F,s,\xi}}$, where ξ is a $(p-1)$-th root of -1, as the corresponding connected component of $\widehat{\Sigma_{1,O_F,e}}$, $\widehat{\Sigma_{1,O_F,s}}$, $\widetilde{\Sigma_{1,O_F,e}}$, $\widetilde{\Sigma_{1,O_F,s}}$. Note that in the notation of Lemma 8.2, $\widetilde{\Sigma_{1,O_F,e,\xi}} = \operatorname{Spf} \widetilde{O_{e,e',\varpi^{p+1}\xi}}[\varpi]$.

In Lemma 8.2, we have an explicit description of $\widetilde{\Sigma_{1,O_F,e}}$. To simplify notation, I will use $\tilde{e}$, $\tilde{e}'$ for e/ϖ, e'/ϖ. Now let s' be an even vertex (for example, the central vertex s_0'). It's not hard to see that

$$\widehat{\Sigma_{1,O_F,s',\xi}} = \widetilde{\Sigma_{1,O_F,s',\xi}} \simeq \operatorname{Spf} O_F\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p+1} + v_1 w_1^{-1}\xi\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big), \tag{23}$$

$$\widehat{\Sigma_{1,O_F,s'}} = \widetilde{\Sigma_{1,O_F,s'}} \simeq \operatorname{Spf} O_F\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p^2-1} + w_1^2\Big(\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big)^{p-1}\Big). \tag{24}$$

If s is an odd vertex, then similarly we have

$$\widehat{\Sigma_{1,O_F,s,\xi}} = \widetilde{\Sigma_{1,O_F,s,\xi}} \simeq \operatorname{Spf} O_F\Big[\zeta, \frac{1}{\zeta^p-\zeta}, \tilde{e}'\Big]\Big/\Big(\tilde{e}'^{p+1} + v_1^{-1} w_1\xi\frac{\zeta^p-\zeta}{(p/\zeta)^{p-1}-1}\Big), \tag{25}$$

$$\widehat{\Sigma_{1,O_F,s}} = \widetilde{\Sigma_{1,O_F,s}} \simeq \operatorname{Spf} O_F\Big[\zeta, \frac{1}{\zeta^p-\zeta}, \tilde{e}'\Big]\Big/\Big(\tilde{e}'^{p^2-1} + w_1^{-2}\Big(\frac{\zeta^p-\zeta}{(p/\zeta)^{p-1}-1}\Big)^{p-1}\Big). \tag{26}$$

Remark 8.5. If we view $\widehat{\Sigma_1}$ as a $\mathbb{Z}_p$-scheme, then $\widehat{\Sigma_1} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F$ has a semistable model over $\operatorname{Spec} O_F$, which I call $\widehat{\Sigma^{(0)}_{1,O_F}}$. It is canonically isomorphic to $\widehat{\Sigma_{1,O_F}} \sqcup \widehat{\Sigma'_{1,O_F}}$, where $\widehat{\Sigma'_{1,O_F}}$ is isomorphic with $\widehat{\Sigma_{1,O_F}}$ as a scheme, but the structure morphism to $\operatorname{Spec} O_F$ is twisted: $O_F \to O_F$ is the unique automorphism that fixes ϖ and acts as Frobenius on O_{F_0}. We use g_φ to denote it as an element in $\operatorname{Gal}(F/\mathbb{Q}_p)$.

From now on, I will use the exponent (0) for everything that is base changed from $\mathbb{Z}_p$ to O_F. For example, we can define $\widehat{\Sigma^{(0)}_{1,O_F,s}}$, $\widehat{\Sigma^{(0)}_{1,O_F,s,\xi}}$, ... Also we use the exponent $'$ for things with same underlying scheme but with twisted structure morphism to O_F. For example $\widehat{\Sigma'_{1,O_F,s}}$, $\widehat{\Sigma'_{1,O_F,s,\xi}}$, ... Under this notation, we have $\widehat{\Sigma^{(0)}_{1,O_F,s}} = \widehat{\Sigma_{1,O_F,s}} \sqcup \widehat{\Sigma'_{1,O_F,s}}$, ...

9. The action of $\mathrm{GL}_2(\mathbb{Z}_p)$, $\mathrm{Gal}(F/F_0)$, $O_D^\times$ on $\widetilde{\Sigma_{1,O_F}}$ and $\widehat{\Sigma_{1,O_F}}$

By acting on the first factor, we have an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\widehat{\Sigma_1} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec} O_F$ which extends naturally to our semistable $\widehat{\Sigma^{(0)}_{1,O_F}}$. Since $\mathrm{GL}_2(\mathbb{Q}_p)$ will interchange $\widehat{\Sigma_{1,O_F}}$ and $\widehat{\Sigma'_{1,O_F}}$, it does not act on $\widehat{\Sigma_{1,O_F}}$. The reason is that $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathbb{Z}_{p^2}$ by $\widetilde{\mathrm{Fr}}^{v_p(\det(g))}$. However, $\mathrm{GL}_2(\mathbb{Z}_p)$ acts on $\widehat{\Sigma_{1,O_F}}$.

So how does $\mathrm{GL}_2(\mathbb{Z}_p)$ act on the central component $\widehat{\Sigma_{1,O_F,s_0'}}$ of $\widehat{\Sigma_{1,O_F}}$? We have an explicit description above (23), (24). We will fix this identification from now on.

$$\widehat{\Sigma_{1,O_F,s_0',\xi}} = \mathrm{Spf}\, O_{F_0}[\varpi]\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p+1} + v_1 w_1^{-1}\xi \frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big), \tag{27}$$

$$\widehat{\Sigma_{1,O_F,s_0'}} = \mathrm{Spf}\, O_{F_0}[\varpi]\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p^2-1} + w_1^2\big(\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\big)^{p-1}\Big). \tag{28}$$

Proposition 9.1. (1) *The action of* $\mathrm{GL}_2(\mathbb{Z}_p)$ *on* $\widetilde{\Sigma_{1,O_F,s_0'}} = \widehat{\Sigma_{1,O_F,s_0'}}$ *is given by*

$$g(\tilde{e}) \equiv \frac{1}{b\eta+d}\tilde{e} \pmod p, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p). \tag{29}$$

So it factors through $\mathrm{GL}_2(\mathbb{F}_p)$ *when acting on the special fiber.*

(2) $g \in \mathrm{GL}_2(\mathbb{Z}_p)$ *maps* $\widetilde{\Sigma_{1,O_F,s_0',\xi}}$ *to* $\widetilde{\Sigma_{1,O_F,s_0',\xi\chi_1(\det(g))}}$.

Proof. Since $\tilde{e} = e/\varpi$, we can apply Proposition 7.1 here and everything is clear except for the claim that how it interchanges connected components. Notice that the "ξ" component is defined by $\tilde{u}_1^{-p-1}\widetilde{\lambda_{\mathcal{L}_1}} - \varpi^{p+1}\xi$. So our claim follows from Proposition 5.9. □

Corollary 9.2. *The identification of the special fiber of* $\widetilde{\Sigma_{1,O_F,s_0'}}$ *with a Deligne–Lusztig variety is* $\mathrm{GL}_2(\mathbb{F}_p)$*-equivariant.*

We will come back to this point later when we review Deligne–Lusztig theory.

For $\widehat{\Sigma_{1,O_F}}$, since we change our base from O_{F_0} to O_F, there is a natural action of $\mathrm{Gal}(F/F_0)$.

Definition 9.3. $\tilde{\omega}_2 : \mathrm{Gal}(F/F_0) \to O_{F_0}^\times$ is the character given by $\tilde{\omega}_2(g) = \dfrac{g(\varpi)}{\varpi}$.

Any other character is a multiple of $\tilde{\omega}_2$.

Remark 9.4. Another equivalent definition of $\tilde{\omega}_2$ is as follows: By local class field theory, it suffices to give a character of $F_0^\times$. This character is trivial on $p^{\mathbb{Z}}$, and on $O_{F_0}^\times$ it is given by first reducing modulo p, then taking the inverse of the Teichmüller character. Our convention on the local Artin map is that uniformizers correspond to arithmetic Frobenius elements.

Remark 9.5. Recall that we defined two characters χ_1, χ_2 of $(O_D/\Pi)^\times$ (see Definition 3.2). Using the above remark, the relation of χ_1 and $\tilde{\omega}_2$ can be described in the following diagram:

$$\begin{tikzcd}
\mathbb{Z}_{p^2}^\times \simeq O_{F_0}^\times \arrow[r, "\mathrm{Art}_{F_0}"] \arrow[d] & \mathrm{Gal}(\overline{F_0}/F_0)^{\mathrm{ab}} \arrow[d, "\tilde{\omega}_2^{-1}"] \\
O_D^\times \arrow[r, "\chi_1"] & \mathbb{Z}_{p^2}^\times \simeq O_{F_0}^\times
\end{tikzcd}$$

where the left arrow is our fixed embedding of $\mathbb{Z}_{p^2}$ into O_D, Art_{F_0} is the Artin map in local class field theory, the isomorphism between $\mathbb{Z}_{p^2}^\times$ and $O_{F_0}^\times$ is the one we fixed in the beginning.

Under the isomorphisms (23)–(26), we have:

Proposition 9.6. *The action of $g \in \mathrm{Gal}(F/F_0)$ is given by*

$$g(\tilde{e}) = \tilde{\omega}_2(g)^{-1}\tilde{e}, \quad g(\tilde{e}') = \tilde{\omega}_2(g)^{-1}\tilde{e}'. \tag{30}$$

This is trivial because $\tilde{e} = e/\varpi$, and $\tilde{e}' = e'/\varpi$.

The last group action we want to consider here is the action of $O_D^\times$.

Proposition 9.7. *Under the isomorphisms (23)–(26), for $d \in O_D^\times$,*

$$d(\tilde{e}) = \chi_1(d)\tilde{e}, \quad d(\tilde{e}') = \chi_2(d)\tilde{e}'. \tag{31}$$

Remark 9.8. The action of $O_D^\times$ on $\widehat{\Sigma'_{1,O_F}}$ is a twist of what we considered above:

$$d(\tilde{e}) = \widetilde{\mathrm{Fr}}(\chi_1(d))\tilde{e} = \chi_2(d)\tilde{e} = \chi_1(d)^p\tilde{e}, \qquad \forall d \in O_D^\times. \tag{32}$$

$$d(\tilde{e}') = \widetilde{\mathrm{Fr}}(\chi_2(d))\tilde{e}' = \chi_1(d)\tilde{e}' = \chi_2(d)^p\tilde{e}', \quad \forall d \in O_D^\times. \tag{33}$$

Here I identify $\widehat{\Sigma'_{1,O_F}}$ with $\widehat{\Sigma_{1,O_F}}$ but with twisted structure morphism. And by saying $\chi_2(d)$ I consider it as an element in the "O_F" coming from the structure map, not the $\mathbb{Z}_{p^2}$ coming from the original scheme $\widehat{\Sigma}_1$. However, the action of $\mathrm{Gal}(F/F_0)$ is the same, not twisted. Another way to see this is using a $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ with $v_p(\det(g))$ odd, then g sends $\widehat{\Sigma_{1,O_F,s}}$ to $\widehat{\Sigma'_{1,O_F,sg}}$. Finally, $g_\varphi \in \mathrm{Gal}(F/\mathbb{Q}_p)$ interchanges $\widehat{\Sigma_{1,O_F}}$ and $\widehat{\Sigma'_{1,O_F}}$ by acting as Frobenius endomorphism on O_{F_0} but fixes other things under the isomorphisms (23)–(26).

10. Another admissible open covering of the Drinfel'd upper half-plane and the generic fiber of $\widehat{\Sigma_{1,O_F}}$

In this section, we work on the generic fiber of everything we considered before. The main result of this section is a description of the generic fiber $\Sigma_{1,F}$ of $\widehat{\Sigma_{1,O_F}}$ (and a similar result for the generic fiber $\Sigma_{1,F}^{(0)}$ of $\widehat{\Sigma_{1,O_F}^{(0)}}$).

Recall that Σ_1 is the generic fiber of $\widehat{\Sigma_1^{\mathrm{nr}}}$. The latter is defined by two line bundles, $\mathcal{L}_1$, $\mathcal{L}_2$, and maps

$$d_1 : \mathcal{L}_1^{\otimes p} \to \mathcal{L}_2, \quad d_2 : \mathcal{L}_2^{\otimes p} \to \mathcal{L}_1 \tag{34}$$

(see the beginning of Section 4). Denote by $\mathcal{L}_{1,\eta}$, $\mathcal{L}_{2,\eta}$, $d_{1,\eta}$, $d_{2,\eta}$ the restriction of the corresponding item to Σ_1, the generic fiber.

First we observe:

Lemma 10.1. *Any line bundle over $\mathcal{X}_0$, the generic fiber of the Drinfel'd upper half-plane (and base changed to $\widehat{\mathbb{Z}_p^{\mathrm{nr}}}$), is trivial.*

To do this we need another admissible open covering of $\mathcal{X}_0$, which is described in [Drinfel'd 1974] ("topological" analog) and in [Schneider and Stuhler 1991] in detail. Let me recall it now.

Define

$$U_n(\mathbb{C}_p) = \{z \in C_p \mid |z| \le p^n,\ |z-a| \ge p^{-n},\ \forall a \in \mathbb{Q}_p\}, \tag{35}$$

where $|\cdot|$ is the canonical norm on $\mathbb{C}_p$ such that $|p| = p^{-1}$. Notice that we only need finitely many a to define this set, so U_n can be identified as an open set of $\mathbb{P}^1$ by removing some open discs. Therefore U_n is an affinoid space. In fact, we can identify it as an affinoid subdomain of a closed unit ball.

Remark 10.2. Another way to construct U_n is by using the formal model we already have. We can define a distance of two vertices of the Bruhat–Tits tree by counting the number of edges on the unique path between these two vertices. For example, two adjacent vertices have distance 1 and any vertex has distance 0 with itself. Now define Z_n as the set of vertices having distance $\le n$ from the central vertex. Let Ω_{U_n} be the union of Ω_e such that e is an edge between two vertices in Z_n and $\Omega_{U_0} = \Omega_{s_0'}$. Then U_n is the generic fiber of Ω_{U_n}.

It is clear $U_n \subset U_{n+1}$ and $\bigcup U_n = \Omega$, the Drinfel'd upper half-plane. Also it's not hard to verify the open covering $\{U_n\}$ is admissible. Let O_{U_n} be the ring of rigid analytic functions on U_n (over $\mathbb{Q}_p$). The key property we need is:

Lemma 10.3. *The image of the canonical inclusion $\phi_n : O_{U_{n+1}} \to O_{U_n}$ is dense under the canonical topology on O_{U_n}.*

Proof. Choose $a_1, \dots, a_m \in \mathbb{Q}_p$ such that $\{B(a_i, p^{-n})\}_i$ is an open covering of $p^{-n}\mathbb{Z}_p$ in $\mathbb{Q}_p$, where $B(a_i, p^{-n})$ is the open ball centered at a_i of radius p^{-n} in $\mathbb{Q}_p$. Now when we define U_n, we can use $a_1, \dots, a_m$ rather than all $a \in \mathbb{Q}_p$. Thus,

$$O_{U_n} = \left\{ F(z) = \sum_{k=0}^{+\infty} b_{0,k}(p^n z)^k + \sum_{i=1}^{m}\sum_{k=0}^{+\infty} b_{i,k}\left(\frac{p^n}{z-a_i}\right)^k \,\middle|\, b_{i,k} \in \mathbb{Q}_p,\ \lim_{k\to+\infty} b_{i,k} = 0, \forall i \right\}.$$

We define a norm $|\cdot|_n$ on O_{U_n} by $|F(z)|_n = \sup_{i,k}|b_{i,k}|$. This is nothing but the supremum norm: $|f|_n = \sup_{x\in \mathrm{Spm}\, O_{U_n}} |f(x)|$. Now the $\mathbb{Q}_p$-algebra generated by $z, 1/(z-a_i)$ $(i = 1, \dots, m)$ is dense in O_{U_n}. But these functions are defined over Ω and so live in $O_{U_{n+1}}$. $\square$

Remark 10.4. Notice that in fact $p^n z$, $p^n/(z-a_i)$ $(i = 1, \dots, m)$ are affinoid generators of O_{U_n} over $\mathbb{Q}_p$ in the sense there exists a surjective map from the Tate algebra $\mathbb{Q}_p\langle T_0, \dots, T_m\rangle$ to O_{U_n} that sends T_0 to $p^n z$ and other T_i to $p^n/(z-a_i)$. If we restrict $p^n z$ or $p^n/(z-a_i)$ to U_{n-1}, by definition of U_{n-1}, its norm is less than 1 (in fact $\le p^{-1}$). From this description, it's easy to see U_{n-1} is relatively compact in U_n. See [Bosch 2014, §6.3] for a precise definition. A direct corollary of this is that the inclusion map $O_{U_n} \to O_{U_{n-1}}$ is a strictly completely continuous map in the sense of [Bosch 2014, §6.4 Definition 1]. Another consequence is that Ω is a Stein-space as defined in [Kiehl 1967].

Now we return to the proof of Lemma 10.1. We still need one more lemma:

Lemma 10.5. *Any line bundle on U_n is trivial.*

Proof. It suffices to prove O_{U_n} is a principal ideal domain. It's obvious that O_{U_n} is regular and hence normal. So we only need to show every maximal ideal of O_{U_n} is principal. But we know U_n is an affinoid subdomain of a (one dimensional) closed unit ball by removing several open discs centered at $\mathbb{Q}_p$-points, with radius$\in p^{\mathbb{Z}}$. Our claim follows from the fact that $\mathbb{Q}_p\langle T\rangle$, the Tate algebra, is a PID [Bosch 2014, §2.2 Corollary 10]. □

Proof of Lemma 10.1. I learned this argument from [Kiehl 1967, proof of Satz 2.4]. Since $\{U_n\}_n$ is an admissible open covering of Ω and every line bundle on U_n is trivial, a line bundle on Ω is equivalent with a $1-$cocycle: $\{f_{ij}\}_{i<j},\ f_{ij} \in O_{U_i}^{\times}$, such that

$$f_{ij}\phi_{ji}(f_{jk}) = f_{ik}$$

for $i < j < k$, where ϕ_{ji} is the canonical inclusion from O_{U_j} to O_{U_i}. It's easy to see that $f_{12}, f_{23}, \ldots$ determine all f_{ij}. Two cocycles $\{f_{i(i+1)}\}, \{f'_{i(i+1)}\}$ define the same line bundle if and only if there exists $\{g_i\},\ g_i \in O_{U_i}^{\times}$, such that

$$f_{i(i+1)}g_i\phi_i(g_{i+1})^{-1} = f'_{i(i+1)}, \quad \forall i \geq 1.$$

Now let $\{f_{i(i+1)}\}$ be a fixed cocycle. Define $g'_1 = 1 \in O_{U_1}$. Thanks to Lemma 10.3, we can find $g'_{i+1} \in O_{U_i},\ i \geq 1$ by induction, satisfying

$$|1 - g'_i f_{i(i+1)}\phi_i(g'_{i+1})^{-1}|_i < \frac{1}{2^i}.$$

This implies, after modifying our cocycle, we can assume $|1 - f_{i(i+1)}|_i < \frac{1}{2^i}$. Now define $g_i = \prod_{j=i}^{\infty} \phi_{ji}(f_{j(j+1)})^{-1}$. Here ϕ_{ii} is the identity map. Notice that $|f|_j \geq |\phi_{ji}(f)|_i$ for $f \in O_{U_j}$; see Remark 10.4. So the infinite product makes sense by our assumption. But now $f_{i(i+1)}g_i\phi_i(g_{i+1})^{-1} = 1$. Therefore it corresponds to a trivial line bundle. □

Although our proof is working over the base field $\mathbb{Q}_p$, the argument still works if we change the base to other fields.

Corollary 10.6. *$\mathcal{L}_{1,\eta}$ and $\mathcal{L}_{2,\eta}$ are trivial line bundles.*

Now let E_1 be a basis of $\mathcal{L}_{1,\eta}$ and E_1^* the dual basis of E_1 under the isomorphism $\lambda_{\mathcal{L}_1}$. Then d_1, d_2 become two elements U_1, U_2 in $H^0(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ such that $\mathcal{X}_1$ is now defined by

$$\mathcal{O}_{\mathcal{X}_0}[E_1, E_1^*]/(E_1^p - U_1E_1^*, (E_1^*)^p - U_2E_1).$$

We know $E_1E_1^* = \widetilde{\lambda_{\mathcal{L}_1}}$, so $U_1U_2 = -w$ (see Corollary 5.5). Σ_1 is

$$\mathcal{O}_{\mathcal{X}_0}[E_1, E_1^*]/\big(E_1^p - U_1E_1^*, (E_1^*)^p - U_2E_1, (E_1E_1^*)^{p-1} + w\big).$$

Since w is invertible on the generic fiber, so is U_1. We can write $E_1^* = E_1^pU_1^{-1}$.

Proposition 10.7. $\Sigma_1 = \mathcal{O}_{\mathcal{X}_0}[E_1]/(E_1^{p^2-1} + U_1^{p-1}w).$

In other words, Σ_1 is $\mathcal{X}_0$ adjoined with a (p^2-1)-th root of a rigid analytic function on $\mathcal{X}_0$.

Remark 10.8. If we are careful enough in the beginning and take E_1 to be $p \in \mathrm{GL}_2(\mathbb{Q}_p)$-invariant, we can descend our description to O_{F_0}. This means we have the same description of the generic fiber $\Sigma_{1,F}$ of $\widehat{\Sigma_{1,O_F}}$.

Corollary 10.9. *$\Sigma_{1,F}$ is a Stein-space.*

Proof. As we remarked before (Remark 10.4), U_n is relatively compact in U_{n+1}. It's easy to see the open set of $\Sigma_{1,F}$ above U_n, which we denote by $V_{n,F}$ is an affinoid space and relatively compact in $V_{n+1,F}$. □

11. De Rham cohomology of $\Sigma_{1,F}$ and $\Sigma^{(0)}_{1,F}$

Let $\Omega^1_{\Sigma_{1,F}}$ be the sheaf of holomorphic differential forms on $\Sigma_{1,F}$ and $\Omega^0_{\Sigma_{1,F}} = \mathcal{O}_{\Sigma_{1,F}}$. Then we can consider the de Rham complex:

$$0 \to \Omega^0_{\Sigma_{1,F}} \xrightarrow{d} \Omega^1_{\Sigma_{1,F}}, \tag{36}$$

where d is the usual derivation. Define the de Rham cohomology:

Definition 11.1. $H^i_{\mathrm{dR}}(\Sigma_{1,F}) \overset{\mathrm{def}}{=} i$-th hypercohomology of the de Rham complex.

Remark 11.2. In a pair of papers Große-Klönne [2000; 2004] introduced a theory of de Rham cohomology for rigid analytic spaces. His approach uses the overconvergent de Rham complex rather than the usual De Rham complex. However in our case, they are the same since $\Omega_{\Sigma_{1,F}}$ is a Stein space [Große-Klönne 2000, Theorem 3.2].

Thanks to Kiehl [1967, Satz 2.4.2], we know that all higher cohomology groups of $\Omega^0_{\Sigma_{1,F}}, \Omega^1_{\Sigma_{1,F}}$ vanish:

Proposition 11.3 (de Rham cohomology).

$$H^0_{\mathrm{dR}}(\Sigma_{1,F}) = \ker(H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}}) \xrightarrow{d} H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})) = F, \tag{37}$$

$$H^1_{\mathrm{dR}}(\Sigma_{1,F}) = \operatorname{coker}(H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}}) \xrightarrow{d} H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})), \tag{38}$$

$$H^i_{\mathrm{dR}}(\Sigma_{1,F}) = 0, \ \forall i \geq 2. \tag{39}$$

We can put a certain topology on $H^1_{\mathrm{dR}}(\Sigma_{1,F})$. This is done by writing:

$$H^0(\Sigma_{1,F}, \Omega^i_{\Sigma_{1,F}}) = \varprojlim_n H^0(V_{n,F}, \Omega^i_{\Sigma_{1,F}}) \quad \text{for } i = 0, 1.$$

See the proof of Corollary 10.9 for the notation. Since each $H^0(V_{n,F}, \Omega^i_{\Sigma_{1,F}})$ is a Banach space and has a canonical topology, we can equip $H^0(\Sigma_{1,F}, \Omega^i_{\Sigma_{1,F}})$ with the projective limit topology. Now $V_{n,F}$ is relatively compact in $V_{n+1,F}$. As we observed in Remark 10.4, the transition map from $H^0(V_{n+1,F}, \Omega^i_{\Sigma_{1,F}})$ to $H^0(V_{n,F}, \Omega^i_{\Sigma_{1,F}})$ is completely continuous. Using Corollary 16.6 of [Schneider 2002], we have (notice that a completely continuous map between two Banach spaces is compact; see Proposition 18.11 of [Schneider 2002]):

Proposition 11.4. *$H^0(\Sigma_{1,F}, \Omega^i_{\Sigma_{1,F}})$, $i = 0, 1$ is a reflexive Fréchet space.*

See page 55 of [Schneider 2002] for the definition of reflexive.

Proposition 11.5 [Große-Klönne 2004, Corollary 3.2]. *The image of the derivation map $d : H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}}) \to H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})$ is closed.*

Corollary 11.6. *$H^1_{\mathrm{dR}}(\Sigma_{1,F})$ is a Fréchet space.*

But how to compute de Rham cohomology? We need our semistable $\widehat{\Sigma_{1,O_F}}$ constructed in Section 8. Let $E(\widehat{\Sigma_{1,O_F}})$ (resp. $V(\widehat{\Sigma_{1,O_F}})$) be the set of singular points (resp. irreducible components) of the special fiber of $\widehat{\Sigma_{1,O_F}}$. By definition, we can identify them as the set of edges (resp. vertices) of the dual graph of the special fiber. Now fix an orientation for each edge $e \in E(\widehat{\Sigma_{1,O_F}})$, and we use $v^+(e)$ (resp. $v^-(e)$) to denote the target (resp. source) vertex of the orientation.

Definition 11.7. Let U_e (resp. U_v) be the tubular neighborhood of the singular point indexed by e (resp. irreducible component indexed by v).

It is clear that $\{U_v\}_v$ is an admissible open covering of Σ_{1,O_F}. Hence:

Lemma 11.8. *We have a long exact sequence of de Rham cohomologies*:

$$0 \to H^0_{\mathrm{dR}}(\Sigma_{1,F}) \longrightarrow \prod_{v \in V(\widehat{\Sigma_{1,O_F}})} H^0_{\mathrm{dR}}(U_v) \xrightarrow{a} \prod_{e \in E(\widehat{\Sigma_{1,O_F}})} H^0_{\mathrm{dR}}(U_e) \xrightarrow{\partial} H^1_{\mathrm{dR}}(\Sigma_{1,F})$$

$$\longrightarrow \prod_{v \in V(\widehat{\Sigma_{1,O_F}})} H^1_{\mathrm{dR}}(U_v) \xrightarrow{b} \prod_{e \in E(\widehat{\Sigma_{1,O_F}})} H^1_{\mathrm{dR}}(U_e),$$

where the arrows without labels are canonical restriction maps, and a, b are the canonical restriction maps to $v^+(e)$ minus the restriction map to $v^-(e)$ for an element indexed by e.

Here the de Rham cohomologies of U_e, U_v are defined by the same method as above. We note that they are not affinoid but Stein spaces.

We first look at U_e, the tubular neighborhood of a singular point. It's not hard to see from the explicit description in Lemma 8.2 that U_e is an annulus $\{T \mid |\varpi| < |T| < 1\}$. So its de Rham cohomology is: $H^0_{\mathrm{dR}}(U_e) = F$, generated by the constant function; $H^1_{\mathrm{dR}}(U_e) \simeq F$, generated by dT/T, where T is a coordinate of U_e.

In Lemma 8.2, although we haven't resolved the singularities there, $d\tilde{e}/\tilde{e}$ still makes sense on the generic fiber, and it generates all of $H^1_{\mathrm{dR}}(U_e)$ for any singular point e above the singularity there. In fact, the process of resolving the singularities $xy - \varpi^n$ is just "dividing" the annulus into several small annuli. For example, the tubular neighborhood of $xy - \varpi^n$ can be thought as the annulus $\{T \mid |\varpi|^n < T < 1\}$. For any e above this singular point, U_e can be identified as $\{T \mid |\varpi|^{l+1} < T < |\varpi|^l\}$ for some $l < n$.

Recall that $O_D^\times$ acts as characters on $\tilde{e}$, so acts trivially on $H^0_{\mathrm{dR}}(U_e)$, $H^1_{\mathrm{dR}}(U_e)$.

What about U_v? There are two possibilities. One is that v corresponds to a rational curve. U_v is an annulus and the result is the same as U_e. In particular $O_D^\times$ acts trivially on their de Rham cohomologies.

The other one is more interesting. We will compute it in the next section. Some notation here: recall that every such vertex can be indexed by (s, ξ), where s is a vertex of the Bruhat–Tits tree and ξ satisfies $\xi^{p-1} = -1$.

Definition 11.9. From now on we will use (s, ξ) to denote these vertices.

Definition 11.10. Denote the irreducible component indexed by (s, ξ) by $\overline{U_{s,\xi}}$ and its generic fiber by $U_{s,\xi}$. We also denote the smooth loci of $\overline{U_{s,\xi}}$ by $\overline{U^0_{s,\xi}}$ (viewed as a subscheme in the special fiber of $\widehat{\Sigma_{1,O_F}}$). Notice that this is nothing but the special fiber of $\widehat{\Sigma_{1,O_F,s,\xi}} = \widetilde{\Sigma_{1,O_F,s,\xi}}$. Define

$$\overline{U_s} = \bigcup_{\xi^{p-1}=-1} \overline{U_{s,\xi}},$$

and $\overline{U^0_s}$ similarly.

Recall that in the beginning, we fix a finite extension E of $\mathbb{Q}_p$ that is large enough and define $\chi(E)$ as the set of characters of $(O_D/\Pi)^\times$ with values in $E^\times$.

$O_D^\times$ acts naturally on $H^1_{\mathrm{dR}}(\Sigma_{1,F}) \otimes_{\mathbb{Q}_p} E$ by acting on the first factor. Since the action of $O_D^\times$ on $\Sigma_{1,F}$ factors through $O_D^\times/(1 + \Pi O_D)$, we can decompose

$H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E$ as

$$H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E = \bigoplus_{\chi\in\chi(E)} (H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi}, \tag{40}$$

where $(H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi} = \{a \mid d(a) = (1\otimes\chi(d))a,\ \forall d\in O_D^{\times}\}$ is the χ-isotypic component.

Now tensor everything in the long exact sequence of Lemma 11.8 with E, and take the χ-isotypic component for a nontrivial character $\chi\in\chi(E)$. As we explained above, $O_D^{\times}$ acts trivially on the cohomology of any annulus, so only the de Rham cohomology of U_s contributes. In other words:

Lemma 11.11. *For a nontrivial character* χ,

$$\begin{aligned}(H^1_{\mathrm{dR}}(\Sigma_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi} &\simeq \prod_s (H^1_{\mathrm{dR}}(U_s)\otimes_{\mathbb{Q}_p} E)^{\chi},\\ (H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi} &\simeq \prod_s (H^1_{\mathrm{dR}}(U^{(0)}_s)\otimes_{\mathbb{Q}_p} E)^{\chi}\\ &= \prod_s \big((H^1_{\mathrm{dR}}(U_s)\oplus H^1_{\mathrm{dR}}(U'_s))\otimes_{\mathbb{Q}_p} E\big)^{\chi},\end{aligned}$$

where s takes value in the set of vertices of the Bruhat–Tits tree.

It's clear that $\mathrm{GL}_2(\mathbb{Q}_p)$ preserves $(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi}$ because the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ commutes with $O_D^{\times}$. Also $g\in\mathrm{GL}_2(\mathbb{Q}_p)$ induces an isomorphism from $U^{(0)}_s$ to $U^{(0)}_{sg}$, hence an isomorphism from $H^1_{\mathrm{dR}}(U^{(0)}_{sg})$ to $H^1_{\mathrm{dR}}(U^{(0)}_s)$. Note that the set of vertices of the Bruhat–Tits tree is nothing but $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^{\times}\backslash\mathrm{GL}_2(\mathbb{Q}_p)$. Thus we have:

Proposition 11.12. *As a representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *over* E, *we have*

$$(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi} \simeq \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^{\times}} (H^1_{\mathrm{dR}}(U^{(0)}_{s'_0})\otimes_{\mathbb{Q}_p} E)^{\chi}$$

for any nontrivial character $\chi\in\chi(E)$. Recall that s'_0 is the central vertex. Here the induction has no restriction on the support.

12. An F_0-structure of $(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^{\chi}$ and the computation of $H^1_{\mathrm{dR}}(U_{s'_0})$

Recall that F_0 is the maximal unramified extension of $\mathbb{Q}_p$ inside F and we fixed an isomorphism between it and $\mathbb{Q}_{p^2}$ in the beginning.

Following Coleman and Iovita [1999], we can define an F_0/Frobenius structure on the de Rham cohomology $H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})$. This means we can find an F_0-linear subspace H_{F_0} equipped with a $\widetilde{\mathrm{Fr}}$-linear Frobenius morphism, such that $H_{F_0}\otimes_{F_0} F\simeq H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})$. Let's recall their construction in our situation now.

By Lemma 11.11, we only need to define an F_0/Frobenius structure on each $(H^1_{\mathrm{dR}}(U^{(0)}_s)\otimes_{\mathbb{Q}_p} E)^{\chi}$, in fact, each $(H^1_{\mathrm{dR}}(U_{s,\xi})\otimes_{\mathbb{Q}_p} E)^{\chi}$ (using the notation from

Definition 11.7). Theorem C of [Große-Klönne 2002] tells us we have a natural isomorphism between $H^1_{\mathrm{dR}}(U_{s,\xi})$ and $H^1_{\mathrm{rig}}(\overline{U^0_{s,\xi}}/F)$, the rigid cohomology of $\overline{U^0_{s,\xi}}$ with coefficients in F defined in [Berthelot 1986]. Recall that $\overline{U^0_{s,\xi}}$ is an open set of $\overline{U_{s,\xi}}$ by removing $(p+1)$ $\mathbb{F}_p$-rational points (each corresponds to an edge connecting s). Then we have the following exact sequence:

$$0 \to H^1_{\mathrm{rig}}(\overline{U_{s,\xi}}/F) \to H^1_{\mathrm{rig}}(\overline{U^0_{s,\xi}}/F) \to F^{\oplus p+1} \to F \to 0. \tag{41}$$

Explicitly, we can construct an isomorphism $\psi_{s,\xi} : U_{s,\xi} \to F_{1,\xi}$, where $F_{1,\xi}$ is defined as

$$\left\{(x, y) \in \mathbb{A}^2_F \;\middle|\; y^{p+1} = v_1^{-1} w_1 \xi (x^p - x),\ |x-k| > p^{-1/(p-1)},\ k = 0, 1, \ldots, p-1, \ |x| < p^{1/(p-1)}\right\}$$

for an odd vertex s (even case is similar). If we restrict this isomorphism to the generic fiber of $\widehat{\Sigma_{1,O_F,s,\xi}}$ and use the description in (25), it is given by

$$x \mapsto \zeta, \quad y \mapsto \tilde{e}'(1-(p/\zeta)^{p-1})^{1/(p+1)},$$

where $(1-(p/\zeta)^{p-1})^{1/(p+1)} = 1 - 1/(p+1)(p/\zeta)^{p-1} + \cdots$ is the binomial expansion. The rigid space $F_{1,\xi}$ is clearly an open set of a projective curve $D_{1,\xi}$ in $\mathbb{P}^2_F$ defined by $y^{p+1} = v_1^{-1} w_1 \xi (x^p - x)$. We note that $D_{1,\xi} - F_{1,\xi}$ is a union of $p+1$ closed discs. Each disc is centered at a point with zero y-coordinate. We denote these points by $C_0, \ldots, C_p$. Then, we have

$$0 \to H^1_{\mathrm{dR}}(D_{1,\xi}) \longrightarrow H^1_{\mathrm{dR}}(F_{1,\xi}) \xrightarrow{\mathrm{Res}} \bigoplus_{i=0}^{p} F \xrightarrow{\mathrm{sum}} F \to 0, \tag{42}$$

where Res is the residue map to each C_i, and sum is taking the sum. A proof of this can be found in Section IV of [Coleman 1989]. Notice that $D_{1,\xi}$ has an obvious formal model over O_F (in fact over O_{F_0}!), and its special fiber is nothing but $\overline{U_{1,\xi}}$. So we have a natural isomorphism between $H^1_{\mathrm{dR}}(D_{1,\xi})$ and $H^1_{\mathrm{rig}}(\overline{U_{1,\xi}})$. Using these isomorphisms, we can identify the two exact sequences (41), (42) with each other.

It is not hard to see $O_D^\times$ acts trivially on the residues. For example, near $x = y = 0$, $t = y/(1-x^{p-1})^{1/(p+1)}$ is a local coordinate. $O_D^\times$ acts as a character on y and acts trivially on x, hence acts trivially on dt/t. Therefore if we tensor the exact sequence (41) with E and take the χ-isotypic component, we obtain:

Lemma 12.1. $(H^1_{\mathrm{dR}}(U_s) \otimes_{\mathbb{Q}_p} E)^\chi \simeq (H^1_{\mathrm{rig}}(\overline{U}_s/F) \otimes_{\mathbb{Q}_p} E)^\chi$.

Since we have a natural isomorphism $H^1_{\mathrm{rig}}(\overline{U}_s/F) \simeq H^1_{\mathrm{crys}}(\overline{U}_s/F_0) \otimes_{F_0} F$, there is an F_0/Frobenius structure on $(H^1_{\mathrm{rig}}(\overline{U}_s/F) \otimes_{\mathbb{Q}_p} E)^\chi$ and thus on $(H^1_{\mathrm{dR}}(U_s) \otimes_{\mathbb{Q}_p} E)^\chi$. Here $H^1_{\mathrm{crys}}(\overline{U}_s/F_0)$ is the first crystalline cohomology of U_s tensored with $\mathbb{Q}_p$. Explicitly, as we mentioned above, $D_{1,\xi}$ can be defined over F_0 and its formal model $\widehat{D_{1,O_{F_0},\xi}}$ over O_{F_0} is a smooth lifting of $\overline{U_{s,\xi}}$. So the de Rham cohomology of $\widehat{D_{1,O_{F_0},\xi}}$ can be identified with the crystalline cohomology of $\overline{U_{s,\xi}}$. Thus we obtain an F_0-linear subspace inside $H^1_{\mathrm{dR}}(D_{1,\xi})$. But to get a Frobenius operator, we need to identify it with the crystalline cohomology.

Remark 12.2. For an even vertex s', we can define similar objects:

$$\psi_{s',\xi} : U_{s',\xi} \to F_{0,\xi},\, D_{0,\xi},\, \widehat{D_{0,O_{F_0},\xi}},\, \dots .$$

In summary, combining the above results with Proposition 11.12, we have:

Proposition 12.3. *$(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E)^{\chi}$ has an F_0/Frobenius structure that comes from the crystalline cohomology of the special fiber of $\widehat{\Sigma^{(0)}_{1,O_F}}$. More precisely, under the identification of $(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E)^{\chi}$ with*

$$\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^{\times}} (H^1_{\mathrm{dR}}(U^{(0)}_{s_0'}) \otimes_{\mathbb{Q}_p} E)^{\chi},$$

the F_0-subspace is

$$\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^{\times}} (H^1_{\mathrm{crys}}(\overline{U^{(0)}_{s_0'}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi},$$

and the Frobenius operator is defined in the obvious way.

Remark 12.4. We can also define a monodromy operator, but for any χ such that $\chi \neq \chi^p$ it is zero on $(H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E)^{\chi}$. The reason is that the definition of monodromy operator uses the cohomologies of the tubes of the singular points, which do not contribute to the cohomology we are interested in. See [Coleman and Iovita 1999] for the precise definition of monodromy operator.

As we remarked before, $\overline{U_{s_0'}}$ has a close relation with the Deligne–Lusztig variety of $\mathrm{GL}_2(\mathbb{F}_p)$ (Corollary 9.2), which we call DL. In fact, the open set

$$\overline{U^0_{s_0'}} \simeq \operatorname{Spec} \mathbb{F}_{p^2}\Big[\eta, \tilde{e}, \frac{1}{\tilde{e}}\Big] \big/ (\tilde{e}^{p^2-1} + w_1^2(\eta^p - \eta)^{p-1})$$

is $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariantly isomorphic with DL over the algebraically closed field (or up to taking a transpose of $\mathrm{GL}_2(\mathbb{F}_p)$). So we can apply Deligne–Lusztig theory (established in [Deligne and Lusztig 1976]). Although Deligne and Lusztig [1976] use l-adic cohomology, their results can be applied directly to crystalline cohomology thanks to Katz and Messing [1974] and Gillet and Messing [1987]. Notice that the action of $O_D^{\times}$ on $\overline{U^0_{s_0'}}$, which factors through $O_D^{\times}/(1+\Pi O_D)$, can be identified with the inverse of the action of a nonsplit torus $(\mathrm{T}(w)^F$ in [Deligne and Lusztig 1976]) of $\mathrm{GL}_2(\mathbb{F}_p)$.

Theorem 12.5. *Let $\chi(F_0)$ be the character group of $O_D^{\times}/(1+\Pi O_D)$ with values in F_0* (it's generated by χ_1; see Definition 3.2). *We can decompose*

$$H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/F_0) = \bigoplus_{\chi' \in \chi(F_0)} H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/F_0)^{\chi'}$$

into the sum of different χ'-isotypic components. Each component has a natural action of $\mathrm{GL}_2(\mathbb{F}_p)$. *Then*:

(1) $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} = 0$ *if and only if* $\chi' = (\chi')^p$.

(2) *If* $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \neq 0$, *it's an irreducible representation of* $\mathrm{GL}_2(\mathbb{F}_p)$.

(3) $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \simeq H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{(\chi')^p}$ *and these are the only isomorphisms among these nonzero representations.*

Definition 12.6. Define ρ_χ as the representation $(H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)\otimes_{F_0} E)^{\chi}$ of $\mathrm{GL}_2(\mathbb{F}_p)$, for any $\chi \in \chi(E)$. The theorem above guarantees that different choices of embedding $F_0 \to E$ give the same representation.

Remark 12.7. $\mathrm{Gal}(F/F_0)$ also acts on $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$. By the results in Section 9, we have

$$H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} = H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\tilde{\omega}_2^{i(\chi')}},$$

the $\tilde{\omega}_2^{i(\chi')}$-isotypic space for $\mathrm{Gal}(F/F_0)$, where $i(\chi') \in \{0, \dots, p^2-2\}$ is defined as the unique integer such that $\chi_1^{-i(\chi')} = \chi'$. Using results in Remark 9.5, another equivalent definition is that $\tilde{\omega}_2^{i(\chi')}$ is the unique character making the following diagram commutative:

$$\begin{tikzcd}
\mathbb{Z}_{p^2}^\times \simeq O_{F_0}^\times \arrow[r, "\mathrm{Art}_{F_0}"] \arrow[d] & \mathrm{Gal}(\bar{F}_0/F_0)^{\mathrm{ab}} \arrow[d, "\tilde{\omega}_2^{i(\chi')}"] \\
O_D^\times \arrow[r, "\chi'"] & \mathbb{Z}_{p^2}^\times \simeq O_{F_0}^\times
\end{tikzcd}$$

Recall that $\tilde{\omega}_2$ is defined in Remark 9.5.

Now I want to translate the theorem above to our situation. Fix an embedding $\tau : F_0 \to E$, and use $\bar{\tau}$ to denote the conjugate embedding. Let $\chi' \in \chi(F_0)$ be the unique character that satisfies $\tau \circ \chi' = \chi$. Recall that $g_\varphi \in \mathrm{Gal}(F/\mathbb{Q}_p)$ is the unique element that fixes ϖ but acts as Frobenius on F_0.

Proposition 12.8. $D_{\mathrm{crys},\chi} \overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_p)}\big(\rho_{\chi'}, (H^1_{\mathrm{crys}}(\overline{U^{(0)}_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi}\big)$ *is a free* $F_0\otimes_{\mathbb{Q}_p}E$*-module of rank* 2. $\mathrm{Gal}(F/\mathbb{Q}_p)$ *and the Frobenius operator* φ *act on it naturally. In fact,* $D_{\mathrm{crys},\chi}$ *is of the form*

$$D_{\mathrm{crys},\chi} = (F_0 \otimes_{\mathbb{Q}_p} E)\cdot \boldsymbol{e}_1 \oplus (F_0 \otimes_{\mathbb{Q}_p} E)\cdot \boldsymbol{e}_2,$$

$$\varphi(\boldsymbol{e}_1) = \boldsymbol{e}_2, \quad \varphi(\boldsymbol{e}_2) = (1\otimes c_x)\boldsymbol{e}_1,$$

$$g\cdot \boldsymbol{e}_1 = (\tilde{\omega}_2(g)^m \otimes 1)\boldsymbol{e}_1, \quad g\cdot \boldsymbol{e}_2 = (\tilde{\omega}_2(g)^{pm} \otimes 1)\boldsymbol{e}_2, \quad \forall g \in \mathrm{Gal}(F/F_0),$$

$$g_\varphi \cdot \boldsymbol{e}_1 = \boldsymbol{e}_1, \quad g_\varphi \cdot \boldsymbol{e}_2 = \boldsymbol{e}_2,$$

with $c_x \in E$ *and* $v_p(c_x) = 1$, $m = i(\chi')$ *defined in Remark 12.7.*

Proof. We can write (using the fact $\overline{U^{(0)}_{s'_0}} = \overline{U_{s'_0}} \sqcup \overline{U'_{s'_0}}$)

$$\begin{aligned}(H^1_{\mathrm{crys}}(\overline{U^{(0)}_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi} &= (H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi} \oplus (H^1_{\mathrm{crys}}(\overline{U'_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi} \\ &= H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E \oplus H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E \\ &\quad \oplus H^1_{\mathrm{crys}}(\overline{U'_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E \oplus H^1_{\mathrm{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E,\end{aligned}$$

where $\overline{\chi'} = (\chi')^p$, the conjugate character, satisfies $\bar{\tau} \circ \overline{\chi'} = \chi$.

Recall that we can identify $\overline{U_{s'_0}}$ with $\overline{U'_{s'_0}}$ but with different structure map to $\operatorname{Spec} \mathbb{F}_{p^2}$. Using Remark 9.8, such an identification induces an isomorphism between $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$ and $H^1_{\mathrm{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}}$. By definition,

$$\operatorname{Hom}_{\mathrm{GL}_2(\mathbb{F}_p)}\big(\rho_{\chi'}, H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E\big) \simeq F_0 \otimes_{F_0,\tau} E$$

and similar results for other factors of $(H^1_{\mathrm{crys}}(\overline{U^{(0)}_{s'_0}}/F_0) \otimes_{\mathbb{Q}_p} E)^{\chi}$ follow from Deligne–Lusztig theory. It's easy to see $D_{\mathrm{crys},\chi} \simeq F_0 \otimes_{\mathbb{Q}_p} E^{\oplus 2}$ from these descriptions.

By Remarks 12.7 and 9.8, $\mathrm{Gal}(F/F_0)$ acts via $\tilde{\omega}_2^m$ (as an F_0-vector space) on $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E$, and $H^1_{\mathrm{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E$ and acts as $\tilde{\omega}_2^{pm}$ on the other two factors since $i(\overline{\chi'}) = i((\chi')^p) = pi(\chi')$. Remark 9.8 also tells us that g_{φ} induces an isomorphism between $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E$ and $H^1_{\mathrm{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E$. Now, choose a generator $\mathbf{f}_1$ of

$$\operatorname{Hom}_{\mathrm{GL}_2(\mathbb{F}_p)}\big(\rho_{\chi'}, H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E\big),$$

and define

$$\boldsymbol{e}_1 = \mathbf{f}_1 + g_{\varphi} \cdot \mathbf{f}_1, \boldsymbol{e}_2 = \varphi(\boldsymbol{e}_1),$$

where φ is the Frobenius operator coming from the crystalline cohomology. We need to verify our claim in the proposition.

First it's easy to see $\boldsymbol{e}_1$ is indeed a generator of

$$\operatorname{Hom}_{\mathrm{GL}_2(\mathbb{F}_p)}\big(\rho_{\chi'}, H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'} \otimes_{F_0,\tau} E \oplus H^1_{\mathrm{crys}}(\overline{U'_{s'_0}}/F_0)^{\overline{\chi'}} \otimes_{F_0,\bar{\tau}} E\big)$$

as a free $F_0 \otimes_{\mathbb{Q}_p} E$-module and satisfies $g \cdot \boldsymbol{e}_1 = (\tilde{\omega}_2(g)^m \otimes 1)\boldsymbol{e}_1,\ g \in \mathrm{Gal}(F/F_0)$. Next we verify the desired property of the Frobenius operator φ. It's induced by the Frobenius endomorphism on $\overline{U_{s'_0}}$, which is nothing but raising anything to its p-th power. So it sends $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$ to $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{(\chi')^p} = H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\overline{\chi'}}$. Therefore everything is clear except our claim for $\varphi(\boldsymbol{e}_2)$. This can be shown by explicit computations. See the next lemma. □

Lemma 12.9. $$c_x = -p\tau(w_1^{-2i}).$$

Proof. This can be done using Gauss sums. Since $H^1_{\mathrm{crys}}(\overline{U_{s'_0}}/F_0)^{\chi'}$ is an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_p)$, φ^2 acts as a scalar $\tilde{c_x}$ on it. It's easy to see $c_x = \tau(\tilde{c_x})$.

To compute $\tilde{c_x}$, we only need to restrict to one component. So let ξ be a root of $\xi^{p-1} = -1$. Then $\overline{U_{s_0',\xi}}$ can be identified as the curve in $\mathbb{P}^2_{\mathbb{F}_{p^2}}$ defined by $y^{p+1} = v_1 w_1^{-1}\xi(x^p - x)$. There is an action of

$$\mu_{p+1}(\mathbb{F}_{p^2}) = \{a \in \mathbb{F}_{p^2}^\times \mid a^{p+1} = 1\}$$

on it given by

$$a \cdot x = x, \quad a \cdot y = ay, \quad a \in \mu_{p+1}(\mathbb{F}_{p^2}^\times).$$

Let $\tilde{\chi} : \mu_{p+1}(\mathbb{F}_{p^2}) \to F_0^\times$ be the Teichmüller character. It's obvious that

$$H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/F_0)^{\chi'} \simeq H^1_{\mathrm{crys}}(\overline{U_{s_0',\xi}}/F_0)^{\tilde{\chi}^{-i}},$$

the $\tilde{\chi}^{-i}$-isotypical component. Here $i \in \{1, \dots, p\}$ is the unique number satisfying $i \equiv m \bmod p+1$.

On the other hand, $\mathbb{F}_p$ also acts on $\overline{U_{s_0',\xi}}$, which comes from the action of an unipotent subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$:

$$b \cdot x = x + 1, \quad b \cdot y = y, \quad b \in \mathbb{F}_p.$$

This action commutes with the action of $\mu_{p+1}(\mathbb{F}_{p^2})$. It's easy to see F_0 contains all p-th roots of unity. Let $\psi_p : \mathbb{F}_p \to F_0^\times$ be a nontrivial additive character. We view $\tilde{\rho} = \tilde{\chi}^{-i} \times \psi_p$ as a one dimensional representation of $\widetilde{G} \overset{\mathrm{def}}{=} \mu_{p+1}(\mathbb{F}_{p^2}) \times \mathbb{F}_p$.

Using Lemma 1.1. of [Katz 1981], we know that the eigenvalue of φ^2 on $(H^1_{\mathrm{crys}}(\overline{U_{s_0',\xi}}/F_0) \otimes_{F_0} F)^{\tilde{\rho}}$ is (we will see later that this lemma indeed can be applied to our situation)

$$-S(\overline{U_{s_0',\xi}}/\mathbb{F}_{p^2}, \tilde{\rho}, 1) \overset{\mathrm{def}}{=} -\frac{1}{\#\widetilde{G}} \sum_{g \in \widetilde{G}} \mathrm{tr}(\tilde{\rho}(g))\, \#\mathrm{Fix}(\mathrm{F}_{p^2} g^{-1}),$$

where F_{p^2} is the Frobenius endomorphism of $\overline{U_{s_0',\xi}}$ relative to $\mathbb{F}_{p^2}$ and $\mathrm{Fix}(\mathrm{F}_{p^2} g^{-1})$ is the subset of $\overline{U_{s_0',\xi}}(\overline{\mathbb{F}_{p^2}})$ fixed by $\mathrm{F}_{p^2} g^{-1}$. Following the strategy of lemma 2.1. of [Katz 1981], we can express $S(\overline{U_{s_0',\xi}}/\mathbb{F}_{p^2}, \tilde{\rho}, 1)$ as the Gauss sum:

$$S(\overline{U_{s_0',\xi}}/\mathbb{F}_{p^2}, \tilde{\rho}, 1) = (v_1 w_1^{-1}\xi)^{-i(p-1)} \sum_{x \in \mathbb{F}_{p^2}^\times} \psi_{p^2}(x) x^{-i(p-1)},$$

where $\psi_{p^2} \overset{\mathrm{def}}{=} \psi_p(\mathrm{tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x)) = \psi_p(x^p + x)$. Notice that for any $x \in \mathbb{F}_{p^2}^\times$,

$$\sum_{a \in \mathbb{F}_p^\times} \psi_{p^2}(ax) = \sum_{a \in \mathbb{F}_p^\times} \psi_p(a(x^p + x)) = \begin{cases} -1 & \text{if } x^p + x \neq 0, \\ p - 1 & \text{if } x^p + x = 0. \end{cases}$$

From this, it's easy to see $S(\overline{U_{s_0',\xi}}/\mathbb{F}_{p^2}, \tilde{\rho}, 1) = w_1^{i(p-1)} p(-1)^i = w_1^{-2i} p$ (recall $v_1^{p-1} = w_1^{p+1} = \xi^{p-1} = -1$). Hence

$$c_x = -p\tau(w_1^{-2i}). \qquad \square$$

Corollary 12.10. *We have a* $\mathrm{Gal}(F/\mathbb{Q}_p)\times O_D^\times\times\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant isomorphism*:

$$F\otimes_{F_0} D_{\mathrm{crys},\chi}\otimes_E \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi \xrightarrow{\sim} (H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^\chi, \tag{43}$$

where $\mathrm{Gal}(F/\mathbb{Q}_p)$ *acts on the first two components*, $O_D^\times$ *acts on the second*, *and* $\mathrm{GL}_2(\mathbb{Q}_p)$ *acts on the third. Moreover*, $D_{\mathrm{crys},\chi}\otimes_E \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi$ *maps to the* F_0 *subspace we constructed in Proposition 12.3.*

Here we extend ρ_χ to a representation of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ by p acting trivially and $\mathrm{GL}_2(\mathbb{Z}_p)$ acting through $\mathrm{GL}_2(\mathbb{F}_p)$.

Remark 12.11. It's easy to see the dual representation of ρ_χ is $\rho_{\chi^{-1}}$, we use $\langle\,\cdot\,,\cdot\,\rangle$ to denote the pairing of them. Then we can construct a pairing:

$$\begin{aligned}\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_{\chi^{-1}}\times \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi &\to E\\ (f_1,f_2)&\mapsto \Sigma_{[g]\in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\backslash \mathrm{GL}_2(\mathbb{Q}_p)}\langle f_1(g),f_2(g)\rangle,\end{aligned}$$

where is the compact induction. More precisely,

$$\begin{aligned}&\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_{\chi^{-1}}\\ &\quad=\Big\{f:\mathrm{GL}_2(\mathbb{Q}_p)\to\rho_{\chi^{-1}}\;\Big|\; f \text{ has compact support mod } \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times,\\ &\qquad\qquad f(kg)=\rho_{\chi^{-1}}(k)f(g),\ k\in\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times,\ g\in\mathrm{GL}_2(\mathbb{Q}_p)\Big\},\end{aligned}$$

and $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi$ is defined similarly without any restrictions on the support. The sum makes sense because it only has finitely many nonzero terms.

This pairing induces an isomorphism $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi\simeq(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_{\chi^{-1}})^\vee$, the algebraic dual representation. We can rewrite the result in Corollary 12.10 as a $\mathrm{Gal}(F/\mathbb{Q}_p)\times O_D^\times\times\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism (Theorem 1.10):

$$F\otimes_{F_0} D_{\mathrm{crys},\chi}\otimes_E (\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_{\chi^{-1}})^\vee \xrightarrow{\sim} (H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F})\otimes_{\mathbb{Q}_p} E)^\chi. \tag{44}$$

By Corollary 11.6, there is a natural Fréchet space structure on the right-hand side of the above map. In fact, we can describe this topology directly on the left-hand side. Choosing a family of representatives of $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\backslash\mathrm{GL}_2(\mathbb{Q}_p)$, we have a noncanonical isomorphism between $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi$ and $\prod_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\backslash\mathrm{GL}_2(\mathbb{Q}_p)}\rho_\chi$ as E-vector spaces. The topology is nothing but the weakest topology on this product such that each projection to ρ_χ is continuous under the canonical (Banach space) topology on ρ_χ.

13. Some considerations from Galois representations

Let's recall what we have on $D_{\mathrm{crys},\chi}$ (see Proposition 12.8 for more details):

- Frobenius operator φ: an F_0-semilinear, E-linear automorphism;

- monodromy operator N, which is zero here;
- an action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, which is F_0-semilinear, E-linear commuting with φ and N.

So if we have a decreasing filtration on $D_F = F \otimes_{F_0} D_{\mathrm{crys},\chi}$, such that $\mathrm{Fil}^i D_F$ is zero if $i \gg 0$ and is equal to D_F if $i \ll 0$ and preserved by the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, $D_{\mathrm{crys},\chi}$ is called a filtered $(\varphi, N, F/\mathbb{Q}_p, E)$-module of rank 2. Moreover, if the underlying (φ, N, F, E)-module is weakly admissible, $D_{\mathrm{crys},\chi}$ is called weakly admissible. See Definitions 2.7 and 2.8 of [Savitt 2005] for the precise definition. The importance of this kind of module is that we have the following result (see [Savitt 2005, Corollary 2.10]).

Theorem 13.1. *The category of E-representations of $G_{\mathbb{Q}_p}$ which become semistable when restricted to G_F and the category of weakly admissible $(\varphi, N, F/\mathbb{Q}_p, E)$-modules are equivalent. Here $G_{\mathbb{Q}_p}$ (resp. G_F) is the absolute Galois group of $\mathbb{Q}_p$ (resp. F).*

Now I want to classify all two dimensional potentially semistable E-representations of $G_{\mathbb{Q}_p}$ that

- have Hodge–Tate weights (0, 1), and
- correspond to $D_{\mathrm{crys},\chi}$ if we forget about the filtration.

Proposition 13.2 [Savitt 2005, Proposition 2.18]. *Any such weakly admissible $(\varphi, N, F/\mathbb{Q}_p, E)$-module is of the form*

$$\mathrm{Fil}^n(D_F) = \begin{cases} D_F, & n \le 0, \\ (F \otimes_{\mathbb{Q}_p} E)\big((\varpi^{(p-1)i} \otimes a)\boldsymbol{e}_1 + (1 \otimes b)\boldsymbol{e}_2\big), & n = 1, \\ 0, & n \ge 2, \end{cases}$$

where $(a, b) \neq (0, 0) \in E^2$, and i, j are defined as follows: write $m = i + (p+1)j$ with $i \in \{1, \dots, p\}$ and $j \in \{0, \dots, p-2\}$.

We denote the filtered module in the above proposition by $D_{\chi,[a,b]}$. It's not hard to see

$$D_{\chi,[a,b]} \simeq D_{\chi^p,[bc_x/p,-a]} \quad \text{and} \quad D_{\chi,[a,b]} = D_{\chi,[ca,cb]}.$$

So we may assume $a = 1$ and $v_p(b) \ge 0$ (recall that c_x is defined in Proposition 12.8). We use $V_{\chi,[1,b]}$ to denote the Galois representation it corresponds to in Theorem 13.1.

Now suppose we have an element f in $D_F = F \otimes_{F_0} D_{\mathrm{crys},\chi}$. How do we check whether or not f is in $\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$ for a given b? First assume $f \in \mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$. Write $f = f_1 + f_2$, $f_1 \in (F \otimes_{\mathbb{Q}_p} E)\cdot \boldsymbol{e}_1$, $f_2 \in (F \otimes_{\mathbb{Q}_p} E)\cdot \boldsymbol{e}_2$. Then we must have

$$f_1 = \Big(\sum a_k \otimes b_k\Big)(\varpi^{(p-1)i} \otimes 1)\boldsymbol{e}_1, \quad f_2 = \Big(\sum a_k \otimes b_k\Big)(1 \otimes b)\boldsymbol{e}_2,$$

for some $a_k \in F,\ b_k \in E$. Notice that $g_\varphi \otimes \varphi$ is well-defined on $F \otimes_{F_0} D_{\mathrm{crys},\chi}$ since g_φ acts as Frobenius on F_0. Here g_φ is considered only acting on F, not on $D_{\mathrm{crys},\chi}$:

$$(g_\varphi \otimes \varphi)(f_1) = \left(\sum g_\varphi(a_k) \otimes b_k\right)(\varpi^{(p-1)i} \otimes 1)\boldsymbol{e}_2.$$

On the other hand, $g_\varphi(f_2) = (\Sigma g_\varphi(a_k) \otimes b_k)(1 \otimes b)\boldsymbol{e}_2$. Therefore,

$$(1 \otimes b)(g_\varphi \otimes \varphi)(f_1) = (\varpi^{(p-1)i} \otimes 1)g_\varphi(f_2).$$

A simple dimension counting shows that this condition is even sufficient. Hence:

Proposition 13.3. *Suppose $f \in F \otimes_{F_0} D_{\mathrm{crys},\chi}$. Write*

$$f = f_1 + f_2, \quad f_1 \in (F \otimes_{\mathbb{Q}_p} E) \cdot \boldsymbol{e}_1, \quad f_2 \in (F \otimes_{\mathbb{Q}_p} E) \cdot \boldsymbol{e}_2.$$

Then $f \in \mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$ if and only if

$$(1 \otimes b)(g_\varphi \otimes \varphi)(f_1) = (\varpi^{(p-1)i} \otimes 1)g_\varphi(f_2).$$

Remark 13.4. In practice, we will assume f is fixed by g_φ; then the condition above is simplified to $(1 \otimes b)(g_\varphi \otimes \varphi)(f_1) = (\varpi^{(p-1)i} \otimes 1) f_2$.

14. Construction of Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this section, I want to construct some Banach space representations $B(\chi, [1, b])$ that should correspond to $V^\vee_{\chi,[1,b]}$ (up to a twist by some character) under the p-adic local Langlands correspondence.

First we define an integral structure ω^1 of $\Omega^1_{\Sigma_{1,F}}$, the sheaf of holomorphic differential forms, on $\widetilde{\Sigma_{1,O_F}}$ defined in Section 8. Recall that $\widetilde{\Sigma_{1,O_F}}$ is a formal model of $\Omega^1_{\Sigma_{1,F}}$ which is not semistable, but only has some mild singularities $(xy - \varpi^{p-1})$. From now on, I will do all computations on this formal model rather than the semistable model.

View $\Omega^1_{\Sigma_{1,F}}$ as a sheaf on $\widetilde{\Sigma_{1,O_F}}$. The coherent sheaf ω^1 will be a subsheaf of it. Recall that there is an open covering $\{\widetilde{\Sigma_{1,O_F,e,\xi}}\}_{e,\xi}$ of $\widetilde{\Sigma_{1,O_F}}$, where e takes value in the set of edges of the Bruhat–Tits tree and $\xi^{p-1} = -1$. Using the explicit description of Lemma 8.2, we define ω^1 on each $\widetilde{\Sigma_{1,O_F,e,\xi}}$ as the trivial line bundle with a basis $d\tilde{e}/\tilde{e} = -d\tilde{e}'/\tilde{e}'$ (recall that $\tilde{e} = e/\varpi$, $\tilde{e}' = e'/\varpi$). It's easy to see that this really defines a line bundle $\widetilde{\Sigma_{1,O_F,e,\xi}}$ which becomes $\Omega^1_{\Sigma_{1,F}}$ if we restrict this line bundle to the generic fiber.

Remark 14.1. We can do exactly the same thing on the semistable model $\widehat{\Sigma_{1,O_F}}$, but this won't give us any extra sections: the sections on $\widetilde{\Sigma_{1,O_F,e}}$ and $\widehat{\Sigma_{1,O_F,e}}$ will be the same. This can be checked locally around the singularities. So I can do all the computations on $\widetilde{\Sigma_{1,O_F,e}}$.

Remark 14.2. We note that ω^1 in fact has an "F_0-structure". In other words, we can define it on $\widehat{\Sigma_1}$. Using the explicit description in Corollary 7.7, locally on $\widehat{\Sigma_{1,e}}$, it is defined as the trivial line bundle generated by de/e. Notice that $de/e = d\tilde{e}/\tilde{e}$ since $e = \tilde{e}\varpi$. Hence its pull-back to $\widetilde{\Sigma_{1,O_F}}$ is ω^1.

Similarly, we can define the same thing on $\widetilde{\Sigma'_{1,O_F}}$, $\widetilde{\Sigma^{(0)}_{1,O_F}}$, which we still denote by ω^1, by abuse of notation. Now if we restrict ω^1 to the special fiber, it becomes the dualizing sheaf (over $\operatorname{Spec}\mathbb{F}_{p^2}$). So there is an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on it. In fact, $\mathrm{GL}_2(\mathbb{Q}_p)$ even acts on ω^1. This can be seen using the explicit description in Section 9. Also, it's clear from the definition that $O_D^\times$ and $\mathrm{Gal}(F/\mathbb{Q}_p)$ act on the global sections of ω^1.

Consider the following maps:

$$H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E \hookrightarrow H^0(\Sigma^{(0)}_{1,F}, \Omega^1_{\Sigma^{(0)}_{1,F}}) \otimes_{\mathbb{Q}_p} E \to H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) \otimes_{\mathbb{Q}_p} E.$$

Both maps are $\mathrm{GL}_2(\mathbb{Q}_p)$, $O_D^\times$, $\mathrm{Gal}(F/\mathbb{Q}_p)$-equivariant. Take the χ-isotypic component, where $\chi \in \chi(E)$ (see Section 11). We get a map (use Corollary 12.10):

$$\begin{aligned} f_\chi : \big(H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E\big)^\chi \to (H^1_{\mathrm{dR}}(\Sigma^{(0)}_{1,F}) &\otimes_{\mathbb{Q}_p} E)^\chi \\ &\simeq F \otimes_{F_0} D_{\mathrm{crys},\chi} \otimes_E \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_\chi. \end{aligned}$$

Now for each two dimensional Galois representation $V_{\chi,[1,b]}$ of $G_{\mathbb{Q}_p}$ defined in the previous section, we have a free $F \otimes_{\mathbb{Q}_p} E$-module $\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$ inside $F \otimes_{F_0} D_{\mathrm{crys},\chi}$. We note that $\mathrm{Gal}(F/\mathbb{Q}_p)$ acts on this $\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]})$. Define

$$\begin{aligned} M(\chi,[1,b]) &= \big(f_\chi^{-1}(\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]}) \otimes_E \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_\chi)\big)^{\mathrm{Gal}(F/\mathbb{Q}_p)} \\ &= f_\chi^{-1}((\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]}))^{\mathrm{Gal}(F/\mathbb{Q}_p)} \otimes_E \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \rho_\chi. \end{aligned}$$

Schneider and Teitelbaum [2002] introduced a category $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(O_E)$ whose objects are all torsion-free and compact, Hausdorff linear-topological O_E-modules, and morphisms are all continuous O_E-linear maps. Our first result about $M(\chi,[1,b])$ is:

Proposition 14.3. *$M(\chi,[1,b])$ with the topology induced from*

$$H^0(\Sigma^{(0)}_{1,F}, \Omega^1_{\Sigma^{(0)}_{1,F}}) \otimes_{\mathbb{Q}_p} E$$

is an object in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(O_E)$.

Proof. I learned this argument from Proposition 4.2.1 of [Breuil 2004]. It is clear that $M(\chi,[1,b])$ is torsion free and Hausdorff. To prove compactness, we use Proposition 15.3(iii) of [Schneider 2002] (c-compactness is equivalent with compactness here since O_E is locally compact [Perez-Garcia and Schikhof 2010, Corollary 6.1.14]). Proposition 11.4 already shows that $H^0(\Sigma^{(0)}_{1,F}, \Omega^1_{\Sigma^{(0)}_{1,F}}) \otimes_{\mathbb{Q}_p} E$ is a reflexive Fréchet space, so it suffices to show $M(\chi,[1,b])$ is closed and bounded (see [Schneider 2002] for the definition of boundedness). In fact it's easy to see we only need to prove closedness and boundedness for $H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)$ in $H^0(\Sigma_{1,F}^{(0)}, \Omega^1_{\Sigma^{(0)}_{1,F}})$. Recall the topology on $H^0(\Sigma_{1,F}^{(0)}, \Omega^1_{\Sigma^{(0)}_{1,F}})$ is defined in Section 11 by:

$$H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}}) = \varprojlim_n H^0(V_{n,F}, \Omega^i_{\Sigma_{1,F}}),$$

where $\{V_{n,F}\}_n$ is an admissible open covering of $\Sigma_{1,F}$ and each $V_{n,F}$ is affinoid and contained in $V_{n+1,F}$. Now $\{\widetilde{\Sigma_{1,O_F,e}}\}_e$ is another admissible open covering. Thus we have:

- Each $\widetilde{\Sigma_{1,O_F,e}}$ is contained in some $V_{n,F}$.
- Each $V_{n,F}$ is covered by finitely many generic fibers of $\widetilde{\Sigma_{1,O_F,e}}$.

Then closedness follows from the first claim above and boundedness follows from the second. □

Suppose M is an object in $\mathrm{Mod}^{\mathrm{fl}}_{\mathrm{comp}}(O_E)$, following [Schneider and Teitelbaum 2002], the E-vector space $M^d \overset{\mathrm{def}}{=} \mathrm{Hom}^{\mathrm{cont}}_{O_E}(M, E)$ with the norm $\|f\| = \max_{m\in M}|f(m)|_E$ is a Banach space.

Definition 14.4. $B(\chi, [1, b]) \overset{\mathrm{def}}{=} (M(\chi, [1, b]))^d = \mathrm{Hom}^{\mathrm{cont}}_{O_E}(M(\chi, [1, b]), E).$

It's clear from the definition that this is a Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Remark 14.5. The relation between $B(\chi, [1, b])$ and the Banach representation $B(\pi, \mathcal{L})$ defined in the introduction (see Definitions 1.3 and 1.4) is as follows: Take $\pi = \mathrm{Ind}^{D^\times}_{O_D^\times\mathbb{Q}_p^\times}\chi$, where χ is viewed as a character of $O_D^\times\mathbb{Q}_p^\times$ trivial on p. Also $\mathrm{Fil}^1(D_{\chi,[1,b]} \otimes F)$ essentially gives a line "$\mathcal{L}_b$" in Definition 1.3 by taking $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariants. Then $B(\chi, [1, b]) = B(\pi, \mathcal{L}_b)$.

Back to the definition of $M(\chi, [1, b])$. By Remark 12.11, we can replace the induced representation by the dual representation of the compact induction. Also by Galois descent, we have $(\mathrm{Fil}^1(F \otimes_{F_0} D_{\chi,[1,b]}))^{\mathrm{Gal}(F/\mathbb{Q}_p)} \simeq E$. Under these isomorphisms, f_χ induces a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map,

$$f_{\chi,[1,b]} : M(\chi, [1, b]) \to (\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho_{\chi^{-1}})^\vee.$$

It is natural to ask whether such a map is injective or not. The answer is positive.

Proposition 14.6. *The composition*

$$H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1)^{\chi'} \hookrightarrow H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})^{\chi'} \to H^1_{\mathrm{dR}}(\Sigma_{1,F})^{\chi'},$$

for a character $\chi' \in \chi(F)$ such that $\chi' \neq \chi'^p$, is injective.

Proof. Since $\chi' \neq \chi'^p$, the kernel of the second map is $H^0(\Sigma_{1,F}, \mathcal{O}_{\Sigma_{1,F}})^{\chi'}$. Consider the intersection of $H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1)^{\chi'}$ and $H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}})^{\chi'}$ in $H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})^{\chi'}$. It can be viewed as a subset in $H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}})^{\chi'}$ and we denote it by H. On the other hand, we use J to denote the same set but viewed in $H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})^{\chi'}$. The induced topology on H and J can be different. Proposition 14.3 tells us that J is compact since $H^0(\Sigma_{1,F}, \Omega^0_{\Sigma_{1,F}})^{\chi'}$ is closed in $H^0(\Sigma_{1,F}, \Omega^1_{\Sigma_{1,F}})^{\chi'}$ (Proposition 11.5). Clearly $\mathrm{SL}_2(\mathbb{Q}_p)$ preserves both H and J.

Let's recall some notation here: For each connected component of $\widetilde{\Sigma_{1,O_F}}$, the dual graph of its special fiber is the Bruhat–Tits tree (see Section 8), and $U_{s,\xi}$ is the tubular neighborhood of $\overline{U_{s,\xi}}$, the irreducible component indexed by (s,ξ) in the special fiber (see Definition 11.10).

Similar to what we did in the beginning of Section 12, we can prove $U_{s'_0,\xi}$ is isomorphic with

$$\{z=(x,y)\in\mathbb{A}^2_F \mid y^{p+1}=v_1w_1^{-1}\xi(x^p-x),\ |x-k|>p^{-1}, k=0,\dots,p-1,\ |x|<p\},$$

and its de Rham cohomology is of finite dimension. Since $U_{s'_0,\xi}$ is a Stein space, $H^1_{\mathrm{dR}}(U_{s'_0,\xi}) = H^0(U_{s'_0,\xi},\Omega^1)/H^0(U_{s'_0,\xi},\Omega^0)$ (we use Ω^i for $\Omega^i_{\Sigma_{1,F}}$ for simplicity).

Fix a ξ. Under the isomorphism above, we can write $U_{s'_0,\xi} = \bigcup_{\rho<p} U_{s'_0,\xi,\rho}$, where $U_{s'_0,\xi,\rho} \subset U_{s'_0,\xi}$ is defined by the same equation but with $|x-k|\geq \rho^{-1}$, $k=0,\dots,p-1$, $|x|\leq\rho$. Then for each $\rho<p$, $H^0(U_{s'_0,\xi,\rho},\Omega^i)$ is a Banach space, and we have $H^0(U_{s'_0,\xi},\Omega^i) = \varprojlim_{\rho\to p} H^0(U_{s'_0,\xi,\rho},\Omega^i)$. So $H^0(U_{s'_0,\xi},\Omega^i)$ is a Fréchet space.

Notice that $O_D^\times$ acts on $U_{s'_0}$, so $H^0(U_{s'_0},\Omega^0)^{\chi'} \hookrightarrow H^0(U_{s'_0},\Omega^1)^{\chi'}$ and the quotient is a finite dimensional space. Thus this inclusion has to be a closed embedding because both of them are Fréchet spaces.

Now consider the canonical maps $H^0(\Sigma_{1,F},\Omega^k)^{\chi'} \to H^0(U_{s'_0},\Omega^k)^{\chi'}$, $k=0,1$. They're clearly continuous and we denote the image of H and J by H_1 and J_1. Since J is compact, J_1 is compact. Hence H_1 is also compact in $H^0(\Sigma_{1,F},\Omega^0)^{\chi'}$ because $H^0(U_{s'_0},\Omega^0)^{\chi'} \hookrightarrow H^0(U_{s'_0},\Omega^1)^{\chi'}$ is a closed embedding. We will show this cannot happen unless $H_1=\{0\}$.

Suppose f is a nonzero rigid function in H. We will prove later that f is unbounded on $\Sigma_{1,F}$ (see the next lemma). For each $U_{s,\xi}$, the maximum principle implies that f must obtain its maximum on the boundary annuli which are the tubes of the singular points on the special fiber. Therefore f is unbounded on $\bigcup_{s' \text{ even}} U_{s',\xi}$. But we know $\mathrm{SL}_2(\mathbb{Q}_p)$ acts on $\Sigma_{1,F}$ and acts transitively on the set of even vertices. Hence using the action of $\mathrm{SL}_2(\mathbb{Q}_p)$, we can get functions in H with arbitrary large norms when restricted to $U_{s'_0,\xi}$ and H_1 cannot be compact. So there is no such f. □

Lemma 14.7. *Any globally bounded function on a connected component of* $\Sigma_{1,F}$ *must be a constant.*

Proof. Fix a connected component ξ. Suppose f is such a function. By multiplying f by some powers of ϖ, we may assume $f\in H^0(\widehat{\Sigma_{1,O_F},\xi}, \mathcal{O}_{\widehat{\Sigma_{1,O_F},\xi}})$. Recall that the special fiber is connected and each irreducible component is a complete curve.

Hence,

$$H^0(\widehat{\Sigma_{1,O_F,\xi}}, \mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}/(\varpi)) = \mathbb{F}_{p^2}.$$

Using induction on n, we can prove $H^0(\widehat{\Sigma_{1,O_F,\xi}}, \mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}/(\varpi^n)) = O_F/(\varpi^n)$. Here we use the fact that $\mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}$ is flat over the constant sheaf O_F. Now the lemma follows from

$$H^0(\widehat{\Sigma_{1,O_F,\xi}}, \mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}) = \varprojlim_n H^0(\widehat{\Sigma_{1,O_F,\xi}}, \mathcal{O}_{\widehat{\Sigma_{1,O_F,\xi}}}/(\varpi^n)) = O_F. \qquad \square$$

Remark 14.8. The proposition is also true if $\chi' \neq \chi'^p$. In this case, it is equivalent to the same result on the Drinfel'd upper half-plane. See Proposition 19 of [Teitelbaum 1993] for a proof.

So we have an injective $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map:

$$f_{\chi,[1,b]} : M(\chi, [1, b]) \to (\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}})^\vee.$$

A simple consideration of the topology (see Remark 12.11) shows that this induces a map

$$\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}} \to B(\chi, [1, b]).$$

It is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant and has to be injective if $B(\chi, [1, b])$ is nonzero since the left-hand side is an irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. If $B(\chi, [1, b])$ is nonzero, or equivalently if $M(\chi, [1, b])$ is nonzero, we can define a lattice inside $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$:

$$\Theta(\chi, [1, b]) = \{X \in \text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}} \mid \langle X, f_{\chi,[1,b]}(Y)\rangle \in O_E, \forall Y \in M(\chi, [1, b])\},$$

where

$$\langle \cdot, \cdot \rangle : \text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}} \times (\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}})^\vee \to O_E$$

denotes the canonical pairing. This is equivalent to the intersection of the unit ball of $B(\chi, [1, b])$ with $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$.

Proposition 14.9. *$B(\chi, [1, b])$ is the completion of $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$ with respect to the lattice $\Theta(\chi, [1, b])$ if $M(\chi, [1, b]) \neq 0$.*

Proof. The argument of Proposition 4.3.5 of [Breuil 2004] works here. I would like to recall it here. By [Schneider and Teitelbaum 2002, Theorem 1.2.], it suffices to prove that the natural map $M(\chi, [1, b]) \to \mathrm{Hom}_{O_E}(\Theta(\chi, [1, b]), O_E)$ is a topological isomorphism. The topology on the right hand side is defined by pointwise convergence. Notice that $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$ can be viewed as the continuous dual space of $(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}})^\vee$ with the topology described in Remark 12.11 and $M(\chi, [1, b])$ is closed in $(\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}})^\vee$ since it's already compact. We can apply Corollary 13.5 of [Schneider 2002] and get the desired isomorphism. It's also clear from the definition that this is a topological isomorphism. □

So if we can show $M(\chi, [1, b])$ is nonzero and moreover admissible as defined in [Schneider and Teitelbaum 2002], we indeed get an admissible Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, which is a completion of the smooth representation $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$. This is the goal of the rest of the paper.

15. Computation of $(H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^{\chi,\mathrm{Gal}(F/\mathbb{Q}_p)}/p$

Our ultimate goal is to prove $M(\chi, [1, b])$ is nonzero and admissible. The method is by explicit computation of its mod p representation. First we review some notation defined in the previous sections that will be used frequently from now on.

Let $\chi \in \chi(E)$ be a character of $(O_D/\Pi)^\times$ such that $\chi^p \neq \chi$. Since we fix an embedding $\tau : F_0 \to E$, we may write $\chi = \tau \circ \chi'$, where χ' is a character of $(O_D/\Pi)^\times$ with values in $F_0^\times$. Then $\chi' = \chi_1^{-m}$, where χ_1 is one of the fundamental characters (Definition 3.2) and $m \in \{1, \dots, p^2 - 2\}$. Write $m = i + (p+1)j$ with $i \in \{1, \dots, p\}$ and $j \in \{0, \dots, p-2\}$. Finally, $g_\varphi \in \mathrm{Gal}(F/\mathbb{Q}_p)$ is the unique element that fixes ϖ and acts as Frobenius on F_0.

Also recall that for any integer n, we use $[n]$ to denote the unique integer in $\{0, 1, \dots, p^2-2\}$ congruent to n modulo p^2-1. For any O_{F_0}-module A, we denote $A \otimes_{O_{F_0},\tau} O_E$ by A_τ and $A \otimes_{O_{F_0},\bar\tau} O_E$ by $A_{\bar\tau}$.

Recall that

$$\widetilde{\Sigma^{(0)}_{1,O_F}} = \widetilde{\Sigma_{1,O_F}} \sqcup \widetilde{\Sigma'_{1,O_F}},$$

and g_φ interchanges $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^\chi$ and $(H^0(\widetilde{\Sigma'_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^\chi$. Hence a g_φ-invariant element in $(H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^\chi$ is determined by its $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^\chi$ component. By definition, $M(\chi, [1, b])$ is g_φ-invariant. Hence it suffices to work on $\widetilde{\Sigma_{1,O_F}}$. This means that we may identify $H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)$ as the g_φ-invariant sections of $H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1)$. Hence there is a natural action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on it: this is nothing but $g_\varphi^{v_p(\det(g))} \circ g$.

Definition 15.1. For any $\chi \in \chi(E)$, $\chi' \in \chi(F_0)$, we define (see Section 8 for the definition of these formal schemes)

$$\begin{aligned}
H^{(0),\chi,\mathbb{Q}_p} &= (H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1/p) \otimes_{\mathbb{Z}_p} O_E)^{\chi,\mathrm{Gal}(F/\mathbb{Q}_p)},\\
H_*^{\chi,F_0} &= (H^0(\widetilde{\Sigma_{1,O_F,*}}, \omega^1/p) \otimes_{\mathbb{Z}_p} O_E)^{\chi,\mathrm{Gal}(F/F_0)},\\
H_*^{\chi',F_0} &= H^0(\widetilde{\Sigma_{1,O_F,*}}, \omega^1/p)^{\chi',\mathrm{Gal}(F/F_0)},\\
H_{*,?}^{\chi',F_0} &= H_*^{\chi',F_0} \otimes_{O_{F_0},?} O_E = H^0(\widetilde{\Sigma_{1,O_F,*}}, \omega^1/p)^{\chi',\mathrm{Gal}(F/F_0)} \otimes_{O_{F_0},?} O_E,
\end{aligned}$$

where $*$ is either a vertex s or an edge e of the Bruhat–Tits tree or nothing, and $? = \tau, \bar\tau$.

It is clear from the definition that if $\chi = \tau \circ \chi'$, then

$$H_*^{\chi, F_0} \simeq H_{*,\tau}^{\chi', F_0} \oplus H_{*,\bar{\tau}}^{(\chi')^p, F_0}. \tag{45}$$

Also, the discussion above shows that we have a canonical isomorphism:

$$H^{(0),\chi,\mathbb{Q}_p} \simeq H^{\chi, F_0}.$$

Definition 15.2. For a vertex s in the Bruhat–Tits tree, we use $A(s)$ to denote the set of vertices adjacent to s.

Now fix $\xi^{p-1} = -1$. We can do all the computation on one ξ-component $\widetilde{\Sigma_{1,O_F,\xi}}$. This is because $O_D^\times$ acts transitively on all connected components.

The goal of this section is to compute $(H^0(\widetilde{\Sigma_{1,O_F}^{(0)}}, \omega^1) \otimes_{\mathbb{Z}_p} O_E)^{\chi, \mathrm{Gal}(F/\mathbb{Q}_p)}/p$. The next lemma implies that this is nothing but $H^{(0),\chi,\mathbb{Q}_p}$.

Lemma 15.3. $H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)/\varpi^n = H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^n)$.

Proof. Clearly there is an injection from the left-hand side to the right-hand side. Since we have

$$H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) = \varprojlim_n H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^n),$$

we only need to prove the canonical map

$$H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^n) \to H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^m), \quad n > m$$

is surjective. Notice that ω^1 is flat over the constant sheaf O_F. It suffices to prove $H^1(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi^n) = 0$ for all $n \in \mathbb{N}^+$. Do induction on n and use the flatness again. It turns out that it's enough to show $H^1(\widetilde{\Sigma_{1,O_F}}, \omega^1/\varpi) = 0$. However, the construction of ω^1 tells us ω^1/ϖ is the dualizing sheaf on the special fiber. This means that if we restrict ω^1/ϖ to each irreducible component V of the special fiber, it is $\Omega_V^1(D_{\mathrm{sing}})$, where Ω_V^1 is the usual sheaf of differential forms on V, D_{sing} is the sum of singular points of D (considered in the whole special fiber) as a divisor. Also, we have the following exact sequence of sheaves:

$$0 \to \omega^1/\varpi \to \prod_V i_{V*}(\Omega_V^1(D_{\mathrm{sing}})) \to \prod_E i_{E*}(\mathbb{F}_{p^2}) \to 0,$$

where E (resp. V) runs through all singular points (resp. irreducible components) of the special fiber, and i_E (resp. i_V) is the corresponding inclusion. Take the long exact sequence of cohomologies of this sequence. H^0 of the third map is surjective since the dual graph of the special fiber of each connected component is a tree. H^1 of the middle term in the exact sequence above vanishes by Riemann–Roch. So we indeed get the vanishing of $H^1(\widetilde{\Sigma_{1,O_F}}, \omega^1)$. $\square$

Hence we only need to compute

$$H^{(0),\chi,\mathbb{Q}_p} \simeq H^{\chi,F_0} \simeq H_{\tau}^{\chi',F_0} \oplus H_{\bar{\tau}}^{(\chi')^p,F_0}. \tag{46}$$

It's not hard to see that we have an injection:

$$H^{\chi',F_0} \hookrightarrow \prod_s H_s^{\chi',F_0},$$

where s takes values in the set of vertices of the Bruhat–Tits tree. Similarly, we have the same injection for $H^{(\chi')^p,F_0}$. Notice that by identifying the sections on $\widetilde{\Sigma_{1,O_F}}$ as the g_φ-invariant sections on $\widetilde{\Sigma^{(0)}_{1,O_F}}$, we have an action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on

$$\prod_s (H_s^{\chi',F_0} \oplus H_s^{(\chi')^p,F_0})$$

(see the beginning of this section). Explicitly, g sends H_s^{χ',F_0} to H_{sg}^{χ',F_0} if $v_p(\det(g))$ ($g \in \mathrm{GL}_2(\mathbb{Q}_p)$) is even and to $H_{sg}^{(\chi')^p,F_0}$ if it is odd. From this description, we have an obvious $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism (recall s'_0 is the central vertex):

$$\prod_s (H_s^{\chi',F_0} \oplus H_s^{(\chi')^p,F_0}) \simeq \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H_{s'_0}^{\chi',F_0} \oplus \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H_{s'_0}^{(\chi')^p,F_0} \otimes_{\mathbb{F}_{p^2},\widetilde{\mathrm{Fr}}} \mathbb{F}_{p^2}.$$

The following lemma basically says that we may identify H_s^{χ',F_0} with sections of ω^1/ϖ on $\overline{U^0_s}$ introduced in Definition 11.10. Notice that ω^1/ϖ is the dualizing sheaf of the special fiber.

Lemma 15.4. *For each vertex s of the Bruhat–Tits tree, we have natural isomorphisms:*

$$\Psi_{s,\chi'} : H_s^{\chi',F_0} \xrightarrow{\sim} H^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1/\varpi)^{\chi'} = H^0(\overline{U^0_s}, \omega^1/\varpi)^{\chi'},$$
$$\Psi_{s,(\chi')^p} : H_s^{(\chi')^p,F_0} \xrightarrow{\sim} H^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1/\varpi)^{(\chi')^p} = H^0(\overline{U^0_s}, \omega^1/\varpi)^{(\chi')^p},$$

such that their product

$$\prod_s (\Psi_{s,\chi'}, \Psi_{s,(\chi')^p}) :$$
$$\prod_s H_s^{\chi',F_0} \oplus H_s^{(\chi')^p,F_0} \to \prod_s H^0(\overline{U^0_s}, \omega^1/\varpi)^{\chi'} \oplus \prod_s H^0(\overline{U^0_s}, \omega^1/\varpi)^{(\chi')^p}$$

is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. As usual, s takes its value in the set of vertices of Bruhat–Tits tree.

Proof. First let's see what happens when $s = s_0'$. Recall that we have a concrete description ((23), (24)) of $\widetilde{\Sigma_{1,O_F,s_0',\xi}}$, $\widetilde{\Sigma_{1,O_F,s_0'}}$ from Section 8:

$$\widetilde{\Sigma_{1,O_F,s_0',\xi}} \simeq \operatorname{Spf} O_{F_0}[\varpi]\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p+1} + v_1 w_1^{-1}\xi\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big),$$
$$\widetilde{\Sigma_{1,O_F,s_0'}} \simeq \operatorname{Spf} O_{F_0}[\varpi]\Big[\eta, \frac{1}{\eta^p-\eta}, \tilde{e}\Big]\Big/\Big(\tilde{e}^{p^2-1} - w_1^2\Big(\frac{\eta^p-\eta}{(p/\eta)^{p-1}-1}\Big)^{p-1}\Big).$$

An element of $H^0(\widetilde{\Sigma_{1,O_F,s_0'}}, \omega^1)^{\chi'}$ is determined by its restriction to $\widetilde{\Sigma_{1,O_F,s_0',\xi}}$. It's easy to see (using the results in Section 9) it must have the form

$$P(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}},$$

where $P(\eta) \in O_F[\eta, 1/(\eta^{p-1}-1)]\widehat{\ }$. Recall (Proposition 13.2) that $\chi' = \chi_1^{-m}$, and $m = i + (p+1)j$, $i \in \{1, \dots, p\}$, $j \in \{0, \dots, p-2\}$. It is $\mathrm{Gal}(F/F_0)$-invariant if and only if

$$P(\eta) = \varpi^{p^2-1-m}F_1(\eta),$$

where $F_1(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]\widehat{\ }$. Similarly, a section of $H^0(\widetilde{\Sigma_{1,O_F,s_0'}}, \omega^1)^{(\chi')^p}$ fixed by $\mathrm{Gal}(F/F_0)$ must have the form

$$\varpi^{[-mp]}F_2(\eta)\tilde{e}^i\,\frac{d\tilde{e}}{\tilde{e}},$$

where $F_2(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]\widehat{\ }$, and $[-mp]$ is defined in the beginning of this section.

Thus any element $\bar{F}$ of $H^0(\widetilde{\Sigma_{1,O_F,s_0'}}, \omega^1/p)^{\chi',\mathrm{Gal}(F/F_0)} = H_s^{\chi',F_0}$ can be written uniquely as

$$\varpi^{p^2-1-m}\bar{F}_1(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}$$

on ξ-components, where $\bar{F}_1(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(\eta^{p-1}-1)]$. Now define $\Psi_{s_0',\chi'}(\bar{F}) = \bar{F}_1(\eta)\tilde{e}^{p+1-i}\,d\tilde{e}/\tilde{e}$. Equivalently, it is "multiplication" by $\varpi^{-(p^2-1-m)}$. It's trivial to see this is indeed an isomorphism. We can define $\Psi_{s_0',(\chi')^p}$ in exactly the same way.

Note that $\Psi_{s_0',\chi'}$, $\Psi_{s_0',(\chi')^p}$ are $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant; we can extend both isomorphisms to any vertex s using the action of $\mathrm{GL}_2(\mathbb{Q}_p)$. Concretely, for an even vertex s', $\Psi_{s',\chi'}$ is "multiplication" by $\varpi^{-(p^2-1-m)}$ and $\Psi_{s',(\chi')^p}$ is "multiplication" by $\varpi^{-[-mp]}$. For an odd vertex s, $\Psi_{s,\chi'}$ is "multiplication" by $\varpi^{-[-mp]}$ and $\Psi_{s,(\chi')^p}$ is "multiplication" by $\varpi^{-(p^2-1-m)}$. $\square$

By abuse of notation, I will identify $H^0(\overline{U_s^0}, \omega^1/\varpi)^{\chi'}$, $H^0(\overline{U_s^0}, \omega^1/\varpi)_\tau^{\chi'}$ with H_s^{χ',F_0}, $H_{s,\tau}^{\chi',F_0}$ via the isomorphisms in Lemma 15.4. Notice that ω^1/ϖ is the sheaf of differential forms on $\overline{U_s^0}$, thus we may view elements in H_s^{χ',F_0} as meromorphic differential forms on $\overline{U_s}$.

From now on, I would like to describe an element of H^{χ,F_0} via its image in $\prod_s H^{\chi',F_0}_{s,\tau} \oplus \prod_s H^{(\chi')^p,F_0}_{s,\bar\tau}$. In other words, using Lemma 15.4, any element $h=(h_1,h_2)$ in $H^{\chi,F_0} \simeq H^{\chi',F_0}_{\tau} \oplus H^{(\chi')^p,F_0}_{\bar\tau}$ corresponds to a family of meromorphic differential forms

$$\{(\omega_{s,\tau},\omega_{s,\bar\tau})\}_s,$$

where $\omega_{s,\tau}=h_1|_{\widetilde{\Sigma_{1,O_F,s}}} \in H^{\chi',F_0}_{s,\tau}$ and $\omega_{s,\bar\tau}=h_2|_{\widetilde{\Sigma_{1,O_F,s}}} \in H^{(\chi')^p,F_0}_{s,\bar\tau}$.

To further determine H^{χ,F_0}, we need to know when such a $\{(\omega_{s,\tau},\omega_{s,\bar\tau})\}_s$ comes from a global section. We will give a necessary condition in Proposition 15.8 and a sufficient condition in Proposition 15.11. To this end, it is crucial to understand the local structure of ω^1 on $\widetilde{\Sigma_{1,O_F,\xi}}$. Recall that $\widetilde{\Sigma_{1,O_F,\xi}}$ has an open covering $\{\widetilde{\Sigma_{1,O_F,e,\xi}}\}_e$ and an explicit description of $\widetilde{\Sigma_{1,O_F,e,\xi}}$ (Lemma 8.2) is:

$$\frac{\operatorname{Spf} O_F\Big[\eta,\zeta,\dfrac{1}{\eta^{p-1}-1},\dfrac{1}{\zeta^{p-1}-1},\tilde e,\tilde e'\Big]^{\wedge}}{\Big(\tilde e^{p+1}+v_1w_1^{-1}\xi\dfrac{\eta^p-\eta}{\zeta^{p-1}-1},\ \tilde e'^{p+1}+v_1^{-1}w_1\xi\dfrac{\zeta^p-\zeta}{\eta^{p-1}-1},\ \tilde e\tilde e'-\varpi^{p-1}\xi\Big)}.$$

Note that e/ϖ, e'/ϖ in Lemma 8.2 is $\tilde e,\tilde e'$ here. Suppose $e=[s,s']$, where s' (resp. s) is an even (resp. odd) vertex and corresponds to η (resp. ζ). It's not too hard to see:

Lemma 15.5. *Any element h of $H^0(\widetilde{\Sigma_{1,O_F,[s,s']}},\omega^1)^{\chi',\mathrm{Gal}(F/F_0)}$, when restricted to $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$, can be written in the following form:*

$$h=\varpi^{p^2-1-m}f(\eta)\tilde e^{p+1-i}\,\frac{d\tilde e}{\tilde e}+\varpi^{[-mp]}g(\zeta)\tilde e'^{i}\,\frac{d\tilde e'}{\tilde e'},$$

where $f(\eta)\in O_{F_0}[\eta,1/(\eta^{p-1}-1)]^{\wedge}$, $g(\zeta)\in O_{F_0}[\zeta,1/(\zeta^{p-1}-1)]^{\wedge}$.

Proof. It suffices to verify this after reducing modulo p. Equivalently, we need to show that any $h\in H^{\chi',F_0}_e$ has the form

$$\varpi^{p^2-1-m}f(\eta)\tilde e^{p+1-i}\,\frac{d\tilde e}{\tilde e}+\varpi^{[-mp]}g(\zeta)\tilde e'^{i}\,\frac{d\tilde e'}{\tilde e'},$$

where $f(\eta)\in\mathbb{F}_{p^2}[\eta,1/(1-\eta^{p-1})]$, $g(\zeta)\in\mathbb{F}_{p^2}[\zeta,1/(1-\zeta^{p-1})]$ when restricted to $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$.

Recall that ω^1 is free over $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$ with a basis $d\tilde e/\tilde e=-d\tilde e'/\tilde e'$ (see the beginning of the previous section). Hence any element h in $H^0(\widetilde{\Sigma_{1,O_F,[s,s'],\xi}},\omega^1/p)$ can be written as

$$\sum_{k=0}^{p} f_{1,k}(\eta,\zeta)\tilde e^{k}\,\frac{d\tilde e}{\tilde e}+\sum_{k=0}^{p} g_{1,k}(\eta,\zeta)\tilde e'^{k}\,\frac{d\tilde e'}{\tilde e'},$$

where $f_{1,k}(\eta,\zeta), g_{1,k}(\eta,\zeta) \in O_F/(p)\big[\eta, \zeta, 1/(1-\eta^{p-1}), 1/(1-\zeta^{p-1})\big]/(\eta\zeta)$. This is because using the explicit description of $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$ above, we see that $\tilde{e}^{p+1}$, $\tilde{e}'^{p+1}$, and $\tilde{e}\tilde{e}'$ can each be written as an element only containing η, ζ.

Using the results in Section 9, we see that such an element comes from an element in the χ'-isotypic component of $H^0(\widetilde{\Sigma_{1,O_F,[s,s']}}, \omega^1/p)$ if and only the coefficients of $\tilde{e}^k$ (resp. $\tilde{e}'^k$) are zero unless $k = p+1-i$ (resp. $k = i$). Hence we may write it as

$$h = f_{1,p+1-i}(\eta,\zeta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + g_{1,i}(\eta,\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}, \tag{47}$$

Next consider the action of $\mathrm{Gal}(F/F_0)$. Using the results in Section 9 once again, it's not hard to see that such an element comes from a Galois-invariant section if and only if

$$f_{1,p+1-i}(\eta,\zeta) = \varpi^{p^2-1-m} f_2(\eta,\zeta), \quad g_{1,i}(\eta,\zeta) = \varpi^{[-mp]} g_2(\eta,\zeta), \tag{48}$$

where $f_2(\eta,\zeta), g_2(\eta,\zeta) \in \mathbb{F}_{p^2}\big[\eta, \zeta, 1/(1-\eta^{p-1}), 1/(1-\zeta^{p-1})\big]/(\eta\zeta)$.

Now in order to prove the lemma, we need to "eliminate" the ζ in $f_2(\eta,\zeta)$ and η in $g_2(\eta,\zeta)$. We will prove this under the following assumption:

$$p^2 - 1 - m \geq [-mp].$$

Equivalently, this means $p^2-1-m = [-mp] + i(p-1)$. The other case is similar.

First we eliminate the η in $g_2(\eta,\zeta)$: We can write

$$g_2(\eta,\zeta) = f_3(\eta) + g_3(\zeta),$$

such that $g_3(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(1-\eta^{p-1})]$ and $g_3(\zeta) \in \mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})]$. This is because we can think $g_2(\eta,\zeta)$ as a regular function on a union of two irreducible smooth affine curves crossing transversally. Such a decomposition is obtained by restricting this function on each irreducible component (with some modification by some constants).

Notice that $f_3(0)$ makes sense here. Replacing $f_3(\eta)$ with $f_3(\eta) - f_3(0)$, we may assume

$$f_3(\eta) = \eta f_4(\eta),$$

where $f_4(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(1-\eta^{p-1})]$. Now in $\mathcal{O}_{\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}}$, we have

$$\eta = C\tilde{e}^{p+1}, \quad \text{where } C = -v_1 w_1 \xi^{-1}\frac{\zeta^{p-1}-1}{\eta^{p-1}-1}.$$

Plug this into (47) and use (48):

$$\begin{aligned}h &= \varpi^{p^2-1-m} f_2(\eta,\zeta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}(\eta f_4(\eta)+g_3(\zeta))\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}\\ &= \varpi^{p^2-1-m} f_2(\eta,\zeta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}\\ &\qquad + f_4(\eta)\varpi^{[-mp]}C\tilde{e}^{p+1}\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}.\end{aligned}$$

Since $\tilde{e}\tilde{e}' = \varpi^{p-1}\xi$, the last term in the above equation is

$$\begin{aligned}f_4(\eta)C\varpi^{[-mp]}\varpi^{i(p-1)}\xi^i\tilde{e}^{p+1-i}\,\frac{d\tilde{e}'}{\tilde{e}'} &= f_4(\eta)C\varpi^{p^2-1-m}\xi^i\tilde{e}^{p+1-i}\,\frac{d\tilde{e}'}{\tilde{e}'}\\ &= -\varpi^{p^2-1-m}Cf_4(\eta)\xi^i\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}},\end{aligned}$$

by our assumption. In other words,

$$h = \varpi^{p^2-1-m}\big(f_2(\eta,\zeta) - Cf_4(\eta)\xi^i\big)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}.$$

Hence in (47), we may assume

$$g_{1,i}(\eta,\zeta) = \varpi^{[-mp]}g_3(\zeta), \quad \text{where } g_3(\zeta)\in\mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})].$$

Now we are going to eliminate the ζ in $f_2(\eta,\zeta)$. As before, write $f_2(\eta,\zeta) = f_5(\eta)+\zeta g_5(\zeta)$ and notice that in $\mathcal{O}_{\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}}$, we can write $\zeta = C'\tilde{e}'^{p+1}$. Plug this into (47):

$$\begin{aligned}h &= \varpi^{p^2-1-m}(f_5(\eta)+\zeta g_5(\zeta))\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}\\ &= \varpi^{p^2-1-m}f_5(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{p^2-1-m}g_5(\zeta)C'\tilde{e}'^{p+1}\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}\\ &\qquad + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}.\end{aligned}$$

Here comes the difference between this case and the former case. The middle term actually vanishes:

$$\begin{aligned}\varpi^{p^2-1-m}g_5(\zeta)C'\tilde{e}'^{p+1}\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} &= \varpi^{p^2-1-m}g_5(\zeta)C'\varpi^{(p+1-i)(p-1)}\xi^{p+1-i}\tilde{e}'^{i}\,\frac{d\tilde{e}}{\tilde{e}}\\ &= 0,\end{aligned}$$

since $\varpi^{p^2-1-m+(p+1-i)(p-1)} = \varpi^{[-mp]+(p+1)(p-1)} = -p\cdot\varpi^{-[mp]} = 0$ by our assumption. Hence we may write

$$h = \varpi^{p^2-1-m}f_5(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g_3(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'},$$

which is exactly what we want. □

Now suppose $h \in H_e^{\chi',F_0}$. We may assume it has the form

$$\varpi^{p^2-1-m}f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}g(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'},$$

where $f(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(1-\eta^{p-1})]$, $g(\zeta) \in \mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})]$ when restricted to $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$. What's its restriction to $\widetilde{\Sigma_{1,O_F,s',\xi}}$? Algebraically, this means that we replace ζ by $p/\eta = 0$ and $\tilde{e}'$ by $\varpi^{p-1}\xi/\tilde{e}$. So we have (notice that $d\tilde{e}/\tilde{e} = -d\tilde{e}'/\tilde{e}'$):

$$h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}} = \varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} - \varpi^{[-mp]} g(0)\varpi^{i(p-1)}\xi^i \tilde{e}^{-i}\,\frac{d\tilde{e}}{\tilde{e}}.$$

We make the following assumption in the rest of this section:

$$p^2 - 1 - m \geq [-mp]. \tag{49}$$

Equivalently, this means $p^2 - 1 - m = [-mp] + i(p-1)$.

On $\widetilde{\Sigma_{1,O_F,s',\xi}}$, we have

$$\tilde{e}^{-i} = \frac{\tilde{e}^{p+1-i}}{\tilde{e}^{p+1}} = -\frac{(p/\eta)^{p-1}-1}{v_1 w_1^{-1}\xi(\eta^p-\eta)}\tilde{e}^{p+1-i} \equiv \frac{1}{v_1 w_1^{-1}\xi(\eta^p-\eta)}\tilde{e}^{p+1-i} \pmod p.$$

Hence,

$$\begin{aligned} h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}} &= \varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} \\ &\qquad - \varpi^{p^2-1-m} g(0)\xi^i \frac{1}{v_1 w_1^{-1}\xi(\eta^p-\eta)}\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} \\ &= \varpi^{p^2-1-m}\Big(f(\eta) + g(0)\xi^{i-1}v_1^{-1}w_1\Big(\frac{1}{\eta} - \frac{\eta^{p-2}}{\eta^{p-1}-1}\Big)\Big)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}. \end{aligned} \tag{50}$$

Write $F(\eta) = f(\eta) - g(0)\xi^{i-1}v_1^{-1}w_1\frac{\eta^{p-2}}{\eta^{p-1}-1}$, $C_1 = g(0)\xi^{i-1}v_1^{-1}w_1$.

Lemma 15.6. *Under the assumption $p^2 - 1 - m \geq [-mp]$,*

$$h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}} = \varpi^{p^2-1-m} C_1 \frac{\tilde{e}^{p+1-i}}{\eta}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{p^2-1-m} F(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}}, \tag{51}$$

where $F(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(1-\eta^{p-1})]$, $C_1 \in \mathbb{F}_{p^2}$.

Now if we view $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ as a differential form on $\overline{U^0_{s',\xi}}$, or what's the same, a meromorphic differential form on $\overline{U_{s',\xi}}$ with poles at the singular points ($\overline{U_{s',\xi}}$ is viewed as a subvariety in the special fiber of $\widetilde{\Sigma_{1,O_F,\xi}}$), the order of the pole at the intersection point of $\overline{U_{s',\xi}}$ and $\overline{U_{s,\xi}}$ must be $i+1$ (if there is a pole) since $1/\eta$ has order $p+1$ at this point ($\eta = \tilde{e} = 0$) and $\tilde{e}$ is a uniformizer of this point.

Now restrict h to $\widetilde{\Sigma_{1,O_F,s,\xi}}$. This time we replace η by $p/\zeta = 0$ and $\tilde{e}$ by $\varpi^{p-1}\xi/\tilde{e}'$.

$$\begin{aligned} h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}} &= -\varpi^{p^2-1-m} f(0)\varpi^{(p+1-i)(p-1)}\xi^{p+1-i}\tilde{e}'^{-(p+1-i)}\,\frac{d\tilde{e}'}{\tilde{e}'} \\ &\qquad + \varpi^{[-mp]} g(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'} \\ &= \varpi^{[-mp]} g(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}. \end{aligned} \tag{52}$$

The first term is zero since $\varpi^{p^2-1-m+(p+1-i)(p-1)} = -p\cdot\varpi^{-[mp]}$ by our assumption.

Lemma 15.7. *Under the assumption $p^2-1-m \geq [-mp]$,*

$$h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}} = \varpi^{[-mp]} g(\zeta)\tilde{e}'^i \frac{d\tilde{e}'}{\tilde{e}'},$$

where $g(\zeta) \in \mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})]$.

Thus if we view $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ as a meromorphic differential form on $\overline{U_{s,\xi}}$, it is holomorphic at the intersection point of $\overline{U_{s,\xi}}$ and $\overline{U_{s',\xi}}$. In summary,

Proposition 15.8. *Assume $p^2-1-m \geq [-mp]$. Under the identification in Lemma 15.4, an element h of $H^{\chi',F_0} = H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1/p)^{\chi',\mathrm{Gal}(F/F_0)}$ has the following description*:

1. *If s is odd, then $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ is a holomorphic differential form on $\overline{U_{s,\xi}}$.*
2. *If s' is even, then $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ can have poles at the intersection points of $\overline{U_{s',\xi}}$ with adjacent components. If there are poles, their order must be $i+1$. Moreover, as an element of the space of meromorphic differential forms on $\overline{U_{s',\xi}}$ modulo holomorphic differential forms, $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ is uniquely determined by the restriction of h to the components adjacent to s'. In other words, $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ is holomorphic on $\overline{U_{s',\xi}}$ if the restriction of h to the components adjacent to s' is zero.*

Proof. The first part is a direct consequence of Lemma 15.7. The assertion for the order of poles follows from Lemma 15.6. As for the last assertion, using the notation before Lemma 15.6, we know that the pole of $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ at the intersection point of $\overline{U_{s',\xi}}$ and $\overline{U_{s,\xi}}$ is determined by $g(0)$ (in fact this pole is given by

$$g(0)\xi^{i-1}v_1^{-1}w_1 \frac{\tilde{e}^{p+1-i}}{\eta}\frac{d\tilde{e}}{\tilde{e}}$$

modulo holomorphic terms). However, $g(0)$ is indeed determined by $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ since $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}} = \varpi^{[-mp]} g(\zeta)\tilde{e}'^i\, d\tilde{e}'/\tilde{e}'$. □

Remark 15.9. Under the assumption $p^2-1-m \geq [-mp]$, we have a similar description for elements in $H^{(\chi')^p,F_0}$ while interchanging the descriptions for odd and even vertices. This is obvious if one uses the action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

If we assume $p^2-1-m \leq [-mp]$, an element h of H^{χ',F_0} has the following similar description:

1. If s' is even, then $h|_{\widetilde{\Sigma_{1,O_F,s',\xi}}}$ is a holomorphic differential form on $\overline{U_{s',\xi}}$.
2. If s is odd, then $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ can have poles at the intersection points of $\overline{U_{s,\xi}}$ with adjacent components. The order of these poles, if they exist, must be $p+2-i$. Moreover, $h|_{\widetilde{\Sigma_{1,O_F,s,\xi}}}$ is holomorphic on $\overline{U_{s,\xi}}$ if the restriction of h to the components adjacent to s is zero.

To get a converse result, we need one more lemma to see when we can glue sections on $\widetilde{\Sigma_{1,O_F,s}}$, $\widetilde{\Sigma_{1,O_F,s'}}$ to a section on $\widetilde{\Sigma_{1,O_F,[s,s']}}$.

Lemma 15.10. *Assume $p^2-1-m \geq [-mp]$, and s' is an even vertex and $s \in A(s')$. Given $h_{s'} \in H_{s'}^{\chi',F_0}$, $h_s \in H_s^{\chi',F_0}$ such that they have the forms in Lemmas 15.6 and 15.7* (under the explicit description in Lemma 8.2):

$$h_{s'}|_{\widetilde{\Sigma_{1,O_F,s',\xi}}} = \varpi^{p^2-1-m} C_1 \frac{\tilde{e}^{p+1-i}}{\eta} \frac{d\tilde{e}}{\tilde{e}} + \varpi^{p^2-1-m} F(\eta)\tilde{e}^{p+1-i} \frac{d\tilde{e}}{\tilde{e}}, \tag{53}$$

$$h_s|_{\widetilde{\Sigma_{1,O_F,s,\xi}}} = \varpi^{[-mp]} g(\zeta)\tilde{e}'^i \frac{d\tilde{e}'}{\tilde{e}'}, \tag{54}$$

where $F(\eta) \in \mathbb{F}_{p^2}[\eta, 1/(1-\eta^{p-1})]$, $C_1 \in \mathbb{F}_{p^2}$, $g(\zeta) \in \mathbb{F}_{p^2}[\zeta, 1/(1-\zeta^{p-1})]$. Moreover assume

$$C_1 = g(0)\xi^{i-1} v_1^{-1} w_1. \tag{55}$$

Then we can find a (*unique*) *section $h \in H_{[s,s']}^{\chi',F_0}$ such that*

$$h|_{\widetilde{\Sigma_{1,O_F,s'}}} = h_{s'}, \quad h|_{\widetilde{\Sigma_{1,O_F,s}}} = h_s.$$

Proof. It is direct to see that the following section h_ξ on $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$ can be extended to an element in $H_{[s,s']}^{\chi',F_0}$ and satisfies all the conditions:

$$h_\xi = \varpi^{p^2-1-m}\Big(F(\eta) + C_1 \frac{\eta^{p-2}}{\eta^{p-1}-1}\Big)\tilde{e}^{p+1-i} \frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]} g(\zeta)\tilde{e}'^i \frac{d\tilde{e}'}{\tilde{e}'}. \qquad \square$$

Proposition 15.11. *Assume $p^2-1-m \geq [-mp]$.*

(1) *Given $h_s \in H_s^{\chi',F_0}$ for each odd vertex s that corresponds to a holomorphic differential form on $\overline{U_{s,\xi}}$, we can find an element h in H^{χ',F_0} such that for any odd vertices s,*

$$h|_{\widetilde{\Sigma_{1,O_F,s}}} = h_s.$$

(2) *Moreover, we have the following freedom of choosing h: given $f_{s'} \in H_{s'}^{\chi',F_0}$ for each even vertex s' that corresponds to a holomorphic differential form on $\overline{U_{s',\xi}}$, we may find a* (*unique*) *element f in H^{χ',F_0} such that*

$$f|_{\widetilde{\Sigma_{1,O_F,s'}}} = f_{s'} \quad \text{for any even vertices } s',$$

$$f|_{\widetilde{\Sigma_{1,O_F,s}}} = 0 \quad \text{for any odd vertices } s.$$

Proof. Both are local questions. The second part is a direct consequence of Lemma 15.10: For any even vertex s' and $s \in A(s')$, applying Lemma 15.10 with $h_s = 0$, $h_{s'} = f_{s'}$ (in this case, $C_1 = 0$), we can glue to a section on $\widetilde{\Sigma_{1,O_F,[s,s']}}$ whose restriction to $\widetilde{\Sigma_{1,O_F,s'}}$ (resp. $\widetilde{\Sigma_{1,O_F,s}}$) is $f_{s'}$ (resp. zero). Hence we can glue to a global section on $\widetilde{\Sigma_{1,O_F}}$.

As for the first part, our strategy is similar. For any even vertex s', we will find a section $h_{s'} \in H_{s'}^{\chi',F_0}$ such that for any vertex $s \in A(s')$, we can use Lemma 15.10 to glue $h_s, h_{s'}$ to a section on $\widetilde{\Sigma_{1,O_F,[s,s']}}$ and obtain a global section on $\widetilde{\Sigma_{1,O_F}}$.

By Lemma 15.4, we may identify elements in $H_{s'}^{\chi',F_0}$ with differential forms on $\overline{U_{s'}^0}$. Since $(O_D/\Pi)^\times \simeq \mathbb{F}_{p^2}^\times$ acts transitively on the connected components of $\overline{U_{s'}^0}$, it is easy to see $\mu_{p+1}(\mathbb{F}_{p^2}) = \{a \in \mathbb{F}_{p^2} \mid a^{p+1} = 1\}$ fixes $\overline{U_{s',\xi}^0}$. As we noted in the proof of Lemma 12.9,

$$H^0(\overline{U_{s'}^0}, \omega^1/\varpi)^{\chi'} \simeq H^0(\overline{U_{s',\xi}^0}, \omega^1/\varpi)^{\mathrm{Id}^{-i}},$$

where we view $\mathrm{Id} : \mu_{p+1}(\mathbb{F}_{p^2}) \to \mathbb{F}_{p^2}^\times$ as a character of $\mu_{p+1}(\mathbb{F}_{p^2})$, and Id^{-i} is its $(-i)$-th power. We denote the intersection point of $\overline{U_{s',\xi}}$ with $\overline{U_{s,\xi}}$, by P_s for $s \in A(s')$.

Now using Lemma 15.10, finding such an $h_{s'} \in H_{s'}^{\chi',F_0}$ is equivalent to finding a meromorphic differential form $\omega_{s'} \in H^0(\overline{U_{s',\xi}^0}, \omega^1/\varpi)^{\mathrm{Id}^{-i}}$ such that:

- It can only have poles at P_s, $s \in A(s')$ with order at most $i+1$ (in fact, it has to be $i+1$ if there is a pole, by considering the action of $\mu_{p+1}(\mathbb{F}_{p^2})$).
- The "leading coefficient" of the pole at P_s is prescribed by h_s for all $s \in A(s')$.

More precisely, using the explicit description in Lemma 8.2, the first condition allows us to write $\omega_{s'}$ into the form (53). Also our condition in the proposition allows us to write h_s into the form (54). Then C_1 in (53) is the leading coefficient in this case and we want it to satisfy (55).

The existence of such a meromorphic differential form follows from:

Lemma 15.12. *Let C be a smooth geometrically connected curve over $\mathbb{F}_{p^2}$ and $\{P_k\}_k$ be a nonempty finite subset of $C(\mathbb{F}_{p^2})$. Then for $n \geq 2$, the restriction map*

$$H^0(C, \Omega^1_C(nD)) \to \bigoplus_k H^0(P_k, \Omega^1_C(nD)|_{P_k})$$

is surjective, where D is the divisor $\sum_k P_k$.

Assume this lemma for the moment. In our case, let $C = \overline{U_{s',\xi}}$, $\{P_k\} = \{P_s\}$ and $n = i+1$. The prescribed leading coefficients become a family of elements $c_s \in H^0(P_s, \Omega^1_C(nD)|_{P_s})$, $s \in A(s')$. Notice that the uniformizer for P_s is either $\tilde{e}$ or $\tilde{e}/\eta$, hence $\mu_{p+1}(\mathbb{F}_{p^2})$ acts on

$$H^0\biggl(P_s, \Omega^1_C\biggl(\sum_k (i+1)P_k\biggr)\bigg|_{P_s}\biggr) = H^0(P_s, \Omega^1_C((i+1)D)|_{P_s})$$

via Id^{-i}. So taking the Id^{-i}-isotypic component of the map in the lemma (which remains surjective since $p+1$ is coprime to p), we may find an element in $H^0\bigl(\overline{U_{s',\xi}}, \Omega^1\bigl(\sum_s (i+1)P_s\bigr)\bigr)^{\mathrm{Id}^{-i}}$ having the correct leading coefficient at each P_s and that's exactly what we want. $\square$

Proof of Lemma 15.12. Consider the following short exact sequence of sheaves:

$$0 \to \Omega^1_C((n-1)D) \to \Omega^1_C(nD) \to \bigoplus_k \Omega^1_C(nD)|_{P_k} \to 0.$$

It suffices to show $H^1(C, \Omega^1_C((n-1)D))$ vanishes. However by Serre duality, this space is dual to $H^0(C, \mathcal{O}_C(-(n-1)D))$, which is zero since we assume $n \geq 2$. $\square$

Now, we can prove the main proposition of this section.

Proposition 15.13. *Assume* $p^2-1-m \geq [-mp]$. *There exists a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant short exact sequence*:

$$0 \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{\chi'}_{\tau} \to H^{(0),\chi,\mathbb{Q}_p} \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}_{\bar{\tau}} \to 0.$$

Proof. Write $\prod_s H_s^{\chi,F_0} = \prod_s (H_{s,\tau}^{\chi',F_0} \oplus H_{s,\bar{\tau}}^{(\chi')^p,F_0})$, where as usual, s runs over the vertices of the Bruhat–Tits tree. Define

$$H_1 = \prod_{s' \text{ even}} H_{s',\tau}^{\chi',F_0} \oplus \prod_{s \text{ odd}} H_{s,\bar{\tau}}^{(\chi')^p,F_0},$$
$$H_2 = \prod_{s \text{ odd}} H_{s,\tau}^{\chi',F_0} \oplus \prod_{s' \text{ even}} H_{s',\bar{\tau}}^{(\chi')^p,F_0}.$$

Notice that $\mathrm{GL}_2(\mathbb{Q}_p)$ actually acts on H_1, H_2. Then we have a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant (split) short exact sequence:

$$0 \to H_1 \to \prod_s H_s^{\chi,F_0} \to H_2 \to 0.$$

Recall that we have an injection of $H^{(0),\chi,\mathbb{Q}_p} \simeq H^{\chi,F_0}$ into $\prod_s H_s^{\chi,F_0}$. So this short exact sequence induces another short exact sequence:

$$0 \to K \to H^{\chi,F_0} \to C \to 0.$$

It remains to determine K, and C.

Let f be an element of H^{χ,F_0}. We will write $f = f_\tau + f_{\bar{\tau}}$ under the decomposition $H^{\chi,F_0} \simeq H_\tau^{\chi',F_0} \oplus H_{\bar{\tau}}^{(\chi')^p,F_0}$ (see (46)).

Suppose f is in K. This means for any odd vertex s and even vertex s',

$$f_\tau|_{\widetilde{\Sigma_{1,O_F,s}}} = 0 \quad \text{and} \quad f_{\bar{\tau}}|_{\widetilde{\Sigma_{1,O_F,s'}}} = 0.$$

By the second part of Proposition 15.8, we know that $f_\tau|_{\widetilde{\Sigma_{1,O_F,s'}}}$ corresponds to a holomorphic differential form on $\overline{U_{s',\xi}}$ for any even vertex s' (tensored with O_E). However the second part of Proposition 15.11 indicates that $f_\tau|_{\widetilde{\Sigma_{1,O_F,s'}}}$ can be any holomorphic differential form inside $H^0(\overline{U_{s'}}, \Omega^1_{\overline{U_{s'}}})^{\chi'}_\tau$. Similarly $f_{\bar{\tau}}|_{\widetilde{\Sigma_{1,O_F,s}}}$ can be any holomorphic differential form inside $H^0(\overline{U_s}, \Omega^1_{\overline{U_s}})^{\chi'}_\tau$, where s is an odd vertex. This certainly implies that

$$K \simeq \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{\chi'}_\tau.$$

By the first part of Proposition 15.8, we know that C is inside

$$\prod_{s:\text{ odd}} H^0(\overline{U_s},\Omega^1_{\overline{U_s}})^{\chi'}_{\tau}\oplus\prod_{s':\text{ even}} H^0(\overline{U_{s'}},\Omega^1_{\overline{U_{s'}}})^{(\chi')^p}_{\bar\tau},$$

as a subset of H_2. However the first part of Proposition 15.11 tells us that in fact C is equal to this set. Clearly this is nothing but $\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau}$. $\square$

Remark 15.14. See the beginning of the paper for the notation here: Under the isomorphism (27), an element of $H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{\chi'}$ must have the form $f(\eta)\tilde{e}^{p+1-i}\,d\tilde{e}/\tilde{e}$ on $\overline{U_{s_0',\xi}}$, where $f(\eta)$ is a polynomial of η of degree at most $i-2$. Using the results in Section 9, it's not hard to construct a $\mathrm{GL}_2(\mathbb{F}_p)$-equivariant isomorphism:

$$H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{\chi'}\to(\mathrm{Sym}^{i-2}\,\mathbb{F}^2_{p^2})\otimes\det{}^{j+1},\tag{56}$$

$$\eta^r\tilde{e}^{p+1-i}\frac{d\tilde{e}}{\tilde{e}}\mapsto x^r y^{i-2-r},\tag{57}$$

where $\mathrm{Sym}^{i-2}\,\mathbb{F}^2_{p^2}$ is the $(i-2)$-th symmetric power of the natural representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on the canonical basis of $\mathbb{F}^2_{p^2}$.

Similarly, we can identify $H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}$ with $(\mathrm{Sym}^{p-1-i}\,\mathbb{F}^2_{p^2})\otimes\det^{i+j}$. Then we can rewrite the exact sequence in Proposition 15.13 as

$$0\to\sigma_{i-2}(j+1)\to H^{(0),\chi,\mathbb{Q}_p}\to\sigma_{p-1-i}(i+j)\to 0.$$

Remark 15.15. If we assume $p^2-1-m\le[-mp]$, then we have the exact sequence of the opposite direction:

$$0\to\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar\tau}\to H^{(0),\chi,\mathbb{Q}_p}\to\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{\chi'}_{\tau}\to 0.$$

16. Computation of $M(\chi,[1,b])/p$, I: $2\le i\le p-1$

In this section, we compute $M(\chi,[1,b])/p$ as a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ when $i\in\{2,\dots,p-1\}$. The strategy is as follows. We first identify the crystalline cohomology with the de Rham cohomology of some formal scheme. Then H^{χ,F_0} will map to some meromorphic differential forms on this formal scheme. Now any cohomology class of the de Rham cohomology can be expressed using 1-hypercocycles and any meromorphic differential form can be naturally viewed as a 1-hypercocycle. The question becomes how to write this 1-hypercocycle into some "good form". This will be done by explicit calculations. We keep the notation from the last section.

Consider the composite of the following maps, which we denote by ι,

$$H^{\chi',F_0}\to H^0(\Sigma_{1,F},\Omega^1)^{\chi'}\to H^1_{\mathrm{dR}}(\Sigma_{1,F})^{\chi'}\simeq\prod_s H^1_{\mathrm{dR}}(U_s)^{\chi'}\simeq\prod_s H^1_{\mathrm{crys}}(\overline{U_s}/F_0)^{\chi'}\otimes_{F_0}F.$$

See Sections 11 and 12 for the notation. Our first result is about the image of ι. We denote the first crystalline cohomology of $\overline{U_s}$ (over Spec $\mathbb{F}_{p^2}$) by $H^1_{\text{crys}}(\overline{U_s}/O_{F_0})$. It is not hard to see that this is a lattice inside $H^1_{\text{crys}}(\overline{U_s}/F_0) = H^1_{\text{crys}}(\overline{U_s}/O_{F_0}) \otimes_{O_{F_0}} F_0$.

Proposition 16.1. $$\iota(H^{\chi', F_0}) \subset \prod_s H^1_{\text{crys}}(\overline{U_s}/O_{F_0})^{\chi'} \otimes_{O_{F_0}} O_F.$$

Proof. We only deal with the even case, that is to say, for an even vertex s', we will prove that the image of H^{χ', F_0} in $H^1_{\text{crys}}(\overline{U_{s'}}/F_0)^{\chi'} \otimes_{F_0} F$ is actually inside $H^1_{\text{crys}}(\overline{U_{s'}}/O_{F_0}) \otimes_{O_{F_0}} O_F$. The odd case is similar.

First we recall some results from Section 12; see the discussion below Lemma 12.1. We constructed an isomorphism $\psi_{s',\xi} : U_{s',\xi} \to F_{0,\xi}$ (recall that $U_{s',\xi}$ is the tubular neighborhood of $\overline{U_{s',\xi}}$ in $\Sigma_{1,F}$), where

$$F_{0,\xi} \stackrel{\text{def}}{=} \Big\{(x, y) \in \mathbb{A}^2_F \,\Big|\, y^{p+1} = v_1 w_1^{-1} \xi(x^p - x),\ |x-k| > p^{-1/(p-1)}, k=0, \ldots, p-1, \\ |x| < p^{1/(p-1)}\Big\}.$$

Cleary $F_{0,\xi}$ is an open set in a projective curve $D_{0,\xi}$ in $\mathbb{P}^2_F$ defined by $y^{p+1} = v_1 w_1^{-1} \xi(x^p - x)$. The curve $D_{0,\xi}$ has an obvious formal model $\widehat{D_{0,O_{F_0},\xi}}$ over O_{F_0}. Its special fiber can be canonically identified with $\overline{U_{s',\xi}}$. Hence we can identify $H^1_{\text{crys}}(\overline{U_{s',\xi}}/O_{F_0})$ with $H^1_{\text{dR}}(\widehat{D_{0,O_{F_0},\xi}})$.

Definition 16.2. For $s \in A(s')$, let $V_{s,\xi}$ be the affine open formal subscheme of $\widehat{D_{0,O_{F_0},\xi}}$ whose underlying space is the union of $\overline{U^0_{s',\xi}}$ and the intersection point of $\overline{U_{s',\xi}}$ and $\overline{U_{s,\xi}}$. Also we define $V_{c,\xi} = \bigcap_{s_v \in A(s')} V_{s_v,\xi}$ (it is equal to $V_{s_1,\xi} \cap V_{s_2,\xi}$ for any $s_1 \neq s_2 \in A(s')$).

Hence $\mathcal{C} = \{V_{s,\xi}\}_{s \in A(s')}$ is an open covering of $\widehat{D_{0,O_{F_0},\xi}}$. Any element in $H^1_{\text{dR}}(\widehat{D_{0,O_{F_0},\xi}})$ can be represented as a 1-hypercocycle $(\{\omega_s\}_{s\in A(s')}, \{f_{s_1,s_2}\}_{s_1,s_2 \in A(s)})$, where $\omega_s \in H^0(V_{s,\xi}, \Omega^1_{V_{s,\xi}})$, and $f_{s_1,s_2} \in H^0(V_{s_1,\xi} \cap V_{s_2,\xi}, \mathcal{O}_{V_{s_1,\xi} \cap V_{s_2,\xi}})$, such that

$$df_{s_1,s_2} = \omega_{s_1}|_{V_{s_1} \cap V_{s_2}} - \omega_{s_2}|_{V_{s_1} \cap V_{s_2}}.$$

Two 1-hypercocycles $(\{\omega_s\}, \{f_{s_1,s_2}\})$, $(\{\omega'_s\}, \{f'_{s_1,s_2}\})$ represent the same cohomology class if and only there exists a family of functions $\{g_s\}_{s\in A(s)}$, $g_s \in H^0(V_{s,\xi}, \mathcal{O}_{V_{s,\xi}})$, such that

$$\omega_s - \omega'_s = dg_s, \quad f_{s_1,s_2} - f'_{s_1,s_2} = g_{s_1}|_{V_{s_1} \cap V_{s_2}} - g_{s_2}|_{V_{s_1} \cap V_{s_2}}.$$

Given a differential form ω on $F_{0,\xi}$, we view it as a cohomology class in $H^1_{\text{dR}}(F_{0,\xi})$. How do we relate it, as above, with a 1-hypercocycle in

$$H^1_{\text{dR}}(D_{0,\xi}) = H^1_{\text{dR}}(\widehat{D_{0,O_{F_0},\xi}}) \otimes_{F_0} F?$$

Definition 16.3. Since the generic fiber of $\widehat{D_{0,O_{F_0},\xi}}$ becomes $D_{0,\xi}$ when tensored with F, the generic fiber of $V_{s,\xi}$ corresponds to an open rigid subspace of $D_{0,\xi}$, which we denote by $W_{s,\xi}$. We also define $Z_{s,\xi} = W_{s,\xi} \cap F_{0,\xi}$.

If ω is in the χ'-isotypic component and $\chi' \neq \chi'^p$, we will see later that we can find a rigid analytic function f_s on $Z_{s,\xi}$ for each $s \in A(s')$ such that $\omega|_{Z_{s,\xi}} - df_s$ can be extended to a holomorphic differential form ω_s on $W_{s,\xi}$. Define

$$f_{s_1,s_2} = f_{s_2}|_{W_{s_1,\xi}\cap W_{s_2,\xi}} - f_{s_1}|_{W_{s_1,\xi}\cap W_{s_2,\xi}}. \tag{58}$$

Then $(\{\omega_s\}, \{f_{s_1,s_2}\})$ is an element in $H^1_{\mathrm{dR}}(D_{0,\xi})$, whose image in $H^1_{\mathrm{dR}}(F_{0,\xi})$ is ω.

Roughly speaking, what we did above is to "remove" the poles of ω so that ω can be extended to a hypercocycle on $D_{0,\xi}$.

Now apply the above abstract discussion to our situation. Let s' be an even vertex and $s \in A(s')$. Then, under the isomorphism in Lemma 8.2,

$$W_{s,\xi} = \left\{(x, y) \in D_{0,\xi} \;\middle|\; |x-k| = 1, k = 1, \dots, p-1,\ |x| \le 1\right\},$$
$$Z_{s,\xi} = \left\{(x, y) \in W_{s,\xi} \;\middle|\; |x| > p^{-1/(p-1)}\right\}.$$

Recall that in Lemma 15.5, we showed that a section ω of $H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{\chi'}$ has the following form when restricted to $\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}$:

$$\omega|_{\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}} = \varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]} g(\zeta)\tilde{e}'^{i}\,\frac{d\tilde{e}'}{\tilde{e}'}, \tag{59}$$

where $f(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]^{\widehat{}}$, $g(\zeta) \in O_{F_0}[\zeta, 1/(\zeta^{p-1}-1)]^{\widehat{}}$.

Hence if we restrict it to $\widetilde{\Sigma_{1,O_F,s',\xi}}$ (replace ζ by p/η and $\tilde{e}'$ by $\varpi^{p-1}\xi/\tilde{e}$):

$$\omega|_{\widetilde{\Sigma_{1,O_F,s',\xi}}} = \varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\,\frac{d\tilde{e}}{\tilde{e}} - \varpi^{[-mp]} g\left(\frac{p}{\eta}\right)\varpi^{i(p-1)}\xi^i\tilde{e}^{-i}\,\frac{d\tilde{e}}{\tilde{e}}, \tag{60}$$

where

$$f(\eta) \in O_{F_0}\left[\eta, \frac{1}{\eta^{p-1}-1}\right]^{\widehat{}}, \quad g\left(\frac{p}{\eta}\right) \in O_{F_0}\left[\frac{p}{\eta}, \frac{1}{(p/\eta)^{p-1}-1}\right]^{\widehat{}} \subset O_{F_0}\left[\left[\frac{p}{\eta}\right]\right].$$

Notice that the restriction of $\psi_{0,\xi}$ to the generic fiber of $\widetilde{\Sigma_{1,O_F,s',\xi}}$ has the form

$$x \mapsto \eta, \quad y \mapsto \tilde{e}(1-(p/\eta)^{p-1})^{1/(p+1)}. \tag{61}$$

Lemma 16.4. *Under the isomorphism $\psi_{0,\xi}$, the 1-form ω has the following form on $Z_{s,\xi}$*

$$\varpi^{p^2-1-m}\left(F(x)y^{p+1-i} + G\left(\frac{p}{x}\right)y^{-i}\right)\frac{dy}{y}, \tag{62}$$

where $F(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^{\widehat{}}$ and $G(p/x) = \sum_{n=0}^{+\infty} a_n(p/x)^n$ with $a_n \in O_{F_0}$ for all n. Moreover, using (60):

(1) $f(x) \equiv F(x) \bmod pO_{F_0}[x, 1/(x^{p-1}-1)]^{\widehat{}}$.

(2) $a_0 \equiv -\xi^i g(0) \bmod pO_{F_0}$ *if* $p^2-1-m \ge [-mp]$ *and* $a_0 \equiv 0 \bmod pO_{F_0}$ *otherwise. When* $i = p$, $a_0/p \equiv -\xi^i g(0) \bmod pO_{F_0}$.

Assume this lemma for the moment. Hence we can write ω on $Z_{s,\xi}$ as

$$\varpi^{p^2-1-m}\Big(F(x)y^{p+1-i}+G\Big(\frac{p}{x}\Big)y^{-i}\Big)\,\frac{dy}{y},$$

where $F(x)\in O_{F_0}[x,1/(x^{p-1}-1)]\widehat{\ }$, $G(p/x)=\sum_{n=0}^{+\infty}a_n(p/x)^n\in O_{F_0}[\![p/x]\!]$. Certainly $F(x)y^{p+1-i}\,dy/y$ extends to $W_{s,\xi}$, so we only need to "remove" the poles of the other term (essentially the pole at $x=y=0$). On $Z^0_{s,\xi}\overset{\text{def}}{=}\{(x,y)\in Z_{s,\xi}\mid |x|<1\}$, we can write

$$x=\sum_{n=1}^{+\infty}c_ny^{(p+1)n},$$

where $c_n\in O_{F_0}$, $c_1=v_1^{-1}w_1\xi^{-1}\in O_{F_0}^{\times}$. Thus a simple computation shows:

Lemma 16.5. *On $Z^0_{s,\xi}$,*

$$\sum_{n=0}^{+\infty}a_n\Big(\frac{p}{x}\Big)^ny^{-i}\frac{dy}{y}=\sum_{n=-\infty}^{+\infty}b_ny^{-n(p+1)-i-1}\,dy,$$

where $b_n\in O_{F_0}$, $\forall n\in\mathbb{Z}$ and for $n\geq 0$, $v_p(b_n)\geq n$. Moreover $b_0\equiv a_0 \bmod p$.

Now define

$$f_s=\varpi^{p^2-1-m}\sum_{n=0}^{+\infty}\frac{b_n}{-n(p+1)-i}y^{-n(p+1)-i}. \tag{63}$$

It can be viewed as a rigid analytic function on $Z_{s,\xi}$. Also, it is clear from the above computation that $\omega-df_s$ can be extended to a holomorphic differential form ω_s on $W_{s,\xi}$. Do the same thing for each $s\in A(s')$; we can define ω_s, f_{s_1,s_2} as explained before. Then $(\{\omega_s\},\{f_{s_1,s_2}\})$ is the 1-hypercocycle in $H^1_{\mathrm{dR}}(D_{0,\xi})\simeq H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}},_{\xi})\otimes_{O_{F_0}}F$ that represents ω.

Notice that for $i\in\{1,\dots,p-1\}$, $v_p(b_n/(-n(p+1)-i))\geq 0$ since $v_p(b_n)\geq n$. When $i=p$, $b_0\equiv a_0\equiv 0 \bmod p$ since we are in the case $p^2-1-m\leq[-mp]$. We still have $v_p(b_n/(-n(p+1)-i))\geq 0$. In fact, equality only can happen when $n=0$. Therefore all the coefficients appearing in ω_s, f_{s_1,s_2} will be integral. In other words,

$$(\{\omega_s\},\{f_{s_1,s_2}\})\in H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}},_{\xi})\otimes_{O_{F_0}}O_F. \qquad\square$$

Proof of Lemma 16.4. We only give a sketch of the computations here. Using the notation in (60), it suffices to deal with the case $g(p/\eta)=0$ and $f(\eta)=0$ separately.

(1) Assume $g(p/\eta)=0$. Plug (61) into (60). A direct computation shows that ω has the form

$$\varpi^{p^2-1-m}f(x)\Big(1+\Big(\frac{p}{x}\Big)^{p-1}G_1(x)\Big)y^{p+1-i}\,\frac{dy}{y},$$

where $G_1(x)\in O_{F_0}\Big[x,\frac{1}{x^{p-1}-1}\Big]\widehat{\ }\Big[\!\Big[\frac{p}{x}\Big]\!\Big]$.

Let $G_2(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^{\wedge}[\![p/x]\!]$ be

$$G_2(x) = v_1 w_1^{-1} \xi (x^{p-1}-1) G_1(x) f(x) \Big(\frac{p}{x}\Big)^{p-2}.$$

Clearly we can decompose $G_2(x)$ as

$$G_2(x) = F_3(x) + G_3(\frac{p}{x}),$$

where $F_3(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^{\wedge}$, $G_3(p/x) \in O_{F_0}[\![p/x]\!]$. Replacing $F_3(x)$ by $F_3(x) - F_3(0)$, we may assume $F_3(x) \in x O_{F_0}[x, 1/(x^{p-1}-1)]^{\wedge}$. Since there is a $(p/x)^{p-2}$ in the definition of $G_2(x)$, it is easy to see (for example, expand $G_2(x)$ as an element in $F_0[\![x, 1/x]\!]$) that the constant term of $G_3(p/x)$ is divisible by p (in fact, by p^{p-2}). Recall that we assume p is odd; hence at least 3.

Now $\varpi^{-(p^2-1-m)}\omega$ can be written as

$$f(x) y^{p+1-i} \frac{dy}{y} + p\Big(\frac{F_3(x)}{x}\Big)\Big(\frac{1}{(x^{p-1}-1)v_1 w_1^{-1}\xi}\Big) y^{p+1-i} \frac{dy}{y} + p\frac{G_3(p/x)}{(x^p - x)v_1 w_1^{-1}\xi} y^{p+1-i} \frac{dy}{y}.$$

Notice that $y^{p+1} = v_1 w_1^{-1} \xi (x^p - x)$. The last term is nothing but

$$pG_3\Big(\frac{p}{x}\Big) y^{-i} \frac{dy}{y}.$$

Now let

$$F(x) = f(x) + p\Big(\frac{F_3(x)}{x}\Big)\Big(\frac{1}{(x^{p-1}-1)v_1 w_1^{-1}\xi}\Big), \quad G\Big(\frac{p}{x}\Big) = pG_3\Big(\frac{p}{x}\Big).$$

It is clear they satisfy all the conditions in the lemma. So we're done in this case.

(2) Assume $f(\eta) = 0$. When $p^2 - 1 - m \geq [-mp]$, we can write $\varpi^{-(p^2-1-m)}\omega$ as

$$-g\Big(\frac{p}{x}\Big)\xi^i \Big(1 + \Big(\frac{p}{x}\Big)^{p-1} H(x)\Big) y^{-i} \frac{dy}{y},$$

where $H(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^{\wedge}[\![p/x]\!]$. Make the decomposition

$$-g\Big(\frac{p}{x}\Big)\xi^i H(x)\Big(\frac{p}{x}\Big)^{p-2} = F_1(x)x^2 + Ax + H_1\Big(\frac{p}{x}\Big),$$

where $F_1(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]^{\wedge}$, $A \in pO_{F_0}$, $H_1(p/x) \in O_{F_0}[\![p/x]\!]$. Notice that A is divisible by p since there is a $(p/x)^{p-2}$ in the expression. Then $\varpi^{-(p^2-1-m)}\omega$ is

$$-g\Big(\frac{p}{x}\Big)\xi^i y^{-i} \frac{dy}{y} + pxF_1(x) y^{-i} \frac{dy}{y} + pAy^{-i} \frac{dy}{y} + \frac{p}{x} H_1\Big(\frac{p}{x}\Big) y^{-i} \frac{dy}{y}.$$

Using $y^{p+1} = v_1 w_1^{-1}\xi(x^p - x)$, the second term is

$$pF_1(x)\frac{1}{v_1 w_1^{-1}\xi(x^{p-1}-1)} y^{p+1-i}\,\frac{dy}{y}.$$

It's easy to see the following $F(x)$, $G(p/x)$ actually work:

$$F(x) = pF_1(x)\frac{1}{v_1 w_1^{-1}\xi(x^{p-1}-1)}, \quad G\Big(\frac{p}{x}\Big) = -g\Big(\frac{p}{x}\Big)\xi^i + pA + \frac{p}{x}H_1\Big(\frac{p}{x}\Big).$$

When $p^2-1-m \geq [-mp]$ does not hold, then

$$\omega = -\varpi^{[-mp]} g\Big(\frac{p}{\eta}\Big)\varpi^{i(p-1)}\xi^i \tilde{e}^{-i}\,\frac{d\tilde{e}}{\tilde{e}} = p\varpi^{p^2-1-m} g\Big(\frac{p}{\eta}\Big)\xi\tilde{e}^{-i}\,\frac{d\tilde{e}}{\tilde{e}}.$$

Repeat the previous argument and it's direct to see the claim in the lemma is true. □

In the previous proposition, we showed how to turn a differential form $\omega \in H^{\chi', F_0}$, when restricted to $U_{s'}$, into a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ inside the de Rham cohomology $H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}}) \otimes_{O_{F_0}} O_F$ (via the isomorphism $\psi_{s',\xi}$). It is crucial to understand the mod p properties of this hypercocycle. Essentially, we need to understand f_s in (63) modulo p (recall that $f_{s_1,s_2} = f_{s_2} - f_{s_1}$; see (58)).

Fix an even vertex s' and $s \in A(s')$. Recall that $V_{c,\xi} = \bigcap_{s_v \in A(s')} V_{s_v,\xi}$. It is clear from our definition that $\varpi^{-(p^2-1-m)} f_s \in H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})$.

Lemma 16.6. (Using notation from the proof of Proposition 16.1.)

(1) *When $i = p$,*

$$\varpi^{-(p^2-1-m)} f_s \equiv \frac{b_0 y^{-p}}{-p} \equiv \frac{a_0 y^{-p}}{-p} \equiv \xi^p g(0) y^{-p} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

(2) *When $i \in \{1, \dots, p-1\}$ and $p^2-1-m \leq [-mp]$,*

$$\varpi^{-(p^2-1-m)} f_s \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

(3) *When $p^2-1-m \geq [-mp]$, we have*

$$\varpi^{-(p^2-1-m)} f_s \equiv \frac{b_0 y^{-i}}{-i} \equiv \frac{a_0 y^{-i}}{-i} \equiv \frac{\xi^i g(0) y^{-i}}{i} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}),$$

except the case $i = p-1$ and the case $p = 3$, $i = 1$. I claim that in these exceptional cases, we can find another 1-hypercocycle $(\{\omega'_{s_v}\}, \{f'_{s_1,s_2}\})$ in the same cohomology class as $(\{\omega_{s_v}\}, \{f_{s_1,s_2}\})$ such that we can write $f'_{s_1,s_2} = f'_{s_2} - f'_{s_1}$ for any $s_1, s_2 \in A(s')$ and

$$\varpi^{-(p^2-1-m)} f'_s \equiv \frac{b_0 y^{-i}}{-i} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}),$$
$$f'_{s_v} = f_s \quad \textit{for any } s_v \neq s.$$

Proof of Lemma 16.6. Everything is clear by Lemma 16.5 and the definition of f_s in (63), except for the exceptional cases. First we assume $i = p-1$, then

$$\varpi^{-(p^2-1-m)} f_s \equiv \frac{b_0 y^{-(p-1)}}{-(p-1)} + \frac{b_1 y^{-2p}}{-2p} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

This makes sense since $v_p(b_1) \geq 1$. Now define $g_{s_v} \in H^0(V_{s_v,\xi}, \mathcal{O}_{V_{s_v,\xi}}),\ s_v \in A(s')$ as:

$$g_{s_v} = \begin{cases} -\dfrac{b_1}{-2pv_1^2 w_1^{-2}\xi^2} \dfrac{(x^{2p-4}-2x^{p-3})y^2}{(x^{p-1}-1)^2} & \text{if } s_v = s, \\ -\dfrac{b_1}{-2pv_1^2 w_1^{-2}\xi^2} \dfrac{y^2}{x^2} & \text{if } s_v \neq s. \end{cases}$$

Hence define $\omega'_{s_v} = \omega_{s_v} + \varpi^{p^2-1-m} dg_{s_v},\ f'_{s_1,s_2} = f_{s_1,s_2} + \varpi^{p^2-1-m}(g_{s_1} - g_{s_2})$; the hypercocycle $(\{\omega_{s_v}\}, \{f_{s_1,s_2}\})$ and $(\{\omega'_{s_v}\}, \{f'_{s_1,s_2}\})$ are in the same cohomology class. A simple computation shows the following identity in $H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})$:

$$-\frac{b_1}{-2pv_1^2 w_1^{-2}\xi^2} \frac{(x^{2p-4}-2x^{p-3})y^2}{(x^{p-1}-1)^2} + \frac{b_1}{-2pv_1^2 w_1^{-2}\xi^2} \frac{y^2}{x^2} = \frac{b_1 y^{-2p}}{-2p}.$$

Thus if we define $f'_s = f_s - b_1 y^{-2p}/(-2p),\ f'_{s_v} = f_{s_v},\ s_v \neq s$, they satisfy

$$f'_{s_1,s_2} = f'_{s_2} - f'_{s_1},$$

and clearly have the property we want.

The case $i = 1,\ p = 3$ can be done by the same method. This time

$$\begin{aligned} \varpi^{-(p^2-1-m)} f_s &\equiv b_0 y^{-1} + \frac{b_2}{-9} y^{-9} \\ &\equiv b_0 y^{-1} + \frac{b_2}{-9v_1^3 w_1^{-3}\xi^3} \left(\frac{x^3 y^3}{(x^2-1)^3} - \frac{y^3}{x^3} \right) \mod 3H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}). \end{aligned}$$

We can define g_{s_v} similarly. I omit the details here. □

Remark 16.7. In the odd case, things are similar. We only restrict ourselves to the case $p^2 - 1 - m \geq [-mp]$. Let s be an odd vertex. We also have $\psi_{s,\xi}$ (see the beginning of Section 12). Let ω be an element of H^{χ',F_0}. Similarly to Lemma 16.4, ω has the form (using (52)):

$$\varpi^{[-mp]} \left(F(x) y^i + pG\left(\frac{p}{x}\right) y^{-p-1+i} \right) \frac{dy}{y}, \tag{64}$$

where $F(x) \in O_{F_0}[x, 1/(x^{p-1}-1)]\widehat{\ }$, $G(p/x) = \sum_{n=0}^{+\infty} a_n (p/x)^n,\ a_n \in O_{F_0}\ \forall n$. All the above arguments work here and we can define a 1-hypercocycle $(\{\omega_{s'}\}, \{f_{s'_1,s'_2}\})$ that represents ω. Notice that there is a "p" in front of $G(p/x)$ in (64). Thus when

$2 \le i \le p$ (resp. $i = 1$),

$$\varpi^{-[-mp]} f_{s'} \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}) \text{ (resp. } H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})\text{)},$$

$$\varpi^{-[-mp]} f_{s'_1,s'_2} \in pH^0(V_{s'_1,\xi} \cap V_{s'_2,\xi}, \mathcal{O}_{V_{s'_1,\xi} \cap V_{s'_2,\xi}}) \text{ (resp. } H^0(V_{s'_1,\xi} \cap V_{s'_2,\xi}, \mathcal{O}_{V_{s'_1,\xi} \cap V_{s'_2,\xi}})\text{)}.$$

Here $V_{s',\xi}$, $s' \in A(s)$ is defined similarly.

Before stating the main result of this section, we still need to do some extra work. Most results here can be found in [Haastert and Jantzen 1990]. Since $\widehat{D_{0,O_{F_0},\xi}}$ is a curve in $\mathbb{P}^2$, the Hodge–de Rham spectral sequence gives us the following exact sequence:

$$0 \to H^0(\widehat{D_{0,O_{F_0},\xi}}, \Omega^1_{\widehat{D_{0,O_{F_0},\xi}}}) \to H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}}) \to H^1(\widehat{D_{0,O_{F_0},\xi}}, \mathcal{O}_{\widehat{D_{0,O_{F_0},\xi}}}) \to 0. \quad (65)$$

And each group in this exact sequence is a finite free O_{F_0}-module. If we use a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ to represent a cohomology class in $H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}})$, then every element in $H^0(\widehat{D_{0,O_{F_0},\xi}}, \Omega^1_{\widehat{D_{0,O_{F_0},\xi}}})$ can be identified as the hypercocycle with all $f_{s_1,s_2} = 0$. And the map to $H^1(\widehat{D_{0,O_{F_0},\xi}}, \mathcal{O}_{\widehat{D_{0,O_{F_0},\xi}}})$ is just mapping the hypercocycle to $\{f_{s_1,s_2}\}$, which is considered as a 1-cocycle. Similarly, we have

$$0 \to H^0(\overline{U_{s'_0,\xi}}, \Omega^1_{\overline{U_{s'_0,\xi}}}) \to H^1_{\mathrm{dR}}(\overline{U_{s'_0,\xi}}) \to H^1(\overline{U_{s'_0,\xi}}, \mathcal{O}_{\overline{U_{s'_0,\xi}}}) \to 0,$$

which can be identified with the reduction mod p of the previous exact sequence.

Recall that the de Rham cohomology of $\widehat{D_{0,O_{F_0},\xi}}$ can be identified as the crystalline cohomology of $\overline{U_{s'_0,\xi}}$. It is equipped with a Frobenius operator φ. It is important to understand the relationship between φ and the above exact sequence. Denote $\bigcup_{\xi^{p-1}=-1} \widehat{D_{0,O_{F_0},\xi}}$ by $\widehat{D_{0,O_{F_0}}}$.

Lemma 16.8. *Under the isomorphism between $H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}})$ and $H^1_{\mathrm{crys}}(\overline{U_{s'_0,\xi}}/O_{F_0})$,*

(1) $\varphi(H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'}) \subset H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}$.

(2) $\varphi(H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'}) \subset H^0(\widehat{D_{0,O_{F_0}}}, \Omega^1_{\widehat{D_{0,O_{F_0}}}})^{(\chi')^p} + pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}$.

(3) *The above inclusion is in fact an equality and φ induces an isomorphism between $H^1(\overline{U_{s'_0}}, \mathcal{O}_{\overline{U_{s'_0}}})^{\chi'}$ and $H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}$.*

Proof. See Section 3 of [Haastert and Jantzen 1990], especially Proposition 3.5. Although our curve is slightly different from the curve in that paper, all arguments in their paper work here. □

Remark 16.9. A variant of Lemma 16.8 is that φ induces an isomorphism

$$H^0(\widehat{D_{0,O_{F_0}}}, \Omega^1_{\widehat{D_{0,O_{F_0}}}})^{\chi'} + pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'} \xrightarrow{\sim} pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}.$$

This follows from the fact that φ^2 is a scalar c_x on these spaces and $v_p(c_x)=1$; see Proposition 12.8. A direct corollary is that φ induces an isomorphism between

$$\big(H^0(\widehat{D_{0,O_{F_0}}},\Omega^1_{\widehat{D_{0,O_{F_0}}}})^{\chi'}+pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'}\big)/pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{\chi'}$$

and

$$pH^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}/\big(pH^0(\widehat{D_{0,O_{F_0}}},\Omega^1_{\widehat{D_{0,O_{F_0}}}})^{(\chi')^p}+p^2H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0}}})^{(\chi')^p}\big),$$

which can be viewed as an isomorphism $H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{\chi'}\xrightarrow{\sim}H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}$.

In fact, we can write down the isomorphism between $H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}$ and $H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}$ explicitly (Lemma 16.13 below). Some notation here: as before (see (27)), we may identify $\overline{U_{s_0',\xi}}$ with the projective curve defined by $\tilde{e}^{p+1}=v_1w_1^{-1}\xi(\eta^p-\eta)$ and the singular points of $\overline{U_{s_0'}}$ (considered in the special fiber of $\widetilde{\Sigma_{1,O_F,\xi}}$) are those points with $\tilde{e}=0$.

Definition 16.10. We write $A(s_0')=\{s_0,\dots,s_{p-1},s_\infty\}$, where for $k=0,\dots,p-1$, s_k is the vertex that corresponds to $\eta=k$, $\tilde{e}=0$ in $\overline{U_{s_0',\xi}}$ and s_∞ corresponds to the point $\eta=\infty$, $\tilde{e}=0$ (equivalently, if we use projective coordinates $[\eta,\tilde{e},1]$, then this point is $[1,0,0]$).

Definition 16.11. Let V_0 be the open set of $\overline{U_{s_0',\xi}}$ that is the complement of the point $\eta=\infty$, $\tilde{e}=0$. We also define V_∞ as the complement of $\eta=\tilde{e}=0$.

Using the notation from Definition 16.2, it is clear that set theoretically, V_0 is the union of $V_{s_0,\xi},\dots,V_{s_{p-1},\xi}$ and V_∞ is the union of $V_{s_1,\xi},\dots,V_{s_{p-1},\xi},V_{s_\infty,\xi}$. By abuse of notation, we also view V_0,V_∞ as open affine formal subschemes of $\widehat{D_{0,O_{F_0},\xi}}$.

Notice that V_0,V_∞ is an open covering of $\widehat{D_{0,O_{F_0},\xi}}$. Hence every cohomology class of $H^1_{\mathrm{dR}}(\widehat{D_{0,O_{F_0},\xi}})$ can be represented by a 1-hypercocycle $(\omega_0,\omega_\infty,f_{0,\infty})$ as before. Every element of $H^1(\overline{U_{s_0',\xi}},\mathcal{O}_{\overline{U_{s_0',\xi}}})$ can be represented by an element in $H^0(V_0\cap V_\infty,\mathcal{O}_{V_0\cap V_\infty})$, viewed as a 1-cocycle. The next lemma is easy to see.

Lemma 16.12. *$H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}$ has a basis, when restricted to $\overline{U_{s_0',\xi}}$, given by*

$$\frac{\tilde{e}^{p+1-i}}{\eta^k},\qquad k=1,\dots,p-i.$$

If $i=p$, then $H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}=0$.

Hence we may view $\tilde{e}^{p+1-i}/\eta^k$ as an element in $H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}$. Then, as a 1-hypercocycle, $\varphi(\tilde{e}^{p+1-i}/\eta^k)$ is $(0,0,\tilde{e}^{(p+1-i)p}/\eta^{pk})$. A direct computation shows:

Lemma 16.13. *$\varphi(\tilde{e}^{p+1-i}/\eta^k)$ is the same as the holomorphic differential form*

$$(v_1w_1^{-1}\xi)^{p-i}(-1)^{p-i-k}k\binom{p-i}{k}\eta^{p-i-k}\,\tilde{e}^{i-1}\,d\tilde{e}.$$

Remark 16.14. We will need to translate a 1-cocycle inside $H^1(\overline{U_{s'_0,\xi}}, \mathcal{O}_{\overline{U_{s'_0,\xi}}})$ using the open covering $\{V_{s,\xi}\}_{s\in A(s'_0)}$ to a 1-cocycle using the open covering $\{V_0, V_\infty\}$. This is done as follows. If we start with a 1-cocycle $\{f'_{s,s''}\}$, we can find another 1-cocycle $\{f_{s,s''}\}$ that represents the same cohomology class and f_{s_0,s_∞} can be extended to a section in $H^0(V_0\cap V_\infty, \mathcal{O}_{V_0\cap V_\infty})$. Then f_{s_0,s_∞} can be viewed as a 1-cocycle of the covering $\{V_0, V_\infty\}$. In fact, this is just what we want.

Example 16.15. Let's compute one example here. Consider the 1-cocycle $\{f'_{s,s''}\}$:

$$f'_{s,s''} = f'_{s''} - f'_s, \quad \text{where } f'_{s_0} = \tilde{e}^{-i},\ f'_s = 0 \text{ for } s \neq s_0.$$

Then clearly f'_{s_0,s_∞} has poles on $V_0\cap V_\infty$. But we can modify this cocycle a little bit: define

$$g_{s_0} = \frac{\eta^{p-2}\tilde{e}^{p+1-i}}{v_1 w_1^{-1}\xi(\eta^{p-1}-1)} \in H^0(V_{s_0,\xi}, \mathcal{O}_{V_{s_0,\xi}}), \quad g_s = 0 \text{ for } s\neq s_0,$$

and let

$$f_{s,s''} = f'_{s,s''} - g_{s''} + g_s.$$

Then $\{f_{s,s''}\}$ and $\{f'_{s,s''}\}$ represent the same cohomology class. Moreover,

$$f_{s_0,s_\infty} = f'_{s_0,s_\infty} + g_{s_0} = -f_{s_0} + g_{s_0} = -\tilde{e}^{-i} + \frac{\eta^{p-2}\tilde{e}^{p+1-i}}{v_1 w_1^{-1}\xi(\eta^{p-1}-1)} = \frac{\tilde{e}^{p+1-i}}{v_1 w_1^{-1}\xi\eta}$$

(using $\tilde{e}^{p+1} = v_1 w_1^{-1}\xi(\eta^p-\eta)$) clearly extends to $V_0\cap V_\infty$. Hence,

$$\frac{\tilde{e}^{p+1-i}}{v_1 w_1^{-1}\xi\eta},$$

viewed as a 1-cocycle of the covering $\{V_0, V_\infty\}$, represents the same cohomology class as $\{f'_{s.s''}\}$.

A combination of Remark 16.7 and the Lemma 16.8 gives:

Lemma 16.16. *Assume $p^2-1-m \geq [-mp]$ and $i\neq 1$. Let s be an odd vertex and $\omega\in H^{\chi',F_0}$.*

(1) *Using the method in the proof of Proposition 16.1, we may view $\varpi^{-[-mp]}\omega$ as a cohomology class inside $H^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})^{\chi'}$. Then*

$$\varphi(\varpi^{-[-mp]}\omega) \in pH^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})^{(\chi')^p},$$

or equivalently (using Remark 16.9),

$$\varpi^{-[-mp]}\omega \in \varphi(H^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})^{(\chi')^p}).$$

(2) *In fact, Proposition 15.8 shows that* $\varpi^{-[-mp]}\omega$ *modulo* p *is a holomorphic differential form inside*

$$H^0(\overline{U}_s, \Omega^1_{\overline{U}_s}) = \varphi(H^1_{\mathrm{dR}}(\overline{U}_s)),$$

which is nothing but $\varpi^{-[-mp]}\omega$ *considered as a cohomology class in* $H^1_{\mathrm{dR}}(\overline{U}_s)$. *In particular, if*

$$\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1),$$

then the cohomology class of $\varpi^{-[-mp]}\omega$ *is inside* $pH^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})$.

Proof. Following Remark 16.7, let $(\{\omega_{s'}\}, \{f_{s'_1,s'_2}\})$ be the 1-hypercocycle that represents ω in $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0)^{\chi'}$ (by identifying the crystalline cohomology of $\overline{U}_s$ with the de Rham cohomology of $\widehat{D_{0,O_{F_0}}}$). Since

$$\varpi^{-[-mp]} f_{s'_1,s'_2} \in pH^0(V_{s'_1,\xi} \cap V_{s'_2,\xi}, \mathcal{O}_{V_{s'_1,\xi}\cap V_{s'_2,\xi}}),$$

all these $\varpi^{-[-mp]} f_{s'_1,s'_2}$ vanish if we reduce modulo p. This means that the image of $\varpi^{-[-mp]}\omega$ in $H^1_{\mathrm{dR}}(\overline{U}_s)$ actually lies inside $H^0(\overline{U}_s, \Omega^1_{\overline{U}_s})$. Now our first claim is a direct consequence of Lemma 16.8. Referring again to Remark 16.7, the rest of the lemma follows from

$$\varpi^{-[-mp]} f_{s'} \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

Thus when we restrict everything to the special fiber of $V_{c,\xi}$ (equivalently, $\overline{U^0_{s,\xi}}$),

$$\varpi^{[-mp]}\omega_{s'} = \varpi^{-[-mp]}\omega - df_{s'} \equiv \varpi^{-[-mp]}\omega \mod pH^0(V_{c,\xi}, \Omega^1_{V_{c,\xi}}).$$

This indicates that the cohomology class of $\varpi^{-[-mp]}\omega$ is just the 1-form $\varpi^{-[-mp]}\omega$ after reducing modulo p. □

Remark 16.17. Using the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, it's not hard to see that if we replace s by an even vertex s' and $\omega \in H^{\chi',F_0}$ by $\omega \in H^{(\chi')^p,F_0}$, we have a similar result:

$$\varpi^{-[-mp]}\omega \in \varphi(H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}),$$

and exactly the same statement for the second part.

Similarly, by combining Lemmas 16.6 and 16.8, we obtain:

Lemma 16.18. *Let* $\omega \in H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)}$ *and* s' *be an even vertex. Assume*

$$\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)} \tag{66}$$

for any $s \in A(s')$.

(1) *The image of* $\varpi^{-(p^2-1-m)}\omega$ *in* $H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}$ *is actually inside*

$$\varphi(H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}).$$

Equivalently, if we view $\varpi^{-(p^2-1-m)}\omega$ *as an element inside* $H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}$,

$$\varphi(\varpi^{-(p^2-1-m)}\omega) \in pH^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}.$$

(2) *Assume* $i \neq p$. Proposition 15.8 *shows that in this case* $\varpi^{-(p^2-1-m)}\omega$ *modulo* p *is a holomorphic differential form inside*

$$H^0(\overline{U_{s'}}, \Omega^1_{\overline{U_s}}) = \varphi(H^1_{\mathrm{dR}}(\overline{U_{s'}})),$$

and we may identify it with the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ *in* $H^1_{\mathrm{dR}}(\overline{U_{s'}})$. *In particular, if*

$$\omega|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1),$$

then the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ *is inside* $pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})$.

(3) *Assume* $i = p$. *We have a slightly weaker result: assume*

$$\omega|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1); \tag{67}$$

then the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ *is inside* $pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})$.

Proof. First we prove the first two parts. If $i = p$, we know that (Lemma 16.12)

$$H^1(\overline{U_{s'}}, \mathcal{O}_{\overline{U_{s'}}})^{\chi'} = H^0(\overline{U_{s'}}, \Omega^1_{\overline{U_{s'}}})^{(\chi')^p} = 0.$$

Hence Lemma 16.8 tells us that

$$\varphi(H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}) = H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}.$$

So in this case, the first part is trivially true.

Now assume $i \neq p$. We need to use some results from Lemma 16.6; see the notation there. We can represent $\omega|_{\widetilde{\Sigma_{1,O_F,s'}}}$ as a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ and for $s \in A(s')$, there exists

$$f_s \in \varpi^{p^2-1-m} H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}),$$

such that $f_{s_1,s_2} = f_{s_2} - f_{s_1}$. Hence, as in the proof of Lemma 16.16, it suffices to prove

$$\varpi^{-(p^2-1-m)} f_s \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

If $p^2 - 1 - m \leq [-mp]$ and $i \neq p$, this already follows from the second part of Lemma 16.6. So we only need to treat the case $p^2-1-m \geq [-mp]$. Then the desired result follows from our condition that $\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)}$.

More precisely, using the notation from the proof of Proposition 16.1 (especially (59)), we can write

$$\omega|_{\widetilde{\Sigma_{1,O_F,[s,s'],\xi}}} = \varpi^{p^2-1-m} f(\eta)\tilde{e}^{p+1-i}\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]} g(\zeta)\tilde{e}'^{i}\frac{d\tilde{e}'}{\tilde{e}'}. \tag{68}$$

Notice that Lemma 16.6 tells us that

$$\varpi^{-(p^2-1-m)} f_s \equiv \frac{\xi^i g(0) y^{-i}}{i} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

It suffices to show $g(0) \in pO_{F_0}$. But by Lemma 15.7,

$$g(\zeta) \in pO_{F_0}\Big[\zeta, \frac{1}{\zeta^{p-1}-1}\Big]^{\widehat{}},$$

since we assume $\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)}$. So we're done for the first two parts.

As for the last claim, we keep using the notation $(\{\omega_s\}, \{f_{s_1,s_2}\})$ and $\{f_s\}$ as above. Notice that we already assume $\omega|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1)$. Hence if we can show

$$\varpi^{-(p^2-1-m)} f_s \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}), \tag{69}$$

then we would know that both ω_s and f_{s_1,s_2} are divisible by p. Thus it suffices to prove (69).

But using the notation (68) above and Lemma 16.6, which says that

$$\varpi^{-(p^2-1-m)} f_s \equiv \xi^p g(0) y^{-p} \mod pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}),$$

we only need to show $g(0)$ is divisible by p. But by our assumption (67),

$$f(\eta) \in pO_{F_0}\Big[\eta, \frac{1}{\eta^{p-1}-1}\Big]^{\widehat{}}.$$

Since we also assume $\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)$,

$$g(\zeta) \in pO_{F_0}\Big[\zeta, \frac{1}{\zeta^{p-1}-1}\Big]^{\widehat{}}.$$

See the computations around Lemma 15.6. Therefore $g(0) \in pO_{F_0}$. $\square$

Remark 16.19. Using the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, we can get a variant of the previous lemma. Let $\omega \in H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{(\chi')^p,\mathrm{Gal}(F/F_0)}$ and s be an odd vertex. Assume

$$\omega|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1)^{(\chi')^p,\mathrm{Gal}(F/F_0)}$$

for any $s' \in A(s)$. Then the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ in $H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}$ is inside $\varphi(H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{\chi'})$. And we have a similar result for the last two parts: if we assume

$$\omega|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1),$$

then the cohomology class of $\varpi^{-(p^2-1-m)}\omega$ is inside $pH^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})$.

Remark 16.20. When $i = p$, we will see in Section 17 that the second part of the lemma is actually still true (Lemma 17.12).

Now let's recall the construction of $M(\chi, [1, b])$ in Section 14. First we write

$$\prod_s (F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0) \otimes_{\mathbb{Q}_p} E)^\chi = F_1 \oplus F_2,$$

where

$$\begin{aligned} F_1 &\stackrel{\text{def}}{=} \prod_{s' \text{ even}} F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U_{s'}}/F_0)^{\chi'}_{\tau} \oplus \prod_{s \text{ odd}} F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0)^{(\chi')^p}_{\bar{\tau}}, \\ F_2 &\stackrel{\text{def}}{=} \prod_{s' \text{ even}} F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U_{s'}}/F_0)^{(\chi')^p}_{\bar{\tau}} \oplus \prod_{s \text{ odd}} F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0)^{\chi'}_{\tau}. \end{aligned} \tag{70}$$

It is clear from Lemma 16.18 that $g_\varphi \otimes \varphi \otimes \mathrm{Id}_E$ sends F_1 to F_2. Let f be an element of $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$. By Proposition 14.6, we have an injective map $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ into $\prod_s \big(F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0) \otimes_{\mathbb{Q}_p} E\big)^\chi$. Let (f_1, f_2) be the decomposition of the image of f into $F_1 \oplus F_2$. Then:

Lemma 16.21. $M(\chi, [1, b]) =$

$$\big\{f \in (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} \,\big|\, (1 \otimes b)(g_\varphi \otimes \varphi \otimes \mathrm{Id}_E)(f_1) = (\varpi^{(p-1)i} \otimes 1) f_2\big\}.$$

Here $1 \otimes b$ and $\varpi^{(p-1)i} \otimes 1$ are viewed as elements in $F \otimes_{\mathbb{Q}_p} E$.

Proof. Considering

$$(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} = (H^0(\widetilde{\Sigma^{(0)}_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/\mathbb{Q}_p)}, \tag{71}$$

the lemma follows from Proposition 13.3 and the remark below it. ☐

Thus we can rewrite $M(\chi, [1, b])$ as the kernel of θ_b, which is defined as the composite of the following maps:

$$(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} \to \prod_s \big(F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0) \otimes_{\mathbb{Q}_p} E\big)^\chi \xrightarrow{L_b} F_2, \tag{72}$$

where $L_b : F_1 \oplus F_2 \to F_2$ is defined as

$$(f_1, f_2) \mapsto -(1 \otimes b)(g_\varphi \otimes \varphi \otimes \mathrm{Id}_E)(f_1) + (\varpi^{(p-1)i} \otimes 1) f_2.$$

To understand the image of θ_b, we introduce:

Definition 16.22.

$$J_1 \stackrel{\text{def}}{=} \prod_{s' \text{ even}} H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}_{\tau} \oplus \prod_{s \text{ odd}} H^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})^{(\chi')^p}_{\bar{\tau}} \subset F_1, \tag{73}$$

$$J_2 \stackrel{\text{def}}{=} (\varpi^{p^2-1-m} g_\varphi \otimes \varphi \otimes \mathrm{Id}_{O_E})(J_1) \subset F_2. \tag{74}$$

Lemma 16.23. *Under the assumption $p^2-1-m \geq [-mp]$ and $2 \leq i \leq p-1$, we have*

$$\theta_b\big((H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\big) \subset J_2.$$

Remark 16.24. In the next section we show that the lemma also holds for $i = 1, p$.

Proof. Let ω be an element in $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$. Write $\omega = (\omega_1, \omega_2)$ as the decomposition into $F_1 \oplus F_2$. By Proposition 16.1, we have $\omega_1 \in \varpi^{p^2-1-m} J_1$. Hence

$$L_b((\omega_1, 0)) = -(g_\varphi \otimes \varphi \otimes b(\omega_1)) \in J_2.$$

It remains to prove that $L_b((0, \omega_2)) \in J_2$, or equivalently, $(\varpi^{(p-1)i} \otimes 1)\omega_2 \in J_2$. Using the action of $\mathrm{GL}_2(\mathbb{Q}_p)$, we only need to check this for one odd vertex. In other words, it suffices to show

$$(\varpi^{(p-1)i} \otimes 1)\omega_{2,s} \in (\varpi^{p^2-1-m} g_\varphi \otimes \varphi \otimes \mathrm{Id}_{O_E})(H^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})^{(\chi')^p}_{\tau}),$$

where s is an odd vertex and $\omega_{2,s}$ is the image of ω inside $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U}_s/F_0)^{\chi'}_{\tau}$. But this is nothing but the first part of Lemma 16.16. $\square$

By abuse of notation, we use θ_b to denote the map

$$(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} \to J_2.$$

Also we use $\bar{\theta}_b$ to denote the modulo p map of θ_b, that is:

$$\bar{\theta}_b : H^{\chi,F_0} = (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E/p)^{\chi,\mathrm{Gal}(F/F_0)} \to J_2/p.$$

Recall that we have an exact sequence (Proposition 15.13):

$$0 \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{U_{s_0'}})^{\chi'}_{\tau} \to H^{\chi,F_0} \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{U_{s_0'}})^{(\chi')^p}_{\tau} \to 0.$$

As for J_2/p, it's obvious that

$$J_2/p \simeq \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} \varphi(H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'}_{\tau})/p.$$

Using Lemma 16.8, the filtration

$$p\varphi(H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'}) \subset pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p} \subset \varphi(H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'})$$

induces the following exact sequence:

$$0 \to H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p} \to \varphi(H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'})/p \to H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p} \to 0.$$

Another way to see this is that J_2/p is canonically isomorphic with J_1/p, and J_1/p has the usual exact sequence for de Rham cohomology. In other words, we have:

$$0 \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\tau} \to J_2/p \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\tau} \to 0.$$

Lemma 16.25. *Assume $p^2-1-m \geq [-mp]$ and $i \in \{2,\dots,p-1\}$. Then $\bar{\theta}_b$ induces the following commutative diagram*:

$$\begin{tikzcd}
\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_\tau \arrow[r] \arrow[d, "\bar{\theta}_{b,1}"'] & H^{\chi,F_0} \arrow[r] \arrow[d, "\bar{\theta}_b"'] & \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}} \arrow[d, "\bar{\theta}_{b,2}"] \\
\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}} \arrow[r] & J_2/p \arrow[r] & \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}}
\end{tikzcd}$$

Proof. Let ω be an element of $(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ whose mod p reduction lies in $\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_\tau$. We need to show that

$$\theta_b(\omega) \in \varpi^{p^2-1-m} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} p H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p}_{\bar{\tau}} \subset J_2.$$

Write $\omega = \omega_\tau + \omega_{\bar{\tau}}$ as in the decomposition

$$\begin{aligned}\big(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes_{\mathbb{Q}_p} O_E\big)^{\chi,\mathrm{Gal}(F/F_0)} \\ = H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{\chi',\mathrm{Gal}(F/F_0)}_\tau \oplus H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{(\chi')^p,\mathrm{Gal}(F/F_0)}_{\bar{\tau}}.\end{aligned}$$

It is clear from the construction in the proof of Proposition 15.13 that ω is in H_1 modulo p (see the notation there). This means that

$$\omega_\tau|_{\widetilde{\Sigma_{1,O_F,s}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s}}, \omega^1)^{\chi'}_\tau \quad \text{and} \quad \omega_{\bar{\tau}}|_{\widetilde{\Sigma_{1,O_F,s'}}} \in pH^0(\widetilde{\Sigma_{1,O_F,s'}}, \omega^1)^{(\chi')^p}_{\bar{\tau}} \tag{75}$$

for any odd vertex s and even vertex s'. Then, by Lemma 16.16, we know that the image of $\varpi^{-[-mp]}\omega_\tau$ in $H^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})^{\chi'}_\tau$ actually lies in $pH^1_{\mathrm{crys}}(\overline{U}_s/O_{F_0})^{\chi'}_\tau$ for any odd vertex s. Similarly the image of $\varpi^{-[-mp]}\omega_{\bar{\tau}}$ will be in $pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\bar{\tau}}$ for any even vertex s'. Let $\omega = (\omega_1, \omega_2)$ be the decomposition of ω into $F_1 \oplus F_2$. Then the discussion before indicates that

$$(\varpi^{(p-1)i} \otimes 1)\omega_2 \in \varpi^{p^2-1-m} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} p H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p}_{\bar{\tau}} \subset J_2.$$

It remains to prove that

$$(g_\varphi \otimes \varphi \otimes \mathrm{Id}_E)(\omega_1) \in \varpi^{p^2-1-m} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} p H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p}_{\bar{\tau}} \subset J_2.$$

This follows from Lemma 16.18 (the condition in that lemma is satisfied since we have (75)). ☐

Now we can state the main theorem of this paper.

Theorem 16.26. *The maps $\bar{\theta}_{b,1}, \bar{\theta}_{b,2}$ are surjective. More precisely, if we identify $H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}}$ with* $(\mathrm{Sym}^{p-1-i}(O_E/p)^2) \otimes \det^{i+j}$ (see Remark 15.14), *and identify*

$$H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_\tau \simeq H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}}$$

with $(\mathrm{Sym}^{i-2}(O_E/p)^2)\otimes\det^{j+1}$, *where the isomorphism is induced by* φ (see Remark 16.9), *then* $\bar\theta_{b,1},\bar\theta_{b,2}$ *are given by*

$$\bar\theta_{b,1}:\sigma_{i-2}(j+1)\to\sigma_{i-2}(j+1),\qquad X\mapsto -bX+((-1)^{j+1}\tau(w_1^i))T(X),$$
$$\bar\theta_{b,2}:\sigma_{p-1-i}(i+j)\to\sigma_{p-1-i}(i+j),\quad X\mapsto X-((-1)^{j+1}\tau(w_1^{-i})b)T(X),$$

where T *is the Hecke operator* (defined in [Breuil 2007]). *See the beginning of the paper for its definition.*

We list some direct consequences of this theorem.

Corollary 16.27. $\bar\theta_b$ *is surjective.*

Corollary 16.28. $\theta_b:(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\to J_2$ *is surjective and we have the following exact sequence*:

$$0\to M(\chi,[1,b])\to(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\to J_2\to 0.\tag{76}$$

Applying the functor $M\mapsto M^d=\mathrm{Hom}^{\mathrm{cont}}_{O_E}(M,E)$ *defined in Section 14, we get*

$$0\to J_2^d\to\big((H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\big)^d\to B(\chi,[1,b])\to 0.\tag{77}$$

Notice that the kernel and the middle term of this exact sequence do not depend on b. In fact, the unitary representation J_2^d *is the completion of* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho_{\chi^{-1}}$ *with respect to the lattice* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho^o_{\chi^{-1}}$, *where* $\rho^o_{\chi^{-1}}\subset\rho_{\chi^{-1}}$ *is an* O_E*-lattice. It is the universal unitary completion of* $\text{c-Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}\rho_{\chi^{-1}}$.

Proof. Recall that $B(\chi,[1,b])=(M(\chi,[1,b]))^d$ defined in Section 14. The surjectivity of θ_b follows from the surjectivity of $\bar\theta_b$ and the fact that J_2 and $(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ are p-adically complete. The explicit description of J_2^d follows from the obvious isomorphism between J_2 and J_1, which is clearly isomorphic to $\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'^{-1}}_\tau$. It is easy to verify that it satisfies the universal property. $\square$

Corollary 16.29. *Under the assumption* $p^2-1-m\ge[-mp]$, $i\in\{2,\dots,p-1\}$, *as a representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$,

$$0\to\big\{X\in\sigma_{i-2}(j+1)\mid c(\chi,b)X=T(X)\big\}\to M(\chi,[1,b])/p$$
$$\to\big\{X\in\sigma_{p-1-i}(i+j)\mid X=c(\chi,b)T(X)\big\}\to 0,$$

where $c(\chi,b)=(-1)^{j+1}\tau(w_1^{-i})b\in O_E/p$. *Thus* $B(\chi,[1,b])$ *is nonzero and admissible.*

Remark 16.30. If we assume $p^2-1-m\le[-mp]$, $i\in\{2,\dots,p-1\}$, the same proof will yield a similar exact sequence:

$$0\to\big\{X\in\sigma_{p-1-i}(i+j)\mid X=c(\chi,b)T(X)\big\}\to M(\chi,[1,b])/p$$
$$\to\big\{X\in\sigma_{i-2}(j+1)\mid c(\chi,b)X=T(X)\big\}\to 0.$$

Proof of Theorem 16.26. First we introduce some notation.

Definition 16.31. Let $\omega \in (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$. Then

(1) $\omega_\tau + \omega_{\bar\tau}$ will be the decomposition of ω in

$$(H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi} = H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{\chi'}_{\tau} \oplus H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)^{(\chi')^p}_{\bar\tau}.$$

We will use $\omega_{\tau,s,\xi}$ (resp. $\omega_{\bar\tau,s,\xi}$) to denote the restriction of ω_τ (resp. $\omega_{\bar\tau}$) to $\widetilde{\Sigma_{1,O_F,s,\xi}}$, where s is a vertex of the Bruhat–Tits tree and $\xi^{p-1} = -1$.

(2) $\omega = \omega_1 + \omega_2$ will be its decomposition into $F_1 \oplus F_2$ (see (70)). For an even vertex s' and odd vertex s, we define $\omega_{1,s'}$ and $\omega_{1,s}$ as the images of ω inside $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U_{s'}}/F_0)^{\chi'}_{\tau}$ and $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U_s}/F_0)^{(\chi')^p}_{\bar\tau}$, respectively. It is clear that $\omega_{2,s'}$, $\omega_{2,s}$ can be defined similarly. We also use $\omega_{1,s,\xi}$ to denote the image of ω in $F \otimes_{F_0} H^1_{\mathrm{crys}}(\overline{U_{s,\xi}}/F_0)^{(\chi')^p}_{\bar\tau}$ and define $\omega_{1,s',\xi}$, $\omega_{2,s,\xi}$, and $\omega_{2,s',\xi}$ similarly.

In fact, Proposition 16.1 tells us that for an even vertex s' and odd vertex s,

$$\omega_{1,s'} \in \varpi^{p^2-1-m} H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}_{\tau}, \quad \omega_{1,s} \in \varpi^{p^2-1-m} H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}_{\bar\tau},$$

and

$$\omega_{2,s'} \in \varpi^{[-mp]} H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\bar\tau}, \quad \omega_{2,s} \in \varpi^{[-mp]} H^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{\chi'}_{\tau}).$$

Now we start to prove the surjectivity of

$$\bar\theta_{b,2} : \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}_{\tau} \to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}_{\bar\tau}.$$

Consider $[\mathrm{Id}, x^k y^{p-1-i-k}]$ in (See the beginning of the paper for the notation here)

$$\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} (\mathrm{Sym}^{p-1-i}(O_E/p)^2) \otimes \det{}^{i+j} \simeq \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times} H^0(\overline{U_{s'_0}}, \Omega^1_{\overline{U_{s'_0}}})^{(\chi')^p}_{\bar\tau}.$$

Let $\bar\omega \in H^{\chi,F_0}$ be a lift of $[\mathrm{Id}, x^k y^{p-1-i-k}]$ in the first row of Lemma 16.25 and let $\omega \in (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ be a lift of $\bar\omega$.

It is clear that we may assume $\omega_\tau = 0$. Then our choice of ω implies:

Lemma 16.32. *Under the identification in (27),*

$$\omega_{\bar\tau,s'_0,\xi} \equiv \varpi^{[-mp]} \eta^k \tilde e^i \frac{d\tilde e}{\tilde e} \mod pH^0(\widetilde{\Sigma_{1,O_F,s'_0,\xi}}, \omega^1)_{\bar\tau}, \tag{78}$$

$$\omega_{\bar\tau,s',\xi} \in pH^0(\widetilde{\Sigma_{1,O_F,s,\xi}}, \omega^1)_{\bar\tau} \quad \textit{for any even vertex } s' \neq s'_0. \tag{79}$$

Using this and Remark 16.17, we know that for any even vertex $s' \neq s_0$,

$$\varpi^{(p-1)i} \omega_{2,s'} \in \varpi^{p^2-1-m} pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\bar\tau},$$

and considered as elements in $H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})\subset H^1_{\mathrm{dR}}(\overline{U_{s_0'}})$,

$$\varpi^{-[-mp]}\omega_{2,s_0',\xi}\equiv\eta^k\tilde{e}^i\,\frac{d\tilde{e}}{\tilde{e}}\mod pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{(\chi')^p}_{\bar\tau}.$$

Similarly, Remark 16.19 tells us that for any odd vertex $s\notin A(s_0')$,

$$\varphi(\varpi^{-(p^2-1-m)}\omega_{2,s})\in pH^1_{\mathrm{crys}}(\overline{U_s}/O_{F_0})^{(\chi')^p}_{\bar\tau}.$$

Hence it is clear from the definition of θ_b that we have:

Lemma 16.33. $$\bar\theta_{b,2}([\mathrm{Id},x^ky^{p-1-i-k}])=[\mathrm{Id},v_{s_0'}]+\sum_{s\in A(s_0')}[g_s^{-1},v_s],$$

where g_s is a chosen representative in the coset defined by s. Recall that we identify the set of vertices of the Bruhat–Tits tree with $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\backslash\mathrm{GL}_2(\mathbb{Q}_p)$.

Since $\omega_{1,s_0'}=0$, it follows from (78) that $v_{s_0'}=x^ky^{p-1-i-k}$.

To determine other terms, we recall some results in Section 7. Recall that s_0 is the vertex that corresponds to $\eta=\tilde{e}=0$. As a coset, it corresponds to $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\cdot w$, where

$$w=\begin{pmatrix}0&-1\\p&0\end{pmatrix}.\tag{80}$$

Then $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$ is isomorphic to

$$\mathrm{Spf}\,\frac{O_F\Big[\eta,\zeta,\dfrac{1}{\eta^{p-1}-1},\dfrac{1}{\zeta^{p-1}-1},\tilde{e},\tilde{e}'\Big]^{\wedge}}{\Big(\tilde{e}^{p+1}+v_1w_1^{-1}\xi\dfrac{\eta^p-\eta}{\zeta^{p-1}-1},\tilde{e}'^{p+1}+v_1^{-1}w_1\xi\dfrac{\zeta^p-\zeta}{\eta^{p-1}-1},\tilde{e}\tilde{e}'-\varpi^{p-1}\xi\Big)}$$

in such a way that the following lemma is true.

Lemma 16.34. *The action of w sends $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$ to*

$$\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi^p}}=\widetilde{\Sigma_{1,O_F,[s_0',s_0],-\xi}}.$$

Explicitly, it is given by (see Corollary 7.7; recall that $\tilde{e}=e/\varpi$):

$$\eta\mapsto-\zeta,\quad\zeta\mapsto-\eta,\quad\tilde{e}\mapsto v_1\tilde{e}',\quad\tilde{e}'\mapsto v_1^{-1}\tilde{e}.$$

Now we come back to our situation. Using Lemma 15.5, the restriction of $\omega_{\bar\tau}$ to $\widetilde{\Sigma_{1,O_F,[s_0',s_0]}}$ can be written as

$$\omega|_{\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}}=\varpi^{[-mp]}f(\eta)\tilde{e}^i\,\frac{d\tilde{e}}{\tilde{e}}+\varpi^{p^2-1-m}g(\zeta)\tilde{e}'^{p+1-i}\,\frac{d\tilde{e}'}{\tilde{e}'},$$

where $f(\eta) \in O_{F_0}[\eta, 1/(\eta^{p-1}-1)]\widehat{\ }$, $g(\zeta) \in O_{F_0}[\zeta, 1/(\zeta^{p-1}-1)]\widehat{\ }$. Since ω is in the $(\chi')^p$-isotypic component, we must have (using results in Section 9):

$$\omega|_{\widetilde{\Sigma_{1,O_F,[s_0',s_0],-\xi}}} = \varpi^{[-mp]} f(\eta)\tilde{e}^i(-1)^{-(i+j)}\frac{d\tilde{e}}{\tilde{e}} + \varpi^{p^2-1-m} g(\zeta)\tilde{e}'^{p+1-i}(-1)^{-(j+1)}\frac{d\tilde{e}'}{\tilde{e}'}.$$

By our construction of ω,

$$\omega_{\bar{\tau},s_0',\xi} = \omega_{\bar{\tau}}|_{\widetilde{\Sigma_{1,O_F,s_0',\xi}}} \equiv \varpi^{[-mp]}\eta^k\tilde{e}^i\frac{d\tilde{e}}{\tilde{e}} \mod pH^0(\widetilde{\Sigma_{1,O_F,s_0',\xi}}, \omega^1).$$

Hence:

Lemma 16.35. $$f(\eta) \equiv \eta^k \mod pO_{F_0}\Big[\eta, \frac{1}{\eta^{p-1}-1}\Big]\widehat{\ }.$$

I would like do all the computations on the central component, so we define

$$h_{s_0} = (w^{-1})^*(\omega_{\bar{\tau}}) \in H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)_\tau. \tag{81}$$

Then (notice that w maps the $(-\xi)$-component to the ξ-component) a direct consequence of Lemma 16.34 is:

Lemma 16.36. $h_{s_0}|_{\widetilde{\Sigma_{1,O_F,s_0',\xi}}} = (w^{-1})^*(\omega_{\bar{\tau},s_0,-\xi})$ *has the form*

$$\varpi^{p^2-1-m}\tilde{g}(-\eta)\tilde{e}^{p+1-i}(-v_1^{-1})^{p+1-i}(-1)^{j+1}\frac{d\tilde{e}}{\tilde{e}} + \varpi^{[-mp]}\tilde{f}(-\zeta)\tilde{e}'^i(-v_1)^i(-1)^{i+j}\frac{d\tilde{e}'}{\tilde{e}'},$$

where $\tilde{f}(-\zeta) = \widetilde{\mathrm{Fr}}(f(-\zeta))$, *applying Frobenius operator on the coefficients, and* $\tilde{g}(-\eta)$ *is defined similarly.*

In fact, by Lemma 16.35, we know that

$$\tilde{f}(-\zeta) \equiv (-\zeta)^k \mod pO_{F_0}\Big[\zeta, \frac{1}{\zeta^{p-1}-1}\Big]\widehat{\ }.$$

We need to compute the cohomology class of $\varpi^{-(p^2-1-m)}h_{s_0}$ in $H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})_\tau$ (modulo p). Following the strategy in the proof of Proposition 16.1 (see the notation there), we may use a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ to represent h_{s_0}. Also recall that $f_{s_1,s_2} = f_{s_2} - f_{s_1}$ (all considered as elements in $\varpi^{p^2-1-m}H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})$). By definition of $\bar{\theta}_{b,2}$, we only need to know the image of $\varphi(\varpi^{-(p^2-1-m)}h_{s_0})$ in

$$H^1_{\mathrm{dR}}(\overline{U_{s_0'}})_{\bar{\tau}} = H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})_{\bar{\tau}}/pH^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})_{\bar{\tau}}.$$

Hence Lemma 16.8 tells us that we only need to know the image of $\varpi^{-(p^2-1-m)}h_{s_0}$ inside

$$H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})_\tau.$$

In other words, we are only concerned with the mod p properties of f_s.

Since w interchanges s_0' and s_0, $w(s)\neq s_0'$ for any odd vertex $s\neq s_0$. Then it follows from Lemma 16.32 that for any $s\in A(s_0')$, $s\neq s_0$,

$$h_{s_0}|_{\widetilde{\Sigma_{1,O_F,s}}}\in pH^0(\widetilde{\Sigma_{1,O_F,s}},\omega^1)_\tau.$$

Therefore the proof of Lemma 16.18 implies that for any $s\in A(s_0')$, $s\neq s_0$,

$$\varpi^{-(p^2-1-m)}f_s\in pH^0(V_{c,\xi},\mathcal{O}_{V_{c,\xi}})_\tau.$$

Moreover, Lemma 16.6 tells us that (compare Lemma 16.36 with (59) and notice that $g(\zeta)$ there is $\tilde f(-\zeta)(-v_1)^i(-1)^{i+j}$ here)

$$\varpi^{-(p^2-1-m)}f_{s_0}\equiv\frac{\xi^i\tilde f(0)(-v_1)^i(-1)^{i+j}y^{-i}}{i}\mod pH^0(V_{c,\xi},\mathcal{O}_{V_{c,\xi}})_\tau.$$

Recall that the identification of $\overline{U_{s_0',\xi}}$ and the special fiber of $\widehat{D_{0,O_{F_0},\xi}}$ is given by

$$x\mapsto\eta,\qquad y\mapsto\tilde e.$$

Lemma 16.37. *The image of $\varpi^{-(p^2-1-m)}h_{s_0}$ in $H^1(\overline{U_{s_0',\xi}},\mathcal{O}_{\overline{U_{s_0',\xi}}})_\tau$ is the following 1-cocycle $\{f'_{s,s''}\}$ if we use the open covering $\{V_{s,\xi}\}_s$:*

$$f'_{s,s''}=f'_{s''}-f'_s,\quad\textit{where}\ \ f'_{s_0}=\xi^i\tilde f(0)(-v_1)^i(-1)^{i+j}i^{-1}\tilde e^{-i},\quad f'_s=0\ \textit{for}\ s\neq s_0.$$

Now we want to write this cohomology class as a 1-cocycle $f_{0,\infty}$ of the open covering $\{V_0,V_\infty\}$ (Definition 16.11). But this is already computed in Example 16.15:

Lemma 16.38. *The image of $\varpi^{-(p^2-1-m)}h_{s_0}$ in $H^1(\overline{U_{s_0',\xi}},\mathcal{O}_{\overline{U_{s_0',\xi}}})_\tau$ is the following 1-cocycle $\{f_{0,\infty}\}$ if we use the open covering $\{V_0,V_\infty\}$:*

$$f_{0,\infty}=\tilde f(0)(-1)^j i^{-1}w_1(v_1\xi)^{i-1}\frac{\tilde e^{p+1-i}}{\eta}.$$

Thanks to Lemma 16.13, a simple computation shows:

Lemma 16.39. *The image of $\varphi(\varpi^{-(p^2-1-m)}h_{s_0})$ in $H^1_{\mathrm{dR}}(\overline{U_{s_0',\xi}})_{\bar\tau}$ is*

$$\varphi\Big(\tilde f(0)(-1)^j i^{-1}w_1(v_1\xi)^{i-1}\frac{\tilde e^{p+1-i}}{\eta}\Big)$$
$$=f(0)(-1)^{i+j+1}w_1^i\eta^{p-1-i}\tilde e^i\,\frac{d\tilde e}{\tilde e}\in H^0(\overline{U_{s_0',\xi}},\Omega^1_{\overline{U_{s_0',\xi}}})_{\bar\tau}.$$

Recall that in the isomorphism

$$H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}\to(\mathrm{Sym}^{p-1-i}\,\mathbb{F}^2_{p^2})\otimes\det{}^{i+j},$$

$\eta^{p-1-i}\tilde{e}^i\, d\tilde{e}/\tilde{e}$ is identified with x^{p-1-i}. By Lemma 16.35, $f(0) = 1$ if $k = 0$ and $f(0) = 0$ otherwise. Hence, considering the definition of $\bar{\theta}_{b,2}$, Lemma 16.39 implies:

Lemma 16.40. $$[w, v_{[w^{-1}]}] = \begin{cases} [w, (-1)^{j+1}\tau(w_1^{-i})bx^{p-1-i}] & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now we compute the same term of $T([\mathrm{Id}, x^k y^{p-1-i-k}])$ (see the beginning of the paper for the notation here):

$$[w, \varphi_r(w^{-1})(x^k y^{p-1-i-k})] = \left[w, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \varphi_r\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)(x^k y^{p-1-i-k})\right],$$

which is nonzero if and only if $k = 0$. When $k = 0$,

$$\begin{aligned}[w, \varphi_r(w^{-1})(y^{p-1-i})] &= \left[w, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \varphi_r\left(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\right)(y^{p-1-i})\right] \\ &= \left[w, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(y^{p-1-i})\right] = [w, x^{p-1-i}].\end{aligned}$$

Lemma 16.41. $$T([\mathrm{Id}, x^k y^{p-1-i-k}]) = \begin{cases} [w, x^{p-1-i}] + \text{other terms}, & k = 0, \\ [w, 0] + \text{other terms}, & k \neq 0. \end{cases}$$

Since $\mathrm{GL}_2(\mathbb{Z}_p)$ acts transitively on $A(s_0')$, the above computation implies

$$\begin{aligned}&\bar{\theta}_{b,2}([\mathrm{Id}, x^k y^{p-1-i-k}]) \\ &\qquad = [\mathrm{Id}, x^k y^{p-1-i-k}] - ((-1)^{j+1}\tau(w_1^{-i})b)T([\mathrm{Id}, x^k y^{p-1-i-k}]).\end{aligned}$$

Notice that $\bar{\theta}_{b,2}$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. Therefore,

$$\bar{\theta}_{b,2} = \mathrm{Id} - ((-1)^{j+1}\tau(w_1^{-i})b)T.$$

As for $\bar{\theta}_{b,1}$, the computation is almost the same. I omit the details here. □

17. Computation of $M(\chi, [1, b])/p$, II: $i = 1, p$

In this section, we deal with the case $i = 1, p$. We keep the notation used in the last two sections. Now Proposition 15.13 becomes:

Proposition 17.1. (1) *If $i = 1$, there exists a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism*

$$H^{\chi, F_0} \xrightarrow{\sim} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})_{\tau}^{(\chi')^p}.$$

(2) *If $i = p$,*

$$H^{\chi, F_0} \xrightarrow{\sim} \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})_{\tau}^{\chi'}.$$

Proof. Notice that when $i = 1$, $H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'} = 0$. So everything follows from Proposition 15.13 and Remark 15.15. □

In fact, we can see the above isomorphisms in the following way. If $i=p$, for any $\bar{h}\in H^{\chi,F_0}$, the restriction of $\bar{h}_\tau$ (resp. $\bar{h}_{\bar{\tau}}$) to $\widetilde{\Sigma_{1,O_F,s'}}$ (resp. $\widetilde{\Sigma_{1,O_F,s}}$) for an odd (resp. even) vertex s' (resp. s) corresponds to a holomorphic differential form on $\overline{U_{s'}}$ (resp. $\overline{U_s}$) under the isomorphism in Lemma 15.4. Hence we can define the above map. The case $i=1$ is similar.

As we promised earlier, we have:

Lemma 17.2. *Assume $i=1$ or p. Then*

$$\theta_b\big((H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\big)\subset J_2.$$

Proof. See (73), (74) for the definitions of J_1, J_2. First we assume $i=p$. Then by Lemma 16.8, we have

$$\varphi(H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'})=pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}.$$

Thus we may identify $J_2=(\varpi^{p^2-1-m}\otimes\varphi\otimes\mathrm{Id}_{O_E})(J_1)$ with (recall F_2 is an $F\otimes_{\mathbb{Q}_p}E$-module)

$$(\varpi^{p^2-1-m}\otimes 1)\Bigg(\prod_{s'\text{ even}}pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}_{\bar{\tau}}\oplus\prod_{s\text{ odd}}pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\tau}\Bigg)\subset F_2.$$

Let $\omega\in(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ and $\omega=\omega_1+\omega_2$ be the decomposition of ω into $F_1\oplus F_2$. By definition $\varphi(\omega_1)\in J_2$. Since

$$[-mp]+i(p-1)=(p^2-1)+p^2-1-m$$

in this case, Proposition 16.1 implies that $\varpi^{i(p-1)}\omega_2\in J_2$. Hence $\theta_b(\omega)\in J_2$.

Now assume $i=1$; then $\varphi(H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'})=pH^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}$. Hence

$$J_2=\varpi^{p^2-1-m}\Bigg(\prod_{s'\text{ even}}H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{\chi'}_{\bar{\tau}}\oplus\prod_{s\text{ odd}}H^1_{\mathrm{crys}}(\overline{U_{s'}}/O_{F_0})^{(\chi')^p}_{\tau}\Bigg).$$

So the lemma follows directly from Proposition 16.1. $\square$

Let $\bar{\theta}_b:H^{\chi,F_0}\to J_2/p$ be the mod p map of θ_b. It is clear that

$$J_2/p\simeq\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}H^1_{\mathrm{dR}}(\overline{U_{s_0'}})^{(\chi')^p}_{\bar{\tau}}\simeq\begin{cases}\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}}, & i=p,\\ \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Q}_p)}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}}, & i=1.\end{cases}$$

We can now state our main results of this section.

Theorem 17.3. *$\bar{\theta}_b$ is surjective. More precisely*:

(1) *Assume $i=p$. If we consider the following isomorphism induced by φ* (Remark 16.9):

$$H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})^{\chi'}_{\tau}\simeq H^1(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}},$$

and use Remark 15.14 to make the identification

$$H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{\chi'}_{\tau} \simeq (\mathrm{Sym}^{p-2}(O_E/p)^2) \otimes \det{}^{j+1},$$

then $\bar{\theta}_b$ is given by

$$\begin{aligned}\bar{\theta}_b : \sigma_{p-2}(j+1) &\to \sigma_{p-2}(j+1),\\ X &\mapsto -bX + (-1)^{j+1}\tau(w_1^p)T(X) - bT^2(X).\end{aligned}$$

(2) *Assume $i = 1$. If we use Remark 15.14 to make the identification*

$$H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}} \simeq (\mathrm{Sym}^{p-2}(O_E/p)^2) \otimes \det{}^{j+1},$$

then $\bar{\theta}_b$ is given by

$$\begin{aligned}\bar{\theta}_b : \sigma_{p-2}(j+1) &\to \sigma_{p-2}(j+1)\\ X &\mapsto X + (-1)^{j+1}b\tau(w_1^{-1})T(X) + T^2(X).\end{aligned}$$

Just like the previous section, we list some corollaries first.

Corollary 17.4. *$\bar{\theta}_b$ is surjective.*

Corollary 17.5. *$\theta_b : (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} \to J_2$ is surjective and we have the following exact sequence*:

$$0 \to M(\chi, [1, b]) \to (H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)} \to J_2 \to 0. \tag{82}$$

Applying the functor $M \mapsto M^d = \mathrm{Hom}^{\mathrm{cont}}_{O_E}(M, E)$, we get

$$0 \to J_2^d \to \big((H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1) \otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}\big)^d \to B(\chi, [1, b]) \to 0. \tag{83}$$

The kernel and the middle term of this exact sequence are independent of b. The kernel J_2^d is the completion of $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$ *with respect to the lattice* $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho^o_{\chi^{-1}}$, *where $\rho^o_{\chi^{-1}} \subset \rho_{\chi^{-1}}$ is an O_E-lattice. It is the universal unitary completion of* $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$.

Corollary 17.6. *Assume $i = p$. As a representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$,

$$M(\chi, [1, b])/p \simeq \big\{X \in \sigma_{p-2}(j+1) \;\big|\; -bX + (-1)^{j+1}\tau(w_1^p)T(X) - bT^2(X) = 0\big\}.$$

When $i = 1$,

$$M(\chi, [1, b])/p \simeq \big\{X \in \sigma_{p-2}(j+1) \;\big|\; X + (-1)^{j+1}b\tau(w_1^{-1})T(X) + T^2(X) = 0\big\}.$$

Thus in any case, $B(\chi, [1, b])$ is nonzero and admissible.

Proof of Theorem 17.3. We only deal with the case $i = 1$. The case where $i = p$ can be treated in almost the same way.

Consider $[\mathrm{Id}, x^k y^{p-2-k}]$ as an element in

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}(\mathrm{Sym}^{p-2}(O_E/p)^2)\otimes\det{}^{j+1}\simeq \mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})_{\bar\tau}^{(\chi')^p}.$$

Let $\omega\in(H^0(\widetilde{\Sigma_{1,O_F}},\omega^1)\otimes O_E)^{\chi,\mathrm{Gal}(F/F_0)}$ be a lift of $[\mathrm{Id}, x^k y^{p-2-k}]$. As before, we may assume $\omega_\tau=0$. It is clear from our construction that for any even vertex $s'\neq s_0'$,

$$\omega_{\bar\tau,s',\xi}\in pH^0(\widetilde{\Sigma_{1,O_F,s',\xi}},\omega^1)_{\bar\tau}. \tag{84}$$

Hence for any odd vertex $s\notin A(s_0')$,

$$\omega_{\bar\tau,s,\xi}\in pH^0(\widetilde{\Sigma_{1,O_F,s,\xi}},\omega^1)_{\bar\tau}. \tag{85}$$

This follows from Remark 15.9 and the fact $H^0(\overline{U_s},\Omega^1_{\overline{U_s}})^{(\chi')^p}=0$. Thus using Lemma 16.18 and Remark 16.19, we know that $\bar\theta_b([\mathrm{Id}, x^k y^{p-2-k}])$ must be of the following form:

Lemma 17.7. $\bar\theta_b([\mathrm{Id}, x^k y^{p-2-k}])=[\mathrm{Id},u_{s_0'}]+\sum_{s\in A(s_0')}[g_s^{-1},u_s]+\sum_{s'\in A^2(s_0')}[g_{s'}^{-1},u_{s'}]$,

where $A^2(s_0')=\{s'\in A(s)\mid s\in A(s_0'),\ s'\neq s_0'\}$.

First we compute $[\mathrm{Id},u_{s_0'}]$. It suffices to compute the image of $\varpi^{-[-pm]}\omega_{\bar\tau}$ in

$$H^1_{\mathrm{dR}}(\overline{U_{s_0'}})_{\bar\tau}^{(\chi')^p}=H^0(\overline{U_{s_0'}},\Omega^1_{\overline{U_{s_0'}}})_{\bar\tau}^{(\chi')^p}.$$

As before, on $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$, we can write (use a variant of Lemma 15.5 and notice that $\omega_{\bar\tau}$ is in the $(\chi')^p$-isotypic component)

$$\omega_{\bar\tau}|_{\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}}=\varpi^{[-mp]}f(\eta)\tilde e\frac{d\tilde e}{\tilde e}+\varpi^{p^2-1-m}g(\zeta)\tilde e'^p\frac{d\tilde e'}{\tilde e'},$$

where $f(\eta)\in O_{F_0}[\eta,1/(\eta^{p-1}-1)]^\wedge$, $g(\zeta)\in O_{F_0}[\zeta,1/(\zeta^{p-1}-1)]^\wedge$. As usual, we identify $\widetilde{\Sigma_{1,O_F,[s_0',s_0],\xi}}$ with

$$\mathrm{Spf}\frac{O_F\Big[\eta,\zeta,\dfrac{1}{\eta^{p-1}-1},\dfrac{1}{\zeta^{p-1}-1},\tilde e,\tilde e'\Big]^\wedge}{\Big(\tilde e^{p+1}+v_1w_1^{-1}\xi\dfrac{\eta^p-\eta}{\zeta^{p-1}-1},\ \tilde e'^{p+1}+v_1^{-1}w_1\xi\dfrac{\zeta^p-\zeta}{\eta^{p-1}-1},\ \tilde e\tilde e'-\varpi^{p-1}\xi\Big)}.$$

Our choice of ω implies:

Lemma 17.8. $$f(\eta)\equiv\eta^k \mod pO_{F_0}\Big[\eta,\frac{1}{\eta^{p-1}-1}\Big]^\wedge.$$

Now restricted to $\widetilde{\Sigma_{1,O_F,s_0,\xi}}$,

$$\omega_{\tau,s_0,\xi}=\omega_\tau|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}=\varpi^{p^2-1-m}\Big(-\xi\tilde e'^{-2}f\Big(\frac{p}{\zeta}\Big)d\tilde e'+g(\zeta)\tilde e'^{p-1}\,d\tilde e'\Big).$$

By (84), we know that for any $s'\in A(s_0)$ that is not s_0',

$$\omega_{\bar\tau,s',\xi}\in pH^0(\widetilde{\Sigma_{1,O_F,s',\xi}},\omega^1)_{\bar\tau}.$$

Then Remark 15.9 implies that the reduction of $\varpi^{-(p^2-1-m)}\omega_{\bar{\tau},s_0,\xi}$ modulo p, as a meromorphic differential form on $\overline{U_{s_0,\xi}}$, can only have poles at $\zeta = \tilde{e}' = 0$. Here we identify $\overline{U_{s_0,\xi}}$ with the projective curve in $\mathbb{P}^2_{\mathbb{F}_{p^2}}$ defined by

$$\tilde{e}'^{p+1} = v^{-1}w_1\xi(\zeta^p - \zeta).$$

Therefore the only possible pole must come from $-\xi f(p/\zeta)\, d\tilde{e}'/\tilde{e}'^2$. Notice that by Lemma 17.8, this term is nonzero modulo p if and only if $k=0$. Thus when $k \neq 0$, the reduction of $\omega_{\bar{\tau},s_0,\xi}$ is a holomorphic differential form on $\overline{U_{s_0,\xi}}$. But

$$H^0(\overline{U_{s_0}}, \Omega^1_{\overline{U_{s_0}}})^{(\chi')^p} = 0,$$

hence $g(\zeta)$ has to be zero modulo p in this case. Therefore we have proved:

Lemma 17.9. *If $k \neq 0$, then $g(\zeta) \in pO_{F_0}[\zeta, 1/(\zeta^{p-1}-1)]^{\widehat{}}$, and*

$$\omega_{\bar{\tau},s_0,\xi} \in pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}}, \omega^1)_{\bar{\tau}}.$$

When $k=0$. Rewrite

$$\begin{aligned}
\tilde{e}'^{-2} f\Big(\frac{p}{\zeta}\Big) &\equiv \frac{1}{\tilde{e}'^2} = \frac{\tilde{e}'^{p-1}}{\tilde{e}'^{p+1}} \equiv \frac{\tilde{e}'^{p-1}}{v_1^{-1}w_1\xi(\zeta^p-\zeta)} \\
&\equiv -\frac{\tilde{e}'^{p-1}}{v_1^{-1}w_1\xi\zeta} + \frac{\tilde{e}'^{p-1}\zeta^{p-2}}{v_1^{-1}w_1\xi(\zeta^{p-1}-1)} \\
&\qquad \Big(\mathrm{mod}\ pO_{F_0}\Big[\tilde{e}', \zeta, \frac{1}{\zeta^p-\zeta}\Big]^{\widehat{}} \Big/ \Big(\tilde{e}'^{p+1} + v_1^{-1}w_1\xi\frac{\zeta^p-\zeta}{(p/\zeta)^{p-1}-1}\Big)\Big).
\end{aligned}$$

Thus

$$\begin{aligned}
&\varpi^{-(p^2-1-m)}\omega_{\bar{\tau},s_0,\xi} \\
&\equiv \frac{\tilde{e}'^{p-1}\,d\tilde{e}'}{v_1^{-1}w_1\zeta} + \Big(-\frac{\tilde{e}'^{p-1}\zeta^{p-2}}{v_1^{-1}w_1(\zeta^{p-1}-1)} + g(\zeta)\tilde{e}'^{p-1}\Big)\,d\tilde{e}' \quad \mathrm{mod}\ pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}}, \omega^1)_{\bar{\tau}}.
\end{aligned}$$

Notice that the first term, $\tilde{e}'^{p-1}\,d\tilde{e}'/(v_1^{-1}w_1\zeta)$, only has a pole at $\tilde{e}' = \zeta = 0$ and the second term is holomorphic at this point. Therefore the second term (modulo p) is a holomorphic differential form on $\overline{U_{s_0}}$, which has to be zero since it is in $H^0(\overline{U_{s_0}}, \Omega^1_{\overline{U_{s_0}}})^{(\chi')^p} = 0$. Hence:

Lemma 17.10. *When $k=0$,*

$$\begin{aligned}
\omega_{\bar{\tau},s_0,\xi} &\equiv \varpi^{p^2-1-m}\frac{\tilde{e}'^{p-1}d\tilde{e}'}{v_1^{-1}w_1\zeta} \quad \mathrm{mod}\ pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}}, \omega^1)_{\bar{\tau}} \\
g(\zeta) &\equiv \frac{\zeta^{p-2}}{v_1^{-1}w_1(\zeta^{p-1}-1)} \quad \mathrm{mod}\ pO_{F_0}\Big[\zeta, \frac{1}{\zeta^{p-1}-1}\Big]^{\widehat{}}.
\end{aligned}$$

A direct corollary of Lemma 17.9 and Lemma 17.10 is:

Lemma 17.11. *For any k, we always have $g(0) \in pO_{F_0}$.*

Now we try to compute the image of ω inside $\varpi^{p^2-1-m} H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})_{\bar{\tau}} \pmod p$. As we did in the previous section, we can use a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ to represent this cohomology class. Moreover, there exists $\{f_s\}_{s\in A(s_0')}$, where $f_s \in \varpi^{p^2-1-m} H^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}})$ such that $f_{s_1,s_2} = f_{s_2} - f_{s_1}$ and $\omega_s = \omega - df_s$. See the proof of Proposition 16.1 for the notation here.

From Lemma 17.11, we know that $g(0)$ is divisible by p. Therefore Lemma 16.6 tells us that

$$\varpi^{-(p^2-1-m)} f_{s_0} \in pH^0(V_{c,\xi}, \mathcal{O}_{V_{c,\xi}}).$$

Using the action of $\mathrm{GL}_2(\mathbb{Z}_p)$, it is easy to see that the above inclusion is also true for other vertex $s \in A(s_0')$. Hence all f_{s_1,s_2} are divisible by p and all ω_s are congruent to ω modulo p. This certainly implies that the image of $\varpi^{-(p^2-1-m)}\omega$ in $H^1_{\mathrm{dR}}(\overline{U_{s_0'}})^{(\chi')^p}_{\bar{\tau}} = H^0(\overline{U_{s_0'}}, \Omega^1_{\overline{U_{s_0'}}})^{(\chi')^p}_{\bar{\tau}}$ is

$$\varpi^{-(p^2-1-m)}\omega \equiv \eta^k \, d\tilde{e},$$

considered as a differential form using Lemma 15.4. In other words:

Lemma 17.12. $$u_{s_0'} = x^k y^{p-2-k}.$$

Next we compute u_{s_0}. As we did in the previous section, we define

$$h'_{s_0} = (w_1^{-1})^*(\omega_{\bar{\tau}}) \in H^0(\widetilde{\Sigma_{1,O_F,s_0',\xi}}, \omega^1)^{\chi'}_{\tau}. \tag{86}$$

Hence Lemma 16.36 tells us that

$$h'_{s_0}|_{\widetilde{\Sigma_{1,O_F,s_0',\xi}}} = \varpi^{p^2-1-m}\tilde{g}(-\eta)\tilde{e}^p(-v_1^{-1})^p(-1)^{j+1}\frac{d\tilde{e}}{\tilde{e}} - \varpi^{[-mp]}\tilde{f}(-\zeta)\tilde{e}'v_1(-1)^{j+1}\frac{d\tilde{e}'}{\tilde{e}'},$$

where $\tilde{f}(-\zeta) = \widetilde{\mathrm{Fr}}(f(-\zeta))$, and $\tilde{g}(-\eta)$ is defined similarly.

We need to compute the image of $\varpi^{-(p^2-1-m)}h'_{s_0}$ in

$$H^1_{\mathrm{dR}}(\overline{U_{s_0'}})^{\chi'}_{\tau} = H^1(\overline{U_{s_0'}}, \mathcal{O}_{\overline{U_{s_0'}}})^{\chi'}_{\tau}.$$

Now the argument becomes exactly the same as the proof of Theorem 16.26: By abuse of notation, we use a 1-hypercocycle $(\{\omega_s\}, \{f_{s_1,s_2}\})$ to represent the cohomology class of $h'_{s_0} \in \varpi^{p^2-1-m} H^1_{\mathrm{crys}}(\overline{U_{s_0'}}/O_{F_0})^{\chi'}_{\tau}$. Also there exists $\{f_s\}$ such that $f_{s_2,s_1} = f_{s_2} - f_{s_1}$. By (84) and Lemma 16.18, we know that all f_s are divisible by p for $s \neq s_0$. As for f_{s_0}, we can compute it using Lemma 16.6 and Lemma 17.8. We omit all the details here but just refer to the arguments from Lemma 16.36 to Lemma 16.38 in the proof of Theorem 16.26.

Lemma 17.13. $u_{s_0} = u_{[w^{-1}]} = \begin{cases} (-1)^{j+1} b\tau(w_1^{-1}) x^{p-2}, & k=0, \\ 0, & k \neq 0. \end{cases}$

Finally we come to the case $s' \in A^2(s'_0)$, which does not exist when $i \in \{2, \ldots, p-1\}$.

Definition 17.14. We define $s''_0 \in A(s_0)$ as the vertex that corresponds to the coset

$$\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}^{-1} \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times \backslash \mathrm{GL}_2(\mathbb{Q}_p).$$

When $k \neq 0$, Lemma 17.9 tells us that $\omega_{\bar\tau, s_0, \xi} \in pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}}, \omega^1)_{\bar\tau}$. Therefore by Lemma 16.18, the cohomology class of $\varpi^{-[-mp]}\omega_{\bar\tau}$ in $H^1_{\mathrm{crys}}(\overline{U_{s''_0}}/O_{F_0})_{\bar\tau}$ is inside $pH^1_{\mathrm{crys}}(\overline{U_{s''_0}}/O_{F_0})_{\bar\tau}$.

Lemma 17.15. *When $k \neq 0$, $u_{s''_0} = 0$.*

So we assume $k = 0$ from now on.

Notice that

$$\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}^{-1}.$$

Hence the (right) action of $\left(\begin{smallmatrix} 1 & 1/p \\ 0 & 1 \end{smallmatrix}\right)$ fixes the vertex s_0 and sends s''_0 to s'_0. This clearly implies that $\left(\begin{smallmatrix} 1 & 1/p \\ 0 & 1 \end{smallmatrix}\right)$ sends the edge $[s''_0, s_0]$ to $[s'_0, s_0]$. In other words, we get an isomorphism

$$\Psi_{s'_0,s''_0} : \widetilde{\Sigma_{1,O_F,[s''_0,s_0]}} \xrightarrow{\sim} \widetilde{\Sigma_{1,O_F,[s'_0,s_0]}}.$$

Restrict $\Psi_{s'_0,s''_0}$ to $\widetilde{\Sigma_{1,O_F,s_0}}$, we thus get an automorphism of $\widetilde{\Sigma_{1,O_F,s_0}}$. As usual, we identify $\widetilde{\Sigma_{1,O_F,s_0,\xi}}$ with

$$\mathrm{Spf}\, O_F\Big[\zeta, \tilde e', \frac{1}{\zeta^p - \zeta}\Big] \Big/ \Big(\tilde e'^{p+1} + v_1^{-1} w_1 \xi \frac{\zeta^p - \zeta}{(p/\zeta)^{p-1} - 1}\Big)^{\widehat{}}.$$

To see $\Psi_{s'_0,s''_0}$ explicitly on it, we use $\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$ to send $\widetilde{\Sigma_{1,O_F,s_0,-\xi}}$ to $\widetilde{\Sigma_{1,O_F,s'_0,\xi}}$ and then apply the results in Section 9. An easy computation shows:

Lemma 17.16. $\Psi_{s'_0,s''_0}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}$ *is*

$$\zeta \mapsto \zeta + 1, \quad \tilde e' \mapsto \tilde e' \bmod pO_F\Big[\zeta, \tilde e', \frac{1}{\zeta^p - \zeta}\Big] \Big/ \Big(\tilde e'^{p+1} + v_1^{-1} w_1 \xi \frac{\zeta^p - \zeta}{(p/\zeta)^{p-1} - 1}\Big)^{\widehat{}}.$$

Now consider

$$h_{s''_0} \stackrel{\mathrm{def}}{=} \left(\begin{pmatrix} 1 & 1/p \\ 0 & 1 \end{pmatrix}^{-1}\right)^* (\omega_{\bar\tau}) \in H^0(\widetilde{\Sigma_{1,O_F}}, \omega^1)_{\bar\tau}^{(\chi')^p}. \tag{87}$$

On $\widetilde{\Sigma_{1,O_F,[s'_0,s_0],\xi}}$, it can be written as

$$h_{s''_0}|_{\widetilde{\Sigma_{1,O_F,[s'_0,s_0],\xi}}} = \varpi^{[-mp]} f_1(\eta) \tilde e \frac{d\tilde e}{\tilde e} + \varpi^{p^2-1-m} g_1(\zeta) \tilde e'^p \frac{d\tilde e'}{\tilde e'}.$$

By our construction (see (84)), $\omega_{\bar\tau,s_0'',\xi}\in pH^0(\widetilde{\Sigma_{1,O_F,s_0'',\xi}},\omega^1)_{\bar\tau}$. Hence,

$$h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,s_0',\xi}}}\in pH^0(\widetilde{\Sigma_{1,O_F,s_0',\xi}},\omega^1)_{\bar\tau}.$$

This implies:

Lemma 17.17. $$f_1(\eta)\in pO_{F_0}\Big[\eta,\frac{1}{\eta^{p-1}-1}\Big]^{\widehat{}}.$$

Restrict $h_{s_0''}$ to $\widetilde{\Sigma_{1,O_F,s_0,\xi}}$. Then we have

$$h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}\equiv\varpi^{p^2-1-m}g_1(\zeta)\tilde e'^p\frac{d\tilde e'}{\tilde e'}\mod pH^0(\widetilde{\Sigma_{1,O_F,s_0,\xi}},\omega^1)_{\bar\tau}.$$

By definition,

$$h_{s_0''}=\left(\begin{pmatrix}1&1/p\\0&1\end{pmatrix}^{-1}\right)^*(\omega_{\bar\tau}).$$

Hence $\Psi_{s_0',s_0''}$ maps $\omega_{\bar\tau}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}$ to $h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}$. Thanks to Lemma 17.16, we can write down this map explicitly (after reducing modulo p). Recall that an explicit expression of $\omega_{\bar\tau}|_{\widetilde{\Sigma_{1,O_F,s_0,\xi}}}$ is given in Lemma 17.10. Thus a simple computation gives:

Lemma 17.18. *When* $k=0$,

$$g_1(\zeta)\equiv\frac{1}{v_1^{-1}w_1(\zeta-1)}\mod O_{F_0}\Big[\zeta,\frac{1}{\zeta^{p-1}-1}\Big]^{\widehat{}}.$$

With this lemma in hand, we can compute the image of $\varpi^{-(p^2-1-m)}h_{s_0''}$ in $H^1_{\mathrm{dR}}(\overline{U_{s_0'}})_{\bar\tau}^{(\chi')^p}=H^0(\overline{U_{s_0'}},\mathcal{O}_{\overline{U_{s_0'}}})_{\bar\tau}^{(\chi')^p}$. We note that $h_{s_0''}|_{\widetilde{\Sigma_{1,O_F,s}}}\in H^0(\widetilde{\Sigma_{1,O_F,s}},\omega^1)_{\bar\tau}$ for any $s\in A(s_0')$ that is not s_0. So the computation is exactly the same as the case where we computed u_{s_0}. I omit the details here. The final result is:

Lemma 17.19. *When* $k=0$, $u_{s_0''}=-x^{p-2}$.

We need to compute the same term of $T^2([\mathrm{Id},x^ky^{p-2-k}])$. Assume $k=0$. When $k\neq 0$, it's easy to see this term is zero. We already computed that

$$T([\mathrm{Id},y^{p-2}])=\left[\begin{pmatrix}0&-1\\p&0\end{pmatrix},x^{p-2}\right]+\text{other terms}.$$

Since

$$\begin{pmatrix}1&1/p\\0&1\end{pmatrix}=\begin{pmatrix}0&-1\\p&0\end{pmatrix}\begin{pmatrix}0&p^{-1}\\-1&-p^{-1}\end{pmatrix},$$

by definition we have,

$$T\left(\left[\begin{pmatrix}0&-1\\p&0\end{pmatrix},x^{p-2}\right]\right)=\left[\begin{pmatrix}1&1/p\\0&1\end{pmatrix},\varphi_{p-2}\left(\begin{pmatrix}0&p^{-1}\\-1&p^{-1}\end{pmatrix}^{-1}\right)(x^{p-2})\right].$$

Write
$$\begin{pmatrix}0 & p^{-1}\\ -1 & -p^{-1}\end{pmatrix}^{-1} = p\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$
Then
$$\begin{aligned}\varphi_{p-2}\left(\begin{pmatrix}0 & p^{-1}\\ -1 & p^{-1}\end{pmatrix}^{-1}\right)(x^{p-2}) &= \varphi_{p-2}\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & p^{-1}\end{pmatrix}\right)((x+y)^{p-2})\\ &= \varphi_{p-2}\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)(y^{p-2})\\ &= -x^{p-2}.\end{aligned}$$
Hence:

Lemma 17.20. $T^2([\mathrm{Id}, y^{p-2}]) = \begin{cases}\left[\begin{pmatrix}1 & 1/p\\ 0 & 1\end{pmatrix}, -x^{p-2}\right] + \textit{other terms}, & k=0,\\ \left[\begin{pmatrix}1 & 1/p\\ 0 & 1\end{pmatrix}, 0\right] + \textit{other terms}, & k\neq 0.\end{cases}$

Combining the results of Lemmas 17.12, 17.13, 17.15, and 17.19 together with Lemmas 16.41 and 17.20:
$$\bar{\theta}_b(X) = X + (-1)^{j+1} b\tau(w_1^{-1})T(X) + T^2(X). \qquad \square$$

18. A conjecture on $B(\chi, [1, b])$

In the previous two sections, we have proved the admissibility of $B(\chi, [1, b])$ and explicitly compute its residue representation (see Corollaries 16.29 and 17.6, and Remark 16.30). Recall that for each data $(\chi, [1, b])$, we associate a two dimensional Galois representation $V_{\chi,[1,b]}$ (Proposition 13.2) and prove that $B(\chi, [1, b])$ is a completion of the smooth representation $\text{c-Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \rho_{\chi^{-1}}$ with respect to the lattice $\Theta(\chi, [1, b])$ (Proposition 14.9). Up to some twist, this smooth representation, via the classical local Langlands correspondence for GL_2, corresponds to the Weil–Deligne representation associated to $V^\vee_{\chi,[1,b]}$ in [Fontaine 1994]. It is natural to make the following:

Conjecture 18.1. *Up to a twist of some character*, $B(\chi, [1, b])$ *is isomorphic to* $\Pi(V^\vee_{\chi,[1,b]})$ *as a Banach space representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$, *where* $\Pi(V^\vee_{\chi,[1,b]})$ *is defined via the* p*-adic local Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Colmez 2010; Colmez et al. 2014]).

The evidence for this conjecture is that we can verify it modulo ϖ_E, the uniformizer of E, namely:

Theorem 18.2. *Via the semisimple modulo-*p *Langlands correspondence defined by Breuil* ([2003] or [2007]), *up to a twist by some character and semisimplification*,

$\Theta(\chi,[1,b])/\varpi_E$ corresponds to the residue representation of $V^\vee_{\chi,[1,b]}$ with respect to some lattice inside.

Proof. The residue representation of $V_{\chi,[1,b]}$ is computed in Theorem 6.12 of [Savitt 2005]. I almost follow his notation except that his w there is my u_x here. $\Theta(\chi,[1,b])/\varpi_E\Theta(\chi,[1,b])$ is computed in Corollaries 16.29 and 17.6, and Remark 16.30. A direct computation shows that they indeed match via Breuil's dictionary. I omit the details here. □

Remark 18.3. There is some duality involved in the conjecture. The reason is that we are using de Rham cohomology rather than its dual, de Rham cohomology with compact support.

Remark 18.4. It seems that this conjecture follows from the work of Dospinescu and Le Bras [2015] by taking the universal unitary completion in their construction. The interested reader is referred to their paper.

List of symbols

Section 2	Ω; $\widehat{\Omega}$; $\widehat{\Omega}_e$; $\widehat{\Omega}_s$; s_0'; X_n; $\mathcal{X}_n$; Σ_n; T_0; T_1
Section 3	χ_1; χ_2; $\mathcal{L}_i$; c_i; d_i
Section 5	λ_1; $\lambda_{\mathcal{L}_1}$; $\widetilde{\lambda_{\mathcal{L}_1}}$; $\widehat{\Sigma_1^{\mathrm{nr}}}$ (Proposition 5.7)
Section 7	$\widehat{\Sigma_{1,e}^{\mathrm{nr}}}$; $\widehat{\Sigma_{1,s}^{\mathrm{nr}}}$; w_1; v_1, $\widehat{\Sigma_1}$ (Proposition 7.5)
Section 8	$\widehat{\Sigma_1'}$; F_0; F, ϖ (Definition 8.1); $\widehat{\Sigma_{1,O_F}}$; $\widehat{\Sigma_{1,O_F,s}}$; $\widetilde{\Sigma_{1,O_F}}$; $\widetilde{\Sigma_{1,O_F,s}}$; $\widetilde{\Sigma_{1,O_F,s,\xi}}$; $\widehat{\Sigma^{(0)}_{1,O_F}}$, …, g_φ (Remark 8.5)
Section 9	$\tilde{\omega}_2$ (Definition 9.3)
Section 10	$\Sigma_{1,F}$; $\Sigma^{(0)}_{1,F}$; U_n
Section 11	$\Omega^i_{\Sigma_{1,F}}$; $H^i_{\mathrm{dR}}(\Sigma_{1,F})$; U_e, U_s (Definition 11.7); (s,ξ) (Definition 11.9); $\overline{U_s}$, $\overline{U_{s,\xi}}$, $\overline{U^0_{s,\xi}}$, $\overline{U^0_s}$, $U_{s,\xi}$ (Definition 11.10)
Section 12	$F_{1,\xi}$; $D_{1,\xi}$; $\psi_{s,\xi}$; $\widehat{D_{1,O_{F_0},\xi}}$; ρ_χ (Definition 12.6); $D_{\mathrm{crys},\chi}$, m, c_x (Proposition 12.8)
Section 13	i, j, $D_{\chi,[a,b]}$, $V_{\chi,[1,b]}$ (Proposition 13.2)
Section 14	ω^1; $M(\chi,[1,b])$; $B(\chi,[1,b])$
Section 15	$H^{(0),\chi,\mathbb{Q}_p}$, H_*^{χ,F_0}, H_*^{χ',F_0}, $H_{*,?}^{\chi',F_0}$ (Definition 15.1); $A(s)$ (Definition 15.2)
Section 16	$F_{0,\xi}$; $D_{0,\xi}$; $\psi_{s',\xi}$; $\widehat{D_{0,O_{F_0},\xi}}$; $V_{s,\xi}$, $V_{c,\xi}$ (Definition 16.2); $W_{s,\xi}$, $Z_{s,\xi}$ (Definition 16.3); ω_s; f_{s_1,s_2}, f_s (58); $A(s_0')$ (Definition 16.10); V_0, V_∞ (Definition 16.11); F_1, F_2 (70); θ_b (72); J_1, J_2 (Definition 16.22); $\bar{\theta}_b$; $\bar{\theta}_{b,1}$, $\bar{\theta}_{b,2}$ (Lemma 16.25); v_s (Lemma 16.33); h_{s_0} (81);

ω_τ, $\omega_{\bar{\tau}}$, $\omega_{\tau,s,\xi}$, ω_1, ω_2, $\omega_{1,s}$, $\omega_{1,s,\xi}$, ... (Definition 16.31)

Section 17 $u_{s_0'}$, u_{s_0}, $u_{s_0''}$ (Lemma 17.7); s_0'' (Definition 17.14); h'_{s_0} (86); $h_{s_0''}$ (87)

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References

[Berthelot 1986] P. Berthelot, "Géométrie rigide et cohomologie des variétés algébriques de caractéristique p", pp. 3, 7–32 in *Introductions aux cohomologies p-adiques* (Luminy, 1984), Mém. Soc. Math. France (N.S.) **23**, 1986. MR Zbl

[Bosch 2014] S. Bosch, *Lectures on formal and rigid geometry*, Lecture Notes in Mathematics **2105**, Springer, Cham, 2014. MR Zbl

[Boutot and Carayol 1991] J.-F. Boutot and H. Carayol, "Uniformisation p-adique des courbes de Shimura: les théorèmes de Čerednik et de Drinfel'd", pp. 7, 45–158 in *Courbes modulaires et courbes de Shimura* (Orsay, 1987/1988), Astérisque **196-197**, Société Mathématique de France, 1991. MR Zbl

[Breuil 2003] C. Breuil, "Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbb{Q}_p)$, II", *J. Inst. Math. Jussieu* **2**:1 (2003), 23–58. MR Zbl

[Breuil 2004] C. Breuil, "Invariant $\mathscr{L}$ et série spéciale p-adique", *Ann. Sci. École Norm. Sup.* (4) **37**:4 (2004), 559–610. MR Zbl

[Breuil 2007] C. Breuil, "Representations of Galois and of GL_2 in characteristic p", course notes, Columbia University, 2007, http://www.math.u-psud.fr/∼breuil/PUBLICATIONS/New-York.pdf.

[Breuil and Mézard 2010] C. Breuil and A. Mézard, "Représentations semi-stables de $\mathrm{GL}_2(\mathbb{Q}_p)$, demi-plan p-adique et réduction modulo p", pp. 117–178 in *p-adic representations of p-adic groups, III : global and geometrical methods*, edited by L. Berger et al., Astérisque **331**, Société Mathématique de France, Paris, 2010. MR Zbl

[Bushnell and Henniart 2006] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Math. Wissenschaften **335**, Springer, Berlin, 2006. MR Zbl

[Coleman 1989] R. F. Coleman, "Reciprocity laws on curves", *Compositio Math.* **72**:2 (1989), 205–235. MR Zbl

[Coleman and Iovita 1999] R. Coleman and A. Iovita, "The Frobenius and monodromy operators for curves and abelian varieties", *Duke Math. J.* **97**:1 (1999), 171–215. MR Zbl

[Colmez 2004] P. Colmez, "Une correspondance de Langlands locale p-adique pour les représentations semi-stables de dimension 2", preprint, 2004, http://webusers.imj-prg.fr/∼pierre.colmez/sst.pdf.

[Colmez 2010] P. Colmez, "Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (ϕ,Γ)-modules", pp. 281–509 in *Représentations p-adiques de groupes p-adiques, II : Représentations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (φ,Γ)-*modules*, edited by L. Berger et al., Astérisque **330**, 2010. MR Zbl

[Colmez et al. 2014] P. Colmez, G. Dospinescu, and V. Paškūnas, "The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$", *Camb. J. Math.* **2**:1 (2014), 1–47. MR Zbl

[Deligne and Lusztig 1976] P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields", *Ann. of Math.* (2) **103**:1 (1976), 103–161. MR Zbl

[Dospinescu and Le Bras 2015] G. Dospinescu and A.-C. Le Bras, "Revêtements du demi-plan de Drinfeld et correspondance de Langlands p-adique", preprint, 2015. arXiv

[Drinfel'd 1974] V. G. Drinfel'd, "Elliptic modules", *Mat. Sb.* (*N.S.*) **94(136)** (1974), 594–627, 656. In Russian; translated in *Math. USSR-Sb.* **23**:4 (1974), 561–592. MR

[Drinfel'd 1976] V. G. Drinfel'd, "Coverings of p-adic symmetric domains", *Funkcional. Anal. i Priložen.* **10**:2 (1976), 29–40. In Russian; translated in *Funct. Anal. Appl.* **10**:2 (1976), 107–115. MR

[Emerton 2005] M. Emerton, "p-adic L-functions and unitary completions of representations of p-adic reductive groups", *Duke Math. J.* **130**:2 (2005), 353–392. MR Zbl

[Fontaine 1994] J.-M. Fontaine, "Représentations l-adiques potentiellement semi-stables", pp. 321–347 in *Périodes p-adiques* (Bures-sur-Yvette, 1988), Astérisque **223**, Société Mathématique de France, Paris, 1994. MR Zbl

[Gillet and Messing 1987] H. Gillet and W. Messing, "Cycle classes and Riemann–Roch for crystalline cohomology", *Duke Math. J.* **55**:3 (1987), 501–538. MR Zbl

[Große-Klönne 2000] E. Große-Klönne, "Rigid analytic spaces with overconvergent structure sheaf", *J. Reine Angew. Math.* **519** (2000), 73–95. MR Zbl

[Große-Klönne 2002] E. Große-Klönne, "Finiteness of de Rham cohomology in rigid analysis", *Duke Math. J.* **113**:1 (2002), 57–91. MR Zbl

[Große-Klönne 2004] E. Große-Klönne, "De Rham cohomology of rigid spaces", *Math. Z.* **247**:2 (2004), 223–240. MR Zbl

[Haastert and Jantzen 1990] B. Haastert and J. C. Jantzen, "Filtrations of the discrete series of $\mathrm{SL}_2(q)$ via crystalline cohomology", *J. Algebra* **132**:1 (1990), 77–103. MR Zbl

[Katz 1981] N. M. Katz, "Crystalline cohomology, Dieudonné modules, and Jacobi sums", pp. 165–246 in *Automorphic forms, representation theory and arithmetic* (Bombay, 1979), Tata Inst. Fund. Res. Studies in Math. **10**, Tata Inst. Fundamental Res., Bombay, 1981. MR Zbl

[Katz and Messing 1974] N. M. Katz and W. Messing, "Some consequences of the Riemann hypothesis for varieties over finite fields", *Invent. Math.* **23** (1974), 73–77. MR Zbl

[Kiehl 1967] R. Kiehl, "Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie", *Invent. Math.* **2** (1967), 256–273. MR Zbl

[Perez-Garcia and Schikhof 2010] C. Perez-Garcia and W. H. Schikhof, *Locally convex spaces over non-Archimedean valued fields*, Cambridge Studies in Advanced Mathematics **119**, Cambridge University Press, 2010. MR Zbl

[Raynaud 1974] M. Raynaud, "Schémas en groupes de type $(p, \ldots, p)$", *Bull. Soc. Math. France* **102** (1974), 241–280. MR Zbl

[Savitt 2005] D. Savitt, "On a conjecture of Conrad, Diamond, and Taylor", *Duke Math. J.* **128**:1 (2005), 141–197. MR Zbl

[Schneider 2002] P. Schneider, *Nonarchimedean functional analysis*, Springer, Berlin, 2002. MR Zbl

[Schneider and Stuhler 1991] P. Schneider and U. Stuhler, "The cohomology of p-adic symmetric spaces", *Invent. Math.* **105**:1 (1991), 47–122. MR Zbl

[Schneider and Teitelbaum 2002] P. Schneider and J. Teitelbaum, "Banach space representations and Iwasawa theory", *Israel J. Math.* **127** (2002), 359–380. MR Zbl

[Teitelbaum 1990] J. Teitelbaum, "Geometry of an étale covering of the p-adic upper half plane", *Ann. Inst. Fourier* (*Grenoble*) **40**:1 (1990), 68–78. MR Zbl

[Teitelbaum 1993] J. T. Teitelbaum, "Modular representations of PGL_2 and automorphic forms for Shimura curves", *Invent. Math.* **113**:3 (1993), 561–580. MR Zbl

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