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John F. R. Duncan and Jeffrey A. Harvey

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The umbral moonshine module for the unique unimodular Niemeier root system

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We use canonically twisted modules for a certain super vertex operator algebra to construct the umbral moonshine module for the unique Niemeier lattice that coincides with its root sublattice. In particular, we give explicit expressions for the vector-valued mock modular forms attached to automorphisms of this lattice by umbral moonshine. We also characterize the vector-valued mock modular forms arising, in which four of Ramanujan's fifth-order mock theta functions appear as components.

1. Introduction

In his Ph.D thesis, Zwegers [2002] gave an intrinsic definition of mock theta functions and provided new insight into three families of such functions, constructed

(1) in terms of Appell–Lerch sums,

(2) as the Fourier coefficients of meromorphic Jacobi forms, and

(3) via theta functions attached to cones in lattices of indefinite signature.

The first two constructions have played a central role in recently observed moonshine connections between finite groups and mock theta functions. These started with the observation in [Eguchi et al. 2011] that the elliptic genus of a K3 surface has a decomposition into characters of the $N=4$ superconformal algebra with multiplicities that at low levels are equal to the dimensions of irreducible representations of the Mathieu group M_{24}. Appell–Lerch sums appear in this analysis in the so called "massless" characters. This Mathieu moonshine connection was conjectured in [Cheng et al. 2014a; 2014b] to be part of a much more general phenomenon, known as umbral moonshine, which attaches a vector-valued mock modular form H^X, a finite group G^X, and an infinite-dimensional graded G^X-module K^X to the root systems of each of the 23 Niemeier lattices. The analysis in [Cheng et al. 2014b] relied heavily on the construction of mock modular forms in terms of meromorphic Jacobi forms and built on the important work in [Dabholkar et al. 2012] extending

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the analysis of [Zwegers 2002] and characterizing special Jacobi forms in terms of growth conditions.

Whilst the existence of the G^X-modules K^X has now been proven [Gannon 2016; Duncan et al. 2015b] for all Niemeier root systems X, no explicit construction of the modules K^X is yet known.

In this paper we construct the G^X-module K^X for the case that $X = E_8^3$. To do so we apply the third characterization of mock theta functions in terms of indefinite theta functions. This enables us to employ the formalism of vertex operator algebras [Borcherds 1986; Frenkel et al. 1988], which has been so fruitfully employed (in [Frenkel et al. 1988; Borcherds 1992] to name just two) in the understanding of monstrous moonshine [Conway and Norton 1979; Thompson 1979a; 1979b].

See [Duncan et al. 2015a] for a recent review of moonshine both monstrous and umbral, and many more references on these subjects.

To explain the methods of this paper in more detail, we first recall the *Pochammer symbol*

$$(x;q)_n := \prod_{k=0}^{n-1}(1-xq^k), \tag{1-1}$$

and the fifth-order mock theta functions

$$\chi_0(q) := \sum_{n\geq 0}\frac{q^n}{(q^{n+1};q)_n}, \qquad \chi_1(q) := \sum_{n\geq 0}\frac{q^n}{(q^{n+1};q)_{n+1}} \tag{1-2}$$

from Ramanujan's last letter to Hardy [Ramanujan 1988; 2000]. The conjectures of [Cheng et al. 2014b] (see also [Cheng et al. $\geq$ 2017]) imply the existence of a bigraded super vector space $K^X = \bigoplus_r K^X_r = \bigoplus_{r,d} K^X_{r,d}$ that is a module for $G^X \simeq S_3$ and satisfies

$$\begin{aligned}\operatorname{sdim}_q K^X_1 &= -2q^{-1/120} + \sum_{n>0}\dim K^X_{1,n-1/120}\,q^{n-1/120}\\ &= 2q^{-1/120}(\chi_0(q)-2),\\ \operatorname{sdim}_q K^X_7 &= \sum_{n>0}\dim K^X_{7,n-49/120}\,q^{n-49/120} = 2q^{71/120}\chi_1(q).\end{aligned} \tag{1-3}$$

Here $\operatorname{sdim}_q V := \sum_n (\dim(V_{\bar{0}})_n - \dim(V_{\bar{1}})_n)q^n$ for V a $\mathbb{Q}$-graded super space with even part $V_{\bar{0}}$ and odd part $V_{\bar{1}}$.

In this article we realize K^X explicitly in terms of canonically twisted modules for a super vertex operator algebra V^X, and we show that the graded trace functions for the resulting bigraded G^X-module are exactly compatible with the predictions of [Cheng et al. 2014b]. Thus we construct the analogue of the moonshine module $V^\natural$ of Frenkel, Lepowsky and Meurman [Frenkel et al. 1984; 1985; 1988], for the

$X = E_8^3$ case of umbral moonshine (although it should be noted that the group for us is $G^X \simeq S_3$, which is of course much less complicated than the monster).

To prove that our construction is indeed the $X = E_8^3$ counterpart to $V^\natural$, we verify the $X = E_8^3$ analogue of the Conway–Norton moonshine conjecture, proven by Borcherds [1992] in the case of the monster, which predicts that the trace functions arising are uniquely determined by their automorphy and their asymptotic behavior near cusps. Thus we verify the $X = E_8^3$ analogues of both of the two major conjectures of monstrous moonshine.

The main motivation for our construction of K^X stems from some q-series identities for $\chi_0(q)$ and $\chi_1(q)$ that were established by Zwegers [2009]. These are

$$\begin{aligned} 2-\chi_0(q) &= \frac{1}{(q;q)_\infty^2}\Bigg(\sum_{k,l,m\geq 0}+\sum_{k,l,m<0}\Bigg)(-1)^{k+l+m} \\ &\qquad\qquad \times q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+(k+l+m)/2}, \\ \chi_1(q) &= \frac{1}{(q;q)_\infty^2}\Bigg(\sum_{k,l,m\geq 0}+\sum_{k,l,m<0}\Bigg)(-1)^{k+l+m} \\ &\qquad\qquad \times q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+3(k+l+m)/2}, \end{aligned} \tag{1-4}$$

where $(x;q)_\infty := \prod_{n\geq 0}(1-xq^n)$. Upon regarding (1-4) we are led to consider a certain indefinite lattice of rank 3 (see Section 2D), which furnishes most of the vertex algebra structure that we use to define V^X. It is a curious circumstance that there seems to be no obvious relationship between the lattice we use and the $X = E_8^3$ structure which underpins the relevant case of umbral moonshine. For this reason it is not yet clear how the construction of K^X presented here should be generalized to the other cases of umbral moonshine.

We now formulate precise statements of our results. To prepare for this, recall that vector-valued functions $H_g^X(\tau) = (H_{g,r}^X(\tau))$ on the upper half plane $\mathbb{H}$ are considered in [Cheng et al. 2014b], for $g \in G^X \simeq S_3$, where the components are indexed by $r \in \mathbb{Z}/60\mathbb{Z}$. Define $o(g)$ to be the order of an element $g \in G^X$. The H_g^X are not uniquely determined in [Cheng et al. 2014b], except for the case that $g = e$ is the identity, $o(g) = 1$. But it is predicted that H_g^X is a mock modular form of weight $\frac{1}{2}$ for $\Gamma_0(o(g))$, with shadow given by a certain vector-valued unary theta function S_g^X (see (3-33)), and specified polar parts at the cusps of $\Gamma_0(o(g))$. In more detail, H_g^X should have the same polar parts as $H^X := H_e^X$ at the infinite cusp of $\Gamma_0(o(g))$, but should have vanishing polar parts at any noninfinite cusps. In practice, this amounts to the statement that we should have

$$\begin{aligned} &H_{g,r}^X(\tau) \\ &\quad = \begin{cases} \mp 2q^{-1/120} + O(q^{119/120}) & \text{if } r = \pm 1, \pm 11, \pm 19, \pm 29 \pmod{60}, \\ O(1) & \text{otherwise,} \end{cases} \end{aligned} \tag{1-5}$$

for $q = e^{2\pi i\tau}$, and all components of $H_g^X(\tau)$ should remain bounded as $\tau \to 0$, if $g \neq e$. (If $g \neq e$ then $o(g) = 2$ or $o(g) = 3$, and then $\Gamma_0(o(g))$ has only one cusp other than the infinite one, and this is the cusp represented by 0.)

Our main result is the following, where the functions T_g^X are defined in Section 3C (see (3-32)) in terms of traces of operators on canonically twisted modules for V^X.

Theorem 1.1. *Let $g \in G^X$. If $o(g) \neq 3$ then $2T_g^X$ is the Fourier expansion of the unique vector-valued mock modular form of weight $\frac{1}{2}$ for $\Gamma_0(o(g))$ whose shadow is S_g^X, and whose polar parts coincide with those of H_g^X. If $o(g) = 3$ then $2T_g^X$ is the Fourier expansion of the unique vector-valued modular form of weight $\frac{1}{2}$ for $\Gamma_0(3)$ which has the multiplier system $\rho_{3|3}\overline{\sigma^X}$ and polar parts coinciding with those of H_g^X.*

Here $\sigma^X : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{60}(\mathbb{C})$ denotes the multiplier system of $S^X := S_e^X$ (see (3-34)), and $\rho_{3|3} : \Gamma_0(3) \to \mathbb{C}^\times$ is defined in (3-38). The shadow function S_g^X is defined in (3-33) and determines the multiplier system of $2T_g^X$ when $o(g) \neq 3$.

Armed with Theorem 1.1, we may now define the H_g^X concretely and explicitly, for $g \in G^X$, by setting

$$H_g^X(\tau) := 2T_g^X(\tau), \tag{1-6}$$

where $T_g^X(\tau)$ denotes the function obtained by substituting $e^{2\pi i\tau}$ for q in the series expression (3-32) for T_g^X.

Expressions for the components of H_g^X are given in §5.4 of [Cheng et al. 2014b], in terms of fifth-order mock theta functions of Ramanujan, for the cases that $o(g) = 1$ and $o(g) = 2$, but it is not verified there that these prescriptions define mock modular forms with the specified shadows. Our work confirms these statements, as the following theorem demonstrates.

Theorem 1.2. *We have the following identities*:

$$H_{1A,r}^X(\tau) = \begin{cases} \pm 2q^{-1/120}(\chi_0(q) - 2) & \text{if } r = \pm 1, \pm 11, \pm 19, \pm 29, \\ \pm 2q^{71/120}\chi_1(q) & \text{if } r = \pm 7, \pm 13, \pm 17, \pm 23, \end{cases} \tag{1-7}$$

$$H_{2A,r}^X(\tau) = \begin{cases} \mp 2q^{-1/120}\phi_0(-q) & \text{if } r = \pm 1, \pm 11, \pm 19, \pm 29, \\ \pm 2q^{-49/120}\phi_1(-q), & \text{if } r = \pm 7, \pm 13, \pm 17, \pm 23. \end{cases} \tag{1-8}$$

The fifth-order mock theta functions ϕ_0 and ϕ_1 were defined by Ramanujan (also in his last letter to Hardy) by setting

$$\phi_0(q) := \sum_{n\geq 0} q^{n^2}(-q;q^2)_n, \qquad \phi_1(q) := \sum_{n\geq 0} q^{(n+1)^2}(-q;q^2)_n. \tag{1-9}$$

The identities (1-7) follow immediately from Theorem 1.1, since the V^X-modules used to define the T_g^X have been constructed specifically so as to make Zwegers' identity (1-4) manifest. By contrast, the $o(g) = 2$ case of Theorem 1.2 requires some work, since the expressions we obtain naturally from our construction of T_g^X do not obviously coincide with (1-8). Thus the proof of Theorem 1.2 entails nontrivial q-series identities which may be of independent interest.

Corollary 1.3. *We have*

$$\left(\sum_{k,m\geq 0}-\sum_{k,m<0}\right)_{k=m \pmod 2}(-1)^m q^{k^2/2+m^2/2+4km+k/2+3m/2} = \prod_{n>0}(1+q^n)\left(\sum_{k,m\geq 0}-\sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+k+m/2}, \tag{1-10}$$

$$\left(\sum_{k,m\geq 0}-\sum_{k,m<0}\right)_{k=m \pmod 2}(-1)^m q^{k^2/2+m^2/2+4km+3k/2+5m/2} = \prod_{n>0}(1+q^n)\left(\sum_{k,m\geq 0}-\sum_{k,m<0}\right)(-1)^{k+m}q^{3k^2+m^2/2+4km+3k+3m/2}. \tag{1-11}$$

The reader who is familiar with modularity results on trace functions attached to vertex operator algebras (see [Zhu 1996; Dong et al. 2000; Miyamoto 2004]) and super vertex operator algebras (see [Dong and Zhao 2005]) may find it surprising that the functions we construct are (generally) mock modular, rather than modular, and have weight $\frac{1}{2}$, rather than weight 0. In light of Zwegers' work [2002; 2009], it is clear that we can obtain trace functions with mock modular behavior by considering vertex algebras constructed according to the usual lattice vertex algebra construction, but with a cone (see Section 2D) taking on the role usually played by a lattice. A suitably chosen cone is the main ingredient for our construction of V^X.

Note however that the cone vertex algebra construction does not, on its own, naturally give rise to trace functions with weight $\frac{1}{2}$. For this we introduce a single "free fermion" to the cone vertex algebra that we use to construct V^X, and we insert the zero mode (i.e., $L(0)$-degree preserving component) of the canonically twisted vertex operator attached to a generator when we compute graded traces on canonically twisted modules for V^X. In practice, this has the effect of multiplying the cone vertex algebra trace functions by $\eta(\tau) := q^{1/24}\prod_{n>0}(1-q^n)$.

We remark that this technique may be profitably applied to other situations. For example, it is known (see, e.g., [Harada and Lang 1998]) that the moonshine module $V^\natural$, when regarded as a module for the Virasoro algebra, is a direct sum of modules $L(h, 24)$, for h ranging over nonnegative integers, satisfying

$$\operatorname{tr}_{L(h,24)} q^{L(0)-c/24} = \begin{cases}(1-q)q^{-23/24}\eta(\tau)^{-1} & \text{for } h=0,\\ q^{h-23/24}\eta(\tau)^{-1} & \text{for } h>0,\end{cases} \tag{1-12}$$

where $c=24$. Also, the multiplicity of $L(0, 24)$ is 1, and the multiplicity of $L(1, 24)$ is 0. Consequently, the weight $\frac{1}{2}$ modular form $\eta(\tau)J(\tau)$, with $J(\tau) = q^{-1} + O(q)$

the (so normalized) elliptic modular invariant, is almost the generating function of the dimensions of the homogeneous spaces of Virasoro highest weight vectors in $V^\natural$. Indeed, the actual generating function is just $q^{1/24}\eta(\tau)J(\tau)+1$.

Certainly $\eta(\tau)J(\tau)$ has nicer modular properties than the Virasoro highest weight generating function of $V^\natural$, and moreover, an even more striking connection to the monster, as four of the dimensions of nontrivial irreducible representations for the monster appear as coefficients:

$$\begin{aligned}\eta(\tau)J(\tau) = \cdots + 196883q^{25/24} + 21296876q^{49/24}\\ + 842609326q^{73/24} + 19360062527q^{97/24} + \cdots \end{aligned} \tag{1-13}$$

(see p. 220 of [Conway et al. 1985]). This function $\eta(\tau)J(\tau)$ can be obtained naturally as a trace function on a canonically twisted module for a super vertex operator algebra. Indeed, if we take V to be the tensor product of $V^\natural$ with the super vertex operator algebra obtained by applying the Clifford module construction to a one-dimensional vector space (see Section 2C for details), then, choosing an irreducible canonically twisted module V_{tw} for V, and denoting by $p(0)$ the coefficient of z^{-1} in the canonically twisted vertex operator attached to a suitably scaled element $p \in V$ with $L(0)p = \frac{1}{2}p$, we have

$$\operatorname{tr}_{V_{\mathrm{tw}}} p(0)q^{L(0)-c/24} = \eta(\tau)J(\tau), \tag{1-14}$$

where now $c = \frac{49}{2}$. (See Section 2C for more detail.)

The importance of trace functions such as (1-14) within the broader context of modularity for super vertex operator algebras is analyzed in detail in [Van Ekeren 2013]; see also [Van Ekeren 2014].

The organization of the paper is as follows. In Section 2 we recall some familiar constructions from the theory of vertex algebras and use these to construct the super vertex operator algebra V^X and its canonically twisted modules $V^{X,\pm}_{\mathrm{tw},a}$, which play the commanding role in this work. We recall the lattice construction of super vertex algebras in Section 2A, modules for lattice super vertex algebras in Section 2B, and the Clifford module super vertex algebra construction in Section 2C. We formulate the construction of V^X and the twisted modules $V^{X,\pm}_{\mathrm{tw},a}$ in Section 2D. We also equip these spaces with G^X-module structure in Section 2D, and compute explicit expressions (see Proposition 2.2) for the graded traces of elements of G^X.

In Section 3 our focus moves from representation theory to number theory, as we seek to determine the properties of the graded traces arising from the action of G^X on the $V^{\pm}_{\mathrm{tw},a}$. We recall the relationship between mock modular forms and harmonic Maass forms in Section 3A, and we recall some results on Zwegers' indefinite theta series in Section 3B. The proofs of our main results, Theorems 1.1 and 1.2, are the content of Section 3C.

We give tables with the first few coefficients of the H^X_g in the Appendix. We frequently employ the notational convention $e(x) := e^{2\pi i x}$.

2. Vertex algebra

This section begins with a review of the lattice (super) vertex algebra construction in Section 2A, and the natural generalization of this, which defines lattice vertex algebra modules, in Section 2B. We review the special case of the Clifford module super vertex algebra construction we require in Section 2C. We put all of this together for the construction of V^X, and its canonically twisted modules, in Section 2D.

2A. ***Lattice vertex algebra.*** We briefly recall, following [Borcherds 1986; Frenkel et al. 1988], the standard construction which associates a super vertex algebra V_L to a central extension of an integral lattice L. We also employ [Frenkel and Ben-Zvi 2004] as a reference. Set $\mathfrak{h} := L \otimes_{\mathbb{Z}} \mathbb{C}$, and extend the bilinear form on L to a symmetric $\mathbb{C}$-bilinear form on $\mathfrak{h}$ in the natural way. Set $\hat{\mathfrak{h}} := \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}\boldsymbol{c}$, for t a formal variable, and define a Lie algebra structure on $\hat{\mathfrak{h}}$ by declaring that $\boldsymbol{c}$ is central, and $[u \otimes t^m, v \otimes t^n] = m\langle u, v\rangle \delta_{m+n,0}\, \boldsymbol{c}$ for $u, v \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. We follow tradition and write $u(m)$ as a shorthand for $u \otimes t^m$. The Lie algebra $\hat{\mathfrak{h}}$ has a triangular decomposition $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}^- \oplus \hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+$, where $\hat{\mathfrak{h}}^\pm := \mathfrak{h}[t^{\pm 1}]t^{\pm 1}$ and $\hat{\mathfrak{h}}^0 := \mathfrak{h} \oplus \mathbb{C}\boldsymbol{c}$.

We require a bilinear function $b : L \times L \to \mathbb{Z}/2\mathbb{Z}$ with the property that

$$b(\lambda, \mu) + b(\mu, \lambda) = \langle \lambda, \mu\rangle + \langle \lambda, \lambda\rangle\langle \mu, \mu\rangle + 2\mathbb{Z}.$$

If $\{\varepsilon_i\}$ is an ordered $\mathbb{Z}$-basis for L then we may take b to be the unique such function for which

$$b(\varepsilon_i, \varepsilon_j) = \begin{cases} 0 + 2\mathbb{Z} & \text{when } i \le j, \\ \langle \lambda, \mu\rangle + \langle \lambda, \lambda\rangle\langle \mu, \mu\rangle + 2\mathbb{Z} & \text{when } i > j. \end{cases} \tag{2-1}$$

Set $\beta(\lambda, \mu) := (-1)^{b(\lambda,\mu)}$, and define $\mathbb{C}_\beta[L]$ to be the ring generated by symbols $\boldsymbol{v}_\lambda$ for $\lambda \in L$ subject to the relations $\boldsymbol{v}_\lambda \boldsymbol{v}_\mu = \beta(\lambda, \mu)\boldsymbol{v}_{\lambda+\mu}$.

Remark 2.1. The algebra $\mathbb{C}_\beta[L]$ is isomorphic to the quotient $\mathbb{C}[\hat{L}]/\langle \kappa + 1\rangle$, where $\hat{L}$ is the unique (up to isomorphism) central extension of L by $\langle \kappa\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ such that

$$aa' = \kappa^{\langle \bar{a}, \bar{a}'\rangle + \langle \bar{a}, \bar{a}\rangle\langle \bar{a}', \bar{a}'\rangle} a'a, \tag{2-2}$$

for $a, a' \in \hat{L}$ lying above $\bar{a}, \bar{a}' \in L$, respectively. See [Frenkel et al. 1988].

Now define an $\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+$-module structure on $\mathbb{C}_\beta[L]$ by setting $\boldsymbol{c}\boldsymbol{v}_\lambda = \boldsymbol{v}_\lambda$ and $u(m)\boldsymbol{v}_\lambda = \delta_{m,0}\langle u, \lambda\rangle \boldsymbol{v}_\lambda$ for $u \in \mathfrak{h}$ and $\lambda \in L$, and define V_L to be the induced $\hat{\mathfrak{h}}$-module,

$$V_L := U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+)} \mathbb{C}_\beta[L]. \tag{2-3}$$

Then, according to §5.4.2 of [Frenkel and Ben-Zvi 2004], for example, V_L admits a unique super vertex algebra structure $Y : V_L \to (\mathrm{End}\, V_L)[\![z, z^{-1}]\!]$ such that $1 \otimes \boldsymbol{v}_0$ is the vacuum vector,

$$Y(u(-1) \otimes \boldsymbol{v}_0, z) = \sum_{n\in\mathbb{Z}} u(n) z^{-n-1} \tag{2-4}$$

for $u \in \mathfrak{h}$, and

$$Y(1 \otimes \boldsymbol{v}_\lambda, z) = \exp\biggl(-\sum_{n<0} \frac{\lambda(n)}{n} z^{-n}\biggr) \exp\biggl(-\sum_{n>0} \frac{\lambda(n)}{n} z^{-n}\biggr) \boldsymbol{v}_\lambda z^{\lambda(0)} \tag{2-5}$$

for $\lambda \in L$. Here $\boldsymbol{v}_\lambda$ denotes the operator $v \otimes \boldsymbol{v}_\mu \mapsto \beta(\lambda, \mu) v \otimes \boldsymbol{v}_{\lambda+\mu}$, and $z^{\lambda(0)}(v \otimes \boldsymbol{v}_\mu)$ is $(v \otimes \boldsymbol{v}_\mu) z^{\langle\lambda,\mu\rangle}$. Note that we have

$$V_L \simeq S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L] \tag{2-6}$$

as modules for $\hat{\mathfrak{h}}^- \oplus \hat{\mathfrak{h}}^0$.

Given that $\{\varepsilon_i\}$ is a basis for L, choose $\varepsilon'_i \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\langle \varepsilon'_i, \varepsilon_j \rangle = \delta_{i,j}$, and define

$$\omega := \frac{1}{2} \sum_{i=1}^{3} \varepsilon'_i(-1)\varepsilon_i(-1) \otimes \boldsymbol{v}_0. \tag{2-7}$$

Then ω is a conformal element for V_L with central charge equal to the rank of L. If we define $L(n) \in \mathrm{End}\, V_L$ so that $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$, then we have $[L(0), v(n)] = -nv(n)$ and $1 \otimes \boldsymbol{v}_\lambda$ is an eigenvector for $L(0)$ with eigenvalue $\frac{1}{2}\langle\lambda, \lambda\rangle$. Note that we do not assume that the bilinear form on L is positive-definite. Vectors of nonpositive length in L give rise to infinite-dimensional eigenspaces for $L(0)$, so in general $(V_L, Y, \boldsymbol{v}_0, \omega_u)$ is a conformal super vertex algebra, but not a super vertex operator algebra.

Automorphisms of L can be lifted to automorphisms of V_L. Indeed, suppose given $g \in \mathrm{Aut}(L)$ and a function $\alpha : L \to \{\pm 1\}$ satisfying

$$\alpha(\lambda + \mu)\beta(\lambda, \mu) = \alpha(\lambda)\alpha(\mu)\beta(g\lambda, g\mu) \tag{2-8}$$

for $\lambda, \mu \in L$. Then we obtain an automorphism $\hat{g}$ of $\mathrm{Aut}(V_L)$ by setting

$$\hat{g}(v \otimes \boldsymbol{v}_\lambda) := \alpha(\lambda)(g \cdot v) \otimes \boldsymbol{v}_{g\lambda} \tag{2-9}$$

for $v \in S(\hat{\mathfrak{h}}^-)$ and $\lambda \in L$, where $g \cdot v$ denotes the natural extension of the action of $\mathrm{Aut}(L)$ on $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ to $S(\hat{\mathfrak{h}}^-)$, determined by $g \cdot u(m) = (gu)(m)$ for $u \in \mathfrak{h}$.

2B. ***Lattice vertex algebra modules.*** Let γ be an element of the dual lattice

$$L^* := \{\lambda \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle \lambda, L \rangle \subset \mathbb{Z}\}.$$

Define $\mathbb{C}_\beta[L+\gamma]$ to be the complex vector space generated by symbols $\boldsymbol{v}_{\mu+\gamma}$ for $\mu \in L$, regarded as a $\mathbb{C}_\beta[L]$-module according to the rule $\boldsymbol{v}_\lambda \cdot \boldsymbol{v}_{\mu+\gamma} = \beta(\lambda, \mu)\boldsymbol{v}_{\lambda+\mu+\gamma}$. Equip $\mathbb{C}_\beta[L+\gamma]$ with a $U(\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+)$-module structure much as before, by letting $\boldsymbol{c}\boldsymbol{v}_{\mu+\gamma} = \boldsymbol{v}_{\mu+\gamma}$ and $u(m)\boldsymbol{v}_{\mu+\gamma} = \delta_{m,0}\langle u, \mu+\gamma\rangle \boldsymbol{v}_{\mu+\gamma}$ for $u \in \mathfrak{h}$ and $\mu \in L$. Let $V_{L+\gamma}$ be the $\hat{\mathfrak{h}}$-module defined by setting $V_{L+\gamma} := U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+)} \mathbb{C}_\beta[L+\gamma]$. Then we have an isomorphism

$$V_{L+\gamma} \simeq S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L+\gamma] \tag{2-10}$$

of modules for $\hat{\mathfrak{h}}^-$. Define vertex operators $Y_\gamma : V_L \to (\operatorname{End} V_{L+\gamma})[\![z, z^{-1}]\!]$ using the same formulas as before, but interpret the operator $\boldsymbol{v}_\lambda$ in (2-5) as $\boldsymbol{v}_\lambda(v \otimes \boldsymbol{v}_{\mu+\gamma}) := \beta(\lambda, \mu) v \otimes \boldsymbol{v}_{\lambda+\mu+\gamma}$, according to the $\mathbb{C}_\beta[L]$-module structure on $\mathbb{C}_\beta[L+\gamma]$ prescribed above. Note that the construction of $V_{L+\gamma}$ depends upon the choice of coset representative $\gamma \in L^*$, so that $V_{L+\gamma}$ might be different from $V_{L+\gamma'}$, as a $\mathbb{C}_\beta[L]$-module, for example, even when $L+\gamma = L+\gamma'$, but different choices of coset representative are guaranteed to define isomorphic V_L-modules according to [Dong 1993].

The construction just described may be generalized so as to realize certain twisted modules for V_L. We give a brief description here, and refer to §3 of [Dong and Mason 1994] for more details.

Choose a vector $h \in \mathfrak{h}$. Then for $v \in S(\hat{\mathfrak{h}}^-)$ and $\lambda \in L$ we have $h(0)v \otimes \boldsymbol{v}_\lambda = \langle h, \lambda\rangle v \otimes \boldsymbol{v}_\lambda$. So if h is chosen to lie in $L \otimes_{\mathbb{Z}} \mathbb{Q}$, then

$$g_h := e^{2\pi i h(0)} \tag{2-11}$$

is a finite order automorphism of V_L, which acts as multiplication by $e^{2\pi i \langle h, \lambda\rangle}$ on the vector $v \otimes \boldsymbol{v}_\lambda$. The kernel of the map $L \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Aut}(V_L)$ given by $h \mapsto g_h$ is exactly L^*, so $(L \otimes_{\mathbb{Z}} \mathbb{Q})/L^*$ is naturally a group of automorphisms of V_L. We may construct all the corresponding twisted modules for V_L explicitly.

To do this choose an h in $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $\mathbb{C}[L+h]$ be the complex vector space generated by symbols $\boldsymbol{v}_{\lambda+h}$ for $\lambda \in L$. Just as before, we define a $U(\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+)$-module structure on $\mathbb{C}[L+h]$ by setting $\boldsymbol{c}\boldsymbol{v}_\mu = \boldsymbol{v}_\mu$ and $u(m)\boldsymbol{v}_\mu = \delta_{m,0}\langle u, \mu\rangle \boldsymbol{v}_\mu$ for $u \in \mathfrak{h}$ and $\mu \in L+h$. Define also the $\hat{\mathfrak{h}}$-module $V_{L+h} := U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^+)} \mathbb{C}[L+h]$, so that we have an isomorphism

$$V_{L+h} \simeq S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L+h] \tag{2-12}$$

of modules for $\hat{\mathfrak{h}}^-$. Taking M to be a positive integer such that $Mh \in L^*$, define vertex operators $Y_h : V_L \to (\operatorname{End} V_{L+h})[\![z^{1/M}, z^{-1/M}]\!]$ using the same formulas as before, but interpret the operator $\boldsymbol{v}_\lambda$ in (2-5) as $\boldsymbol{v}_\lambda(v \otimes \boldsymbol{v}_{\mu+h}) = \beta(\lambda, \mu) v \otimes \boldsymbol{v}_{\lambda+\mu+h}$. Then $V_{L+h} = (V_{L+h}, Y_h)$ is an irreducible g_h-twisted module for V_L, and any g_h-twisted module for V_L is of the form $V_{L+h'}$ for some $h' \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ that is congruent to h modulo L^*.

Note that the action of $L\otimes_{\mathbb{Z}}\mathbb{Q}$ on V_L, given by $h\mapsto g_h$, extends to the $g_{h'}$-twisted module $V_{L+h'}$, for $h'\in L\otimes_{\mathbb{Z}}\mathbb{Q}$. For given $h, h'\in L\otimes_{\mathbb{Z}}\mathbb{Q}$, we may define

$$g_h(v\otimes \boldsymbol{v}_{\lambda+h'}) := e^{2\pi i\langle h,\lambda\rangle}(v\otimes \boldsymbol{v}_{\lambda+h'}) \tag{2-13}$$

for $v\in S(\hat{\mathfrak{h}}^-)$ and $\lambda\in L$. Then we have $g_hY_{h'}(v,z)v' = Y_{h'}(g_hv,z)g_hv'$ for $v\in V_L$ and $v'\in V_{L+h'}$.

2C. ***Clifford module vertex algebra.*** We also require the standard procedure which attaches a Clifford module super vertex operator algebra to a vector space equipped with a symmetric bilinear form; see [Feingold et al. 1991] for a general treatment, and [Feingold et al. 1996] for the special, one-dimensional case we consider here.

So let $\mathfrak{p}$ be a one-dimensional complex vector space equipped with a nondegenerate symmetric bilinear form $\langle\cdot\,,\cdot\rangle$. Set $\hat{\mathfrak{p}}=\mathfrak{p}[t,t^{-1}]t^{1/2}$ and write $a(r)$ for $a\otimes t^r$. Extend the bilinear form from $\mathfrak{p}$ to $\hat{\mathfrak{p}}$ by requiring that $\langle a(r), b(s)\rangle = \langle a,b\rangle\delta_{r+s,0}$. Set $\hat{\mathfrak{p}}^{\pm}=\mathfrak{p}[t^{\pm1}]t^{\pm1/2}$, write $\langle\hat{\mathfrak{p}}^{\pm}\rangle$ for the subalgebra of $\mathrm{Cliff}(\hat{\mathfrak{p}})$ generated by $\hat{\mathfrak{p}}^{\pm}$, and define a one-dimensional $\langle\hat{\mathfrak{p}}^{+}\rangle$-module $\mathbb{C}\boldsymbol{v}$ by requiring that $\mathbf{1}\boldsymbol{v}=\boldsymbol{v}$ and $a(r)\boldsymbol{v}=0$ for $a\in\mathfrak{p}$ and $r>0$. Here $\mathrm{Cliff}(\hat{\mathfrak{p}})$ denotes the *Clifford algebra* attached to $\hat{\mathfrak{p}}$, which we take to be the quotient of the tensor algebra $T(\hat{\mathfrak{p}})=\mathbb{C}\mathbf{1}\oplus\hat{\mathfrak{p}}\oplus\hat{\mathfrak{p}}^{\otimes 2}\oplus\cdots$ by the ideal generated by expressions of the form $u\otimes u+\frac{1}{2}\langle u,u\rangle\mathbf{1}$ for $u\in\hat{\mathfrak{p}}$.

Observe that the induced $\mathrm{Cliff}(\hat{\mathfrak{p}})$-module, $A(\mathfrak{p})=\mathrm{Cliff}(\hat{\mathfrak{p}})\otimes_{\langle\hat{\mathfrak{p}}^+\rangle}\mathbb{C}\boldsymbol{v}$, is isomorphic to $\bigwedge(\hat{\mathfrak{p}}^-)\boldsymbol{v}$ as a $\langle\hat{\mathfrak{p}}^-\rangle$-module. We obtain a super vertex algebra structure on $A(\mathfrak{p})$ by setting

$$Y\big(a\big(-\tfrac{1}{2}\big)\boldsymbol{v},z\big)=\sum_{n\in\mathbb{Z}}a\big(n+\tfrac{1}{2}\big)z^{-n-1} \tag{2-14}$$

for $a\in\mathfrak{p}$, for the reconstruction theorem of [Frenkel and Ben-Zvi 2004] ensures that this rule extends uniquely to a super vertex algebra structure $Y: A(\mathfrak{p})\otimes A(\mathfrak{p})\to A(\mathfrak{p})((z))$ with $Y(\boldsymbol{v},z)=\mathrm{Id}$.

Let $p\in\mathfrak{p}$ such that $\langle p,p\rangle=-2$. We obtain a super vertex operator algebra structure, with central charge $c=\frac{1}{2}$, by taking

$$\omega=\tfrac{1}{4}p\big(-\tfrac{3}{2}\big)p\big(-\tfrac{1}{2}\big)\boldsymbol{v} \tag{2-15}$$

to be the conformal element.

To construct canonically twisted modules for $A(\mathfrak{p})$ set $\hat{\mathfrak{p}}_{\mathrm{tw}}=\mathfrak{p}[t,t^{-1}]$ and extend the bilinear form from $\mathfrak{p}$ to $\hat{\mathfrak{p}}_{\mathrm{tw}}$ as before, by requiring that $\langle a(r),b(s)\rangle=\langle a,b\rangle\delta_{r+s,0}$. Set $\hat{\mathfrak{p}}_{\mathrm{tw}}^{>}=\mathfrak{p}[t]t$ and $\hat{\mathfrak{p}}_{\mathrm{tw}}^{\leq}=\mathfrak{p}[t^{-1}]$, and define a one-dimensional $\langle\hat{\mathfrak{p}}_{\mathrm{tw}}^{>}\rangle$-module $\mathbb{C}\boldsymbol{v}_{\mathrm{tw}}$ by requiring, much as before, that $\mathbf{1}\boldsymbol{v}_{\mathrm{tw}}=\boldsymbol{v}_{\mathrm{tw}}$ and $a(r)\boldsymbol{v}_{\mathrm{tw}}=0$ for $a\in\mathfrak{p}$ and $r>0$. Then for the induced $\mathrm{Cliff}(\hat{\mathfrak{p}}_{\mathrm{tw}})$-module

$$A(\mathfrak{p})_{\mathrm{tw}}:=\mathrm{Cliff}(\hat{\mathfrak{p}}_{\mathrm{tw}})\otimes_{\langle\hat{\mathfrak{p}}_{\mathrm{tw}}^{>}\rangle}\mathbb{C}\boldsymbol{v}_{\mathrm{tw}}, \tag{2-16}$$

there is a unique linear map $Y_{\mathrm{tw}} : A(\mathfrak{p}) \otimes A(\mathfrak{p})_{\mathrm{tw}} \to A(\mathfrak{p})_{\mathrm{tw}}((z^{1/2}))$ such that

$$Y_{\mathrm{tw}}\big(u\big(-\tfrac{1}{2}\big)\boldsymbol{v}, z\big) = \sum_{n\in\mathbb{Z}} u(n) z^{-n-1/2} \tag{2-17}$$

for $u \in \mathfrak{p}$, and $(A(\mathfrak{p})_{\mathrm{tw}}, Y_{\mathrm{tw}})$ is a canonically twisted module for $A(\mathfrak{p})$. One may use a suitably modified formulation of the reconstruction theorem of [Frenkel and Ben-Zvi 2004] to see this; see [Frenkel and Szczesny 2004]. We refer to [Feingold et al. 1996] for a concrete and detailed description of Y_{tw}. Note that $A(\mathfrak{p})$ is isomorphic to $\bigwedge(\hat{\mathfrak{p}}_{\mathrm{tw}}^{\leq})\boldsymbol{v}$ as a $\langle\hat{\mathfrak{p}}_{\mathrm{tw}}^{\leq}\rangle$-module.

With $p \in \mathfrak{p}$ as above, such that $\langle p, p\rangle = -2$, we have $p(0)^2 = \mathbf{1}$ in $\mathrm{Cliff}(\mathfrak{p})$. Set

$$\boldsymbol{v}_{\mathrm{tw}}^{\pm} := (\mathbf{1} \pm p(0))\boldsymbol{v}_{\mathrm{tw}}, \tag{2-18}$$

so that $p(0)\boldsymbol{v}_{\mathrm{tw}}^{\pm} = \pm\boldsymbol{v}_{\mathrm{tw}}^{\pm}$. Then $A(\mathfrak{p})_{\mathrm{tw}} = A(\mathfrak{p})_{\mathrm{tw}}^{+} \oplus A(\mathfrak{p})_{\mathrm{tw}}^{-}$ is a decomposition of $A(\mathfrak{p})_{\mathrm{tw}}$ into irreducible canonically twisted $A(\mathfrak{p})$-modules, where $A(\mathfrak{p})_{\mathrm{tw}}^{\pm}$ denotes the submodule of $A(\mathfrak{p})_{\mathrm{tw}}$ generated by $\boldsymbol{v}_{\mathrm{tw}}^{\pm}$:

$$A(\mathfrak{p})_{\mathrm{tw}}^{\pm} := \mathrm{Cliff}(\hat{\mathfrak{p}}_{\mathrm{tw}}) \otimes_{\langle\hat{\mathfrak{p}}_{\mathrm{tw}}^{>}\rangle} \mathbb{C}\boldsymbol{v}_{\mathrm{tw}}^{\pm}. \tag{2-19}$$

From (2-17) we see that the $L(0)$-degree preserving component of $Y_{\mathrm{tw}}\big(p\big(-\tfrac{1}{2}\big)\boldsymbol{v}, z\big)$ is $p(0)$. Computing the graded-trace of $p(0)$ on $A(\mathfrak{p})_{\mathrm{tw}}^{\pm}$, we find

$$\mathrm{tr}_{A(\mathfrak{p})_{\mathrm{tw}}^{\pm}}\, p(0) q^{L(0)-c/24} = \pm q^{1/24} \prod_{n>0} (1-q^n), \tag{2-20}$$

where the factor $q^{1/24}$ appears because $L(0)\boldsymbol{v}_{\mathrm{tw}}^{\pm} = \frac{1}{16}\boldsymbol{v}_{\mathrm{tw}}^{\pm}$ and $c = \frac{1}{2}$.

2D. ***Main construction.*** We now take $L = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + \mathbb{Z}\varepsilon_3$ to be the rank 3 lattice with bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ determined by

$$\langle\varepsilon_i, \varepsilon_j\rangle = 2 - \delta_{i,j}. \tag{2-21}$$

This is an integral, noneven lattice with signature $(1, 2)$. Set $\rho := \frac{1}{5}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ and observe that

$$\langle\lambda, \rho\rangle = k + l + m \tag{2-22}$$

for $\lambda = k\varepsilon_1 + l\varepsilon_2 + m\varepsilon_3$, so ρ belongs to the dual L^* of L. In fact, L^*/L is cyclic of order 5, and $\rho + L$ is a generator. If we set

$$L^j := \{\lambda \in L \mid \langle\lambda, \rho\rangle = j \pmod 2\}, \tag{2-23}$$

then $L = L^0 \cup L^1$ is the decomposition of L into its even and odd parts, by which we mean that $\langle\lambda, \lambda\rangle$ is even or odd according as λ lies in L^0 or L^1.

Let V_L be the super vertex operator algebra attached to L via the construction of Section 2A, where the bilinear function $b : L \times L \to \mathbb{Z}/2\mathbb{Z}$ is determined by setting

$$b(\varepsilon_i, \varepsilon_j) := \begin{cases} 0 & \text{when } i \le j, \\ 1 & \text{when } i > j. \end{cases} \tag{2-24}$$

There is an obvious action of the symmetric group S_3 on L, by permutations of the basis vectors ε_i. We lift this action to V_L in the following way. Recall from Section 2A that a lift $\hat{g} \in \operatorname{Aut}(V_L)$ of an automorphism $g \in \operatorname{Aut}(L)$ is determined by a choice of function $\alpha : L \to \{\pm 1\}$ satisfying (2-8). Taking $\mu = k\lambda$ in (2-8) we have $\alpha((k+1)\lambda) = \alpha(\lambda)\alpha(k\lambda)\beta(\lambda, \lambda)^k \beta(g\lambda, g\lambda)^k$, since β is bimultiplicative. So given (2-24) we see that $\beta(\lambda, \lambda) = k_1k_2 + k_2k_3 + k_3k_1$ for $\lambda = k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3$, which is invariant under the action of S_3. So actually $\beta(\lambda, \lambda) = \beta(g\lambda, g\lambda)$, and thus we may assume $\alpha(k\lambda) = \alpha(\lambda)^k$ in (2-8) for $\lambda \in L$ and k a positive integer, when g acts by permuting the ε_i. Observe also that for $\lambda, \mu, \nu \in L$ we have

$$\begin{aligned} &\alpha(\lambda + \mu + \nu)\beta(\lambda, \mu)\beta(\mu, \nu)\beta(\nu, \lambda) \\ &\qquad = \alpha(\lambda)\alpha(\mu)\alpha(\nu)\beta(g\lambda, g\mu)\beta(g\mu, g\nu)\beta(g\nu, g\lambda) \end{aligned} \tag{2-25}$$

according to (2-8), which specializes to

$$\begin{aligned} &\alpha(\lambda)\beta(\varepsilon_1, \varepsilon_2)^{k_1k_2}\beta(\varepsilon_2, \varepsilon_3)^{k_2k_3}\beta(\varepsilon_3, \varepsilon_1)^{k_3k_1} \\ &\quad = \alpha(\varepsilon_1)^{k_1}\alpha(\varepsilon_2)^{k_2}\alpha(\varepsilon_3)^{k_3}\beta(g\varepsilon_1, g\varepsilon_2)^{k_1k_2}\beta(g\varepsilon_2, g\varepsilon_3)^{k_2k_3}\beta(g\varepsilon_3, g\varepsilon_1)^{k_3k_1} \end{aligned} \tag{2-26}$$

for $\lambda = k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3$.

Consider the case that $g = \sigma$ is the cyclic permutation (123). From (2-26) we see that we may lift σ to $\operatorname{Aut}(V_L)$ by taking $\alpha(\varepsilon_i) = 1$ for $i \in \{1, 2, 3\}$, and more generally $\alpha(k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3) = (-1)^{k_2k_3 + k_3k_1}$, in the construction of Section 2A. We denote the corresponding automorphism of V_L by $\hat{\sigma}$. In the notation of (2-9) we have

$$\hat{\sigma}(v \otimes \boldsymbol{v}_{k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3}) := (-1)^{k_2k_3 + k_3k_1}(\sigma \cdot v) \otimes \boldsymbol{v}_{k_3\varepsilon_1 + k_1\varepsilon_2 + k_2\varepsilon_3}. \tag{2-27}$$

Next consider $g = \tau := (12)$. Applying (2-26) we see that we may lift τ to $\operatorname{Aut}(V_L)$ by taking $\alpha(\varepsilon_i) = 1$ as before, and more generally $\alpha(k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3) = (-1)^{k_1k_2}$. We denote the corresponding automorphism of V_L by $\hat{\tau}$, so that

$$\hat{\tau}(v \otimes \boldsymbol{v}_{k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3}) := (-1)^{k_1k_2}(\tau \cdot v) \otimes \boldsymbol{v}_{k_2\varepsilon_1 + k_1\varepsilon_2 + k_3\varepsilon_3}. \tag{2-28}$$

Using (2-27) and (2-28) one can check that $\hat{\sigma}^3 = \hat{\tau}^2 = (\hat{\tau}\hat{\sigma})^2 = \mathrm{Id}$ in $\operatorname{Aut}(V_L)$, so $\hat{\sigma}$ and $\hat{\tau}$ generate a group

$$\hat{G} := \langle \hat{\sigma}, \hat{\tau} \rangle \simeq S_3 \tag{2-29}$$

in $\operatorname{Aut}(V_L)$.

Note that $V_L = V_{L^0} \oplus V_{L^1}$ is the decomposition of V_L into its even and odd parity subspaces, where L^j is defined by (2-23). Thus the canonical involution of V_L, acting as $+1$ on the even subspace V_{L^0} and -1 on the odd subspace V_{L^1}, is realized by $g_{\rho/2}$ in the notation of (2-11). So the canonically twisted modules for V_L are exactly the $V_{L+a\rho/2}$, for $a \in \{1, 3, 5, 7, 9\}$ (see Section 2B).

The prescription (2-13) furnishes an extension of the action of the canonical involution $g_{\rho/2}$ from V_L to $V_{L+a\rho/2}$. Since ρ is S_3-invariant we may also extend the actions of $\hat{\sigma}$ and $\hat{\tau}$ to $V_{L+a\rho/2}$ by setting

$$\begin{aligned}\hat{\sigma}(v \otimes \boldsymbol{v}_{\lambda+a\rho/2}) &:= (-1)^{k_2k_3+k_3k_1}(\sigma \cdot v) \otimes \boldsymbol{v}_{\sigma\lambda+a\rho/2},\\ \hat{\tau}(v \otimes \boldsymbol{v}_{\lambda+a\rho_2}) &:= (-1)^{k_1k_2+(a-1)/2}(\tau \cdot v) \otimes \boldsymbol{v}_{\tau\lambda+a\rho/2},\end{aligned} \tag{2-30}$$

for $v \in S(\hat{\mathfrak{h}}^-)$ and $\lambda = k_1\varepsilon_1 + k_2\varepsilon_2 + k_3\varepsilon_3$. We use these rules to equip the $V_{L+a\rho/2}$ with $\hat{G}$- and $\langle g_{\rho/2}\rangle$-module structures.

Now observe that if $K \subset L$ is closed under addition and contains 0 then the $\hat{\mathfrak{h}}$-submodule $V_K < V_L$ generated by the $\boldsymbol{v}_\lambda$ for $\lambda \in K$,

$$V_K \simeq S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[K], \tag{2-31}$$

is a sub-super vertex algebra of V_L, and the ω of (2-7) is a conformal element for V_K. Furthermore, if $K' \subset L + \frac{1}{2}a\rho$ satisfies $K + K' \subset K'$ then the restriction of the twisted vertex operators $u \otimes v \mapsto Y_{a\rho/2}(u, z^{1/2})v$ to $V_K \otimes V_{K'} < V_L \otimes V_{L+a\rho/2}$ equips $V_{K'}$ with a canonically twisted module structure over V_K (for a odd). To describe the choices of K and K' that are relevant to us, define P to be the monoid of nonnegative integer combinations of the ε_i,

$$P := \left\{ \sum_i k_i\varepsilon_i \in L \;\middle|\; k_i \geq 0,\ \forall i \right\}. \tag{2-32}$$

Then V_P is naturally a conformal super vertex algebra, and for a odd, $V_{P+a\rho/2}$ is a canonically twisted module for it. By our choice of basis $\{\varepsilon_i\}$, all nontrivial vectors of the monoid P have strictly positive norm. So the eigenspaces for the action of $L(0)$ on V_P are finite-dimensional, and the eigenvalues of $L(0)$ are contained in $\frac{1}{2}\mathbb{Z}$ and bounded from below. Thus V_P is actually a super vertex operator algebra.

Observe that the actions (2-30) restrict to $V_{P+a\rho/2}$ for any a, since P and ρ are invariant under coordinate permutations. The canonical involution $g_{\rho/2}$, acting on $V_{L+a\rho/2}$ according to (2-13), also preserves the subspace $V_{P+a\rho/2}$.

We now let V^X denote the tensor product super vertex operator algebra

$$V^X := A(\mathfrak{p}) \otimes V_P. \tag{2-33}$$

For a an odd integer we define $V^{\pm}_{\mathrm{tw},a}$ to be the canonically twisted V^X-module

$$V^{\pm}_{\mathrm{tw},a} := A(\mathfrak{p})^{\pm}_{\mathrm{tw}} \otimes V_{P+a\rho/2} \oplus A(\mathfrak{p})^{\mp}_{\mathrm{tw}} \otimes V_{P+(10-a)\rho/2}. \tag{2-34}$$

We extend the action of $\hat{G} \simeq S_3$ to V^X and $V^{\pm}_{\mathrm{tw},a}$ by letting $\hat{G}$ act trivially on the Clifford module factors, setting

$$\hat{\sigma}(u \otimes v) := u \otimes \hat{\sigma}(v), \qquad \hat{\tau}(u \otimes v) := u \otimes \hat{\tau}(v) \tag{2-35}$$

for $u \in A(\mathfrak{p})$ and $v \in V_P$, and for $u \in A(\mathfrak{p})^{\pm}_{\mathrm{tw}}$ and $v \in V_{P+a\rho/2}$.

Given $g \in \hat{G}$ and a an odd integer, we now define $T^{\pm}_{g,a}$ to be the trace of the operator $g g_{\rho/2} p(0) q^{L(0)-c/24}$ on the canonically twisted V^X-module $V^{\pm}_{\mathrm{tw},a}$,

$$T^{\pm}_{g,a} := \mathrm{tr}_{V^{\pm}_{\mathrm{tw},a}} g g_{\rho/2} p(0) q^{L(0)-c/24}. \tag{2-36}$$

Note that $c = \frac{7}{2}$ here. Also, the identities

$$T^{\pm}_{g,a} = -T^{\mp}_{g,a} = T^{\mp}_{g,10-a} \tag{2-37}$$

follow directly from (2-34) and (2-36). In particular, we have $T^{\pm}_{g,5} = 0$ for all g.

Recall that $(q;q)_\infty = \prod_{n>0}(1-q^n)$ (see (1-1)). Our concrete construction allows us to compute explicit formulas for the trace functions $T^{\pm}_{g,a}$.

Proposition 2.2. *The trace functions $T^{\pm}_{g,a}$ admit the following expressions*:

$$T^{\pm}_{e,a} = \pm \frac{q^{-1/12}}{(q;q)^2_\infty} \biggl(\sum_{k,l,m \geq 0} + \sum_{k,l,m<0} \biggr) (-1)^{k+l+m} \times q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+a(k+l+m)/2+3a^2/40}, \tag{2-38}$$

$$T^{\pm}_{\hat{\tau},a} = \pm(-1)^{(a-1)/2} \frac{q^{-1/12}}{(q^2;q^2)_\infty} \biggl(\sum_{k,m \geq 0} - \sum_{k,m<0} \biggr) (-1)^{k+m} \times q^{3k^2+m^2/2+4km+a(2k+m)/2+3a^2/40}, \tag{2-39}$$

$$T^{\pm}_{\hat{\sigma},a} = \pm q^{-1/12} \frac{(q;q)_\infty}{(q^3;q^3)_\infty} \sum_{k \in \mathbb{Z}} (-1)^k q^{15k^2/2+3ak/2+3a^2/40}. \tag{2-40}$$

Proof. First consider the case that $g = e$ is the identity. From the definition (2-36) of $T^{\pm}_{e,a}$ we derive

$$T^{\pm}_{e,a} = \pm \frac{1}{(q;q)^2_\infty} \sum_{\mu \in P+a\rho/2} (-1)^{\langle \mu - a\rho/2, \rho \rangle} q^{\langle \mu,\mu \rangle/2-1/12} \mp \frac{1}{(q;q)^2_\infty} \sum_{\mu \in P+(10-a)\rho/2} (-1)^{\langle \mu-(10-a)\rho/2,\rho \rangle} q^{\langle \mu,\mu \rangle/2-1/12}. \tag{2-41}$$

We replace μ with $k\varepsilon_1 + l\varepsilon_2 + m\varepsilon_3 + \frac{1}{2}a\rho$ in the first summation, where $k, l, m \geq 0$, and obtain

$$\pm\frac{q^{-1/12}}{(q;q)_\infty^2}\sum_{k,l,m\geq 0}(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+a(k+l+m)/2+3a^2/40} \tag{2-42}$$

using (2-21) and (2-22) and the fact that $\langle\rho,\rho\rangle = \frac{3}{5}$. For the second summation we replace $P + \frac{1}{2}(10-a)\rho$ with $-P - \frac{1}{2}(10-a)\rho = -P^+ + \frac{1}{2}a\rho$, where

$$P^+ := \left\{\sum_i k_i\varepsilon_i \in L \;\middle|\; k_i > 0,\ \forall i\right\}, \tag{2-43}$$

and replace μ with $-\mu$ in the summands. We obtain

$$\pm\frac{q^{-1/12}}{(q;q)_\infty^2}\sum_{k,l,m<0}(-1)^{k+l+m}q^{(k^2+l^2+m^2)/2+2(kl+lm+mk)+a(k+l+m)/2+3a^2/40}, \tag{2-44}$$

which together with (2-42) gives (2-38) as required.

Next take $g = \hat{\tau}$. Using (2-36) and the formula (2-30) we compute

$$\begin{aligned} T^{\pm}_{\hat{\tau},a} &= \pm(-1)^{(a-1)/2}\frac{1}{(q^2;q^2)_\infty} \\ &\quad\times\sum_{\substack{\mu\in P+a\rho/2\\ \tau\mu=\mu}}(-1)^{\langle\mu-a\rho/2,\rho+\varepsilon_1'\rangle}q^{\langle\mu,\mu\rangle/2-1/12} \mp (-1)^{(a-1)/2}\frac{1}{(q^2;q^2)_\infty} \\ &\quad\quad\times\sum_{\substack{\mu\in P+(10-a)\rho/2\\ \tau\mu=\mu}}(-1)^{\langle\mu-(10-a)\rho/2,\rho+\varepsilon_1'\rangle}q^{\langle\mu,\mu\rangle/2-1/12}. \end{aligned} \tag{2-45}$$

We now replace μ with $k\varepsilon_1 + k\varepsilon_2 + m\varepsilon_3 + \frac{1}{2}a\rho$ in the first summation, where $k, m \geq 0$, and obtain

$$\pm(-1)^{(a-1)/2}\frac{q^{-1/12}}{(q^2;q^2)_\infty}\sum_{k,m\geq 0}(-1)^{k+m}q^{3k^2+m^2/2+4km+a(2k+m)/2+3a^2/40}. \tag{2-46}$$

Note that the factor $(-1)^k$ in $(-1)^{k+m}$, corresponding to $(-1)^{\langle\mu-a\rho/2,\varepsilon_1'\rangle}$ in (2-45), arises from the factor $(-1)^{k_1k_2} = (-1)^{k^2} = (-1)^k$ in (2-30). For the second summation we proceed as before, replacing $P + \frac{1}{2}(10-a)\rho$ with $-P^+ + \frac{1}{2}a\rho$ and μ with $-\mu$, and obtain

$$\mp(-1)^{(a-1)/2}\frac{q^{-1/12}}{(q^2;q^2)_\infty}\sum_{k,m<0}(-1)^{k+m}q^{3k^2+m^2/2+4km+a(2k+m)/2+3a^2/40}. \tag{2-47}$$

This taken together with (2-46) gives the required expression (2-39).

Finally we consider $g=\hat{\sigma}$ (see (2-30)). Then the appropriate analogue of (2-41) and (2-45) is

$$T^{\pm}_{\hat{\sigma},a}=\pm\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\sum_{\substack{\mu\in P+a\rho/2\\ \sigma\mu=\mu}}(-1)^{\langle\mu-a\rho/2,\rho\rangle}q^{\langle\mu,\mu\rangle/2-1/12}$$
$$\mp\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\sum_{\substack{\mu\in P+(10-a)\rho/2\\ \sigma\mu=\mu}}(-1)^{\langle\mu-(10-a)\rho/2,\rho\rangle}q^{\langle\mu,\mu\rangle/2-1/12}. \quad (2\text{-}48)$$

We obtain (2-40) from (2-48) in much the same way as for $g=e$, so we omit the remaining details. □

3. Mock theta functions

In this section we consider the modular properties of the trace functions defined in Section 2D, computed explicitly in Proposition 2.2. We recall some basic facts about Maass forms in Section 3A, including their relationship to mock modular forms. We require some facts about theta series of cones in indefinite lattices due to Zwegers [2002], which we recall in Section 3B. The proof of our main result, Theorem 1.1, appears in Section 3C. In particular, we identify the umbral McKay–Thompson series attached to $X=E_8^3$ as trace functions arising from the action of G^X on canonically twisted modules for V^X in Section 3C.

3A. *Harmonic Maass forms.* Define the weight $\frac{1}{2}$ *Casimir operator* $\Omega_{\frac{1}{2}}$, a differential operator on smooth functions $H:\mathbb{H}\to\mathbb{C}$, by setting

$$(\Omega_{\frac{1}{2}}H)(\tau):=-4\Im(\tau)^2\frac{\partial^2 H}{\partial\tau\partial\bar{\tau}}(\tau)+i\Im(\tau)\frac{\partial H}{\partial\bar{\tau}}(\tau)+\frac{3}{16}H(\tau). \quad (3\text{-}1)$$

Note that $\Omega_{\frac{1}{2}}=\Delta_{\frac{1}{2}}+\frac{3}{16}$, where Δ_k is the hyperbolic Laplace operator in weight k.

Following [Bruinier and Funke 2004] (see also [Ono 2009; Zagier 2009]), a *harmonic weak Maass form* of weight $\frac{1}{2}$ for $\Gamma<\mathrm{SL}_2(\mathbb{Z})$ is defined to be a smooth function $H:\mathbb{H}\to\mathbb{C}$ that transforms as a (not necessarily holomorphic) modular form of weight $\frac{1}{2}$ for Γ, is an eigenfunction for $\Omega_{\frac{1}{2}}$ with eigenvalue $\frac{3}{16}$, and has at most exponential growth as τ approaches cusps of Γ.

Define $\beta(x)$ for $x\in\mathbb{R}_{\geq 0}$ by setting

$$\beta(x):=\int_x^\infty u^{-1/2}e^{-\pi u}\,\mathrm{d}u. \quad (3\text{-}2)$$

Note that β is related to the incomplete gamma function by $\sqrt{\pi}\beta(x)=\Gamma\big(\frac{1}{2},\pi x\big)$. If H is a harmonic weak Maass form of weight $\frac{1}{2}$ then we can canonically decompose

H into its *holomorphic* and *nonholomorphic* parts, $H = H^+ + H^-$, where

$$H^+(\tau) = \sum_{n \gg -\infty} c_H^+(n) q^n, \tag{3-3}$$

$$H^-(\tau) = 2i c_H^-(0) \sqrt{2\Im(\tau)} - i \sum_{n>0} c_H^-(n) \frac{1}{\sqrt{2n}} \beta(4n\Im(\tau)) q^{-n}, \tag{3-4}$$

for some uniquely determined values $c_H^{\pm}(n) \in \mathbb{C}$. (See §3 of [Bruinier and Funke 2004]. See also §5 of [Zagier 2009] and §7.1 of [Dabholkar et al. 2012].) Note that n should be allowed to range over rational values in (3-3) and (3-4).

We may define the *mock modular forms* of weight $\frac{1}{2}$ to be those holomorphic functions $H^+ : \mathbb{H} \to \mathbb{C}$ which arise as the holomorphic parts of harmonic weak Maass forms of weight $\frac{1}{2}$. For $H^{\pm}$ as above, the *shadow* of H^+ is defined, up to a choice of scaling factor C, by

$$g(\tau) := C \sqrt{2\Im(\tau)} \, \overline{\frac{\partial H^-}{\partial \bar{\tau}}} = C \sum_{n \geq 0} \overline{c_H^-(n)} q^n. \tag{3-5}$$

Then so long as $c_H^-(0) = 0$ (i.e., g is a cusp form), the function H^- is the *Eichler integral* of g,

$$H^-(\tau) = \frac{e\left(-\frac{1}{8}\right)}{C} \int_{-\bar{\tau}}^{\infty} \frac{\overline{g(-\bar{z})}}{\sqrt{z+\tau}} \, \mathrm{d}z. \tag{3-6}$$

In this setting, the weak harmonic Maass form $H = H^+ + H^-$ is called the *completion* of H^+.

Various choices for C can be found in the literature. In [Cheng et al. 2014b] we find $C = \sqrt{2m}$ in the case that $H = (H_r)$ is a $2m$-vector-valued Maass form for some $\Gamma_0(N)$, such that

$$(H \cdot \theta)(\tau, z) := \sum_r H_r(\tau) \theta_{m,r}(\tau, z) \tag{3-7}$$

transforms like a (not necessarily holomorphic in τ) Jacobi form of weight 1 and index m for $\Gamma_0(N)$, where

$$\theta_{m,r}(\tau, z) := \sum_{k \in \mathbb{Z}} q^{(2km+r)^2/4m} e^{2\pi i z(2km+r)}. \tag{3-8}$$

The cases of relevance to us here all have $m = 30$, so we take $C = \sqrt{60}$ henceforth in (3-5) and (3-6). All the shadows arising in this work are linear combinations of the unary theta functions

$$S_{m,r}(\tau) := \frac{1}{2\pi i} \frac{\partial}{\partial z} \theta_{m,r}(\tau, z) \Big|_{z=0} = \sum_{k \in \mathbb{Z}} (2km+r) q^{(2km+r)^2/4m}, \tag{3-9}$$

where $m = 30$ and $r \neq 0 \pmod{30}$. In particular, we do not encounter any examples for which the shadow g (see (3-5)) is not a cusp form.

3B. ***Indefinite theta series.*** Even though our main construction uses a lattice of signature $(1, 2)$, it will develop in Section 3C that the trace functions (2-38) and (2-39) can be analyzed in terms of theta series of indefinite lattices with signature $(1, 1)$, whereas (2-40) is essentially a theta series with rank 1 and consequently can be handled by classical methods. With this in mind we now recall some results from [Zwegers 2002] on theta series for lattices of signature $(r-1, 1)$, restricting to the case that $r = 2$.

Given a symmetric 2×2 matrix A, we define a quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$, by setting

$$Q(\boldsymbol{x}) := \tfrac{1}{2}(\boldsymbol{x}, A\boldsymbol{x}), \tag{3-10}$$

where $(\,\cdot\,,\cdot\,)$ denotes the usual Euclidean inner product on $\mathbb{R}^2$. The associated bilinear form is

$$B(\boldsymbol{x}, \boldsymbol{y}) := (\boldsymbol{x}, A\boldsymbol{y}) = Q(\boldsymbol{x}+\boldsymbol{y}) - Q(\boldsymbol{x}) - Q(\boldsymbol{y}). \tag{3-11}$$

Henceforth assume that A has signature $(1, 1)$. Then the set of vectors $\boldsymbol{c} \in \mathbb{R}^2$ with $Q(\boldsymbol{c}) < 0$ is nonempty and has two components. Let C_Q be one of these components. Two vectors $\boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)}$ belong to the same component if $B(\boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)}) < 0$. Thus, picking a vector $\boldsymbol{c}_0$ in C_Q, we may identify

$$C_Q = \{\boldsymbol{c} \in \mathbb{R}^2 \mid Q(\boldsymbol{c}) < 0,\ B(\boldsymbol{c}, \boldsymbol{c}_0) < 0\}. \tag{3-12}$$

Zwegers also defines a set of representatives of *cusps*,

$$S_Q := \{\boldsymbol{c} \in \mathbb{Z}^2 \mid \boldsymbol{c} \text{ primitive},\ Q(\boldsymbol{c}) = 0,\ B(\boldsymbol{c}, \boldsymbol{c}_0) < 0\}. \tag{3-13}$$

Define the *indefinite theta function* with characteristics $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$, with respect to $\boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)} \in C_Q$, by setting

$$\begin{aligned}\vartheta_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{c}^{(1)},\boldsymbol{c}^{(2)}}(\tau) := \sum_{\nu \in \boldsymbol{a}+\mathbb{Z}^2} \Biggl(& E\biggl(\frac{B(\boldsymbol{c}^{(1)}, \nu)}{\sqrt{-Q(\boldsymbol{c}^{(1)})}}\sqrt{\Im(\tau)}\biggr) \\ & - E\biggl(\frac{B(\boldsymbol{c}^{(2)}, \nu)}{\sqrt{-Q(\boldsymbol{c}^{(2)})}}\sqrt{\Im(\tau)}\biggr)\Biggr) q^{Q(\nu)} e^{2\pi i B(\nu, \boldsymbol{b})},\end{aligned} \tag{3-14}$$

where $E(z) := \operatorname{sgn}(z)(1 - \beta(z^2))$. Corollary 2.9 of [Zwegers 2002] (see also Theorem 3.1 of [Zagier 2009]) shows that $\vartheta_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{c}^{(1)},\boldsymbol{c}^{(2)}}(\tau)$ is a nonholomorphic modular form of weight 1.

Presently we will see that these indefinite theta functions can be used to define harmonic Maass forms whose nonholomorphic parts can be written in terms of the functions

$$R_{a,b}(\tau) := \sum_{\nu \in a+\mathbb{Z}} \operatorname{sgn}(\nu)\beta(2\nu^2\Im(\tau))q^{-\nu^2/2}e^{-2\pi i \nu b}. \tag{3-15}$$

Note that the $R_{a,b}$ are Eichler integrals (see (3-6)) of unary theta functions of weight $\frac{3}{2}$. Indeed, we have

$$R_{a,b}(\tau) = e\left(-\tfrac{1}{8}\right) \int_{-\bar{\tau}}^{i\infty} \frac{g_{a,-b}(z)}{\sqrt{z+\tau}}\,\mathrm{d}z, \tag{3-16}$$

for

$$g_{a,b}(\tau) := \sum_{\nu \in a+\mathbb{Z}} \nu q^{\nu^2/2} e^{2\pi i \nu b}. \tag{3-17}$$

Note that $\overline{g_{a,b}(-\bar{z})} = g_{a,-b}(z)$. Observe also that

$$g_{r/2m,0}(m\tau) = \frac{1}{2m} S_{m,r}(\tau) \tag{3-18}$$

(see (3-9)), which is useful for comparing the results of [Zwegers 2002] to those of [Cheng et al. 2014b].

Define $\langle \boldsymbol{c} \rangle_{\mathbb{Z}}^{\perp} := \{\xi \in \mathbb{Z}^r \mid B(\boldsymbol{c}, \xi) = 0\}$. For future use we quote the $r = 2$ case of Proposition 4.3 from [Zwegers 2002].

Proposition 3.1 (Zwegers). *Let $\boldsymbol{c} \in C_Q \cap \mathbb{Z}^2$ be primitive. Let $P_0 \subset \mathbb{R}^2$ be any finite set such that*

$$\left\{\mu \in \boldsymbol{a} + \mathbb{Z}^2 \;\middle|\; 0 \le \frac{B(\boldsymbol{c}, \mu)}{2Q(\boldsymbol{c})} < 1\right\} = \bigsqcup_{\mu_0 \in P_0} \left(\mu_0 + \langle \boldsymbol{c} \rangle_{\mathbb{Z}}^{\perp}\right). \tag{3-19}$$

Then we have

$$\begin{aligned} &\sum_{\nu \in \boldsymbol{a}+\mathbb{Z}^2} \operatorname{sgn}(B(\boldsymbol{c}, \nu))\beta\left(-\frac{B(\boldsymbol{c}, \nu)^2}{Q(\boldsymbol{c})}\Im(\tau)\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu, \boldsymbol{b})} \\ &\quad = -\sum_{\mu_0 \in P_0} R_{B(\boldsymbol{c},\mu_0)/2Q(\boldsymbol{c}),B(\boldsymbol{c},\boldsymbol{b})}(-2Q(\boldsymbol{c})\tau) \cdot \sum_{\xi \in \mu_0^{\perp} + \langle \boldsymbol{c} \rangle_{\mathbb{Z}}^{\perp}} e^{2\pi i Q(\xi)\tau + 2\pi i B(\xi, \boldsymbol{b}^{\perp})}, \end{aligned} \tag{3-20}$$

where $\mu_0^{\perp} = \mu_0 - \dfrac{B(\boldsymbol{c}, \mu_0)}{2Q(\boldsymbol{c})}\boldsymbol{c}$ and $\boldsymbol{b}^{\perp} = \boldsymbol{b} - \dfrac{B(\boldsymbol{c}, \boldsymbol{b})}{2Q(\boldsymbol{c})}\boldsymbol{c}$.

Note that the term

$$\sum_{\xi \in \mu_0^{\perp} + \langle \boldsymbol{c} \rangle_{\mathbb{Z}}^{\perp}} e^{2\pi i Q(\xi)\tau + 2\pi i B(\xi, \boldsymbol{b}^{\perp})} \tag{3-21}$$

is a classical (positive-definite) theta function of weight $\frac{1}{2}$.

The indefinite theta function construction (3-14) is applied in [Zwegers 2002] to mock theta functions of Ramanujan (other than χ_0 and χ_1, which are treated in [Zwegers 2009]). Amongst those appearing are the four functions F_0, F_1, ϕ_0 and

ϕ_1, where ϕ_0 and ϕ_1 are defined in (1-9), and

$$\begin{aligned} F_0(q) &:= \sum_{n\geq 0} \frac{q^{2n^2}}{(q;q^2)_n}, \\ F_1(q) &:= \sum_{n\geq 0} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}}. \end{aligned} \tag{3-22}$$

These are amongst the fifth-order mock theta functions introduced by Ramanujan in his last letter to Hardy.

To study these functions, 6-vector-valued mock modular forms

$$F_{5,1}(\tau) = (F_{5,1,r}(\tau)), \quad F_{5,2}(\tau) = (F_{5,2,r}(\tau)) \tag{3-23}$$

are introduced on pages 74 and 79, respectively, of [Zwegers 2002]. Inspecting their definitions, and substituting 2τ for τ, we find that

$$F_{5,1,3}(2\tau) = q^{-1/120}(F_0(q)-1), \qquad F_{5,2,3}(2\tau) = q^{-1/120}\phi_0(-q), \tag{3-24}$$

$$F_{5,1,4}(2\tau) = q^{71/120}F_1(q), \qquad F_{5,2,4}(2\tau) = -q^{-49/120}\phi_1(-q). \tag{3-25}$$

The content of Proposition 4.10 of [Zwegers 2002] is that

$$H_{5,1}(\tau) = F_{5,1}(\tau) - G_{5,1}(\tau), \tag{3-26}$$

where the vector-valued functions $H_{5,1}$ and $G_{5,1}$ are such that the components of $2\eta(\tau)H_{5,1}(\tau)$ are nonholomorphic indefinite theta functions of the form $\vartheta_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{c}^{(1)},\boldsymbol{c}^{(2)}}(\tau)$ (see (3-14)), and the third and fourth components of $G_{5,1}$ satisfy

$$G_{5,1,3}(2\tau) = -\tfrac{1}{2}\big(R_{\frac{19}{60},0} + R_{\frac{29}{60},0} - R_{\frac{49}{60},0} - R_{\frac{59}{60},0}\big)(60\tau), \tag{3-27}$$

$$G_{5,1,4}(2\tau) = -\tfrac{1}{2}\big(R_{\frac{13}{60},0} + R_{\frac{23}{60},0} - R_{\frac{43}{60},0} - R_{\frac{53}{60},0}\big)(60\tau). \tag{3-28}$$

(See (3-15) for $R_{a,b}$.) Moreover, $H_{5,1}(\tau)$ is an eigenfunction for $\Omega_{\frac{1}{2}}$ with eigenvalue $\frac{3}{16}$ (see (3-1)). In other words, the components of $H_{5,1} = (H_{5,1,r})$ are harmonic weak Maass forms of weight $\frac{1}{2}$ (see Section 3A).

Proposition 4.13 of [Zwegers 2002] establishes a similar result for $F_{5,2}$, namely

$$H_{5,2}(\tau) = F_{5,2}(\tau) - G_{5,2}(\tau), \tag{3-29}$$

where $H_{5,2}$ is again a harmonic weak Maass form of weight $\frac{1}{2}$, and $G_{5,2} = -G_{5,1}$.

The left-hand sides of (3-26) and (3-29) are harmonic weak Maass forms of weight $\frac{1}{2}$, so they admit canonical decompositions into holomorphic (see (3-3)) and nonholomorphic (see (3-4)) parts. The summands $F_{5,1}$ and $F_{5,2}$ on the right-hand sides are holomorphic by construction, and the $R_{a,b}$ are of the same form as (3-4) by construction (see (3-15)), so the right-hand sides of (3-26) and (3-29) are precisely the decompositions of $H_{5,1}$ and $H_{5,2}$ into their holomorphic and nonholomorphic parts.

Equivalently, the four functions $F_{5,j,r}$ are mock modular forms of weight $\frac{1}{2}$ with completions given by the $H_{5,j,r}$, and the $G_{5,j,r}$ are the Eichler integrals of their shadows. Thus we can describe their shadows explicitly. Applying (3-16), (3-17) and (3-18), and the identities $g_{1-a,0} = g_{-a,0} = -g_{a,0}$, we see that $F_{5,1,3}(2\tau)$ and $-F_{5,2,3}(2\tau)$ have the same shadow

$$\tfrac{1}{2}(S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29})(\tau), \tag{3-30}$$

while $F_{5,1,4}(2\tau)$ and $-F_{5,2,4}(2\tau)$ both have shadow

$$\tfrac{1}{2}(S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23})(\tau). \tag{3-31}$$

3C. ***McKay–Thompson series.*** We now prove our main result, Theorem 1.1, that the trace functions arising from the action of G^X on the $V^{\pm}_{\mathrm{tw},a}$ recover the Fourier expansions of the mock modular forms H^X_g attached to $g \in G^X \simeq S_3$ by umbral moonshine at $X = E_8^3$.

To formulate this precisely, let $T^X_g = (T^X_{g,r})$ be the vector of Laurent series in $q^{1/120}$, with components indexed by $\mathbb{Z}/60\mathbb{Z}$, such that

$$T^X_{g,r} := \begin{cases} T^{\mp}_{g,1} & \text{for } r = \pm 1, \pm 11, \pm 19, \pm 29 \pmod{60}, \\ T^{\mp}_{g,7} & \text{for } r = \pm 7, \pm 13, \pm 17, \pm 23 \pmod{60}, \\ 0 & \text{else.} \end{cases} \tag{3-32}$$

(Recall from Section 2D that $T^{\mp}_{g,a} = -T^{\mp}_{g,10-a}$ and in particular $T^{\mp}_{g,5} = 0$ for all g.) Define the *polar part at infinity* of T^X_g to be the vector of polynomials in $q^{-1/120}$ obtained by removing all nonnegative rational powers of q in each component $T^X_{g,r}$. Let $g \mapsto \bar{\chi}^X_g$ be the natural permutation character of G^X, so that $\bar{\chi}_g$ is 3, 1 or 0, according as g has order 1, 2 or 3, and define a vector $S^X_g = (S^X_{g,r})$ of theta series, with components indexed by $\mathbb{Z}/60\mathbb{Z}$, by setting

$$S^X_{g,r} := \begin{cases} \pm\bar{\chi}_g(S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29}) \\ \qquad\qquad\qquad \text{for } r = \pm 1, \pm 11, \pm 19, \pm 29 \pmod{60}, \\ \pm\bar{\chi}_g(S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23}) \\ \qquad\qquad\qquad \text{for } r = \pm 7, \pm 13, \pm 17, \pm 23 \pmod{60}, \\ 0 \qquad\qquad\qquad \text{else.} \end{cases} \tag{3-33}$$

(See (3-9) for $S_{m,r}$.)

Set $S^X := S^X_e$, and let $\sigma^X : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{60}(\mathbb{C})$ denote the multiplier system of S^X, so that

$$\sigma^X(\gamma) S^X(\gamma\tau)(c\tau + d)^{-3/2} = S^X(\tau) \tag{3-34}$$

for $\tau \in \mathbb{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, when (c, d) is the lower row of γ. Our next goal (to be realized in Proposition 3.2) is to show that $2T_g^X$ is a mock modular form with shadow S_g^X for $g \in G^X$. This condition tells us what the multiplier system of T_g^X must be, at least when $o(g)$ is 1 or 2 (as S_g^X is identically zero when $o(g) = 3$). For the convenience of the reader we describe this multiplier system in more detail now.

It is cumbersome to work with matrices in $\mathrm{GL}_{60}(\mathbb{C})$, but we can avoid this since any nonzero component of T_g^X is ± 1 times $T_{g,1}^X$ or $T_{g,7}^X$. In other words, we can work with the 2-vector-valued functions $\check{T}_g^X := (T_{g,1}^X, T_{g,7}^X)$ and $\check{S}_g^X := (S_{g,1}^X, S_{g,7}^X)$. If $h = (h_r)$ is a modular form of weight $\frac{1}{2}$ with multiplier system conjugate to that of S^X, and satisfying

$$h_r := \begin{cases} h_1 & \text{for } r = \pm 1, \pm 11, \pm 19, \pm 29 \pmod{60}, \\ h_7 & \text{for } r = \pm 7, \pm 13, \pm 17, \pm 23 \pmod{60}, \\ 0 & \text{else}, \end{cases} \tag{3-35}$$

then, setting $\check{h} = (h_1, h_7)$, we have

$$\check{h}\Big(\frac{a\tau+b}{c\tau+d}\Big)\check{\nu}\Big(\frac{a\tau+b}{c\tau+d}\Big)(c\tau+d)^{-1/2} = \check{h}(\tau), \tag{3-36}$$

where $\check{\nu} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{C})$ is determined by the rules

$$\begin{aligned} \check{\nu}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} e\big(-\frac{1}{120}\big) & 0 \\ 0 & e\big(-\frac{49}{120}\big) \end{pmatrix}, \\ \check{\nu}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= \frac{2e\big(\frac{3}{8}\big)}{\sqrt{15}} \begin{pmatrix} \sin\big(\frac{1}{30}\pi\big) + \sin\big(\frac{11}{30}\pi\big) & \sin\big(\frac{7}{30}\pi\big) + \sin\big(\frac{13}{30}\pi\big) \\ \sin\big(\frac{7}{30}\pi\big) + \sin\big(\frac{13}{30}\pi\big) & -\sin\big(\frac{1}{30}\pi\big) - \sin\big(\frac{11}{30}\pi\big) \end{pmatrix}. \end{aligned} \tag{3-37}$$

We now return to our main objective: the determination of the modularity of T_g^X for $g \in G^X$. To describe the multiplier system for T_g^X when $o(g) = 3$, we require the function $\rho_{3|3} : \Gamma_0(3) \to \mathbb{C}^\times$, defined by setting

$$\rho_{3|3}\begin{pmatrix} a & b \\ c & d \end{pmatrix} := e\Big(\frac{cd}{9}\Big). \tag{3-38}$$

Evidently $\rho_{3|3}$ has order 3, and restricts to the identity on $\Gamma_0(9)$.

Proposition 3.2. *Let $g \in G^X$. Then $2T_g^X$ is the Fourier series of a mock modular form for $\Gamma_0(o(g))$, whose shadow is S_g^X. The polar part at infinity of $2T_g^X$ is given by*

$$T_{g,r}^X = \begin{cases} \mp 2q^{-1/120} + O(1) & \textit{for } r = \pm 1, \pm 11, \pm 19, \pm 29 \pmod{60}, \\ O(1) & \textit{otherwise}, \end{cases} \tag{3-39}$$

and $2T_g^X$ has vanishing polar part at all noninfinite cusps of $\Gamma_0(o(g))$. If $o(g) = 3$ then the multiplier system of $2T_g^X$ is given by $\gamma \mapsto \rho_{3|3}(\gamma)\overline{\sigma^X(\gamma)}$.

Proof. According to our definition (3-32), the components of T_g^X are $T_{g,1}^{\pm}$ or $T_{g,7}^{\pm}$. In practice it is more convenient to work with $T_{g,3}^{\pm}$ than $T_{g,7}^{\pm}$, and we may do so because $T_{g,7}^{\pm} = -T_{g,3}^{\pm}$ according to (2-37).

We now verify that the series T_g^X are Fourier expansions of vector-valued mock modular forms, and determine their shadows. For the case $g = e$ we compute $\frac{3}{40} - \frac{1}{12} = -\frac{1}{120}$ and $\frac{27}{40} - \frac{1}{12} = \frac{71}{120}$, and see, upon comparison of (2-38) with (1-4), that $T_{e,1}^{\pm}(q) = \pm q^{-1/120}(2 - \chi_0(q))$ and $T_{e,3}^{\pm} = \pm q^{71/120}\chi_1(q)$. In particular,

$$\begin{aligned} 2T_{e,1}^{-} &= 2q^{-1/120}(\chi_0(q) - 2), \\ 2T_{e,7}^{-} &= 2q^{71/120}\chi_1(q). \end{aligned} \tag{3-40}$$

Note that identities $H_{e,1}^X = 2q^{-1/120}(\chi_0(q) - 2)$ and $H_{e,7}^X = 2q^{71/120}\chi_1(q)$ are predicted in §5.4 of [Cheng et al. 2014b], but it is not verified there that this specification yields a mock modular form with shadow $S^X = S_e^X$.

We determine the modular properties of $2T_{e,1}^{-}$ and $2T_{e,7}^{-}$ by applying the results of Zwegers on F_0, F_1, ϕ_0 and ϕ_1 that we summarized in Section 3B. To apply these results we first recall the expressions

$$\begin{aligned} \chi_0(q) &= 2F_0(q) - \phi_0(-q), \\ \chi_1(q) &= 2F_1(q) + q^{-1}\phi_1(-q), \end{aligned} \tag{3-41}$$

which are proven in §3 of [Watson 1937]. (The first of these was given by Ramanujan in his last letter to Hardy, where he also mentioned the existence of a similar formula relating χ_1, F_1 and ϕ_1.) Thus we obtain

$$2T_{e,1}^{-} = 4F_{5,1,3}(2\tau) - 2F_{5,2,3}(2\tau), \tag{3-42}$$

$$2T_{e,7}^{-} = 4F_{5,1,4}(2\tau) - 2F_{5,2,4}(2\tau), \tag{3-43}$$

upon comparison of (3-24), (3-25), (3-40) and (3-41).

Applying the results of Zwegers on $F_{5,1}$ and $F_{5,2}$ recalled in Section 3B, and the equations (3-30) and (3-31) in particular, we conclude that $2T_{e,1}^{-}$ and $2T_{e,7}^{-}$ are mock modular forms of weight $\frac{1}{2}$, with respective shadows given by

$$3(S_{30,1} + S_{30,11} + S_{30,19} + S_{30,29})(\tau), \tag{3-44}$$

$$3(S_{30,7} + S_{30,13} + S_{30,17} + S_{30,23})(\tau). \tag{3-45}$$

In other words, the shadow of T_e^X is precisely S_e^X, as required. The modular transformation formulas for $H_{5,1}(\tau)$ and $H_{5,2}(\tau)$ given in Propositions 4.10 and 4.13 of [Zwegers 2002], respectively, show that T_e^X transforms in the desired way under $\mathrm{SL}_2(\mathbb{Z})$.

We now consider the case $o(g) = 2$. We may take $g = \hat{\tau}$. We again begin by using the results recalled in Section 3B to analyze the components $T_{\hat{\tau},1}^{-}$ and $T_{\hat{\tau},7}^{-}$

separately. For $T^-_{\hat{\tau},1}$ let

$$A=\begin{pmatrix}6&4\\4&1\end{pmatrix},\quad \boldsymbol{a}=\begin{pmatrix}\frac{1}{10}\\ \frac{1}{10}\end{pmatrix},\quad \boldsymbol{b}=\begin{pmatrix}\frac{3}{20}\\ -\frac{2}{20}\end{pmatrix},\quad \boldsymbol{c}^{(1)}=\begin{pmatrix}-1\\4\end{pmatrix},\quad \boldsymbol{c}^{(2)}=\begin{pmatrix}-2\\3\end{pmatrix}. \tag{3-46}$$

Then a direct computation using

$$\begin{gathered}\nu=\begin{pmatrix}k+\frac{1}{10}\\ m+\frac{1}{10}\end{pmatrix},\qquad Q(\nu)=3k^2+\tfrac{1}{2}m^2+4km+k+\tfrac{1}{2}m+\tfrac{3}{40},\\ B(\nu,\boldsymbol{b})=\tfrac{1}{2}(k+m)+\tfrac{1}{10},\\ \operatorname{sgn}(B(\boldsymbol{c}^{(1)},\nu))=\operatorname{sgn}\bigl(k+\tfrac{1}{10}\bigr),\quad \operatorname{sgn}(B(\boldsymbol{c}^{(2)},\nu))=\operatorname{sgn}\bigl(-m-\tfrac{1}{10}\bigr),\end{gathered} \tag{3-47}$$

gives

$$\begin{aligned}2T^-_{\hat{\tau},1}=-\frac{e\bigl(-\frac{1}{10}\bigr)}{\eta(2\tau)}\sum_{\nu\in\boldsymbol{a}+\mathbb{Z}^2}\bigl(&\operatorname{sgn}(B(\boldsymbol{c}^{(1)},\nu))\\ &-\operatorname{sgn}(B(\boldsymbol{c}^{(2)},\nu))\bigr)e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,\boldsymbol{b})}.\end{aligned} \tag{3-48}$$

Comparing this to the indefinite theta function construction (3-14) we find that

$$\begin{aligned}\vartheta^{\boldsymbol{c}^{(1)},\boldsymbol{c}^{(2)}}_{\boldsymbol{a},\boldsymbol{b}}(\tau)&=-e\bigl(\tfrac{1}{10}\bigr)\eta(2\tau)2T^-_{\hat{\tau},1}(\tau)\\ &+\sum_{\nu\in\boldsymbol{a}+\mathbb{Z}^2}\biggl(\sum_{k=1}^{2}(-1)^k\operatorname{sgn}(B(\boldsymbol{c}^{(k)},\nu))\beta\biggl(-\frac{B(\boldsymbol{c}^{(k)},\nu)^2\Im(\tau)}{Q(\boldsymbol{c}^{(k)})}\biggr)\biggr)\\ &\qquad\qquad\times q^{Q(\nu)}e^{2\pi i B(\nu,\boldsymbol{b})}.\end{aligned} \tag{3-49}$$

We now use Proposition 3.1 to rewrite the terms involving $\boldsymbol{c}^{(1)}$ and $\boldsymbol{c}^{(2)}$ in the second line of (3-49). For the term with $\boldsymbol{c}^{(1)}$ the set P_0 of Proposition 3.1 has one element, $\mu_0=\frac{1}{10}\binom{-9}{1}$, and we find

$$\langle\boldsymbol{c}^{(1)}\rangle^\perp_{\mathbb{Z}}=\bigl\{\tbinom{0}{m}\mid m\in\mathbb{Z}\bigr\},\quad \boldsymbol{b}^\perp=\tfrac{1}{2}\tbinom{0}{1},\quad \mu_0^\perp=\tfrac{1}{2}\tbinom{0}{-7}. \tag{3-50}$$

Thus

$$\sum_{\xi\in\mu_0^\perp+\langle c\rangle^\perp_{\mathbb{Z}}}e^{2\pi i Q(\xi)\tau+2\pi i B(\xi,b^\perp)}=e\bigl(-\tfrac{1}{4}\bigr)\sum_{m\in\mathbb{Z}}(-1)^m q^{(m-1/2)^2/2}=0, \tag{3-51}$$

so this term vanishes.

For the term with $\boldsymbol{c}^{(2)}$ the set P_0 has three elements, $\mu_0=\frac{1}{10}\binom{1}{1},\frac{1}{10}\binom{1}{11},\frac{1}{10}\binom{1}{21}$, and we have $B(\boldsymbol{c}^{(2)},\mu_0)/2Q(\boldsymbol{c}^{(2)})=\frac{1}{30},\frac{11}{30},\frac{21}{30}$, in the respective cases. The last value of μ_0 also leads to a vanishing contribution, while the other two lead to

$$-e\bigl(\tfrac{1}{12}\bigr)R_{\frac{1}{30},-\frac{1}{2}}(15\tau)\eta(2\tau)-e\bigl(-\tfrac{1}{12}\bigr)R_{\frac{11}{30},-\frac{1}{2}}(15\tau)\eta(2\tau), \tag{3-52}$$

which we see by applying Euler's identity

$$q^{1/12}\sum_{k\in\mathbb{Z}}(-1)^k q^{3k^2+k}=\eta(2\tau). \tag{3-53}$$

We thus have

$$-e\big(-\tfrac{1}{10}\big)\frac{\vartheta^{\boldsymbol{c}^{(1)},\boldsymbol{c}^{(2)}}_{\boldsymbol{a},\boldsymbol{b}}(\tau)}{\eta(2\tau)} = 2T^-_{\hat{\tau},1}-e\big(-\tfrac{1}{60}\big)R_{\frac{1}{30},-\frac{1}{2}}(15\tau)-e\big(-\tfrac{11}{60}\big)R_{\frac{11}{30},-\frac{1}{2}}(15\tau). \tag{3-54}$$

In particular, $T^-_{\hat{\tau},1}$ is the Fourier expansion of a holomorphic function on $\mathbb{H}$, which we henceforth denote $T^-_{\hat{\tau},1}(\tau)$.

Since $T^-_{\hat{\tau},1}(\tau)$ is holomorphic, the function (3-54) is a harmonic weak Maass form of weight $\frac{1}{2}$, according to Proposition 4.2 of [Zwegers 2002] (see also Section 3A). Thus we are in a directly similar situation to that encountered at the end of Section 3B. Namely, we have that $T^-_{\hat{\tau},1}(\tau)$ is a mock modular form of weight $\frac{1}{2}$ for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and the second and third summands of the right-hand side of (3-54) comprise the Eichler integral of its shadow. Applying (3-16), (3-17) and (3-18), and also

$$e\big(-\tfrac{1}{60}\big)g_{\frac{1}{30},\frac{1}{2}}(15\tau)+e\big(-\tfrac{11}{60}\big)g_{\frac{11}{30},\frac{1}{2}}(15\tau) = \tfrac{1}{30}(S_{30,1}+S_{30,11}+S_{30,19}+S_{30,29})(\tau), \tag{3-55}$$

we conclude that the shadow of $2T^-_{\hat{\tau},1}(\tau)$ is indeed $S^X_{\hat{\tau},1}(\tau)$ (see (3-33)).

For $T^-_{\hat{\tau},7}$ we take A, $\boldsymbol{b}$, $\boldsymbol{c}^{(1)}$, $\boldsymbol{c}^{(2)}$ as before but set $\boldsymbol{a}=\frac{1}{10}\binom{3}{3}$. We now have

$$\begin{aligned} \nu=\begin{pmatrix}k+\frac{3}{10}\\ m+\frac{3}{10}\end{pmatrix},\quad & Q(\nu)=3k^2+\tfrac{1}{2}m^2+4km+3k+\tfrac{3}{2}m-\tfrac{27}{40},\\ & B(\nu,\boldsymbol{b})=\tfrac{1}{2}(k+m)+\tfrac{3}{10},\\ \operatorname{sgn}(B(\boldsymbol{c}^{(1)},\nu))=\operatorname{sgn}\big(k+\tfrac{3}{10}\big),\quad & \operatorname{sgn}(B(\boldsymbol{c}^{(2)},\nu))=\operatorname{sgn}\big(-m-\tfrac{3}{10}\big). \end{aligned} \tag{3-56}$$

Proceeding as we did for $T^-_{\hat{\tau},1}$, the contribution from the $\boldsymbol{c}^{(1)}$ term vanishes again. For the $\boldsymbol{c}^{(2)}$ term, P_0 consists of the three values $\mu_0=\frac{1}{10}\binom{3}{3}$, $\frac{1}{10}\binom{3}{13}$, $\frac{1}{10}\binom{3}{23}$, and we have

$$\frac{B(\boldsymbol{c}^{(2)},\mu_0)}{2Q(\boldsymbol{c}^{(2)})}=\frac{3}{30},\ \frac{13}{30},\ \frac{23}{30}, \tag{3-57}$$

respectively. The first value of μ_0 leads to a vanishing contribution while the other two terms lead to

$$-e\left(-\tfrac{3}{10}\right)\frac{\vartheta^{\boldsymbol{c}^{(1)},\boldsymbol{c}^{(2)}}_{\boldsymbol{a},\boldsymbol{b}}(\tau)}{\eta(2\tau)} = 2T^-_{\hat{\tau},7} - e\left(-\tfrac{13}{60}\right)R_{\frac{13}{30},-\frac{1}{2}}(15\tau) - e\left(-\tfrac{23}{60}\right)R_{\frac{23}{30},-\frac{1}{2}}(15\tau). \quad (3\text{-}58)$$

We conclude thus that $T^-_{\hat{\tau},7}$ is the Fourier expansion of a mock modular form of weight $\frac{1}{2}$, and using

$$e\left(-\tfrac{13}{60}\right)g_{\frac{13}{30},\frac{1}{2}}(15\tau) + e\left(-\tfrac{23}{60}\right)g_{\frac{23}{30},\frac{1}{2}}(15\tau) = \tfrac{1}{30}(S_{30,7}+S_{30,13}+S_{30,17}+S_{30,23})(\tau), \quad (3\text{-}59)$$

we see that the shadow of $2T^-_{\hat{\tau},1}(\tau)$ is $S^X_{\hat{\tau},1}(\tau)$ (see (3-33)). So we have verified that the shadow of $2T^-_g = (2T^-_{g,r})$ is $S^X_g = (S^X_{g,r})$ for $o(g) = 2$.

Corollary 2.9 of [Zwegers 2002] details the modular transformation properties of the indefinite theta functions $\vartheta^{\boldsymbol{c}^{(1)},\boldsymbol{c}^{(2)}}_{\boldsymbol{a},\boldsymbol{b}}(\tau)$. Applying these formulas, much as in the proofs of Propositions 4.10 and 4.13. in [Zwegers 2002], we see that $2T^-_{\hat{\tau}}$ transforms in the desired way under the action of $\Gamma_0(2)$.

Corollary 2.9 also enables us to compute the expansion of $2T^-_{\hat{\tau}}$ at the cusp of $\Gamma_0(2)$ represented by 0. We ultimately find that both $T^-_{\hat{\tau},1}(\tau)$ and $T^-_{\hat{\tau},7}(\tau)$ vanish as $\tau \to 0$. Thus $2T^-_{\hat{\tau}}$ has no poles away from the infinite cusp.

It remains to consider the case $o(g) = 3$, but this can be handled by applying classical results on positive-definite theta functions, since the formula (2-40) gives $T^-_{\hat{\sigma},1}$ and $T^-_{\hat{\sigma},7}$ explicitly in terms of the Dedekind eta function and the theta series of a rank 1 lattice. We easily check that these functions transform in the desired way under $\Gamma_0(3)$, and have no poles away from the infinite cusp of $\Gamma_0(3)$. In particular, $2T^-_{\hat{\sigma}}$ is modular, and has vanishing shadow. □

We are now ready to prove our main results.

Proof of Theorem 1.1. Proposition 3.2 demonstrates that the functions $2T^X_g$ are mock modular forms of weight $\frac{1}{2}$ with the claimed shadows, multiplier systems and polar parts. It remains to verify that they are the unique such functions.

The uniqueness in case $g = e$ is shown in Corollary 4.2 of [Cheng et al. 2014b], using the fact that there are no weak Jacobi forms of weight 1 (see Theorem 9.7 in [Dabholkar et al. 2012]). We give a different (but certainly related) argument here.

Consider first the case that $o(g)$ is 1 or 2. It suffices to show that if $h = (h_r)$ is a modular form of weight $\frac{1}{2}$, transforming with the same multiplier system as H^X under $\Gamma_0(2)$, with h_r vanishing whenever r does not belong to

$$\{\pm 1, \pm 7, \pm 11, \pm 13, \pm 17, \pm 19, \pm 23, \pm 29\}, \quad (3\text{-}60)$$

then h vanishes identically. The multiplier system for H^X is trivial when restricted to $\Gamma(120)$, so the components h_r are modular forms for $\Gamma_0(2)\cap\Gamma(120) = \Gamma(120)$.

Satz 5.2 of [Skoruppa 1985] is an effective version of the celebrated theorem of Serre and Stark [1977] on modular forms of weight $\frac{1}{2}$ for congruence subgroups of $SL_2(\mathbb{Z})$. It tells us that the space of modular forms of weight $\frac{1}{2}$ for $\Gamma(120)$ is spanned by certain linear combinations of the thetanullwerte $\theta^0_{n,r}(\tau) := \theta_{n,r}(\tau, 0)$, and the only n that can appear are those that divide 30. On the other hand, the restriction (3-60) implies that any nonzero component h_r must belong to one of $q^{-1/120}\mathbb{C}[\![q]\!]$ or $q^{71/120}\mathbb{C}[\![q]\!]$. We conclude that all the h_r are necessarily zero by checking, using

$$\theta^0_{n,r}(\tau) = \sum_{k\in\mathbb{Z}} q^{(2kn+r)^2/4n}, \tag{3-61}$$

that none of the $\theta^0_{n,r}$ belong to either space, for n a divisor of 30.

The case that $o(g) = 3$ is very similar, except that the h_r are now modular forms on $\Gamma_0(9)\cap\Gamma(120)$, which contains $\Gamma(360)$, and the relevant thetanullwerte are those $\theta^0_{n,r}$ with n a divisor of 90. We easily check using (3-61) that there are nonzero possibilities for h_r, and this completes the proof. □

Proof of Theorem 1.2. Taking now (1-6) as the definition of H^X_g, the identities (1-7) follow directly from the definition (3-32) of T^X_g, and the explicit expressions (2-38) for the components of T^X_e.

The identities (1-8) follow from the characterization of H^X_g for $o(g) = 2$ that is entailed in Theorem 1.1. Indeed, using Zwegers' results (viz., Propositions 4.10 and 4.13 in [Zwegers 2002]) on the modularity of $\phi_0(-q)$ and $\phi_1(-q)$, we see that the function defined by the right-hand side of (1-8) is a vector-valued mock modular form with exactly the same shadow as $2T^X_{\hat{\tau}}$, transforming with the same multiplier system under $\Gamma_0(2)$, and having the same polar parts at both the infinite and noninfinite cusps of $\Gamma_0(2)$. So it must coincide with $H^X_{2A,1} = 2T^X_{\hat{\tau}}$ according to Theorem 1.1. This completes the proof. □

Proof of Corollary 1.3. Andrews [1986] established Hecke-type "double sum" identities for ϕ_0 and ϕ_1. Rewriting these slightly, we find

$$\phi_0(-q) = \frac{(q;q)_\infty}{(q^2;q^2)^2_\infty}\Bigg(\sum_{k,m\geq 0} - \sum_{k,m<0}\Bigg)_{k=m \pmod 2} (-1)^m \times q^{k^2/2+m^2/2+4km+k/2+3m/2}, \tag{3-62}$$

$$-q^{-1}\phi_1(-q) = \frac{(q;q)_\infty}{(q^2;q^2)^2_\infty}\Bigg(\sum_{k,m\geq 0} - \sum_{k,m<0}\Bigg)_{k=m \pmod 2} (-1)^m \times q^{k^2/2+m^2/2+4km+3k/2+5m/2}. \tag{3-63}$$

Armed with the identities (1-8), we obtain (1-10) and (1-11) by comparing (3-62) and (3-63) with the explicit expression (2-39) for the components of $T^X_{\hat{\tau}}$. □

Appendix: Coefficients

$[g]$	1A	2A	3A
Γ_g	1|1	2|1	3|3
−1	−2	−2	−2
119	2	2	2
239	2	−2	2
359	4	0	−2
479	2	−2	2
599	6	2	0
719	4	0	−2
839	6	2	0
959	6	−2	0
1079	10	2	−2
1199	6	−2	0
1319	12	0	0
1439	10	−2	−2
1559	14	2	2
1679	14	−2	2
1799	18	2	0
1919	14	−2	2
2039	24	4	0
2159	22	−2	−2
2279	26	2	2
2399	26	−2	2
2519	34	2	−2
2639	30	−2	0
2759	42	2	0
2879	40	−4	−2
2999	48	4	0
3119	48	−4	0
3239	58	2	−2
3359	56	−4	2
3479	72	4	0
3599	70	−2	−2
3719	80	4	2
3839	84	−4	0
3959	100	4	−2
4079	96	−4	0
4199	116	4	2
4319	116	−4	−4
4439	134	6	2
4559	140	−4	2

Table 1. $H^X_{g,1}$, $X = E_8^3$

$[g]$	1A	2A	3A
Γ_g	1|1	2|1	3|3
71	2	−2	2
191	4	0	−2
311	4	0	−2
431	6	2	0
551	6	−2	0
671	8	0	2
791	8	0	2
911	12	0	0
1031	10	−2	−2
1151	14	2	2
1271	16	0	−2
1391	18	2	0
1511	18	−2	0
1631	24	0	0
1751	24	0	0
1871	30	2	0
1991	30	−2	0
2111	36	0	0
2231	38	−2	2
2351	46	2	−2
2471	46	−2	−2
2591	54	2	0
2711	60	0	0
2831	66	2	0
2951	68	−4	2
3071	82	2	−2
3191	84	0	0
3311	98	2	2
3431	102	−2	0
3551	114	2	0
3671	122	−2	2
3791	138	2	0
3911	144	−4	0
4031	162	2	0
4151	174	−2	0
4271	192	4	0
4391	200	−4	2
4511	226	2	−2
4631	238	−2	−2

Table 2. $H^X_{g,7}$, $X = E_8^3$

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john.duncan@emory.edu *Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, United States*

j-harvey@uchicago.edu *Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, IL 60637, United States*

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