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On an analogue of the Ichino–Ikeda conjecture for Whittaker coefficients on the metaplectic group

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In previous papers we formulated an analogue of the Ichino–Ikeda conjectures for Whittaker–Fourier coefficients of automorphic forms on quasisplit classical groups and the metaplectic group of arbitrary rank. In the latter case we reduced the conjecture to a local identity. In this paper we prove the local identity in the p-adic case, and hence the global conjecture under simplifying conditions at the archimedean places.

1. Introduction

Let G be a quasisplit group over a number field F with ring of adeles $\mathbb{A}$. In [Lapid and Mao 2015a], we formulated (under some hypotheses) a conjecture relating Whittaker–Fourier coefficients of cusp forms on $G(F)\backslash G(\mathbb{A})$ to the Petersson inner product. This conjecture is in the spirit of conjectures of Sakellaridis and Venkatesh [2012] and Ichino and Ikeda [2010], which attempt to generalize the classical

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work of Waldspurger [1985; 1981]. In the case of (quasisplit) classical groups, as well as the metaplectic group (i.e., the metaplectic double cover of the symplectic group), we made this conjecture explicit using the descent construction of Ginzburg, Rallis and Soudry [Ginzburg et al. 2011] and the functorial transfer of generic representations of classical groups by Cogdell, Kim, Piatetski-Shapiro and Shahidi [Cogdell et al. 2001; 2004; 2011].

In a follow-up paper [Lapid and Mao 2017], we reduced the global conjecture in the metaplectic case to a local conjectural identity. We also gave a *purely formal* argument for the case of $\widetilde{\mathrm{Sp}}_1$ (i.e., ignoring convergence issues). In this paper we prove the local identity in the p-adic case, justifying the heuristic analysis (and extending it to the general case).

Let us recall the conjecture of [Lapid and Mao 2015a] in the case of the metaplectic group $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$, the double cover of $\mathrm{Sp}_n(\mathbb{A})$ with the standard (Rao) cocycle. We view $\mathrm{Sp}_n(F)$ as a subgroup of $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$. For any genuine function φ on $\mathrm{Sp}_n(F)\backslash\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ we consider the Whittaker coefficient

$$\widetilde{\mathcal{W}}(\tilde{\varphi}) = \widetilde{\mathcal{W}}^{\psi_{\widetilde{N}}}(\tilde{\varphi}) := (\mathrm{vol}(N'(F)\backslash N'(\mathbb{A})))^{-1}\int_{N'(F)\backslash N'(\mathbb{A})} \tilde{\varphi}(u)\psi_{\widetilde{N}}(u)^{-1}\,du.$$

Here $\psi_{\widetilde{N}}$ is a nondegenerate character on $N'(\mathbb{A})$ which is trivial on $N'(F)$, where N' is the standard maximal unipotent subgroup of Sp_n. (We view $N'(\mathbb{A})$ as a subgroup of $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$.) We also consider the inner product

$$(\tilde{\varphi}, \tilde{\varphi}^\vee)_{\mathrm{Sp}_n(F)\backslash \mathrm{Sp}_n(\mathbb{A})} = (\mathrm{vol}(\mathrm{Sp}_n(F)\backslash \mathrm{Sp}_n(\mathbb{A})))^{-1}\int_{\mathrm{Sp}_n(F)\backslash \mathrm{Sp}_n(\mathbb{A})} \tilde{\varphi}(g)\tilde{\varphi}^\vee(g)\,dg$$

of two square-integrable genuine functions on $\mathrm{Sp}_n(F)\backslash\widetilde{\mathrm{Sp}}_n(\mathbb{A})$.

Another ingredient in the conjecture of [Lapid and Mao 2015a] is a regularized integral

$$\int^{\mathrm{st}}_{N'(F_S)} f(u)\,du$$

for a finite set of places S and for a suitable class of smooth functions f on $N'(F_S)$. Suffice it to say that if S consists solely of nonarchimedean places then

$$\int^{\mathrm{st}}_{N'(F_S)} f(u)\,du = \int_{N'_1} f(u)\,du$$

for any sufficiently large compact open subgroup N'_1 of $N'(F_S)$. (The definition of the regularized integral is different in the archimedean case, however in this paper we do not use regularized integrals over archimedean fields.)

The conjecture of [Lapid and Mao 2015a] is applicable for $\psi_{\widetilde{N}}$-generic representations which are not exceptional, in the sense that their theta ψ-lift to $\mathrm{SO}(2n+1)$ is cuspidal (or equivalently, their theta ψ-lift to $\mathrm{SO}(2n-1)$ vanishes; here ψ is determined by $\psi_{\widetilde{N}}$). By [Ginzburg et al. 2011, Chapter 11], which is also based on [Cogdell et al. 2001], there is a one-to-one correspondence between these representations and automorphic representations π of $\mathrm{GL}_{2n}(\mathbb{A})$ which are the isobaric sum $\pi_1 \boxplus \cdots \boxplus \pi_k$ of pairwise inequivalent irreducible cuspidal representations π_i of $\mathrm{GL}_{2n_i}(\mathbb{A})$, for $i = 1, \dots, k$ (with $n_1 + \cdots + n_k = n$), such that $L^S\big(\frac{1}{2}, \pi_i\big) \neq 0$ and $L^S(s, \pi_i, \wedge^2)$ has a pole (necessarily simple) at $s = 1$ for all i. Here $L^S(s, \pi_i)$ and $L^S(s, \pi_i, \wedge^2)$ are the standard and exterior square (partial) L-functions, respectively. More specifically, to any such π one constructs a $\psi_{\widetilde{N}}$-generic representation $\tilde{\pi}$ of $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$, which is called the $\psi_{\widetilde{N}}$-descent of π. The theta ψ-lift of $\tilde{\pi}$ is the unique irreducible generic cuspidal representation of $\mathrm{SO}(2n+1)$ which lifts to π.

Conjecture 1.1 [Lapid and Mao 2015a, Conjecture 1.3]. Assume that $\tilde{\pi}$ is the $\psi_{\widetilde{N}}$-descent of π as above. Then for any $\tilde{\varphi} \in \tilde{\pi}$ and $\tilde{\varphi}^\vee \in \tilde{\pi}^\vee$ and for any sufficiently large finite set S of places of F, we have

$$\widetilde{\mathcal{W}}^{\psi_{\widetilde{N}}}(\tilde{\varphi})\widetilde{\mathcal{W}}^{\psi_{\widetilde{N}}^{-1}}(\tilde{\varphi}^\vee) = 2^{-k}\left(\prod_{i=1}^{n} \zeta_F^S(2i)\right)\frac{L^S\big(\frac{1}{2}, \pi\big)}{L^S(1, \pi, \mathrm{sym}^2)}$$
$$\times \big(\mathrm{vol}(N'(\mathcal{O}_S)\backslash N'(F_S))\big)^{-1}\int^{\mathrm{st}}_{N'(F_S)} (\tilde{\pi}(u)\tilde{\varphi}, \tilde{\varphi}^\vee)_{\mathrm{Sp}_n(F)\backslash \mathrm{Sp}_n(\mathbb{A})}\psi_{\widetilde{N}}(u)^{-1}\, du. \quad (1\text{-}1)$$

Here $\zeta_F^S(s)$ is the partial Dedekind zeta function, $\mathcal{O}_S$ is the ring of S-integers of F, and $L^S(s, \pi, \mathrm{sym}^2)$ is the symmetric square partial L-function of π.

The main result in [Lapid and Mao 2017] is the following.

Theorem 1.2 [Lapid and Mao 2017, Theorem 6.2]. *In the above setup we have*

$$\widetilde{\mathcal{W}}^{\psi_{\widetilde{N}}}(\tilde{\varphi})\widetilde{\mathcal{W}}^{\psi_{\widetilde{N}}^{-1}}(\tilde{\varphi}^\vee) = 2^{-k}\left(\prod_{i=1}^{n} \zeta_F^S(2i)\right)\frac{L^S\big(\frac{1}{2}, \pi\big)}{L^S(1, \pi, \mathrm{sym}^2)}\left(\prod_{v\in S} c_{\pi_v}^{-1}\right)$$
$$\times \big(\mathrm{vol}(N'(\mathcal{O}_S)\backslash N'(F_S))\big)^{-1}\int^{\mathrm{st}}_{N'(F_S)} (\tilde{\pi}(u)\tilde{\varphi}, \tilde{\varphi}^\vee)_{\mathrm{Sp}_n(F)\backslash \mathrm{Sp}_n(\mathbb{A})}\psi_{\widetilde{N}}(u)^{-1}\, du, \quad (1\text{-}2)$$

where c_{π_v}, $v \in S$ are certain nonzero constants which depend only on the local representations π_v.

The main result of this paper specifies a value for the c_{π_v}.

Theorem 1.3. *In Theorem 1.2 we have $c_{\pi_v} = \epsilon\big(\frac{1}{2}, \pi_v, \psi_v\big)$ (the root number of π_v) for all finite places v.*

We also show in Proposition 8.6 that the root number of π_v equals the central sign of $\tilde{\pi}_v$. In [Ichino et al. 2017] it is shown that Theorem 1.3 implies the formal degree conjecture of Hiraga, Ichino and Ikeda [Hiraga et al. 2008] (or more precisely, its metaplectic analogue) for generic square-integrable representations of $\widetilde{\mathrm{Sp}}_n$. Conversely, in the real case (where the formal degree conjecture is a reformulation of classical results of Harish-Chandra) it is shown that $c_{\pi_v} = \epsilon\bigl(\frac{1}{2}, \pi_v, \psi_v\bigr)$ if $\tilde{\pi}_v$ is square-integrable. (Note that in the square-integrable case, matrix coefficients are integrable on $N'(F_v)$ and no regularization is necessary.) We conclude:

Corollary 1.4. *Conjecture 1.1 holds if F is totally real and $\tilde{\pi}_\infty$ is a discrete series.*

Theorem 1.3 is the culmination of the series of papers [Lapid and Mao 2014; 2015b; 2017]. More precisely, the theorem can be formulated as an identity — the main identity (MI) made explicit in Section 3F (based on [Lapid and Mao 2017]) between integrals of Whittaker functions in the induced spaces of π (in a local setting). In principle, formal manipulations using the functional equations of [Lapid and Mao 2014] reduce the identity to the results of [Lapid and Mao 2015b]. Such an argument was described heuristically for the case $n = 1$ in [Lapid and Mao 2017, §7]. However, making this rigorous (even in the case $n = 1$ and for π supercuspidal) seems nontrivial because the integrals only converge as iterated integrals. This is the main task of the present paper.

In Section 3F we reduce the theorem to the cases where π is tempered and satisfies some good properties. Here we rely on the classification, due to Matringe [2015], of the generic representations admitting a nontrivial $\mathrm{GL}_n \times \mathrm{GL}_n$-invariant functional. We also use a globalization result from [Ichino et al. 2017, Appendix A] which is based on a result of [Sakellaridis and Venkatesh 2012].

The rest of the argument is purely local. In Section 5 we start the manipulation of the left-hand side of the main identity. It is technically important to restrict oneself to certain special sections in the induced space. This is possible by a nonvanishing result on the Bessel function of generic representations proved in the Appendix. Another useful idea is to write the left-hand side of the main identity (for W special) as $B\bigl(W, M\bigl(\pi, \frac{1}{2}\bigr)W^\wedge, \frac{1}{2}\bigr)$, where $B(W, W^\vee, s)$ (for $s \in \mathbb{C}$) is an analytic family of bilinear forms on $I(\pi, s) \times I(\pi^\vee, -s)$, and $M(\pi, s)$ is the intertwining operator on $I(\pi, s)$. This relies on results of Baruch [2005], generalized to the present context in [Lapid and Mao 2013].

The reason for introducing this analytic family is that because of convergence issues, we can only apply the functional equations of [Lapid and Mao 2014] for $\operatorname{Re} s \ll 0$. This is the most delicate step, and is described in Section 7. It entails a further restriction on $W^\wedge$ (which is fortunately harmless for our purpose). The upshot is an expression (for special W and $W^\wedge$ and for $\operatorname{Re} s \ll 0$)

$$B(W, M(\pi, s)W^\wedge, s) = \int E^{\psi}(M_s^* W, t) E^{\psi^{-1}}(W_s^\wedge, t) \, \frac{dt}{|\det t|},$$

where t is integrated over a certain n-dimensional torus and E^{ψ} is a certain integral of W. The restriction on $W^\wedge$ ensures that the integrand is compactly supported. Moreover, by [Lapid and Mao 2015b], as a distribution in t, $E^{\psi}(W_s, t)$ extends to an entire function on s. Therefore, the above identity is meaningful for all s

where $M(\pi, s)$ is holomorphic. Specializing to $s = \frac{1}{2}$, the remaining assertions are that $E^{\psi}(M^* W, t)$ is constant in t (and in particular, is a function), whose value can be determined, while the integral of $E^{\psi^{-1}}(W^{\wedge}_{1/2}, t)$ factors through $M\big(\pi, \frac{1}{2}\big) W^{\wedge}$, again in an explicit way. This is the content of Corollaries 8.2 and 8.5, respectively, which follow from the representation-theoretic results established in [Lapid and Mao 2015b]. We refer the reader to Section 4 below for a more detailed sketch of the proof.

2. Notation and preliminaries

For the convenience of the reader we introduce in this section the most common notation that will be used throughout.

We fix a positive integer $\mathbf{n}$ (not to be confused with a running variable n). In the following m is either $\mathbf{n}$ or $2\mathbf{n}$. Let F be a local field of characteristic 0.

2A. ***Groups, homomorphisms and group elements.*** All algebraic groups are defined over F. We typically denote algebraic varieties (or groups) over F by boldface letters (e.g., $\mathbf{X}$) and denote their set (or group) of F-points by the corresponding plain letter (e.g., X). (In most cases $\mathbf{X}$ will be clear from the context.)

- I_m is the identity matrix in GL_m, w_m is the $m \times m$ matrix with ones on the nonprincipal diagonal and zeros elsewhere.
- For any group Q, Z_Q is the center of Q; e is the identity element of Q. We denote the modulus function of Q (i.e., the quotient of a right Haar measure by a left Haar measure) by δ_Q.
- Mat_m is the vector space of $m \times m$ matrices over F.
- $x \mapsto x^t$ is the transpose on Mat_m; $x \mapsto \check{x}$ is the twisted transpose map on Mat_m given by $\check{x} = w_m x^t w_m$; $g \mapsto g^*$ is the outer automorphism of GL_m given by $g^* = w_m^{-1} (g^t)^{-1} w_m$.
- $\mathfrak{s}_m = \{x \in \mathrm{Mat}_m : \check{x} = x\}$.
- $\mathbb{M} = \mathrm{GL}_{2\mathbf{n}}$, $\mathbb{M}' = \mathrm{GL}_{\mathbf{n}}$.
- $G = \mathrm{Sp}_{2\mathbf{n}} = \left\{g \in \mathrm{GL}_{4\mathbf{n}} : g^t \begin{pmatrix} & w_{2\mathbf{n}} \\ -w_{2\mathbf{n}} & \end{pmatrix} g = \begin{pmatrix} & w_{2\mathbf{n}} \\ -w_{2\mathbf{n}} & \end{pmatrix}\right\}$.
- $G' = \mathrm{Sp}_{\mathbf{n}} = \left\{g \in \mathrm{GL}_{2\mathbf{n}} : g^t \begin{pmatrix} & w_{\mathbf{n}} \\ -w_{\mathbf{n}} & \end{pmatrix} g = \begin{pmatrix} & w_{\mathbf{n}} \\ -w_{\mathbf{n}} & \end{pmatrix}\right\}$.
- G' is embedded as a subgroup of G via $g \mapsto \eta(g) = \mathrm{diag}(I_{\mathbf{n}}, g, I_{\mathbf{n}})$.
- $P = M \ltimes U$ and $P' = M' \ltimes U'$ are the Siegel parabolic subgroups of G and G', respectively, with the standard Levi decomposition.
- $\overline{P} = P^t$ is the opposite parabolic of P, with unipotent radical $\overline{U} = U^t$.
- We use the isomorphism $\varrho(g) = \mathrm{diag}(g, g^*)$ to identify $\mathbb{M}$ with $M \subset G$. Similarly for $\varrho' : \mathbb{M}' \to M' \subset G'$.

- We use the embeddings $\eta_{\mathbb{M}}(g) = \mathrm{diag}(g, I_{\mathbf{n}})$ and $\eta^{\vee}_{\mathbb{M}}(g) = \mathrm{diag}(I_{\mathbf{n}}, g)$ to identify $\mathbb{M}'$ with subgroups of $\mathbb{M}$. We also set $\eta_M = \varrho \circ \eta_{\mathbb{M}}$ and $\eta^{\vee}_M = \varrho \circ \eta^{\vee}_{\mathbb{M}} = \eta \circ \varrho'$.
- K is the standard maximal compact subgroup of G. (In the p-adic case it consists of the matrices with integral entries.)
- N is the standard maximal unipotent subgroup of G consisting of upper unitriangular matrices; T is the maximal torus of G consisting of diagonal matrices; $B = T \ltimes N$ is the Borel subgroup of G.
- For any subgroup X of G, we set $X' = \eta^{-1}(X)$, $X_M = X \cap M$ and $X_{\mathbb{M}} = \varrho^{-1}(X_M)$; similarly, $X'_{M'} = X' \cap M'$ and $X'_{\mathbb{M}'} = \varrho'^{-1}(X'_{M'})$.
- $\ell_{\mathbb{M}} : \mathrm{Mat}_{\mathbf{n}} \to N_{\mathbb{M}}$ is the group embedding given by $\ell_{\mathbb{M}}(x) = \left(\begin{smallmatrix} I_{\mathbf{n}} & x \\ & I_{\mathbf{n}} \end{smallmatrix}\right)$ and $\ell_M = \varrho \circ \ell_{\mathbb{M}}$.
- $\ell : \mathfrak{s}_{2\mathbf{n}} \to U$ is the isomorphism given by $\ell(x) = \left(\begin{smallmatrix} I_{2\mathbf{n}} & x \\ & I_{2\mathbf{n}} \end{smallmatrix}\right)$.
- $\widetilde{G} = \widetilde{\mathrm{Sp}}_{\mathbf{n}}$ is the metaplectic group, i.e., the nontrivial two-fold cover of G' (unless F is complex). We write elements of $\widetilde{G}$ as pairs (g, ϵ), for $g \in G$ and $\epsilon = \pm 1$, with the multiplication given by Rao's cocycle. See [Ginzburg et al. 2011].
- When $g \in G'$, we write $\tilde{g} = (g, 1) \in \widetilde{G}$. Of course, $g \mapsto \tilde{g}$ is not a group homomorphism, but we do have $\widetilde{gn} = \tilde{g}\tilde{n}$ and $\widetilde{ng} = \tilde{n}\tilde{g}$ for any $g \in G'$ and $n \in N'$.
- $\widetilde{N}$ and $\widetilde{P}$ are the inverse images of N' and P' under the canonical projection $\widetilde{G} \to G'$. We identify N' with a subgroup of $\widetilde{N}$ via $n \mapsto \tilde{n}$.
- $\xi_m = (0, \dots, 0, 1) \in F^m$.
- $\mathcal{P}$ is the mirabolic subgroup of $\mathbb{M}$ consisting of the elements g such that $\xi_{2\mathbf{n}} g = \xi_{2\mathbf{n}}$.
- $E = \mathrm{diag}(1, -1, \dots, 1, -1) \in \mathbb{M}$. H is the centralizer of $\varrho(E)$ in G. It is isomorphic to $\mathrm{Sp}_{\mathbf{n}} \times \mathrm{Sp}_{\mathbf{n}}$.
- $H_{\mathbb{M}}$ is then the centralizer of E in $\mathbb{M}$. It is isomorphic to $\mathrm{GL}_{\mathbf{n}} \times \mathrm{GL}_{\mathbf{n}}$.
- $w'_0 = \left(\begin{smallmatrix} & w_{\mathbf{n}} \\ -w_{\mathbf{n}} & \end{smallmatrix}\right) \in G'$ represents the longest Weyl element of G'.
- $w_U = \left(\begin{smallmatrix} & I_{2\mathbf{n}} \\ -I_{2\mathbf{n}} & \end{smallmatrix}\right) \in G$ represents the longest M-reduced Weyl element of G.
- $w'_{U'} = \left(\begin{smallmatrix} & I_{\mathbf{n}} \\ -I_{\mathbf{n}} & \end{smallmatrix}\right) \in G'$ represents the longest M'-reduced Weyl element of G'.
- $w_0^{\mathbb{M}} = w_{2\mathbf{n}} \in \mathbb{M}$ represents the longest Weyl element of $\mathbb{M}$; $w_0^M = \varrho(w_0^{\mathbb{M}})$.
- $w_0^{\mathbb{M}'} = w_{\mathbf{n}} \in \mathbb{M}'$ represents the longest Weyl element of $\mathbb{M}'$; $w_0^{M'} = \varrho'(w_0^{\mathbb{M}'})$.
- $w_{2\mathbf{n},\mathbf{n}} = \left(\begin{smallmatrix} & I_{\mathbf{n}} \\ I_{\mathbf{n}} & \end{smallmatrix}\right) \in \mathbb{M}$, $w'_{2\mathbf{n},\mathbf{n}} = \left(\begin{smallmatrix} & I_{\mathbf{n}} \\ w_0^{\mathbb{M}'} & \end{smallmatrix}\right) \in \mathbb{M}$.
- $\gamma = w_U \eta(w'_{U'})^{-1} = \begin{pmatrix} & I_{\mathbf{n}} & & \\ & & & I_{\mathbf{n}} \\ -I_{\mathbf{n}} & & & \\ & & I_{\mathbf{n}} & \end{pmatrix} \in G.$
- $\mathfrak{d} = \mathrm{diag}(1, -1, \dots, (-1)^{\mathbf{n}-1}) \in \mathrm{Mat}_{\mathbf{n}}$, $\boldsymbol{\epsilon}_1 = \ell_M((-1)^{\mathbf{n}}\mathfrak{d}) \in N_M$, $\boldsymbol{\epsilon}_2 = \ell_{\mathbb{M}}(\mathfrak{d}) \in N_{\mathbb{M}}$, $\boldsymbol{\epsilon}_3 = w'_{2\mathbf{n},\mathbf{n}}\boldsymbol{\epsilon}_2 \in \mathbb{M}$, $\boldsymbol{\epsilon}_4 = \ell_{\mathbb{M}}\left(-\tfrac{1}{2}\mathfrak{d} w_0^{\mathbb{M}'}\right) \in N_{\mathbb{M}}$.

• V and $V^\sharp$ are the unipotent radicals of the standard parabolic subgroup of G with Levi $\mathrm{GL}_1^{\mathbf{n}} \times \mathrm{Sp}_{\mathbf{n}}$ and $\mathrm{GL}_1^{\mathbf{n}-1} \times \mathrm{Sp}_{\mathbf{n}+1}$, respectively. Thus, $N = \eta(N') \ltimes V$, $V^\sharp$ is normal in V and $V/V^\sharp$ is isomorphic to the Heisenberg group of dimension $2\mathbf{n}+1$ (see below). Also, $V = V_M \ltimes V_U$, where $V_U = V \cap U = \left\{\ell\left(\left(\begin{smallmatrix} x & y \\ & \check{x}\end{smallmatrix}\right)\right) : x \in \mathrm{Mat}_{\mathbf{n}},\ y \in \mathfrak{s}_{\mathbf{n}}\right\}$.

• $V_- = V_M^\sharp \ltimes V_U$. (Recall $V_M^\sharp = V^\sharp \cap M$ by our convention.)

• $V_\gamma = V \cap \gamma^{-1} N \gamma = \eta(w'_{U'}) V_M \eta(w'_{U'})^{-1} = \eta_M(N'_{\mathbb{M}'}) \ltimes \left\{\ell\left(\left(\begin{smallmatrix} x & \\ & \check{x}\end{smallmatrix}\right)\right) : x \in \mathrm{Mat}_{\mathbf{n}}\right\} \subset V_-$.

• $V_+ \subset V$ is the image under ℓ_M of the space of $\mathbf{n} \times \mathbf{n}$ matrices whose rows are zero except possibly for the last one. Thus, $V = V_+ \ltimes V_-$. For $c = \ell_M(x) \in V_+$ we denote by $\underline{c} \in F^{\mathbf{n}}$ the last row of x.

• $N^\sharp = V_- \rtimes \eta(N')$ is the stabilizer in N of the character ψ_U defined below.

• $N_{\mathbb{M}}^\flat = (N_{\mathbb{M}}^\sharp)^*$.

• J is the subspace of $\mathrm{Mat}_{\mathbf{n}}$ consisting of the matrices whose first column is zero.

• $\bar{R} = \left\{\left(\begin{smallmatrix} I_{\mathbf{n}} & \\ x & n^t\end{smallmatrix}\right) : x \in J,\ n \in N'_{\mathbb{M}'}\right\}$.

• $T'' = Z_{\mathbb{M}} \times \eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'}) = \{\mathrm{diag}(t_1, \dots, t_{2\mathbf{n}}) : t_1 = \dots = t_{\mathbf{n}}\} \subset T_{\mathbb{M}}$.

2B. *Characters.* We fix a nontrivial unitary character ψ of F. For each of the unipotent groups X listed below we assign a character ψ_X of X as follows:

$$\psi_{N_{\mathbb{M}}}(u) = \psi(u_{1,2} + \dots + u_{2\mathbf{n}-1,2\mathbf{n}}) \quad \text{(nondegenerate)},$$

$$\psi_{N_M} \circ \varrho = \psi_{N_{\mathbb{M}}},$$

$$\psi_{N'_{\mathbb{M}'}}(u') = \psi(u'_{1,2} + \dots + u'_{\mathbf{n}-1,\mathbf{n}}) \quad \text{(nondegenerate)},$$

$$\psi_{N'_{M'}} \circ \varrho' = \psi_{N'_{\mathbb{M}'}} \quad \text{(i.e., } \psi_{N'_{M'}}(n) = \psi_{N_M}(\gamma\eta(n)\gamma^{-1})),$$

$$\psi_{N'}(nu) = \psi_{N'_{M'}}(n)\psi\left(\tfrac{1}{2}u_{\mathbf{n},\mathbf{n}+1}\right)^{-1}, \quad n \in N'_{M'},\ u \in U',$$

$$\psi_{\widetilde{N}} \text{ is the extension of } \psi_{N'} \text{ to a genuine character of } \widetilde{N},$$

$$\psi_N(nu) = \psi_{N_M}(n), \quad n \in N_M,\ u \in U, \quad \text{(degenerate)},$$

$$\psi_U(\ell(v)) = \psi\left(\tfrac{1}{2}(v_{\mathbf{n},\mathbf{n}+1} - v_{2\mathbf{n},1})\right),$$

$$\psi_{V_-}(vu) = \psi_{N_M}(v)^{-1}\psi_U(u), \quad v \in V_M^\sharp,\ u \in V_U,$$

$$\psi_{N^\sharp}(\eta(n)v) = \psi_{N'}(n)\psi_{V_-}(v), \quad n \in N',\ v \in V_-,$$

$$\psi_{N_{\mathbb{M}}^\sharp} = \psi_{N^\sharp} \circ \varrho,$$

$$\psi_{N_{\mathbb{M}}^\flat}(m) = \psi_{N_{\mathbb{M}}^\sharp}(m^*), \quad m \in N_{\mathbb{M}}^\flat,$$

$$\psi_{\overline{U}}(\ell(v)^t) = \psi(v_{1,1}), \quad v \in \mathfrak{s}_{2\mathbf{n}}.$$

2C. *Other notation.* • We use the notation $a \ll_d b$ to mean that $a \le cb$ with $c > 0$ a constant depending on d.

• For any $g \in G$, define $\nu(g) \in \mathbb{R}_{>0}$ by $\nu(u\varrho(m)k) = |\det m|$ for any $u \in U$, $m \in \mathbb{M}$, $k \in K$. Let $\nu'(g) = \nu(\eta(g))$ for $g \in G'$.

• $\mathcal{CSGR}(Q)$ is the set of compact open subgroups of a topological group Q.

• $\mathcal{CSGR}^s(G')$ is the subset of $\mathcal{CSGR}(G')$ consisting of the K_0 for which $k \in K_0 \mapsto \tilde{k}$ is a group isomorphism. Any sufficiently small $K_0 \in \mathcal{CSGR}(G')$ belongs to $\mathcal{CSGR}^s(G')$. We identify any $K_0 \in \mathcal{CSGR}^s(G')$ with its image under $k \mapsto \tilde{k}$ (an element of $\mathcal{CSGR}(\widetilde{G})$).

• For an ℓ-group Q, let $C(Q)$ be the space of continuous functions on Q, and $\mathcal{S}(Q)$ the space of Schwartz functions on Q.

• When F is p-adic, if Q' is a closed subgroup of Q and χ is a character of Q', we denote by $C(Q'\backslash Q, \chi)$ (resp. $C^{\mathrm{sm}}(Q'\backslash Q, \chi)$, $C_c^\infty(Q'\backslash Q, \chi)$) the spaces of continuous (resp. Q-smooth,[1] smooth and compactly supported modulo Q') complex-valued left (Q', χ)-equivariant functions on Q.

• For an ℓ-group Q, we write $\operatorname{Irr} Q$ for the set of equivalence classes of irreducible representations of Q. If Q is reductive, we also write $\operatorname{Irr}_{\mathrm{sqr}} Q$ and $\operatorname{Irr}_{\mathrm{temp}} Q$ for the subsets of irreducible unitary square-integrable (modulo center) and tempered representations, respectively. We write $\operatorname{Irr}_{\mathrm{gen}} \mathbb{M}$ and $\operatorname{Irr}_{\mathrm{meta}} \mathbb{M}$ for the subsets of irreducible generic representations of $\mathbb{M}$ and representations of metaplectic type (see below), respectively. For the set of irreducible generic representations of $\widetilde{G}$, we use the notation $\operatorname{Irr}_{\mathrm{gen},\psi_{\tilde{N}}} \widetilde{G}$ to emphasize the dependence on the character $\psi_{\tilde{N}}$.

• For $\pi \in \operatorname{Irr} Q$, let $\pi^\vee$ be the contragredient of π.

• For $\pi \in \operatorname{Irr}_{\mathrm{gen}} \mathbb{M}$, we denote by $\mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)$ the (uniquely determined) Whittaker space of π with respect to the character $\psi_{N_{\mathbb{M}}}$. We use the notation $\mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}$, $\mathbb{W}^{\psi_{N_M}}$, $\mathbb{W}^{\psi_{N_M}^{-1}}$, $\mathbb{W}^{\psi_{\tilde{N}}}$, $\mathbb{W}^{\psi_{\tilde{N}}^{-1}}$ similarly.

• For $\pi \in \operatorname{Irr}_{\mathrm{gen}} M$, let $\operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ be the space of smooth left U-invariant functions $W : G \to \mathbb{C}$ such that for all $g \in G$, the function $m \mapsto \delta_P(m)^{-1/2} W(mg)$ on M belongs to $\mathbb{W}^{\psi_{N_M}}(\pi)$. Similarly define $\operatorname{Ind}(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi))$.

• If a group G_0 acts on a vector space W and H_0 is a subgroup of G_0, we denote by W^{H_0} the subspace of H_0-fixed points.

• We use the following bracket notation for iterated integrals: $\iint \bigl(\iint \dots\bigr) \dots$ implies that the inner integrals converge as a double integral and after evaluating them, the outer double integral is absolutely convergent.

2D. ***Measures.*** We take the self-dual Haar measure on F with respect to ψ. We use the following convention for Haar measures for algebraic subgroups of G. (We consider G', $\mathbb{M}$, $\mathbb{M}'$ as subgroups of G through the embeddings η, ϱ and $\eta \circ \varrho'$.)

[1] That is, right-invariant under an open subgroup of Q.

When F is a p-adic field, let $\mathcal{O}$ be its ring of integers. The Lie algebra $\mathfrak{M}$ of $\mathrm{GL}_{4\mathbf{n}}$ consists of the $4\mathbf{n}\times 4\mathbf{n}$ matrices X over F. Let $\mathfrak{M}_{\mathcal{O}}$ be the lattice of integral matrices in $\mathfrak{M}$. For any algebraic subgroup $\mathbf{Q}$ of $\mathrm{GL}_{4\mathbf{n}}$ defined over F (e.g., an algebraic subgroup of $\mathbf{G}$), let $\mathfrak{q}\subset\mathfrak{M}$ be the Lie algebra of $\mathbf{Q}$. The lattice $\mathfrak{q}\cap\mathfrak{M}_{\mathcal{O}}$ of $\mathfrak{q}$ gives rise to a gauge form of $\mathbf{Q}$ (determined up to multiplication by an element of $\mathcal{O}^*$) and we use it to define a Haar measure on Q by the recipe of [Kneser 1967].

When $F=\mathbb{R}$, the measures are fixed similarly, except that we use $\mathfrak{M}_{\mathbb{Z}}$ in place of $\mathfrak{M}_{\mathcal{O}}$. When $F=\mathbb{C}$ we only consider subgroups of G which are defined over $\mathbb{R}$ and take the gauge forms induced from the above recipe for $\mathbb{R}$.

2E. ***Weil representation.*** Let $\mathbb{V}$ be a symplectic space over F with a symplectic form $\langle\cdot,\cdot\rangle$. Let $\mathcal{H}=\mathcal{H}_{\mathbb{V}}$ be the Heisenberg group of $(\mathbb{V},\langle\cdot,\cdot\rangle)$. Recall that $\mathcal{H}_{\mathbb{V}}=\mathbb{V}\oplus F$ with the product rule

$$(x,t)\cdot(y,z)=\big(x+y,\,t+z+\tfrac{1}{2}\langle x,y\rangle\big).$$

Fix a polarization $\mathbb{V}=\mathbb{V}_+\oplus\mathbb{V}_-$. The group $\mathrm{Sp}(\mathbb{V})$ acts on the right on $\mathbb{V}$. We write a typical element of $\mathrm{Sp}(\mathbb{V})$ as $\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$, where $A\in\mathrm{Hom}(\mathbb{V}_+,\mathbb{V}_+)$, $B\in\mathrm{Hom}(\mathbb{V}_+,\mathbb{V}_-)$, $C\in\mathrm{Hom}(\mathbb{V}_-,\mathbb{V}_+)$ and $D\in\mathrm{Hom}(\mathbb{V}_-,\mathbb{V}_-)$. Let $\widetilde{\mathrm{Sp}}(\mathbb{V})$ be the metaplectic two-fold cover of $\mathrm{Sp}(\mathbb{V})$ with respect to the Rao cocycle determined by the splitting. Consider the Weil representation ω_ψ of the group $\mathcal{H}_{\mathbb{V}}\rtimes\widetilde{\mathrm{Sp}}(\mathbb{V})$ on $\mathcal{S}(\mathbb{V}_+)$. Explicitly, for any $\Phi\in\mathcal{S}(\mathbb{V}_+)$ and $X\in\mathbb{V}_+$, the action of $\mathcal{H}_{\mathbb{V}}$ is given by

$$\omega_\psi(a,0)\Phi(X)=\Phi(X+a),\qquad a\in\mathbb{V}_+,\tag{2-1a}$$
$$\omega_\psi(b,0)\Phi(X)=\psi(\langle X,b\rangle)\Phi(X),\qquad b\in\mathbb{V}_-,\tag{2-1b}$$
$$\omega_\psi(0,t)\Phi(X)=\psi(t)\Phi(X),\qquad t\in F,\tag{2-1c}$$

while the action of $\widetilde{\mathrm{Sp}}(\mathbb{V})$ is (partially) given by

$$\omega_\psi\big(\left(\begin{smallmatrix}g&0\\0&g^*\end{smallmatrix}\right),\epsilon\big)\Phi(X)=\epsilon\gamma_\psi(\det g)|\det g|^{1/2}\Phi(Xg),\qquad g\in\mathrm{GL}(\mathbb{V}_+),\tag{2-2a}$$
$$\omega_\psi\big(\left(\begin{smallmatrix}I&B\\0&I\end{smallmatrix}\right),\epsilon\big)\Phi(X)=\epsilon\psi\big(\tfrac{1}{2}\langle X,XB\rangle\big)\Phi(X),\qquad B\in\mathrm{Hom}(\mathbb{V}_+,\mathbb{V}_-)\text{ self-dual},\tag{2-2b}$$

where γ_ψ is Weil's factor.

We now take $\mathbb{V}=F^{2\mathbf{n}}$ with the standard symplectic form

$$\langle(x_1,\dots,x_{2\mathbf{n}}),(y_1,\dots,y_{2\mathbf{n}})\rangle=\sum_{i=1}^{\mathbf{n}}x_iy_{2\mathbf{n}+1-i}-\sum_{i=1}^{\mathbf{n}}y_ix_{2\mathbf{n}+1-i}$$

and the standard polarization

$$\mathbb{V}_+=\{(x_1,\dots,x_{\mathbf{n}},0,\dots,0)\},\qquad\mathbb{V}_-=\{(0,\dots,0,y_1,\dots,y_{\mathbf{n}})\}.$$

(We identify $\mathbb{V}_+$ and $\mathbb{V}_-$ with $F^{\mathbf{n}}$.) The corresponding Heisenberg group is isomorphic to the quotient $V/V^\sharp$ (with V, $V^\sharp$ as defined in Section 2A) via $v \mapsto v_{\mathcal{H}} := \big((v_{\mathbf{n},\mathbf{n}+j})_{j=1,\dots,2\mathbf{n}}, \tfrac{1}{2}v_{\mathbf{n},3\mathbf{n}+1}\big)$.

For $X = (x_1, \dots, x_{\mathbf{n}})$, $X' = (x'_1, \dots, x'_{\mathbf{n}}) \in F^{\mathbf{n}}$, define

$$\langle X, X'\rangle' = x_1 x'_{\mathbf{n}} + \cdots + x_{\mathbf{n}} x'_1.$$

For $\Phi \in \mathcal{S}(F^{\mathbf{n}})$, define the Fourier transform

$$\hat{\Phi}(X) = \int_{F^{\mathbf{n}}} \Phi(X')\psi(\langle X, X'\rangle')\,dX'.$$

Then, realized on $\mathcal{S}(F^{\mathbf{n}})$, the Weil representation satisfies

$$\omega_\psi(\widetilde{\varrho'(h)})\Phi(X) = |\det(h)|^{1/2}\beta_\psi(\varrho'(h))\Phi(Xh), \quad h \in \mathbb{M}', \tag{2-3a}$$

$$\omega_\psi\Big(\widetilde{\begin{pmatrix} 1 & B \\ & 1 \end{pmatrix}}\Big)\Phi(X) = \psi\big(\tfrac{1}{2}\langle X, XB\rangle'\big)\Phi(X), \quad B \in \mathfrak{s}_{\mathbf{n}}, \tag{2-3b}$$

$$\omega_\psi(\widetilde{w'_{U'}})\Phi(X) = \beta_\psi(w'_{U'})\hat{\Phi}(X). \tag{2-3c}$$

Here $\beta_\psi(g)$ for $g \in G'$ is a certain root of unity; moreover, $\beta_\psi(\varrho'(h)) = \gamma_\psi(\det h)$.

We extend ω_ψ to $V \rtimes \widetilde{G}$ by setting

$$\omega_\psi(v\tilde{g})\Phi = \psi(v_{1,2} + \cdots + v_{\mathbf{n}-1,\mathbf{n}})^{-1}\omega_\psi(v_{\mathcal{H}})(\omega_\psi(\tilde{g})\Phi), \quad v \in V,\ g \in G'. \tag{2-4}$$

Then for any $g \in G'$, $v \in V$ we have

$$\omega_\psi((\eta(g)v\eta(g)^{-1})\tilde{g})\Phi = \omega_\psi(\tilde{g})(\omega_\psi(v)\Phi). \tag{2-5}$$

2F. *Stable integral.* For the rest of the section, we assume F is p-adic.

Suppose that U_0 is a unipotent group over F with a fixed Haar measure du. Recall that the group generated by a relatively compact subset of U_0 is relatively compact. In particular, the set $\mathcal{CSGR}(U_0)$ is directed. Recall the following definition of stable integral from [Lapid and Mao 2015a].

Definition 2.1. Let f be a smooth function on U_0. We say that f has a *stable integral* over U_0 if there exists $U_1 \in \mathcal{CSGR}(U_0)$ such that for any $U_2 \in \mathcal{CSGR}(U_0)$ containing U_1, we have

$$\int_{U_2} f(u)\,du = \int_{U_1} f(u)\,du. \tag{2-6}$$

In this case, we write $\int^{\mathrm{st}}_{U_0} f(u)\,du$ for the common value of (2-6) and say that $\int^{\mathrm{st}}_{U_0} f(u)\,du$ stabilizes at U_1. In other words, $\int^{\mathrm{st}}_{U_0} f(u)\,du$ is the limit of the net

$$\left(\int_{U_1} f(u)\,du\right)_{U_1 \in \mathcal{CSGR}(U_0)}$$

with respect to the discrete topology of $\mathbb{C}$.

Given a family of functions $f_x \in C^{\mathrm{sm}}(U_0)$, we say that the integral $\int_{U_0}^{\mathrm{st}} f_x(u)\,du$ *stabilizes uniformly in* x if U_1 as above can be chosen independently of x. Similarly, if x ranges over a topological space X, then we say that $\int_{U_0}^{\mathrm{st}} f_x(u)\,du$ stabilizes locally uniformly in x if any $y \in X$ admits a neighborhood on which $\int_{U_0}^{\mathrm{st}} f_x(u)\,du$ stabilizes uniformly.

2G. ***A remark on convergence.*** We frequently make use of the following elementary remark.

Remark 2.2. Let $\mathbf{H}$ be any algebraic group over F and $\mathbf{H}'$ a closed subgroup. Assume that $\delta_H|_{H'} \equiv \delta_{H'}$. Suppose that $f \in C^{\mathrm{sm}}(H)$ and that the integral $\int_H f(h)\,dh$ converges absolutely. Then the same is true for $\int_{H'} f(h')\,dh'$.

3. Statement of main result

3A. ***Local Fourier–Jacobi transform.*** For any $f \in C(G)$ and $s \in \mathbb{C}$, we define $f_s(g) = f(g)\nu(g)^s$ for $g \in G$. Let $\pi \in \mathrm{Irr}_{\mathrm{gen}}\, M$ with Whittaker model $\mathbb{W}^{\psi_{N_M}}(\pi)$. Let $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ be the space of G-smooth left U-invariant functions $W : G \to \mathbb{C}$ such that for all $g \in G$, the function $\delta_P(m)^{-1/2} W(mg)$ on M belongs to $\mathbb{W}^{\psi_{N_M}}(\pi)$. For any $s \in \mathbb{C}$ we have a representation $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi), s)$ on the space $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ given by $(I(s, g)W)_s(x) = W_s(xg)$ for $x, g \in G$. It is equivalent to the induced representation of $\pi \otimes \nu^s$ from P to G. The family W_s, $s \in \mathbb{C}$ is a holomorphic section of this family of induced representations.

Let F be a p-adic field. As in [Ginzburg et al. 1998], for any $W \in C^{\mathrm{sm}}(N\backslash G, \psi_N)$ and $\Phi \in \mathcal{S}(F^{\mathbf{n}})$ define a genuine function on $\widetilde{G}$ by

$$A^{\psi}(W, \Phi, \tilde{g}) = \int_{V_\gamma \backslash V} W(\gamma v \eta(g)) \omega_{\psi^{-1}}(v\tilde{g}) \Phi(\xi_{\mathbf{n}})\,dv, \quad g \in G', \tag{3-1}$$

where the element $\gamma \in G$ and the groups V and V_γ are as defined in Section 2A. Its properties were studied in [Lapid and Mao 2017, §4]. In particular, the integrand is always compactly supported and A^{ψ} gives rise to a $V \rtimes \widetilde{G}$-intertwining map

$$A^{\psi} : C^{\mathrm{sm}}(N\backslash G, \psi_N) \otimes \mathcal{S}(F^{\mathbf{n}}) \to C^{\mathrm{sm}}(N'\backslash \widetilde{G}, \psi_{\widetilde{N}}), \tag{3-2}$$

where $V \rtimes \widetilde{G}$ acts via $V \rtimes \eta(G')$ by right translation on $C^{\mathrm{sm}}(N\backslash G, \psi_N)$, through $\omega_{\psi^{-1}}$ on $\mathcal{S}(F^{\mathbf{n}})$ and via the projection to $\widetilde{G}$ by right translation on $C^{\mathrm{sm}}(N'\backslash \widetilde{G}, \psi_{\widetilde{N}})$.

Let $V_+ \subset V$ be the image under ℓ_M of the space of $\mathbf{n} \times \mathbf{n}$ matrices whose rows are zero except possibly for the last one. For $c = \ell_M(x) \in V_+$, we denote by $\underline{c} \in F^{\mathbf{n}}$ the last row of x. Then

$$A^{\psi}(W(\cdot\, c), \Phi(\cdot + \underline{c}), \tilde{g}) = A^{\psi}(W, \Phi, \tilde{g}), \quad c \in V_+,\ g \in G'.$$

It follows that the function $A^{\psi}(W, \Phi, \cdot)$ factors through $W \otimes \Phi \mapsto \Phi * W$, where for any function $f \in C^{\infty}(G)$ we set

$$\Phi * f(g) = \int_{V_+} f(gc)\Phi(\underline{c})\, dc.$$

We denote by $A^{\psi}_{\sharp}$ the map

$$A^{\psi}_{\sharp} : C^{\mathrm{sm}}(N\backslash G, \psi_N) \to C^{\mathrm{sm}}(\widetilde{N}\backslash\tilde{G}, \psi_{\widetilde{N}})$$

such that

$$A^{\psi}(W, \Phi, \cdot) = A^{\psi}_{\sharp}(\Phi * W, \cdot).$$

(Informally, $A^{\psi}_{\sharp}(W, \cdot) = A^{\psi}(W, \delta_0, \cdot)$, where δ_0 is the delta function at 0.) The map $A^{\psi}_{\sharp}$ is no longer $\widetilde{G}$ or V-equivariant since neither $\widetilde{G}$ nor V (acting through $\omega_{\psi^{-1}}$) stabilizes δ_0. However, $A^{\psi}_{\sharp}$ satisfies the following equivariance property. Let $V_- = V_U \rtimes V^{\sharp}_M$, where $V_U = V \cap U$ and $V^{\sharp}_M \subset V_M$ is defined in Section 2A. Thus, V_- is the preimage of $\mathbb{V}_-$ under the composition $V \to V/V^{\sharp} \simeq \mathcal{H} \to \mathbb{V}$ and we have $V = V_+ \ltimes V_-$. Note that V_- is normalized by P'. Let ψ_{V_-} be the character on V_- given by

$$\psi_{V_-}(vu) = \psi_{N_M}(v)^{-1}\psi_U(u), \quad v \in V^{\sharp}_M,\ u \in V_U.$$

Lemma 3.1. *For any $v \in V_-$ and $p = mu \in P'$ where $m \in M'$ and $u \in U'$, we have*

$$A^{\psi}_{\sharp}(W(\cdot\, v\eta(p)), \tilde{g}) = \nu'(m)^{1/2}\beta_{\psi^{-1}}(m)^{-1}\psi_{V_-}(v)A^{\psi}_{\sharp}(W, \tilde{g}\tilde{p}).$$

Proof. The statement simply reflects the fact that $V_- \rtimes \widetilde{P}$ acts on δ_0 by multiplication by the character

$$\widetilde{m'}\widetilde{u'}v \mapsto \nu'(m')^{-1/2}\beta_{\psi^{-1}}(m')\psi^{-1}_{V_-}(v), \quad v \in V_-,\ m' \in M',\ u' \in U'.$$

More rigorously, fix W, p and v and let C be a small neighborhood of 0 in $F^{\mathbf{n}}$. Suppose that $\Phi \in C^{\infty}_c(F^{\mathbf{n}})$ is supported in C and that $\int_{F^{\mathbf{n}}} \Phi(c)\, dc = 1$. Then $A^{\psi}(W, \Phi, \tilde{g}) = A^{\psi}_{\sharp}(W, \tilde{g})$ for all $g \in G'$ provided that C is sufficiently small. On the other hand, if $\Phi' = \omega_{\psi^{-1}}(v\tilde{p})\Phi$ then

$$A^{\psi}(W(\cdot\, v\eta(p)), \Phi', \tilde{g}) = A^{\psi}(W, \Phi, \tilde{g}\tilde{p}).$$

By (2-1b), (2-1c), (2-3a), (2-3b) and (2-4), $\operatorname{supp}\Phi' \subset Cm^{-1}$ and, when C is sufficiently small,

$$\int_{F^{\mathbf{n}}} \Phi'(c)\, dc = \psi_{V_-}(v)^{-1}\beta_{\psi^{-1}}(m)\nu'(m)^{-1/2}.$$

Thus, if C is small enough then

$$\Phi' * W(\cdot\, vp) = \psi_{V_-}(v)^{-1}\beta_{\psi^{-1}}(m)\nu'(m)^{-1/2}W(\cdot\, vp).$$

Hence, $A^{\psi}(W(\cdot\, v\eta(p)), \Phi', \tilde g) = \psi_{V_-}(v)^{-1}\beta_{\psi^{-1}}(m)\nu'(m)^{-1/2}A^{\psi}_{\sharp}(W(\cdot\, v\eta(p)), \tilde g)$. The lemma follows. □

For later reference we record the following result, which is implicit in the proof of [Lapid and Mao 2017, Lemma 4.5].

Lemma 3.2. *Suppose that F is p-adic. Then for any $K_0 \in \mathcal{CSGR}(G)$, there exists $\Omega \in \mathcal{CSGR}(V_U)$ such that for any $W \in C(N\backslash G, \psi_N)^{K_0}$, the support of $W(\gamma\,\cdot)|_{V_-}$ is contained in $V_\gamma\eta(w'_{U'})\Omega\eta(w'_{U'})^{-1}$.*

Remark 3.3. When F is an archimedean field, the above discussion still holds for $W \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ (or more generally if $W \in C^{\mathrm{sm}}(N\backslash G, \psi_N)$ is of moderate growth). From [Lapid and Mao 2017, Lemma 4.9], the integral (3-1) converges. Moreover, $A^{\psi}(W, \Phi, \cdot)$ factors through $W \otimes \Phi \mapsto \Phi * W$, and the induced linear form $A^{\psi}_{\sharp}$ satisfies the equivariance property in Lemma 3.1 (by an approximate identity argument).

3B. ***Explicit local descent.*** Let F be a local field of characteristic 0. Define the intertwining operator

$$M(\pi, s) = M(s) : \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi), s) \to \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi^\vee), -s)$$

by (the analytic continuation of)

$$M(s)W(g) = \nu(g)^s \int_U W_s(\varrho(\mathfrak{t})w_U ug)\, du, \tag{3-3}$$

where $\mathfrak{t} = \operatorname{diag}(1, -1, \dots, 1, -1)$ is introduced in order to preserve the character ψ_{N_M}. By abuse of notation we also denote by $M(\pi, s)$ the intertwining operator $\operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi), s\big) \to \operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi^\vee), -s\big)$ defined in the same way.

For simplicity, we define $M^*_s W := (M(s)W)_{-s}$, so that

$$M^*_s W = \int_U W_s(\varrho(\mathfrak{t})w_U u\,\cdot)\, du$$

for $\operatorname{Re} s \gg_\pi 1$. Set $M^*W := M^*_{1/2}W$.

Recall that $H_{\mathbb{M}}$ is the centralizer of $E = \operatorname{diag}(1, -1, \dots, 1, -1) (= \mathfrak{t})$, isomorphic to $\mathrm{GL}_{\mathbf{n}} \times \mathrm{GL}_{\mathbf{n}}$. The involution $w_0^{\mathbb{M}}$ lies in the normalizer of $H_{\mathbb{M}}$. We consider the class $\operatorname{Irr}_{\mathrm{meta}} \mathbb{M}$ of irreducible representations of $\mathbb{M}$ which admit a continuous nonzero $H_{\mathbb{M}}$-invariant linear form ℓ on the space of π. It is known that any such π is self-dual and ℓ is unique up to a scalar [Jacquet and Rallis 1996; Aizenbud and Gourevitch 2009]. Thus, $\ell \circ \pi(w_0^{\mathbb{M}}) = \epsilon_\pi \ell$, where $\epsilon_\pi \in \{\pm 1\}$ does not depend on the choice of ℓ. By [Lapid and Mao 2017, Theorem 3.2], when F is p-adic, for any $\pi \in \operatorname{Irr}_{\mathrm{meta,gen}} \mathbb{M}$ we have

$$\epsilon_\pi = \epsilon\big(\tfrac{1}{2}, \pi, \psi\big), \tag{3-4}$$

where $\epsilon(s, \pi, \psi)$ is the standard ϵ-factor attached to π.

Let $\pi \in \mathrm{Irr}_{\mathrm{gen,meta}}\,\mathbb{M}$, considered also as a representation of M via ϱ. By [Lapid and Mao 2017, Proposition 4.1], $M(s)$ is holomorphic at $s=\frac{1}{2}$. Denote by $\mathcal{D}_\psi(\pi)$ the space of Whittaker functions on $\widetilde{G}$ generated by $A^\psi(M^*W,\Phi,\cdot)$, $W\in\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$, $\Phi\in\mathcal{S}(F^{\mathbf{n}})$, i.e.,

$$\mathcal{D}_\psi(\pi)=\{A^\psi_\sharp(M^*W,\cdot):W\in\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))\}.$$

This defines an explicit descent map $\pi\mapsto\mathcal{D}_\psi(\pi)$ on $\mathrm{Irr}_{\mathrm{gen,meta}}\,\mathbb{M}$. By [Ginzburg et al. 1999, theorem in §1.3], $\mathcal{D}_\psi(\pi)\neq 0$. It can be shown that $\mathcal{D}_\psi(\pi)$ is admissible but we do not do it here since we do not use this fact directly.

3C. Good *representations.* Consider $\pi\in\mathrm{Irr}_{\mathrm{gen}}\,M$ and $\tilde\sigma\in\mathrm{Irr}_{\mathrm{gen},\psi^{-1}_{\widetilde{N}}}\,\widetilde{G}$ with Whittaker model $\mathbb{W}^{\psi^{-1}_{\widetilde{N}}}(\tilde\sigma)$. Following [Ginzburg et al. 1998], for any

$$\widetilde{W}\in\mathbb{W}^{\psi^{-1}_{\widetilde{N}}}(\tilde\sigma),\qquad W\in\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi)),$$

define the local Shimura type integral

$$\tilde J(\widetilde{W},W,s):=\int_{N'\backslash G'}\widetilde{W}(\tilde g)A^\psi_\sharp(W_s,\tilde g)\,dg. \tag{3-5}$$

By [Ginzburg et al. 1998, §6.3; 1999], $\tilde J$ converges for $\mathrm{Re}\,s\gg_{\pi,\tilde\sigma}1$ and admits a meromorphic continuation in s. Moreover, for any $s\in\mathbb{C}$ we can choose $\widetilde{W}$ and W such that $\tilde J(\widetilde{W},W,s)\neq 0$.

Let $\pi\in\mathrm{Irr}_{\mathrm{gen,meta}}\,\mathbb{M}$. We say that π is *good* if the following conditions are satisfied for all ψ:

(1) $\mathcal{D}_\psi(\pi)$ is irreducible.

(2) $\tilde J(\widetilde{W},W,s)$ is holomorphic at $s=\frac{1}{2}$ for $\widetilde{W}\in\mathcal{D}_{\psi^{-1}}(\pi)$, $W\in\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$. (We do not assume that the integral defining $\tilde J$ converges at $s=\frac{1}{2}$.)

(3) There is a nondegenerate $\widetilde{G}$-invariant pairing $[\cdot,\cdot]$ on $\mathcal{D}_{\psi^{-1}}(\pi)\times\mathcal{D}_\psi(\pi)$ such that
$$\tilde J\big(\widetilde{W},W,\tfrac{1}{2}\big)=[\widetilde{W},A^\psi_\sharp(M^*W,\cdot)]$$
for any $\widetilde{W}\in\mathcal{D}_{\psi^{-1}}(\pi)$, $W\in\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$. In particular, $\mathcal{D}_\psi(\pi)^\vee\simeq\mathcal{D}_{\psi^{-1}}(\pi)$.

This property was introduced and discussed in [Lapid and Mao 2017, §5]. In particular, if π is good and $\tilde\pi=\mathcal{D}_{\psi^{-1}}(\pi)$ then there is a constant c_π such that for any $\widetilde{W}\in\mathbb{W}^{\psi^{-1}_{\widetilde{N}}}(\tilde\pi^\vee)$, $\widetilde{W}^\vee\in\mathbb{W}^{\psi_{\widetilde{N}}}(\tilde\pi)$, we have

$$\int^{\mathrm{st}}_{N'}[\tilde\pi(\tilde n)\widetilde{W},\widetilde{W}^\vee]\psi_{\widetilde{N}}(n)\,dn=c_\pi\widetilde{W}(e)\widetilde{W}^\vee(e). \tag{3-6}$$

More explicitly, for any $W\in\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ and $W^\wedge\in\mathrm{Ind}(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi))$ we have

$$\int_{N'}^{\mathrm{st}} \tilde{J}\big(A_{\sharp}^{\psi^{-1}}(M^* W^{\wedge}, \cdot\, \tilde{n}), W, \tfrac{1}{2}\big)\psi_{\widetilde{N}}(n)\, dn \\ = c_\pi A_{\sharp}^{\psi^{-1}}(M^* W^{\wedge}, e) A_{\sharp}^{\psi}(M^* W, e). \quad (3\text{-}7)$$

Let F be a p-adic field. In the rest of the paper, we prove the following statement:

Theorem 3.4. *For any unitarizable $\pi \in \mathrm{Irr}_{\mathrm{gen,meta}}\, \mathbb{M}$ which is good we have $c_\pi = \epsilon_\pi$.*

Remark 3.5. Of course, we expect Theorem 3.4 to hold in the archimedean case as well. However, we do not deal with the archimedean case in this paper.

The following special case is of particular importance (see Remark 3.9 below).

Corollary 3.6. *Suppose $\pi = \tau_1 \times \cdots \times \tau_l$, where $\tau_j \in \mathrm{Irr}_{\mathrm{sqr,meta}}\, \mathrm{GL}_{2m_j}$, $j = 1, \dots, l$, are distinct and $\mathbf{n} = m_1 + \cdots + m_l$. Then π is good and $\tilde{\pi} := \mathcal{D}_{\psi^{-1}}(\pi)$ is square-integrable. Moreover,*

$$\int_{N'} \tilde{J}\big(\tilde{\pi}(\tilde{n})\widetilde{W}, W, \tfrac{1}{2}\big)\psi_{\widetilde{N}}(n)\, dn = \epsilon_\pi \widetilde{W}(e) A_{\sharp}^{\psi}(M^* W, e)$$

for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ and $\widetilde{W} \in \mathbb{W}^{\psi_{\widetilde{N}}^{-1}}(\tilde{\pi})$.

3D. *Relation to global statement.* Suppose now that F is a number field and $\mathbb{A}$ is its ring of adeles. We say that an irreducible cuspidal representation π of $\mathbb{M}$ is of *metaplectic type* if

$$\int_{H_{\mathbb{M}}(F)\backslash H_{\mathbb{M}}(\mathbb{A}) \cap \mathbb{M}(\mathbb{A})^1} \varphi(h)\, dh \neq 0$$

for some φ in the space of π. (Here, $\mathbb{M}(\mathbb{A})^1 = \{m \in \mathbb{M}(\mathbb{A}) : |\det m| = 1\}$.) Equivalently, $L^S\big(\frac{1}{2}, \pi\big) \operatorname{res}_{s=1} L^S(s, \pi, \wedge^2) \neq 0$ [Friedberg and Jacquet 1993].[2] In particular, π is self-dual and admits a trivial central character. We write $\mathrm{Cusp}_{\mathrm{meta}}\, \mathbb{M}$ for the set of irreducible cuspidal representations of metaplectic type.

Consider the set $\mathrm{MCusp}\, \mathbb{M}$ of automorphic representations π of $\mathbb{M}(\mathbb{A})$ which are realized on Eisenstein series induced from $\pi_1 \otimes \cdots \otimes \pi_k$, where $\pi_i \in \mathrm{Cusp}_{\mathrm{meta}}\, \mathrm{GL}_{2n_i}$, $i = 1, \dots, k$, are distinct and $\mathbf{n} = n_1 + \cdots + n_k$. The representation π is irreducible: it is equivalent to the parabolic induction $\pi_1 \times \cdots \times \pi_k$. Moreover, π determines $\pi_1, \dots, \pi_k$ uniquely up to permutation [Jacquet and Shalika 1981a; 1981b].

Recall that Conjecture 1.1 is pertaining to the representations $\pi \in \mathrm{MCusp}\, \mathbb{M}$. The following fact is crucial for us.

Proposition 3.7 [Lapid and Mao 2017, Theorem 6.2]. *If $\pi \in \mathrm{MCusp}\, \mathbb{M}$, then its local components π_v are good.*

Thus, Theorem 3.4 implies Theorem 1.3, our main result.

[2]We can replace the partial L-function by the completed one since the local factors are holomorphic and nonzero.

3E. ***Relation to Bessel functions.*** We record here a *purely formal* argument that relates Theorem 3.4 to an identity of Bessel functions, which are defined in [Lapid and Mao 2013]. We do not worry about convergence issues in this subsection.

Using the functional equation (see [Kaplan 2015], noting that the central character of π is trivial)

$$\tilde{J}(\widetilde{W}, M(s)W, -s) = |2|^{2\mathbf{n}s} \frac{\gamma\big(\tilde{\pi} \otimes \pi, s+\frac{1}{2}, \psi\big)}{\gamma(\pi, s, \psi)\gamma(\pi, \wedge^2, 2s, \psi)} \tilde{J}(\widetilde{W}, W, s), \tag{3-8}$$

(3-6) becomes

$$\int_{N'} \tilde{J}\big(\tilde{\pi}(\tilde{n})\widetilde{W}, M\big(\tfrac{1}{2}\big)W, -\tfrac{1}{2}\big)\psi_{\widetilde{N}}(n)\,dn \\ = |2|^{\mathbf{n}} \frac{\gamma\big(\tilde{\pi} \otimes \pi, s+\frac{1}{2}, \psi\big)}{\gamma(\pi, s, \psi)\gamma(\pi, \wedge^2, 2s, \psi)}\bigg|_{s=1/2} c_\pi \widetilde{W}(e) A_\sharp^\psi(M^* W, e).$$

Explicitly, the left-hand side is

$$\int_{N'} \int_{N' \backslash G'} \widetilde{W}(\tilde{g}\tilde{n}) A_\sharp^\psi(M^* W, \tilde{g})\,dg\,\psi_{\widetilde{N}}(n)\,dn \\ = \int_{N'} \int_{N' \backslash G'} \widetilde{W}(\tilde{g}\tilde{n}) \widetilde{W}'(\tilde{g})\,dg\,\psi_{\widetilde{N}}(n)\,dn,$$

where $\widetilde{W}' = A_\sharp^\psi(M^* W, \cdot)$. Using Bruhat decomposition, this is

$$\int_{T'} \int_{N'} \int_{N'} \delta_{B'}(t) \widetilde{W}\big(\widetilde{w_0'}\tilde{t}\widetilde{n'}\tilde{n}\big) \widetilde{W}'\big(\widetilde{w_0'}\tilde{t}\widetilde{n'}\big)\psi_{\widetilde{N}}(n)\,dn'\,dn\,dt \\ = \int_{T'} \int_{N'} \int_{N'} \delta_{B'}(t) \widetilde{W}\big(\widetilde{w_0'}\tilde{t}\tilde{n}\big) \widetilde{W}'\big(\widetilde{w_0'}\tilde{t}\widetilde{n'}\big)\psi_{\widetilde{N}}(n)\psi_{\widetilde{N}}^{-1}(n')\,dn'\,dn\,dt.$$

By definition of Bessel functions $\mathbb{B}_{\tilde{\pi}}^{\psi_{\widetilde{N}}^{-1}}$ (see (5-1)), this is

$$\widetilde{W}(e)\widetilde{W}'(e) \int_{T'} \mathbb{B}_{\tilde{\pi}}^{\psi_{\widetilde{N}}^{-1}}\big(\widetilde{w_0'}\tilde{t}\big) \mathbb{B}_{\tilde{\pi}^\vee}^{\psi_{\widetilde{N}}}\big(\widetilde{w_0'}\tilde{t}\big)\delta_{B'}(t)\,dt.$$

Here $\tilde{\pi}^\vee = \mathcal{D}_\psi(\pi)$. Thus, on a formal level, Theorem 3.4 becomes the following inner product identity:

$$\int_{T'} \mathbb{B}_{\tilde{\pi}}^{\psi_{\widetilde{N}}^{-1}}\big(\widetilde{w_0'}\tilde{t}\big) \mathbb{B}_{\tilde{\pi}^\vee}^{\psi_{\widetilde{N}}}\big(\widetilde{w_0'}\tilde{t}\big)\delta_{B'}(t)\,dt = |2|^{\mathbf{n}} \frac{\gamma\big(\tilde{\pi} \otimes \pi, s+\frac{1}{2}, \psi\big)}{\gamma(\pi, \wedge^2, 2s, \psi)}\bigg|_{s=1/2}. \tag{3-9}$$

(It is easy to determine the order of the poles in the numerator and denominator on the right-hand side in terms of the data of Theorem 3.8 below.) Of course, the above manipulations are purely formal as the integrals are not absolutely convergent.

3F. ***A reduction of the main theorem.*** For the rest of the paper, let F be a p-adic field. The proof of Theorem 3.4 is essentially local under the additional assumption that $\pi \in \mathrm{Irr}_{\mathrm{temp}}\,\mathbb{M}$ and $\tilde{\pi} := \mathcal{D}_{\psi^{-1}}(\pi) \in \mathrm{Irr}_{\mathrm{temp}}\,\widetilde{G}$. We first show that it indeed suffices to prove Theorem 3.4 under these additional assumptions. This uses a global argument as well as the following classification result due to Matringe. We denote by $\times$ parabolic induction for GL_m.

Theorem 3.8 [Matringe 2015]. *The set* $\mathrm{Irr}_{\mathrm{gen,meta}}\,\mathbb{M}$ *consists of the irreducible representations of the form*

$$\pi = \sigma_1 \times \sigma_1^\vee \times \cdots \times \sigma_k \times \sigma_k^\vee \times \tau_1 \times \cdots \times \tau_l,$$

with $\sigma_1, \dots, \sigma_k$ *essentially square-integrable,*[3] $\tau_1, \dots, \tau_l$ *square-integrable and* $L(0, \tau_i, \wedge^2) = \infty$ *for all* i.

(We expect the same result to hold in the archimedean case as well.)

Denote by ω_σ the central character of $\sigma \in \mathrm{Irr}\,\mathrm{GL}_m$. We also write $\sigma[s]$ for the twist of σ by $|\det\cdot|^s$.

Let $\tau_j \in \mathrm{Irr}_{\mathrm{sqr,meta}}\,\mathrm{GL}_{2m_j}$, $j = 1, \dots, l$, be distinct, and let $\delta_i \in \mathrm{Irr}_{\mathrm{sqr}}\,\mathrm{GL}_{n_i}$, $i = 1, \dots, k$ with $\mathbf{n} = n_1 + \cdots + n_k + m_1 + \cdots + m_l$. (Possibly $k = 0$ or $l = 0$.) For $\underline{s} = (s_1, \dots, s_k) \in \mathbb{C}^k$ we consider the representation

$$\pi(\underline{s}) = \delta_1[s_1] \times \delta_1^\vee[-s_1] \times \cdots \times \delta_k[s_k] \times \delta_k^\vee[-s_k] \times \tau_1 \times \cdots \times \tau_l.$$

Suppose that our given p-adic field is the completion at a place v of a number field. We claim that for a dense set of $\underline{s} \in \mathrm{i}\mathbb{R}^k$, $\pi(\underline{s})$ is the local component at v of an element of $\mathrm{MCusp}\,\mathbb{M}$. This follows from [Ichino et al. 2017, Appendix A].[4] Indeed, let $m = m_1 + \cdots + m_l$, let $\rho \in \mathrm{Irr}_{\mathrm{sqr,gen}}\,\mathrm{SO}(2m+1)$ be the representation corresponding to $\tau_1 \times \cdots \times \tau_l$ under Jiang and Soudry [2004] and let $\tilde{\rho} \in \mathrm{Irr}_{\mathrm{sqr,gen}}\,\widetilde{\mathrm{Sp}}_m$ be the theta lift of ρ. Let $\sigma(\underline{s}) = \delta_1[s_1] \times \cdots \times \delta_k[s_k] \rtimes \rho$ (parabolic induction) and $\tilde{\sigma}(\underline{s}) = \delta_1[s_1] \times \cdots \times \delta_k[s_k] \rtimes \tilde{\rho}$. Then for $\underline{s}$ in a dense open subset of $\mathrm{i}\mathbb{R}^k$ we have $\sigma(\underline{s}) \in \mathrm{Irr}\,\mathrm{SO}(2\mathbf{n}+1)$ and $\tilde{\sigma}(\underline{s}) \in \mathrm{Irr}\,\widetilde{G}$ and moreover, by [Gan and Savin 2012], $\tilde{\sigma}(\underline{s})$ is the theta lift of $\sigma(\underline{s})$. By [Ichino et al. 2017, Corollary A.8], for a dense set of $\underline{s} \in \mathrm{i}\mathbb{R}^k$, $\tilde{\sigma}(\underline{s})$ is the local component at v of a generic cuspidal automorphic representation of $\widetilde{\mathrm{Sp}}_{\mathbf{n}}(\mathbb{A})$ whose theta lift to $\mathrm{SO}(2\mathbf{n}+1)$ is cuspidal. The Cogdell–Kim–Piatetski-Shapiro–Shahidi lift [Cogdell et al. 2004] of the latter to $\mathbb{M}$ is the required representation in $\mathrm{MCusp}\,\mathbb{M}$.

It follows from Proposition 3.7 that for a dense set of $\underline{s} \in \mathrm{i}\mathbb{R}^k$, $\pi(\underline{s}) \in \mathrm{Irr}_{\mathrm{temp,meta}}\,\mathbb{M}$ is good. Moreover, by [Ichino et al. 2017, Proposition 4.6], $\mathcal{D}_{\psi^{-1}}(\pi(\underline{s})) = \tilde{\sigma}(\underline{s})$, and in particular, it is tempered.

[3] That is, a twist of a square-integrable representation by a quasicharacter.

[4] We remark that the appendices and §4 of [Ichino et al. 2017], and in particular the proof of [Ichino et al. 2017, Theorem 3.1], are independent of the results of the current paper.

Suppose that $\underline{s}$ is in the domain

$$\mathfrak{D} = \left\{(s_1, \dots, s_k) \in \mathbb{C}^k : -\tfrac{1}{2} < \operatorname{Re} s_i < \tfrac{1}{2} \text{ for all } i\right\}.$$

We recall that by [Lapid and Mao 2017, Lemma 4.12], the integral defining $\tilde{J}\big(A_\sharp^{\psi^{-1}}(M^* W^\wedge, \cdot), W, s\big)$ converges and is holomorphic at $s = \frac{1}{2}$ for any

$$W \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi(\underline{s}))), \qquad W^\wedge \in \operatorname{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi(\underline{s}))\big). \tag{3-10}$$

Moreover, by the properties of A^ψ and $\widetilde{J}$ (see [Lapid and Mao 2017, (4.4) and (4.13)]), the function $g \mapsto \tilde{J}\big(A_\sharp^{\psi^{-1}}(M^* W^\wedge, \cdot\, \tilde{g}), W, \frac{1}{2}\big)$ is bi-K_0-invariant, provided that $K_0 \in \mathcal{CSGR}^s(G')$ is such that $I\big(\frac{1}{2}, \eta(k)\big)W = W$ and $I\big(\frac{1}{2}, \eta(k)\big)W^\wedge = W^\wedge$ for all $k \in K_0$. It follows from [Lapid and Mao 2015a, Proposition 2.11] (applied to $\widetilde{G}$) that the stable integral on the left-hand side of (3-7) can be written as an integral over a compact open subgroup of N' depending only on K_0. Thus, taking the Whittaker functions in (3-10) to be defined through Jacquet integrals, both sides of (3-7) for $\pi(\underline{s})$ are holomorphic functions of $\underline{s} \in \mathfrak{D}$. Note that by [Lapid and Mao 2017, Lemma 3.6], we have that $\epsilon_{\pi(\underline{s})} = \omega_{\delta_1}(-1) \cdots \omega_{\delta_k}(-1)\epsilon_{\tau_1} \cdots \epsilon_{\tau_l}$ is independent of $\underline{s}$. Thus, in order to prove (3-7) for $\underline{s} \in \mathfrak{D}$, it is enough to show it for a dense set of $\underline{s} \in \mathrm{i}\mathbb{R}^k$. Since every unitarizable $\pi \in \operatorname{Irr}_{\mathrm{meta}} \mathbb{M}$ is of the form $\pi(\underline{s})$ for some $\tau_1, \dots, \tau_l, \delta_1, \dots, \delta_k$ as above and $\underline{s} \in \mathfrak{D}$, the reduction step follows.

Remark 3.9. In the case $k = 0$, we can globalize π itself. Thus, by Proposition 3.7, π is good; $\mathcal{D}_{\psi^{-1}}(\pi)$ is irreducible and square integrable (see [Ichino et al. 2017, Theorem 3.1]).[4] This yields Corollary 3.6 from Theorem 3.4.

In the remainder of the paper we prove Theorem 3.4, i.e., the main identity

$$\int_{N'}^{\mathrm{st}} \left(\int_{N' \backslash G'} A_\sharp^\psi(W_s, \tilde{g}) A_\sharp^{\psi^{-1}}(M^* W^\wedge, \tilde{g}\tilde{u})\, dg \right) \psi_{\widetilde{N}}(\tilde{u})\, du \Big|_{s=1/2} = \epsilon_\pi A_\sharp^{\psi^{-1}}(M^* W^\wedge, e) A_\sharp^\psi(M^* W, e) \tag{MI}$$

under the assumptions that $\pi \in \operatorname{Irr}_{\mathrm{temp,meta}} \mathbb{M}$ is good and $\tilde{\pi} := \mathcal{D}_{\psi^{-1}}(\pi)$ is tempered.

Remark 3.10. As already pointed out before (and used in the reduction step), by [Lapid and Mao 2017, Lemma 4.12] the inner integral on the left-hand side of (MI) converges (and is analytic) for $\operatorname{Re} s \geq \frac{1}{2}$ (for any unitary π). We will not need to use this fact explicitly any further.

4. An informal sketch

Before embarking on the rigorous proof of the relation (MI), let us give a brief sketch of the argument. For convenience we momentarily ignore all convergence issues (which are of course at the heart of the argument, and will be discussed below). Thus, the discussion of this section is *purely formal*.

Define (assuming convergent)

$$B(W,W^\vee)=\int_{N'}\left(\int_{N'\backslash G'}A^{\psi}_{\sharp}(W,\tilde g)A^{\psi^{-1}}_{\sharp}(W^\vee,\tilde g\tilde u)\,dg\right)\psi_{\widetilde N}(\tilde u)\,du,$$

so that the left-hand side of (MI) is $B(W_{1/2}, M^*W)$.

We use the following formal identity (which follows from the Bruhat decomposition for G'): for (suitable) $\widetilde W\in C^{\mathrm{sm}}(\widetilde N\backslash\widetilde G,\psi_{\widetilde N})$ and $\widetilde W^\vee\in C^{\mathrm{sm}}(\widetilde N\backslash\widetilde G,\psi_{\widetilde N}^{-1})$,

$$\begin{aligned}\int_{N'}&\left(\int_{N'\backslash G'}\widetilde W(\tilde g)\widetilde W^\vee(\tilde g\tilde u)\,dg\right)\psi_{\widetilde N}(\tilde u)\,du\\&=\int_{T'}\delta_{B'}(t)\int_{N'}\widetilde W(\widetilde{w_0't}\tilde n_1)\psi_{\widetilde N}(\tilde n_1)^{-1}\,dn_1\int_{N'}\widetilde W^\vee(\widetilde{w_0't}\tilde n_2)\psi_{\widetilde N}(\tilde n_2)\,dn_2\,dt.\qquad(4\text{-}1)\end{aligned}$$

(In the ensuing discussion it would be more natural to replace the right-hand side by

$$\begin{aligned}\int_{N'_{M'}}\int_{N'_{M'}\backslash M'}&\delta_{P'}(m)\int_{U'}\widetilde W(\widetilde{w_{U'}'}\tilde m\tilde n\widetilde{u_1})\psi_{\widetilde N}(\widetilde{u_1})^{-1}\,du_1\\&\int_{U'}\widetilde W^\vee(\widetilde{w_{U'}'}\tilde m\widetilde{u_2})\psi_{\widetilde N}(\widetilde{u_2})\,du_2\,dm\,\psi_{N'_{M'}}(n)\,dn.\qquad(4\text{-}2)\end{aligned}$$

However, it is analytically advantageous to work with the former expression.)

Applying this to $\widetilde W=A^{\psi}_{\sharp}(W,\cdot)$ and $\widetilde W^\vee=A^{\psi^{-1}}_{\sharp}(W^\vee,\cdot)$ we get

$$B(W,W^\vee)=\int_{T'}Y^{\psi}(W,t)Y^{\psi^{-1}}(W^\vee,t)\delta_{B'}(t)\,dt,\qquad(4\text{-}3)$$

where

$$Y^{\psi}(W,t):=\int_{N'}A^{\psi}_{\sharp}(W,\widetilde{w_0'tn})\psi_{\widetilde N}(\tilde n)^{-1}\,dn.$$

We note the following equivariance property of $Y^{\psi}(W,t)$. Let $N^\sharp=V_-\rtimes\eta(N')$, which is the stabilizer of ψ_U in N. Let $\psi_{N^\sharp}$ be the character

$$\psi_{N^\sharp}(\eta(n)v)=\psi_{N'}(n)\psi_{V_-}(v),\quad n\in N',\ v\in V_-$$

on $N^\sharp$. Then by Lemma 3.1, $Y^{\psi}(W,t)$ is $(N^\sharp,\psi_{N^\sharp})$-equivariant in W.

Next, we consider $A^{\psi}_{\sharp}(W,\widetilde{w_0'tn})$. By Lemma 3.1 it is enough to determine $A^{\psi}_{\sharp}(W,\widetilde{w_{U'}'})$. We have (up to an immaterial root of unity, which we suppress)

$$A^{\psi}_{\sharp}(W,\widetilde{w_{U'}'})=*\int_{V_M^\sharp\backslash V_-}W(w_Uv)\psi_{V_-}(v)^{-1}\,dv=*\int_{V_U}W(w_Uv)\psi_U(v)^{-1}\,dv.$$

(See Remark 6.5 for a more precise statement.) We also mention that

$$A^{\psi}_{\sharp}(W,e)=\int_{\epsilon_1^{-1}(V_\gamma\rtimes\eta(N'))\epsilon_1\backslash N^\sharp}W(\gamma\epsilon_1n)\psi_{N^\sharp}(n)^{-1}\,dn$$

for a suitable unipotent element ϵ_1 (this will be used for the right-hand side of (MI); see Lemma 6.1). This expression is clearly $(N^\sharp,\psi_{N^\sharp})$-equivariant in W.

Remark 4.1. By [Ginzburg et al. 1999, theorem in §4.4; Lapid and Mao 2015b, Lemma 6.1], for $\pi \in \mathrm{Irr}_{\mathrm{meta,temp}}\,\mathbb{M}$ the space of $(N^\sharp, \psi_{N^\sharp})$-equivariant linear forms on the Langlands quotient of $\mathrm{Ind}\big(\mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi), \tfrac12\big)$ is one-dimensional. This implies that $Y^\psi(M^*W, t)$ is proportional to $A^\psi_\sharp(M^*W, e)$. The constant of proportionality is a complicated function of t which is probably related to the Bessel function. At best, the putative equality becomes something like (3-9), which seems difficult to approach directly. Instead, we take a different approach.

It follows from the formula for $A^\psi_\sharp(W, \widetilde{w'_{U'}})$ above that

$$Y^\psi(W,t) = *\nu'(t)^{\mathbf{n}-1/2}\int_{N'_{M'}}\int_U W(w_U\eta(w_0^{M'}tn)v)\psi_{N'_{M'}}(n)^{-1}\psi_U(u)^{-1}\,du\,dn. \tag{4-4}$$

Substituting this in (4-3) and using the Bruhat decomposition for $\mathbb{M}'$, we obtain

$$B(W,W^\vee) = \int_U\int_U\int_{N'_{\mathbb{M}'}}\int_{N'_{\mathbb{M}'}\backslash\mathbb{M}'} W(\eta_M(gn)w_Uu_1)W^\vee(\eta_M(g)w_Uu_2) \times \delta_P(\eta_M(g))^{-1}|\det g|^{1-\mathbf{n}}\,dg\,\psi_{N'_{\mathbb{M}'}}(n)\,dn\,\psi_U(u_1)^{-1}\psi_U(u_2)\,du_1\,du_2. \tag{4-5}$$

For the inner integral, we will use the following functional equation proved in [Lapid and Mao 2014]: for any $W \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)$ and $W^\vee \in \mathbb{W}^{\psi^{-1}_{N_{\mathbb{M}}}}(\pi^\vee)$, we have

$$\int_{N'_{\mathbb{M}'}\backslash\mathbb{M}'} W(\eta_{\mathbb{M}}(g))W^\vee(\eta_{\mathbb{M}}(g))|\det g|^{1-\mathbf{n}}\,dg = \int_J\int_J\int_{N'_{\mathbb{M}'}\backslash\mathbb{M}'} W(w_{2\mathbf{n},\mathbf{n}}\eta_{\mathbb{M}}(g)\ell_{\mathbb{M}}(X))W^\vee(w_{2\mathbf{n},\mathbf{n}}\eta_{\mathbb{M}}(g)\ell_{\mathbb{M}}(Y)) \times |\det g|^{\mathbf{n}-1}\,dg\,dX\,dY,$$

where J is the subspace of $\mathrm{Mat}_{\mathbf{n}}$ consisting of the matrices whose first column is zero. From this it is easy to infer that

$$\int_{N'_{\mathbb{M}'}}\left(\int_{N'_{\mathbb{M}'}\backslash\mathbb{M}'} W(\eta_{\mathbb{M}}(gn))W^\vee(\eta_{\mathbb{M}}(g))|\det g|^{1-\mathbf{n}}\,dg\right)\psi_{N'_{\mathbb{M}'}}(n)\,dn = \int_{\eta^\vee_{\mathbb{M}}(N'_{\mathbb{M}'})\backslash N^\flat_{\mathbb{M}}}\int_{\eta^\vee_{\mathbb{M}}(N'_{\mathbb{M}'})\backslash N^\flat_{\mathbb{M}}}\int_{T'_{\mathbb{M}'}} W(\eta^\vee_{\mathbb{M}}(t)\boldsymbol{\epsilon}_3r_1)W^\vee(\eta^\vee_{\mathbb{M}}(t)\boldsymbol{\epsilon}_3r_2) \times |\det t|^{\mathbf{n}-1}\delta_{B'_{\mathbb{M}'}}(t)^{-1}\psi_{N^\flat_{\mathbb{M}}}(r_1^{-1}r_2)\,dt\,dr_2\,dr_1, \tag{4-6}$$

where $N^\flat_{\mathbb{M}} = (N^\sharp_{\mathbb{M}})^*$, $\psi_{N^\flat_{\mathbb{M}}}$ is the character on $N^\flat_{\mathbb{M}}$ with

$$\psi_{N^\flat_{\mathbb{M}}}(m) = \psi_{N^\sharp_{\mathbb{M}}}(m^*)$$

and $\boldsymbol{\epsilon}_3 \in \mathbb{M}$ is a suitable element to be introduced in Section 7B. Substituting this in (4-5), we obtain

$$B(W,W^\vee)=\int_{\eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'})} E^\psi(W,t)E^{\psi^{-1}}(W^\vee,t)\,\frac{dt}{|\det t|},$$

where

$$E^\psi(W,t):=\delta_B^{-1/2}(\varrho(t))\int_{\eta_M(N'_{\mathbb{M}'})\backslash N^\sharp} W(\varrho(t\boldsymbol{\epsilon}_4\boldsymbol{\epsilon}_3)w_U v)\psi_{N^\sharp}(v)^{-1}\,dv,\quad t\in\eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'}).$$

Here, $\boldsymbol{\epsilon}_4\in\mathbb{M}$ is an auxiliary unipotent element chosen so that $\varrho(\boldsymbol{\epsilon}_4\boldsymbol{\epsilon}_3)w_U$ conjugates ψ_U to a character which is supported on a single (short negative) root. The sought-after identity (MI) is now a consequence of the following two results concerning $(N^\sharp,\psi_{N^\sharp})$-equivariant linear forms in W, which are proved in [Lapid and Mao 2015b].

(1) $E^\psi(M^*W,t)$ is the constant function $\epsilon_\pi^{\mathbf{n}}A_e^\psi(M^*W)$.

(2) $\displaystyle\int_{\eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'})} E^\psi(W_{1/2},t)\,\frac{dt}{|\det t|}=\epsilon_\pi^{\mathbf{n}+1}A_e^\psi(M^*W).$

It is here that we use in an essential way that $\pi\in\operatorname{Irr}_{\mathrm{meta}}\mathbb{M}$ (and where the factor ϵ_π shows up).

This concludes the sketch of the argument. We finish with a few technical remarks about the rigorous justification of the above argument, and also give a few pointers to the upcoming sections. Hopefully, this will shed some light on (if not motivate) the contents of the remaining sections.

(1) To justify (4-1), with $\int_{N'}$ replaced by $\int_{N'}^{\mathrm{st}}$ on both sides, we are forced to assume that $\widetilde{W}(\widetilde{w_0'tn})$ is compactly supported on $T'\times N'$ (see Lemma 5.4). This entails imposing a certain support condition on the section W (see Definition 5.5 and Lemma 5.6). Fortunately (and this is a key point in our argument), it is enough to prove (MI) for a single pair $(W,W^\wedge)$ for which the right-hand side is nonzero. For the special sections as above, the nonvanishing is guaranteed by that of the Bessel function of $\tilde{\pi}$, which is proved in the Appendix, following ideas of Ichino and Zhang [2014].

(2) The fact that Y^ψ is well-defined (with $\int_{N'}^{\mathrm{st}}$ instead of $\int_{N'}$) relies on results and techniques of Baruch [2005] (modified to the case at hand in [Lapid and Mao 2013]) on the stability of Bessel functions (Corollary 5.3). This is the reason why we use the Bruhat decomposition with respect to B' (i.e., (4-1)) rather than P' (i.e., (4-2)).

(3) Since we cannot control the support of $M^*W^\wedge$, it is necessary to justify (4-4) for general sections. To that end, we introduce a complex parameter s and consider the family

$$B(W,W^\wedge,s)=B(W_s,W^\wedge_{-s}).$$

The equality (4-4) is then justified for W_s with $\operatorname{Re}s\gg1$. (See Lemma 6.6.) This yields the analogue of (4-5) for $B(W, W^\wedge, s)$, provided that $-\operatorname{Re} s \gg 1$ and W is special (Lemma 6.11).

(4) A similar issue arises with (4-6): we can only justify it if at least one of the Whittaker functions satisfies an additional support constraint (which is in some sense dual to the support condition considered in the first remark). (See Definition 7.5 and Proposition 7.6.) Thus, we take W to be special of the first kind and $W^\wedge$ to be special of the second kind (see Definition 7.8).

(5) Ultimately, we need to consider $B(W, M(s)W^\wedge, s)$ at $s = \frac{1}{2}$. In order to exploit the assumptions on W and $W^\wedge$, we need to know that $M(s)$ is self-adjoint with respect to the bilinear form B. Such a relation was proved in [Lapid and Mao 2014, Appendix B] for a variant of B (where the order of integration is slightly different). (See Corollary 7.2.) For $-\operatorname{Re} s \gg 1$ and W special of the first kind, this agrees with the original B.

(6) The upshot is an identity

$$B(W, M(s)W^\wedge, s) = \int_{\eta^\vee_{\mathbb{M}}(T'_{\mathbb{M}'})} E^\psi(M_s^* W, t) E^{\psi^{-1}}(W_s^\wedge, t)\, \frac{dt}{|\det t|}$$

for special sections W, $W^\vee$ and for $-\operatorname{Re} s \gg 1$ (see Proposition 7.11). To finish the argument, we need to show that the right-hand side has analytic continuation at s and compute its value at $s = \frac{1}{2}$. These steps are carried out in [Lapid and Mao 2015b]. See Section 8, where it is also explained why the restriction on $W^\wedge$ is harmless from the point of view of the relation (MI).

5. A bilinear form

We begin the analysis of the main identity. A key ingredient is the stability of the integral defining a Bessel function, which was proved in [Lapid and Mao 2013] following ideas of Baruch [2005].

5A. We recall the main result of [Lapid and Mao 2013] for the group $\widetilde{G}$.

Theorem 5.1 [Lapid and Mao 2013]. *Let $K_0 \in \mathcal{CSGR}^s(G')$. Then the integral*

$$\int_{N'}^{\mathrm{st}} \widetilde{W}(\widetilde{w'_{U'}}\,\widetilde{w_0^{M'} t n})\psi_{\widetilde{N}}(n)^{-1}\, dn$$

stabilizes uniformly for $\widetilde{W} \in C(\widetilde{N}\backslash\widetilde{G}, \psi_{\widetilde{N}})^{K_0}$ and locally uniformly in $t \in T'$. In particular, if $\widetilde{W} \in \mathbb{W}^{\psi_{\tilde{N}}}(\tilde{\pi})$ with $\tilde{\pi} \in \operatorname{Irr}_{\mathrm{gen},\psi_{\widetilde{N}}} \widetilde{G}$ then

$$\int_{N'}^{\mathrm{st}} \widetilde{W}(\widetilde{w'_{U'}}\,\widetilde{w_0^{M'} t n})\psi_{\widetilde{N}}(n)^{-1}\, dn = \mathbb{B}_{\tilde{\pi}}^{\psi_{\widetilde{N}}}(\widetilde{w'_{U'}}\,\widetilde{w_0^{M'} t})\tilde{W}(e), \tag{5-1}$$

where $\mathbb{B}_{\tilde{\pi}}^{\psi_{\widetilde{N}}}$ is the Bessel function of $\tilde{\pi}$ (see (A-1)).

Remark 5.2. Note that $w_0' = w_{U'}' w_0^{M'}$. Of course, one can state the theorem above (equivalently) for

$$\int_{N'}^{\mathrm{st}} \widetilde{W}(\widetilde{w_0' tn}) \psi_{\widetilde{N}}(n)^{-1}\, dn.$$

The original formulation will be slightly more convenient for computation.

We apply this to the function $\widetilde{W}(g) = A_{\sharp}^{\psi}(W, \tilde{g})$.

Corollary 5.3. *Let $K_0 \in \mathcal{CSGR}(G)$. Then the integral*

$$Y^{\psi}(W, t) := \int_{N'}^{\mathrm{st}} A_{\sharp}^{\psi}(W, \widetilde{w_{U'}'} \widetilde{w_0^{M'} tn}) \psi_{\widetilde{N}}(n)^{-1}\, dn, \quad t \in T',$$

stabilizes uniformly for $W \in C(N \backslash G, \psi_N)^{K_0}$ and locally uniformly in $t \in T'$. In particular, $Y^{\psi}(W_s, t)$ is entire in $s \in \mathbb{C}$, and if $\pi \in \mathrm{Irr}_{\mathrm{gen}}\, \mathbb{M}$ and $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ then $Y^{\psi}(M_s^* W, t)$ is meromorphic in s. Both $Y^{\psi}(W_s, t)$ and $Y^{\psi}(M_s^* W, t)$ are locally constant in t, uniformly in $s \in \mathbb{C}$.

Finally, if we assume that $\pi \in \mathrm{Irr}_{\mathrm{meta,gen}}\, \mathbb{M}$ and that $\tilde{\pi} = \mathcal{D}_{\psi^{-1}}(\pi)$ is irreducible then for any $W^{\wedge} \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)$ we have

$$Y^{\psi^{-1}}(M^* W^{\wedge}, t) = \mathbb{B}_{\tilde{\pi}}^{\psi_{\widetilde{N}}^{-1}}(\widetilde{w_{U'}'} \widetilde{w_0^{M'} t}) A_{\sharp}^{\psi^{-1}}(M^* W^{\wedge}, e).$$

Another useful consequence of Theorem 5.1 is the following.

Lemma 5.4. *Let $\widetilde{W} \in C^{\mathrm{sm}}(\widetilde{N} \backslash \widetilde{G}, \psi_{\widetilde{N}})$ and $\widetilde{W}^{\vee} \in C^{\mathrm{sm}}(\widetilde{N} \backslash \widetilde{G}, \psi_{\widetilde{N}}^{-1})$. Assume that the function*

$$(t, n) \in T' \times N' \mapsto \widetilde{W}(\widetilde{w_0' tn})$$

is compactly supported. (In particular, $\widetilde{W} \in C_c^{\mathrm{sm}}(\widetilde{N} \backslash \widetilde{G}, \psi_{\widetilde{N}})$.) Then the iterated integral

$$\int_{N'}^{\mathrm{st}} \left(\int_{N' \backslash G'} \widetilde{W}(\tilde{g}) \widetilde{W}^{\vee}(\tilde{g}\tilde{u})\, dg \right) \psi_{\widetilde{N}}(\tilde{u})\, du$$

is well defined and is equal to

$$\int_{T'} \int_{N'} \delta_{B'}(t) \widetilde{W}(\widetilde{w_{U'}'} \widetilde{w_0^{M'} tn}) \psi_{\widetilde{N}}(\tilde{n}^{-1}) \left(\int_{N'}^{\mathrm{st}} \widetilde{W}^{\vee}(\widetilde{w_{U'}'} \widetilde{w_0^{M'} tu}) \psi_{\widetilde{N}}(\tilde{u})\, du \right) dn\, dt.$$

Proof. Using the Bruhat decomposition we can write the left-hand side as

$$\int_{N'}^{\mathrm{st}} \left(\int_{T'} \int_{N'} \widetilde{W}(\widetilde{w_{U'}'} \widetilde{w_0^{M'} tn}) \widetilde{W}^{\vee}(\widetilde{w_{U'}'} \widetilde{w_0^{M'} tn\tilde{u}}) \delta_{B'}(t) \psi_{\widetilde{N}}(\tilde{u})\, dn\, dt \right) du.$$

Note that by the assumption on $\widetilde{W}$, the integrand is compactly supported in t, n. Consider

$$\int_\Omega \left(\int_{T'} \int_{N'} \widetilde{W}(\widetilde{w'_{U'}} \widetilde{w_0^{M'} tn}) \widetilde{W}^\vee (\widetilde{w'_{U'}} \widetilde{w_0^{M'} tn}\tilde{u}) \delta_{B'}(t) \psi_{\widetilde{N}}(\tilde{u})\, dn\, dt \right) du$$
$$= \int_{T'} \int_{N'} \int_\Omega \delta_{B'}(t) \widetilde{W}(\widetilde{w'_{U'}} \widetilde{w_0^{M'} tn}) \widetilde{W}^\vee (\widetilde{w'_{U'}} \widetilde{w_0^{M'} tn}\tilde{u}) \psi_{\widetilde{N}}(\tilde{u})\, du\, dn\, dt$$

for $\Omega \in \mathcal{CSGR}(N')$. By Theorem 5.1 this is equal to

$$\int_{T'} \int_{N'} \delta_{B'}(t) \widetilde{W}(\widetilde{w'_{U'}} \widetilde{w_0^{M'} tn}) \left(\int_{N'}^{\mathrm{st}} \widetilde{W}^\vee (\widetilde{w'_{U'}} \widetilde{w_0^{M'} tn}\tilde{u}) \psi_{\widetilde{N}}(\tilde{u})\, du \right) dn\, dt,$$

provided that Ω is sufficiently large (independently of t and n, since they can be confined to compact sets by the assumption on $\widetilde{W}$). Making a change of variable $u \mapsto n^{-1}u$, we get

$$\int_{T'} \int_{N'} \delta_{B'}(t) \widetilde{W}(\widetilde{w'_{U'}} \widetilde{w_0^{M'} tn}) \psi_{\widetilde{N}}(\tilde{n}^{-1}) \left(\int_{N'}^{\mathrm{st}} \widetilde{W}^\vee (\widetilde{w'_{U'}} \widetilde{w_0^{M'} tu}) \psi_{\widetilde{N}}(\tilde{u})\, du \right) dn\, dt,$$

as required. □

5B. In order to apply Lemma 5.4 for $A_\sharp^\psi(W_s, \cdot)$, we make a special choice of W. Consider the P-invariant subspace $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ$ of $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ consisting of functions supported in the big cell $Pw_U P = Pw_U U$. Any element of $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ$ is a linear combination of functions of the form

$$W(u'mw_U u) = \delta_P(m)^{1/2} W^M(m) \phi(u), \quad m \in M,\ u, u' \in U, \tag{5-2}$$

with $W^M \in \mathbb{W}^{\psi_{N_M}}(\pi)$ and $\phi \in C_c^\infty(U)$. Let $\eta_{\mathbb{M}}$ be the embedding $\eta_{\mathbb{M}}(g) = \begin{pmatrix} g & \\ & I_{\mathbf{n}} \end{pmatrix}$ of $\mathbb{M}'$ into $\mathbb{M}$. Also let $\eta_M = \varrho \circ \eta_{\mathbb{M}}$, so that

$$\eta_M(g) = \begin{pmatrix} g & & \\ & I_{2\mathbf{n}} & \\ & & g^* \end{pmatrix}.$$

Definition 5.5. Let $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ be the linear subspace of $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ$ generated by those W as in (5-2) that satisfy the additional property that the function $(t, n) \mapsto W^M(\eta_M(tw_0^{\mathbb{M}'} n))$ is compactly supported on $T'_{\mathbb{M}'} \times N'_{\mathbb{M}'}$, or equivalently, that the function $W^M \circ \eta_M$ on $\mathbb{M}'$ is supported in the big cell $B'_{\mathbb{M}'} w_0^{\mathbb{M}'} N'_{\mathbb{M}'}$ and its support is compact modulo $N'_{\mathbb{M}'}$.

This space is nonzero; see the proof of Lemma 6.13 below. It is also invariant under $\eta(T') \ltimes N$.

Lemma 5.6. *For any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$, the function $A_\sharp^\psi(W_s, \widetilde{w'_{U'}} \widetilde{w_0^{M'} tn})$ is compactly supported in $t \in T'$ and $n \in N'$ uniformly in $s \in \mathbb{C}$.*

Proof. Since $\Phi * W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ whenever $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ (for any $\Phi \in C_c^\infty(F^{\mathbf{n}})$), it is enough to show (in view of the definition of $A_\sharp^\psi$, upon replacing the domain of integration in (3-1) by $\eta(w'_{U'})V_U\eta(w'_{U'})^{-1}$), that for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$, the function $(v, t, n) \mapsto W(\gamma\eta(w'_{U'})v\eta(w_0^{M'}tn))$ is compactly supported in $V_U \times T' \times N'$. Write $t = \varrho'(t')$ and $n \in N'$ as $n = \varrho'(n'_1)n_2$ with $n_1 \in N'_{\mathbb{M}'}$ and $n_2 \in U'$. Also let $m' = w_0^{\mathbb{M}'}t'n'_1 \in \mathbb{M}'$ and $m = \eta(\varrho'(m')) = \eta(w_0^{M'}t\varrho'(n'_1))$. Then

$$W(\gamma\eta(w'_{U'})v\eta(w_0^{M'}tn)) = W(w_U vm\eta(n_2)) = W(\eta_M((m')^*)w_U m^{-1}vm\eta(n_2)).$$

It follows immediately from the definition of the space $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ that this function is compactly supported in $(v, n_1, n_2, t') \in V_U \times N'_{\mathbb{M}'} \times U' \times T'_{\mathbb{M}'}$. $\square$

5C. For $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ and $W^\vee \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi^\vee)\big)$, we define

$$B(W, W^\vee, s) := \int_{N'}^{\mathrm{st}}\left(\int_{N'\backslash G'} A_\sharp^\psi(W_s, \tilde{g})A_\sharp^{\psi^{-1}}(W_{-s}^\vee, \tilde{g}\tilde{u})\,dg\right)\psi_{\widetilde{N}}(\tilde{u})\,du.$$

By Lemma 5.6 and the proof of Lemma 5.4, $B(W, W^\vee, s)$ is an entire function of s and we have

$$B(W, W^\vee, s) = \int_{T'} Y^\psi(W_s, t)Y^{\psi^{-1}}(W_{-s}^\vee, t)\delta_{B'}(t)\,dt \tag{5-3}$$

for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ and $W^\vee \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)$.

Assume that $\pi \in \mathrm{Irr}_{\mathrm{gen,meta}}\,\mathbb{M}$ and $\tilde{\pi} = \mathcal{D}_{\psi^{-1}}(\pi)$ is irreducible. Then for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ and $W^\wedge \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)$,

$$\text{the left-hand side of the main identity (MI) is } B\big(W, M\big(\tfrac{1}{2}\big)W^\wedge, \tfrac{1}{2}\big). \tag{5-4}$$

We obtain a more explicit form of $B(W, W^\vee, s)$ for $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ in the next section.

6. Further analysis

6A. ***The function $A_\sharp^\psi(W_s, \tilde{g})$.*** At this point it is necessary to investigate $A_\sharp^\psi(W_s, \cdot)$ on the big cell.

We start with $A_\sharp^\psi(W, e)$, which is easier and also useful for the right-hand side of (6-12).

Fix an element $\boldsymbol{\epsilon}_1 \in V$ of the form $\ell_M(X)$, where $X \in \mathrm{Mat}_{\mathbf{n}}$ and the last row of X is $-\xi_{\mathbf{n}}$. For any $W \in C^{\mathrm{sm}}(N\backslash G, \psi_N)$, define

$$A_e^\psi(W) := \int_{V_\gamma\backslash V_-} W(\gamma v\boldsymbol{\epsilon}_1)\psi_{V_-}(\boldsymbol{\epsilon}_1^{-1}v\boldsymbol{\epsilon}_1)^{-1}\,dv. \tag{6-1}$$

Since $\psi_{V_-}(\epsilon_1^{-1}v\epsilon_1)=\psi_{V_-}(v)\psi(v_{\mathbf{n},2\mathbf{n}+1})$, this expression is clearly independent of the choice of ϵ_1 as above. Moreover, we have

$$A_e^{\psi}(W)=\int_{(V_\gamma\rtimes\eta(N'))\backslash N^\sharp}W(\gamma n\epsilon_1)\psi_{N^\sharp}(\epsilon_1^{-1}n\epsilon_1)^{-1}\,dn$$
$$=\int_{\epsilon_1^{-1}(V_\gamma\rtimes\eta(N'))\epsilon_1\backslash N^\sharp}W(\gamma\epsilon_1 n)\psi_{N^\sharp}(n)^{-1}\,dn. \tag{6-2}$$

The integrand in the first equation is invariant under $\eta(N')$ since the character $\psi_{N^\sharp}(\epsilon_1^{-1}\eta(u)\epsilon_1)$ is trivial on U' and agrees with $\psi_{N'_{M'}}$ on $N'_{M'}$.

Lemma 6.1. *For any $W\in C^{\mathrm{sm}}(N\backslash G,\psi_N)$, the integrand in* (6-1) *is compactly supported on $V_\gamma\backslash V_-$ and we have $A_\sharp^{\psi}(W,e)=A_e^{\psi}(W)$.*

Remark 6.2. Note that by Lemma 3.1, the linear form $W\mapsto A_\sharp^{\psi}(W,e)$ is $(N^\sharp,\psi_{N^\sharp})$-equivariant. By (6-2) the same is true for $A_e^{\psi}(W)$.

Proof. The support condition follows from Lemma 3.2. By (3-1) we have

$$A^{\psi}(W,\Phi,e)=\int_{V_\gamma\backslash V}W(\gamma v)\omega_{\psi^{-1}}(v)\Phi(\xi_{\mathbf{n}})\,dv.$$

We can write this as

$$\int_{V_+}\int_{V_\gamma\backslash V_-}W(\gamma vc)\omega_{\psi^{-1}}(vc)\Phi(\xi_{\mathbf{n}})\,dv\,dc,$$

which by (2-1a)–(2-1c) is equal to

$$\int_{V_+}\int_{V_\gamma\backslash V_-}W(\gamma vc)\psi_{V_-}(v)^{-1}\psi(v_{\mathbf{n},2\mathbf{n}+1})^{-1}\Phi(\underline{c}+\xi_{\mathbf{n}})\,dv\,dc.$$

Changing variables in c and v, we can rewrite this as

$$\int_{V_+}\int_{V_\gamma\backslash V_-}W(\gamma v\epsilon_1 c)\Phi(\underline{c})\psi_{V_-}(v)^{-1}\psi(v_{\mathbf{n},2\mathbf{n}+1})^{-1}\,dv\,dc$$
$$=\int_{V_\gamma\backslash V_-}(\Phi*W)(\gamma v\epsilon_1)\psi_{V_-}(v)^{-1}\psi(v_{\mathbf{n},2\mathbf{n}+1})^{-1}\,dv\,dc.$$

This is $A_e^{\psi}(\Phi*W)$, as $\psi_{V_-}(\epsilon_1^{-1}v\epsilon_1)=\psi_{V_-}(v)\psi(v_{\mathbf{n},2\mathbf{n}+1})$. □

We now turn to $A_\sharp^{\psi}(W_s,\cdot)$ on the big cell. By Lemma 3.1, it is enough to consider the element $w'_{U'}$.

Lemma 6.3. *Let $\pi\in\mathrm{Irr}_{\mathrm{gen}}M$. Then for $\mathrm{Re}\,s\gg_\pi 1$ and any $W\in\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$, we have*

$$A^{\psi}_{\sharp}(W_s,\widetilde{w'_{U'}})=\beta_{\psi^{-1}}(w'_{U'})\int_{V_U}W_s(w_U v)\psi_U(v)^{-1}\,dv$$
$$=\beta_{\psi^{-1}}(w'_{U'})\int_{V^{\sharp}_{M}\backslash V_-}W_s(w_U v)\psi_{V_-}(v)^{-1}\,dv. \tag{6-3}$$

Remark 6.4. It is easy to see that both sides of (6-3) are $\bigl(V_-\rtimes\eta(N'_{M'}),\,\psi_{V_-}\psi^{-1}_{N'_{M'}}\bigr)$-equivariant in W.

Proof. From (3-1),

$$A^{\psi}(W_s,\Phi,\widetilde{w'_{U'}})=\int_{V_\gamma\backslash V}W_s(\gamma v\eta(w'_{U'}))\omega_{\psi^{-1}}(v\widetilde{w'_{U'}})\Phi(\xi_{\mathbf{n}})\,dv.$$

Make a change of variable $v\mapsto\eta(w'_{U'})v\eta(w'_{U'})^{-1}$. Note that

$$V_\gamma=\eta(w'_{U'})V_M\eta(w'_{U'})^{-1},$$

$\eta(w'_{U'})$ normalizes V, and $V=V_M\ltimes V_U$. By (2-5) we infer that

$$A^{\psi}(W_s,\Phi,\widetilde{w'_{U'}})=\int_{V_U}W_s(w_U v)\omega_{\psi^{-1}}(\widetilde{w'_{U'}}v)\Phi(\xi_{\mathbf{n}})\,dv,$$

and using (2-3c) we get

$$\beta_{\psi^{-1}}(w'_{U'})\int_{V_U}\left(\int_{F^{\mathbf{n}}}W_s(w_U v)\omega_{\psi^{-1}}(v)\Phi(Y)\psi(Y_1)^{-1}\,dY\right)dv.$$

We claim that the double integral converges absolutely when $\operatorname{Re}s$ is large enough. Note that $V_U=\left\{\ell\left(\left(\begin{smallmatrix}x & y\\ & \check{x}\end{smallmatrix}\right)\right):x\in\mathrm{Mat}_{\mathbf{n}},\,y\in\mathfrak{s}_{\mathbf{n}}\right\}$ and hence $|\omega_{\psi^{-1}}(v)\Phi(Y)|=|\Phi(Y)|$ for $v\in V_U$. Thus,

$$\int_{V_U}\int_{F^{\mathbf{n}}}|W_s(w_U v)\omega_{\psi^{-1}}(v)\Phi(Y)\psi(Y_1)^{-1}|\,dY\,dv=\widehat{|\Phi|}(0)\int_{V_U}|W_s(w_U v)|\,dv.$$

The integration on the right-hand side is absolutely convergent when $\operatorname{Re}s\gg_\pi 1$. This follows from the convergence of $\int_U|W_s(w_U u)|\,du$ and Remark 2.2.

For $c\in V_+$ such that $\underline{c}=Y\in F^{\mathbf{n}}$, we have $W_s(w_U cg)=\psi(Y_1)^{-1}W_s(w_U g)$ by the equivariance of W, and $\omega_{\psi^{-1}}(c)\Phi(0)=\Phi(Y)$ by (2-1a). Thus,

$$\beta_{\psi^{-1}}(w'_{U'})^{-1}A^{\psi}(W_s,\Phi,\widetilde{w'_{U'}})=\int_{V_U}\int_{V_+}W_s(w_U cv)\omega_{\psi^{-1}}(cv)\Phi(0)\,dc\,dv.$$

Since V_+ normalizes V_U and the double integral converges we can write the last integral as

$$\int_{V_U}\int_{V_+} W_s(w_U vc)\omega_{\psi^{-1}}(vc)\Phi(0)\,dc\,dv$$
$$= \int_{V_U}\int_{V_+} W_s(w_U vc)\psi_{V_-}(v)^{-1}\Phi(\underline{c})\,dc\,dv = \int_{V_U}(\Phi * W_s)(w_U v)\psi_{V_-}(v)^{-1}\,dv.$$

This gives the first equality in (6-3), and the second one follows from it since the integrand is left $V_M^\sharp$-invariant. □

Remark 6.5. One can show that for any $W \in C^{\mathrm{sm}}(N\backslash G, \psi_N)$, we have

$$A_\sharp^\psi(W, \widetilde{w'_{U'}}) = \beta_{\psi^{-1}}(w'_{U'})\int_{V''\backslash V_-}\left(\int_{V_M^\sharp\backslash V''} W(w_U vx)\psi_{V_-}(vx)^{-1}\,dv\right)dx, \quad (6\text{-}4)$$

where on the right-hand side V'' is the preimage of the center of the Heisenberg group under $v \mapsto v_{\mathcal{H}}$ and in both the inner and outer integral the integrand is compactly supported, thus convergent. We do not give the details since we are not going to use this result.

6B. We can now compute $Y^\psi(W_s, t)$ for $\operatorname{Re} s \gg 1$.

Lemma 6.6. *Let $\pi \in \operatorname{Irr}_{\mathrm{gen}} M$. For $\operatorname{Re} s \gg_\pi 1$ and any $W \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$, $t \in T'$, we have the identity*

$$Y^\psi(W_s, t) = \nu'(t)^{\mathbf{n}-1/2}\beta_{\psi^{-1}}(w'_{U'})\beta_{\psi^{-1}}(w_0^{M'}t) \times \int_{V_M^\sharp\backslash N^\sharp} W_s(w_U\eta(w_0^{M'}t)v)\psi_{N^\sharp}(v)^{-1}\,dv, \quad (6\text{-}5)$$

where the right-hand side is absolutely convergent.

Remark 6.7. Clearly, both sides of (6-5) are $(N^\sharp, \psi_{N^\sharp})$-equivariant in W.

We first need the following convergence result.

Lemma 6.8. *Let $\pi \in \operatorname{Irr}_{\mathrm{gen}} M$. Then for $\operatorname{Re} s \gg_\pi 1$, we have*

$$\int_{N_{\mathbb{M}}^t\cap\mathcal{P}}\int_U |W_s(\varrho(n)w_U ug)|\,du\,dn < \infty, \quad (6\text{-}6a)$$
$$\int_{N_{\mathbb{M}}^t\cap\mathcal{P}^*}\int_U |W_s(\varrho(n)w_U ug)|\,du\,dn < \infty, \quad (6\text{-}6b)$$

for any $W \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$, $g \in G$.

Proof. We may assume without loss of generality that $g = e$. Denote by $\varrho(a(x))$ the torus part in the Iwasawa decomposition of $x \in G$. Let $\Delta_i(x)$ be the set of nonzero $i \times i$ minors in the last i rows of x (with the convention that $\Delta_0(x) = \{1\}$). Then $\Delta_i(\varrho(a(x)))$ is (obviously) a singleton and its absolute value is $\max_{\delta\in\Delta_i(x)}|\delta|$.

For any $n \in N^t_{\mathbb{M}}$ and $u \in U$ consider

$$a(\varrho(n) w_U u) = \operatorname{diag}(a_1, \dots, a_{2\mathbf{n}}).$$

Notice that

$$\prod_{i=1}^{2\mathbf{n}} |a_i| = |\det(a(\varrho(n) w_U u))| = |\det(a(w_U u))| \le 1 \tag{6-7}$$

and for all $1 \le j \le i \le 2\mathbf{n}$ we have $n_{i,j} \in \{0\} \cup \Delta_{i-1}(\varrho(n) w_U u)$ (since $n_{i,j}$ is an entry in the adjugate of n^*). Hence

$$|n_{i,j}| \le \prod_{k=1}^{i-1} |a_k|^{-1}. \tag{6-8}$$

We show below that for all $n \in N^t_{\mathbb{M}} \cap (\mathcal{P} \cup \mathcal{P}^*)$ and $u \in U$ such that $W(\varrho(n) w_U u) \ne 0$, we have

$$|\det(a(w_U u))| \ll_W |a_i| \ll_W |\det(a(w_U u))|^{2-2\mathbf{n}}, \quad i = 1, \dots, 2\mathbf{n}. \tag{6-9}$$

Assuming this, we first show the lemma. It follows from (6-7) and (6-9) that there exists $\lambda \in \mathbb{R}$, depending only on π such that

$$|W_s(\varrho(n) w_U u)| \ll_W |\det(a(w_U u))|^{\operatorname{Re} s - \lambda} \tag{6-10}$$

for any $u \in U$ and $n \in N^t_{\mathbb{M}} \cap (\mathcal{P} \cup \mathcal{P}^*)$. Together with (6-8) and (6-9) we obtain $|n_{i,j}| \ll_W |\det(a(w_U u))|^{1-i}$ whenever $W(\varrho(n) w_U u) \ne 0$. Thus,

$$\int_{N^t_{\mathbb{M}} \cap \mathcal{P}} \int_U |W_s(\varrho(n) w_U u)| \, du \, dn \ll_W \int_U |\det(a(w_U u))|^{\operatorname{Re} s - \lambda - \sum_{i=1}^{2\mathbf{n}-1} i^2} \, du$$

$\big($and similarly for $\int_{N^t_{\mathbb{M}} \cap \mathcal{P}^*}\big)$, where the last integral is finite when $\operatorname{Re} s \gg_\pi 1$ by a standard result on intertwining operators.

We are left to prove (6-9). First, consider the case where $n \in N^t_{\mathbb{M}} \cap \mathcal{P}$. Then the $2\mathbf{n}$-th row of $\varrho(n) w_U u$ is $\xi_{4\mathbf{n}}$. Thus, $\Delta_{2\mathbf{n}+1}(\varrho(n) w_U u) \subset \Delta_{2\mathbf{n}}(\varrho(n) w_U u)$, and hence

$$|a_{2\mathbf{n}}| \prod_{i=1}^{2\mathbf{n}} |a_i|^{-1} \le \prod_{i=1}^{2\mathbf{n}} |a_i|^{-1},$$

or equivalently $|a_{2\mathbf{n}}| \le 1$. Thus, if $W(\varrho(n) w_U u) \ne 0$ then $|a_1| \ll_W \cdots \ll_W |a_{2\mathbf{n}}| \le 1$. The relation (6-9) follows (since $1 \le |\det(a(w_U u))|^{2-2\mathbf{n}}$).

Next, consider the case $n \in N^t_{\mathbb{M}} \cap \mathcal{P}^*$. The last row of $\varrho(n) w_U u$ is the same as the last row of $w_U u$. Thus, $|a_1|^{-1}$ is bounded above by the norm of the entries in u, which in turn is bounded above by $|\det(a(w_U u))|^{-1}$ (since each nonzero entry of u belongs to $\Delta_{2\mathbf{n}}(w_U u)$). Again, if $W(\varrho(n) w_U u) \ne 0$ then $|a_1|_W \ll \cdots \ll_W |a_{2\mathbf{n}}|$.

From (6-7) we conclude that $|\det(a(\varrho(n)w_U u))| \ll |a_1| \ll_W |a_i|$ for all i. On the other hand,

$$\begin{aligned} |\det(a(w_U u))|^{2\mathbf{n}-2} &= |a_1|^{2\mathbf{n}-2}|a_2\cdots a_{2\mathbf{n}-1}|^{2\mathbf{n}-2}|a_{2\mathbf{n}}|^{2\mathbf{n}-2} \\ &\ll_W |a_2\cdots a_{2\mathbf{n}-1}|^{2\mathbf{n}-1}|a_{2\mathbf{n}}|^{2\mathbf{n}-2} \\ &= |a_2\cdots a_{2\mathbf{n}}|^{2\mathbf{n}-1}|a_{2\mathbf{n}}|^{-1} \le |a_{2\mathbf{n}}|^{-1}, \end{aligned}$$

and the relation (6-9) follows. □

Remark 6.9. From the proof it is clear that we can take a uniform region of convergence $\operatorname{Re} s \gg 1$ for all unitarizable π.

Lemma 6.6 is an immediate consequence of Lemmas 6.3, 3.1 and 6.8. Indeed, the right-hand side of (6-5) can be viewed as a "partial integration" of the double integral (6-6a) for $g = \eta(w_0^{M'}t)$. Thus, by Remark 2.2, it is absolutely convergent for $\operatorname{Re} s \gg_\pi 1$. Meanwhile, by Lemma 3.1,

$$\begin{aligned} Y^\psi(W_s, t) &= \int_{N'} A_\sharp^\psi(W_s, \widetilde{w'_{U'}}\widetilde{w_0^{M'}tn})\psi_{\widetilde{N}}(n)^{-1}\,dn \\ &= \nu'(t)^{-1/2}\beta_{\psi^{-1}}(w_0^{M'}t)\int_{N'} A_\sharp^\psi(W_s(\cdot\,\eta(w_0^{M'}tn)), \widetilde{w'_{U'}})\psi_{\widetilde{N}}(n)^{-1}\,dn, \end{aligned}$$

assuming the integral converges. From (6-3), this is

$$\nu'(t)^{-1/2}\beta_{\psi^{-1}}(w_0^{M'}t)\beta_{\psi^{-1}}(w'_{U'}) \times \int_{N'}\int_{V_U} W_s(w_U v\eta(w_0^{M'}tn))\psi_U(v)^{-1}\psi_{\widetilde{N}}(n)^{-1}\,dv\,dn.$$

Conjugating v over $\eta(w_0^{M'}t)$ and combining the two integrals (which converge by the above), we get Lemma 6.6. (We note that the integrand on the right-hand side of (6-5) is indeed left $V_M^\sharp$-invariant.)

6C. We now go back to the bilinear form B defined in Section 5C. Recall the definition of the space $\operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ in Section 5B.

Lemma 6.10. *For $W \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$, the integrand on the right-hand side of* (6-5) *is compactly supported in t, v uniformly in s (i.e., the support in (t, v) is contained in a compact set which is independent of s). In particular, the identity* (6-5) *holds for all $s \in \mathbb{C}$.*

Proof. Suppose that $W \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$ is of the form (5-2). Then for $n \in N'_{\mathbb{M}'}$, $u \in U$ and $t \in T'_{\mathbb{M}'}$, we have $\eta(\varrho'(w_0^{\mathbb{M}'}tn)) \in M$ and

$$W(w_U\eta(\varrho'(w_0^{\mathbb{M}'}tn))u) = \delta_P(\eta(t))^{-1/2}W^M(\eta_M((w_0^{\mathbb{M}'}tn)^*))\phi(u).$$

The lemma follows from the definition of $\operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\sharp^\circ$. □

Lemma 6.11. *Let $\pi \in \mathrm{Irr}_{\mathrm{gen}}\, M$. Then for $-\operatorname{Re} s \gg 1$, we have*

$$B(W, W^\vee, s) = \int_U \int_{V_M^\sharp \backslash N^\sharp} \int_{N'_{\mathbb{M}'} \backslash \mathbb{M}'} W_s(\eta_M(g) w_U v) W^\vee_{-s}(\eta_M(g) w_U u) \\ \delta_P(\eta_M(g))^{-1} |\det g|^{1-\mathbf{n}} \psi_{N^\sharp}(v)^{-1} \psi_U(u)\, dg\, dv\, du$$

for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$, $W^\vee \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi^\vee)\big)$, with the integral being absolutely convergent.

Proof. Suppose that $-\operatorname{Re} s \gg 1$ and $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$. Then by (5-3), Lemmas 6.6 and 6.10, and the fact that $\delta_{B'}(t) = \delta_{B'_{M'}}(t)\nu'(t)^{\mathbf{n}+1}$, $B(W, W^\vee, s)$ equals

$$\int_{T'_{\mathbb{M}'}} \int_{\eta(N'_{M'}) \ltimes U} \int_{\eta(N'_{M'}) \ltimes U} W_s(\eta_M((w_0^{\mathbb{M}'} t)^*) w_U v_1) W^\vee_{-s}(\eta_M((w_0^{\mathbb{M}'} t)^*) w_U v_2) \\ \times |\det t|^{3\mathbf{n}} \delta_{B'_{\mathbb{M}'}}(t) \psi_{N^\sharp}(v_1)^{-1} \psi_{N^\sharp}(v_2)\, dv_1\, dv_2\, dt,$$

where the integral is absolutely convergent. Here we use $\eta(N'_{M'}) \ltimes U$ as a section of $V_M^\sharp \backslash N^\sharp$.

Making a change of variable $v_1 \mapsto v_2 v_1$ and writing $v_2 = \eta(\varrho'(n))u$ with $n \in N'_{\mathbb{M}'}$ and $u \in U$, we get that $B(W, W^\vee, s)$ is equal to

$$\int_{T'_{\mathbb{M}'}} \int_{N'_{\mathbb{M}'}} \int_U \int_{\eta(N'_{M'}) \ltimes U} W_s(\eta_M((w_0^{\mathbb{M}'} tn)^*) w_U u v_1) W^\vee_{-s}(\eta_M((w_0^{\mathbb{M}'} tn)^*) w_U u) \\ \times |\det t|^{3\mathbf{n}} \delta_{B'_{\mathbb{M}'}}(t) \psi_{N^\sharp}(v_1)^{-1}\, dv_1\, du\, dn\, dt.$$

Make a further change of variable $v_1 \mapsto u^{-1} v_1$. It remains to use the Bruhat decomposition for $\mathbb{M}'$ and to note that $\delta_P(\eta_M(g)) = |\det g|^{2\mathbf{n}+1}$ for any $g \in \mathbb{M}'$ and that as a function of v, the integrand in the statement of the lemma is left $V_M^\sharp$-invariant. □

Define when convergent

$$\{W, W^\vee\} := \int_{N'_{\mathbb{M}'} \backslash \mathbb{M}'} W(\eta_M(g)) W^\vee(\eta_M(g)) \delta_P(\eta_M(g))^{-1} |\det g|^{1-\mathbf{n}}\, dg.$$

Then we get, for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$, $W^\vee \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi^\vee)\big)$ and when $-\operatorname{Re} s \gg 1$,

$$B(W, W^\vee, s) \\ = \int_U \int_{V_M^\sharp \backslash N^\sharp} \{W_s(\cdot\, w_U v), W^\vee_{-s}(\cdot\, w_U u)\} \psi_{N^\sharp}(v)^{-1} \psi_U(u)\, dv\, du. \quad (6\text{-}11)$$

Remark 6.12. If π is unitarizable then the integral defining $\{W, W^\vee\}$ is absolutely convergent for any $(W, W^\vee) \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi)) \times \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi^\vee)\big)$ by [Lapid and Mao 2014, Lemma 1.2].

6D. We need a nonvanishing result which follows from Theorem A.1.

Lemma 6.13. *Assume that* $\pi \in \mathrm{Irr}_{\mathrm{gen,meta}} M$ *and* $\tilde{\pi} = \mathcal{D}_{\psi^{-1}}(\pi)$ *is irreducible and tempered. Then the bilinear form* $B\big(W, M(\frac{1}{2})W^\wedge, \frac{1}{2}\big)$ *does not vanish identically on* $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp \times \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)$.

Proof. Since the image of the restriction map $\mathbb{W}^{\psi_{N_M}}(\pi) \to C(N_M\backslash\varrho(\mathcal{P}), \psi_{N_M})$ contains $C_c^\infty(N_M\backslash\varrho(\mathcal{P}), \psi_{N_M})$, it follows that for any $\varphi \in C_c^\infty(T'_{\mathbb{M}'} \times N'_{\mathbb{M}'})$ and $\phi \in C_c^\infty(U)$ there exists $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ$ (necessarily in $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$) of the form (5-2) such that $\varphi(t, n) = W^M(\eta_M((tw_0^{\mathbb{M}'}n)^*))$. It follows from Lemma 6.10 that the linear map $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp \to C_c^\infty(T')$ given by $W \mapsto Y^\psi(W_{1/2}, \cdot)$ is onto. Therefore, by (5-3), the lemma amounts to the nonvanishing of the linear form $W^\wedge \mapsto Y^{\psi^{-1}}(M^*W^\wedge, t)$ on $\mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)$ for some $t \in T'$. This follows from Theorem A.1 and the last part of Corollary 5.3. □

6E. By (5-4), Lemma 6.1 and Lemma 6.13, we conclude the following corollary.

Corollary 6.14. *Suppose that* $\pi \in \mathrm{Irr}_{\mathrm{meta,temp}} M$ *is good and* $\tilde{\pi} = \mathcal{D}_{\psi^{-1}}(\pi)$ *is tempered. Then*

$$B\big(W, M(\tfrac{1}{2})W^\wedge, \tfrac{1}{2}\big) = c_\pi A_e^\psi(M^*W)A_e^{\psi^{-1}}(M^*W^\wedge)$$

for all $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$ *and* $W^\wedge \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)$. *Moreover, the linear form* $A_e^\psi(M^*W)$ *does not vanish identically on* $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$.

In other words, taking into account the reduction step of Section 3F we have reduced Theorem 3.4 to the statement below.

Proposition 6.15. *Assume* $\pi \in \mathrm{Irr}_{\mathrm{meta,temp}} M$ *is good and* $\tilde{\pi} = \mathcal{D}_{\psi^{-1}}(\pi)$ *is tempered. Then for any* $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$ *and* $W^\wedge \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)$, *we have*

$$B\big(W, M(\tfrac{1}{2})W^\wedge, \tfrac{1}{2}\big) = \epsilon_\pi A_e^\psi(M^*W)A_e^{\psi^{-1}}(M^*W^\wedge). \tag{6-12}$$

The proposition will eventually be proved in Section 8 after further reductions, using the results of [Lapid and Mao 2014; 2015b].

7. Applications of functional equations

In this section we use the functional equations established in [Lapid and Mao 2014] to analyze $B(W, M(s)W^\wedge, s)$.

7A. We first apply a functional equation proved in [Lapid and Mao 2014, Appendix B]. To that end we introduce a variant of B. We define $\underline{B}(W, W^\vee, s)$ to be the right-hand side of (6-11) whenever the integral defining $\{\cdot,\cdot\}$ and the double integrals over $V_M^\sharp\backslash N^\sharp$ and U are absolutely convergent.

Clearly for $g \in \mathbb{M}'$, with $|\det g| = 1$,

$$\{W(\cdot\,\eta_M(g)),\, W^\vee(\cdot\,\eta_M(g))\} = \{W, W^\vee\}.$$

Thus, $\underline{B}(W, W^\vee, s)$ is equal to

$$\int_{N'_{\mathbb{M}'}}\int_U\int_U \{W_s(\cdot\,\eta_M(n)w_U v),\, W^\vee_{-s}(\cdot\,w_U u)\}\psi_U(v)^{-1}\psi_U(u)\psi_{N'_{\mathbb{M}'}}(n)\,dv\,du\,dn$$
$$= \int_{N'_{\mathbb{M}'}}\int_U\int_U \{W_s(\cdot\,w_U v),\, W^\vee_{-s}(\cdot\,\eta_M(n)w_U u)\}\psi_U(v)^{-1}\psi_U(u)\psi_{N'_{\mathbb{M}'}}(n)^{-1}\,dv\,du\,dn$$
$$= \int_{V_M^\sharp\backslash N^\sharp}\int_U \{W_s(\cdot\,w_U v),\, W^\vee_{-s}(\cdot\,w_U v_2)\}\psi_U(v)^{-1}\psi_{N^\sharp}(v_2)\,dv\,dv_2. \tag{7-1}$$

By (6-11), for any $\pi \in \mathrm{Irr}_{\mathrm{gen}}\, M$, $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$, $W^\vee \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi^\vee))$ and $-\operatorname{Re} s \gg 1$, we have

$$\underline{B}(W, W^\vee, s) = B(W, W^\vee, s).$$

On the other hand, we have the following result.

Proposition 7.1 [Lapid and Mao 2014, Appendix B]. *Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}\, M$.*

(1) *For $\operatorname{Re} s \gg 1$, $\underline{B}(W, W^\vee, s)$ is well defined for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$, $W^\vee \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi^\vee)\big)^\circ$.*

(2) *For $-\operatorname{Re} s \gg 1$, $\underline{B}(W, W^\vee, s)$ is well defined for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ$, $W^\vee \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi^\vee)\big)$.*

(3) *For $-\operatorname{Re} s \gg 1$, we have*

$$\underline{B}(W, M(s)W^\wedge, s) = \underline{B}(M(s)W, W^\wedge, -s)$$

for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ$, $W^\wedge \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)^\circ$.

Combined with the above we obtain:

Corollary 7.2. *For $-\operatorname{Re} s \gg 1$, we have*

$$B(W, M(s)W^\wedge, s) = \underline{B}(M(s)W, W^\wedge, -s)$$

for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$, $W^\wedge \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)^\circ$.

7B. ***An identity of Whittaker functions on*** $\mathbf{GL_{2n}}$**.** A key fact in the formal argument for the case $\mathbf{n} = 1$ in [Lapid and Mao 2017, §7] was that for any unitarizable $\pi \in \mathrm{Irr}_{\mathrm{gen}}\,\mathrm{GL}_2$, the expression

$$\int_{F^*} W\big(\big(\begin{smallmatrix} t & \\ & 1\end{smallmatrix}\big)\big)W^\vee\big(\big(\begin{smallmatrix} t & \\ & 1\end{smallmatrix}\big)\big)\,dt$$

defines a GL_2-invariant bilinear form on $\mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi) \times \mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi^\vee)$.

In the general case, we encountered (in the definition of $\{\cdot,\cdot\}$) a similar integral

$$A_{\mathbf{n}}(W, W^\vee) = \int_{N'_{\mathbb{M}'}\backslash\mathbb{M}'} W(\eta_{\mathbb{M}}(g))W^\vee(\eta_{\mathbb{M}}(g))|\det g|^{1-\mathbf{n}}\,dg,$$

where $W \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)$ and $W^\vee \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi^\vee)$.

Let J be the subspace of $\mathrm{Mat}_{\mathbf{n}}$ consisting of the matrices whose first column is zero. Note that for any $g \in \mathbb{M}'$, $X \in \mathrm{Mat}_{\mathbf{n}}$, $Y \in J$, $n_1, n_2 \in N'_{\mathbb{M}'}$, we have

$$A_{\mathbf{n}}\big(\pi(\eta_{\mathbb{M}}(g)\ell_{\mathbb{M}}(X+Y)\eta^\vee_{\mathbb{M}}(n_1))W, \pi^\vee(\eta_{\mathbb{M}}(g)\ell_{\mathbb{M}}(X)\eta^\vee_{\mathbb{M}}(n_2))W^\vee\big) \\ = \psi_{N'_{\mathbb{M}'}}(n_1n_2^{-1})|\det g|^{\mathbf{n}-1}A_{\mathbf{n}}(W, W^\vee) \quad (7\text{-}2)$$

(with $\ell_{\mathbb{M}}$, $\eta_{\mathbb{M}}$ and $\eta^\vee_{\mathbb{M}}$ defined in Section 2A). We also have the following relation, for $w_{2\mathbf{n},\mathbf{n}} := \begin{pmatrix} & I_{\mathbf{n}} \\ I_{\mathbf{n}} & \end{pmatrix}$.

Theorem 7.3 [Lapid and Mao 2014, Theorem 1.3]. *Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}\,\mathbb{M}$. Then for any $W \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)$, $W^\vee \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi^\vee)$, we have*

$$A_{\mathbf{n}}(W, W^\vee) = \int_J\int_J\int_{N'_{\mathbb{M}'}\backslash\mathbb{M}'} W(w_{2\mathbf{n},\mathbf{n}}\eta_{\mathbb{M}}(g)\ell_{\mathbb{M}}(X)) \\ \times W^\vee(w_{2\mathbf{n},\mathbf{n}}\eta_{\mathbb{M}}(g)\ell_{\mathbb{M}}(Y))|\det g|^{\mathbf{n}-1}\,dg\,dX\,dY. \quad (7\text{-}3)$$

The integrals on both sides are absolutely convergent.

Note that it is not a priori clear that the right-hand side of (7-3) satisfies (7-2) for $X \in \mathrm{Mat}_{\mathbf{n}} - J$.

We can slightly rephrase Theorem 7.3 as follows. First, we write the right-hand side of (7-3) as

$$\int_J\int_J\int_{N'_{\mathbb{M}'}\backslash\mathbb{M}'} W(\eta^\vee_{\mathbb{M}}(g)w_{2\mathbf{n},\mathbf{n}}\ell_{\mathbb{M}}(X))W^\vee(\eta^\vee_{\mathbb{M}}(g)w_{2\mathbf{n},\mathbf{n}}\ell_{\mathbb{M}}(Y))|\det g|^{\mathbf{n}-1}\,dg\,dX\,dY.$$

We may multiply $w_{2\mathbf{n},\mathbf{n}}$ on the left by any element $x \in \eta^\vee_{\mathbb{M}}(\mathbb{M}')$ with $|\det x| = 1$. In particular, we take $w'_{2\mathbf{n},\mathbf{n}} := \begin{pmatrix} & I_{\mathbf{n}} \\ w_0^{\mathbb{M}'} & \end{pmatrix}$ instead of $w_{2\mathbf{n},\mathbf{n}}$. Let $\boldsymbol{\epsilon}_2 \in \ell_{\mathbb{M}}(\epsilon_{1,1}+J)$, where $\epsilon_{1,1}$ is the matrix in $\mathrm{Mat}_{\mathbf{n}}$ with 1 in the upper left corner and zero elsewhere. Since

$$A_{\mathbf{n}}(\pi(\boldsymbol{\epsilon}_2)W, \pi^\vee(\boldsymbol{\epsilon}_2)W^\vee) = A_{\mathbf{n}}(W, W^\vee),$$

we infer that $A_{\mathbf{n}}(W, W^\vee)$ equals

$$\int_J\int_J\int_{N'_{\mathbb{M}'}\backslash\mathbb{M}'} W(\eta^\vee_{\mathbb{M}}(g)w'_{2\mathbf{n},\mathbf{n}}\ell_{\mathbb{M}}(X)\boldsymbol{\epsilon}_2) \\ \times W^\vee(\eta^\vee_{\mathbb{M}}(g)w'_{2\mathbf{n},\mathbf{n}}\ell_{\mathbb{M}}(Y)\boldsymbol{\epsilon}_2)|\det g|^{\mathbf{n}-1}\,dg\,dX\,dY.$$

Using Bruhat decomposition for g, the integral can be rewritten as

$$\int_J \int_J \int_{N'_{\mathbb{M}'}} \int_{T'_{\mathbb{M}'}} W(\eta^\vee_{\mathbb{M}}(t) w'_{\mathbf{2n,n}} \eta_{\mathbb{M}}(n) \ell_{\mathbb{M}}(X) \boldsymbol{\epsilon}_2) \\ \times W^\vee(\eta^\vee_{\mathbb{M}}(t) w'_{\mathbf{2n,n}} \eta_{\mathbb{M}}(n) \ell_{\mathbb{M}}(Y) \boldsymbol{\epsilon}_2) |\det t|^{\mathbf{n}-1} \delta_{B'_{\mathbb{M}'}}(t)^{-1} \, dt \, dn \, dX \, dY.$$

Since $\ell_{\mathbb{M}}(\epsilon_{1,1}+J)$ is invariant under conjugation by $\eta_{\mathbb{M}}(N'_{\mathbb{M}'})$, by changing variables in X and Y we get

$$A_{\mathbf{n}}(W, W^\vee) = \int_J \int_J \int_{N'_{\mathbb{M}'}} \int_{T'_{\mathbb{M}'}} W(\eta^\vee_{\mathbb{M}}(t) \boldsymbol{\epsilon}_3 \ell_{\mathbb{M}}(X) \eta_{\mathbb{M}}(n)) \\ \times W^\vee(\eta^\vee_{\mathbb{M}}(t) \boldsymbol{\epsilon}_3 \ell_{\mathbb{M}}(Y) \eta_{\mathbb{M}}(n)) |\det t|^{\mathbf{n}-1} \delta_{B'_{\mathbb{M}'}}(t)^{-1} \, dt \, dn \, dX \, dY, \quad (7\text{-}4)$$

where $\boldsymbol{\epsilon}_3 = w'_{\mathbf{2n,n}} \boldsymbol{\epsilon}_2$.

7C. In the first expression of (7-1) for $\underline{B}$, we have an inner integral of the form

$$\int_{N'_{\mathbb{M}'}} \{W_s(\cdot\, \eta_M(n)), W^\vee_{-s}\} \psi_{N'_{\mathbb{M}'}}(n) \, dn.$$

This leads us to apply the above functional equation in the following setting. Let $\pi \in \mathrm{Irr}_{\mathrm{temp}}\, \mathbb{M}$. Define the bilinear form $D(W, W^\vee)$ on $\mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi) \times \mathbb{W}^{\psi^{-1}_{N_{\mathbb{M}}}}(\pi^\vee)$ by

$$D(W, W^\vee) := \int_{N'_{\mathbb{M}'}} A_{\mathbf{n}}(\pi(\eta_{\mathbb{M}}(n)) W, W^\vee) \psi_{N'_{\mathbb{M}'}}(n) \, dn. \quad (7\text{-}5)$$

It is shown in [Lapid and Mao 2014, Appendix A] that the integral is absolutely convergent.

Remark 7.4. For $(W, W^\vee) \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi)) \times \mathrm{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi^\vee)\big)$, we have

$$D(\delta_P^{-1/2} W \circ \varrho, \delta_P^{-1/2} W^\vee \circ \varrho) = D(\delta_P^{-1/2} W_s \circ \varrho, \delta_P^{-1/2} W^\vee_{-s} \circ \varrho) \\ = \int_{N'_{\mathbb{M}'}} \{W_s(\cdot\, \eta_M(n)), W^\vee_{-s}\} \psi_{N'_{\mathbb{M}'}}(n) \, dn. \quad (7\text{-}6)$$

For convenience, set

$$\bar{R} = \boldsymbol{\epsilon}_3 (\eta_{\mathbb{M}}(N'_{\mathbb{M}'}) \ltimes \ell_{\mathbb{M}}(J)) \boldsymbol{\epsilon}_3^{-1} = w'_{\mathbf{2n,n}} (\eta_{\mathbb{M}}(N'_{\mathbb{M}'}) \ltimes \ell_{\mathbb{M}}(J)) w'^{-1}_{\mathbf{2n,n}} \\ = \left\{ \begin{pmatrix} I_{\mathbf{n}} & \\ x & n^t \end{pmatrix} : x \in J, n \in N'_{\mathbb{M}'} \right\} \subset N^t_{\mathbb{M}} \cap \mathcal{P}^*.$$

Definition 7.5. Let $\mathbb{W}^{\psi^{-1}_{N_{\mathbb{M}}}}(\pi)_\natural$ be the linear subspace of $\mathbb{W}^{\psi^{-1}_{N_{\mathbb{M}}}}(\pi)$ consisting of W such that

$$W(\cdot\, \boldsymbol{\epsilon}_3)|_{\mathcal{P}^*} \in C_c^\infty(N_{\mathbb{M}} \backslash \mathcal{P}^*, \psi^{-1}_{N_{\mathbb{M}}}) \quad \text{and} \quad W(\cdot\, \boldsymbol{\epsilon}_3)|_{\eta^\vee_{\mathbb{M}}(T'_{\mathbb{M}'}) \ltimes \bar{R}} \in C_c^\infty(\eta^\vee_{\mathbb{M}}(T'_{\mathbb{M}'}) \ltimes \bar{R}).$$

It is easy to see that this definition is independent of the choice of $\boldsymbol{\epsilon}_2$ (and correspondingly $\boldsymbol{\epsilon}_3$).

We note that $\mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi)_{\natural}$ is nonzero, since the restriction of $\mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi)$ to $\mathcal{P}^*$ contains $C_c^\infty(N_{\mathbb{M}}\backslash\mathcal{P}^*, \psi_{N_{\mathbb{M}}}^{-1})$ and $\eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'}) \ltimes \bar{R} \subset B_{\mathbb{M}}^t \cap \mathcal{P}^*$. On the other hand, even if we assume that π is supercuspidal, $\mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi)_{\natural}$ is a proper subspace of $\mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi)$ since the set $N_{\mathbb{M}} \cdot (\eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'}) \ltimes \bar{R})$ is not closed.

Let $N_{\mathbb{M}}^\flat = (N_{\mathbb{M}}^\sharp)^*$ and let $\psi_{N_{\mathbb{M}}^\flat}$ be the character on $N_{\mathbb{M}}^\flat$ given by

$$\psi_{N_{\mathbb{M}}^\flat}(m) = \psi_{N_{\mathbb{M}}^\sharp}(m^*).$$

Note that $N_{\mathbb{M}}^\flat$ consists of the elements of $N_{\mathbb{M}}$ whose $(\mathbf{n}+1)$-st column vanishes above the diagonal.

Proposition 7.6. *Let $\pi \in \operatorname{Irr}_{\text{temp}} \mathbb{M}$. For any $W \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)$ and $W^\vee \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi^\vee)_{\natural}$, $D(W, W^\vee)$ is equal to the absolutely convergent integral*

$$\int_{\eta_{\mathbb{M}}^\vee(N'_{\mathbb{M}'})\backslash N_{\mathbb{M}}^\flat} \int_{\eta_{\mathbb{M}}^\vee(N'_{\mathbb{M}'})\backslash N_{\mathbb{M}}^\flat} \int_{T'_{\mathbb{M}'}} W(\eta_{\mathbb{M}}^\vee(t)\boldsymbol{\epsilon}_3 r_1) W^\vee(\eta_{\mathbb{M}}^\vee(t)\boldsymbol{\epsilon}_3 r_2) \\ \times |\det t|^{\mathbf{n}-1} \delta_{B'_{\mathbb{M}'}}(t)^{-1} \psi_{N_{\mathbb{M}}^\flat}(r_1^{-1} r_2)\, dt\, dr_2\, dr_1,$$

where the integrand is compactly supported in all variables.

Proof. From (7-4) and (7-5), we get that $D(W, W^\vee)$ is equal to

$$\int_{N'_{\mathbb{M}'}} \left(\int_J \int_J \int_{N'_{\mathbb{M}'}} \int_{T'_{\mathbb{M}'}} W(\eta_{\mathbb{M}}^\vee(t)\boldsymbol{\epsilon}_3 \ell_{\mathbb{M}}(X) \eta_{\mathbb{M}}(n_2 n_1)) W^\vee(\eta_{\mathbb{M}}^\vee(t)\boldsymbol{\epsilon}_3 \ell_{\mathbb{M}}(Y) \eta_{\mathbb{M}}(n_2)) \\ \times |\det t|^{\mathbf{n}-1} \delta_{B'_{\mathbb{M}'}}(t)^{-1}\, dt\, dn_2\, dX\, dY \right) \psi_{N'_{\mathbb{M}'}}(n_1)\, dn_1.$$

By the condition on $W^\vee$, the integrand is compactly supported in t, Y and n_2. Since $\eta_{\mathbb{M}}^\vee(t)$ normalizes $\bar{R}$, it is also compactly supported in X, n_1 in view of Lemma 7.7 below. A change of variable $n_1 \mapsto n_2^{-1} n_1$ gives the identity in the proposition. ☐

Lemma 7.7. *The restriction to $N_{\mathbb{M}}^t \cap (\mathcal{P} \cup \mathcal{P}^*)$ of any $W \in C^{\text{sm}}(N_{\mathbb{M}}\backslash\mathbb{M}, \psi_{N_{\mathbb{M}}})$ is compactly supported.*

Proof. This is standard. By passing to $W(\cdot^*)$, it is enough to consider the support in $N_{\mathbb{M}}^t \cap \mathcal{P}$. Let $n = uak$ with $a = \operatorname{diag}(a_1, \dots, a_{2\mathbf{n}})$ be the Iwasawa decomposition of $n \in N_{\mathbb{M}}^t \cap \mathcal{P}$. Then $a_{2\mathbf{n}} = 1$ and $|n_{i,j}| \le |a_i \dots a_{2\mathbf{n}}|$ for all $1 \le j \le i \le 2\mathbf{n}$. Thus, if $W(n) \ne 0$ then the a_i are bounded in terms of W, and hence n is bounded. The lemma follows. ☐

7D. We apply Proposition 7.6 to get a new expression for $\underline{B}(W, W^\vee, s)$. Before stating the result, we first introduce an integral that appears in the new expression.

As in [Lapid and Mao 2015b, §7.4], define

$$\Delta(t) := |t_1|^{-\mathbf{n}} \delta_B^{1/2}(\varrho(t)), \quad t = \operatorname{diag}(t_1, \dots, t_{2\mathbf{n}}) \in T_{\mathbb{M}}.$$

In particular, when $t = \eta_{\mathbb{M}}^\vee(t')$ with $t' \in T'_{\mathbb{M}'}$, we have $\Delta(t) = \delta_{B'}(\varrho'(t))^{1/2}$.

For now, let ϵ_4 be an arbitrary element in $N_{\mathbb{M}}$. (We will fix it in the next section in order to make the formulas look nicer.) Let $T'' = \eta_{\mathbb{M}}^{\vee}(T'_{\mathbb{M}'}) \times Z_{\mathbb{M}}$. For $W \in C^{\mathrm{sm}}(N\backslash G, \psi_N)$, $t \in T''$, let

$$\begin{aligned} E^{\psi}(W,t) &:= \Delta(t)^{-1} \int_{\eta_M(N'_{\mathbb{M}'})\backslash N^{\sharp}} W(\varrho(t\epsilon_4\epsilon_3)w_U v)\psi_{N^{\sharp}}(v)^{-1}\,dv \\ &= \Delta(t)^{-1}\psi_{N_{\mathbb{M}}}(t\epsilon_4 t^{-1}) \int_{\eta_M(N'_{\mathbb{M}'})\backslash N^{\sharp}} W(\varrho(t\epsilon_3)w_U v)\psi_{N^{\sharp}}(v)^{-1}\,dv \\ &= \Delta(t)^{-1}\psi_{N_{\mathbb{M}}}(t\epsilon_4 t^{-1}) \int_U \int_{\bar{R}} W(\varrho(tv\epsilon_3)w_U u) \\ &\qquad\qquad \times \psi_{N^{\flat}_{\mathbb{M}}}(\epsilon_3^{-1}v\epsilon_3)^{-1}\psi_U(u)^{-1}\,dv\,du, \end{aligned} \tag{7-7}$$

provided that the integral converges. (It is easy to check that the integrand above is invariant under $\eta_M(N'_{\mathbb{M}'})$ when $t \in T''$.) Similarly, define $E^{\psi^{-1}}(W^{\wedge}, t)$ for $W^{\wedge} \in C^{\mathrm{sm}}(N\backslash G, \psi_N^{-1})$.

Note that for any $t \in T''$, $E^{\psi}(W,t)$ is $(N^{\sharp}, \psi_{N^{\sharp}})$-equivariant in W. It is also clear from the definition that whenever defined,

$$E^{\psi}(W_s, tz) = E^{\psi}(W_s, t)|\det z|^{s+1/2}\omega_\pi(z), \quad z \in Z_{\mathbb{M}},\ t \in T'', \tag{7-8}$$

for $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ with $\pi \in \mathrm{Irr}_{\mathrm{gen}}\,\mathbb{M}$, where ω_π is the central character of π.

Definition 7.8. Let $\mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)^{\circ}_{\natural}$ be the linear subspace of $\mathrm{Ind}(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi))^{\circ}$ spanned by the functions which vanish outside Pw_UN and on the big cell are given by

$$W(u'mw_Uu) = \delta_P^{1/2}(m)W^M(m)\phi(u), \quad m \in M,\ u, u' \in U,$$

with $\phi \in C_c^{\infty}(U)$ and $W^M \circ \varrho \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi)_{\natural}$.

This space is clearly nonzero since $\mathbb{W}^{\psi_{N_{\mathbb{M}}}^{-1}}(\pi)_{\natural}$ is nonzero.

Lemma 7.9. *Let $\pi \in \mathrm{Irr}_{\mathrm{gen}}\,M$. For $\mathrm{Re}\,s \gg 1$, the integral (7-7) defining $E^{\psi}(W_s,t)$ converges for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ and $t \in T''$ uniformly for (s,t) in a compact set. Hence, $E^{\psi}(W_s,t)$ is holomorphic for $\mathrm{Re}\,s \gg 1$ and continuous in t. If $W^{\wedge} \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)^{\circ}$ then $E^{\psi^{-1}}(W_s^{\wedge}, t)$ is entire in s and locally constant in t, uniformly in s. If moreover $W^{\wedge} \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)^{\circ}_{\natural}$ then $E^{\psi^{-1}}(W_s^{\wedge}, t)$ is compactly supported in $t \in \eta_{\mathbb{M}}^{\vee}(T'_{\mathbb{M}'})$, uniformly in s.*

Proof. For the first assertion, since $T_{\mathbb{M}}$ normalizes the group $\bar{R}$ and U is normalized by M, we need to check the convergence, locally uniformly in $m \in M$, of

$$\int_{\bar{R}} \int_U |W_s(\varrho(v)w_U um)|\,du\,dv.$$

This expression is clearly locally constant in m, so we can assume $m = e$. Since $\bar{R} \subset N^t_{\mathbb{M}} \cap \mathcal{P}^*$, the integral above is a "partial integration" of the double integral (6-6b) in Lemma 6.8 (with $g = e$), thus converges by Remark 2.2.

By Lemma 7.7, when $W^\wedge \in \operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi)\big)^\circ$, the integrand in the last expression of (7-7) (with $W^\wedge_s$ in place of W) is compactly supported in v and u, uniformly in s and locally uniformly in t. It is then clear that $E^{\psi^{-1}}(W^\wedge_s, t)$ is locally constant in t. The second part follows. Furthermore, if $W^\wedge \in \operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi)\big)^\circ_\natural$, it is clear from the definition of the latter space that the integrand in (7-7) (with $W^\wedge$ in place of W) is compactly supported in v and $t \in \eta^\vee_{\mathbb{M}}(T'_{\mathbb{M}'})$. The lemma follows. □

7E.

Proposition 7.10. *Let* $\pi \in \operatorname{Irr}_{\mathrm{temp}} M$. *Then for* $\operatorname{Re} s \gg 1$, *we have*

$$\underline{B}(W, W^\vee, s) = \int_{\eta^\vee_{\mathbb{M}}(T'_{\mathbb{M}'})} E^{\psi}(W_s, t) E^{\psi^{-1}}(W^\vee_{-s}, t) \,\frac{dt}{|\det t|} \tag{7-9}$$

for any $W \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$ *and* $W^\vee \in \operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi^\vee)\big)^\circ_\natural$, *where* (by Lemma 7.9) *the integrand on the right-hand side is continuous and compactly supported.*

Proof. First note that the element ϵ_4 has no effect on the validity of (7-9), so we may ignore it from the consideration. By Proposition 7.1, if $W^\vee \in \operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi^\vee)\big)^\circ$, the integrals defining $\underline{B}(W, W^\vee, s)$ are absolutely convergent when $\operatorname{Re} s \gg 1$. If moreover $W^\vee \in \operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi^\vee)\big)^\circ_\natural$, then by the relation (7-6), (the first expression of) (7-1) and Proposition 7.6, $\underline{B}(W, W^\vee, s)$ is equal to

$$\begin{aligned}\int_U \int_U \Bigg(\int_{T'_{\mathbb{M}'}} \int_{\eta^\vee_{\mathbb{M}}(N'_{\mathbb{M}'})\backslash N^\flat_{\mathbb{M}}} \int_{\eta^\vee_{\mathbb{M}}(N'_{\mathbb{M}'})\backslash N^\flat_{\mathbb{M}}} & W_s(\varrho(\eta^\vee_{\mathbb{M}}(t)\epsilon_3 r_1) w_U u_1) \\ \times W^\vee_{-s}(\varrho(\eta^\vee_{\mathbb{M}}(t)\epsilon_3 r_2) w_U u_2) & \delta_{B'_{\mathbb{M}'}}(t)^{-1} \delta_P(\eta^\vee_M(t))^{-1} |\det t|^{\mathbf{n}-1} \\ \times \psi_U(u_1)^{-1} \psi_{N^\flat_{\mathbb{M}}}(r_1)^{-1} & \psi_U(u_2) \psi_{N^\flat_{\mathbb{M}}}(r_2)\, dr_1\, dr_2\, dt \Bigg) du_1\, du_2.\end{aligned}$$

Using the fact that for $t \in T'_{\mathbb{M}'}$

$$\delta_{B'_{\mathbb{M}'}}(t)^{-1} \delta_P(\eta^\vee_M(t))^{-1} |\det t|^{\mathbf{n}-1} = \delta_{B'}(\varrho'(t))^{-1} |\det t|^{-1},$$

to finish the proof of Proposition 7.10 it suffices to show the convergence of

$$\begin{aligned}\int_{T'_{\mathbb{M}'}} \Bigg(\int_{\eta_M(N'_{\mathbb{M}'})\backslash N^\sharp} & |W_s(\varrho(\eta^\vee_{\mathbb{M}}(t)\epsilon_3) w_U v_1)|\, dv_1 \\ & \times \int_{\eta_M(N'_{\mathbb{M}'})\backslash N^\sharp} |W^\vee_{-s}(\varrho(\eta^\vee_{\mathbb{M}}(t)\epsilon_3) w_U v_2)|\, dv_2 \Bigg) \delta_{B'}(\varrho'(t))^{-1} \,\frac{dt}{|\det t|}.\end{aligned}$$

It follows from Lemma 7.9 that the integrals over v_1 and v_2 converge for a fixed t and that the integral over v_2 is compactly supported as a function of t. The argument in Lemma 7.9 also shows that the resulting integrals over v_1 and v_2 are smooth in t. Thus, the integral converges. □

Combining Proposition 7.10 with Corollary 7.2 and (7-8) we get the following.

Proposition 7.11. *Let $\pi \in \mathrm{Irr}_{\mathrm{temp}} M$. Then for $-\operatorname{Re} s \gg 1$ and any*

$$W \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}}(\pi)\big)^{\circ}_{\sharp},$$
$$W^{\wedge} \in \mathrm{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi)\big)^{\circ}_{\natural},$$

we have

$$B(W, M(s)W^{\wedge}, s) = \int_{Z_{\mathbb{M}}\backslash T''} E^{\psi}(M_s^{*}W, t)E^{\psi^{-1}}(W_s^{\wedge}, t)\,\frac{dt}{|\det t|}, \tag{7-10}$$

where the integrand is continuous and compactly supported.

8. Proof of Proposition 6.15

Proposition 7.11 is an important (and hard-earned) step towards the proof of Proposition 6.15. However, it is still insufficient. First, it is not valid at the point $s = \frac{1}{2}$, which is pertinent for the left-hand side of (6-12). Second, the functions E^{ψ} have to be investigated in order to match the right-hand side of (6-12). Also, so far we have only used the fact that $\pi \in \mathrm{Irr}_{\mathrm{meta}}\,\mathbb{M}$ to reduce to special sections, but it should be clear from the nature of our formula that this fact has to enter the computation in a more substantial and quantitative way. Fortunately, these issues were taken up in [Lapid and Mao 2015b, §11]. We recall these results and explain how they are used to conclude the proof of Proposition 6.15 from Proposition 7.11.

8A. Let $\mathfrak{d} = \mathrm{diag}(1, -1, \dots, (-1)^{\mathbf{n}-1}) \in \mathrm{Mat}_{\mathbf{n}}$. We now fix

$$\boldsymbol{\epsilon}_4 = \ell_{\mathbb{M}}\big(-\tfrac{1}{2}\mathfrak{d}w_0^{\mathbb{M}'}\big) \in N_{\mathbb{M}}.$$

This element is denoted by ϵ' in [Lapid and Mao 2015b, p. 9550] with the parameter $\mathfrak{a} = -\frac{1}{2}$ in the notation of that paper. We also fix $\boldsymbol{\epsilon}_2 = \ell_{\mathbb{M}}(\mathfrak{d})$ (and correspondingly $\boldsymbol{\epsilon}_3 = w'_{2\mathbf{n},\mathbf{n}}\boldsymbol{\epsilon}_2$; see Section 7B). The reason for fixing the elements this way is (partly) due to the special form of the character $\psi_{\overline{U}}$ on $\overline{U}$ given by

$$\psi_{\overline{U}}(\bar{v}) = \psi_U((\varrho(\boldsymbol{\epsilon}_4\boldsymbol{\epsilon}_3)w_U)^{-1}\bar{v}\varrho(\boldsymbol{\epsilon}_4\boldsymbol{\epsilon}_3)w_U)^{-1} = \psi(\bar{v}_{2\mathbf{n}+1,1}), \quad \bar{v} \in \overline{U}. \tag{8-1}$$

We rewrite the first expression on the right-hand side of (7-7) by splitting the integral into integrals over U and $\eta_M(N'_{\mathbb{M}'})\backslash N^{\sharp}_M$, making the change of variables

$$v \mapsto (\varrho(\boldsymbol{\epsilon}_4\boldsymbol{\epsilon}_3)w_U)^{-1}\bar{v}\varrho(\boldsymbol{\epsilon}_4\boldsymbol{\epsilon}_3)w_U$$

in the integral over U, and conjugating the variable in the second integral by w_U. We get, for $\operatorname{Re} s \gg 1$,

$$E^{\psi}(W_s,t)=\Delta(t)^{-1}\int_{\eta_{\mathbb{M}}^{\vee}(N'_{\mathbb{M}'})\backslash N^{\flat}_{\mathbb{M}}}\int_{\overline{U}} W_s(\varrho(t)\bar{v}\varrho(\boldsymbol{\epsilon}_4\boldsymbol{\epsilon}_3 r)w_U)\psi_{\overline{U}}(\bar{v})\psi_{N^{\flat}_{\mathbb{M}}}(r)^{-1}\,d\bar{v}\,dr$$

$$=\Delta(t)^{-1}\int_{\bar{R}}\int_{\overline{U}} W_s(\varrho(t)\bar{v}\varrho(\boldsymbol{\epsilon}_4 r\boldsymbol{\epsilon}_3)w_U)\psi_{\overline{U}}(\bar{v})\psi_{\bar{R}}(r)\,d\bar{v}\,dr, \tag{8-2}$$

where $\psi_{\bar{R}}(r)=\psi_{N^{\flat}_{\mathbb{M}}}(\boldsymbol{\epsilon}_3^{-1}r\boldsymbol{\epsilon}_3)^{-1}$.

We now quote the first pertinent result from [Lapid and Mao 2015b, §11]. Let S be the subtorus $\prod_{i=1}^{\mathbf{n}-1}T_i$ of T'' (of codimension two), where T_i is the one-dimensional torus

$$T_i:=\{\operatorname{diag}(\overbrace{z^{-1},\dots,z^{-1}}^{2\mathbf{n}-i},\overbrace{z,\dots,z}^{i}):z\in F^*\}.$$

For any $f\in C_c^{\infty}(S)$ and $g\in C(T'')$, we write $f*g(\cdot)=\int_S f(t)g(\cdot\, t)\,dt$.

Theorem 8.1. *For any $W\in C^{\mathrm{sm}}(N\backslash G,\psi_N)$ which is left-invariant under a compact open subgroup of Z_M and any $f\in C_c^{\infty}(S)$, the function $f*E^{\psi}(W_s,t)$ extends to an entire function in s which is locally constant in t, uniformly in s. Moreover, if $\pi\in\operatorname{Irr}_{\mathrm{meta,temp}}\mathbb{M}$ then*

$$f*E^{\psi}(M_s^*W,t)\big|_{s=1/2}=\epsilon_{\pi}^{\mathbf{n}}A_e^{\psi}(M^*W)\int_S f(t')\,dt'$$

for any $W\in\operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))$, $t\in T''$ and $f\in C_c^{\infty}(S)$.

The first part is [Lapid and Mao 2015b, Corollary 11.9], except for a slight correction which we now explain. Recall some notation from that paper. As in Section 6 of [ibid.] let $\mathcal{Z}$ be the unipotent subgroup of $\mathbb{M}$ given by

$$\mathcal{Z}=\{m\in\mathbb{M}:m_{i,i}=1\ \forall i,\ m_{i,j}=0 \text{ if either } (j>i \text{ and } i+j>2\mathbf{n}) \text{ or } (i>j \text{ and } i+j\le 2\mathbf{n}+1)\},$$

and let $\psi_{\mathcal{Z}}$ be its character

$$\psi_{\mathcal{Z}}(m)=\psi(m_{1,2}+\cdots+m_{\mathbf{n}-1,\mathbf{n}}-m_{\mathbf{n}+2,\mathbf{n}+1}-\cdots-m_{2\mathbf{n},2\mathbf{n}-1}),\quad m\in\mathcal{Z}. \tag{8-3}$$

The group $\varrho(\mathcal{Z})$ stabilizes the character $\psi_{\overline{U}}$. Let $\mathfrak{E}=\varrho(\mathcal{Z})\ltimes\overline{U}$ with the character

$$\psi_{\mathfrak{E}}(\varrho(m)\bar{u})=\psi_{\mathcal{Z}}(m)\psi_{\overline{U}}^{-1}(\bar{u}),\quad m\in\mathcal{Z},\ \bar{u}\in\overline{U}.$$

Also, let $\mathcal{Z}^+=\mathcal{Z}\cap N_{\mathbb{M}}$, $V_{\Delta}=\mathcal{Z}\cap N_{\mathbb{M}}^t$ and

$$N_{\mathbb{M},\Delta}=\{\ell_{\mathbb{M}}(X)^t:X\in J,\ X_{i,j}=0 \text{ if } i+j>\mathbf{n}+1\}.$$

We have $\mathcal{Z}=\mathcal{Z}^+\cdot V_{\Delta}$ and $\bar{R}=V_{\Delta}\cdot N_{\mathbb{M},\Delta}$. In Section 10.4 of [ibid.] it was erroneously claimed that the second product is semidirect. Fortunately, this does not have any bearing on the argument. All what matters is that the character $\psi_{\bar{R}}$ is given by $\psi_{\bar{R}}(vn) = \psi_{V_\Delta}(v)$, $v \in V_\Delta$, $n \in N_{\mathbb{M},\Delta}$, where $\psi_{V_\Delta}(r)$ is defined right after (7.11) of [ibid.]. Thus, the middle integral in (11.11) of [ibid.] should be taken over $V_\Delta \cdot N_{\mathbb{M},\Delta}$, i.e., over $\bar{R}$ in our notation. (Also, the variable v should be replaced by r.) With this correction, the expression in (11.11) of [ibid.], evaluated at $\varrho(\boldsymbol{\epsilon}_3)w_U$, is $(f\Delta) * E^\psi(W_s, t)$ and its analytic continuation is given by the expression in (11.10) of [ibid.], which amounts to a finite sum.

Meanwhile, from the definition of A_e^ψ in (6-1) we have

$$A_e^\psi(W) = \int_{\varrho(V_{\mathbb{M}}^*)\backslash\gamma V_- \gamma^{-1}} W(v\gamma\boldsymbol{\epsilon}_1)\psi_{V_-}((\gamma\boldsymbol{\epsilon}_1)^{-1}v\gamma\boldsymbol{\epsilon}_1)^{-1}\,dv.$$

We can integrate instead over the group $\overline{U}^{\urcorner}$ in Section 11.5 of [ibid.], consisting of elements $\bar{u}$ in $\overline{U}$ such that $\bar{u}_{2\mathbf{n}+i,j} = 0$ whenever $i \geq \mathbf{n}$ and $j \leq \mathbf{n}$. The character $\hat{\psi}_{\overline{U}^{\urcorner}}(\bar{u}) := \psi_{V_-}((\gamma\boldsymbol{\epsilon}_1)^{-1}\bar{u}\gamma\boldsymbol{\epsilon}_1)^{-1}$ on $\overline{U}^{\urcorner}$ also matches the one defined in Sections 11.3 and 11.5 of [ibid.] (with the parameter $\mathfrak{a} = -\frac{1}{2}$). Thus, $A_e^\psi(W)$ is $\mathcal{T}'(W)(\gamma\boldsymbol{\epsilon}_1)$ in the notation there. The second part of Theorem 8.1 amounts to the first statement of [ibid., Corollary 11.10] upon taking

$$\boldsymbol{\epsilon}_1 = (\hat{w}\gamma)^{-1}\varrho(\boldsymbol{\epsilon}_3)w_U = \ell_M((-1)^{\mathbf{n}}\mathfrak{d}),$$

where

$$\hat{w} := \eta(w'_{U'})\varrho(w'_{2\mathbf{n},\mathbf{n}}) = \begin{pmatrix} & I_{\mathbf{n}} & & \\ & & & w_{\mathbf{n}} \\ -w_{\mathbf{n}} & & & \\ & & I_{\mathbf{n}} & \end{pmatrix}$$

is as in the bottom of p. 9523 of [ibid.].

Theorem 8.1 is a crucial step. Recall that the Langlands quotient of

$$\mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}}(\pi), \tfrac{1}{2}\big),$$

i.e., the image under M^*, admits a unique $(N^\sharp, \psi_{N^\sharp})$-equivariant functional. Thus, $E^\psi(M^*W, t)$ (which is technically not defined) is a priori proportional to $A_e^\psi(M^*W)$ and the constant of proportionality is a function depending on t. Theorem 8.1 essentially says that in fact this function is a constant which can be computed explicitly. The main input is that the Langlands quotient admits a realization in $C^{\mathrm{sm}}(H\backslash G)$, where $H \simeq \mathrm{Sp}_{\mathbf{n}} \times \mathrm{Sp}_{\mathbf{n}}$ is the centralizer of $\varrho(E)$ in G. (Recall that $E = \mathrm{diag}(1, -1, \dots, 1, -1)$.) We refer to [Lapid and Mao 2015b] for more details.

We may thus conclude the following corollary.

Corollary 8.2. *Suppose that $\pi \in \mathrm{Irr}_{\mathrm{meta,temp}}\,\mathbb{M}$. Then for any $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$, $W^\wedge \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}^{-1}}(\pi)\big)^\circ_\natural$ we have*

$$B\big(W, M(\tfrac{1}{2})W^\wedge, \tfrac{1}{2}\big) = \epsilon_\pi^{\mathbf{n}} A_e^\psi(M^*W) \int_{Z_{\mathbb{M}}\backslash T''} E^{\psi^{-1}}(W^\wedge_{1/2}, t)\,\frac{dt}{|\det t|}. \tag{8-4}$$

Proof. By Lemma 7.9, for any $W^\wedge \in \operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi)\big)^\circ_\natural$ there exists $K_0 \in \mathcal{CSGR}(T'')$ such that $E^{\psi^{-1}}(W^\wedge_s, \cdot) \in C(T'')^{K_0}$ for all s and $E^{\psi^{-1}}(W^\wedge_s, \cdot)$ is compactly supported modulo $\mathbb{Z}_{\mathbb{M}}$ uniformly in s. Suppose that $f \in C_c^\infty(S)$ is supported in $S \cap K_0$ and let $f^\vee(t) := f(t^{-1})$. By Proposition 7.11, for $-\operatorname{Re} s \gg 1$ and $W \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp$ we have

$$
\begin{aligned}
B(W, M(s)W^\wedge, s) \int_S f(t)\,dt
&= \int_{Z_{\mathbb{M}}\backslash T''} E^\psi(M_s^* W, t) f^\vee * E^{\psi^{-1}}(W_s^\wedge, t)\,\frac{dt}{|\det t|}\\
&= \int_{Z_{\mathbb{M}}\backslash T''} f * E^\psi(M_s^* W, t) E^{\psi^{-1}}(W_s^\wedge, t)\,\frac{dt}{|\det t|}.
\end{aligned} \tag{8-5}
$$

From the first part of Theorem 8.1 (i.e., [Lapid and Mao 2015b, Corollary 11.9]) we infer that both sides of (8-5) are meromorphic functions in s and the identity holds whenever $M(s)$ is holomorphic. Specializing to $s = \frac{1}{2}$ and using the second part of Theorem 8.1 (i.e., [ibid., Corollary 11.10]) the corollary follows. □

8B. It remains to compute the integral on the right-hand side of (8-4). Again, this is essentially done in [Lapid and Mao 2015b] and it relies heavily on the fact that $\pi \in \operatorname{Irr}_{\mathrm{meta}} \mathbb{M}$.

Let

$$
\begin{aligned}
\mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)_{\natural\natural} = \{W \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi) : W|_{\mathcal{P}^*} \in C_c^\infty(N_{\mathbb{M}}\backslash \mathcal{P}^*, \psi_{N_{\mathbb{M}}}) \text{ and}\\
W|_{\eta^\vee_{\mathbb{M}}(T'_{\mathbb{M}'})\ltimes \mathcal{Z}} \in C_c^\infty(\mathcal{Z}^+\backslash \eta^\vee_{\mathbb{M}}(T'_{\mathbb{M}'}) \ltimes \mathcal{Z}, \psi_{\mathcal{Z}})\}.
\end{aligned}
$$

Then $\mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)_{\natural\natural}$ is invariant under $\pi(\boldsymbol{\epsilon}_4)$ and $\pi(u\boldsymbol{\epsilon}_3)W \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)_{\natural\natural}$ for any $W \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)_\natural$ and $u \in N_{\mathbb{M},\Delta}$. Note that $\boldsymbol{\epsilon}_4$ normalizes $\mathcal{Z}$, $\mathcal{Z}^+$ and $\psi_{\mathcal{Z}}$ (by direct computation). Similarly $\boldsymbol{\epsilon}_3(w'_{2\mathbf{n},\mathbf{n}})^{-1}$ stabilizes $(\mathcal{Z}, \psi_{\mathcal{Z}})$.

Theorem 8.3. *Let $\pi \in \operatorname{Irr}_{\mathrm{meta,temp}} \mathbb{M}$. Then*

(1) [Lapid and Mao 2015b, Proposition 3.2; Matringe 2015]. *The integral*

$$\mathfrak{P}^{H_{\mathbb{M}}}(W) := \int_{N_{\mathbb{M}}\cap H_{\mathbb{M}}\backslash \mathcal{P}\cap H_{\mathbb{M}}} W(p)\,dp$$

converges and defines a nonzero $H_{\mathbb{M}}$-invariant functional on $\mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)$.

(2) [Lapid and Mao 2015b, Proposition 11.1]. *For any $W \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)_{\natural\natural}$ we have*

$$
\begin{aligned}
\int_{\mathcal{Z}^+\backslash\mathcal{Z}} \int_{\eta^\vee_{\mathbb{M}}(T'_{\mathbb{M}'})} \Delta(t)^{-1} |\det t|^{\mathbf{n}} W(tr) \psi_{\mathcal{Z}}(r)^{-1}\,dt\,dr\\
= \int_{\mathcal{Z}\cap H_{\mathbb{M}}\backslash \mathcal{Z}} \mathfrak{P}^{H_{\mathbb{M}}}(\pi(n)W)\psi_{\mathcal{Z}}(n)^{-1}\,dn.
\end{aligned}
$$

(3) [Lapid and Mao 2015b, Lemma 4.3]. *The integral*

$$L_W(g) := \int_{P\cap H\backslash H} \int_{N_{\mathbb{M}}\cap H_{\mathbb{M}}\backslash \mathcal{P}\cap H_{\mathbb{M}}} W(\varrho(p)hg)|\det p|^{-(\mathbf{n}+1)}\,dp\,dh$$
$$= \int_{H\cap \bar{U}} \mathfrak{P}^{H_{\mathbb{M}}}\big(\big(\delta_P^{-1/2} I\big(\tfrac{1}{2}, \bar{u}g\big)W\big)\circ\varrho\big)\,d\bar{u}$$

converges for any $W \in \mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}}(\pi), \tfrac{1}{2}\big)$ *and defines an intertwining map*

$$\mathrm{Ind}\big(\mathbb{W}^{\psi_{N_M}}(\pi), \tfrac{1}{2}\big) \to C^{\mathrm{sm}}(H\backslash G).$$

(4) [Lapid and Mao 2015b, second statement of Corollary 11.10]. *We have*

$$A_e^{\psi}(M^*W) = \epsilon_\pi^{\mathbf{n}+1} \int_{N_{\mathbb{M},\Delta}} \left(\int_{H\cap\mathfrak{E}\backslash\mathfrak{E}} L_W(v\varrho(\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3)w_U)\psi_{\mathfrak{E}}^{-1}(v)\,dv \right) du. \tag{8-6}$$

The first part is based on a variant of Bernstein's theorem [Bernstein 1984] on $\mathcal{P}$-invariant distributions. The second part is proved by "root killing", a method which goes back at least to [Jacquet et al. 1979] and has been used constantly ever since. The third part is essentially a formal consequence of the first part. The last (and most complicated) part is closely related to Theorem 8.1 (with $\boldsymbol{\epsilon}_1$ as before). We refer to [Lapid and Mao 2015b] for more details.

Remark 8.4. While the above formula for $A_e^{\psi}(M^*W)$ is convenient for computing the right-hand side of (8-4), it is also helpful to write the integration with the variables at the right. We do that in (8-9) below.

Corollary 8.5. *Let* $\pi \in \mathrm{Irr}_{\mathrm{meta,temp}}\,\mathbb{M}$. *Then for any* $W \in \mathrm{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))_\natural^\circ$ *we have*

$$\int_{Z_{\mathbb{M}}\backslash T''} E^{\psi}(W_{1/2}, t)\,\frac{dt}{|\det t|} = \epsilon_\pi^{\mathbf{n}+1} A_e^{\psi}(M^*W). \tag{8-7}$$

Note it is not a priori clear that the left-hand side of (8-7) factors through M^*W.

Proof. We may assume without loss of generality that for $m \in \mathbb{M}$, $u, u' \in U$,

$$W_{1/2}(u'\varrho(m)w_U u) = W^{\mathbb{M}}(m)|\det m|^{1/2}\delta_P(\varrho(m))^{1/2}\phi(u) \tag{8-8}$$

with $W^{\mathbb{M}} \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)_\natural$ and $\phi \in C_c^\infty(U)$. We evaluate the left-hand side I of (8-7) using the last expression in (7-7) (where we recall that the integrand is compactly supported by Lemma 7.9). Thus,

$$I = I' \int_U \phi(v)\psi_U^{-1}(v)\,dv$$

where

$$I' = \int_{\eta_{\mathbb{M}}^{\vee}(T'_{\mathbb{M}'})} \int_{\bar{R}} \Delta(t)^{-1}|\det t|^{\mathbf{n}} W^{\mathbb{M}}(t\boldsymbol{\epsilon}_4 r \boldsymbol{\epsilon}_3)\psi_{\bar{R}}(r)\,dr\,dt.$$

The integrand in I' is compactly supported because $W^{\mathbb{M}} \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)_{\natural}$. Note that for $r \in V_{\Delta} = \mathcal{Z} \cap \bar{R}$, we have $\psi_{\bar{R}}(r) = \psi_{\mathcal{Z}}(r)^{-1}$. We write

$$
\begin{aligned}
I' &= \int_{N_{\mathbb{M},\Delta}} \left(\int_{V_\Delta} \int_{\eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'})} \Delta(t)^{-1} |\det t|^{\mathbf{n}} W^{\mathbb{M}}(t\boldsymbol{\epsilon}_4 r u \boldsymbol{\epsilon}_3) \psi_{\mathcal{Z}}(r)^{-1} \, dt \, dr \right) du \\
&= \int_{N_{\mathbb{M},\Delta}} \left(\int_{\mathcal{Z}^+ \backslash \mathcal{Z}} \int_{\eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'})} \Delta(t)^{-1} |\det t|^{\mathbf{n}} W^{\mathbb{M}}(t\boldsymbol{\epsilon}_4 r u \boldsymbol{\epsilon}_3) \psi_{\mathcal{Z}}(r)^{-1} \, dt \, dr \right) du \\
&= \int_{N_{\mathbb{M},\Delta}} \left(\int_{\mathcal{Z}^+ \backslash \mathcal{Z}} \int_{\eta_{\mathbb{M}}^\vee(T'_{\mathbb{M}'})} \Delta(t)^{-1} |\det t|^{\mathbf{n}} W^{\mathbb{M}}(t r \boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) \psi_{\mathcal{Z}}(r)^{-1} \, dt \, dr \right) du
\end{aligned}
$$

(since $\boldsymbol{\epsilon}_4$ stabilizes $\psi_{\mathcal{Z}}$). For the double integral in the brackets we apply part (2) of the theorem above to $\pi(\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) W^{\mathbb{M}}$ (which is applicable since $W^{\mathbb{M}} \in \mathbb{W}^{\psi_{N_{\mathbb{M}}}}(\pi)_{\natural}$). We get

$$
I' = \int_{N_{\mathbb{M},\Delta}} \left(\int_{\mathcal{Z} \cap H_{\mathbb{M}} \backslash \mathcal{Z}} \mathfrak{P}^{H_{\mathbb{M}}}(\pi(n\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) W^{\mathbb{M}}) \psi_{\mathcal{Z}}(n)^{-1} \, dn \right) du.
$$

Thus,

$$
I = \int_{N_{\mathbb{M},\Delta}} \left(\int_{\mathcal{Z} \cap H_{\mathbb{M}} \backslash \mathcal{Z}} \int_U \mathfrak{P}^{H_{\mathbb{M}}}(\pi(n\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) W^{\mathbb{M}}) \psi_{\mathcal{Z}}(n)^{-1} \phi(v) \psi_U^{-1}(v) \, dv \, dn \right) du.
$$

From (8-8), $\big(\delta_P^{-1/2} I\big(\tfrac{1}{2}, \varrho(m) w_U v\big) W\big) \circ \varrho = \phi(v) \delta_P^{1/2}(\varrho(m)) |\det m|^{1/2} \pi(m) W^{\mathbb{M}}$ for any $v \in U$, $m \in \mathbb{M}$. Thus, I equals

$$
\begin{aligned}
\int_{N_{\mathbb{M},\Delta}} \bigg(\int_{\mathcal{Z} \cap H_{\mathbb{M}} \backslash \mathcal{Z}} \int_U \mathfrak{P}^{H_{\mathbb{M}}} \big(\big(\delta_P^{-1/2} I\big(\tfrac{1}{2}, \varrho(n\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) w_U v\big) W\big) \circ \varrho \big) \\
\psi_{\mathcal{Z}}(n)^{-1} \psi_U(v)^{-1} \, dv \, dn \bigg) du.
\end{aligned}
$$

Because $\boldsymbol{\epsilon}_3^{-1} N_{\mathbb{M},\Delta} \boldsymbol{\epsilon}_3 \subset N_{\mathbb{M}}^{\flat}$, the group $\varrho(\boldsymbol{\epsilon}_3^{-1} N_{\mathbb{M},\Delta} \boldsymbol{\epsilon}_3)$ stabilizes the character $\psi_U(w_U^{-1} \cdot w_U)$ on $\overline{U}$. Making a change of variable

$$
v \mapsto (\varrho(n\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) w_U)^{-1} \bar{v} \varrho(n\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) w_U
$$

on U we obtain

$$
I = \int_{N_{\mathbb{M},\Delta}} \left(\int_{\varrho(\mathcal{Z} \cap H_{\mathbb{M}}) \backslash \mathfrak{E}} \mathfrak{P}^{H_{\mathbb{M}}} \big(\big(\delta_P^{-1/2} I\big(\tfrac{1}{2}, v \varrho(\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) w_U\big) W\big) \circ \varrho \big) \psi_{\mathfrak{E}}^{-1}(v) \, dv \right) du.
$$

Integrating over $H \cap \overline{U}$ first, we get from part (3) of the theorem that

$$
I = \int_{N_{\mathbb{M},\Delta}} \left(\int_{H \cap \mathfrak{E} \backslash \mathfrak{E}} L_W(v \varrho(\boldsymbol{\epsilon}_4 u \boldsymbol{\epsilon}_3) w_U) \psi_{\mathfrak{E}}^{-1}(v) \, dv \right) du.
$$

Finally, by part (4) of the theorem, this is equal to $\epsilon_\pi^{\mathbf{n}+1} A_e^{\psi}(M^* W)$ as required. $\square$

8C.

Proof of Proposition 6.15. From Corollaries 8.2 and 8.5 we get that (6-12) holds for $(W, W^\wedge) \in \operatorname{Ind}(\mathbb{W}^{\psi_{N_M}}(\pi))^\circ_\sharp \times \operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi)\big)^\circ_\natural$, namely

$$B\big(W, M(\tfrac{1}{2})W^\wedge, \tfrac{1}{2}\big) = \epsilon_\pi A_e^\psi(M^* W) A_e^{\psi^{-1}}(M^* W^\wedge).$$

On the other hand, as in the proof of Lemma 6.13, it follows from the definition (7-7) and the fact that the image of the restriction map $\mathbb{W}^{\psi^{-1}_{N_M}}(\pi) \to C(N_M\backslash\varrho(\mathcal{P}^*), \psi^{-1}_{N_M})$ contains $C_c^\infty(N_M\backslash\varrho(\mathcal{P}^*), \psi^{-1}_{N_M})$ that the linear map $\operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi)\big)^\circ_\natural \to C_c^\infty(T'_{\mathbb{M}'})$ given by $W^\wedge \mapsto E^{\psi^{-1}}(W^\wedge_{1/2}, \cdot)$ is onto. Therefore, by Corollary 8.5 the linear form $A_e^{\psi^{-1}}(M^* W^\wedge)$ is nonvanishing on $\operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi)\big)^\circ_\natural$. By Corollary 6.14 we conclude that (6-12) holds for all $W^\wedge$ $\big($not necessarily in $\operatorname{Ind}\big(\mathbb{W}^{\psi^{-1}_{N_M}}(\pi)\big)^\circ_\natural\big)$. Proposition 6.15 follows. □

By the discussion before Proposition 6.15, this concludes the proof of Theorem 3.4 and thus the proof of Theorem 1.3.

8D. ***Central character of the descent.*** Assume that $\pi \in \operatorname{Irr}_{\text{meta,temp}} \mathbb{M}$. Let us give another expression for $A_e^\psi(M^* W)$, which we use to prove a formula for the so-called "central sign" of the descent $\mathcal{D}_\psi(\pi)$.

Denote by $\boldsymbol{\epsilon}_5$ the element $\varrho(\boldsymbol{\epsilon}_4\boldsymbol{\epsilon}_3)w_U$. A change of variables in u and v in (8-6) gives

$$A_e^\psi(M^* W) = \epsilon_\pi^{\mathbf{n}+1} \int_{N^\sharp_{\mathbb{M},\Delta}} \bigg(\int_{(H\cap\mathfrak{E})^{\boldsymbol{\epsilon}_5}\backslash\mathfrak{E}^{\boldsymbol{\epsilon}_5}} L_W(\boldsymbol{\epsilon}_5 v\varrho(u)) \psi^{-1}_{\mathfrak{E}^{\boldsymbol{\epsilon}_5}}(v)\, dv \bigg)\, du. \tag{8-9}$$

Here

$$\begin{aligned} N^\sharp_{\mathbb{M},\Delta} &= \{\ell_{\mathbb{M}}(X) : X_{i,j} = 0 \text{ if either } j > i \text{ or } i = \mathbf{n}\},\\ \mathfrak{E}^{\boldsymbol{\epsilon}_5} &= \boldsymbol{\epsilon}_5^{-1}\mathfrak{E}\boldsymbol{\epsilon}_5 = \varrho(\mathcal{Z}^\sharp) \ltimes U,\\ \psi_{\mathfrak{E}^{\boldsymbol{\epsilon}_5}}(\varrho(m)u) &= \psi_{\mathcal{Z}^\sharp}(m)\psi_U(u), \quad m \in \mathcal{Z}^\sharp,\ u \in U, \end{aligned}$$

with

$$\begin{aligned} \mathcal{Z}^\sharp &= \Big\{ \big(\begin{smallmatrix} n_1 & v_1 \\ v_2 & n_2 \end{smallmatrix}\big) : n_1, n_2 \in N'_{\mathbb{M}'},\ I_{\mathbf{n}} + v_1,\ I_{\mathbf{n}} + v_2 \in N'_{\mathbb{M}'} \Big\},\\ \psi_{\mathcal{Z}^\sharp}(m) &= \psi(-m_{1,2} - \cdots - m_{\mathbf{n}-1,\mathbf{n}} + m_{\mathbf{n}+1,\mathbf{n}+2} + \cdots + m_{2\mathbf{n}-1,2\mathbf{n}}), \quad m \in \mathcal{Z}^\sharp. \end{aligned}$$

From this expression we get the next proposition.

Proposition 8.6. *Let $\pi \in \operatorname{Irr}_{\text{gen,meta}} \mathbb{M}$, and $\tilde{\pi} = \mathcal{D}_\psi(\pi)$. Then*

$$\tilde{\pi}(\widetilde{-I_{2\mathbf{n}}}) = \gamma_{\psi^{-1}}((-1)^{\mathbf{n}})\epsilon_\pi.$$

Recall that $\tilde{\pi}(\widetilde{-I_{2\mathbf{n}}})/\gamma_{\psi^{-1}}((-1)^{\mathbf{n}})$ is the central sign of $\tilde{\pi}$ (with respect to ψ^{-1}) introduced by Gan and Savin [2012]. (However, we do not assume $\tilde{\pi}$ is irreducible.)

Proof. We need to show $A^{\psi}_{\sharp}(M^*W, \widetilde{-I_{2\mathbf{n}}}) = \gamma_{\psi^{-1}}((-1)^{\mathbf{n}})\epsilon_\pi A^{\psi}_{e}(M^*W)$. For the moment assume that π is tempered. By Lemma 3.1 and (8-9), $A^{\psi}_{\sharp}(M^*W, \widetilde{-I_{2\mathbf{n}}})$ equals

$$\gamma_{\psi^{-1}}((-1)^{\mathbf{n}})\epsilon_\pi^{\mathbf{n}+1}\int_{N^{\sharp}_{\mathbb{M},\Delta}}\left(\int_{(H\cap\mathfrak{E})^{\epsilon_5}\backslash\mathfrak{E}^{\epsilon_5}} L_W(\boldsymbol{\epsilon}_5 v\varrho(u)\eta(-I_{2\mathbf{n}}))\psi^{-1}_{\mathfrak{E}^{\epsilon_5}}(v)\,dv\right)du.$$

Conjugating $v\varrho(u)$ by $\eta(-I_{2\mathbf{n}})$ we obtain

$$\gamma_{\psi^{-1}}((-1)^{\mathbf{n}})\epsilon_\pi^{\mathbf{n}+1}\int_{N^{\sharp}_{\mathbb{M},\Delta}}\left(\int_{(H\cap\mathfrak{E})^{\epsilon_5}\backslash\mathfrak{E}^{\epsilon_5}} L_W(\boldsymbol{\epsilon}_5 \eta(-I_{2\mathbf{n}})v\varrho(u))\psi^{-1}_{\mathfrak{E}^{\epsilon_5}}(v)\,dv\right)du.$$

Here we need to verify that $\eta(-I_{2\mathbf{n}})$ stabilizes $(\mathfrak{E}^{\epsilon_5}, \psi_{\mathfrak{E}^{\epsilon_5}})$ (which is clear) and normalizes H^{ϵ_5}. The second fact follows from the observation that $\boldsymbol{\epsilon}_5\eta(-I_{2\mathbf{n}}) = \varrho(\boldsymbol{a})\boldsymbol{\epsilon}_5$, where

$$\boldsymbol{a} = \operatorname{diag}\left(\tfrac{1}{2}\mathfrak{d}, 2\mathfrak{d}^*\right)w_0^{\mathbb{M}} \in H_{\mathbb{M}}w_0^{\mathbb{M}}.$$

Since $L_W(\varrho(w_0^{\mathbb{M}})\,\cdot\,) = \epsilon_\pi L_W(\,\cdot\,)$ [Lapid and Mao 2015b, (4.8)], we get

$$A^{\psi}_{\sharp}(M^*W, \widetilde{-I_{2\mathbf{n}}}) = \gamma_{\psi^{-1}}((-1)^{\mathbf{n}})\epsilon_\pi^{\mathbf{n}}\int_{N^{\sharp}_{\mathbb{M},\Delta}}\left(\int_{(H\cap\mathfrak{E})^{\epsilon_5}\backslash\mathfrak{E}^{\epsilon_5}} L_W(\boldsymbol{\epsilon}_5 v\varrho(u))\psi^{-1}_{\mathfrak{E}^{\epsilon_5}}(v)\,dv\right)du.$$

The claim now follows from the comparison with (8-9). Finally, the same argument as in Section 3F using the classification result Theorem 3.8 gives the proposition for all $\pi \in \operatorname{Irr}_{\mathrm{gen,meta}}\mathbb{M}$. □

Appendix: Nonvanishing of Bessel functions

Let G be a split group over a p-adic field.[5] Let $B = A \ltimes N$ be a Borel subgroup of G. Let $G^\circ = Bw_0B$ be the open Bruhat cell where w_0 is the longest element of the Weyl group. Fix a nondegenerate continuous character ψ_N of N. For any $\pi \in \operatorname{Irr}_{\mathrm{gen},\psi_N}(G)$, the Bessel function $\mathbb{B}_\pi = \mathbb{B}_\pi^{\psi_N}$ of π with respect to ψ_N is the locally constant function on G° given by the relation

$$\int_N^{\mathrm{st}} W(gn)\psi_N(n)^{-1}\,dn = \mathbb{B}_\pi(g)W(e) \tag{A-1}$$

for any $W \in \mathbb{W}^{\psi_N}(\pi)$; see [Lapid and Mao 2013]. In this section we prove the following result.

Theorem A.1. *For any tempered $\pi \in \operatorname{Irr}_{\mathrm{gen},\psi_N}(G)$ the function $\mathbb{B}_\pi$ is not identically zero on G°.*

[5] The notation in the appendix is different from the body of the paper.

The argument is similar to the one in [Ichino and Zhang 2014]. Fix a tempered $\pi \in \mathrm{Irr}_{\mathrm{gen},\psi_N}(G)$ and realize it on its Whittaker model $\mathbb{W}^{\psi_N}(\pi)$. Similarly realize $\pi^\vee$ on $\mathbb{W}^{\psi_N^{-1}}(\pi^\vee)$. Thus, we get a pairing $(\cdot\,,\cdot)$ on $\mathbb{W}^{\psi_N}(\pi) \times \mathbb{W}^{\psi_N^{-1}}(\pi^\vee)$.

Fix $W_0^\vee \in \mathbb{W}^{\psi_N^{-1}}(\pi^\vee)$ such that

$$\int_N^{\mathrm{st}} (\pi(n)W, W_0^\vee)\psi_N(n)^{-1}\, dn = W(e)$$

for all $W \in \mathbb{W}^{\psi_N}(\pi)$. This is possible by [Lapid and Mao 2015a, Propositions 2.3 and 2.10]. Then for any $g \in G$ we have

$$W(g) = \int_N^{\mathrm{st}} (\pi(ng)W, W_0^\vee)\psi_N(n)^{-1}\, dn. \tag{A-2}$$

Similarly, fix $W_0 \in \mathbb{W}^{\psi_N}(\pi)$ such that for all $W^\vee \in \mathbb{W}^{\psi_N^{-1}}(\pi^\vee)$,

$$\int_N^{\mathrm{st}} (W_0, \pi^\vee(n)W^\vee)\psi_N(n)\, dn = W^\vee(e). \tag{A-3}$$

Set $\Phi(g) = (\pi(g)W_0, W_0^\vee)$. Let N^{der} be the derived group of N. Also let Ξ be the Harish-Chandra function on G (see, e.g., [Waldspurger 2003]).

Lemma A.2. *The function*

$$g \mapsto \int_{N^{\mathrm{der}}}\int_{N^{\mathrm{der}}} \Phi(n_1 g n_2)\, dn_1\, dn_2$$

on G° is locally L^1 on G. Moreover,

$$g \mapsto \int_{N^{\mathrm{der}}}\int_{N^{\mathrm{der}}} \Xi(n_1 g n_2)\, dn_1\, dn_2$$

is locally L^1 on G.

Proof. The argument is exactly as in [Ichino and Zhang 2014, Lemma A.4] using the convergence of $\int_{N^{\mathrm{der}}} \Xi(n)\, dn$ [Sakellaridis and Venkatesh 2012, Lemma 6.3.1]. □

Remark A.3. Note that $g \mapsto \int_{N^{\mathrm{der}}}\int_{N^{\mathrm{der}}} \Xi(n_1 g n_2)\, dn_1\, dn_2$ is locally constant on G°. Thus, its local integrability on G implies its convergence for any $g \in G^\circ$.

For any $f \in C_c^\infty(G)$ let $L_f(W) = \int_G f(g)W(g)\, dg$ for $W \in \mathbb{W}^{\psi_N}(\pi)$. Then $L_f \in \pi^\vee$. Let L_f^* be the corresponding element in $\mathbb{W}^{\psi_N^{-1}}(\pi^\vee)$ and set $B_\pi(f) = L_f^*(e)$. The distribution $f \mapsto B_\pi(f)$ is called the *Bessel distribution*. It is nonzero: we can choose $f \in C_c^\infty(G)$ such that L_f is nontrivial, and then, by translating f if necessary we can arrange that $L_f^*(e) \neq 0$.

Note that by (A-3),

$$B_\pi(f) = \int_N^{\mathrm{st}} (W_0, \pi^\vee(n)L_f^*)\psi_N(n)\, dn = \int_N^{\mathrm{st}} \left(\int_G f(g)W_0(gn^{-1})\, dg\right)\psi_N(n)\, dn,$$

and therefore by (A-2) we have

$$B_\pi(f)=\int_N^{\mathrm{st}}\left(\int_G f(g)\left(\int_N^{\mathrm{st}}\Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1}\,dn_1\right)dg\right)\psi_N(n_2)\,dn_2. \quad \text{(A-4)}$$

Let A^1 be the maximal compact subgroup of A. Fix $\Omega_0\in\mathcal{CSGR}(N)$, which is invariant under conjugation by A^1 (e.g., take $\Omega_0=N\cap K$). Fix an element $a\in A$ such that $|\alpha(a)|>1$ for all $\alpha\in\Delta_0$. Consider the sequence $\Omega_n=a^n\Omega_0a^{-n}\in\mathcal{CSGR}(N)$, $n=1,2,\dots$. Any Ω_n is invariant under conjugation by A^1, $\Omega_1\subset\Omega_2\subset\cdots$ and $\bigcup\Omega_n=N$.

Let $A^d=A\cap G^{\mathrm{der}}$. Consider the family

$$A_n=\{t\in A^d:|\alpha(t)-1|\le q^{-n}\text{ for all }\alpha\in\Delta_0\}\in\mathcal{CSGR}(A^d),$$

which forms a basis of neighborhoods of 1 for A^d. We only consider n sufficiently large so that the image of A_n under the homomorphism $t\in A\mapsto(\alpha(t))_{\alpha\in\Delta_0}\in(F^*)^{\Delta_0}$ is $\prod_{\alpha\in\Delta_0}(1+\varpi^n\mathcal{O})$, where ϖ is a uniformizer of $\mathcal{O}$.

For any $\alpha\in\Delta_0$ choose a parameterization $x_\alpha:F\to N_\alpha$ such that $\psi_N\circ x_\alpha$ is trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$. Let N_n be the group generated by $\langle x_\alpha(\varpi^{-n}\mathcal{O}),\alpha\in\Delta_0\rangle$ and the derived group N^{der} of N. The following lemma is clear.

Lemma A.4. *For any $u\in N$ we have*

$$(\mathrm{vol}\,A_n)^{-1}\int_{A_n}\psi_N(tut^{-1})\,dt=\begin{cases}\psi_N(u) & \textit{for } u\in N_n,\\ 0 & \textit{otherwise.}\end{cases}$$

Next we prove:

Lemma A.5. *Suppose that f and Φ are bi-invariant under A_n. Then we have*

$$B_\pi(f)=\int_G f(g)\alpha_n(g)\,dg$$

where

$$\alpha_n(g)=\int_{N_n}\int_{N_n}\Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1}\psi_N(n_2)\,dn_2\,dn_1.$$

Remark A.6. Of course we cannot conclude from the lemma by itself that B_π is given by a locally L^1 function (though this is conjectured to be the case.)

Proof. We start with (A-4). Let m be such that

$$B_\pi(f)=\int_{\Omega_m}\int_G\left(\int_N^{\mathrm{st}}f(g)\Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1}\,dn_1\right)dg\,\psi_N(n_2)\,dn_2.$$

Then for any $t \in A_n$ we have

$$\begin{aligned} B_\pi(f) &= \int_{\Omega_m} \int_G \left(\int_N^{\mathrm{st}} f(gt)\Phi(n_1 g n_2^{-1} t)\psi_N(n_1)^{-1}\, dn_1 \right) dg\, \psi_N(n_2)\, dn_2 \\ &= \int_{\Omega_m} \int_G \left(\int_N^{\mathrm{st}} f(gt)\Phi(n_1 g t n_2^{-1})\psi_N(n_1)^{-1}\, dn_1 \right) dg\, \psi_N(t n_2 t^{-1})\, dn_2 \\ &= \int_{\Omega_m} \int_G \left(\int_N^{\mathrm{st}} f(g)\Phi(n_1 g n_2^{-1})\psi_N(n_1)^{-1}\, dn_1 \right) dg\, \psi_N(t n_2 t^{-1})\, dn_2. \end{aligned}$$

Averaging over $t \in A_n$ we get

$$B_\pi(f) = \int_{N_n \cap \Omega_m} \int_G \left(\int_N^{\mathrm{st}} f(g)\Phi(n_1 g n_2^{-1})\psi_N(n_1)^{-1}\, dn_1 \right) dg\, \psi_N(n_2)\, dn_2$$

by Lemma A.4. Thus, for m' sufficiently large (depending on f and m) we have

$$B_\pi(f) = \int_{N_n \cap \Omega_m} \int_G \int_{\Omega_{m'}} f(g)\Phi(n_1 g n_2^{-1})\psi_N(n_1)^{-1}\, dn_1\, dg\, \psi_N(n_2)\, dn_2.$$

Once again, for any $t \in A_n$,

$$\begin{aligned} B_\pi(f) &= \int_{N_n \cap \Omega_m} \int_G \int_{\Omega_{m'}} f(tg)\Phi(t n_1 g n_2^{-1})\psi_N(n_1)^{-1}\, dn_1\, dg\, \psi_N(n_2)\, dn_2 \\ &= \int_{N_n \cap \Omega_m} \int_G \int_{\Omega_{m'}} f(tg)\Phi(n_1 t g n_2^{-1})\psi_N(t^{-1} n_1 t)^{-1}\, dn_1\, dg\, \psi_N(n_2)\, dn_2 \\ &= \int_{N_n \cap \Omega_m} \int_G \int_{\Omega_{m'}} f(g)\Phi(n_1 g n_2^{-1})\psi_N(t^{-1} n_1 t)^{-1}\, dn_1\, dg\, \psi_N(n_2)\, dn_2. \end{aligned}$$

As before, averaging over A_n we get

$$B_\pi(f) = \int_{N_n \cap \Omega_m} \int_G \int_{N_n \cap \Omega_{m'}} f(g)\Phi(n_1 g n_2^{-1})\psi_N(n_1)^{-1}\, dn_1\, dg\, \psi_N(n_2)\, dn_2.$$

Since m and m' can be chosen arbitrarily large, the lemma follows from the convergence of

$$\int_{N_n} \int_G \int_{N_n} |f(g)\Phi(n_1 g n_2^{-1})|\, dn_1\, dg\, dn_2,$$

i.e., Lemma A.2. □

We prove the following analogously.

Lemma A.7. *For $g \in G^\circ$, let $\mathbb{B}_\pi^{A_n}(g) := \mathrm{vol}(A_n)^{-2} \int_{A_n} \int_{A_n} \mathbb{B}_\pi(t_1 g t_2)\, dt_1\, dt_2$. Suppose that Φ is bi-invariant under A_n. Then*

$$\mathbb{B}_\pi^{A_n}(g) W_0(e) = \alpha_n(g)$$

for all $g \in G^\circ$.

Proof. First note that for any compact subset C of G° we have

$$\int_{\Omega_m}\left(\int_N^{\mathrm{st}}\Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1}\,dn_1\right)\psi_N(n_2)\,dn_2=\mathbb{B}_\pi(g)W_0(e)$$

for all $g\in C$ and m sufficiently large. Indeed, by (A-2) the inner stable integral is $W_0(gn_2^{-1})$. Thus, the relation above follows from [Lapid and Mao 2013] and (A-1).

Now, for any $t\in A_n$ we have

$$\begin{aligned}\mathbb{B}_\pi(gt)W_0(e)&=\int_{\Omega_m}\left(\int_N^{\mathrm{st}}\Phi(n_1gtn_2^{-1})\psi_N(n_1)^{-1}\,dn_1\right)\psi_N(n_2)\,dn_2\\&=\int_{\Omega_m}\left(\int_N^{\mathrm{st}}\Phi(n_1gn_2^{-1}t)\psi_N(n_1)^{-1}\,dn_1\right)\psi_N(t^{-1}n_2t)\,dn_2\\&=\int_{\Omega_m}\left(\int_N^{\mathrm{st}}\Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1}\,dn_1\right)\psi_N(t^{-1}n_2t)\,dn_2.\end{aligned}$$

Thus, by Lemma A.4, we have

$$\begin{aligned}&\mathrm{vol}(A_n)^{-1}\int_{A_n}\mathbb{B}_\pi(gt)\,dt\;W_0(e)\\&\qquad=\int_{N_n\cap\Omega_m}\left(\int_N^{\mathrm{st}}\Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1}\,dn_1\right)\psi_N(n_2)\,dn_2.\end{aligned}$$

For g in a compact set C of G° and for m' sufficiently large (depending on C and m) we can write this as

$$\int_{N_n\cap\Omega_m}\int_{\Omega_{m'}}\Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1}\,dn_1\;\psi_N(n_2)\,dn_2.$$

Now, for any $t_1\in A_n$,

$$\begin{aligned}&\mathrm{vol}(A_n)^{-1}\int_{A_n}\mathbb{B}_\pi(t_1gt_2)\,dt_2\;W_0(e)\\&\qquad=\int_{N_n\cap\Omega_m}\int_{\Omega_{m'}}\Phi(n_1t_1gn_2^{-1})\psi_N(n_1)^{-1}\,dn_1\;\psi_N(n_2)\,dn_2\\&\qquad=\int_{N_n\cap\Omega_m}\int_{\Omega_{m'}}\Phi(t_1n_1gn_2^{-1})\psi_N(t_1n_1t_1^{-1})^{-1}\,dn_1\;\psi_N(n_2)\,dn_2\\&\qquad=\int_{N_n\cap\Omega_m}\int_{\Omega_{m'}}\Phi(n_1gn_2^{-1})\psi_N(t_1n_1t_1^{-1})^{-1}\,dn_1\;\psi_N(n_2)\,dn_2.\end{aligned}$$

Averaging over $t_1\in A_n$ we get

$$\mathbb{B}_\pi^{A_n}(g)W_0(e)=\int_{N_n\cap\Omega_m}\int_{N_n\cap\Omega_{m'}}\Phi(n_1gn_2^{-1})\psi_N(n_1)^{-1}\psi_N(n_2)\,dn_1\,dn_2.$$

The lemma follows from the convergence of

$$\int_{N_n}\int_{N_n} |\Phi(n_1 g n_2^{-1})|\, dn_1\, dn_2,$$

i.e., Remark A.3. □

Proof of Theorem A.1. Since $B_\pi(f) \neq 0$, we have $\alpha_n|_{G^\circ} \not\equiv 0$ for $n \gg 1$ by Lemma A.5. Hence, by Lemma A.7, $\mathbb{B}_\pi^{A_n}\big|_{G^\circ} \not\equiv 0$ and therefore $\mathbb{B}_\pi\big|_{G^\circ} \not\equiv 0$. □

Remark A.8. Theorem A.1 and its proof are also valid for $\widetilde{\mathrm{Sp}}_n$.

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