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Cup products of line bundles on homogeneous varieties and generalized PRV components of multiplicity one

Ivan Dimitrov and Mike Roth

Let $X = G/B$ and let L_1 and L_2 be two line bundles on X. Consider the cup-product map

$$H^{d_1}(X, L_1) \otimes H^{d_2}(X, L_2) \xrightarrow{\cup} H^{d}(X, L),$$

where $L = L_1 \otimes L_2$ and $d = d_1 + d_2$. We answer two natural questions about the map above: When is it a nonzero homomorphism of representations of G? Conversely, given generic irreducible representations V_1 and V_2, which irreducible components of $V_1 \otimes V_2$ may appear in the right hand side of the equation above? For the first question we find a combinatorial condition expressed in terms of inversion sets of Weyl group elements. The answer to the second question is especially elegant: the representations V appearing in the right hand side of the equation above are exactly the generalized PRV components of $V_1 \otimes V_2$ of stable multiplicity one. Furthermore, the highest weights $(\lambda_1, \lambda_2, \lambda)$ corresponding to the representations (V_1, V_2, V) fill up the generic faces of the Littlewood–Richardson cone of G of codimension equal to the rank of G. In particular, we conclude that the corresponding Littlewood–Richardson coefficients equal one.

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1. Introduction

1.1. ***Main problems.*** The main object of study of this paper is the cup-product map

$$\mathrm{H}^{d_1}(\mathrm{X}, \mathrm{L}_1) \otimes \cdots \otimes \mathrm{H}^{d_k}(\mathrm{X}, \mathrm{L}_k) \xrightarrow{\cup} \mathrm{H}^{d}(\mathrm{X}, \mathrm{L}), \tag{1.1.1}$$

where $\mathrm{X} = \mathrm{G}/\mathrm{B}$, G is a semisimple algebraic group over an algebraically closed field of characteristic zero, B is a Borel subgroup of G, $\mathrm{L}_1, \ldots, \mathrm{L}_k$ are arbitrary line bundles on X, $\mathrm{L} = \mathrm{L}_1 \otimes \cdots \otimes \mathrm{L}_k$, $d_1, \ldots, d_k$ are nonnegative integers, and $d = d_1 + \cdots + d_k$.

We assume that both sides of (1.1.1) are nonzero for otherwise the cup-product map is the zero map. Without loss of generality we may also assume that the line bundles $\mathrm{L}_1, \ldots, \mathrm{L}_k$, and L are G-equivariant; then both sides of (1.1.1) carry a natural G-module structure and the cup-product map is G-equivariant. Furthermore by the Borel–Weil–Bott theorem there are irreducible representations $\mathrm{V}_{\mu_1}, \ldots, \mathrm{V}_{\mu_k}$, and V_μ so that $\mathrm{H}^{d_i}(\mathrm{X}, \mathrm{L}_i) = \mathrm{V}^*_{\mu_i}$ for $i = 1, \ldots, k$, and $\mathrm{H}^d(\mathrm{X}, \mathrm{L}) = \mathrm{V}^*_\mu$ as representations of G. The dual of (1.1.1) is thus a G-homomorphism

$$\mathrm{V}_\mu \to \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}. \tag{1.1.2}$$

Since $\mathrm{V}_{\mu_1}, \ldots, \mathrm{V}_{\mu_k}$, and V_μ are irreducible representations, (1.1.1) is either surjective or zero; respectively, (1.1.2) is either injective or zero. This leads us naturally to the two main problems of this paper.

Problem I. When is (1.1.1) a surjection of nontrivial representations?

Problem II. For which $(k+1)$-tuples $(\mathrm{V}_{\mu_1}, \ldots, \mathrm{V}_{\mu_k}, \mathrm{V}_\mu)$ of irreducible representations of G can V_μ be realized as a component of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ via (1.1.2) for appropriate line bundles $\mathrm{L}_1, \ldots, \mathrm{L}_k$ on X?

We call an irreducible representation V_μ that can be embedded into $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ via (1.1.2) a *cohomological component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$. A variation of Problem II is to determine the cohomological components of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$, for $\mathrm{V}_{\mu_1}, \ldots, \mathrm{V}_{\mu_k}$ fixed.

With the exception of some quite degenerate cases for Problem II, we provide a complete solution to both problems.

1.2. ***Solution of Problem I.*** Fix a maximal torus $\mathrm{T} \subseteq \mathrm{B}$. The G-equivariant line bundles on X are in one-to-one correspondence with the characters of T. For a character λ of T, we denote by L_λ the line bundle on X corresponding to the one dimensional representation of B on which T acts via $-\lambda$.

The affine action of the Weyl group $\mathcal{W}$ of G on the lattice of T-characters Λ is defined as

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

where ρ, as usual, denotes the half-sum of the roots of B. A character $\lambda \in \Lambda$ is *regular* if there exists a (necessarily unique) element $w \in \mathcal{W}$ such that $w \cdot \lambda$ is a dominant character. Following Kostant [1961, Definition 5.10], we define *the inversion set* Φ_w *of* $w \in \mathcal{W}$ as the set $\Phi_w = w^{-1}\Delta^- \cap \Delta^+$, where $\Delta^- = -\Delta^+$ is the set of negative roots of G.

Let $\lambda_1, \dots, \lambda_k \in \Lambda$ be the (regular) characters such that $\mathrm{L}_i = \mathrm{L}_{\lambda_i}$ for $1 \leqslant i \leqslant k$. Then $\mathrm{L} = \mathrm{L}_\lambda$, where $\lambda = \sum_{i=1}^k \lambda_i$. Assume that λ is also regular and denote by w and $w_1, \dots, w_k$ the Weyl group elements for which $w \cdot \lambda$ and $w_i \cdot \lambda_i$, for $1 \leqslant i \leqslant k$, are dominant. With this notation we prove the following criterion for surjectivity of (1.1.1).

Theorem I. *For any semisimple* G, *if* $\mathrm{H}^d(\mathrm{X}, \mathrm{L}_\lambda) \neq 0$, *then the cup-product map* (1.1.1) *is surjective if and only if*

$$\Phi_w = \bigsqcup_{i=1}^{k} \Phi_{w_i}. \tag{1.2.1}$$

Studying the structure of $(k+1)$-tuples $(w_1, \dots, w_k, w)$ satisfying (1.2.1) is an interesting combinatorial problem which we do not address here. A recursive description of such $(k+1)$-tuples in types A, B, and C is given in [Dewji et al. 2017]. For some open questions concerning (1.2.1) see the expository article [Dimitrov and Roth 2009].

1.3. ***Solution of Problem II.*** We say that a component V_μ of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ has *stable multiplicity one* if the multiplicity of $\mathrm{V}_{m\mu}$ in $\mathrm{V}_{m\mu_1} \otimes \cdots \otimes \mathrm{V}_{m\mu_k}$ is one for all $m \gg 0$. We say that V_μ is a *generalized PRV component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ if there exist $w_1, \dots, w_k$, and $w \in \mathcal{W}$ such that $w^{-1}\mu = w_1^{-1}\mu_1 + \cdots + w_k^{-1}\mu_k$. (See Sections 2.3 and 6.1 for further discussion of these conditions.)

Theorem II. (a) *Let* V_μ *be a cohomological component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$. *Then* V_μ *is a generalized PRV component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ *of stable multiplicity one.*

(b) *Conversely, assume that* V_μ *is a generalized PRV component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ *of stable multiplicity one. If, in addition, one of the following holds:*

(i) *at least one of* $\mu_1, \dots, \mu_k$ *or* μ *is strictly dominant,*

(ii) G *is a simple classical group or a product of simple classical groups,*

then V_μ *is a cohomological component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$.

It is unfortunate that in part (b) above we require condition (i) or (ii). Indeed, we believe that we do not need these conditions but we impose them due to our inability to overcome a combinatorial problem.

Remark. In type A a conjecture of Fulton, proved by Knutson, Tao, and Woodward [Knutson et al. 2004, §6.1, §7] states that if V_μ is a component of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$

of multiplicity one, then $V_{m\mu}$ has multiplicity one in $V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k}$ for all $m \geqslant 1$. Together with Theorem II, this means that in type A a component V_μ is a cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if and only if V_μ is a PRV component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ of multiplicity one.[1]

1.4. ***Representation-theoretic implications of Theorem II.*** The representation-theoretic significance of Theorem II is twofold: it provides both a geometric construction of special components of a tensor product via the Bott theorem, and a new way of generalizing the classical PRV component.

The Borel–Weil–Bott theorem provides a geometric realization of every irreducible representation of G as the cohomology (in any degree) of an appropriate line bundle on X. In particular, every irreducible representation equals the space of global sections of a unique line bundle on X. In this sense the Borel–Weil theorem (the statement about cohomology in degree zero) suffices since the Bott theorem (the statement about higher cohomology) yields the same representations. However, in addition to being representations, the cohomology groups carry a ring structure induced from the cup product. Theorem II employs this structure to give a geometric realization of certain components of a tensor product of representations. As far as we know this is the first use of the Bott theorem for a geometric construction of representations in the case when G is a semisimple algebraic group over a field of characteristic zero.

We are borrowing the term “generalized PRV component” from the case when $k = 2$. Parthasarathy, Ranga Rao, and Varadarajan [Parthasarathy et al. 1967] established that if μ is in the $\mathcal{W}$-orbit of $\mu_1 + w_0\mu_2$ (where w_0 denotes the longest element of $\mathcal{W}$), then V_μ is a component of $V_{\mu_1} \otimes V_{\mu_2}$. Moreover, they proved that V_μ has multiplicity one in, and is the smallest component of, $V_{\mu_1} \otimes V_{\mu_2}$. It is true more generally that if μ is in the $\mathcal{W}$-orbit of $\mu_1 + v\mu_2$ (where v is now an arbitrary element of $\mathcal{W}$) then V_μ is again a component of $V_{\mu_1} \otimes V_{\mu_2}$; this was established independently by Kumar [1988] and Mathieu [1989].

Unlike the original PRV component, a generalized PRV component V_μ may have multiplicity greater than one in $V_{\mu_1} \otimes V_{\mu_2}$. However, by Theorem II, every cohomological component is a generalized PRV component of stable multiplicity one. The cohomological components also retain an aspect of the minimality of the original PRV component: every cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ is extreme among all components of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$. These properties of cohomological components suggest that they may be viewed as the “true” analog of the original PRV component.

Figure 1 illustrates Theorem II in the case $k = 2$.

[1] This is not true in other types; see the component $V_{1,0}$ of $V_{1,1} \otimes V_{1,1}$ in the middle example of Figure 1.

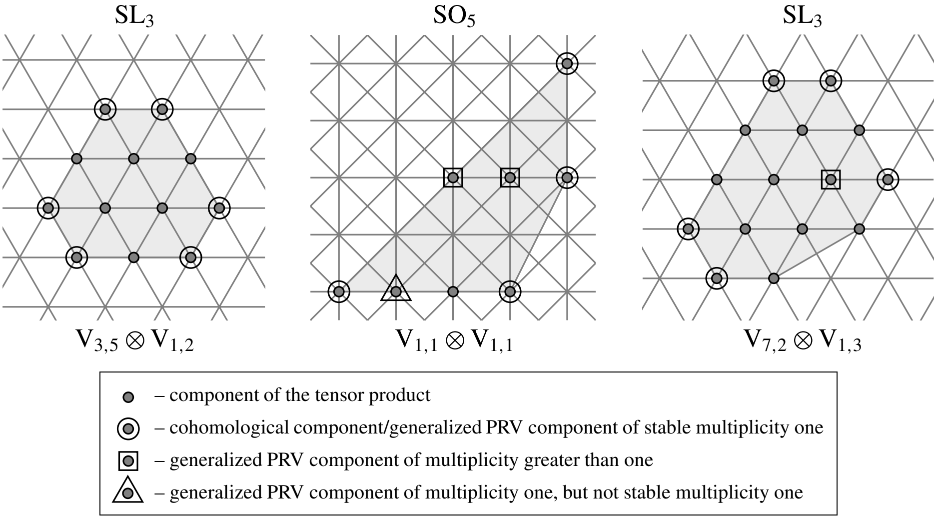

Figure 1. Illustration of Theorem II when $k = 2$.

1.5. ***Other results.*** In conclusion we mention several other results which may be of independent interest.

The cup product and Schubert calculus. Recall that a basis for the cohomology ring $\mathrm{H}^*(\mathrm{X}, \mathbb{Z})$ of $\mathrm{X} = \mathrm{G}/\mathrm{B}$ is given by the classes of the Schubert cycles $\{[\mathrm{X}_w]\}_{w\in\mathcal{W}}$ indexed by the elements of the Weyl group $\mathcal{W}$. The dual basis $\{[\Omega_w]\}_{w\in\mathcal{W}}$, is given by $\Omega_w := \mathrm{X}_{w_0 w}$. With the notation of Section 1.2 we prove the following:

Theorem III. *For any semisimple algebraic group* G,

(a) *if* $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] = 1$ *then the cup-product map* (1.1.1) *is surjective*;

(b) *if* $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] = 0$ *then the cup-product map* (1.1.1) *is zero.*

We use Theorem III as stated above and a variation of its proof to prove Theorem I. In general it is not known if condition (1.2.1) implies that $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] = 1$. In [Dimitrov and Roth $\geq$ 2017] we show that this is the case when G is a classical group or G_2; We do not know if condition (1.2.1) implies that the intersection number is one in the other exceptional cases.

Diagonal Bott–Samelson–Demazure–Hansen varieties. We construct a class of varieties which generalize the Bott–Samelson–Demazure–Hansen varieties. One way to understand these varieties is a resolution of singularities of the total space of intersections of translates of Schubert varieties; see Theorem 3.7.4. Other notable results related to this construction include Lemma 3.8.1, which controls the multiplicity of cohomological components, and Theorem 3.9.1, which provides a new proof of the necessity of the inequalities determining the Littlewood–Richardson cone. These varieties have applications outside this paper. For instance, in a future paper we use them to establish multiplicity bounds for the Littlewood–Richardson coefficients generalizing the Klymik bound, each of which has the same asymptotic order of growth as the multiplicity function, with each "centered" around a particular cohomological component (in a way that the Klymik bound appears as the version for the highest weight component). These varieties are also used in [Roth 2011] to prove reduction rules for Littlewood–Richardson coefficients.

1.6. ***Related Work.*** After the initial version of this paper appeared on the arXiv, other authors have worked on related ideas. V. Tsanov [2013], considers the more general situation of an embedding $G_1 \hookrightarrow G_2$ of complex semisimple Lie groups, inducing an embedding

$$X_1 := G_1/B_1 \hookrightarrow X_2 := G_2/B_2,$$

where B_1 and B_2 are nested Borel subgroups. The main result of [Tsanov 2013] extends Theorem I to this setting, giving necessary and sufficient conditions for the pullback map $H^d(X_1, L|_X) \leftarrow H^d(X_2, L)$ to be nonzero, when L is an equivariant bundle on X_2; see [Tsanov 2013, Theorem 2.2]. The arguments in [Tsanov 2013] use Lie algebra cohomology, and are quite different in character from the arguments of this paper.

In a preprint, N. Ressayre [2009, Theorem 1] states that every generalized PRV component of stable multiplicity one is a cohomological component. That is, this result states that part (b) of Theorem II holds without requiring either of the conditions (i) or (ii) of (b).

Finally, the varieties $X = G/B$ considered in this paper have the property that they are projective varieties acted on transitively by an algebraic group. There is another natural class of varieties also fitting this description, namely Abelian varieties. Here Mumford's index theorem and the theorem on irreducibility of the theta-group representation take the place of the Borel–Weil–Bott theorem. N. Grieve [2014] proves results on the surjectivity of cup-product maps between cohomology of line bundles on Abelian varieties, again subject to certain combinatorial restrictions.

2. Notation and background results

2.1. ***Notation and conventions.*** The ground field is algebraically closed of characteristic zero. Throughout the paper we fix a semisimple connected algebraic group G, a Borel subgroup $B \subset G$, and a maximal torus $T \subset B$. All parabolic subgroups we consider contain T. The Lie algebras of algebraic groups are denoted by Fraktur letters, e.g., $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, etc. We use the term "G-module" instead of "representation of G" to avoid differentiating between representations of algebraic groups and modules over the respective Lie algebras; likewise, since T is fixed, we use the term "weight" both for characters of T and weights of $\mathfrak{t}$; in particular we only consider integral weights of $\mathfrak{t}$.

The point

$$wB/B \in X_w \subseteq X = G/B,$$

where $w \in \mathcal{W}$ and X_w is the corresponding Schubert variety, is denoted by w for short. If $M = G/P$ for some parabolic P we similarly use w to indicate the point $wP/P \in M$.

If Λ is the lattice of weights of T we denote the group ring of Λ by $\mathbb{Z}[\Lambda]$, i.e.,

$$\mathbb{Z}[\Lambda] = \left\{ \sum_{i=1}^{k} c_i e^{\lambda_i} \;\middle|\; c_i \in \mathbb{Z}, \lambda_i \in \Lambda \right\}.$$

For a T-module $\mathcal{M}$, the *formal character of* $\mathcal{M}$ is

$$\operatorname{Ch} \mathcal{M} = \sum_{\lambda \in \Lambda} \dim \mathcal{M}^{\lambda} e^{\lambda} \in \mathbb{Z}[\Lambda],$$

where

$$\mathcal{M}^{\lambda} = \{x \in \mathcal{M} \mid t \cdot x = \lambda(t)x \text{ for every } t \in \mathfrak{t}\}.$$

All formal characters discussed in this paper are contained in $\mathbb{Z}[\Delta]$. For a subset $\Phi \subseteq \Delta$, the formal character of $\bigoplus_{\alpha \in \Phi} \mathfrak{g}^{\alpha}$ is denoted by $\langle \Phi \rangle$, i.e.,

$$\langle \Phi \rangle = \sum_{\alpha \in \Phi} e^{\alpha}.$$

If w is an element of the Weyl group $\mathcal{W}$, then $\ell(w)$ means the length of any minimal expression giving w as a product of simple reflections. If $\underline{v}$ is a word in the simple reflections, then $\ell(\underline{v})$ is the number of reflections in the word. Note that, if $\underline{v}$ is a word in simple reflections, and $v \in \mathcal{W}$ is the corresponding element of the Weyl group, then $\ell(\underline{v}) = \ell(v)$ if and only if $\underline{v}$ is a reduced word. If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is a nonempty word, we denote by $\underline{v}_R$ the word $s_{i_1} \cdots s_{i_{m-1}}$ obtained from $\underline{v}$ by dropping the rightmost reflection in $\underline{v}$. If $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ is a sequence of words then we set $\ell(\underline{\boldsymbol{v}}) = \sum_{i=1}^{k} \ell(\underline{v}_i)$.

A list of symbols used in the paper can be found on page 812.

2.2. ***Inversion sets.*** Let Δ^+ be the set of positive roots of $\mathfrak{g}$ (with respect to B). Following Kostant [1961, Definition 5.10], for any element w of the Weyl group $\mathcal{W}$ we define Φ_w, the *inversion set* of w, to be the set of positive roots sent to negative roots by w, i.e.,

$$\Phi_w := w^{-1}\Delta^- \cap \Delta^+. \tag{2.2.1}$$

For a subset Φ of Δ^+, we set $\Phi^c := \Delta^+ \setminus \Phi$. We will need the following formulas, which follow easily from the definition:

$$\Phi_{w_0 w} = \Phi_w^c, \tag{2.2.2}$$

$$w^{-1}\Delta^+ = \Phi_w^c \sqcup -\Phi_w, \tag{2.2.3}$$

$$w^{-1} \cdot 0 = w^{-1}\rho - \rho = -\sum_{\alpha \in \Phi_w} \alpha. \tag{2.2.4}$$

2.3. *Generalized PRV components.* For fixed dominant weights μ_1, μ_2, and μ it is clear that the two conditions,

(a) there exist w_1, w_2, and w in $\mathcal{W}$ such that $w^{-1}\mu = w_1^{-1}\mu_1 + w_2^{-1}\mu_2$,

(b) there exists v in $\mathcal{W}$ such that μ is in the $\mathcal{W}$-orbit of $\mu_1 + v\mu_2$,

are equivalent. If these conditions are satisfied we call V_μ a *generalized PRV component of* $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$.

As is suggested by the name, but is far from obvious from the definition, every generalized PRV component of $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$ is in fact a component of the tensor product $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$ of G-modules. This was first proved when $v = w_0$ (i.e., when μ is in the $\mathcal{W}$-orbit of $\mu_1 + w_0\mu_2$) in [Parthasarathy et al. 1967]. In the literature this component is referred to simply as the *PRV component*. The general case, that V_μ is a component of $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$ for an arbitrary v, became known as the PRV conjecture, and was established independently by Kumar [1988] and Mathieu [1989].

In the present paper we extend the notion of generalized PRV component to components of the tensor product of k irreducible G-modules for $k \geqslant 2$. We call V_μ a *generalized PRV component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ if there exist $w_1, \ldots, w_k$, and w in $\mathcal{W}$ such that $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$. A straightforward induction from the case $k = 2$ implies that every generalized PRV component of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ is a component of the tensor product $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ of G-modules. We record the special case when $\mu = 0$ for use in the proof of Theorem I.

Lemma 2.3.1. *For any dominant weights $\mu_1, \ldots, \mu_k$, and Weyl group elements $w_1, \ldots, w_k$, if $\sum_{i=1}^k w_i^{-1}\mu_i = 0$ then $(\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k})^{\mathrm{G}} \neq 0$.*

2.4. *Borel–Weil–Bott theorem.* Suppose that λ is a regular weight, so there is a unique $w \in \mathcal{W}$ with $w \cdot \lambda \in \Lambda^+$. The Borel–Weil–Bott theorem identifies the cohomology of the line bundle L_λ on X as G-modules:

$$\mathrm{H}^d(\mathrm{X}, \mathrm{L}_\lambda) = \begin{cases} \mathrm{V}_{w\cdot\lambda}^* & \text{if } d = \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

If λ is not a regular weight then the cohomology of L_λ is zero in all degrees.

2.5. ***Serre duality on*** **X.** For any weight λ set $S(\lambda) = -\lambda - 2\rho$. Since the canonical bundle K_X of X is equal to $L_{-2\rho}$ we see that $L_{S(\lambda)} = K_X \otimes L_\lambda^*$. In other words, S is the function that for each weight λ returns the weight $S(\lambda)$ of the line bundle Serre dual to L_λ; the map S is clearly an involution. Let w be any element of the Weyl group and λ any weight. A straightforward computation shows that S commutes with the affine action of the Weyl group, i.e., that $w \cdot S(\lambda) = S(w \cdot \lambda)$.

Lemma 2.5.1. *If λ is a regular weight and w the unique element of the Weyl group with $w \cdot \lambda \in \Lambda^+$ then $(w_0 w) \cdot S(\lambda) \in \Lambda^+$.*

Proof. If μ is a dominant weight then $V_\mu^* = V_{-w_0\mu}$. Therefore if $w \cdot \lambda = \mu \in \Lambda^+$ then

$$(w_0 w) \cdot S(\lambda) = w_0 \cdot S(w \cdot \lambda) = w_0 \cdot S(\mu) = -w_0\mu \in \Lambda^+, \tag{2.5.2}$$

concluding the proof. □

Since $\ell(w_0 w) = N - \ell(w)$, the calculation above fits in neatly with the Borel–Weil–Bott theorem (BWB) and Serre duality. If λ is a regular weight and w an element of the Weyl group with $w \cdot \lambda = \mu \in \Lambda^+$ then we have

$$\begin{aligned} V_\mu &= \left(H^{\ell(w)}(X, L_\lambda)\right)^* && \text{(by BWB)} \\ &= H^{N-\ell(w)}(X, K_X \otimes L_\lambda^*) && \text{(by Serre duality)} \\ &= H^{\ell(w_0 w)}(X, L_{S(\lambda)}) && \text{(see Section 2.5)} \\ &= V_{-w_0\mu}^*, && \text{(by BWB and (2.5.2)).} \end{aligned}$$

2.6. ***Schubert varieties.*** For an element $w \in \mathcal{W}$ of the Weyl group the *Schubert variety* X_w is defined by

$$X_w := \overline{BwB/B} \subseteq G/B = X.$$

Recall that the classes of the Schubert cycles $\{[X_w]\}_{w \in \mathcal{W}}$ give a basis for the cohomology ring $H^*(X, \mathbb{Z})$ of X. Each $[X_w]$ is a cycle of complex dimension $\ell(w)$. The dual Schubert cycles $\{[\Omega_w]\}_{w \in \mathcal{W}}$, given by $\Omega_w := X_{w_0 w}$, also form a basis. Each $[\Omega_w]$ is a cycle of complex codimension $\ell(w)$. The work of Demazure [1974], Kempf [1976], Ramanathan [1985], and Seshadri [1987] shows that each Schubert variety X_w is normal with rational singularities.

Remark. If $w_1, \ldots, w_k$, and $w \in \mathcal{W}$ are such that $\ell(w) = \sum \ell(w_i)$, then the intersection $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [X_w]$ is a number. The number is the coefficient of $[\Omega_w]$ when writing the product $\bigcap_{i=1}^k [\Omega_{w_i}]$ in terms of the basis $\{[\Omega_v]\}_{v \in \mathcal{W}}$.

To reduce notation we use w to also refer to the point $wB/B \in X_w \subseteq X$. In particular, for the identity $e \in \mathcal{W}$, $X_e = \{e\}$. Note that $e \in X$ is also the image of 1_G under the projection from G onto X.

Definition. The *Bruhat order* on the Weyl group $\mathcal{W}$ is the partial order given by the relation $v \leqslant w$ if and only if $X_v \subseteq X_w$. The minimum element in this order is e and the maximum element is w_0, corresponding to the subvarieties $X_e = \{e\}$ and $X_{w_0} = X$ respectively.

The following result will be used several times throughout the paper.

Lemma 2.6.1. *Suppose that $w_1, \dots, w_k$ are elements of the Weyl group such that $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$. Then $\bigcap_{i=1}^k [\Omega_{w_i}] \neq 0$.*

Proof. Each class $[\Omega_{w_i}]$ is represented by any translation of the cycle Ω_{w_i}, so to understand $\bigcap_{i=1}^k [\Omega_{w_i}]$ we can study the intersection of schemes

$$\bigcap_{i=1}^{k} (w_0 w_i)^{-1} \Omega_{w_i}. \tag{2.6.2}$$

Each of the schemes $(w_0 w_i)^{-1}\Omega_{w_i}$ passes through $e \in X$. The tangent space to $(w_0 w)^{-1}\Omega_w$ at e is

$$\mathrm{Lie}\big((w_0 w)^{-1} \mathrm{B} (w_0 w)\big) / \mathrm{Lie}(\mathrm{B}) = \bigoplus_{\alpha \in -\Phi_{w_0 w}} \mathfrak{g}^\alpha \overset{(2.2.2)}{=} \bigoplus_{\alpha \in -\Phi_w^{\mathrm{c}}} \mathfrak{g}^\alpha \subseteq \mathfrak{b}^- = \mathrm{T}_e X,$$

where we have identified $\mathrm{T}_e X$ with $\mathfrak{b}^-$ via the projection $G \to X$. Noting that

$$\bigcap_{i=1}^{k} \Phi_w^{\mathrm{c}} = \left(\bigcup_{i=1}^{k} \Phi_w \right)^{\mathrm{c}} = (\Delta^+)^{\mathrm{c}} = \varnothing,$$

we conclude that the intersection of the tangent spaces of the varieties $(w_0 w_i)^{-1}\Omega_{w_i}$ at $e \in X$ is 0. Hence the intersection (2.6.2) is transverse at the identity. By Kleiman's transversality theorem [1974, Corollary 4(ii)], small translations of each of the varieties $(w_0 w_i)^{-1}\Omega_{w_i}$ will intersect properly and compute the intersection number. Small translations of varieties cannot remove transverse points of intersection and thus $\bigcap_{i=1}^k [\Omega_{w_i}] \neq 0$. □

2.7. ***Symmetric and nonsymmetric forms.*** Most questions we consider, including Problem I and Problem II, can be stated in nonsymmetric and symmetric forms and it is frequently convenient to switch from one to the other. We illustrate this procedure by showing how to switch from the nonsymmetric to the symmetric form of Problem I.

In the nonsymmetric form we are given $w_1, \dots, w_k$, and w, such that $\ell(w) = \sum \ell(w_i)$, and $\lambda_1, \dots, \lambda_k$, and λ, such that $\lambda = \sum \lambda_i$, satisfying the additional conditions that $w_i \cdot \lambda_i \in \Lambda^+$ for $i = 1, \dots, k$, and $w \cdot \lambda \in \Lambda^+$. This corresponds to the data of a cup-product problem:

$$\mathrm{H}^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(X, L_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^{\ell(w)}(X, L_\lambda). \tag{2.7.1}$$

Set $\mu_i = w_i \cdot \lambda_i$ for $i = 1, \dots, k$, and $\mu = w \cdot \lambda$ to keep track of the modules which appear as cohomology groups. By the Borel–Weil–Bott theorem the map (2.7.1) corresponds to a G-equivariant map

$$\mathrm{V}^*_{\mu_1} \otimes \mathrm{V}^*_{\mu_2} \otimes \cdots \otimes \mathrm{V}^*_{\mu_k} \to \mathrm{V}^*_{\mu}.$$

By Serre duality $\mathrm{H}^{\mathrm{N}-\ell(w)}(\mathrm{X}, \mathrm{K_X} \otimes \mathrm{L}^*_\lambda) \neq 0$ and the cup-product map

$$\mathrm{H}^{\ell(w)}(\mathrm{X}, \mathrm{L}_\lambda) \otimes \mathrm{H}^{\mathrm{N}-\ell(w)}(\mathrm{X}, \mathrm{K_X} \otimes \mathrm{L}^*_\lambda) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K_X})$$

is a perfect pairing. Since $\mathrm{H}^{\ell(w)}(\mathrm{X}, \mathrm{L}_\lambda)$ is an irreducible G-module, the surjectivity of (2.7.1) is equivalent to the surjectivity of the cup-product map

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \otimes \mathrm{H}^{\mathrm{N}-\ell(w)}(\mathrm{X}, \mathrm{K_X} \otimes \mathrm{L}^*_\lambda) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K_X}). \quad (2.7.2)$$

To get the symmetric form of this problem, we set $w_{k+1} = w_0 w$, $\lambda_{k+1} = \mathrm{S}(\lambda) = -\lambda - 2\rho$, and $\mu_{k+1} = -w_0\mu = w_{k+1} \cdot \lambda_{k+1}$. Then $\mathrm{L}_{\lambda_{k+1}} = \mathrm{K_X} \otimes \mathrm{L}^*_\lambda$ by Section 2.5, $w_{k+1} \cdot \lambda_{k+1} \in \Lambda^+$ by Lemma 2.5.1, and $\ell(w_{k+1}) = \mathrm{N} - \ell(w)$, so that (2.7.2) becomes

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \otimes \mathrm{H}^{\ell(w_{k+1})}(\mathrm{X}, \mathrm{L}_{\lambda_{k+1}}) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K_X}).$$

Since

$$\sum_{i=1}^{k+1} \lambda_i = \lambda + (-\lambda - 2\rho) = -2\rho$$

and $\mathrm{L}_{-2\rho} = \mathrm{K_X}$, this is again a cup-product problem of the type we consider, but now all weights $\lambda_1, \dots, \lambda_{k+1}$ and Weyl group elements $w_1, \dots, w_{k+1}$ play equal roles.

By (2.2.2) $\Phi_{w_{k+1}} = \Phi^c_w$ and therefore the condition that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$ is equivalent to the condition $\Delta^+ = \bigsqcup_{i=1}^{k+1} \Phi_{w_i}$. Since $[\Omega_{w_{k+1}}] = [\mathrm{X}_{w_0 w_{k+1}}] = [\mathrm{X}_w]$, the intersection numbers $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w]$ and $\bigcap_{i=1}^{k+1} [\Omega_{w_i}]$ are the same. Finally, the multiplicity of V_μ in $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ is the same as the multiplicity of the trivial module in $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k} \otimes \mathrm{V}_{\mu_{k+1}}$ because $\mathrm{V}_{\mu_{k+1}} = \mathrm{V}_{-w_0\mu} = \mathrm{V}^*_\mu$.

To go from the symmetric form to the nonsymmetric form we simply reverse the above procedure, although of course we are free to desymmetrize with respect to any of the indices $i = 1, \dots, k+1$, and not just the last one.

For convenience we list in Table 1 the symmetric and nonsymmetric forms of some formulas and expressions we are interested in. Since k is an arbitrary positive integer, after switching to the symmetric form we often use k in place of $k+1$ to reduce notation.

2.8. ***Demazure reflections.*** Suppose that W and M are varieties and $\pi : \mathrm{W} \to \mathrm{M}$ is a $\mathbb{P}^1$-fibration, i.e., a smooth morphism with fibers isomorphic to $\mathbb{P}^1$. Let L be a line bundle on W and b be the degree of L on the fibers of π. Demazure [1976,

Nonsymmetric	Symmetric
$\bigotimes_{i=1}^{k} \mathrm{H}^{\ell(w_i)}(\mathrm{X}, \mathrm{L}_{\lambda_i}) \to \mathrm{H}^{\ell(w)}(\mathrm{X}, \mathrm{L}_{\lambda})$	$\bigotimes_{i=1}^{k+1} \mathrm{H}^{\ell(w_i)}(\mathrm{X}, \mathrm{L}_{\lambda_i}) \to \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}})$
$\sum_{i=1}^{k} \ell(w_i) = \ell(w)$	$\sum_{i=1}^{k+1} \ell(w_i) = \mathrm{N}$
$\sum_{i=1}^{k} \lambda_i = \lambda$	$\sum_{i=1}^{k+1} \lambda_i = -2\rho$
$\sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu$	$\sum_{i=1}^{k+1} w_i^{-1}\mu_i$
$\sum_{i=1}^{k} w_i^{-1} \cdot 0 - w^{-1} \cdot 0$	$\sum_{i=1}^{k+1} w_i^{-1} \cdot 0 + 2\rho$
$\Phi_w = \bigsqcup_{i=1}^{k} \Phi_{w_i}$	$\Delta^+ = \bigsqcup_{i=1}^{k+1} \Phi_{w_i}$
$\bigcap_{i=1}^{k} [\Omega_{w_i}] \cdot [\mathrm{X}_w]$	$\bigcap_{i=1}^{k+1} [\Omega_{w_i}]$

Table 1. Nonsymmetric and symmetric forms of some formulas.

Theorem 1] proves the following isomorphism of vector bundles on M:

$$\mathrm{R}^i \pi_* \mathrm{L} \cong \mathrm{R}^{1-i} \pi_* (\mathrm{L} \otimes \omega_\pi^{b+1}) \quad \text{for } i = 0, 1, \tag{2.8.1}$$

where ω_π is the relative cotangent bundle of π. The line bundle $\mathrm{L} \otimes \omega_\pi^{b+1}$ is called the *Demazure reflection of* L *with respect to* π.

Note that there is at most one value of i for which the resulting vector bundles are nonzero: $i = 0$ if $b \geqslant 0$, $i = 1$ if $b \leqslant -2$, and neither if $b = -1$. Equation (2.8.1) and the corresponding Leray spectral sequence give the isomorphisms

$$\mathrm{H}^j(\mathrm{W}, \mathrm{L}) \cong \begin{cases} \mathrm{H}^{j+1}(\mathrm{W}, \mathrm{L} \otimes \omega_\pi^{b+1}) & \text{if } b \geqslant 0, \\ \mathrm{H}^{j-1}(\mathrm{W}, \mathrm{L} \otimes \omega_\pi^{b+1}) & \text{if } b \leqslant -2, \end{cases} \quad \text{for all} \quad j.$$

Link between Demazure reflections and the affine action. Let α_i be any simple root, P_{α_i} the parabolic associated to α_i, and $\pi_i : \mathrm{X} \to \mathrm{M}_i := \mathrm{G}/\mathrm{P}_{\alpha_i}$ the corresponding $\mathbb{P}^1$-fibration. The relative cotangent bundle ω_{π_i} of π_i is the line bundle $\mathrm{L}_{-\alpha_i}$. Given any $\lambda \in \Lambda$, the degree of the line bundle L_λ on the fibers of π_i is $\lambda(\alpha_i^\vee)$, where $\alpha_i^\vee$ is the coroot corresponding to α_i. We thus obtain that the Demazure reflection of L_λ with respect to the fibration π_i is the line bundle

$$\mathrm{L}_\lambda \otimes \mathrm{L}_{-\alpha_i}^{\lambda(\alpha_i^\vee)+1} = \mathrm{L}_{\lambda - (\lambda(\alpha_i^\vee)+1)\alpha_i} = \mathrm{L}_{s_i\lambda - \alpha_i} = \mathrm{L}_{s_i \cdot \lambda},$$

where s_i is the simple reflection corresponding to α_i. The combinatorics of performing Demazure reflections with respect to the various $\mathbb{P}^1$-fibrations of X is therefore kept track of by the affine action of the Weyl group on Λ. In particular, if $v = s_{i_1} \cdots s_{i_m} \in \mathcal{W}$ and $\lambda \in \Lambda$, the result of applying the Demazure reflections with respect to the fibrations $\pi_{i_m}, \pi_{i_{m-1}}, \ldots, \pi_{i_1}$ in that order to L_λ is $\mathrm{L}_{v \cdot \lambda}$.

Demazure reflections and base change. Given any morphism $h : \mathrm{Y}_2 \to \mathrm{M}$ we can form the fiber product diagram

$$\begin{tikzcd} \mathrm{Y}_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & \mathrm{W} \arrow[d, "\pi"] \\ \mathrm{Y}_2 \arrow[r, "h"] & \mathrm{M} \end{tikzcd}$$

If π is a $\mathbb{P}^1$-fibration then so is π_1, and $\omega_{\pi_1} = f^*\omega_\pi$. Therefore, for any line bundle L on V, we have

$$f^*(\mathrm{L} \otimes \omega_\pi^{b+1}) = (f^*\mathrm{L}) \otimes \omega_{\pi_1}^{b+1},$$

where b is the degree of L on the fibers of π. The degree of $f^*\mathrm{L}$ on π_1 is also b and therefore the formula above shows that the pullback of the Demazure reflection of L with respect to π is the Demazure reflection of the pullback of L with respect to π_1. Furthermore, by the theorem on cohomology and base change, the natural morphisms

$$\mathrm{R}^i\pi_{1*}(f^*\mathrm{L}) \xleftarrow{\simeq} h^*(\mathrm{R}^i\pi_*\mathrm{L}),$$
$$\mathrm{R}^{1-i}\pi_{1*}((f^*\mathrm{L}) \otimes \omega_{\pi_1}^{b+1}) \xleftarrow{\simeq} h^*(\mathrm{R}^{1-i}\pi_*(\mathrm{L} \otimes \omega_\pi^{b+1}))$$

are isomorphisms for $i = 0, 1$.

2.9. E$_2$-*terms and computation of maps on cohomology.* Suppose that we have a commutative diagram of varieties

$$\begin{tikzcd} \mathrm{W}' \arrow[r, hook, "\gamma"] \arrow[d, "\pi'"] & \mathrm{W} \arrow[d, "\pi"] \\ \mathrm{M}' \arrow[r, hook] & \mathrm{M} \end{tikzcd}$$

where the vertical maps are proper and the horizontal maps are closed immersions. Suppose further that we have coherent sheaves $\mathcal{F}$ on W and $\mathcal{F}'$ on W$'$, and a map $\varphi : \gamma^*\mathcal{F} \to \mathcal{F}'$ of sheaves on W$'$. The map φ induces maps $\varphi_d : \mathrm{H}^d(\mathrm{W}, \mathcal{F}) \to \mathrm{H}^d(\mathrm{W}', \mathcal{F}')$ on cohomology and maps $\varphi_{d,k} : \mathrm{H}^{d-k}(\mathrm{M}, \mathrm{R}^k_{\pi*}\mathcal{F}) \to \mathrm{H}^{d-k}(\mathrm{M}', \mathrm{R}^k_{\pi*}\mathcal{F}')$ on the E$_2$-terms of the Leray spectral sequences for $\mathcal{F}$ and $\mathcal{F}'$ with respect to π and π'. Assume that both spectral sequences degenerate at the E$_2$-term. In Section 5.4 we will need to know when we can compute φ_d by knowing the maps $\varphi_{d,k}$.

By the definition of convergence of a spectral sequence there are increasing filtrations

$$0 = \mathrm{U}_{-1} \subseteq \mathrm{U}_0 \subseteq \cdots \subseteq \mathrm{U}_d = \mathrm{H}^d(\mathrm{W}, \mathcal{F}),$$
$$0 = \mathrm{U}'_{-1} \subseteq \mathrm{U}'_0 \subseteq \cdots \subseteq \mathrm{U}'_d = \mathrm{H}^d(\mathrm{W}', \mathcal{F}'),$$

such that $\mathrm{U}_k/\mathrm{U}_{k-1} = \mathrm{H}^{d-k}(\mathrm{M}, \mathrm{R}^k_{\pi*}\mathcal{F})$ and $\mathrm{U}'_k/\mathrm{U}'_{k-1} = \mathrm{H}^{d-k}(\mathrm{M}', \mathrm{R}^k_{\pi*}\mathcal{F}')$ for $k = 0, \dots, d$. Since the map φ_d on the cohomology groups is compatible with the filtrations (in the sense that $\varphi_d(\mathrm{U}_k) \subseteq \mathrm{U}'_k$ for $k = -1, \dots, d$), φ_d induces maps between the associated graded pieces of the filtrations; these maps are exactly the maps $\varphi_{d,k}$.

We will need to know that φ_d can be computed from the maps $\varphi_{d,k}$ in an elementary case. Suppose there is a unique k such that $\mathrm{U}_k/\mathrm{U}_{k-1}$ is nonzero (and so $\mathrm{U}_k/\mathrm{U}_{k-1} = \mathrm{H}^d(\mathrm{W}, \mathcal{F})$), and a unique k' such that $\mathrm{U}'_{k'}/\mathrm{U}'_{k'-1}$ is nonzero (and so $\mathrm{U}'_{k'}/\mathrm{U}'_{k-1} = \mathrm{H}^d(\mathrm{W}', \mathcal{F}')$). Then we can compute φ_d from the maps $\varphi_{d,k}$ if and only if $k = k'$; if this occurs then $\varphi_d = \varphi_{d,k}$.

In order to show that we must check the condition $k = k'$ above, i.e., that the map on E_2-terms does not always determine the map φ_d, we give the following example of a nonzero map between cohomology groups of sheaves where the induced map on E_2-terms is zero. This example is also a cup-product map.

Example 2.9.1. Let $\mathrm{W} = \mathbb{P}^m \times \mathbb{P}^m$ for some $m \geqslant 1$, $\mathcal{F} = \mathcal{O}_{\mathbb{P}^m}(1) \boxtimes \mathcal{O}_{\mathbb{P}^m}(-r)$ with $r \geqslant m+2$, and let $\mathcal{G} = \mathcal{O}_\Delta(1-r)$ be the restriction of $\mathcal{F}$ to the diagonal of W. We have $\mathrm{H}^m(\mathrm{W}, \mathcal{F}) = \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)) \otimes \mathrm{H}^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(-r))$ and $\mathrm{H}^m(\mathrm{W}, \mathcal{G}) = \mathrm{H}^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1-r))$. The natural restriction map $\varphi : \mathcal{F} \to \mathcal{G}$ induces the cup-product map

$$\varphi_m : \mathrm{H}^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1)) \otimes \mathrm{H}^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(-r)) \xrightarrow{\cup} \mathrm{H}^m(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1-r)),$$

which is a surjective map of nonzero groups.

If $\pi : \mathrm{W} \to \mathrm{M} = \mathbb{P}^m$ is the projection onto the first factor then both of the Leray spectral sequences degenerate at the E_2 term with only one nonzero entry in each sequence. We have

$$\mathrm{H}^m(\mathrm{W}, \mathcal{F}) = \mathrm{H}^0(\mathrm{M}, \mathrm{R}^m_{\pi*}\mathcal{F}) \quad \text{(i.e., } k = m\text{)},$$
$$\mathrm{H}^m(\mathrm{W}, \mathcal{G}) = \mathrm{H}^m(\mathrm{M}, \pi_*\mathcal{G}) \quad \text{(i.e., } k' = 0\text{)}.$$

The maps $\varphi_{m,k}$ on the E_2-terms are clearly zero, even though φ_m is nonzero.

2.10. ***Bott–Samelson–Demazure–Hansen varieties.*** Let $\underline{v} = s_{i_1} \cdots s_{i_m}$ be a word, not necessarily reduced, of simple reflections. Associated to $\underline{v}$ is a variety $\mathrm{Z}_{\underline{v}}$, a left action of B on $\mathrm{Z}_{\underline{v}}$, and a B-equivariant map $f_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{X}$. If $\underline{v}$ is nonempty there is also a B-equivariant map $\pi_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{Z}_{\underline{v}_R}$ expressing $\mathrm{Z}_{\underline{v}}$ as a $\mathbb{P}^1$-bundle over $\mathrm{Z}_{\underline{v}_R}$ together with a B-equivariant $\sigma_{\underline{v}} : \mathrm{Z}_{\underline{v}_R} \to \mathrm{Z}_{\underline{v}}$ section such that $f_{\underline{v}_R} = f_{\underline{v}} \circ \sigma_{\underline{v}}$.

These varieties were originally constructed by Demazure [1974] and Hansen [1973] following an analogous construction by Bott and Samelson [1958] in the compact case. In this subsection we recall their construction and several related facts. We give two different descriptions of the construction; both will be used in the constructions in Section 3.

Recursive construction. Recall that e is unique point of X fixed by B. If the word $\underline{v}$ is empty we define $\mathrm{Z}_{\underline{v}}$ to be e, the map $f_{\underline{v}}$ to be the inclusion $e \hookrightarrow \mathrm{X}$, and the B-action on $\mathrm{Z}_{\underline{v}}$ to be trivial.

If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is nonempty, let $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$ be the word obtained by dropping the rightmost reflection of $\underline{v}$. By induction we have already constructed $\mathrm{Z}_{\underline{u}}$ and the map $f_{\underline{u}} : \mathrm{Z}_{\underline{u}} \to \mathrm{X}$. Set $h = \pi_{i_m} \circ f_{\underline{u}}$, where π_{i_m} is the G-equivariant projection (and $\mathbb{P}^1$-fibration) $\mathrm{X} \to \mathrm{M}_{i_m} = \mathrm{G}/\mathrm{P}_{\alpha_{i_m}}$. We then define $\mathrm{Z}_{\underline{v}}$ to be the fiber product $\mathrm{Z}_{\underline{u}} \times_{\mathrm{M}_{i_m}} \mathrm{X}$, and $f_{\underline{v}}$ and $\pi_{\underline{v}}$ to be the maps from the fiber product to X and to $\mathrm{Z}_{\underline{u}}$ respectively. Since $h = \pi_{i_m} \circ f_{\underline{u}}$, by the universal property of the fiber product there exists a unique map $\sigma_{\underline{v}} : \mathrm{Z}_{\underline{u}} \to \mathrm{Z}_{\underline{v}}$ such that $f_{\underline{u}} = f_{\underline{v}} \circ \sigma_{\underline{v}}$ and $\mathrm{id}_{\mathrm{Z}_{\underline{u}}} = \pi_{\underline{v}} \circ \sigma_{\underline{v}}$. These maps are summarized in the following diagram, where the square is a fiber product:

$$
\begin{tikzcd}
\mathrm{Z}_{\underline{v}} \arrow[r, "f_{\underline{v}}"] \arrow[d, "\pi_{\underline{v}}"] & \mathrm{X} \arrow[d, "\pi_{i_m}"] \\
\mathrm{Z}_{\underline{u}} \arrow[r, "h"] \arrow[ur, "f_{\underline{u}}"] \arrow[u, bend left, "\sigma_{\underline{v}}"] \arrow[ur, phantom, "\square"', pos=0.5] & \mathrm{M}_{i_m}
\end{tikzcd}
\tag{2.10.1}
$$

Since B acts on $\mathrm{Z}_{\underline{u}}$ and on X, and the maps $f_{\underline{u}}$, π_{i_m}, and h are B-equivariant, by the universal property of the fiber product, the diagram (2.10.1) induces a B-action on $\mathrm{Z}_{\underline{v}}$ such that $f_{\underline{v}}$ and $\sigma_{\underline{v}}$ are B-equivariant maps. Since each morphism $\sigma_{\underline{v}}$ is a $\mathbb{P}^1$-fibration it follows immediately that each $\mathrm{Z}_{\underline{v}}$ is a smooth proper variety of dimension $\ell(\underline{v})$.

Direct construction. For any word $\underline{v}$ set

$$
\mathrm{P}_{\underline{v}} := \begin{cases} e & \text{if } \underline{v} \text{ is empty,} \\ \mathrm{P}_{\alpha_{i_1}} \times \cdots \times \mathrm{P}_{\alpha_{i_m}} & \text{if } \underline{v} = s_{i_1} \cdots s_{i_m} \text{ is nonempty.} \end{cases}
$$

If $\underline{v}$ is empty we define $\mathrm{Z}_{\underline{v}}$, $f_{\underline{v}}$, and the B-action as in the direct construction.

If $\underline{v} = s_{i_1} \cdots s_{i_m}$ is nonempty then $\mathrm{Z}_{\underline{v}}$ is the quotient of $\mathrm{P}_{\underline{v}}$ by B^m, where an element $(b_1, \ldots, b_m)$ of B^m acts on the right on $(p_1, \ldots, p_m)$ by

$$
(p_1, \ldots, p_m) \cdot (b_1, \ldots, b_m) = \big(p_1 b_1,\ b_1^{-1} p_2 b_2,\ b_2^{-1} p_3 b_3, \ldots, b_{m-1}^{-1} p_m b_m\big).
$$

The left action of B on $\mathrm{P}_{\underline{v}}$ given by

$$
b \cdot (p_1, p_2, \ldots, p_m) = (b p_1, p_2, \ldots, p_m)
$$

commutes with the right action of B^m and therefore descends to a left action of B on $\mathrm{Z}_{\underline{v}}$. We denote the corresponding B-equivariant quotient map by $\psi_{\underline{v}} : \mathrm{P}_{\underline{v}} \to \mathrm{Z}_{\underline{v}}$.

The product map $\mathrm{P}_{\underline{v}} \xrightarrow{\phi_{\underline{v}}} \mathrm{G}$ given by $(p_1, \ldots, p_m) \mapsto p_1 \cdots p_m$ is equivariant for the left B-action described above and left multiplication of G by B. Under the homomorphism of groups $\mathrm{B}^m \to \mathrm{B}$ given by the projection $(b_1, \ldots, b_m) \mapsto b_m$ the product map $\phi_{\underline{v}}$ is also equivariant for the right action of B^m on $\mathrm{P}_{\underline{v}}$ and the right multiplication of G by B. The product map therefore descends to a left B-equivariant morphism $f_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{X}$.

Let $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$ be the word obtained by dropping the rightmost reflection in $\underline{v}$. The projection map $\mathrm{pr}_{\underline{v}} : \mathrm{P}_{\underline{v}} \to \mathrm{P}_{\underline{u}}$ sending $(p_1, \ldots, p_m)$ to $(p_1, \ldots, p_{m-1})$ is equivariant with respect to the projection $\mathrm{B}^m \to \mathrm{B}^{m-1}$ sending $(b_1, \ldots, b_m)$ to $(b_1, \ldots, b_{m-1})$. Similarly the inclusion map $j_{\underline{v}} : \mathrm{P}_{\underline{u}} \hookrightarrow \mathrm{P}_{\underline{v}}$ sending $(p_1, \ldots, p_{m-1})$ to $(p_1, \ldots, p_{m-1}, 1_{\mathrm{G}})$ is equivariant with respect to the inclusion $\mathrm{B}^{m-1} \hookrightarrow \mathrm{B}^m$ sending $(b_1, \ldots, b_{m-1})$ to $(b_1, \ldots, b_{m-1}, b_{m-1})$. The maps $\mathrm{pr}_{\underline{v}}$ and $j_{\underline{v}}$ respect the left B-action on $\mathrm{P}_{\underline{v}}$ and $\mathrm{P}_{\underline{u}}$, and therefore descend to B-equivariant maps $\pi_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{Z}_{\underline{u}}$ and $\sigma_{\underline{v}} : \mathrm{Z}_{\underline{u}} \to \mathrm{Z}_{\underline{v}}$. Since $\mathrm{pr}_{\underline{v}} \circ j_{\underline{v}} = \mathrm{id}_{\mathrm{P}_{\underline{u}}}$ and $\phi_{\underline{v}} \circ j_{\underline{v}} = \phi_{\underline{u}}$, taking quotients we obtain $\pi_{\underline{v}} \circ \sigma_{\underline{v}} = \mathrm{id}_{\mathrm{Z}_{\underline{u}}}$ and $f_{\underline{v}} \circ \sigma_{\underline{v}} = f_{\underline{u}}$. Finally, the fibers of $\pi_{\underline{v}}$ are isomorphic to $\mathrm{P}_{\alpha_{i_m}}/\mathrm{B} \cong \mathbb{P}^1$.

We record the following well-known facts about the construction above.

Proposition 2.10.2. (a) *The varieties* $\mathrm{Z}_{\underline{v}}$ *produced by the recursive and direct constructions above are isomorphic over* X.

(b) *If* $\underline{v} = s_{i_1} \cdots s_{i_m}$ *is a reduced word with product* v *then the image of* $f_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{X}$ *is* X_v *and* $f_{\underline{v}}$ *is a resolution of singularities of* X_v.

Proof. Part (b) is proved in [Demazure 1974] and [Hansen 1973]. To show (a) it is enough to show that the varieties produced by the direct construction satisfy the fiber product diagram (2.10.1). This is most easily checked after pulling back (2.10.1) via the maps $\mathrm{G} \to \mathrm{X}$ and $\mathrm{P}_{\underline{u}} = \mathrm{P}_{i_1} \times \cdots \times \mathrm{P}_{i_{m-1}} \to \mathrm{Z}_{\underline{u}}$; the details are omitted here. □

Maximum points. Let $\underline{v} = s_{i_1} \cdots s_{i_m}$ be a reduced word with product v. The image of $\mathrm{Z}_{\underline{v}}$ under $f_{\underline{v}}$ is X_v, by Proposition 2.10.2(b), and one can check that there is a unique point $p_{\underline{v}}$ of $\mathrm{Z}_{\underline{v}}$ which maps to $v \in \mathrm{X}_v$. More specifically, from the point of view of the direct construction, the point $(s_{i_1}, \ldots, s_{i_m})$ is a point of $\mathrm{P}_{\underline{v}}$ and its image under the quotient map $\mathrm{P}_{\underline{v}} \to \mathrm{Z}_{\underline{v}}$ is $p_{\underline{v}}$. From the point of view of the recursive construction one starts with $p_{\varnothing} = e$, and recursively defines $p_{\underline{v}}$ to be unique torus fixed point in the $\mathbb{P}^1$-fiber of $\pi_{\underline{v}} : \mathrm{Z}_{\underline{v}} \to \mathrm{Z}_{\underline{u}}$ over $p_{\underline{u}}$ which is not equal to $\sigma_{\underline{v}}(p_{\underline{u}})$, where $\underline{u} = \underline{v}_R = s_{i_1} \cdots s_{i_{m-1}}$. Note that $p_{\underline{v}}$ is the unique torus fixed point of $\mathrm{Z}_{\underline{v}}$ whose image in X_v is the largest in the Bruhat order among torus-fixed points of X_v. We call $p_{\underline{v}}$ the *maximum point* of $\mathrm{Z}_{\underline{v}}$.

Since $p_{\underline{v}}$ is a torus fixed point, the torus acts on the tangent space $\mathrm{T}_{p_{\underline{v}}}\mathrm{Z}_{\underline{v}}$ and it will be important for us to know the formal character of $\mathrm{T}_{p_{\underline{v}}}\mathrm{Z}_{\underline{v}}$. It follows inductively from the recursive construction that

$$\mathrm{Ch}(\mathrm{T}_{p_{\underline{v}}}\mathrm{Z}_{\underline{v}}) = \langle \Phi_{v^{-1}} \rangle. \tag{2.10.3}$$

2.11. ***Semistability of torus fixed points.*** The following lemma is due to Kostant.

Lemma 2.11.1. *Let* W *be a projective variety with a* G*-action and* L *a* G*-equivariant ample line bundle on* W*. A torus fixed point $q \in \mathrm{W}$ is semistable with respect to* L *if and only if the weight of L_q is zero. In this case the orbit of q is closed in the semistable locus.*

Proof. If the action of the torus on the fiber L_q is nontrivial then it is easy to see (for instance using the Hilbert–Mumford criterion for semistability, [Mumford et al. 1994, Theorem 2.1, p. 49]) that q is not a stable point.

Conversely, suppose that the weight of L_q is zero. Replacing L by a multiple we may assume that L is very ample and gives an embedding $\mathrm{W} \hookrightarrow \mathbb{P}^r$ for some r. Let $\mathbb{A}^{r+1}$ be the affine space corresponding to $\mathbb{P}^r$ and $\mathbb{A}^{r+1} \setminus \{0\} \to \mathbb{P}^r$ be the quotient map. Then G acts linearly on $\mathbb{A}^{r+1}$ inducing an action on $\mathbb{P}^r$ compatible with the action on W. Let $\tilde{q}$ be any lift to $\mathbb{A}^{r+1}$ of the image of q in $\mathbb{P}^r$. The condition that the torus act trivially on L_q is equivalent to the condition that $\tilde{q}$ be fixed by T under the G-action on $\mathbb{A}^{r+1}$. Kostant ([1963, p. 354, Remark 11]) proves that for any finite dimensional module of a reductive group G and any point $\tilde{q}$ fixed by T, the G-orbit of $\tilde{q}$ is closed; this result was also later generalized by Luna [1975, Theorem (∗∗)]. Since G is reductive and the orbit of $\tilde{q}$ does not meet zero, there is a G-invariant homogeneous form of some degree m which is nonzero on $\tilde{q}$. This corresponds to a G-invariant section $s \in \mathrm{H}^0(\mathrm{W}, \mathrm{L}^m)^{\mathrm{G}}$ such that $s(q) \neq 0$. We thus see that if the weight of L_q is zero then q is a semistable point, and the orbit of q is closed in the semistable locus. □

3. Diagonal Bott–Samelson–Demazure–Hansen–Kumar varieties

In this section we give a generalization of the varieties from Section 2.10. The construction is a variation of a construction of Kumar [1988]; see Section 3.10 for a comparison. These varieties are obtained by applying the idea of the Bott–Samelson resolution to the diagonal inclusion $\mathrm{X} \hookrightarrow \mathrm{X}^k$. They can also be thought of as a desingularization of the total space of the variety of intersections of translates of Schubert cycles. This alternate description is established in Theorem 3.7.4.

More specifically, for each sequence $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ of words we construct a smooth variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ of dimension $\mathrm{N} + \ell(\underline{\boldsymbol{v}})$ with a G-action together with a proper map $f_{\underline{\boldsymbol{v}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}^k$ which is G-equivariant for the diagonal action of G on X^k. If $\underline{\boldsymbol{u}}$ is the sequence obtained by dropping a single simple reflection from the right of one of the $\underline{v}_j$'s then $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ is a $\mathbb{P}^1$-fibration over $\mathrm{Y}_{\underline{\boldsymbol{u}}}$, and there is a section $\mathrm{Y}_{\underline{\boldsymbol{u}}} \hookrightarrow \mathrm{Y}_{\underline{\boldsymbol{v}}}$ compatible with the maps $f_{\underline{\boldsymbol{v}}}$ and $f_{\underline{\boldsymbol{u}}}$ to X^k. The fibration and section maps are

G-equivariant; moreover they are compatible with the $\mathbb{P}^1$-fibrations on factors of X^k. These relationships are summarized in diagram (3.1.2).

3.1. ***Recursive construction.*** Let $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ be a sequence of words. If all $\underline{v}_j$ are empty, i.e., if $\underline{\boldsymbol{v}} = (\varnothing, \ldots, \varnothing)$, we set $\mathrm{Y}_{\underline{\boldsymbol{v}}} = \mathrm{X}$ and let $f_{\underline{\boldsymbol{v}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}^k$ be the diagonal embedding.

Otherwise suppose that $\underline{v}_j$ is nonempty. Let

$$\underline{u}_l := \begin{cases} \underline{v}_l & \text{if } l \neq j, \\ (\underline{v}_j)_R & \text{if } l = j, \end{cases} \qquad l = 1, \ldots, k, \tag{3.1.1}$$

and set $\underline{\boldsymbol{u}} = (\underline{u}_1, \ldots, \underline{u}_k)$. By induction on $\ell(\underline{\boldsymbol{v}})$ we may assume that $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ and the map $f_{\underline{\boldsymbol{u}}} : \mathrm{Y}_{\underline{\boldsymbol{u}}} \to \mathrm{X}^k$ have been constructed. If $\underline{v}_j = s_{i_1} \cdots s_{i_m}$, so that $\underline{u}_j = s_{i_1} \cdots s_{i_{m-1}}$ then we define $\mathrm{Y}_{\underline{\boldsymbol{v}}}$, the map $f_{\underline{\boldsymbol{v}}}$, the projection $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$, and the section $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$ by the following fiber product square:

$$\begin{tikzcd}
\mathrm{Y}_{\underline{\boldsymbol{v}}} \arrow[rr, "f_{\underline{\boldsymbol{v}}}"] \arrow[d, "\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}"] & & \mathrm{X}^k \arrow[d, "(\mathrm{id}_{\mathrm{X}})^{j-1} \times \pi_{i_m} \times (\mathrm{id}_{\mathrm{X}})^{k-j}"] \\
\mathrm{Y}_{\underline{\boldsymbol{u}}} \arrow[rr] \arrow[u, bend left, "\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}"] \arrow[urr, "f_{\underline{\boldsymbol{u}}}"] & \Box & \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}
\end{tikzcd} \tag{3.1.2}$$

Here $\pi_{i_m} : \mathrm{X} \to \mathrm{M}_{i_m} := \mathrm{G}/\mathrm{P}_{\alpha_{i_m}}$ is the natural projection, and $\mathrm{X}^k \to \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}$ is the projection π_{i_m} on the j-th factor and the identity on all others. The bottom map

$$\mathrm{Y}_{\underline{\boldsymbol{u}}} \to \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}$$

is the map $f_{\underline{\boldsymbol{u}}}$ to X^k followed by the map $\mathrm{X}^k \to \mathrm{X}^{j-1} \times \mathrm{M}_{i_m} \times \mathrm{X}^{k-j}$ above.

Since $\mathrm{X}^k \to \mathrm{X}^{j-1} \times \pi_{i_m} \times \mathrm{X}^{k-j}$ is a $\mathbb{P}^1$-fibration the same is true of $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$. We conclude by induction that the variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ is smooth, proper, and irreducible of dimension $\mathrm{N} + \ell(\underline{\boldsymbol{v}})$. The maps $f_{\underline{\boldsymbol{u}}}$ and $\mathrm{id}_{\mathrm{Y}_{\underline{\boldsymbol{u}}}}$ from $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ to X^k and $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ respectively give rise to the section $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$. By construction we have

$$f_{\underline{\boldsymbol{u}}} = f_{\underline{\boldsymbol{v}}} \circ \sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} \quad \text{and} \quad \mathrm{id}_{\mathrm{Y}_{\underline{\boldsymbol{u}}}} = \pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} \circ \sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}.$$

This construction is well defined. Indeed, assume that we had dropped a simple reflection from the right of $\underline{v}_{j'}$, $j' \neq j$ to obtain a sequence of words $\underline{\boldsymbol{u}}'$ and used $\mathrm{Y}_{\underline{\boldsymbol{u}}'}$ instead of $\mathrm{Y}_{\underline{\boldsymbol{u}}}$ to construct $\mathrm{Y}_{\underline{\boldsymbol{v}}}$. We claim that the resulting variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ is the same. This follows easily by induction on $\ell(\underline{\boldsymbol{v}})$ and the fact that the diagram expressing the commutativity of the projections on the different factors is a fiber square:

$$\begin{tikzcd}
X^k \arrow[r, "(\mathrm{id}_X)^{j'-1}\times\pi_{i_m}\times(\mathrm{id}_X)^{k-j'}"] \arrow[d, "(\mathrm{id}_X)^{j-1}\times\pi_{i_m}\times(\mathrm{id}_X)^{k-j}"'] \arrow[dr, phantom, "\square"] & X^{j'-1}\times M_{i'_m}\times X^{k-j'} \arrow[d] \\
X^{j-1}\times M_{i_m}\times X^{k-j} \arrow[r] & X^{j'-1}\times M_{i_{m'}}\times X^{j-j'-1}\times M_{i_m}\times X^{k-j}
\end{tikzcd}$$

Here, by symmetry, we have assumed that $j' < j$.

3.2. ***Direct construction.*** Let $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ be a sequence of words. The group B acts diagonally on $Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ on the left. We define $Y_{\underline{\boldsymbol{v}}}$ to be the quotient of $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ by the left B-action

$$b \cdot (g, z_1, \ldots, z_k) = (gb^{-1}, b \cdot z_1, \ldots, b \cdot z_k). \tag{3.2.1}$$

Since $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ is smooth and B acts without fixed points, the quotient $Y_{\underline{\boldsymbol{v}}}$ is smooth.

The group G acts on $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ by left multiplication on the first factor. Since this action commutes with the action of B, it descends to an action of G on $Y_{\underline{\boldsymbol{v}}}$. The map from $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ to X^k given by

$$(g, z_1, \ldots, z_k) \mapsto \big(g\cdot f_{\underline{v}_1}(z_1),\ g\cdot f_{\underline{v}_2}(z_2), \ldots, g\cdot f_{\underline{v}_k}(z_k)\big) \tag{3.2.2}$$

is invariant under the B-action. If we let G act on X^k diagonally then (3.2.2) is also G-equivariant and hence descends to a G-equivariant morphism $f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \to X^k$.

As in the direct construction, we suppose that $\underline{v}_j$ is nonempty, define $\underline{u}_l$ by (3.1.1) and set $\underline{\boldsymbol{u}} = (\underline{u}_1, \ldots, \underline{u}_k)$. The B-equivariant morphisms $\pi_{\underline{v}} : Z_{\underline{v}_j} \to Z_{\underline{u}_j}$ and $\sigma_{\underline{v}_j} : Z_{\underline{u}_j} \to Z_{\underline{v}_j}$ from Section 2.10 give rise to B-equivariant morphisms between $G \times (Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k})$ and $G \times (Z_{\underline{u}_1} \times \cdots \times Z_{\underline{u}_k})$ and hence to a G-equivariant $\mathbb{P}^1$-fibration $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : Y_{\underline{\boldsymbol{v}}} \to Y_{\underline{\boldsymbol{u}}}$ and a G-equivariant section $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : Y_{\underline{\boldsymbol{u}}} \to Y_{\underline{\boldsymbol{v}}}$. These maps fit together to give diagram (3.1.2).

3.3. ***Expanded version of the direct construction.*** Combining the formulas for $P_{\underline{v}}$ from Section 2.10 with the direct construction above we obtain a more explicit expression for $Y_{\underline{\boldsymbol{v}}}$. If $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ with $\underline{v}_j = s_{i_{1,j}} \cdots s_{i_{m_j,j}}$ for $j = 1, \ldots, k$ then we define $Y_{\underline{\boldsymbol{v}}}$ to be the quotient of

$$G \times P_{\underline{v}_1} \times \cdots \times P_{\underline{v}_k} = G \times (P_{i_{1,1}} \times \cdots \times P_{i_{m_1,1}}) \times \cdots \times (P_{i_{1,k}} \times \cdots \times P_{i_{m_k,k}})$$

by the right action of $B \times B^{m_1} \times \cdots \times B^{m_k}$, where an element

$$(b_0 \mid b_{1,1}, \ldots, b_{m_1,1} \mid b_{1,2}, \ldots, b_{m_2,2} \mid \cdots \mid b_{1,k}, \ldots, b_{m_k,k})$$

acts from the right on

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \ldots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \ldots, p_{i_{m_2,2}} \mid \cdots \mid p_{i_{1,k}}, \ldots, p_{i_{m_k,k}})$$

to give

$$\big(gb_0 \mid b_0^{-1} p_{i_{1,1}} b_{1,1},\ b_{1,1}^{-1} p_{i_{2,1}} b_{2,1}, \ldots, b_{m_1-1,1}^{-1} p_{i_{m_1,1}} b_{m_1,1} \mid \cdots \\ \cdots \mid b_0^{-1} p_{i_{1,k}} b_{1,k}, \ldots, b_{m_k-1,k}^{-1} p_{i_{m_k,k}} b_{m_k,k}\big).$$

(In the expressions above the vertical lines "|" are used to indicate logical groupings, but otherwise have no significance.) The group G acts on $\mathrm{G} \times \mathrm{P}_{\underline{v}_1} \times \cdots \times \mathrm{P}_{\underline{v}_k}$ by left multiplication on the G factor, this action descends to a left action on $\mathrm{Y}_{\underline{\boldsymbol{v}}}$.

The map $f_{\underline{\boldsymbol{v}}}$ is induced by the map sending an element

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \ldots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \ldots, p_{i_{m_2,2}} \mid \cdots \mid p_{i_{1,k}}, \ldots, p_{i_{m_k,k}})$$

of $\mathrm{G} \times \mathrm{P}_{\underline{v}_1} \times \cdots \times \mathrm{P}_{\underline{v}_k}$ to

$$(gp_{i_{1,1}} p_{i_{2,1}} \cdots p_{i_{m_1,1}} \mid gp_{i_{1,2}} p_{i_{2,2}} \cdots p_{i_{m_2,2}} \mid \cdots \mid gp_{i_{1,k}} \cdots p_{i_{m_k,k}}) \tag{3.3.1}$$

in X^k. From the explicit formulas this is clearly a G-equivariant map.

Finally, if $\underline{\boldsymbol{v}}$ is a sequence of words, and $\underline{\boldsymbol{u}}$ is a sequence obtained by dropping the rightmost reflection of a single word in $\underline{\boldsymbol{v}}$ (as in Section 3.2) then the G-equivariant $\mathbb{P}^1$-fibration $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{Y}_{\underline{\boldsymbol{u}}}$ and the G-equivariant section $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : \mathrm{Y}_{\underline{\boldsymbol{u}}} \to \mathrm{Y}_{\underline{\boldsymbol{v}}}$ are constructed using the obvious formulas analogous to those in Section 2.10. It again follows easily from these formulas that $f_{\underline{\boldsymbol{u}}} = f_{\underline{\boldsymbol{v}}} \circ \sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$.

Remark. Note that the variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ depends on the sequence of words $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ and not just on the corresponding sequence $(v_1, \ldots, v_k)$ of Weyl group elements. If we choose a different reduced factorization of each v_i the resulting variety is birational to $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ over X^k. The proof is omitted because we do not need this fact.

3.4. ***The map $f_\circ$.*** As before, let $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ be a sequence of words. Besides the map $f_{\underline{\boldsymbol{v}}}$ to X^k, each $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ comes with a G-equivariant map $f_\circ$ to X expressing $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ as a $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$-bundle over X.

From the point of view of the construction in Section 3.1 $f_\circ$ is the composite map

$$\mathrm{Y}_{\underline{\boldsymbol{v}}} \xrightarrow{\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}} \mathrm{Y}_{\underline{\boldsymbol{u}}} \to \cdots \to \mathrm{Y}_{\underline{\varnothing}} = \mathrm{X}$$

obtained by dropping the elements in the entries of $\underline{\boldsymbol{v}}$ one at a time. The fiber over e in X is then the result of applying the recursive construction in Section 2.10 separately for each $\underline{v}_i$, $i = 1, \ldots, k$, and so the fiber is $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$.

From the point of view of the construction in Section 3.2 one starts with the projection $\mathrm{G} \times (\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}) \to \mathrm{G}$ onto the first factor. This is B-equivariant for the right action of B on G and hence descends to a morphism $f_\circ : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}$ expressing $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ as a $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$-bundle over X.

Let $\underline{\boldsymbol{u}} = (\underline{\varnothing}, \underline{v}_1, \ldots, \underline{v}_k)$. Since the action of B on the point $\mathrm{Z}_{\underline{\varnothing}} = e$ is trivial, we have an isomorphism

$$\mathrm{G} \times \mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k} \simeq \mathrm{G} \times \mathrm{Z}_{\underline{\varnothing}} \times \mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$$

of B-varieties and hence a G-isomorphism $\phi : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{Y}_{\underline{\boldsymbol{u}}}$. From the explicit description in (3.2.2) we see that the composite map $f_{\underline{\boldsymbol{u}}} \circ \phi : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}^{k+1}$ followed by projection onto the first factor is $f_\circ$, and that $f_{\underline{\boldsymbol{u}}} \circ \phi$ followed by projection onto the last k factors is $f_{\underline{\boldsymbol{v}}}$.

Thus the map $f_\circ \times f_{\underline{\boldsymbol{v}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X} \times \mathrm{X}^k$ is equal to the map

$$f_{(\underline{\varnothing}, \underline{v}_1, \ldots, \underline{v}_k)} : \mathrm{Y}_{(\underline{\varnothing}, \underline{v}_1, \ldots, \underline{v}_k)} \to \mathrm{X}^{k+1}$$

under the isomorphism ϕ. This will be used in the proof of Theorem 3.7.4.

3.5. *Maximum point.* Let $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ be a sequence of words. We define the *maximum point* $p_{\underline{\boldsymbol{v}}}$ of $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ to be the product maximum point (Section 2.10) $p_{\underline{v}_1} \times \cdots \times p_{\underline{v}_k}$ in the fiber $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$ of $f_\circ$ over e in X. Alternatively, if $\underline{v}_j = (s_{i_{1,j}}, \ldots, s_{i_{m_j,j}})$ for $j = 1, \ldots, k$ then (in the notation of Section 3.3) the point

$$(e \mid s_{i_{1,1}}, s_{i_{2,1}}, \ldots, s_{i_{m_1,1}} \mid \cdots \mid s_{i_{1,k}}, \ldots, s_{i_{m_k,k}})$$

is a point of

$$\mathrm{G} \times (\mathrm{P}_{i_{1,1}} \times \cdots \times \mathrm{P}_{i_{m_1,1}}) \times \cdots \times (\mathrm{P}_{i_{1,k}} \times \cdots \times \mathrm{P}_{i_{m_k,k}})$$

and its image in $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ under the quotient map by $\mathrm{B} \times \mathrm{B}^{m_1} \times \cdots \times \mathrm{B}^{m_k}$ is the maximum point $p_{\underline{\boldsymbol{v}}}$. If each $\underline{v}_j$ is a factorization of some $v_j \in \mathcal{W}$, then the image $f_{\underline{\boldsymbol{v}}}(p_{\underline{\boldsymbol{v}}})$ of the maximum point in X^k is the point $q_{\underline{\boldsymbol{v}}} := (v_1, \ldots, v_k)$.

3.6. *Tangent space formulas.* We will need to know the formal character (see Section 2.1) of the tangent space of $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ at the maximum point $p_{\underline{\boldsymbol{v}}}$. If each $\underline{v}_j$ is a reduced word with product v_j, then the formal character of the tangent space to $\mathrm{Z}_{\underline{v}_j}$ at $p_{\underline{v}_j}$ is $\langle \Phi_{v_j^{-1}} \rangle$ and the formal character of the tangent space of X at e is $\langle \Delta^- \rangle$.

Since the fibration $f_\circ$ is smooth, the formal character of $\mathrm{T}_{p_{\underline{\boldsymbol{v}}}} \mathrm{Y}_{\underline{\boldsymbol{v}}}$ is the sum of these formal characters, i.e.,

$$\mathrm{Ch}(\mathrm{T}_{p_{\underline{\boldsymbol{v}}}} \mathrm{Y}_{\underline{\boldsymbol{v}}}) = \langle \Delta^- \rangle + \sum_{i=1}^{k} \langle \Phi_{v_i^{-1}} \rangle.$$

If $v_j = w_j^{-1} w_0$ for $j = 1, \ldots, k$, then by (2.2.2) this is the same as

$$\mathrm{Ch}(\mathrm{T}_{p_{\underline{\boldsymbol{v}}}} \mathrm{Y}_{\underline{\boldsymbol{v}}}) = \Delta^- + \sum_{i=1}^{k} \langle \Phi^{\mathrm{c}}_{w_i} \rangle. \tag{3.6.1}$$

3.7. *Fibers and images of $f_{\underline{\boldsymbol{v}}}$.*

Lemma 3.7.1. *Let $\underline{\boldsymbol{v}} = (\varnothing, \underline{v}_2, \ldots, \underline{v}_k)$ be a sequence of words, with each $\underline{v}_i$ a reduced factorization of v_i, and let $\mathrm{X}_{\underline{\boldsymbol{v}}}$ be the (reduced) image of $f_{\underline{\boldsymbol{v}}}$ in X^k. Then:*

(a) *Projection onto the first factor of X^k endows $\mathrm{X}_{\underline{\boldsymbol{v}}}$ with the structure of a fiber bundle over X with fiber isomorphic to $\mathrm{X}_{v_2} \times \cdots \times \mathrm{X}_{v_k}$.*

(b) *The variety $\mathrm{X}_{\underline{\boldsymbol{v}}}$ is normal with rational singularities of dimension $\mathrm{N} + \ell(\underline{\boldsymbol{v}})$, and the induced map $\mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}_{\underline{\boldsymbol{v}}}$ is birational with connected fibers.*

Proof. Projection on the first factor of X^k gives a G-equivariant morphism $\mathrm{X}_{\underline{\boldsymbol{v}}} \xrightarrow{\eta} \mathrm{X}$. Since G acts transitively on X this morphism is surjective and all fibers are isomorphic, i.e., this expresses $\mathrm{X}_{\underline{\boldsymbol{v}}}$ as a fiber bundle over X. To study the fibers we look at the fiber $\eta^{-1}(e)$ over the B-fixed point e of X.

Consider the diagram

$$\begin{tikzcd}
\mathrm{G} \times e \times \mathrm{Z}_{\underline{v}_2} \times \cdots \times \mathrm{Z}_{\underline{v}_k} \arrow[r, "\psi_{\underline{\boldsymbol{v}}}"] \arrow[d, "\mathrm{id}_{\mathrm{G}} \times \mathrm{id}_{x_o} \times f_{\underline{v}_1} \times \cdots \times f_{\underline{v}_k}"'] & \mathrm{Y}_{\underline{\boldsymbol{v}}} \arrow[d, "f_{\underline{\boldsymbol{v}}}"] \\
\mathrm{G} \times e \times \mathrm{X}_{v_2} \times \cdots \times \mathrm{X}_{v_k} \arrow[r, "\phi"] & \mathrm{X}^k
\end{tikzcd} \tag{3.7.2}$$

where ϕ is given by $\phi(g, e, x_2, \ldots, x_k) = (g \cdot e, g \cdot x_2, \ldots, g \cdot x_k) \in \mathrm{X}^k$. Since $\psi_{\underline{\boldsymbol{v}}}$ and the leftmost vertical map are surjective, the image of $f_{\underline{\boldsymbol{v}}}$ is the same as the image of ϕ. Since B is the stabilizer of e, the fiber $\eta^{-1}(e)$ is the image of $\mathrm{B} \times e \times \mathrm{X}_{v_2} \times \cdots \times \mathrm{X}_{v_k}$ under ϕ. But each Schubert variety X_w is stable under the action of B and therefore the image above is just $e \times \mathrm{X}_{v_2} \times \cdots \times \mathrm{X}_{v_k}$, proving (a).

From the fibration η it is clear that

$$\dim(\mathrm{X}_{\underline{\boldsymbol{v}}}) = \dim(\mathrm{X}) + \sum_{i=2}^{k} \dim(\mathrm{X}_{v_i}) = \mathrm{N} + \sum_{i=2}^{k} \ell(v_i) = \mathrm{N} + \ell(\underline{\boldsymbol{v}}),$$

because each $\underline{v}_i$ is reduced and hence $\ell(v_i) = \ell(\underline{v}_i)$ for $i \geqslant 2$.

The product of normal varieties is again normal, and the product of varieties with rational singularities also has rational singularities. Since each X_w is normal with rational singularities (Section 2.6), the fibers also have this property, and therefore so does $\mathrm{X}_{\underline{\boldsymbol{v}}}$ (since the properties of being normal or having rational singularities are local, and $\mathrm{X}_{\underline{\boldsymbol{v}}}$ is locally the product of the fiber and a smooth variety).

Since each map $f_{\underline{v}_i} : \mathrm{Z}_{\underline{v}_i} \to \mathrm{X}_{v_i}$ is a resolution of singularities of a normal variety, each $f_{\underline{v}_i}$ is birational with connected fibers. It follows that the map $\mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}_{\underline{\boldsymbol{v}}}$, which is the quotient of the leftmost vertical map in (3.7.2) by the action of B, is also birational with connected fibers. This proves (b). ☐

Definition 3.7.3. If X_w is any Schubert subvariety of X and q is any point of X, we define the subvariety $q\mathrm{X}_w$ of X to be the result of translating X_w by any element in the B-coset corresponding to q. Since X_w is B-stable the result is independent of the choice of representative for q.

The following theorem gives more precise information about the image and fibers of $f_{\underline{\boldsymbol{v}}}$.

Theorem 3.7.4. *Let $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$ be a sequence of reduced words with corresponding Weyl group elements $(v_1, \ldots, v_k)$. Then there exists a factorization $f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \xrightarrow{\tau} Q_{\underline{\boldsymbol{v}}} \xrightarrow{h} X^k$ such that*

(a) $Q_{\underline{\boldsymbol{v}}}$ *is normal with rational singularities*;

(b) *the map $\tau : Y_{\underline{\boldsymbol{v}}} \to Q_{\underline{\boldsymbol{v}}}$ is proper and birational with connected fibers*;

(c) *for each point $(q_1, \ldots, q_k)$ of X^k there is a natural inclusion*

$$h^{-1}(q_1, \ldots, q_k) \hookrightarrow \bigcap_{i=1}^{k} q_i X_{v_i^{-1}}$$

of the scheme-theoretic fiber $h^{-1}(q_1, \ldots, q_k)$ into the scheme-theoretic intersection $\bigcap_{i=1}^{k} q_i X_{v_i^{-1}}$;

(d) *the inclusion of schemes in* (c) *induces an isomorphism at the level of reduced schemes*, *or in other words*, *the set-theoretic fiber $h^{-1}(q_1, \ldots, q_k)$ is equal to the set-theoretic intersection $\bigcap_{i=1}^{k} q_i X_{v_i^{-1}}$.*

Proof. Let $f_\circ \times f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \to X \times X^k$ be the product of $f_{\underline{\boldsymbol{v}}}$ and the map $f_\circ : Y_{\underline{\boldsymbol{v}}} \to X$ from Section 3.4 expressing $Y_{\underline{\boldsymbol{v}}}$ as a $Z_{\underline{\boldsymbol{v}}_1} \times \cdots \times Z_{\underline{\boldsymbol{v}}_k}$-bundle over X. We define $Q_{\underline{\boldsymbol{v}}}$ to be the image of $f_\circ \times f_{\underline{\boldsymbol{v}}}$ with the reduced scheme structure, τ to be the map from $Y_{\underline{\boldsymbol{v}}}$ onto $Q_{\underline{\boldsymbol{v}}}$, and h to be the map from $Q_{\underline{\boldsymbol{v}}}$ to X^k induced by the projection $X \times X^k \to X^k$. By construction $f_{\underline{\boldsymbol{v}}} = h \circ \tau$.

Letting $\psi_{\underline{\boldsymbol{v}}}$ be the map (from Section 3.2) defining $Y_{\underline{\boldsymbol{v}}}$ as a quotient of $B \times Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k}$ and $\phi : G \times X_{v_1} \times \cdots \times X_{v_k} \to Q_{\underline{\boldsymbol{v}}} \subseteq X \times X^k$ as the map sending $(g, x_1, \ldots, x_k)$ to $(gB/B, g \cdot x_1, \ldots, g \cdot x_k)$ in $X \times X^k$, we obtain a refinement of diagram (3.7.2):

$$\begin{tikzcd}
G \times Z_{\underline{v}_1} \times \cdots \times Z_{\underline{v}_k} \arrow[r, "\psi_{\underline{\boldsymbol{v}}}"] \arrow[d] \arrow[dr, phantom, "\square"] & Y_{\underline{\boldsymbol{v}}} \arrow[d, "\tau"'] \arrow[dr, "f_\circ \times f_{\underline{\boldsymbol{v}}}"] & \\
G \times X_{v_1} \times \cdots \times X_{v_k} \arrow[r, "\phi"] & Q_{\underline{\boldsymbol{v}}} \arrow[r, hook] \arrow[d, "h"'] & X \times X^k \arrow[d] \\
& X^k \arrow[r, equal] & X^k
\end{tikzcd}$$

Since $\mathrm{Y}_{\underline{\boldsymbol{v}}} \simeq \mathrm{Y}_{(\varnothing,\underline{v}_1,\ldots,\underline{v}_k)}$ (see Section 3.4) and under this isomorphism the map $f_\circ \times f_{\underline{\boldsymbol{v}}}$ is the map $f_{(\varnothing,\underline{v}_1,\ldots,\underline{v}_k)}$, it follows from Lemma 3.7.1(b) that $\mathrm{Q}_{\underline{\boldsymbol{v}}}$ is normal with rational singularities and that $\tau : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{Q}_{\underline{\boldsymbol{v}}}$ is birational with connected fibers, proving (a) and (b).

The composite map

$$\mathrm{G} \times \mathrm{P}_{\underline{v}_1} \times \cdots \times \mathrm{P}_{\underline{v}_k} \to \mathrm{G} \times \mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k} \xrightarrow{\psi_{\underline{\boldsymbol{v}}}} \mathrm{Y}_{\underline{\boldsymbol{v}}} \xrightarrow{\tau} \mathrm{Q}_{\underline{\boldsymbol{v}}}$$

is given (in the notation of Section 3.3) by sending

$$(g \mid p_{i_{1,1}}, p_{i_{2,1}}, \ldots, p_{i_{m_1,1}} \mid p_{i_{1,2}}, p_{i_{2,2}}, \ldots, p_{i_{m_2,2}} \mid \cdots \mid p_{i_{1,k}}, \ldots, p_{i_{m_k,k}})$$

to

$$(g \mid g p_{i_{1,1}} p_{i_{2,1}} \cdots p_{i_{m_1,1}} \mid g p_{i_{1,2}} p_{i_{2,2}} \cdots p_{i_{m_2,2}} \mid \cdots \mid g p_{i_{1,k}} \cdots p_{i_{m_k,k}})$$

in $\mathrm{X} \times \mathrm{X}^k$. A point q of X is therefore in the fiber

$$h^{-1}(q_1, \ldots, q_k) \subseteq \mathrm{X} \times q_1 \times \cdots \times q_k = \mathrm{X}$$

if for any B-coset representatives $g, g_1, \ldots, g_k$ of $q, q_1, \ldots, q_k$, there exist elements $\{p_{i,j}\}$ in the respective parabolic subgroups such that we can solve the equations

$$\begin{aligned} g p_{i_{1,1}} p_{i_{2,1}} \cdots p_{i_{m_1,1}} &= g_1, \\ \vdots \qquad\qquad & \ \vdots \ \ \vdots \\ g p_{i_{1,k}} p_{i_{2,k}} \cdots p_{i_{m_k,k}} &= g_k. \end{aligned}$$

Moving the $p_{i,j}$'s to the right hand side, the system above becomes

$$\begin{aligned} g &= g_1 p_{i_{m_1,1}}^{-1} \cdots p_{i_{2,1}}^{-1} p_{i_{1,1}}^{-1}, \\ \vdots \ & \ \vdots \qquad\quad \vdots \\ g &= g_k p_{i_{m_k,k}}^{-1} \cdots p_{i_{2,k}}^{-1} p_{i_{1,k}}^{-1}, \end{aligned}$$

which is equivalent to q belonging in the intersection $\bigcap_{i=1}^{k} q_i \mathrm{X}_{v_i^{-1}}$, proving (d).

Let $\underline{v}$ be a reduced word with product v. By part (d) the set

$$\mathrm{Q}'_{\underline{v}} := \{(q, p) \in \mathrm{X} \times \mathrm{X} \mid p \in q\mathrm{X}_v\}$$

is the image of $f_{(\varnothing,\underline{v})} : \mathrm{Y}_{(\varnothing,\underline{v})} \to \mathrm{X} \times \mathrm{X}$ and is therefore a closed subvariety of $\mathrm{X} \times \mathrm{X}$. Alternatively $\mathrm{Q}'_{\underline{v}}$ is the Zariski closure of the set $\{(g, g \cdot v) \mid g \in \mathrm{G}\} \subseteq \mathrm{X} \times \mathrm{X}$.

For $i = 1, \ldots, k$, let $p_i : \mathrm{X} \times \mathrm{X}^k \to \mathrm{X} \times \mathrm{X}$ be the map which is the product of id_X with projection $\mathrm{X}^k \to \mathrm{X}$ onto the i-th factor. The intersection

$$\mathrm{Q}'_{\underline{\boldsymbol{v}}} := \bigcap_{i=1}^{k} p_i^{-1}(\mathrm{Q}'_{\underline{v}_i})$$

is a closed subscheme of $X \times X^k$ which, by (d), agrees set theoretically with $Q_{\underline{v}}$. Since $Q_{\underline{v}}$ is reduced, we have the inclusion of schemes $Q_{\underline{v}} \subseteq Q'_{\underline{v}}$. If h' is the map $h' : Q'_{\underline{v}} \to X^k$ induced by projection, then the scheme-theoretic fibers of h are naturally a subscheme of the scheme-theoretic fibers of h' (and both are naturally subschemes of X). The scheme-theoretic fiber of h' is the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$, proving (c). □

The image $X_{\underline{v}}$ is therefore the set of translations $(q_1, \ldots, q_k)$ in X^k for which the intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ of translated Schubert varieties is nonempty, and the set-theoretic fibers of h are the intersections themselves. Moreover, $Q_{\underline{v}}$ is the incidence correspondence of intersections of translates of Schubert varieties (the first coordinate in $X \times X^k$ is the intersection, the remaining k coordinates are the parameters $(q_1, \ldots, q_k)$ controlling the translates). Theorem 3.7.4 shows that $Y_{\underline{v}}$ is a resolution of singularities of $Q_{\underline{v}}$.

Corollary 3.7.5. *Let $\underline{v} = (\underline{v}_1, \ldots, \underline{v}_k)$ be a sequence of reduced words with corresponding Weyl group elements $(v_1, \ldots, v_k)$ such that $\sum_{i=1}^k \ell(\underline{v}_i) = (k-1)N$. Then the degree of the map $f_{\underline{v}} : Y_{\underline{v}} \to X^k$ is given by the intersection number $\bigcap_{i=1}^k [\Omega_{w_0 v_i^{-1}}] = \bigcap_{i=1}^k [X_{v_i^{-1}}]$.*

Remark. The dimension of $Y_{\underline{v}}$ in this case is $N + \sum \ell(\underline{v}_i) = kN = \dim(X^k)$ so it is reasonable to ask for the degree of the map.

Proof. Since we are working in characteristic zero, the degree of $f_{\underline{v}}$ is given by the number of points in a generic fiber. By Theorem 3.7.4 the map $p : Y_{\underline{v}} \to Q_{\underline{v}}$ is birational, and so the generic fiber of $f_{\underline{v}}$ is the same as the generic fiber of $h : Q_{\underline{v}} \to X^k$. By the Kleiman transversality theorem, if $q_1, \ldots, q_k$ are generic, the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ is reduced and finite, and the number of points is equal to the intersection number $\bigcap_{i=1}^k [X_{v_i^{-1}}] = \bigcap_{i=1}^k [\Omega_{w_0 v_i^{-1}}]$ in $H^*(X, \mathbb{Z})$. By Theorem 3.7.4(c–d) if the scheme-theoretic intersection $\bigcap_{i=1}^k q_i X_{v_i^{-1}}$ is reduced it is equal to the scheme-theoretic fiber $h^{-1}(q_1, \ldots, q_k)$, proving the corollary. □

3.8. *Key lemma.* We now prove an important lemma which will allow us to derive several results necessary for the proofs of Theorems I and II. The lemma itself will also be used in the proof of Theorem I.

Lemma 3.8.1. *Let $\underline{v}$ be a sequence of reduced words, L be a G-equivariant line bundle on $Y_{\underline{v}}$ and $s \in H^0(Y_{\underline{v}}, L)^G$ be a nonzero G-invariant section. Then:*

(a) *the weight of L at the T-fixed maximum point (Section 3.5) $p = p_{\underline{v}} \in Y_{\underline{v}}$ belongs to $\operatorname{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$;*
(b) *the weight of L at p is zero if and only if s does not vanish at p;*
(c) *without supposing that L has a G-invariant section, if L is an equivariant bundle on $Y_{\underline{v}}$ and the weight of L at p is zero, then $\dim H^0(Y_{\underline{v}}, L)^G \leqslant 1$.*

Remark. Part (c) will be used often to control the size of the G-invariant sections.

Proof. Let $f_\circ : \mathrm{Y}_{\underline{v}} \to \mathrm{X}$ be the map from Section 3.4 expressing $\mathrm{Y}_{\underline{v}}$ as a $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$-bundle over X. The section s cannot vanish on any fiber of $f_\circ$ since (by G-invariance and transitivity of G-action on X) s would vanish on all of $\mathrm{Y}_{\underline{v}}$. We can thus restrict s to get a nonzero section on the fiber $\mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$ of $f_\circ$ over $e \in \mathrm{X}$; this fiber contains the maximum point p.

The formal character of the tangent space at the maximum point p_i of $\mathrm{Z}_{\underline{v}_i}$ is $\langle \Phi_{v_i^{-1}} \rangle$; i.e., all the weights of this space are positive roots. Since the maximum point $p = p_1 \times \cdots \times p_k \in \mathrm{Z} := \mathrm{Z}_{\underline{v}_1} \times \cdots \times \mathrm{Z}_{\underline{v}_k}$ is the product of the maximum points of the factors, each of the weights on the tangent space of p in Z is also a positive root.

Let $\mathfrak{m}_p$ be the maximal ideal of p in $\mathcal{O}_{\mathrm{Z},p}$. For every $r \geqslant 0$ we get a T-equivariant restriction map

$$\mathrm{H}^0(\mathrm{Z}, \mathrm{L}|_{\mathrm{Z}}) \to \mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p^{r+1}) = \mathrm{L} \otimes \left(\mathcal{O}_\mathrm{Z}/\mathfrak{m}_p \oplus \mathfrak{m}_p/\mathfrak{m}_p^2 \oplus \cdots \oplus \mathfrak{m}_p^r/\mathfrak{m}_p^{r+1}\right),$$

which is an injection for r sufficiently large. In particular, for sufficiently large r, the section s restricts to a nonzero element of $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} \mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p^{r+1}$. Since s is an invariant section, this means that the zero weight is a weight of $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p^{r+1})$, and so must appear in one of the factors $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathfrak{m}_p^i/\mathfrak{m}_p^{i+1}) = \mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} \mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ for $i = 0, \ldots, r$.

Since $\mathfrak{m}_p/\mathfrak{m}_p^2$ is dual to the tangent space at p, all weights of $\mathfrak{m}_p/\mathfrak{m}_p^2$ are negative roots, and therefore the weights of $\mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ belong to $\mathrm{span}_{\mathbb{Z}_{\leqslant 0}} \Delta^+$. Tensoring with L multiplies the formal character of $\mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ by the weight of L at p. Thus the zero weight is a weight of $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} \mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ only if the weight of L at p belongs to $\mathrm{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$. This proves (a).

The value of s at p is the restriction of s to the factor $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p) = \mathrm{L}_p$. If s does not vanish at p the weight of L_p is therefore zero. Conversely, if the weight of L_p is zero then the weights of $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} \mathrm{Sym}^i(\mathfrak{m}_p/\mathfrak{m}_p^2)$ are nonzero for $i \geqslant 1$. Hence the only possibility for the invariant section s under the restriction map is to have nonzero restriction to $\mathrm{L} \otimes_{\mathcal{O}_\mathrm{Z}} (\mathcal{O}_{\mathrm{Z},p}/\mathfrak{m}_p) = \mathrm{L}_p$, proving (b).

Suppose that the weight of L at p is zero. If there were two linearly independent sections $s_1, s_2 \in \mathrm{H}^0(\mathrm{Y}_{\underline{v}}, \mathrm{L})^\mathrm{G}$ then some nonzero linear combination would vanish at p contradicting (b). Hence if the weight is zero we must have $\dim \mathrm{H}^0(\mathrm{Y}_{\underline{v}}, \mathrm{L}) \leqslant 1$, giving (c). □

3.9. *Applications of Lemma 3.8.1.*

Theorem 3.9.1. *Suppose that $w_1, \ldots, w_k$, and w are elements of the Weyl group such that*

$$\ell(w) = \sum_{i=1}^{k} \ell(w_i) \quad \textit{and} \quad \bigcap_{i=1}^{k} [\Omega_{w_i}] \cdot [\mathrm{X}_w] \neq 0 \textit{ in } \mathrm{H}^*(\mathrm{X}, \mathbb{Z}).$$

Then:

(a) *For any dominant weights* $\mu_1, \ldots, \mu_k$, *and* μ *such that the irreducible module* V_μ *is a component of* $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$, *the weight* $\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu$ *belongs to* $\operatorname{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$.

(b) *If* $\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu = 0$ *then* $\operatorname{mult}(V_\mu, V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}) = 1$.

(c) $\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0 = \sum_{i=1}^k (w_k^{-1}\rho - \rho) - (w^{-1}\rho - \rho)$ *belongs to* $\operatorname{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$.

(d) *If* $\sum_{i=1}^k w_i^{-1} \cdot 0 = w^{-1} \cdot 0$ *then* $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$.

Note that the action of the Weyl group in parts (a) *and* (b) *is the homogeneous action*, *while the action in parts* (c) *and* (d) *is the affine action.*

Proof. Let $v_i = w_i^{-1}w_0$ for $i = 1, \ldots, k$, $v_{k+1} = w^{-1}$, let $\underline{v}_i$ be a reduced word with product v_i, for $i = 1, \ldots, k+1$, and set $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_{k+1})$. Then $\sum \ell(\underline{v}_i) = (k+1-1)N$ and so, by Corollary 3.7.5, the degree of $f_{\underline{\boldsymbol{v}}} : Y_{\underline{\boldsymbol{v}}} \to X^{k+1}$ is given by the intersection number

$$\bigcap_{i=1}^{k+1} [\Omega_{w_0 v_i^{-1}}] = \bigcap_{i=1}^{k} [\Omega_{w_i}] \cdot [X_w].$$

By hypothesis this intersection number is nonzero and therefore $f_{\underline{\boldsymbol{v}}}$ is surjective.

Given dominant weights $\mu_1, \ldots, \mu_k$, and μ let $\lambda_i = -w_0\mu_i$ for $i = 1, \ldots, k$ and $\lambda_{k+1} = \mu$. Set L to be the line bundle $L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_{k+1}}$ on X^{k+1}, so that $H^0(X^{k+1}, L) = V_{\mu_1} \otimes \cdots \otimes V_{\mu_k} \otimes V_\mu^*$ and $\dim H^0(X^{k+1}, L)^G$ is the multiplicity of V_μ in the tensor product $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$.

Since $f_{\underline{\boldsymbol{v}}}$ is surjective, pullback induces an inclusion

$$H^0(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L) \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} H^0(X^{k+1}, L)$$

and, in particular, $\dim H^0(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L)^G \geqslant \dim H^0(X^{k+1}, L)^G$. We know, by applying Lemma 3.8.1(a), that if $f_{\underline{\boldsymbol{v}}}^* L$ has a nonzero G-invariant section then the weight of $f_{\underline{\boldsymbol{v}}}^* L$ at the maximum point $p_{\underline{\boldsymbol{v}}}$ belongs to $\operatorname{span}_{\mathbb{Z}_{\geqslant 0}} \Delta^+$. This weight is

$$\sum_{i=1}^{k+1} v_i(-\lambda_i) = \sum_{i=1}^{k} (w_i^{-1}w_0)(w_0\mu_i) + w^{-1}(-\mu) = \sum_{i=1}^{k} w_i^{-1}\mu_i - w^{-1}\mu, \quad (3.9.2)$$

proving (a).

If the weight in (3.9.2) is zero then $\dim H^0(X^{k+1}, L)^G \leqslant \dim H^0(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L)^G \leqslant 1$ by Lemma 3.8.1(c), and so if V_μ is a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ then it is of multiplicity at most one. The fact that V_μ actually is a component of the tensor product is a consequence of the solution of the PRV conjecture — see Section 2.3 for a discussion. This proves (b).

The map $f_{\underline{v}} : \mathrm{Y}_{\underline{v}} \to \mathrm{X}^{k+1}$ induces a natural map $f_{\underline{v}}^* \mathrm{K}_{\mathrm{X}^{k+1}} \to \mathrm{K}_{\mathrm{Y}_{\underline{v}}}$ which is given by a global section s of $\mathrm{H}^0(\mathrm{Y}_{\underline{v}}, (f_{\underline{v}}^* \mathrm{K}_{\mathrm{X}^{k+1}})^* \otimes \mathrm{K}_{\mathrm{Y}_{\underline{v}}})$. Since $\mathrm{Y}_{\underline{v}}$ and X^{k+1} have the same dimension and since $f_{\underline{v}}$ is surjective, this section is nonzero. Because the pullback morphism is natural, the section s is G-invariant. By Lemma 3.8.1(a) the weight of the line bundle $\mathrm{K}_{\mathrm{Y}_{\underline{v}}/\mathrm{X}^{k+1}} := (f_{\underline{v}}^* \mathrm{K}_{\mathrm{X}^{k+1}})^* \otimes \mathrm{K}_{\mathrm{Y}_{\underline{v}}}$ at the maximum point $p_{\underline{v}}$ belongs to $\mathrm{span}_{\mathbb{Z}\geqslant 0} \Delta^+$.

By (3.6.1), (2.2.2), and (2.2.3) the formal characters of the tangent spaces at $p_{\underline{v}}$ in $\mathrm{Y}_{\underline{v}}$ and $q_{\underline{v}} := f_{\underline{v}}(p_{\underline{v}})$ in X^{k+1} are, respectively

$$\mathrm{Ch}(\mathrm{T}_{p_{\underline{v}}} \mathrm{Y}_{\underline{v}}) = \langle \Phi_w \rangle + \langle \Delta^- \rangle + \sum_{i=1}^{k} \langle \Phi_{w_i}^{\mathrm{c}} \rangle \tag{3.9.3}$$

and

$$\mathrm{Ch}(\mathrm{T}_{q_{\underline{v}}} \mathrm{X}^{k+1}) = \big(\langle \Phi_w \rangle + \langle -\Phi_w^{\mathrm{c}} \rangle\big) + \sum_{i=1}^{k} \big(\langle \Phi_{w_i}^{\mathrm{c}} \rangle + \langle -\Phi_{w_i} \rangle\big). \tag{3.9.4}$$

A short calculation using formula (2.2.4) shows that the weight of $\mathrm{K}_{\mathrm{Y}_{\underline{v}}/\mathrm{X}^{k+1}}$ at $p_{\underline{v}}$ is $\sum_{i=1}^{k} (w_k^{-1}\rho - \rho) - (w^{-1}\rho - \rho)$, proving (c).

If the weight $\sum_{i=1}^{k} (w_k^{-1}\rho - \rho) - (w^{-1}\rho - \rho)$ is zero then, by Lemma 3.8.1(b), the section s is nonzero at $p_{\underline{v}}$. This means that $f_{\underline{v}}$ is unramified at $p_{\underline{v}}$ and therefore the tangent space map $\mathrm{T}_{\mathrm{Y}_{\underline{v}},p} \xrightarrow{df_{\underline{v}}} \mathrm{T}_{\mathrm{X}^{k+1},q}$ is an isomorphism. Hence both spaces must have the same formal characters. Comparing the negative roots and their multiplicities in (3.9.3) and (3.9.4) gives $\Delta^- = \big(\bigsqcup_{i=1}^{k} -\Phi_{w_i}\big) \bigsqcup -\Phi_w^{\mathrm{c}}$ which is equivalent to $\Phi_w = \bigsqcup_{i=1}^{k} \Phi_{w_i}$, proving (d). □

3.10. ***Relation with existing results.*** Part (a) of Theorem 3.9.1 is due to Berenstein and Sjamaar [2000] . A theorem of this type was first proved by Klyachko [1998] for GL_n. This was later extended to all semisimple groups by Berenstein and Sjamaar [2000] and by Kapovich, Leeb, and Millson [Kapovich et al. 2009]. Parts (c) and (d) are due to Belkale and Kumar [2006]: part (c) is their Theorem 29 and (d) is their Theorem 15, both in the case when the parabolic group P is the Borel group B.

Part (b) is new and crucial for controlling the multiplicities of cohomological components. The remaining statements have been included because Lemma 3.8.1 allows us to give a new, short, and unified proof of these results. In particular, we obtain a new proof of the necessity of the inequalities determining the Littlewood–Richardson cone. Namely, these inequalities are obtained by requiring that the weights in Theorem 3.9.1(a) (for all $w_1, \dots, w_k, w$ satisfying the conditions of the theorem) belong to $\mathrm{span}_{\mathbb{Z}\geqslant 0} \Delta^+$. (The proof that these inequalities are sufficient requires a separate GIT argument.)

Relation with a construction of Kumar. Given a sequence $\underline{u}$ of simple reflections, Kumar [1988, §1.1] defined a variety $\widetilde{Z}_{\underline{u}}$ along with a map $\theta_{\underline{u}}$ from $\widetilde{Z}_{\underline{u}}$ to X^2. For any pair of words $\underline{\boldsymbol{v}} = (\underline{v}_1, \underline{v}_2)$ let $\underline{u} = \underline{v}_1^{-1}\underline{v}_2$ be the word obtained by reversing $\underline{v}_1$ and concatenating it onto the left of $\underline{v}_2$. By comparing the construction of $Y_{\underline{\boldsymbol{v}}}$ and $\widetilde{Z}_{\underline{u}}$ it is not hard to find an isomorphism $\widetilde{Z}_{\underline{u}} = Y_{\underline{\boldsymbol{v}}}$ over X^2 (i.e., such that $\theta_{\underline{u}} = f_{\underline{\boldsymbol{v}}}$ under the isomorphism). Therefore when $k = 2$ the varieties produced by our construction are the same as the ones constructed in [Kumar 1988, §1.1].

4. Proof of Theorem III

4.1. We will prove Theorem III in its symmetric form. After applying the symmetrization procedure from Section 2.7 (and replacing $k+1$ by k) we obtain:

Theorem 4.1.1 (symmetric form of Theorem III). *Let $w_1, \dots, w_k$ be elements of the Weyl group $\mathcal{W}$ such that $\sum_i \ell(w_i) = N$, and let $\lambda_1, \dots, \lambda_k$ be weights such that $w_i \cdot \lambda_i$ are dominant weights for $i = 1, \dots, k$, and $\sum_{i=1}^k \lambda_i = -2\rho$.*

(a) *If $\bigcap_{i=1}^k [\Omega_{w_i}] = 1$ then the cup-product map*

$$H^{\ell(w_1)}(X, L_{\lambda_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{\lambda_k}) \xrightarrow{\cup} H^N(X, K_X) \tag{4.1.2}$$

is surjective.

(b) *If $\bigcap_{i=1}^k [\Omega_{w_i}] = 0$ then (4.1.2) is zero.*

The proof of Theorem 4.1.1 is given in Section 4.3. We will use the following common notation. For any sequence $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ of weights let $L_{\underline{\lambda}}$ be the line bundle

$$L_{\underline{\lambda}} := L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_k} = \mathrm{pr}_1^* L_{\lambda_1} \otimes \cdots \otimes \mathrm{pr}_k^* L_{\lambda_k}$$

on X^k, where $\mathrm{pr}_i : X^k \to X$ denotes projection onto the i-th factor.

4.2. ***Inductive lemma.*** Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ be a sequence of weights and $\underline{\boldsymbol{v}} = (\underline{v}_1, \dots, \underline{v}_k)$ a sequence of words. Let $\underline{\boldsymbol{u}}$ be a sequence of words as in (3.1.1), i.e., $\underline{\boldsymbol{u}}$ is a sequence of words obtained by dropping a simple reflection from the right of a single member of $\underline{\boldsymbol{v}}$. The following lemma lets us propagate information about the pullback map

$$H^{N+\ell(\underline{\boldsymbol{v}})}(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\lambda}}) \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} H^{N+\ell(\underline{\boldsymbol{v}})}(X^k, L_{\underline{\lambda}}) \tag{4.2.1}$$

on the top degree cohomology of $Y_{\underline{\boldsymbol{v}}}$ to information about an analogous pullback map to the top degree cohomology of $Y_{\underline{\boldsymbol{u}}}$. If $\underline{v}_j = s_{i_1} \cdots s_{i_m}$, so that we are dropping s_{i_m} from $\underline{v}_j$ to get $\underline{u}_j$, we denote by $\underline{\mu}$ the sequence

$$\underline{\mu} := (\lambda_1, \dots, \lambda_{j-1}, s_{i_m} \cdot \lambda_j, \lambda_{j+1}, \dots, \lambda_k).$$

Finally, we assume that the degree of $L_{\underline{\lambda}}$ is negative on the fibers of the $\mathbb{P}^1$-fibration $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}} : Y_{\underline{\boldsymbol{v}}} \to Y_{\underline{\boldsymbol{u}}}$.

Lemma 4.2.2. *Under the conditions above, the pullback map*

$$H^{N+\ell(\underline{\boldsymbol{u}})}(Y_{\underline{\boldsymbol{u}}}, f_{\underline{\boldsymbol{u}}}^{*}L_{\underline{\mu}}) \xleftarrow{f_{\underline{\boldsymbol{u}}}^{*}} H^{N+\ell(\underline{\boldsymbol{u}})}(X^k, L_{\underline{\mu}})$$

is (a) *surjective*, (b) *zero*, *or* (c) *surjective on the space of* G*-invariants*, *if the pullback map* (4.2.1) *has the corresponding property* (a), (b), *or* (c).

Here "surjective on the space of G*-invariants" means* (*in the case of* $Y_{\underline{\boldsymbol{v}}}$) *that*

$$H^{N+\ell(\underline{\boldsymbol{v}})}(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^{*}L_{\underline{\lambda}})^{G} \xleftarrow{f_{\underline{\boldsymbol{v}}}^{*}} H^{N+\ell(\underline{\boldsymbol{v}})}(X^k, L_{\underline{\lambda}})^{G}$$

is surjective.

Proof. To reduce notation set

$$M_{\underline{\boldsymbol{u}}} = X^{j-1} \times M_{i_m} \times X^{k-j}$$

and let $\pi : X^k \to M_{\underline{\boldsymbol{u}}}$ be the map

$$\pi = (\mathrm{id}_X)^{j-1} \times \pi_{i_m} \times (\mathrm{id}_X)^{k-j}.$$

The fiber product diagram (3.1.2) relating $Y_{\underline{\boldsymbol{v}}}$, $Y_{\underline{\boldsymbol{u}}}$, X^k, and $M_{\underline{\boldsymbol{u}}}$ is

$$\begin{tikzcd}
Y_{\underline{\boldsymbol{v}}} \arrow[r, "f_{\underline{\boldsymbol{v}}}"] \arrow[d, "\pi_{\underline{\boldsymbol{v}}}"] \arrow[dr, phantom, "\square"] & X^k \arrow[d, "\pi"] \\
Y_{\underline{\boldsymbol{u}}} \arrow[r, "h"] \arrow[u, bend left, "\sigma_{\underline{\boldsymbol{v}}}"] & M_{\underline{\boldsymbol{u}}}
\end{tikzcd} \tag{4.2.3}$$

where $h = \pi \circ f_{\underline{\boldsymbol{u}}}$ and where we use $\pi_{\underline{\boldsymbol{v}}}$ and $\sigma_{\underline{\boldsymbol{v}}}$ in place of $\pi_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$ and $\sigma_{\underline{\boldsymbol{v}},\underline{\boldsymbol{u}}}$ to reduce notation.

Note that $L_{\underline{\mu}}$ is the Demazure reflection of $L_{\underline{\lambda}}$ with respect to π. By Section 2.8 this means that we have natural isomorphisms

$$\pi_{\underline{\boldsymbol{v}}*}(f_{\underline{\boldsymbol{v}}}^{*}L_{\underline{\mu}}) \cong R^1\pi_{\underline{\boldsymbol{v}}*}(f_{\underline{\boldsymbol{v}}}^{*}L_{\underline{\lambda}}) \quad \text{and} \quad \pi_{*}L_{\underline{\mu}} \cong R^1\pi_{*}L_{\underline{\lambda}} \tag{4.2.4}$$

valid on $Y_{\underline{\boldsymbol{u}}}$ and $M_{\underline{\boldsymbol{u}}}$ respectively. Diagram (4.2.3), the Leray spectral sequences for $L_{\underline{\lambda}}$ and $L_{\underline{\mu}}$ relative to π and $\pi_{\underline{\boldsymbol{v}}}$, and the isomorphisms (4.2.4) then give the commutative diagram of cohomology groups:

$$
\begin{tikzcd}
H^{N+\ell(\underline{\boldsymbol{v}})}(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\lambda}}) \arrow[d, equal, "\wr"', "\text{Leray}"] & H^{N+\ell(\underline{\boldsymbol{v}})}(X^k, L_{\underline{\lambda}}) \arrow[l, "f_{\underline{\boldsymbol{v}}}^*"', "(4.2.1)"] \arrow[d, equal, "\wr"', "\text{Leray}"] \\
H^{N+\ell(\underline{\boldsymbol{v}})-1}(Y_{\underline{\boldsymbol{u}}}, R^1\pi_{\underline{\boldsymbol{v}}*} f_{\underline{\boldsymbol{v}}}^* L_{\underline{\lambda}}) \arrow[d, equal, "\wr"', "(4.2.4)"] & H^{N+\ell(\underline{\boldsymbol{v}})-1}(M_{\underline{\boldsymbol{u}}}, R^1\pi_* L_{\underline{\lambda}}) \arrow[l, "h^*"'] \arrow[d, equal, "\wr"', "(4.2.4)"] \\
H^{N+\ell(\underline{\boldsymbol{v}})-1}(Y_{\underline{\boldsymbol{u}}}, \pi_{\underline{\boldsymbol{v}}*} f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}) \arrow[d, equal, "\wr"', "\text{Leray}"] & H^{N+\ell(\underline{\boldsymbol{v}})-1}(M_{\underline{\boldsymbol{u}}}, \pi_* L_{\underline{\mu}}) \arrow[l, "h^*"'] \arrow[d, equal, "\wr"', "\text{Leray}"] \\
H^{N+\ell(\underline{\boldsymbol{v}})-1}(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}) & H^{N+\ell(\underline{\boldsymbol{v}})-1}(X^k, L_{\underline{\mu}}) \arrow[l, "f_{\underline{\boldsymbol{v}}}^*"']
\end{tikzcd}
\tag{4.2.5}
$$

We conclude that the bottom pullback map

$$
H^{N+\ell(\underline{\boldsymbol{v}})-1}(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}) \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} H^{N+\ell(\underline{\boldsymbol{v}})-1}(X^k, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}})
$$

is surjective, zero, or surjective on the space of G-invariants if (4.2.1) is.

On $Y_{\underline{\boldsymbol{v}}}$ we have the exact sequence of bundles

$$
0 \to f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}(-Y_{\underline{\boldsymbol{u}}}) \to f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}} \to f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}|_{Y_{\underline{\boldsymbol{u}}}} \to 0, \tag{4.2.6}
$$

where we consider $Y_{\underline{\boldsymbol{u}}}$ to be a divisor in $Y_{\underline{\boldsymbol{v}}}$ via the section $\sigma_{\underline{\boldsymbol{v}}}$. The degree of $f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}(-Y_{\underline{\boldsymbol{u}}})$ is at least -1 on the fibers of $\pi_{\underline{\boldsymbol{v}}}$ so the corresponding Leray spectral sequence gives

$$
H^{N+\ell(\underline{\boldsymbol{v}})}\big(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}(-Y_{\underline{\boldsymbol{u}}})\big) = H^{N+\ell(\underline{\boldsymbol{v}})}\big(Y_{\underline{\boldsymbol{u}}}, \pi_{\underline{\boldsymbol{v}}*}(f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}(-Y_{\underline{\boldsymbol{u}}}))\big) = 0,
$$

where the second cohomology group above equals zero by reason of dimension:

$$
N + \ell(\underline{\boldsymbol{v}}) = N + \ell(\underline{\boldsymbol{u}}) + 1 = \dim(Y_{\underline{\boldsymbol{u}}}) + 1.
$$

The end of the long exact cohomology sequence associated to (4.2.6) is therefore

$$
H^{N+\ell(\underline{\boldsymbol{v}})-1}(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}) \xrightarrow{\sigma_{\underline{\boldsymbol{v}}}^*} H^{N+\ell(\underline{\boldsymbol{v}})-1}(Y_{\underline{\boldsymbol{u}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}|_{Y_{\underline{\boldsymbol{u}}}}) \to 0. \tag{4.2.7}
$$

Since $\ell(\underline{\boldsymbol{u}}) = \ell(\underline{\boldsymbol{v}}) - 1,\ f_{\underline{\boldsymbol{u}}} = f_{\underline{\boldsymbol{v}}} \circ \sigma_{\underline{\boldsymbol{v}}}$, and all maps are G-equivariant, we conclude that the pullback map $f_{\underline{\boldsymbol{u}}}^*$, being the composite map

$$
\begin{aligned}
H^{N+\ell(\underline{\boldsymbol{u}})}(X^k, L_{\underline{\mu}}) \xrightarrow{f_{\underline{\boldsymbol{v}}}^*} H^{N+\ell(\underline{\boldsymbol{u}})}(Y_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}) \xrightarrow{\sigma_{\underline{\boldsymbol{v}}}^*} H^{N+\ell(\underline{\boldsymbol{u}})}(Y_{\underline{\boldsymbol{u}}}, f_{\underline{\boldsymbol{v}}}^* L_{\underline{\mu}}|_{Y_{\underline{\boldsymbol{u}}}}) \\
= H^{N+\ell(\underline{\boldsymbol{u}})}(Y_{\underline{\boldsymbol{u}}}, f_{\underline{\boldsymbol{u}}}^* L_{\underline{\mu}}),
\end{aligned}
$$

is (a) surjective, (b) zero, or (c) surjective on the space of G-invariants, if the pullback map $f_{\underline{\boldsymbol{v}}}^*$ in (4.2.1) has the corresponding property (a), (b), or (c). $\square$

Remark. In part (c) of Lemma 4.2.2 we can replace the statement about G-invariants with a statement about any isotypic component; the proof above goes through without change. We will only need the case of G-invariants as part of the proof of Theorem I in Section 5 below.

4.3. ***Proof of Theorem 4.1.1 and variation.*** For the rest of this section, we fix the following notation. Let $w_1, \ldots, w_k$ and $\lambda_1, \ldots, \lambda_k$ be as in Theorem 4.1.1. For each $i = 1, \ldots, k$ set $v_i := w_i^{-1} w_0$ and $\lambda'_i := v_i^{-1} \cdot \lambda_i$. Let $\underline{v}_i$ be a reduced factorization of v_i and let $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$. Finally, set $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and $\underline{\lambda}' = (\lambda'_1, \ldots, \lambda'_k)$.

Proof of Theorem 4.1.1. Since

$$\dim(\mathrm{Y}_{\underline{\boldsymbol{v}}}) = \mathrm{N} + \sum_{i=1}^{k} \ell(v_i) = \mathrm{N} + \sum_{i=1}^{k} (\mathrm{N} - \ell(w_i)) = k\mathrm{N} = \dim(\mathrm{X}^k),$$

Corollary 3.7.5 implies that the degree of $f_{\underline{\boldsymbol{v}}} : \mathrm{Y}_{\underline{\boldsymbol{v}}} \to \mathrm{X}^k$ is given by the intersection number $\bigcap_{i=1}^{k} [\Omega_{w_i}]$. Therefore the pullback map

$$\mathrm{H}^{k\mathrm{N}}(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* \mathrm{L}_{\underline{\lambda}'}) \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^{k\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'})$$

is a surjection if $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 1$ and is zero if $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 0$: If $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 1$ then $f_{\underline{\boldsymbol{v}}}$ is a birational map between the smooth varieties $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ and X^k in characteristic zero, and so the pullback map

$$\mathrm{H}^{j}(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* \mathrm{L}_{\underline{\lambda}'}) \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^{j}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'})$$

is an isomorphism in all degrees, and in particular is a surjection in degree $j = k\mathrm{N}$. On the other hand, if $\bigcap_{i=1}^{k} [\Omega_{w_i}] = 0$ then the image $\mathrm{X}_{\underline{\boldsymbol{v}}}$ of $f_{\underline{\boldsymbol{v}}}$ is subvariety of X^k of dimension strictly less than $k\mathrm{N}$ and therefore the pullback map $f_{\underline{\boldsymbol{v}}}^*$ in top cohomology, which factors through $\mathrm{H}^{k\mathrm{N}}(\mathrm{X}_{\underline{\boldsymbol{v}}}, \mathrm{L}_{\underline{\lambda}}|_{\mathrm{X}_{\underline{\boldsymbol{v}}}}) = 0$, is the zero map.

Consider a sequence

$$\underline{\boldsymbol{v}} =: \underline{\boldsymbol{v}}^0, \underline{\boldsymbol{v}}^1, \ldots, \underline{\boldsymbol{v}}^{(k-1)\mathrm{N}} := \underline{\varnothing} = (\varnothing, \ldots, \varnothing)$$

of sequences of words which reduces $\underline{\boldsymbol{v}}$ to the empty sequence, and where at each step $\underline{\boldsymbol{v}}^{j+1}$ is obtained by dropping a simple reflection from the right of a single member of $\underline{\boldsymbol{v}}^j$. Set $\underline{\lambda}^j = (\underline{\boldsymbol{v}}^j)^{-1} \cdot \underline{\lambda}$ where (by slight abuse of notation) $\underline{\boldsymbol{v}}^j$ is considered as an element of $\mathcal{W}^k$ and the action is componentwise. Note that $\underline{\lambda}^0 = \underline{\lambda}'$ and $\underline{\lambda}^{(k-1)\mathrm{N}} = \underline{\lambda}$. The construction of $\underline{\boldsymbol{v}}^j$ and $\underline{\lambda}^j$ implies that the degree of $\mathrm{L}_{\underline{\lambda}^j}$ is negative on the fibers of the $\mathbb{P}^1$-fibration $\pi_{\underline{\boldsymbol{v}}^j, \underline{\boldsymbol{v}}^{j+1}} : \mathrm{Y}_{\underline{\boldsymbol{v}}^j} \to \mathrm{Y}_{\underline{\boldsymbol{v}}^{j+1}}$.

Applying Lemma 4.2.2 to the pairs $(\underline{\boldsymbol{v}}^j, \underline{\boldsymbol{v}}^{j+1})$ for $j = 0, \ldots, (k-1)\mathrm{N} - 1$ we conclude that

$$\mathrm{H}^{\mathrm{N}}(\mathrm{Y}_{\underline{\varnothing}}, f_{\underline{\varnothing}}^* \mathrm{L}_{\underline{\lambda}}) \xleftarrow{f_{\underline{\varnothing}}^*} \mathrm{H}^{\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}}) \tag{4.3.1}$$

is surjective if $\bigcap_{i=1}^k [\Omega_{w_i}] = 1$ and zero if $\bigcap_{i=1}^k [\Omega_{w_i}] = 0$. By construction $f_{\underline{\varnothing}} : \mathrm{Y}_{\underline{\varnothing}} = \mathrm{X} \to \mathrm{X}^k$ is the diagonal embedding of X into X^k and the pullback map (4.3.1) is the cup-product map. This proves Theorem 4.1.1 and completes the proof of Theorem III. □

We record a statement that will be used in the proof of Theorem I below.

Proposition 4.3.2. *If the pullback map*

$$\mathrm{H}^{k\mathrm{N}}(\mathrm{Y}_{\underline{v}},\, f_{\underline{v}}^* \mathrm{L}_{\underline{\lambda}'}) \xleftarrow{f_{\underline{v}}^*} \mathrm{H}^{k\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'})$$

is surjective on the space of G*-invariants then the cup-product map* (4.1.2) *is surjective.*

Proof. We repeat the inductive reduction in the proof of Theorem 4.1.1 above with part (c) of Lemma 4.2.2 in place of parts (a) and (b). As a result we conclude that the cup-product map (4.1.2) is surjective on the space of G-invariants. Since $\mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}})$ is the trivial G-module we conclude that (4.1.2) is surjective. □

5. Proof of Theorem I and corollaries

In this section we use Theorem III and Proposition 4.3.2 to prove Theorem I. The proof that (1.2.1) is necessary for the surjectivity of the cup-product map appears in Section 5.1 and the proof that (1.2.1) is sufficient appears in Section 5.3.

5.1. ***Proof that*** $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$ ***is a necessary condition for surjectivity.*** We assume the notation of Section 1.2, and set $\mu_i = w_i \cdot \lambda_i$ for $i = 1, \dots, k$, and $\mu = w \cdot \lambda$. By assumption the weights $\mu_1, \dots, \mu_k$, and μ are dominant. By the Borel–Weil–Bott theorem each $\mathrm{H}^{\ell(w_i)}(\mathrm{X}, \mathrm{L}_{\lambda_i}) = \mathrm{V}_{\mu_i}^*$ and $\mathrm{H}^d(\mathrm{X}, \mathrm{L}_\lambda) = \mathrm{V}_\mu^*$.

Since $w_i^{-1}\mu_i = w_i^{-1} \cdot \mu_i - w_i^{-1} \cdot 0$ and $w^{-1}\mu_i = w^{-1} \cdot \mu_i - w^{-1} \cdot 0$, we have

$$\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu = \Big(\sum_{i=1}^k w_i^{-1} \cdot \mu_i - w^{-1} \cdot \mu\Big) - \Big(\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0\Big).$$

Furthermore $\sum_{i=1}^k w_i^{-1} \cdot \mu_i - w^{-1} \cdot \mu = \sum \lambda_i - \lambda = 0$ and so the equation above becomes

$$\sum_{i=1}^k w_i^{-1}\mu_i - w^{-1}\mu = -\Big(\sum_{i=1}^k w_i^{-1} \cdot 0 - w^{-1} \cdot 0\Big). \tag{5.1.1}$$

If the cup-product map $\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^d(\mathrm{X}, \mathrm{L}_\lambda)$ is surjective, then (after dualizing) V_μ must be a component of the tensor product $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ and by Theorem III(b), the intersection $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] \neq 0$ in $\mathrm{H}^*(\mathrm{X}, \mathbb{Z})$; we may therefore apply Theorem 3.9.1.

By Theorem 3.9.1(a) the left hand side of (5.1.1) belongs to $\operatorname{span}_{\mathbb{Z}\geqslant 0}\Delta^+$ and by part (c) of the same theorem the right hand side belongs to $\operatorname{span}_{\mathbb{Z}\leqslant 0}\Delta^+$. We conclude that both sides are zero and so, by Theorem 3.9.1(d), $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$. □

Remark. In the first half of the argument above, the hypothesis that the cup-product map is surjective was used, along with Theorem III, to conclude that V_μ is a component of $\mathrm{V}_{\mu_1}\otimes\cdots\otimes\mathrm{V}_{\mu_k}$ and $\bigcap_{i=1}^k[\Omega_{w_i}]\cdot[\mathrm{X}_w]\neq 0$. If, on the other hand, we assume the latter two conditions then the second half of the argument still applies to give $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$. We will use this observation in Corollary 5.4.7 below.

5.2. ***Setup for the proof of sufficiency.*** For convenience, we collect some of the consequences of condition (1.2.1) in its symmetric form which have effectively appeared in previous arguments, and which we will use in the proof of sufficiency.

Proposition 5.2.1. *Suppose that $w_1,\dots,w_k$ are elements of the Weyl group such that $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$.*

Combinatorial Consequences:

(a) $\sum_{i=1}^k w_i^{-1}\cdot 0 = -2\rho$.

(b) *Suppose that $\lambda_1,\dots,\lambda_k$ are weights such that $\sum\lambda_i = -2\rho$, and set $\mu_i = w_i\cdot\lambda_i$ for $i = 1,\dots,k$. Then $\sum_{i=1}^k w_i^{-1}\mu_i = 0$.*

Geometric Consequences: *For each $i = 1,\dots,k$, let $v_i = w_i^{-1}w_0$ and let $\underline{v}_i$ be a word which is a reduced factorization of v_i. We set $\underline{\boldsymbol{v}} = (\underline{v}_1,\dots,\underline{v}_k)$ and construct as usual the variety $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ and the map $f_{\underline{\boldsymbol{v}}}:\mathrm{Y}_{\underline{\boldsymbol{v}}}\to\mathrm{X}^k$.*

Then

(c) $\deg(f_{\underline{\boldsymbol{v}}})\neq 0$.

(d) *The weight of the relative canonical bundle $\mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{K}_{\mathrm{X}^k}}$ at $p_{\underline{\boldsymbol{v}}}$ is zero.*

Proof. Part (a) is immediate from the condition $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$ and formula (2.2.4). Part (b) reverses the argument used to arrive at (5.1.1) in Section 5.1:

$$\sum_{i=1}^k \mu^{-1}\mu_i = \left(\sum_{i=1}^k w_i^{-1}\cdot\mu_i\right) - \left(\sum_{i=1}^k w_i^{-1}\cdot 0\right) = \sum_{i=1}^k \lambda_i - (-2\rho) = 0.$$

Part (c) is Corollary 3.7.5 combined with Lemma 2.6.1. Part (d) is the symmetric version of the computation in the proof of Theorem 3.9.1(d): the weight of the relative canonical bundle $\mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}$ at $p_{\underline{\boldsymbol{v}}}$ is $\sum_{i=1}^k w_i^{-1}\cdot 0 + 2\rho$, which is zero by part (a). □

5.3. ***Proof that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$ is a sufficient condition for surjectivity.*** Consider the symmetric version of the problem as in Section 2.7. It suffices to show the surjectivity of a cup-product map

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}}), \tag{5.3.1}$$

where $w_1, \ldots, w_k$ are elements of the Weyl group such that $\sum \ell(w_i) = \mathrm{N}$, $\lambda_1, \ldots, \lambda_k$ are weights such that $w_i \cdot \lambda_i \in \Lambda^+$ for $i = 1, \ldots, k$ and $\sum \lambda_i = -2\rho$. After this reduction condition (1.2.1) becomes $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$. We recall the notation from Section 4.3: $v_i := w_i^{-1} w_0$, $\lambda'_i := v_i^{-1} \cdot \lambda_i$, $\underline{v}_i$ is a reduced factorization of v_i, $\underline{\boldsymbol{v}} = (\underline{v}_1, \ldots, \underline{v}_k)$, and $\underline{\lambda}' = (\lambda'_1, \ldots, \lambda'_k)$.

By Proposition 4.3.2, to show the surjectivity of (5.3.1) it is enough to show that the pullback map

$$\mathrm{H}^{k\mathrm{N}}(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^* \mathrm{L}_{\underline{\lambda}'})^{\mathrm{G}} \xleftarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^{k\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'})^{\mathrm{G}} \tag{5.3.2}$$

on the space of G-invariants is surjective. We will show that both spaces of G-invariants are one-dimensional, and that the induced map is an isomorphism. Note that by Proposition 5.2.1(c) $\deg(f_{\underline{\boldsymbol{v}}}) \neq 0$ and so $f_{\underline{\boldsymbol{v}}}$ is surjective.

The pullback map on top cohomology is Serre dual to the trace map:

$$\mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, (f_{\underline{\boldsymbol{v}}}^* \mathrm{L}_{\underline{\lambda}'})^* \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}}) = \mathrm{H}^0\big(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*(\mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}\big) \xrightarrow{\mathrm{Tr}_{f_{\underline{\boldsymbol{v}}}}} \mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}).$$

Let $s \in \mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k})^{\mathrm{G}}$ be the nonzero G-invariant section giving the map $f_{\underline{\boldsymbol{v}}}^* \mathrm{K}_{\mathrm{X}^k} \to \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}}$ induced by $f_{\underline{\boldsymbol{v}}}$. The composition

$$\mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) \xrightarrow{f_{\underline{\boldsymbol{v}}}^*} \mathrm{H}^0\big(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*(\mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k})\big) \xrightarrow{\cdot s} \mathrm{H}^0\big(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*(\mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}\big)$$
$$\xrightarrow{\mathrm{Tr}_{f_{\underline{\boldsymbol{v}}}}} \mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k})$$

of pullback, multiplication by s, and the trace map is multiplication by $\deg(f_{\underline{\boldsymbol{v}}})$, which is nonzero. This gives us the inequality

$$\dim \mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k})^{\mathrm{G}} \leqslant \dim \mathrm{H}^0\big(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f_{\underline{\boldsymbol{v}}}^*(\mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}\big)^{\mathrm{G}} \tag{5.3.3}$$

and shows that in order to prove that the trace map induces an isomorphism on G-invariants it is sufficient to prove that we have equality of dimensions in (5.3.3).

Set $\mu_i = w_i \cdot \lambda_i = w_0 \cdot \lambda'_i$ for $i = 1, \ldots, k$. By the Borel–Weil–Bott Theorem we have $\mathrm{H}^{k\mathrm{N}}(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}) = \mathrm{V}_{\mu_1}^* \otimes \cdots \otimes \mathrm{V}_{\mu_k}^*$ and so (by Serre duality) $\mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\lambda}'}^* \otimes \mathrm{K}_{\mathrm{X}^k}) = \mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$. Now set $\nu_i = -w_0 \mu_i$ for $i = 1, \ldots, k$ so that $\mathrm{V}_{\nu_i} = \mathrm{V}_{\mu_i}^*$ and let $\underline{\nu} = (\nu_1, \ldots, \nu_k)$. By the calculation

$$\mathrm{S}(\lambda'_i) = -\lambda'_i - 2\rho = -w_0 \cdot \mu_i - 2\rho = -(w_0 \mu_i - 2\rho) - 2\rho = -w_0 \mu_i$$

in each coordinate factor (as in Section 2.5), we conclude that $\mathrm{L}^*_{\underline{\lambda}'} \otimes \mathrm{K}_{\mathrm{X}^k} = \mathrm{L}_{\underline{\nu}}$.

The weight of $\mathrm{L}_{\underline{\nu}}$ at $q := f_{\underline{\boldsymbol{v}}}(p_{\underline{\boldsymbol{v}}}) = (v_1, \ldots, v_k)$ is

$$-\sum_{i=1}^{k} v_i \nu_i = \sum_{i=1}^{k} (w_i^{-1} w_0)(w_0 \mu_i) = \sum_{i=1}^{k} w_i^{-1} \mu_i = 0,$$

where the last equality is due to Proposition 5.2.1(b). Since by Proposition 5.2.1(d) the weight of $\mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}$ at $p_{\underline{\boldsymbol{v}}}$ is zero, the weight of $f^*_{\underline{\boldsymbol{v}}}\mathrm{L}_{\underline{\nu}} \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k}$ at $p_{\underline{\boldsymbol{v}}}$ in $\mathrm{Y}_{\underline{\boldsymbol{v}}}$ is also zero and hence

$$\dim \mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f^*_{\underline{\boldsymbol{v}}}\mathrm{L}_{\underline{\nu}} \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k})^{\mathrm{G}} \leqslant 1$$

by Lemma 3.8.1(c). On the other hand, Lemma 2.3.1 implies that

$$(\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k})^{\mathrm{G}} \neq 0$$

so we conclude that $\dim \mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\nu}})^{\mathrm{G}} \geqslant 1$. This gives us

$$1 \leqslant \dim \mathrm{H}^0(\mathrm{X}^k, \mathrm{L}_{\underline{\nu}})^{\mathrm{G}} \leqslant \dim \mathrm{H}^0(\mathrm{Y}_{\underline{\boldsymbol{v}}}, f^*_{\underline{\boldsymbol{v}}}\mathrm{L}_{\underline{\nu}} \otimes \mathrm{K}_{\mathrm{Y}_{\underline{\boldsymbol{v}}}/\mathrm{X}^k})^{\mathrm{G}} \leqslant 1.$$

Therefore the inequality in (5.3.3) is an equality, and the cup-product map in (5.3.1) is surjective. □

5.4. *Corollaries of Theorem I and its proof.*

Corollary 5.4.1. *The cup-product map*

$$\mathrm{H}^0(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \mathrm{H}^d(\mathrm{X}, \mathrm{L}_{\lambda_2}) \to \mathrm{H}^d(\mathrm{X}, \mathrm{L}_{\lambda_1} \otimes \mathrm{L}_{\lambda_2})$$

is surjective whenever both sides are nonzero.

Proof. If w_2 is the element of the Weyl group so that $w_2 \cdot \lambda_2 \geqslant 0$ then the conditions that λ_1 is dominant and that $\mathrm{L}_{\lambda_1+\lambda_2}$ has cohomology in the same degree d as L_{λ_2} imply that $w_2 \cdot (\lambda_1 + \lambda_2) \geqslant 0$, and so the corollary follows from Theorem I and the obvious statement that $\Phi_{w_2} = \Phi_{w_2} \sqcup \Phi_e$. □

Corollary 5.4.2 (compatibility with Leray spectral sequence). *Suppose that* λ_1, λ_2, *and* $\lambda = \lambda_1 + \lambda_2$ *are regular weights and that the cup-product map*

$$\mathrm{H}^{d_1}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \mathrm{H}^{d_2}(\mathrm{X}, \mathrm{L}_{\lambda_2}) \xrightarrow{\cup} \mathrm{H}^d(\mathrm{X}, \mathrm{L}_{\lambda})$$

is nonzero. Let P *be any parabolic subgroup of* G *containing* B, *and*

$$\pi : \mathrm{X} \to \mathrm{M} := \mathrm{G}/\mathrm{P}$$

be the corresponding projection. Then the cup-product map on X *factors as a composition*

$$\begin{tikzcd}
\mathrm{H}^{d_1}(\mathrm{X},\mathrm{L}_{\lambda_1})\otimes\mathrm{H}^{d_2}(\mathrm{X},\mathrm{L}_{\lambda_2}) \arrow[rr, "\cup_{\mathrm{X}}"] \arrow[d, equal] & & \mathrm{H}^{d}(\mathrm{X},\mathrm{L}_{\lambda}) \arrow[d, equal] \\
\mathrm{H}^{d_1-i}(\mathrm{M},\mathrm{R}^i_{\pi*}\mathrm{L}_{\lambda_1})\otimes\mathrm{H}^{d_2-j}(\mathrm{M},\mathrm{R}^j_{\pi*}\mathrm{L}_{\lambda_2}) \arrow[dr, "\cup_{\mathrm{M}}"] & & \mathrm{H}^{d-i-j}(\mathrm{M},\mathrm{R}^{i+j}_{\pi*}\mathrm{L}_{\lambda}) \\
& \mathrm{H}^{d-i-j}(\mathrm{M},\mathrm{R}^i_{\pi*}\mathrm{L}_{\lambda_1}\otimes\mathrm{R}^j_{\pi*}\mathrm{L}_{\lambda_2}) \arrow[ur, "\cup_{\pi}"] &
\end{tikzcd}$$

of the cup product on M *followed by the map induced on cohomology by the relative cup-product map* $\mathrm{R}^i_{\pi*}\mathrm{L}_{\lambda_1}\otimes\mathrm{R}^j_{\pi*}\mathrm{L}_{\lambda_2}\to\mathrm{R}^{i+j}_{\pi*}\mathrm{L}_{\lambda}$ *on the fibers of* π*. A similar statement holds for the cup product of an arbitrary number of factors.*

Proof. The factorization statement amounts to a numerical condition on the cohomology degrees of the line bundles on the fibers of π ensuring that the cup-product map is computed by the map on E_2-terms of the Leray spectral sequence. This numerical condition is immediately implied by (1.2.1). We explain this in more detail below.

Set $\mathrm{L}=\mathrm{L}_{\lambda_1}\boxtimes\mathrm{L}_{\lambda_2}$ on $\mathrm{X}\times\mathrm{X}$ and consider the following factorization of the diagonal map $\delta_{\mathrm{X}}:\mathrm{X}\hookrightarrow\mathrm{X}\times\mathrm{X}$:

$$\begin{tikzcd}
\mathrm{X} \arrow[r, hook, "s"] \arrow[d, "\pi"] & \mathrm{X}\times_{\mathrm{M}}\mathrm{X} \arrow[r, hook, "t"] \arrow[d, "\psi"] \arrow[dr, phantom, "\square"] & \mathrm{X}\times\mathrm{X} \arrow[d, "\pi\times\pi"] \\
\mathrm{M} \arrow[r, equal] & \mathrm{M} \arrow[r, hook, "\delta_{\mathrm{M}}"] & \mathrm{M}\times\mathrm{M}
\end{tikzcd} \tag{5.4.3}$$

The cup-product map then factors as

$$\mathrm{H}^d(\mathrm{X},\mathrm{L}_\lambda)\xleftarrow{s^*}\mathrm{H}^d(\mathrm{X}\times_{\mathrm{M}}\mathrm{X},t^*\mathrm{L})\xleftarrow{t^*}\mathrm{H}^{d_1}(\mathrm{X},\mathrm{L}_{\lambda_1})\otimes\mathrm{H}^{d_2}(\mathrm{X},\mathrm{L}_{\lambda_2})=\mathrm{H}^d(\mathrm{X}\times\mathrm{X},\mathrm{L}) \tag{5.4.4}$$

and we claim that (5.4.4) induces the factorization claimed above.

By the Borel–Weil–Bott theorem applied to the fibers of π, for each of the line bundles L_{λ_1}, L_{λ_2}, and L_λ there is precisely one degree for which the higher direct image sheaf is nonzero. Suppose i is the degree such that $\mathrm{R}^i_{\pi*}\mathrm{L}_{\lambda_1}\neq 0$, j is the degree such that $\mathrm{R}^j_{\pi*}\mathrm{L}_{\lambda_2}\neq 0$, and k is the degree such that $\mathrm{R}^k_{\pi*}\mathrm{L}_\lambda\neq 0$. The Leray spectral sequence for the cohomology of these bundles degenerates at the E_2 term and we have the isomorphisms $\mathrm{H}^{d_1}(\mathrm{X},\mathrm{L}_{\lambda_1})=\mathrm{H}^{d_1-i}(\mathrm{M},\mathrm{R}^i_{\pi*}\mathrm{L}_{\lambda_1})$, $\mathrm{H}^{d_2}(\mathrm{X},\mathrm{L}_{\lambda_2})=\mathrm{H}^{d_2-j}(\mathrm{M},\mathrm{R}^j_{\pi*}\mathrm{L}_{\lambda_2})$, and $\mathrm{H}^d(\mathrm{X},\mathrm{L}_{\lambda_1})=\mathrm{H}^{d-k}(\mathrm{M},\mathrm{R}^k_{\pi*}\mathrm{L}_\lambda)$.

Since $R^{i+j}_{\pi\times\pi *}L = R^i_{\pi *}L_{\lambda_1} \boxtimes R^j_{\pi *}L_{\lambda_2}$ is a vector bundle on $M\times M$, the theorem on cohomology and base change gives us

$$R^{i+j}_{\psi *}t^*L = \delta_M^*(R^i_{\pi *}L_{\lambda_1} \boxtimes R^j_{\pi *}L_{\lambda_2}) = R^i_{\pi *}L_{\lambda_1} \otimes R^j_{\pi *}L_{\lambda_2} \quad \text{on } M,$$

and therefore we have $H^d(X\times_M X, t^*L) = H^{d-i-j}(M, R^i_{\pi *}L_{\lambda_1} \otimes R^j_{\pi *}L_{\lambda_2})$. The Leray spectral sequences for L and t^*L with respect to ψ and $\pi\times\pi$ also degenerate at the E_2-terms and have nonzero terms in the same degree. The discussion in Section 2.9 implies that the map on E_2-terms computes the pullback map t^*. Therefore t^* in (5.4.4) is equal to the map

$$H^{d-i-j}(M, R^i_{\pi *}L_{\lambda_1} \otimes R^j_{\pi *}L_{\lambda_2}) \xleftarrow{\delta_M^*} H^{d_1-i}(M, R^i_{\pi *}L_{\lambda_1}) \otimes H^{d_2-j}(M, R^j_{\pi *}L_{\lambda_2}),$$

which shows that t^* is the first part of the factorization claimed.

We now study s^*. The map s includes X as the relative diagonal of $X\times_M X$ over M. It follows that s^* induces the relative cup-product map on the higher direct image sheaves of t^*L and L_λ. Therefore the map associated to s^* on the E_2-terms of the Leray spectral sequences for t^*L and L_λ is given by the relative cup-product map

$$H^{d-i-j}(M, R^{i+j}L_\lambda) = H^{d-i-j}(M, R^{i+j}(L_{\lambda_1}\otimes L_{\lambda_2})) \xleftarrow{\cup_\pi} H^{d-i-j}(M, R^i_{\pi *}L_{\lambda_1} \otimes R^j_{\pi *}L_{\lambda_2}).$$

All that is needed to demonstrate the factorization claimed is to demonstrate the condition $k = i + j$ which ensures the map on the associated graded pieces in the E_2-terms agrees with the global map on the cohomology groups (see Section 2.9).

Suppose that w_1, w_2, and w are the elements of the Weyl group such that $w_1\cdot\lambda_1$, $w_2\cdot\lambda_2$, and $w\cdot\lambda$ are dominant. Then

$$k = \#(\Phi_w \cap -\Delta_P), \quad i = \#(\Phi_{w_1} \cap -\Delta_P), \quad j = \#(\Phi_{w_2} \cap -\Delta_P),$$

where the symbol # indicates the cardinality of a set. The condition $k = i + j$ guaranteeing the factorization thus amounts to the condition

$$\#(\Phi_w \cap -\Delta_P) = \#(\Phi_{w_1} \cap -\Delta_P) + \#(\Phi_{w_2} \cap -\Delta_P). \tag{5.4.5}$$

Since the original cup-product map was assumed surjective we must have $\Phi_w = \Phi_{w_1} \sqcup \Phi_{w_2}$ by Theorem I; this immediately implies that (5.4.5) holds. □

Corollary 5.4.6. *Suppose that $w_1, \dots, w_k$ are elements of the Weyl group such that $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$, and that $\mu_1, \dots, \mu_k$ are dominant weights satisfying the condition $\sum_{i=1}^k w_i^{-1}\mu_i = 0$. Then* $\dim(V_{\mu_1}\otimes\cdots\otimes V_{\mu_k})^G = 1$.

Proof. Set $\lambda_i = w_i^{-1}\cdot\mu_i$ for $i = 1, \dots, k$. Then $\sum\lambda_i = -2\rho$ and we have a cup-product problem as in (5.3.1). As part of the proof of Theorem I in Section 5.3 it was established that $\dim(V_{\mu_1}\otimes\cdots\otimes V_{\mu_k})^G = 1$. Alternatively, the corollary is simply Theorem 3.9.1(b) applied in symmetric form, with Lemma 2.6.1 used to ensure that the hypotheses of the theorem are satisfied. □

Corollary 5.4.7. *Suppose that we have a cup-product map*

$$\mathrm{H}^{\ell(w_1)}(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(w_k)}(\mathrm{X}, \mathrm{L}_{\lambda_k}) \xrightarrow{\cup} \mathrm{H}^d(\mathrm{X}, \mathrm{L}_\lambda)$$

and, as above, Weyl group elements $w_1, \ldots, w_k$, *and* w *such that* $\mu_i := w_i \cdot \lambda_i$, $i = 1, \ldots, k$, *and* $\mu := w \cdot \mu$ *are dominant weights. Then if* $\bigcap_{i=1}^k [\Omega_{w_i}] \cdot [\mathrm{X}_w] \neq 0$ *the cup-product map is surjective if and only if* V_μ *is a component of* $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$.

Proof. If V_μ is not a component of the tensor product the map is clearly not surjective. Conversely, if V_μ is a component, the assumption on the intersection number and the argument in Section 5.1 for the necessity of condition (1.2.1) show that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$, and therefore we conclude that the map is surjective by the sufficiency of condition (1.2.1). □

The following example illustrates Corollary 5.4.7 and provides an example which shows that condition (1.2.1) is not necessary in order to have a cup-product problem for which both sides are nonzero.

Example 5.4.8. Let $\mathrm{G} = \mathrm{SL}_6$ and $w_1 = w_2 = s_2 s_4 s_3$. For any integers $a_i, b_i \geqslant 0$ $(i = 1, 2)$ set

$$\mu_i = (0, a_i, 0, b_i, 0) \quad \text{and} \quad \lambda_i = w_i^{-1} \cdot \mu_i = (a_i+1, b_i+1, -4-a_i-b_i, a_i+1, b_i+1).$$

(The weights are written in terms of the fundamental weights of SL_6.) Finally, let $w = s_1 s_3 s_5 s_2 s_4 s_3$ and set

$$\mu = w \cdot (\lambda_1 + \lambda_2) = (0, a_1 + a_2 + 1, 0, b_1 + b_2 + 1, 0) \in \Lambda^+.$$

We therefore get a cup-product problem:

$$\mathrm{H}^3(\mathrm{X}, \mathrm{L}_{\lambda_1}) \otimes \mathrm{H}^3(\mathrm{X}, \mathrm{L}_{\lambda_2}) \xrightarrow{\cup} \mathrm{H}^6(\mathrm{X}, \mathrm{L}_{\lambda_1+\lambda_2}).$$

By Theorem I, this cup product cannot be surjective, since $\Phi_{w_1} = \Phi_{w_2}$; alternatively, the map cannot be surjective since V_μ is clearly not a component of $\mathrm{V}_{\mu_1} \otimes \mathrm{V}_{\mu_2}$. The intersection number $([\Omega_{w_1}] \cap [\Omega_{w_2}]) \cdot \mathrm{X}_w$ is two.

Corollary 5.4.9. *If* $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{w_i}$ *then for any subset* $\mathrm{I} \subseteq \{1, \ldots, k\}$ *there is an element* w *of the Weyl group such that* $\Phi_w = \bigsqcup_{i \in \mathrm{I}} \Phi_{w_i}$.

Proof. Let $\lambda_i = w_i^{-1} \cdot 0$ so that we get a cup-product problem as in (5.3.1). (Here each $\mathrm{H}^{\ell(w_i)}(\mathrm{X}, \mathrm{L}_{\lambda_i})$ is the trivial G-module). By Theorem I and the assumption on $w_1, \ldots, w_k$ this cup product is surjective. It can be factored by first taking the cup product of any subset $\mathrm{I} \subseteq \{1, \ldots, k\}$ of the factors and the resulting cup-product problem must also be nonzero since the larger problem is. Hence by Theorem I there is a $w \in \mathcal{W}$ with $w \cdot \left(\sum_{i \in \mathrm{I}} \lambda_i\right) \in \Lambda^+$ and such that $\Phi_w = \bigsqcup_{i \in \mathrm{I}} \Phi_{w_i}$. □

5.5. ***Comments.*** (1) Corollary 5.4.9 can also be proved independently of any of the constructions in this paper by using a similar argument in nilpotent cohomology. We are grateful to Olivier Mathieu for pointing this out to us.

(2) Using the result of Corollary 5.4.9 and induction, to prove Theorem I it is sufficient to prove it in the case $k = 2$ of the cup product of two cohomology groups into a third. We have chosen to develop the description of the varieties $\mathrm{Y}_{\underline{v}}$ for arbitrary k partly since this is the natural generality of the construction, partly because it makes no difference in our proofs, but also because some of the applications (e.g., the multiplicity bounds) do not follow by induction. Note that by the methods of this paper, even to prove the case $k = 2$ of the cup product it would be necessary to consider the case of the cup product of three factors into $\mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}})$, and hence we would need the construction of $\mathrm{Y}_{\underline{v}}$ for three factors.

(3) As Example 5.4.8 shows, the natural numerical condition $\ell(w_1)+\ell(w_2)=\ell(w)$ does not imply condition (1.2.1) even if there is a nontrivial cup-product problem corresponding to w_1, w_2, and w. On the other hand, condition (5.4.5) imposes further necessary numerical conditions for (1.2.1). Namely,

$$\ell(w_1^{\mathrm{P}}) + \ell(w_2^{\mathrm{P}}) = \ell(w^{\mathrm{P}}) \quad \text{for every parabolic subgroup } \mathrm{P} \supseteq \mathrm{B} \text{ of } \mathrm{G}, \tag{5.5.1}$$

where w_1^{P}, w_2^{P}, w^{P} denote the minimal length representatives in $w_1\mathcal{W}_{\mathrm{P}}$, $w_2\mathcal{W}_{\mathrm{P}}$, $w\mathcal{W}_{\mathrm{P}}$. In the case when $\mathrm{G} = \mathrm{SL}_{n+1}$ one can show that condition (5.5.1) is sufficient for (1.2.1). The simple inductive argument relies on the fact that if $\mathrm{G} = \mathrm{SL}_{n+1}$ it is possible to assign a parabolic $\mathrm{P}_\alpha \supset \mathrm{B}$ to every root $\alpha \in \Delta^+$ in such a way that $-\alpha$ is a root of P_α but not a root of any proper parabolic subgroup of P_α containing B. We do not know if (5.5.1) is sufficient to imply (1.2.1) for general G.

(4) Corollary 5.4.2 establishes the following factorization property: any nonzero cup-product map on X factors as a cup product on G/P and fibers of $\pi : \mathrm{X} \to \mathrm{G/P}$ for all $\mathrm{P} \supset \mathrm{B}$. We know of no a priori reason why this should hold. The factorization property is equivalent to (5.4.5) holding for all $\mathrm{P} \supset \mathrm{B}$ which is equivalent to (5.5.1). Hence, in the case $\mathrm{G} = \mathrm{SL}_{n+1}$ the factorization property is equivalent to (1.2.1).

6. Cohomological components and proof of Theorem II

6.1. ***Conditions on components of tensor products.*** We begin by introducing two relevant conditions. We also recall the notion of generalized PRV component from Section 2.3 for convenience.

Definitions 6.1.1. Suppose that $\mu_1, \dots, \mu_k$, and μ are dominant weights.

(a) We say that V_μ is a *generalized PRV component* of $\mathrm{V}_{\mu_1} \otimes \cdots \otimes \mathrm{V}_{\mu_k}$ if there exist $w_1, \dots, w_k$, and w in $\mathcal{W}$ such that $w^{-1}\mu = \sum_{i=1}^{k} w_i^{-1}\mu_i$.

(b) We say that V_μ is a component of *stable multiplicity one* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if we have $\dim(V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k} \otimes V^*_{m\mu})^G = 1$ for all $m \gg 0$.

(c) We say that V_μ is a *cohomological component* of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ if there exist $w_1, \ldots, w_k$, and w in $\mathcal{W}$ such that $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$ and such that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$.

Under the hypothesis that $\Phi_w = \bigsqcup_{i=1}^k \Phi_{w_i}$, the condition $w^{-1}\mu = \sum_{i=1}^k w_i^{-1}\mu_i$ is equivalent to the condition $w^{-1} \cdot \mu = \sum_{i=1}^k w_i^{-1} \cdot \mu_i$. Therefore by Theorem I condition (c) is equivalent to having a surjective cup-product map

$$H^{\ell(w_1)}(X, L_{w_1^{-1}\cdot\mu_1}) \otimes \cdots \otimes H^{\ell(w_k)}(X, L_{w_k^{-1}\cdot\mu_k}) \xrightarrow{\cup} H^{\ell(w)}(X, L_{w^{-1}\cdot\mu})$$

which, after dualizing, gives an injective map

$$V_\mu \to V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}.$$

In other words, we obtain a construction of V_μ as a component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ realized through the cohomology of X.

Note that the conditions in Definitions 6.1.1 are *homogeneous*: if V_μ is a generalized PRV component, a component of stable multiplicity one, or a cohomological component of $V_{\mu_1} \otimes \cdots \otimes V_{\mu_k}$ then the same is true of $V_{m\mu}$ as a component of $V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k}$ for all $m \geqslant 1$. This follows immediately from the definitions.

6.2. *Proof of Theorem II*(a) *and restatement of Theorem II*(b).

Proof of Theorem II(a). Every cohomological component has multiplicity one by Theorem 3.9.1(b) (the condition on nonzero intersection holds by the nonsymmetric version of Lemma 2.6.1). By homogeneity we conclude that homological components are of stable multiplicity one. From Definitions 6.1.1(a, c) it is clear that every cohomological component is a generalized PRV component. Thus every cohomological component is a generalized PRV component of stable multiplicity one. □

For the proof of part (b) it will be more convenient to work with the symmetric form of the problem. Applying the symmetrization procedure from Section 2.7 (and replacing $k+1$ by k) we obtain the following reformulation of Theorem II(b).

Proposition 6.2.1. *Let $\mu_1, \ldots, \mu_k$ be dominant weights such that*

$$\dim(V_{m\mu_1} \otimes \cdots \otimes V_{m\mu_k})^G = 1 \quad \textit{for } m \gg 0$$

and suppose that we have elements $w_1, \ldots, w_k$ such that $\sum w_i^{-1}\mu_i = 0$. Then in either of the following two cases:

(i) *at least one of $\mu_1, \ldots, \mu_k$ is strictly dominant*,

(ii) G *is a classical simple group or product of classical simple groups*, *there exist $\bar{w}_1, \dots, \bar{w}_k \in \mathcal{W}$ such that*

$$\sum_{i=1}^{k} \bar{w}_i^{-1}\mu_i = 0 \quad \textit{and} \quad \Delta^+ = \bigsqcup_{i=1}^{k} \Phi_{\bar{w}_i}. \tag{6.2.2}$$

The proof of Proposition 6.2.1 will be given in Section 6.8 after some preliminary reduction steps.

For the rest of this section we assume that we have fixed dominant weights $\mu_1, \dots, \mu_k$ and Weyl group elements $w_1, \dots, w_k$ satisfying the conditions of Proposition 6.2.1.

6.3. ***Outline of the proof of Proposition 6.2.1.*** For $i = 1, \dots, k$, let P_i be the parabolic subgroup of G such that L_{μ_i} is the pullback to X of an ample line bundle $\mathrm{L}_{\tilde{\mu}_i}$ on G/P_i. Set $\mathrm{M} = \mathrm{G}/\mathrm{P}_1 \times \cdots \times \mathrm{G}/\mathrm{P}_k$ and $\mathrm{L} = \mathrm{L}_{\tilde{\mu}_1} \boxtimes \cdots \boxtimes \mathrm{L}_{\tilde{\mu}_k}$. The condition that $\dim(\mathrm{V}_{m\mu_1} \otimes \cdots \otimes \mathrm{V}_{m\mu_k})^{\mathrm{G}} = 1$ for all $m \gg 1$ implies that the GIT quotient $\mathrm{M}/\!\!/\mathrm{G}$ with respect to L is a point.

If $w_1, \dots, w_k$ are elements such that $\sum_{i=1}^{k} w_i^{-1}\mu_i = 0$ then by Lemma 2.11.1, the point $q = (w_1^{-1}, \dots, w_k^{-1})$ is a semistable point of M with a closed orbit. Let $\mathrm{H} \subseteq \mathrm{G}$ be the stabilizer subgroup of q, and $\mathcal{N}_q$ be the normal space to the orbit at q. By the Luna slice theorem and the fact that the GIT quotient $\mathrm{M}/\!\!/\mathrm{G}$ is a point we conclude that $\mathrm{Sym}^{\cdot}(\mathcal{N}_q)^{\mathrm{H}}$ is one-dimensional.

The explicit combinatorial formula for the weights appearing in $\mathcal{N}_q$ shows that a necessary condition for a solution of (6.2.2) to exist is that there is $v \in \mathcal{W}$ such that the weights of $v\mathcal{N}_q$ are contained in Δ^-. In Proposition 6.6.1 below we formulate a condition which, together with the necessary condition above, guarantees the existence of a solution of (6.2.2). Together these two conditions are equivalent to the existence of a parabolic subalgebra $\mathfrak{p}$ with reductive part Lie(H) such that the weights of $\mathcal{N}_q$ are contained in $\mathfrak{p}$.

Finally, we use the restriction that $\mathrm{Sym}^{\cdot}(\mathcal{N}_q)^{\mathrm{H}}$ is one-dimensional to show the existence of such a parabolic subalgebra when G is a classical group, or for any semisimple group G under a genericity condition.

6.4. ***Stabilizer subgroup of a semistable* T*-fixed point.*** Let P_i be the parabolic with roots $\Delta_{\mathrm{P}_i} = \{\alpha \in \Delta \mid \kappa(\alpha, \mu_i) \geqslant 0\}$, and let $\mathrm{M}_i = \mathrm{G}/\mathrm{P}_i$. The stabilizer subgroup of the point w_i^{-1} in M_i is $w_i^{-1}\mathrm{P}_i w_i$, whose roots are

$$\Delta_{w_i^{-1}\mathrm{P}_i w_i} = \left\{\alpha \in \Delta \mid \kappa(w_i\alpha, \mu_i) \geqslant 0\right\} = \left\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) \geqslant 0\right\}. \tag{6.4.1}$$

Let $\mathrm{M} = \mathrm{M}_1 \times \cdots \times \mathrm{M}_k$ and let q be the point $q = (w_1^{-1}, \dots, w_k^{-1})$ of M. We set $\mathrm{H} = \bigcap_{i=1}^{k} w_i^{-1}\mathrm{P}w_i$ to be the stabilizer subgroup of q. The condition $\sum_{i=1}^{k} w_i^{-1}\mu_i = 0$ in combination with (6.4.1) shows that the roots of H are given by

$$\Delta_{\mathrm{H}} = \left\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) = 0 \text{ for } i = 1, \dots, k\right\}. \tag{6.4.2}$$

We conclude from (6.4.2) that H is a reductive subgroup of G. Noting that $\mathrm{T} \subseteq \mathrm{H}$, the following lemma is another immediate consequence of (6.4.2).

Lemma 6.4.3. *We have* $\mathrm{H} = \mathrm{T}$ *if and only if the span of* $\{w_i^{-1}\mu_i\}_{i=1}^k$ *intersects the interior of some Weyl chamber. This happens, for instance, if any one of the weights* μ_i *is strictly dominant.*

6.5. *Torus action at fixed points of* M *and combinatorial deductions.* Let $\mathcal{W}_i = \{w \in \mathcal{W} \mid w\mu_i = \mu_i\} \subseteq \mathcal{W}$ be the stabilizer subgroup of μ_i; this is the Weyl group of P_i. We will need the formula for the formal character of the tangent space of M_i at a torus fixed point. Because of the way that the inverses of group elements enter into our formulas we make the following convention: For any element w of $\mathcal{W}$ and any i we let $w_{s(i)}$ and $w_{l(i)}$ be respectively the shortest and longest elements in the coset $\mathcal{W}_i w$. Recall also that for $\Phi \subseteq \Delta$, $\langle\Phi\rangle$ denotes the formal character $\sum_{\alpha\in\Phi} e^{\alpha}$.

With this convention, if w_i is any element of $\mathcal{W}$, the formal character of the tangent space of M_i at the torus fixed point corresponding to the coset $w_i^{-1}\mathcal{W}_i$ is

$$\mathrm{Ch}(\mathrm{T}_{w_i^{-1}}\mathrm{M}_i) = \langle\Phi_{w_{i,s(i)}}\rangle + \langle-\Phi^{\mathrm{c}}_{w_{i,l(i)}}\rangle = \big\langle\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) < 0\}\big\rangle.$$

The formal character of the tangent space of M at q is therefore

$$\mathrm{Ch}(\mathrm{T}_q\mathrm{M}) = \sum_{i=1}^{k}\big(\langle\Phi_{w_{i,s(i)}}\rangle + \langle-\Phi^{\mathrm{c}}_{w_{i,l(i)}}\rangle\big) = \sum_{i=1}^{k}\big\langle\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) < 0\}\big\rangle. \quad (6.5.1)$$

Note that the multiplicity of each root α in the equations above is the number of i for which $\kappa(\alpha, w_i^{-1}\mu_i) < 0$.

If $\alpha \notin \Delta_{\mathrm{H}}$ then there is some i for which $\kappa(\alpha, w_i^{-1}\mu_i) \neq 0$ and hence, by the condition $\sum_{i=1}^{k} w_i^{-1}\mu_i = 0$, there is some i for which $\kappa(\alpha, w_i^{-1}\mu_i) < 0$, i.e., α must appear as a weight in $\mathrm{T}_q\mathrm{M}$. By looking at the positive roots of $\mathrm{T}_q\mathrm{M}$ we therefore conclude that

$$(\Delta^+ \setminus \Delta^+_{\mathrm{H}}) = \bigcup \Phi_{w_{i,s(i)}}. \quad (6.5.2)$$

Let O_q be the G-orbit of q in M. Since H is the stabilizer of q, the formal character of the tangent space $\mathrm{T}_q\mathrm{O}_q$ is

$$\mathrm{Ch}(\mathrm{T}_q\mathrm{O}_q) = \langle\Delta^+ \setminus \Delta^+_{\mathrm{H}}\rangle + \langle\Delta^- \setminus \Delta^-_{\mathrm{H}}\rangle. \quad (6.5.3)$$

If $\mathcal{N}_q = \mathrm{T}_q\mathrm{M}/\mathrm{T}_q\mathrm{O}_q$ is the normal space to the orbit at q, then the union in (6.5.2) is disjoint if and only if the formal character of $\mathcal{N}_q$ contains no positive root.

Let $\mathcal{M}$ be the subspace of $\mathfrak{g}$ spanned by the root spaces corresponding to the roots appearing in $\mathcal{N}_q$. Comparing the multiplicities in (6.5.1) and (6.5.3) we conclude that the roots of $\mathcal{M}$ are

$$\Delta_{\mathcal{M}} = \big\{\alpha \in \Delta \mid \kappa(\alpha, w_i^{-1}\mu_i) < 0 \text{ for at least two } i \in \{1, \ldots, k\}\big\}. \quad (6.5.4)$$

Let $\mathfrak{s} = \mathrm{Lie(H)}$; equations (6.5.4) and (6.4.2) show that $\mathcal{M}$ is an $\mathfrak{s}$-submodule of $\mathfrak{g}$.

The point q is not the only torus fixed point in its orbit; for any $v \in \mathcal{W}$ we can act on the left to get the torus fixed point $vq = (vw_1^{-1}, \ldots, vw_k^{-1})$. The weights of the normal space $\mathcal{N}_{vq}$ to the G-orbit at vq are the result of acting on the weights of $\mathcal{N}_q$ by v and are hence the roots appearing in $\mathrm{Ch}(v\mathcal{M})$.

Repeating the previous arguments with the new point vq and the new stabilizer group $v\mathrm{H}v^{-1} = \mathrm{Stab}(vq)$, gives the following result.

Lemma 6.5.5. *For any $v \in \mathcal{W}$ we have*

$$(\Delta^+ \setminus \Delta^+_{v\mathrm{H}v^{-1}}) = \bigsqcup_{i=1}^{k} \Phi_{(w_i v^{-1})_{s(i)}}$$

if and only if $v\mathcal{M} \subseteq \mathfrak{b}^-$.

6.6. *Reduction to the existence of* $\mathfrak{p}_{\mathcal{M}}$.

Proposition 6.6.1. *Suppose that there exists $v \in \mathcal{W}$ satisfying the conditions*

(i) $v\mathcal{M} \subseteq \mathfrak{b}^-$,

(ii) *there is an element $w \in \mathcal{W}$ such that $\Phi_w = \Delta^+_{v\mathrm{H}v^{-1}}$.*

Then there exist $\bar{w}_1, \ldots, \bar{w}_k \in \mathcal{W}$ such that $\sum_{i=1}^k \bar{w}_i^{-1}\mu_i = 0$ and $\Delta^+ = \bigsqcup_{i=1}^k \Phi_{\bar{w}_i}$.

Proof. By condition (i) and Lemma 6.5.5 we have $(\Delta^+ \setminus \Delta^+_{v\mathrm{H}v^{-1}}) = \bigsqcup_{i=1}^k \Phi_{(w_i v^{-1})_{s(i)}}$. Set $\widetilde{w}_{k+1} = w$, $\mu_{k+1} = 0$, and $\widetilde{w}_i = (w_i v^{-1})_{s(i)}$ for $i = 1, \ldots, k$. Conditions (i) and (ii) above and the original assumption about $w_1, \ldots, w_k$ imply

$$\sum_{i=1}^{k+1} \widetilde{w}_i^{-1}\mu_i = 0 \quad \text{and} \quad \Delta^+ = \bigsqcup_{i=1}^{k+1} \Phi_{\widetilde{w}_i}. \tag{6.6.2}$$

Equation (6.6.2) and Theorem I show that there is a surjective cup-product map

$$\mathrm{H}^{\ell(\tilde{w}_1)}(\mathrm{X}, \mathrm{L}_{\tilde{w}_1^{-1}\cdot\mu_1}) \otimes \cdots \otimes \mathrm{H}^{\ell(\tilde{w}_{k+1})}(\mathrm{X}, \mathrm{L}_{\tilde{w}_{k+1}^{-1}\cdot\mu_{k+1}}) \xrightarrow{\cup} \mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}}).$$

Since $\mathrm{H}^{\ell(\tilde{w}_{k+1})}(\mathrm{X}, \mathrm{L}_{\tilde{w}_{k+1}^{-1}\cdot\mu_{k+1}})$ is the trivial module, if we factor the map above by cupping the k-th and $(k+1)$-st factors together first, we obtain a surjective cup-product map onto $\mathrm{H}^{\mathrm{N}}(\mathrm{X}, \mathrm{K}_{\mathrm{X}})$ only involving the modules $\mathrm{V}^*_{\mu_1}, \ldots, \mathrm{V}^*_{\mu_k}$. By invoking Theorem I again we conclude that there are $\bar{w}_1, \ldots, \bar{w}_k$ such that

$$\sum_{i=1}^{k} \bar{w}_i^{-1}\mu_i = 0 \quad \text{and} \quad \Delta^+ = \bigsqcup_{i=1}^{k} \Phi_{\bar{w}_i}, \tag{6.6.3}$$

proving Proposition 6.6.1. □

Remark. If there do exist $\bar{w}_1, \ldots, \bar{w}_k$ satisfying the conclusion of Proposition 6.2.1 it is not hard to show that there must exist $v \in \mathcal{W}$ so that (i) of Proposition 6.6.1 holds. As a consequence of our method of proof we see *a posteriori* that there must be a v so that both (i) and (ii) hold when G is a classical group or under a genericity condition. We do not know if condition (ii) is necessary in general.

It is useful to rephrase the conditions of Proposition 6.6.1 in terms of the existence of a particular parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$.

Lemma 6.6.4. *Let* $\mathfrak{s} = \mathrm{Lie}(\mathrm{H})$. *Suppose that there exists a parabolic subalgebra* $\mathfrak{p}_{\mathcal{M}}$ *with reductive part* $\mathfrak{s}$ *such that* $\mathcal{M} \subseteq \mathfrak{p}_{\mathcal{M}}$. *Then conditions* (i) *and* (ii) *of Proposition 6.6.1 hold.*

Proof. Let $\mathfrak{p}_{\mathcal{M}}$ be such a parabolic subalgebra. Acting by an element $v \in \mathcal{W}$ we can conjugate $\mathfrak{p}_{\mathcal{M}}$ so that $\mathfrak{b}^- \subseteq v\mathfrak{p}_{\mathcal{M}}$. This implies that $v\mathcal{M} \subseteq \mathfrak{b}^-$. Since $v\mathfrak{s}$ is the radical of a parabolic subalgebra containing $\mathfrak{b}^-$, if w is the longest element of the Weyl group of $v\mathfrak{s}$ then $\Phi_w = \Delta^+_{v\mathfrak{s}} = \Delta^+_{v\mathrm{H}v^{-1}}$. □

Remark. If there exists $v \in \mathcal{W}$ such that condition (ii) of Proposition 6.6.1 holds then one can show that $\mathfrak{p} := \mathfrak{b}^- + v\mathfrak{s}$ is a parabolic subalgebra of $\mathfrak{g}$. If condition (i) also holds for this v then $\mathfrak{p}_{\mathcal{M}} := v^{-1}\mathfrak{p}$ is a parabolic subalgebra satisfying the conditions of Lemma 6.6.4. Therefore the existence of the parabolic $\mathfrak{p}_{\mathcal{M}}$ is equivalent to the conditions in Proposition 6.6.1. Since we will not need this direction of the equivalence we omit the justification of the first assertion.

6.7. ***GIT consequences of the stable multiplicity one condition.*** Let L be the line bundle on M whose pullback to X^k is $\mathrm{L}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{L}_{\mu_k}$. Then L is a G-equivariant ample line bundle on M. By the stable multiplicity one condition we have $\dim(\mathrm{M}, \mathrm{L}^m)^{\mathrm{G}} = 1$ for all $m \gg 1$, and so the GIT quotient $\mathrm{M}/\!\!/\mathrm{G}$ is a point.

The weight of L at q is $\sum_{i=1}^k w_i^{-1}\mu_i = 0$. By Lemma 2.11.1 this means that q is a semistable point with a closed orbit. By the Luna slice theorem [1973, théorèm du slice étale, p. 97], $\mathrm{Spec}(\mathrm{Sym}(\mathcal{N}_q^*)^{\mathrm{H}})$ and the image of q in the GIT quotient $\mathrm{M}/\!\!/\mathrm{G}$ have a common étale neighborhood. Hence $\dim(\mathcal{N}_q/\mathrm{H}) = \dim(\mathrm{M}/\!\!/\mathrm{G}) = 0$, i.e., $\dim \mathrm{Sym}^{\cdot}(\mathcal{N}_q^*)^{\mathrm{H}} = 1$. Passing to the level of Lie algebras and dualizing we obtain $\dim \mathrm{Sym}^{\cdot}(\mathcal{N}_q)^{\mathfrak{s}} = 1$.

Since $\mathcal{M}$ is isomorphic to an $\mathfrak{s}$-submodule of $\mathcal{N}_q$ we arrive at the following consequence of the stable multiplicity one condition:

Lemma 6.7.1. *Under the hypotheses of Proposition 6.2.1 and with the notation of Section 6.5, we have* $\dim \mathrm{Sym}^{\cdot}(\mathcal{M})^{\mathfrak{s}} = 1$, *i.e.*, $\mathrm{Sym}^{\cdot}(\mathcal{M})^{\mathfrak{s}}$ *consists of just the constants.*

6.8. ***Proof of Proposition 6.2.1.*** By Proposition 6.6.1 and Lemma 6.6.4, to prove Proposition 6.2.1 it is enough to show the existence of the parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$. By Lemma 6.7.1 we may assume that $\dim \mathrm{Sym}^{\cdot}(\mathcal{M})^{\mathfrak{s}} = 1$.

Proof of 6.2.1(i). If any one of the weights $\mu_1, \ldots, \mu_k$ is strictly dominant, or more generally, if the span of $\{w_i^{-1}\mu_i\}_{i=1}^k$ intersects the interior of some Weyl chamber, then by Lemma 6.4.3 $\mathrm{H} = \mathrm{T}$ and so $\mathfrak{s} = \mathrm{Lie}(\mathrm{T}) = \mathfrak{t}$ and $\Delta_{\mathfrak{t}}^{+} = \varnothing$. The condition that $\dim(\mathrm{Sym}^{\cdot}(\mathcal{M})^{\mathfrak{t}}) = 1$ is then equivalent to the condition that no nontrivial nonnegative combination of weights of $\mathcal{M}$ is zero. Hence by Farkas's lemma the weights of $\mathcal{M}$ all lie strictly on one side of a hyperplane and the cone dual to the cone they span is open. We may therefore pick a weight in the interior of the dual cone which is not on any hyperplane of the Weyl chambers. The roots lying on the positive side of this hyperplane give the parabolic subalgebra $\mathfrak{p}_{\mathcal{M}}$. □

Proof of Proposition 6.2.1(ii). Equation (6.4.2) shows that the roots of $\mathfrak{s}$ are given by the vanishing of linear forms and hence $\mathfrak{s}$ is the reductive part of a parabolic subalgebra. Let $\mathfrak{a}$ be the center of $\mathfrak{s}$. For any $\nu \in \mathfrak{a}^* \setminus \{0\}$ set

$$\mathfrak{g}^{\nu} = \big\{x \in \mathfrak{g} \mid [t, x] = \nu(t)x \text{ for all } t \in \mathfrak{a}\big\}.$$

Following Kostant [2010], we call $\nu \in \mathfrak{a}^* \setminus \{0\}$ an $\mathfrak{a}$-root if $\mathfrak{g}^{\nu} \neq 0$. Let $\mathcal{R}$ be the set of $\mathfrak{a}$-roots of $\mathfrak{g}$ and $\mathcal{S}$ the subset of those $\mathfrak{a}$-roots appearing in $\mathcal{M}$, so that $\mathcal{M} = \bigoplus_{\nu \in \mathcal{S}} \mathfrak{g}^{\nu}$.

A subset $\mathcal{R}'$ of $\mathcal{R}$ is called *saturated* if whenever $\nu \in \mathcal{R}'$ and $r\nu \in \mathcal{R}$ for some $r \in \mathbb{Q}_{+}$ then $r\nu \in \mathcal{R}'$ as well. It follows from (6.5.4) that $\mathcal{S}$ is a saturated subset of $\mathcal{R}$.

As part of the main theorem of [Dimitrov and Roth 2017] we establish the following result.[2]

Theorem. *Let $\mathfrak{g}$ be a classical Lie algebra, $\mathfrak{s}$ be a subalgebra which is the reductive part of a parabolic subalgebra of $\mathfrak{g}$, $\mathcal{S}$ be a saturated subset of the $\mathfrak{a}$-roots $\mathcal{R}$, and $\mathcal{M} = \bigoplus_{\nu \in \mathcal{S}} \mathfrak{g}^{\nu}$. If $\dim(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = 1$, then there exists a parabolic subalgebra $\mathfrak{p}_{\mathcal{M}} \subseteq \mathfrak{g}$, with reductive part $\mathfrak{s}$, such that $\mathcal{M} \subseteq \mathfrak{p}_{\mathcal{M}}$.*

Thus when G is a simple classical group or a product of simple classical groups, the above theorem along with the previous reductions establish Proposition 6.2.1 and finish the proof of Theorem II. □

List of symbols

$\bigsqcup_{i=1}^{k}$	disjoint union
$\kappa(\cdot\,,\cdot)$	the Killing form of G
Λ, Λ^{+}	weight lattice and cone of dominant weights
V_{μ}	irreducible G-module of highest weight μ
$\mathrm{mult}(\mathrm{V}_{\mu}, \mathrm{V})$	the multiplicity of V_{μ} in V
$\{\alpha_1, \dots, \alpha_n\}$	base of simple roots of B
$\mathcal{W}$	Weyl group of $\mathfrak{g}$

[2]The proof is rather technical, and involves a case-by-case analysis of the different types, a characterization of the desired parabolics in terms of certain linearly ordered data, and an argument that the hypothesis $\dim(\mathrm{Sym}^{\cdot}(\mathcal{M}))^{\mathfrak{s}} = 1$ allows one to take a partial order constructed from $\mathcal{S}$ and extend it to a linear one.

$w \cdot \lambda$	$w(\lambda+\rho)-\rho$, the result of the affine action of $w \in \mathcal{W}$ on $\lambda \in \Lambda$
s_i	simple reflection along α_i
P_{α_i}	the minimal parabolic subgroup of G associated to α_i
P_{I}	the minimal parabolic subgroup of G associated to a set I of simple roots
$\mathcal{W}_{\mathrm{P}}$	the Weyl group of a parabolic subgroup $\mathrm{P} \subseteq \mathrm{G}$
$\mathrm{span}_{\mathbb{Z}_{\geqslant 0}} \Phi$	the set of nonnegative integer combinations of elements of $\Phi \subseteq \Delta$
$\underline{u}$ or $\underline{v}$	a word $s_{i_1} \cdots s_{i_m}$ in the simple reflections of the Weyl group
u, v	the element of $\mathcal{W}$ corresponding to $\underline{u}$ or $\underline{v}$
$\underline{v}_R$	the word obtained by dropping the rightmost reflection of $\underline{v}$
$\underline{\boldsymbol{u}}$ or $\underline{\boldsymbol{v}}$	a sequence $(\underline{u}_1, \ldots, \underline{u}_k)$ or $(\underline{v}_1, \ldots, \underline{v}_k)$ of words
Φ_w	$w^{-1}\Delta^- \cap \Delta^+$, the inversion set of $w \in \mathcal{W}$
$\langle \Phi \rangle$	$\sum_{\alpha \in \Phi} e^{\alpha}$, the formal character of $\bigoplus_{\alpha \in \Phi} \mathfrak{g}^{\alpha}$, where $\Phi \subset \Delta$
$\ell(w)$	the length of $w \in \mathcal{W}$
L_λ	the line bundle on X corresponding to the B-module on which T acts via $-\lambda$
N	the dimension of X
π_i	the projection $\pi_i : \mathrm{X} \to \mathrm{G}/\mathrm{P}_{\alpha_i}$ (a $\mathbb{P}^1$-fibration)

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