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A new equivariant in nonarchimedean dynamics

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Let K be a complete, algebraically closed nonarchimedean valued field, and let $\varphi(z) \in K(z)$ have degree $d \geq 2$. We show there is a canonical way to assign nonnegative integer weights $w_\varphi(P)$ to points of the Berkovich projective line over K in such a way that $\sum_P w_\varphi(P) = d-1$. When φ has bad reduction, the set of points with nonzero weight forms a distributed analogue of the unique point which occurs when φ has potential good reduction. Using this, we characterize the minimal resultant locus of φ in analytic and moduli-theoretic terms: analytically, it is the barycenter of the weight-measure associated to φ; moduli-theoretically, it is the closure of the set of points where φ has semistable reduction, in the sense of geometric invariant theory.

Let K be a complete, algebraically closed nonarchimedean valued field with absolute value $|\cdot|$ and associated valuation $\mathrm{ord}(\cdot)$. Write $\mathcal{O}$ for the ring of integers of K, $\mathfrak{m}$ for its maximal ideal, and $\tilde{k}$ for its residue field. Let $\mathbf{P}^1_K$ be the Berkovich projective line over K: a compact, path-connected Hausdorff space which contains $\mathbb{P}^1(K)$ as a dense subset (a nonmetric tree in the sense of [Favre and Jonsson 2004]).

Our main result is:

Theorem A. *Let $\varphi(z) \in K(z)$ be a rational function with degree $d \geq 2$. There is a canonical way, determined by the variation in the resultant under conjugation of φ, to assign nonnegative integer weights $w_\varphi(P)$ to points of $\mathbf{P}^1_K$ in such a way that*

$$\sum_P w_\varphi(P) = d-1,$$

where $w_\varphi(P) = 0$ for each $P \in \mathbb{P}^1(K)$.

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We call the finite set of points which receive weight the *crucial set* of φ. Write δ_P for the Dirac measure at a point $P \in \mathbf{P}^1_K$. We call the discrete probability measure

$$\nu_\varphi := \frac{1}{d-1} \sum_P w_\varphi(P) \delta_P$$

the *crucial measure* of φ.

The crucial set and crucial measure are conjugation equivariants of φ which live in the interior of the Berkovich projective line. It is natural to ask about their meaning. The crucial set, decorated with the weights, adjacency data, and the reductions of φ at its points, appears to be a kind of "analytic special fiber" for φ, analogous to the algebraic special fiber for a model of a curve over a nonarchimedean valued field. When φ has potential good reduction in the sense of nonarchimedean dynamics, the crucial set consists of the unique point where φ has good reduction and that point has weight $d-1$. When φ has bad reduction, the single point of good reduction splits into a collection of points with total weight $d-1$.

Explicit formulas for the weights $w_\varphi(P)$ are given Section 6, and there are only finitely many ways a point can receive weight. The crucial set, decorated with adjacency data and information about how each point receives weight, appears to the classify possible reduction types for a functions φ of given degree: the author and students in a University of Georgia VIGRE group have shown that for quadratic rational functions and cubic polynomials [Doyle et al. 2016; Rumely et al. $\geq$ 2017], the finitely many possible configurations of the crucial set lead to a stratification of the dynamical moduli space which governs where φ lies in the moduli space, and is compatible with reduction $(\bmod\, \mathfrak{m})$. This is analogous to the fact that for an elliptic curve E over a nonarchimedean local field, E has good reduction if and only if its j-invariant satisfies $|j_E| \leq 1$, while E has bad reduction (with semistable reduction a loop of curves), if and only if $|j_E| > 1$. It seems likely that such a stratification holds in general.

The crucial measure appears to be an intrinsic discrete approximation to the canonical invariant measure μ_φ supported on the Berkovich Julia set: Kenneth Jacobs [2014] has shown that the crucial measures of the iterates of φ converge weakly to μ_φ; for a given test function, the convergence is geometrically fast.

Our construction of the crucial set is an analytic one. It depends on the theory of the function $\operatorname{ordRes}_\varphi(\cdot)$ introduced in [Rumely 2015]. We associate to φ a canonical, finitely generated tree $\Gamma_{\mathrm{Fix,Repel}} \subset \mathbf{P}^1_K$. The weights $w_\varphi(P)$ and the crucial set are obtained by taking the Laplacian of $\operatorname{ordRes}_\varphi(\cdot)$ restricted to $\Gamma_{\mathrm{Fix,Repel}}$. It would be interesting to have a more algebraic construction.

Under conjugation of φ by an element $\gamma \in \mathrm{GL}_2(K)$ (given by $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$), the crucial set $\mathrm{Cr}(\varphi)$ and the crucial measure ν_φ have the same equivariance properties as fixed points: P is a fixed point of φ if and only if $\gamma^{-1}(P)$ is a fixed point of φ^γ.

Analogously, $\mathrm{Cr}(\varphi^\gamma) = \gamma^{-1}(\mathrm{Cr}(\varphi))$ and $\nu_{\varphi^\gamma} = \gamma^*(\nu_\varphi)$ (see Proposition 6.6). The construction of the crucial set applies in arbitrary characteristic. The crucial set is unchanged under enlargement of K (see Proposition 6.7), and is Galois-equivariant (see Proposition 6.8).

One application of the theory is as follows: by the formulas in Section 6, each repelling fixed point of φ in $\mathbf{H}_K^1 := \mathbf{P}_K^1 \setminus \mathbb{P}^1(K)$ receives weight. Hence φ can have at most $d-1$ nonclassical repelling fixed points (Corollary 6.2); previously it seems not to have been known that there were finitely many. In [Rumely 2014] it is shown that the bound $d-1$ is sharp.

A second application concerns the *minimal resultant locus* of φ, the set of points in $\mathbf{P}_K^1$ where, under change of coordinates, φ achieves its minimal resultant. We show that the minimal resultant locus is the *barycenter* of the crucial measure: the set of points P such that each component of $\mathbf{P}_K^1 \setminus \{P\}$ has ν_φ-mass at most $\frac{1}{2}$. We also show that the minimal resultant locus characterizes the elements $\gamma \in \mathrm{GL}_2(K)$ for which φ^γ has semistable reduction in the sense of geometric invariant theory (see Theorems B and C below). Thus, the minimal resultant locus forms a bridge between analytic and moduli-theoretic properties of φ.

A third application is a new fixed point theorem, showing the existence of fixed points in balls which φ maps onto $\mathbf{P}_K^1$ (Theorem 4.6).

We will now give an overview of the paper. Suppose (F, G) is a normalized representation for φ: a pair of homogeneous functions $F(X, Y), G(X, Y) \in \mathcal{O}[X, Y]$ of degree d such that $\varphi(z) = F(z, 1)/G(z, 1)$ and some coefficient of F or G is a unit in $\mathcal{O}$. Such a pair (F, G) is unique up to scaling by a unit. Let $\mathrm{Res}(F, G)$ be the resultant of F and G; then $\mathrm{ordRes}(\varphi) := \mathrm{ord}(\mathrm{Res}(F, G))$ is well-defined and nonnegative.

Let $\mathbf{P}_K^1$ be the Berkovich projective line over K. By Berkovich's classification theorem (see [Berkovich 1990, p. 18]), points of $\mathbf{P}_K^1 \setminus \{\infty\}$ correspond to discs $D(a, r) \subset K$ or to nested sequences of discs; points corresponding to discs with radius $r \in |K^\times|$ are said to be of type II. The point ζ_G corresponding to $D(0, 1)$ is called the *Gauss point*. The action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ extends continuously to $\mathbf{P}_K^1$, and $\mathrm{GL}_2(K)$ acts transitively on type II points.

In [Rumely 2015] it is shown that the map $\gamma \mapsto \mathrm{ordRes}(\varphi^\gamma)$ factors through a function $\mathrm{ordRes}_\varphi(\cdot)$ on $\mathbf{P}_K^1$, given on type II points by

$$\mathrm{ordRes}_\varphi(\gamma(\zeta_G)) = \mathrm{ordRes}(\varphi^\gamma).$$

By [Rumely 2015, Theorem 0.1] the function $\mathrm{ordRes}_\varphi(\cdot)$ is piecewise affine and convex upwards on paths in $\mathbf{P}_K^1$ and it achieves a minimum. The set $\mathrm{MinResLoc}(\varphi)$ where the minimum occurs is called the *minimal resultant locus* of φ. For each $a \in \mathbb{P}^1(K)$, $\mathrm{MinResLoc}(\varphi)$ is contained in the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ spanned by the classical fixed points of φ and the preimages of a. It is either a type II point or a segment joining two type II points.

In this paper, we show that the intersection of the trees $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ for all $a \in \mathbb{P}^1(K)$ is the tree $\Gamma_{\mathrm{Fix,Repel}}$ spanned by the classical fixed points of φ and the repelling fixed points in $\mathbf{H}^1_K$ (Theorem 4.2). Although $\mathrm{ordRes}_\varphi(\,\cdot\,)$ does not belong to the domain of the Laplacian on $\mathbf{P}^1_K$ (it is increasing too fast, in too many directions), one can compute its graph-theoretic Laplacian on the finitely generated tree $\Gamma_{\mathrm{Fix,Repel}}$. Comparing this Laplacian with the so-called "branching measure" μ_{Br} of $\Gamma_{\mathrm{Fix,Repel}}$ (which depends only on the topology of the tree), leads to the construction of the weights $w_\varphi(P)$ and the proof of Theorem A. An intrinsic formula for ν_φ is given in Corollary 6.5. Only points of type II can receive weight. The weights $w_\varphi(P)$, the crucial set, and the crucial measure are canonical because $\mathrm{ordRes}_\varphi(\,\cdot\,)$, $\Gamma_{\mathrm{Fix,Repel}}$, and μ_{Br} are canonical.

The following pair of theorems justify our claim that the minimal resultant locus forms a bridge between analytic and moduli-theoretic properties of φ:

Theorem B (analytic characterization of $\mathrm{MinResLoc}(\varphi)$). *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then* $\mathrm{MinResLoc}(\varphi)$ *is the barycenter of ν_φ in $\mathbf{P}^1_K$. If d is even,* $\mathrm{MinResLoc}(\varphi)$ *is a vertex of the tree Γ_φ spanned by the crucial set. If d is odd,* $\mathrm{MinResLoc}(\varphi)$ *is either a vertex or an edge of Γ_φ.*

Using geometric invariant theory (GIT), Silverman [1998] has constructed a moduli space $\mathcal{M}_d/\mathrm{Spec}(\mathbb{Z})$ for rational functions of degree d, modulo conjugacy. The construction involves the notions of semistable and stable reduction for φ, explained in Section 7. Building on work of Szpiro, Tepper, and Williams [2014], we show

Theorem C (moduli-theoretic characterization of $\mathrm{MinResLoc}(\varphi)$). *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Suppose $\gamma \in \mathrm{GL}_2(K)$ and put $P = \gamma(\zeta_G)$, then*:

(A) *P belongs to* $\mathrm{MinResLoc}(\varphi)$ *if and only if φ^γ has semistable reduction in the sense of* GIT.

(B) $\mathrm{MinResLoc}(\varphi)$ *consists of the single point P if and only if φ^γ has stable reduction in the sense of* GIT.

The plan of the paper is as follows. In Section 1 we recall basic facts and definitions. In Sections 2 and 3 we establish some preliminaries concerning surplus multiplicities $s_\varphi(P, \vec{v})$ and repelling fixed points and in Section 4 we prove the tree intersection theorem (Theorem 4.2). In Section 5 we carry out slope computations needed to find the Laplacian of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ on $\Gamma_{\mathrm{Fix,Repel}}$ and in Section 6 we prove Theorem A as Theorem 6.1. In Section 7, we prove Theorems B and C as Theorems 7.1 and 7.4, respectively.

1. Background

The Berkovich projective line is a path-connected Hausdorff space containing $\mathbb{P}^1(K)$. We denote the Berkovich projective line by $\mathbf{P}^1_K$ (in [Baker and Rumely 2010], it was written $\mathbb{P}^1_{\mathrm{Berk}}$). By Berkovich's classification theorem (see e.g., [Baker and Rumely 2010, p. 5]), points of $\mathbf{P}^1_K \setminus \{\infty\}$ correspond to discs, or nested sequences of discs, in K. There are four kinds of points: Type I points are the points of $\mathbb{P}^1(K)$, which we regard as discs of radius 0. Type II points correspond to discs $D(a,r) = \{z \in K : |z-a| \le r\}$ such that r belongs to the value group $|K^\times|$. Type III points correspond to discs $D(a,r)$ with $r \notin |K^\times|$. We write $\zeta_{a,r}$ for the point corresponding to $D(a,r)$. The type II point $\zeta_{0,1}$ plays a special role; it is called the *Gauss point* and is denoted ζ_G. Type IV points serve to complete $\mathbf{P}^1_K$; they correspond to (cofinal equivalence classes of) sequences of nested discs with empty intersection. If $\{D(a_i, r_i)\}_{i \ge 1}$ is such a sequence, by abuse of notation we continue to write $\zeta_{a,r}$ for the associated point in $\mathbf{P}^1_K$.

Paths in $\mathbf{P}^1_K$ correspond to ascending or descending chains of discs or unions of chains sharing an endpoint. For example the path from 0 to 1 in $\mathbf{P}^1_K$ corresponds to the chains $\{D(0,r) : 0 \le r \le 1\}$ and $\{D(1,r) : 1 \ge r \ge 0\}$; here $D(0,1) = D(1,1)$. Topologically, $\mathbf{P}^1_K$ is a tree: there is a unique path $[P, Q]$ between any two points $P, Q \in \mathbf{P}^1_K$. We write (P, Q) for the interior of that path, with similar notation for half-open segments.

If $P \in \mathbf{P}^1_K$, the *tangent space* T_P is the set of equivalence classes of paths $(P, x]$ as x varies over $\mathbf{P}^1_K \setminus \{P\}$; we call paths $(P, x]$ and $(P, y]$ *equivalent* if they share a common initial segment. We call elements of T_P *tangent vectors* or *directions* and denote them by vectors; given $\vec{v} \in T_P$, we write

$$B_P(\vec{v})^- = \{x \in \mathbf{P}^1_K : (P, x] \in \vec{v}\}$$

for the associated path-component of $\mathbf{P}^1_K \setminus \{P\}$. If $x \in B_P(\vec{v})^-$, we will say that x lies in the direction $\vec{v}$ at P. If P is of type I or type IV, then T_P has one element; if P is of type III, T_P has two elements; and if P is of type II then T_P is infinite. When $P = \zeta_G$, there is a natural one-to-one correspondence between elements of T_{ζ_G} and $\mathbb{P}^1(\tilde{k})$. More generally, for any type II point P, a map $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(\zeta_G) = P$ induces a parametrization of T_P by $\mathbb{P}^1(\tilde{k})$; if $a \in \mathbb{P}^1(\tilde{k})$ we write $\vec{v}_a \in T_P$ for the corresponding direction.

The set $\mathbf{H}^1_K = \mathbf{P}^1_K \setminus \mathbb{P}^1(K)$ (written $\mathbb{H}_{\mathrm{Berk}}$ in [Baker and Rumely 2010]) is called the *Berkovich upper half-space*; it carries a metric $\rho(x, y)$ called the *logarithmic path distance*, for which the length of the path corresponding to $\{D(a,r) : R_1 \le r \le R_2\}$ is $\log(R_2/R_1)$. (We normalize the function $\log t$ so that $\mathrm{ord}(x) = -\log|x|$.) There are two natural topologies on $\mathbf{P}^1_K$, called the *weak* and *strong* topologies. The weak topology on $\mathbf{P}^1_K$ is the coarsest one which makes the evaluation functionals $z \to |f(z)|$ continuous for all $f(z) \in K(z)$; under the weak topology, $\mathbf{P}^1_K$ is compact and $\mathbb{P}^1(K)$ is dense in it. The basic open sets for the weak topology are the path-components of $\mathbf{P}^1_K \setminus \{P_1, \dots, P_n\}$ as $\{P_1, \dots, P_n\}$ ranges over finite subsets of $\mathbf{H}^1_K$. The strong topology on $\mathbf{P}^1_K$ (which is finer than the weak topology) restricts to the topology on $\mathbf{H}^1_K$ induced by $\rho(x, y)$. The basic open sets for the strong topology are the $\rho(x, y)$-balls in $\mathbf{H}^1_K$ together with the basic open sets from the weak topology. Type II points are dense in $\mathbf{P}^1_K$ for both topologies.

The action of φ on $\mathbb{P}^1(K)$ extends functorially to $\mathbf{P}^1_K$. If φ is nonconstant, the induced map $\varphi : \mathbf{P}^1_K \to \mathbf{P}^1_K$ is surjective, open and continuous for both topologies and takes points of a given type to points of the same type; if $\varphi(P) = Q$, there is an induced surjection $\varphi_* : T_P \to T_Q$. The action of $\mathrm{GL}_2(K)$ on $\mathbb{P}^1(K)$ extends to an action on $\mathbf{P}^1_K$ which is continuous for both topologies and preserves the type of each point. $\mathrm{GL}_2(K)$ acts transitively on type II points. It preserves the logarithmic path distance:

$$\rho(\gamma(x), \gamma(y)) = \rho(x, y),$$

for all $x, y \in \mathbf{H}^1_K$ and $\gamma \in \mathrm{GL}_2(K)$.

At each $P \in \mathbf{P}^1_K$, the map φ has a *local degree* $\deg_\varphi(P)$ (called the multiplicity $m_\varphi(P)$ in [Baker and Rumely 2010]), which is a positive integer in the range $1 \le \deg_\varphi(P) \le d$. It has the property that for each $Q \in \mathbf{P}^1_K$,

$$\sum_{\varphi(P)=Q} \deg_\varphi(P) = d.$$

When $P \in \mathbb{P}^1(K)$, $\deg_\varphi(P)$ coincides with the classical algebraic multiplicity of φ at P.

For additional facts about $\mathbf{P}^1_K$, see [Berkovich 1990; Rivera-Letelier 2003b; Favre and Rivera-Letelier 2006; 2010; Baker and Rumely 2010; Faber 2013a; 2013b; Benedetto et al. 2014].

We will use two notions of "reduction" for φ. The first, which we simply call *the reduction of* φ, is defined as follows. If (F, G) is a normalized representation of φ, the reduction $\tilde{\varphi}$ is the rational map on $\mathbb{P}^1(\tilde{k})$ obtained by reducing F and G $(\mathrm{mod}\, \mathfrak{m})$ and eliminating common factors. Explicitly, let $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be the reductions of F, G $(\mathrm{mod}\, \mathfrak{m})$ and put $\widetilde{A}(X, Y) = \mathrm{GCD}(\widetilde{F}(X, Y), \widetilde{G}(X, Y))$. Write $\widetilde{F}(X, Y) = \widetilde{A}(X, Y)\widetilde{F}_0(X, Y)$ and $\widetilde{G}(X, Y) = \widetilde{A}(X, Y)\widetilde{G}_0(X, Y)$. Then $\tilde{\varphi} : \mathbb{P}^1(\tilde{k}) \to \mathbb{P}^1(\tilde{k})$ is the map defined by $(X, Y) \mapsto (\widetilde{F}_0(X, Y) : \widetilde{G}_0(X, Y))$. If $\tilde{\varphi}$ has degree $d = \deg(\varphi)$, then φ is said to have *good reduction*.

The second, which we call the *reduction of* φ *at* P, is obtained by fixing a type II point $P \in \mathbb{H}_{\mathrm{Berk}}$, choosing a $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(\zeta_G) = P$, and taking the reduction $\tilde{\varphi}_P$ of $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$ in the sense above. (If (F_P, G_P) is a normalized representation of φ^γ, we call (F_P, G_P) a *normalized representation of* φ *at* P.) When $P = \zeta_G$ and $\gamma = \mathrm{id}$, this notion of reduction coincides with previous one. The map $\tilde{\varphi}_P$ depends on the choice of γ, but if $\gamma' \in \mathrm{GL}_2(K)$ also satisfies $\gamma'(\zeta_G) = \zeta$ and $\tilde{\varphi}'_P$ is the reduction of φ at P corresponding to γ', there is an $\tilde{\eta} \in \mathrm{GL}_2(\tilde{k})$ such that $\tilde{\varphi}'_P = \tilde{\eta}^{-1} \circ \tilde{\varphi}_P \circ \tilde{\eta}$. Thus $\deg(\tilde{\varphi}_P)$, and the properties that $\tilde{\varphi}_P$ is constant, is nonconstant or is the identity map, are independent of the choice of γ with

$\gamma(\zeta_G) = P$. If φ does not have good reduction but after a change of coordinates by some $\gamma \in \mathrm{GL}_2(K)$ the map φ^γ has good reduction, then φ is said to have *potential good reduction*.

A theorem of Rivera-Letelier [2003a] (see [Baker and Rumely 2010, Corollary 9.27]) says that when P is of type II, then $\varphi(P) = P$ if and only if $\tilde{\varphi}_P$ is nonconstant. Rivera-Letelier shows that in this situation $\deg_\varphi(P) = \deg(\tilde{\varphi}_P)$; he calls P a *repelling fixed point* if $\deg(\tilde{\varphi}_P) \geq 2$ and an *indifferent fixed point* if $\deg(\tilde{\varphi}_P) = 1$. When $\varphi(P) = P$, the induced map $\varphi_* : T_P \to T_P$ on the tangent space comes from the action of $\tilde{\varphi}_P$ on $\mathbb{P}^1(\tilde{k})$. If $\varphi(P) \neq P$, the map $\tilde{\varphi}_P$ is constant, and in that case, if $\tilde{\varphi}_P(z) \equiv a \in \mathbb{P}^1(\tilde{k})$, then $\varphi(P)$ belongs to the ball $B_P(\vec{v}_a)^-$.

Another result of Rivera-Letelier [2003a, Lemma 2.1] (see [Faber 2013a, Proposition 3.10] for a definitive version) says that for each $P \in \mathbf{P}^1_K$ and each $\vec{v} \in T_P$, there are integers $m = m_\varphi(P, \vec{v}) \geq 1$ and $s = s_\varphi(P, \vec{v}) \geq 0$ such that

(a) for each $y \in B_{\varphi(P)}(\varphi_*(\vec{v}))^-$ there are exactly $m + s$ solutions to $\varphi(x) = y$ in $B_P(\vec{v})^-$ (counted with multiplicities), and

(b) for each $y \in \mathbf{P}^1_K \setminus \big(B_{\varphi(P)}(\varphi_*(\vec{v}))^- \cup \{\varphi(P)\}\big)$, there are exactly s solutions to $\varphi(x) = y$ in $B_P(\vec{v})^-$ (counted with multiplicities).

The number $m_\varphi(P, \vec{v})$ is called the *directional multiplicity* of φ at P in the direction $\vec{v}$, and $s_\varphi(P, \vec{v})$ is called the *surplus multiplicity* of φ at P in the direction $\vec{v}$. Several formulas relating $m_\varphi(P, \vec{v})$ to geometric quantities are given in [Baker and Rumely 2010, Theorem 9.26]. In particular, when P is of type II and $\varphi(P) = P$, then $m_\varphi(P, \vec{v}_a)$ is the algebraic multiplicity of $\tilde{\varphi}_P$ at a, for each for $a \in \mathbb{P}^1(\tilde{k})$. An important result due to Faber [2013a, Lemma 3.17] says that if P is of type II and $\varphi(P) = P$, if (F, G) is a normalized representation of φ at P and if $\widetilde{F}(X, Y) = \widetilde{A}(X, Y)\widetilde{F}_0(X, Y)$ and $\widetilde{G}(X, Y) = \widetilde{A}(X, Y)\widetilde{G}_0(X, Y)$, then $s_\varphi(P, \vec{v}_a)$ is the multiplicity of a as a root of $\widetilde{A}(X, Y)$.

Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let (F, G) be a normalized representation of φ. Writing $F(X, Y) = f_d X^d + f_{d-1} X^{d-1} Y + \cdots + f_0 Y^d$ and $G(X, Y) = g_d X^d + g_{d-1} X^{d-1} Y + \cdots + g_0 Y^d$ the resultant of F and G is

$$\mathrm{Res}(F, G) = \det\left(\begin{bmatrix} f_d & f_{d-1} & \cdots & f_1 & f_0 & & & \\ & f_d & f_{d-1} & \cdots & f_1 & f_0 & & \\ & & & & \vdots & & & \\ & & & f_d & f_{d-1} & \cdots & f_1 & f_0 \\ g_d & g_{d-1} & \cdots & g_1 & g_0 & & & \\ & g_d & g_{d-1} & \cdots & g_1 & g_0 & & \\ & & & & \vdots & & & \\ & & & g_d & g_{d-1} & \cdots & g_1 & g_0 \end{bmatrix}\right). \tag{1}$$

The quantity $\operatorname{ordRes}(\varphi) := \operatorname{ord}(\operatorname{Res}(F, G))$ is independent of the choice of normalized representation; by construction, it is nonnegative. It is well-known (see e.g., [Silverman 2007, Theorem 2.15]) that φ has good reduction if and only if $\operatorname{ordRes}(\varphi) = 0$.

The starting point for the investigation in [Rumely 2015] was the following observation. By standard formulas for the resultant (see for example [Silverman 2007, p. 75]), for each $\gamma \in \mathrm{GL}_2(K)$ and each $\tau \in K^\times \cdot \mathrm{GL}_2(\mathcal{O})$, one has

$$\operatorname{ordRes}(\varphi^\gamma) = \operatorname{ordRes}(\varphi^{\gamma\tau}).$$

On the other hand, $K^\times \cdot \mathrm{GL}_2(\mathcal{O})$ is the stabilizer of the Gauss point. Since $\mathrm{GL}_2(K)$ acts transitively on the type II points in $\mathbf{P}^1_K$, the map $\gamma \mapsto \operatorname{ordRes}(\varphi^\gamma)$ factors through a well-defined function $\operatorname{ordRes}_\varphi(\,\cdot\,)$ on type II points given by

$$\operatorname{ordRes}_\varphi(\gamma(\zeta_G)) := \operatorname{ordRes}(\varphi^\gamma). \tag{2}$$

The main theorem in [Rumely 2015] (Theorem 0.1 of that paper) is:

Theorem 1.1. *Let $\varphi(z) \in K(z)$ and suppose $d = \deg(\varphi) \geq 2$. The function $\operatorname{ordRes}_\varphi(\,\cdot\,)$ on type II points extends to a function $\operatorname{ordRes}_\varphi : \mathbf{P}^1_K \to [0, \infty]$ which is continuous for the strong topology, is finite on $\mathbf{H}^1_K$, and takes the value ∞ on $\mathbb{P}^1(K)$. The extended function $f(\,\cdot\,) = \operatorname{ordRes}_\varphi(\,\cdot\,)$ has the following properties:*

(A) *$f(\,\cdot\,)$ is piecewise affine and convex upwards on each path in $\mathbf{P}^1_K$ relative to the logarithmic path distance; the slope of each affine piece is an integer m in the range $-(d^2+d) \leq m \leq d^2+d$ satisfying $m \equiv d^2+d \pmod{2d}$; the breaks between affine pieces occur at type II points.*

(B) *$f(\,\cdot\,)$ achieves a minimum value on $\mathbf{P}^1_K$. The set $\operatorname{MinResLoc}(\varphi)$ on which the minimum is taken is a single type II point if d is even and is either a single type II point or a segment with type II endpoints if d is odd.*

(C) *For each $a \in \mathbb{P}^1(K)$, the extended function $f(\,\cdot\,)$ is strictly increasing away from the tree $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)} \subset \mathbf{P}^1_K$ spanned by the classical fixed points of φ and the preimages of a in $\mathbb{P}^1(K)$.*

In particular, $\operatorname{MinResLoc}(\varphi)$ belongs to $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$, for each $a \in \mathbb{P}^1(K)$.

2. Identification lemmas

The surplus multiplicity $s_\varphi(P, \vec{v})$ plays an important role in the study of $\operatorname{ordRes}_\varphi(\,\cdot\,)$. It turns out that there is a relation between directions $\vec{v} \in T_P$ for which $s_\varphi(P, \vec{v}) > 0$ and directions containing type I fixed points. This is expressed in two lemmas which we call "identification lemmas", because they identify the directions which can have $s_\varphi(P, \vec{v}) > 0$.

If $P \in \mathbf{H}_K^1$ is a type II fixed point of φ and $\tilde{\varphi}_P$ is the reduction of φ at P, we say that P is id-indifferent for φ if $\tilde{\varphi}_P$ is the identity map. (This is part of a more general classification of fixed points given in Section 3.)

Definition 1 (directional and reduced fixed point multiplicities). Suppose $\varphi \in K(z)$ is nonconstant:

(A) For each $P \in \mathbf{H}_K^1$ and each $\vec{v} \in T_P$, we define the *directional fixed point multiplicity* $\#F_\varphi(P, \vec{v})$ to be the number of classical fixed points of φ in $B_P(\vec{v})^-$ (counting multiplicities).

(B) If $P \in \mathbf{H}_K^1$ is a type II fixed point of φ, which is not id-indifferent for φ, let $\tilde{\varphi}_P$ be the reduction of φ at P and parametrize T_P by $\mathbb{P}^1(\tilde{k})$ in such a way that $\varphi_*(\vec{v}_a) = \vec{v}_{\tilde{\varphi}_P(a)}$ for each $a \in \mathbb{P}^1(\tilde{k})$. If $\vec{v} = \vec{v}_a$, we define the *reduced fixed point multiplicity* $\#\widetilde{F}_\varphi(P, \vec{v})$ to be the multiplicity of a as a fixed point of $\tilde{\varphi}_P$.

Clearly $\sum_{\vec{v} \in T_P} F_\varphi(P, \vec{v}) = \deg(\varphi) + 1$. If P is a type II point with $\varphi(P) = P$ and P is not id-indifferent for φ, then $\sum_{\vec{v} \in T_P} \#\widetilde{F}_\varphi(P, \vec{v}) = \deg_\varphi(P) + 1$.

Lemma 2.1 (first identification lemma). *Suppose $P \in \mathbf{H}_K^1$ is of type II, and that $\varphi(P) = P$, but P is not* id-*indifferent. Let $\vec{v} \in T_P$; then*

$$\#F_\varphi(P, \vec{v}) = s_\varphi(P, \vec{v}) + \#\widetilde{F}_\varphi(P, \vec{v}). \tag{3}$$

In particular, $B_P(\vec{v})^-$ contains a type I fixed point of φ if and only if at least one of the following hold:

$$\varphi_*(\vec{v}) = \vec{v} \quad \textit{or} \quad s_\varphi(P, \vec{v}) > 0.$$

Proof. Choose $\gamma \in \mathrm{GL}_2(K)$ with $\gamma(P) = \zeta_G$ and put $\Phi = \varphi^\gamma$; then $\Phi(\zeta_G) = \zeta_G$. Note that $\xi \in \mathbb{P}^1(K)$ is fixed by φ if and only if $\gamma^{-1}(\xi)$ is fixed by Φ and that $\vec{v} \in T_P$ is fixed by φ_* if and only if $(\gamma^{-1})_*(\vec{v}) \in T_{\zeta_G}$ is fixed by Φ_*. Hence it suffices to prove the result under the assumption that $P = \zeta_G$.

Choose $F(X, Y), G(X, Y) \in \mathcal{O}[X, Y]$ so (F, G) is a normalized representation of φ. Let $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be the reductions of F and G and put

$$\widetilde{A}(X, Y) = \mathrm{GCD}(\widetilde{F}(X, Y), \widetilde{G}(X, Y)).$$

Write $\widetilde{F}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{F}_0(X, Y)$ and $\widetilde{G}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{G}_0(X, Y)$. Then $\tilde{\varphi}$ is the map on $\mathbb{P}^1(\tilde{k})$ defined by $(X, Y) \mapsto (\widetilde{F}_0(X, Y) : \widetilde{G}_0(X, Y))$.

Put $H(X, Y) = XG(X, Y) - YF(X, Y)$ and $\widetilde{H}_0(X, Y) = X\widetilde{G}_0(X, Y) - Y\widetilde{F}_0(X, Y)$. Here $H(X, Y) \neq 0$ since $\deg(\varphi) > 1$ and $\widetilde{H}_0(X, Y) \neq 0$ since $\tilde{\varphi}(z) \not\equiv z$ by assumption. The fixed points of Φ are the zeros of $H(X, Y)$ in $\mathbb{P}^1(K)$ and the fixed points of $\tilde{\varphi}$ are the zeros of $\widetilde{H}_0(X, Y)$ in $\mathbb{P}^1(\tilde{k})$.

Reducing $H(X, Y)$ modulo $\mathfrak{m}$, we see that

$$\widetilde{H}(X, Y) = X\widetilde{G}(X, Y) - Y\widetilde{F}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{H}_0(X, Y). \tag{4}$$

Since K is algebraically closed, $H(X, Y)$ factors over $K[X, Y]$ as a product of linear factors. After scaling the factors if necessary, we can assume that the factorization has the form

$$H(X, Y) = C \cdot \prod_{i=1}^{d+1} (b_i X - a_i Y), \tag{5}$$

where $\max(|a_i|, |b_i|) = 1$ for each $i = 1, \ldots, d+1$. We claim that $|C| = 1$ as well. To see this, note that if we choose u, v in $\mathcal{O}$ so that $(\tilde{u} : \tilde{v})$ is not a zero of either $\widetilde{A}(X, Y)$ or $\widetilde{H}_0(X, Y)$, then $\widetilde{H}(\tilde{u}, \tilde{v}) \neq 0$ by (4).

It follows that

$$\widetilde{H}(X, Y) = \widetilde{C} \cdot \prod_{i=1}^{d+1} (\tilde{b}_i X - \tilde{a}_i Y) \not\equiv 0. \tag{6}$$

Since $\tilde{k}[X, Y]$ is a unique factorization domain, comparing (4) and (6) we see that after reordering factors if necessary, there are constants $\widetilde{C}_1, \widetilde{C}_2 \in \tilde{k}^\times$ and an integer $0 \leq n \leq d$ such that

$$\widetilde{A}(X, Y) = \widetilde{C}_1 \cdot \prod_{i=1}^{n} (\tilde{b}_i X - \tilde{a}_i Y) \quad \text{and} \quad \widetilde{H}_0(X, Y) = \widetilde{C}_2 \cdot \prod_{i=n+1}^{d+1} (\tilde{b}_i X - \tilde{a}_i Y). \tag{7}$$

The fixed points of $\Phi(X, Y)$ are the points $(a_i : b_i) \in \mathbb{P}^1(K)$, $i = 1, \ldots, d+1$. By (4) and (7), each fixed point of $\Phi(X, Y)$ specializes to a zero of $\widetilde{H}_0(X, Y)$ or $\widetilde{A}(X, Y)$ and conversely each such zero is the specialization of a fixed point of $\Phi(X, Y)$. For a given $\vec{v} \in T_{\zeta_G}$, if $\tilde{a} \in \mathbb{P}^1(\tilde{k})$ is such that $\vec{v} = \vec{v}_{\tilde{a}}$, then $\#F_\varphi(P, \vec{v})$ is the multiplicity of $\tilde{a}$ as a zero of $\widetilde{H}(X, Y)$, $s_\varphi(P, \vec{v})$ is the multiplicity of $\tilde{a}$ as a zero of $\widetilde{A}(X, Y)$, and $\#\widetilde{F}_\varphi(P, \vec{v})$ is the multiplicity of $\tilde{a}$ as a root of $\widetilde{H}_0(X, Y)$. Thus $\#F_\varphi(P, \vec{v}) = s_\varphi(P, \vec{v}) + \#\widetilde{F}_\varphi(P, \vec{v})$. □

Lemma 2.2 (second identification lemma). *Suppose $P \in \mathbf{H}^1_K$ is of type II and that $\varphi(P) = Q \neq P$. Let $\vec{v} \in T_P$. Then $B_P(\vec{v})^-$ contains a type I fixed point of φ if and only if at least one of the following hold*

(A) $Q \in B_P(\vec{v})^-$,

(B) $P \in B_Q(\varphi_*(\vec{v}))^-$, *or*

(C) $s_\varphi(P, \vec{v}) > 0$.

Proof. Let $Q = \varphi(P)$. We claim that there is a $\gamma \in \mathrm{GL}_2(K)$ such that $\gamma(\zeta_G) = P$ and $\gamma^{-1}(Q) = \zeta_{0,r}$ for some $0 < r < 1$. To see this, take any $\tau_0 \in \mathrm{GL}_2(K)$ with $\tau_0(\zeta_G) = P$. Then $\tau_0^{-1}(Q) \neq \zeta_G$, since τ_0 is one-to-one. Let $\alpha \in \mathbb{P}^1(K)$ be such that the path $[\alpha, \zeta_G]$ contains $\tau_0^{-1}(Q)$ and choose $\tau_1 \in \mathrm{GL}_2(\mathcal{O})$ with $\tau_1(0) = \tau_0^{-1}(\alpha)$. (Such a τ_1 exists because $\mathrm{GL}_2(\mathcal{O})$ acts transitively on $\mathbb{P}^1(K)$). Setting $\gamma = \tau_0 \circ \tau_1$, we see that $\gamma(\zeta_G) = P$ and $\gamma(0) = \gamma_0(\tau_1(0)) = \alpha$. This means that

$\gamma^{-1}(Q) = \tau_1^{-1}(\tau_0^{-1}(Q))$ belongs to $\tau_1^{-1}([\alpha, \zeta_G]) = [0, \zeta_G]$, so $\gamma^{-1}(Q) = \zeta_{0,r}$ for some $0 < r < 1$. Here $r \in |K^\times|$, since $\zeta_{0,r} = \gamma^{-1}(\varphi(P))$ is of type II. Since the fixed points of φ and conditions (A), (B), (C) are equivariant under conjugation, replacing φ with $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$ we are reduced to the case where $P = \zeta_G$ and $Q = \zeta_{0,r}$.

Take any $c \in K^\times$ with $|c| = r$. Put $\gamma_1(z) = \mathrm{id}$ and $\gamma_2(z) = cz$; then $\gamma_1(\zeta_G) = \zeta_G = P$ and $\gamma_2(\zeta_G) = \zeta_{0,r} = Q$. Put $\Phi = \gamma_2^{-1} \circ \varphi \circ \gamma_1$, noting that $\varphi(z) = c \cdot \Phi(z)$.

Choose $F(X, Y), G(X, Y) \in \mathcal{O}[X, Y]$ so (F, G) is a normalized representation of Φ. Let $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be their reductions and put

$$\widetilde{A}(X, Y) = \mathrm{GCD}(\widetilde{F}(X, Y), \widetilde{G}(X, Y)).$$

Write $\widetilde{F}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{F}_0(X, Y)$ and $\widetilde{G}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{G}_0(X, Y)$. The reduction $\widetilde{\Phi}$ of Φ is the map $(X, Y) \mapsto (\widetilde{F}_0(X, Y) : \widetilde{G}_0(X, Y))$, so by Faber's theorem [2013a, Lemma 3.17], the directions $\vec{v}_a \in T_P$ with $s_\varphi(P, \vec{v}_a) > 0$ (which are the same as the ones with $s_\Phi(P, \vec{v}_a) > 0$) are precisely those corresponding to points $a \in \mathbb{P}^1(\tilde{k})$ with $\widetilde{A}(a) = 0$.

Since $\varphi = c \cdot \Phi$, the pair (cF, G) is a normalized representation of φ. The fixed points of φ in $\mathbb{P}^1(K)$ are the zeros $(a_i : b_i)$ of $H(X, Y) = XG(X, Y) - Y \cdot cF(X, Y)$. As in the proof of Lemma 2.1, we can write

$$H(X, Y) = C \cdot \prod_{i=1}^{d+1} (b_i X - a_i Y), \tag{8}$$

where $\max(|a_i|, |b_i|) = 1$ for each $i = 1, \ldots, d+1$ and $|C| = 1$. Absorbing C into the coefficients a_1, b_1, we can assume that $C = 1$. Reducing $H(X, Y)$ modulo $\mathfrak{m}$ gives

$$\widetilde{H}(X, Y) = \prod_{i=1}^{d+1} (\tilde{b}_i X - \tilde{a}_i Y) \not\equiv 0. \tag{9}$$

On the other hand, since $H(X, Y) = XG(X, Y) - Y \cdot cF(X, Y)$ with $|c| < 1$, we also have

$$\widetilde{H}(X, Y) = X\widetilde{G}(X, Y) = X \cdot \widetilde{A}(X, Y) \cdot \widetilde{G}_0(X, Y). \tag{10}$$

Comparing (9) and (10), we see that after reordering the factors if necessary, there are constants $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3 \in \tilde{k}^\times$ and an integer $1 \le n \le d$ such that in $\tilde{k}[X, Y]$

$$X = \widetilde{C}_1 \cdot (\tilde{b}_1 X - \tilde{a}_1 Y), \tag{11}$$

$$\widetilde{A}(X, Y) = \widetilde{C}_2 \cdot \prod_{i=2}^{n} (\tilde{b}_i X - \tilde{a}_i Y), \tag{12}$$

$$\widetilde{G}_0(X, Y) = \widetilde{C}_3 \cdot \prod_{i=n+1}^{d+1} (\tilde{b}_i X - \tilde{a}_i Y), \tag{13}$$

and each fixed point of φ corresponds to a factor of one of these terms.

From (11) it follows that $|a_1| < 1$ and $|b_1| = 1$. Thus the fixed point $(a_1 : b_1)$ belongs to $B_P(\vec{v}_0)^-$, where $\vec{v}_0 \in T_P$ is the tangent direction corresponding to $0 \in \mathbb{P}^1(\tilde{k})$. We can express this in a coordinate-free way by noting that $B_P(\vec{v}_0)^-$ is the ball containing $Q = \zeta_{0,r} = \varphi(P)$.

From (12) and the fact that the zeros of $\widetilde{A}(X, Y)$ correspond to the directions $\vec{v}_a$ for which $s_\varphi(P, \vec{v}_a) > 0$, we see that each direction $\vec{v} \in T_P$ with $s_\varphi(P, \vec{v}) > 0$ contains a fixed point of φ.

Finally, from (13), we see that each direction $\vec{v}_a \in T_P$ corresponding to a zero of $\widetilde{G}_0(X, Y)$ contains a fixed point of φ. However, the zeros of $\widetilde{G}_0(X, Y)$ are the poles of $\widetilde{\Phi}$ and if $\widetilde{\Phi}(a) = \infty$ then $\Phi_*(\vec{v}_a) = \vec{v}_\infty \in T_P$. Since $\varphi = c \cdot \Phi$ and $|c| < 1$, it follows that $\varphi_*(\vec{v}_a) \in T_Q = T_{\zeta_{0,r}}$ is the "upwards" direction $\vec{v}_{Q,\infty} \in T_Q$. This can be expressed a coordinate-free manner by noting that $B_Q(\varphi_*(\vec{v}_a))^- = B_Q(\vec{v}_{Q,\infty})^-$ is the unique ball containing $P = \zeta_G$. $\square$

As a consequence of Lemmas 2.1 and 2.2 we obtain

Corollary 2.3. *Let P be of type II and suppose P is not* id*-indifferent for φ. Then for each $\vec{v} \in T_P$ such that $s_\varphi(P, \vec{v}) > 0$, the ball $B_P(\vec{v})^-$ contains a type I fixed point of φ.*

Remark. Later, when we have proved the tree intersection theorem (Theorem 4.2) we will establish a third identification lemma for id-indifferent points P (Lemma 4.5) and we will strengthen Corollary 2.3, obtaining a fixed point theorem for arbitrary balls $B_P(\vec{v})^-$ which φ maps onto $\mathbf{P}^1_K$ (Theorem 4.6).

3. Classification of fixed points in $\mathbf{H}^1_K$

Recall that a fixed point P of φ in $\mathbf{H}^1_K$ is called *indifferent* if $\deg_\varphi(P) = 1$ and *repelling* if $\deg_\varphi(P) > 1$. We begin by introducing two trees which will play an important role:

Definition 2 (the trees Γ_{Fix} and $\Gamma_{\text{Fix,Repel}}$). Suppose $\varphi(z) \in K(z)$ has degree $d \geq 2$. Let Γ_{Fix} be the tree in $\mathbf{P}^1_K$ spanned by the type I (classical) fixed points of φ. Let $\Gamma_{\text{Fix,Repel}}$ be the tree in $\mathbf{P}^1_K$ spanned by the type I fixed points of φ and the type II repelling fixed points of φ in $\mathbf{H}^1_K$.

Clearly $\Gamma_{\text{Fix}} \subset \Gamma_{\text{Fix,Repel}}$. Since φ has at most $d+1$ distinct type I fixed points, Γ_{Fix} is a finitely generated tree. It is possible that φ has a single type I fixed point of multiplicity $d+1$, in which case Γ_{Fix} is reduced to a point.

Below, we refine the notions of indifferent and repelling fixed points in $\mathbf{H}^1_K$.

Definition 3 (refined classification of indifferent fixed points in $\mathbf{H}^1_K$). Let $P \in \mathbf{H}^1_K$ be a type II indifferent fixed point of φ and let $\tilde{\varphi}_P$ be the reduction of φ at P. After a change of coordinates, $\tilde{\varphi}_P$ is of one of three types:

(A) $\tilde{\varphi}_P(z) = z$; in this case we will say P is id-*indifferent* for φ.

(B) $\tilde{\varphi}_P(z) = \tilde{a} \cdot z$ for some $\tilde{a} \in \tilde{k}^\times$ with $\tilde{a} \neq 1$; in this case we will say P is *multiplicatively indifferent* for φ and that $\tilde{\varphi}_P$ has reduced rotation number $\tilde{a}$. (We remark that the reduced rotation number is only well-defined as an element of $\{\tilde{a}, \tilde{a}^{-1}\}$: if coordinates on $\mathbb{P}^1(\tilde{k})$ are changed by conjugating with $1/z$, then $\tilde{a}$ is replaced by $\tilde{a}^{-1}$. If we need to be more precise, we will speak as follows. Note that the directions $\vec{v}_0, \vec{v}_\infty \in T_P$ corresponding to $0, \infty \in \mathbb{P}^1(\tilde{k})$ are the only directions $\vec{v} \in T_P$ fixed by φ_*. Let points $P_0 \in B_P(\vec{v}_0)^-$, $P_\infty \in B_P(\vec{v}_\infty)^-$ be given. We will say that φ has reduced rotation number $\tilde{a}$ for the axis (P_0, P_∞) and that it has reduced rotation number $\tilde{a}^{-1}$ for the axis (P_∞, P_0).)

(C) $\tilde{\varphi}_P(z) = z + \tilde{a}$ for some $\tilde{a} \in \tilde{k}$ with $\tilde{a} \neq 0$; in this case we will say that P is *additively indifferent* for φ, with reduced translation number $\tilde{a}$. The reduced translation number is well-defined: it is the same for any choice of coordinates on $\mathbb{P}^1(\tilde{k})$ with $\tilde{\varphi}_P(\infty) = \infty$.

Definition 4 (reduced multipliers for fixed directions). Suppose $P \in \mathbf{H}^1_K$ is a type II fixed point of φ and let $\vec{v} \in T_P$ be a direction with $\varphi_*(\vec{v}) = \vec{v}$ and $m_\varphi(P, \vec{v}) = 1$. Let $\tilde{\varphi}_P$ be the reduction of φ at P; without loss, assume coordinates have been chosen so that $\vec{v} = \vec{v}_0$. Then 0 is a fixed point of $\tilde{\varphi}_P$. Put $\tilde{a} = \tilde{\varphi}'_P(0) \in \tilde{k}$. We will call $\tilde{a}$ the *reduced multiplier* of φ in the direction $\vec{v}$.

Definition 5 (refined classification of repelling fixed points in $\mathbf{H}^1_K$). Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let $P \in \mathbf{H}^1_K$ be a repelling fixed point of φ. The directions $\vec{v}_1, \dots, \vec{v}_m \in T_P$ such that $B_P(\vec{v}_i)^-$ contains a classical fixed point of φ, will be called the *focal directions* of φ at P:

(A) P will be called *unifocused* (or just *focused*) if it has a unique focal direction $\vec{v}_1$.

(B) P will be called *bifocused* if it has exactly two distinct focal directions $\vec{v}_1, \vec{v}_2$.

(C) P will be called *multifocused* if it has $m \geq 3$ focal directions.

Below are some properties of unifocused repelling fixed points.

Proposition 3.1. *A repelling fixed point of φ in $\mathbf{H}^1_K$ is a focused repelling fixed point if and only if it does not belong to Γ_{Fix}. Each focused repelling fixed point is an endpoint of $\Gamma_{\mathrm{Fix,Repel}}$. If P is a focused repelling fixed point, with focus $\vec{v}_1$, then*:

(A) $\varphi_*(\vec{v}_1) = \vec{v}_1$, $m_\varphi(P, \vec{v}_1) = 1$, *and* $\#\widetilde{F}_\varphi(P, \vec{v}_1) = \deg_\varphi(P) + 1 \geq 3$.

(B) $s_\varphi(P, \vec{v}_1) = d - \deg_\varphi(P)$ *and* $\vec{v}_1$ *is the only direction* $\vec{v} \in T_P$ *for which one could have* $s_\varphi(P, \vec{v}) > 0$.

(C) *For each* $\vec{v} \in T_P$ *with* $\vec{v} \neq \vec{v}_1$ *we have* $\varphi_*(\vec{v}) \neq \vec{v}$ *and* $\varphi(B_P(\vec{v})^-)$ *is the ball* $B_P(\varphi_*(\vec{v}))^-$.

(D) *For each* $\vec{w} \in T_P$, *there is at least one* $\vec{v} \in T_P$ *with* $\vec{v} \neq \vec{v}_1$ *such that* $\varphi_*(\vec{v}) = \vec{w}$.

Proof. Let P be a repelling fixed point of φ. If $P \notin \Gamma_{\mathrm{Fix}}$ all the type I fixed points of φ lie in a single ball $B_P(\vec{v}_1)^-$, so P is a focused repelling fixed point. Conversely, suppose P is a focused repelling fixed point with focus $\vec{v}_1$. By assumption, all the type I fixed points of φ belong to $B_P(\vec{v}_1)^-$, so $\Gamma_{\mathrm{Fix}} \subset B_P(\vec{v}_1)^-$. Thus, $P \notin \Gamma_{\mathrm{Fix}}$.

We next show that if P is a focused repelling fixed point, it must be an endpoint of $\Gamma_{\mathrm{Fix,Repel}}$, and the only $\vec{v} \in T_P$ fixed by φ_* is $\vec{v}_1$. After a change of coordinates, we can assume that $P = \zeta_G$. Index directions $\vec{v} \in T_P$ by points $\tilde{\alpha} \in \mathbb{P}^1(\tilde{k})$ and choose coordinates so that $\vec{v}_1$ corresponds to $\tilde{\alpha} = 1$. Take any $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1$. Since $B_P(\vec{v})^-$ contains no type I fixed points, Lemma 2.1 shows that $s_\varphi(P, \vec{v}) = 0$ and $\varphi_*(\vec{v}) \neq \vec{v}$. Thus $\varphi(B_P(\vec{v})^-)$ is a ball, necessarily $B_P(\varphi_*(\vec{v})^-)$. If P were not an endpoint of $\Gamma_{\mathrm{Fix,Repel}}$, there would be a direction $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1$ such that $B_P(\vec{v})^-$ contained a repelling fixed point Q of φ. This is impossible, since $\varphi(B_P(\vec{v})^-) = B_P(\varphi_*(\vec{v}))^-$ where $\varphi_*(\vec{v}) \neq \vec{v}$, yet $Q \in B_P(\vec{v})^-$ and $\varphi(Q) = Q$. From the fact that $\varphi_*(\vec{v}) \neq \vec{v}$ for all $\vec{v} \neq \vec{v}_1$, it follows that $\varphi_*(\vec{v}_1) = \vec{v}_1$, since the action of φ_* on T_P corresponds to the action of the reduction $\tilde{\varphi}$ on $\mathbb{P}^1(\tilde{k})$ and $\tilde{\varphi}$ has at least one fixed point.

Since P is a repelling fixed point, necessarily $\deg_\varphi(P) \geq 2$. Since $\vec{v}_1$ is the only direction fixed by φ_*, $\tilde{\alpha} = 1$ is the only point of $\mathbb{P}^1(\tilde{k})$ fixed by $\tilde{\varphi}(z)$. Thus, $\tilde{\alpha} = 1$ is a fixed point of $\tilde{\varphi}$ of multiplicity $\deg(\tilde{\varphi}) + 1 = \deg_\varphi(P) + 1$. This means $\varphi_*(\vec{v}_1) = \vec{v}_1$ and $\#\widetilde{F}_\varphi(P, \vec{v}_1) = \deg_\varphi(P) + 1 \geq 3$. Since $\tilde{\alpha} = 1$ is a fixed point of $\tilde{\varphi}$ of multiplicity > 1, α is not a critical point of $\tilde{\varphi}$, so $m_\varphi(P, \vec{v}_1) = 1$.

Since all the classical fixed points belong to $B_P(\vec{v}_1)^-$, one has $\#F_\varphi(P, \vec{v}_1) = d+1$. By Lemma 2.1, $\#F_\varphi(P, \vec{v}_1) = s_\varphi(P, \vec{v}_1) + \#\widetilde{F}_\varphi(P, \vec{v}_1)$. We have seen above that $\#\widetilde{F}_\varphi(P, \vec{v}_1) = \deg_\varphi(P) + 1$, so $s_\varphi(P, \vec{v}_1) = d - \deg_\varphi(P)$. Since $\#F_\varphi(P, \vec{v}) = 0$ for all $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1$, Lemma 2.1 shows that $\vec{v}_1$ is the only direction in T_P for which one could have $s_\varphi(P, \vec{v}) > 0$.

Finally, we show that for each $\vec{w} \in T_P$, there is a $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1$ satisfying $\varphi_*(\vec{v}) = \vec{w}$. If $\vec{w} \neq \vec{v}_1$, this is trivial, since $\varphi_* : T_P \to T_P$ is surjective and $\varphi_*(\vec{v}_1) = \vec{v}_1$. If $\vec{w} = \vec{v}_1$, it follows from the fact that $m_\varphi(P, \vec{v}_1) = 1$, since

$$\sum_{\vec{v} \in T_P,\, \varphi_*(\vec{v}) = \vec{v}_1} m_\varphi(P, \vec{v}) = \deg_\varphi(P) \geq 2. \qquad \square$$

It may be worthwhile to note that focused repelling fixed points can exist. Here is a class of maps with a focused repelling fixed point at ζ_G, with focal direction $\vec{v}_0$:

Example (maps with a focused repelling fixed point at ζ_G). Fix $d \geq 2$ and let $\widetilde{L}(X, Y) \in \tilde{k}[X, Y]$ be a homogeneous form of degree $d-1$ with $\widetilde{L}(X, Y) \neq \widetilde{C} X^{d-1}$. Let $L_1(X, Y), L_2(X, Y) \in \mathcal{O}[X, Y]$ be homogeneous forms of degree $d-1$ whose reductions satisfy

$$\widetilde{L}_1(X, Y) \equiv \widetilde{L}_2(X, Y) \equiv \widetilde{L}(X, Y) \pmod{\mathfrak{m}}.$$

Put

$$F(X,Y) = XL_1(X,Y) \quad \text{and} \quad G(X,Y) = X^d + YL_2(X,Y).$$

For generic $L_1(X,Y)$ and $L_2(X,Y)$ we will have $\mathrm{GCD}(F,G) = 1$; if this fails, perturb $L_1(X,Y)$ or $L_2(X,Y)$ slightly. Henceforth we will assume $\mathrm{GCD}(F,G) = 1$.

Consider the map φ with representation (F,G). Write

$$\widetilde{F}(X,Y) = \widetilde{A}(X,Y)\widetilde{F}_0(X,Y) \quad \text{and} \quad \widetilde{G}(X,Y) = \widetilde{A}(X,Y)\widetilde{G}_0(X,Y).$$

The reduced map $\tilde{\varphi}$ is represented by $(\widetilde{F}_0, \widetilde{G}_0)$. Since $\widetilde{F}(X,Y) = X\widetilde{L}(X,Y)$ and $\widetilde{G}(X,Y) = X^d + Y\widetilde{L}(X,Y)$, we have

$$\widetilde{A}(X,Y) = \mathrm{GCD}(\widetilde{F}(X,Y), \widetilde{G}(X,Y)) = \mathrm{GCD}(X^{d-1}, \widetilde{L}(X,Y)).$$

Since $\widetilde{L}(X,Y) \neq \widetilde{C}X^{d-1}$, we must have $\widetilde{A}(X,Y) = X^s$ for some $0 \le s \le d-2$. Putting $n = d-s$, it follows that $\deg(\tilde{\varphi}) = n \ge 2$. Thus ζ_G is a repelling fixed point of φ and $\deg_\varphi(\zeta_G) = n$. Let $\vec{v}_0 \in T_{\zeta_G}$ be the direction corresponding to $0 \in \tilde{k} \subset \mathbb{P}^1(\tilde{k})$.

The fixed points of φ in $\mathbb{P}^1(K)$ are the zeros of $H(X,Y) = XG(X,Y) - YF(X,Y)$. The reduction of $H(X,Y)$ is

$$\begin{aligned} \widetilde{H}(X,Y) &= X\widetilde{G}(X,Y) - Y\widetilde{F}(X,Y) \\ &= X \cdot (X^d + Y\widetilde{L}_2(X,Y)) - Y \cdot (X\widetilde{L}_1(X,Y)) = X^{d+1}, \end{aligned} \tag{14}$$

since $\widetilde{L}_1(X,Y) = \widetilde{L}_2(X,Y)$. By the theory of Newton polygons, the type I fixed points all belong to $B_{\zeta_G}(\vec{v}_0)^-$. We claim that ζ_G is a focused repelling fixed point for φ. To see this, note that Γ_{Fix} is contained in $B_{\zeta_G}(\vec{v}_0)^-$, so $\zeta_G \notin \Gamma_{\mathrm{Fix}}$. Since ζ_G is a repelling fixed point of φ, it belongs to $\Gamma_{\mathrm{Fix,Repel}}$. By Proposition 3.1, ζ_G must be a focused repelling fixed point.

By taking $\widetilde{L}(X,Y) = X^sY^{d-1-s}$ for a given integer $0 \le s \le d-2$, we can arrange that $\widetilde{A}(X,Y) = X^s$. Thus for any pair (n,s) with $2 \le n \le d$ and $n+s = d$, there is a $\varphi \in K(z)$ of degree d which has a focused repelling fixed point at ζ_G with $\deg_\varphi(\zeta_G) = n$ and $s_\varphi(\zeta_G, \vec{v}_0) = s$. □

We next consider the properties of bifocused repelling fixed points. First, we will need a lemma.

Lemma 3.2. *Suppose $P \in \mathbf{H}^1_K$ is of type II and $\vec{v} \in T_P$. If there is a focused repelling fixed point $Q \in B_P(\vec{v})^-$ whose focus $\vec{v}_1$ points towards P, then $s_\varphi(P,\vec{v}) > 0$.*

Proof. We must show that $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$. Since $(\mathbf{P}^1_K \setminus B_Q(\vec{v}_1)^-) \subset B_P(\vec{v})^-$, it suffices to show that

$$\varphi\big(\mathbf{P}^1_K \setminus B_Q(\vec{v}_1)^-\big) = \mathbf{P}^1_K.$$

To see this, note that $\mathbf{P}^1_K \setminus B_Q(\vec{v}_1)^- = \{Q\} \cup \big(\bigcup_{\vec{v} \in T_Q, \vec{v} \neq \vec{v}_1} B_Q(\vec{v})^-\big)$. By assumption $\varphi(Q) = Q$. Since $\varphi_* : T_Q \to T_Q$ is surjective, Proposition 3.1 shows that for each

$\vec{w} \in T_Q$ there is at least one $\vec{v} \in T_Q$ with $\vec{v} \neq \vec{v}_1$ for which $\varphi_*(B_Q(\vec{v})^-) = B_Q(\vec{w})^-$. Since $\mathbf{P}^1_K = \{Q\} \cup \big(\bigcup_{\vec{w} \in T_Q} B_Q(\vec{w})^-\big)$, the claim follows. $\square$

Proposition 3.3. *A repelling fixed point of φ in $\mathbf{H}^1_K$ is a bifocused repelling fixed point if and only if it belongs Γ_{Fix} but is not a branch point of $\Gamma_{\mathrm{Fix,Repel}}$. Suppose P is a bifocused repelling fixed point and let $\vec{v}_1, \vec{v}_2 \in T_P$ be its focal directions, then:*

(A) *$\varphi_*(\vec{v}) = \vec{v}$, $m_\varphi(P, \vec{v}) = 1$, and $\#\widetilde{F}_\varphi(P, \vec{v}) \geq 2$, for at least one $\vec{v} \in \{\vec{v}_1, \vec{v}_2\}$.*

(B) *For each $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_1, \vec{v}_2$, we have $\varphi_*(\vec{v}) \neq \vec{v}$ and $s_\varphi(P, \vec{v}) = 0$.*

Proof. Suppose P is a bifocused repelling fixed point. By definition, there are type I fixed points α_1, α_2 of φ belonging to distinct directions in T_P, so $P \in \Gamma_{\mathrm{Fix}}$. However, P cannot be a branch point of Γ_{Fix}, because if it were there would be at least three distinct directions in T_P containing type I fixed points. It also cannot be a branch point of $\Gamma_{\mathrm{Fix,Repel}}$ which is not a branch point of Γ_{Fix}, because if it were there would be at least one branch of $\Gamma_{\mathrm{Fix,Repel}} \setminus \Gamma_{\mathrm{Fix}}$ off P. If $\vec{v} \in T_P$ is the corresponding direction, then $B_P(\vec{v})^-$ would contain a type II repelling fixed point Q but no type I fixed points. Since Q is a focused repelling fixed point, whose focus $\vec{v}_1 \in T_Q$ points towards Γ_{Fix}, by Lemma 3.2 we would have $s_\varphi(P, \vec{v}) > 0$. However, this contradicts Lemma 2.1 since $B_P(\vec{v})^-$ contains no type I fixed points.

Conversely, suppose $P \in \Gamma_{\mathrm{Fix}}$ is a type II repelling fixed point but is not a vertex of $\Gamma_{\mathrm{Fix,Repel}}$. Then there are exactly two directions $\vec{v}_1, \vec{v}_2 \in T_P$ containing type I fixed points, so P is a bifocused repelling fixed point.

To prove assertions (A) and (B), let P be any bifocused repelling fixed point. After a change of coordinates, we can assume that $P = \zeta_G$ and that 0 and ∞ are fixed points of φ. Let (F, G) be a normalized representation of φ. Since $\varphi(z)$ fixes P, it has nonconstant reduction. Thus there are nonzero homogeneous polynomials

$$\widetilde{A}(X, Y), \widetilde{F}_0(X, Y), \widetilde{G}_0(X, Y) \in \tilde{k}[X, Y]$$

such that $(\widetilde{F}, \widetilde{G}) = (\widetilde{A} \cdot \widetilde{F}_0, \widetilde{A} \cdot \widetilde{G}_0)$, with $\mathrm{GCD}(\widetilde{F}_0, \widetilde{G}_0) = 1$. Since P is a repelling fixed point, we have $\delta := \deg_\varphi(P) \geq 2$ and $\deg(\widetilde{F}_0) = \deg(\widetilde{G}_0) = \delta$. Write $H(X, Y) = XG(X, Y) - YF(X, Y)$ and put $\widetilde{H}_0(X, Y) = X\widetilde{G}_0(X, Y) - Y\widetilde{F}_0(X, Y)$. Then $\widetilde{H}(X, Y) = \widetilde{A}(X, Y) \cdot \widetilde{H}_0(X, Y)$. Since $0, \infty \in \mathbb{P}^1(K)$ are fixed points of φ and P is a bifocused repelling fixed point, $\vec{v}_0, \vec{v}_\infty \in T_P$ are the only directions $\vec{v} \in T_P$ for which the balls $B_P(\vec{v})^-$ can contain type I fixed points. It follows from Lemma 2.1 that $\widetilde{H}(X, Y) = \tilde{c} \cdot X^\ell Y^{d+1-\ell}$ for some $\tilde{c} \in \tilde{k}^\times$ and some ℓ. Since $\widetilde{H}_0(X, Y) \mid \widetilde{H}(X, Y)$, we must have $\widetilde{H}_0(X, Y) = \tilde{h} \cdot X^{\ell_0} Y^{\delta+1-\ell_0}$ for some $\tilde{h} \in \tilde{k}^\times$ and some $0 \leq \ell_0 \leq \delta + 1$. Since $\delta + 1 \geq 3$, either $\#\widetilde{F}_\varphi(P, \vec{v}_0) \geq 2$ or $\#\widetilde{F}_\varphi(P, \vec{v}_\infty) \geq 2$. If $i \in \{1, 2\}$ is such that $\#\widetilde{F}_\varphi(P, \vec{v}_i) \geq 2$ then necessarily $\varphi_*(\vec{v}_i) = \vec{v}_i$ and $m_\varphi(P, \vec{v}_i) = 1$. This proves assertion (A).

For each $\vec{v} \in T_P$ with $\vec{v} \neq \vec{v}_0, \vec{v}_\infty$, there are no type I fixed point in $B_P(\vec{v})^-$, so Lemma 2.1 shows that $\varphi_*(\vec{v}) \neq \vec{v}$ and $s_\varphi(P, \vec{v}) = 0$. Thus assertion (B) holds. $\square$

Finally, we consider multifocused repelling fixed points. Recall that the valence of a point P in a graph Γ is the number of edges of Γ emanating from P.

Proposition 3.4. *A fixed point of φ in $\mathbf{H}_K^1$ is a multifocused repelling fixed point if and only if it is a repelling fixed point and is a branch point of Γ_{Fix}. If P is a multifocused repelling fixed point, then its valence in $\Gamma_{\text{Fix,Repel}}$ is the same as its valence in Γ_{Fix}.*

Remark. Branch points of Γ_{Fix} need not be repelling fixed points: there can be branch points of Γ_{Fix} which are indifferent fixed points of φ or are moved by φ.

Proof of Proposition 3.4. If P is a multifocused repelling fixed point, then at least three directions in T_P contain type I fixed points of φ, so P is a branch point of Γ_{Fix}. Conversely, if a repelling fixed point P is a branch point of Γ_{Fix}, at least three directions in T_P contain type I fixed points of φ, so P is multifocused.

The second assertion can be reformulated as saying that if P is a multifocused repelling fixed point of φ, then there are no branches of $\Gamma_{\text{Fix,Repel}} \setminus \Gamma_{\text{Fix}}$ which fork off Γ_{Fix} at P. Suppose to the contrary that there were such a branch and let $\vec{v} \in T_P$ be the corresponding direction. By the same argument as in the proof of Proposition 3.3, $B_P(\vec{v})^-$ would contain a type II repelling fixed point Q but no type I fixed points. Since Q is a focused repelling fixed point, whose focus $\vec{v}_1$ points towards P, by Lemma 3.2 we would have $s_\varphi(P, \vec{v}) > 0$. However, this contradicts Lemma 2.1 since $B_P(\vec{v})^-$ contains no type I fixed points. $\square$

The fact that focused repelling fixed points are endpoints of $\Gamma_{\text{Fix,Repel}}$, while bifocused repelling fixed points and multifocused repelling fixed points belong to Γ_{Fix}, leads one to ask about the nature of points of $\Gamma_{\text{Fix,Repel}} \setminus \Gamma_{\text{Fix}}$ which are not endpoints of $\Gamma_{\text{Fix,Repel}}$.

Proposition 3.5. *Suppose P is a focused repelling fixed point of φ and let P_0 be the nearest point to P in Γ_{Fix}. Then each type II point $Q \in (P, P_0]$ is an* id*-indifferent fixed point of φ.*

Proof. Let Q be a type II point in $(P, P_0]$. Let $\vec{w} \in T_Q$ be the direction for which $P \in B_Q(\vec{w})^-$. The focus $\vec{v}_1$ of P points towards Γ_{Fix} and hence towards Q. By Lemma 3.2, we have $s_\varphi(Q, \vec{w}) > 0$. If $\varphi(Q) = Q$ but Q is not id-indifferent, this contradicts Lemma 2.1 since $B_Q(\vec{w})^-$ does not contain any type I fixed points of φ. If $\varphi(Q) \neq Q$, it contradicts Lemma 2.2 for the same reason. Thus, Q must be id-indifferent. $\square$

4. The tree intersection theorem

In this section we study the tree $\Gamma_{\text{Fix,Repel}}$. We first note that it is never a single point.

Lemma 4.1. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then either φ has at least two type I fixed points or it has one type I fixed point and one or more type II repelling fixed points. In either case*, $\Gamma_{\mathrm{Fix,Repel}}$ *has at least one edge.*

Proof. If φ has at least two type I fixed points, we are done. If it has only one, that point is necessarily a fixed point of multiplicity $d+1$, hence has multiplier 1 and is an indifferent fixed point. By a theorem of Rivera-Letelier (see [Rivera-Letelier 2003b, Theorem B] or [Baker and Rumely 2010, Theorem 10.82]), φ has at least one repelling fixed point in $\mathbf{P}^1_K$ (which may be in either $\mathbb{P}^1(K)$ or $\mathbf{H}^1_K$), so in this case φ must have a repelling fixed point in $\mathbf{H}^1_K$. $\square$

Theorem 4.2 (the tree intersection theorem). *Let $\varphi \in K(z)$ have degree $d \geq 2$. Then* $\Gamma_{\mathrm{Fix,Repel}}$ *is the intersection of the trees* $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$, *for all $a \in \mathbb{P}^1(K)$.*

Proof. Write Γ_0 for the intersection of the trees $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$, for all $a \in \mathbb{P}^1(K)$.

We first show that $\Gamma_{\mathrm{Fix,Repel}} \subseteq \Gamma_0$. For this, it is enough to show that each repelling fixed point of φ in $\mathbf{H}^1_K$ belongs to Γ_0, since Γ_0 is connected and clearly contains the type I fixed points. Let P be a repelling fixed point of φ in $\mathbf{H}^1_K$ and fix $a \in \mathbb{P}^1(K)$. We claim that P belongs to $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$.

By a theorem of Rivera-Letelier (see [Rivera-Letelier 2000, Proposition 5.1] or [Baker and Rumely 2010, Lemma 10.80]), P is of type II. By [Baker and Rumely 2010, Corollary 2.13(B)], there is a $\gamma \in \mathrm{GL}_2(K)$ for which $\gamma(\infty) = a$ and $\gamma(\zeta_G) = P$. After conjugating φ by γ, we can assume that $a = \infty$ and $P = \zeta_G$. Let (F, G) be a normalized representation of φ. The poles of φ are the zeros of $G(X, Y)$ and the fixed points of φ are the zeros of $H(X, Y) := YF(X, Y) - XG(X, Y)$. Let $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be the reductions of F and G. Put

$$\widetilde{A} = \mathrm{GCD}(\widetilde{F}, \widetilde{G}).$$

Write $\widetilde{F} = \widetilde{A} \cdot \widetilde{F}_0$, and $\widetilde{G} = \widetilde{A} \cdot \widetilde{G}_0$. Then $\tilde{\varphi}$ is the map $(X, Y) \mapsto (\widetilde{F}_0(X, Y), \widetilde{G}_0(X, Y))$ on $\mathbb{P}^1(\tilde{k})$. Since P is a repelling fixed point of φ, we have $\tilde{d} := \deg(\tilde{\varphi}) \geq 2$.

Put $\widetilde{H}_0(X, Y) = Y\widetilde{F}_0(X, Y) - X\widetilde{G}_0(X, Y)$, so $\deg(\widetilde{H}_0) = \tilde{d} + 1$. The fixed points of $\tilde{\varphi}$ are the zeros of $\widetilde{H}_0(X, Y)$ in $\mathbb{P}^1(\tilde{k})$, listed with multiplicities. Since $\mathrm{GCD}(\widetilde{F}_0, \widetilde{G}_0) = 1$ and $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = \mathrm{GCD}(Y\widetilde{F}_0, \widetilde{G}_0)$, we must have either $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = 1$ or $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = Y$.

If $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = 1$, the fixed points and poles of $\tilde{\varphi}$ are disjoint. Since each fixed point of $\tilde{\varphi}$ is the reduction of at least one fixed point of φ (Lemma 2.1) and each pole of $\tilde{\varphi}$ is the reduction of at least one pole of φ, we conclude that φ has a fixed point z_0 and a pole z_1 lying in different directions in T_P. Thus P belongs to $[z_0, z_1]$ and $P \in \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$.

If $\mathrm{GCD}(\widetilde{H}_0, \widetilde{G}_0) = Y$, then Y divides $\widetilde{G}_0$, so $\tilde{\infty}$ is a pole of $\tilde{\varphi}$. This means φ has a pole z_1 in the ball $B_{\zeta_G}(\vec{v}_\infty)^- \subset \mathbb{P}^1(K)$. On the other hand, Y^2 cannot divide both $\widetilde{H}_0$ and $\widetilde{G}_0$. If Y^2 does not divide $\widetilde{H}_0$, then $\tilde{\varphi}$ has at least one fixed point in $\tilde{k}$, so φ has a fixed point z_0 in $\mathcal{O}$. Since z_0 and z_1 lie in different tangent directions at ζ_G, we conclude that $\zeta_G \in \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ in this case. On the other hand, if Y^2 does not divide $\widetilde{G}_0$, then $\tilde{\varphi}$ has at least one pole in $\tilde{k}$, so φ has a pole $z_0 \in \mathcal{O}$, and again $P = \zeta_G \in \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$.

We next show that $\Gamma_{\mathrm{Fix,Repel}} = \Gamma_0$. Since $\Gamma_{\mathrm{Fix,Repel}} \subseteq \Gamma_0$, it will suffice to show that each endpoint of $\Gamma_{\mathrm{Fix,Repel}}$ which does not belong to Γ_{Fix} is an endpoint of the intersection of some set of trees $\{\Gamma_{\mathrm{Fix},\varphi^{-1}(a_i)} : i \in I\}$, for an appropriate index set I.

By Proposition 3.1, each endpoint of $\Gamma_{\mathrm{Fix,Repel}}$ not in Γ_{Fix} is a focused repelling fixed point of φ. Let $P \in \mathbf{H}_K^1$ be a focused repelling fixed point, with focus $\vec{v}_1$. Choose distinct directions $\vec{v}_2, \vec{v}_3 \in T_P \setminus \{\vec{v}_1\}$ and take $a_2 \in \mathbb{P}^1(K) \cap B_P(\vec{v}_2)^-$ and $a_3 \in \mathbb{P}^1(K) \cap B_P(\vec{v}_3)^-$. We claim that P is an endpoint of $\Gamma_{\mathrm{Fix},\varphi^{-1}(a_2)} \cap \Gamma_{\mathrm{Fix},\varphi^{-1}(a_3)}$. To see this, note that if $\xi_2 \in \mathbb{P}^1(K) \cap (\mathbf{P}_K^1 \setminus B_P(\vec{v}_1)^-)$ is a solution to $\varphi(\xi_2) = a_2$ and $\xi_3 \in \mathbb{P}^1(K) \cap (\mathbf{P}_K^1 \setminus B_P(\vec{v}_1)^-)$ is a solution to $\varphi(\xi_3) = a_3$, the directions $\vec{w}_2, \vec{w}_3 \in T_P$ such that $\xi_2 \in B_P(\vec{w}_2)^-$ and $\xi_3 \in B_P(\vec{w}_3)^-$ are necessarily distinct. This follows from Proposition 3.1, which asserts that for each $\vec{w} \in T_P$ with $\vec{w} \neq \vec{v}_1$, the image $\varphi(B_P(\vec{w})^-)$ is precisely $B_P(\varphi_*(\vec{w}))^-$. □

The tree $\Gamma_{\mathrm{Fix,Repel}}$ is spanned by finitely many points.

Corollary 4.3. *If $\varphi(z) \in K(z)$ has degree $d \geq 2$, then*

(A) *φ has at most d focused repelling fixed points, and*

(B) *$\Gamma_{\mathrm{Fix,Repel}}$ is a finitely generated tree with at most $2d + 1$ endpoints. Each endpoint of $\Gamma_{\mathrm{Fix,Repel}}$ is either a type I fixed point or a type II focused repelling fixed point.*

Proof. If $\varphi(z)$ has no focused repelling fixed points, part (A) holds trivially. Otherwise, choose any focused repelling fixed point P. By Proposition 3.1, it is an endpoint of $\Gamma_{\mathrm{Fix,Repel}}$; let $\vec{v} \in T_P$ be a direction pointing away from $\Gamma_{\mathrm{Fix,Repel}}$. Fix a point $\alpha \in \mathbb{P}^1(K) \cap B_P(\vec{v})^-$ and consider the solutions $\xi_1, \ldots, \xi_d$ to $\varphi(z) = \alpha$. By Proposition 3.1, for each focused repelling fixed point Q there is a direction $\vec{w}_Q \in T_Q$ pointing away from $\Gamma_{\mathrm{Fix,Repel}}$ such that $\alpha \in \varphi(B_Q(\vec{w}_Q)^-)$; it follows that some ξ_i belongs to $B_Q(\vec{w}_Q)^-$. For distinct Q, the balls $B_Q(\vec{w}_Q)^-$ are pairwise disjoint. Thus φ has at most d focused repelling fixed points.

Since φ has at most $d + 1$ type I fixed points, the tree $\Gamma_{\mathrm{Fix,Repel}}$ is spanned by at most $2d + 1$ points. Its endpoints are clearly as claimed. □

We will sharpen Corollary 4.3 in Corollary 6.2. Below are some additional consequences of Theorem 4.2. A trivial but important one is:

Corollary 4.4. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then* $\mathrm{MinResLoc}(\varphi)$ *is contained in $\Gamma_{\mathrm{Fix,Repel}}$.*

Proof. By Theorem 1.1, $\mathrm{MinResLoc}(\varphi)$ is contained in $\Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$ for each $a \in \mathbb{P}^1(K)$. Hence it is contained in the intersection of those trees, which is $\Gamma_{\mathrm{Fix,Repel}}$. □

Another is an "identification lemma" for id-indifferent fixed points.

Lemma 4.5 (third identification lemma). *Suppose P is a type II* id-*indifferent fixed point of φ. Then for each $\vec{v} \in T_P$ such that $s_\varphi(P, \vec{v}) > 0$, the ball $B_P(\vec{v})^-$ contains either a type I fixed point or a type II focused repelling fixed point of φ.*

Proof. Suppose $s_\varphi(P, \vec{v}) > 0$. This means $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$. If $B_P(\vec{v})^-$ contains a type I fixed point, we are done.

If not, then there must be some $\vec{v}_0 \in T_P$ with $\vec{v}_0 \neq \vec{v}$ such that $B_P(\vec{v}_0)^-$ contains a type I fixed point. Since $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$, for each $a \in \mathbb{P}^1(K)$ there is a solution to $\varphi(x) = a$ in $B_P(\vec{v})^-$. The path from this solution to the type I fixed point passes through P, so $P \in \Gamma_{\mathrm{Fix},\varphi^{-1}(a)}$. Letting a vary, we see that

$$P \in \bigcap_{a \in \mathbb{P}^1(K)} \Gamma_{\mathrm{Fix},\varphi^{-1}(a)} = \Gamma_0 = \Gamma_{\mathrm{Fix,Repel}}.$$

Since P is id-indifferent, we must have $m_\varphi(P, \vec{v}) = 1$. Hence by a theorem of Rivera-Letelier (see [Rivera-Letelier 2003b, §4] or [Baker and Rumely 2010, Theorem 9.46]) there is a $Q \in B_P(\vec{v})^-$ such that φ maps the annulus $\mathrm{Ann}(P, Q)$ to itself and fixes each point in $[P, Q]$. Take any type II point $Z \in (P, Q)$ and let $\vec{w} \in T_Z$ be the direction towards Q. We claim that $s_\varphi(Z, \vec{w}) > 0$. If not, φ maps $B_Z(\vec{w})^-$ to a ball and that ball must be $B_Z(\vec{w})^-$ since $\varphi(Z) = Z$ and $\varphi_*(\vec{w}) = \vec{w}$. However, this means that

$$\begin{aligned} \varphi(B_P(\vec{v})^-) &= \varphi(\mathrm{Ann}(P, Q) \cup B_Z(\vec{w})^-) \\ &= \varphi(\mathrm{Ann}(P, Q)) \cup \varphi(B_Z(\vec{w})^-) \\ &= \mathrm{Ann}(P, Q) \cup B_Z(\vec{w})^- \\ &= B_P(\vec{v})^-, \end{aligned}$$

which contradicts that $s_\varphi(P, \vec{v}) > 0$. Hence it must be that $s_\varphi(Z, \vec{w}) > 0$ and, by the same argument as above, it follows that $Z \in \Gamma_{\mathrm{Fix,Repel}}$.

Thus $\Gamma_{\mathrm{Fix,Repel}}$ contains points in $B_P(\vec{v})^-$, so it has an endpoint in $B_P(\vec{v})^-$. Since $B_P(\vec{v})^-$ does not contain type I fixed points, the endpoint must be a focused repelling fixed point. □

As Lemma 4.5 shows, when P is a type II id-indifferent fixed point, the linkage between directions $\vec{v} \in T_P$ for which $s_\varphi(P, \vec{v}) > 0$ and directions containing a type I fixed point breaks down. It turns out that for id-indifferent fixed points it is the "primary terms" in a normalized representation at P which determine when $s_\varphi(P, \vec{v}) > 0$ and the "secondary terms" which determine the type I fixed points. This is made precise by the following class of examples:

Example (maps with an id-indifferent fixed point at ζ_G). Fix $d \geq 2$ and let $\widetilde{A}(X, Y) \in \tilde{k}[X, Y]$ be a nonzero homogeneous form of degree $d-1$. Lift $\widetilde{A}(X, Y)$

to $A(X, Y) \in \mathcal{O}[X, Y]$ and fix $\pi \in \mathcal{O}$ with $\mathrm{ord}(\pi) > 0$. Let $F_1(X, Y), G_1(X, Y) \in \mathcal{O}[X, Y]$ be arbitrary homogeneous forms of degree d. Put

$$F(X, Y) = A(X, Y) \cdot X + \pi F_1(X, Y),$$
$$G(X, Y) = A(X, Y) \cdot Y + \pi G_1(X, Y).$$

For generic $F_1(X, Y)$ and $G_1(X, Y)$ we will have $\mathrm{GCD}(F, G) = 1$; if that is the case, let φ be the map with normalized representation (F, G). Then

(1) $(\widetilde{F}(X, Y), \widetilde{G}(X, Y)) = \widetilde{A}(X, Y) \cdot (X, Y)$, while

(2) $H(X, Y) = XG(X, Y) - Y F(X, Y) = \pi(XG_1(X, Y) - Y F_1(X, Y))$.

Thus the directions $\vec{v} \in T_{\zeta_G}$ with $s_\varphi(\zeta_G, \vec{v}) > 0$ come from the roots of $\widetilde{A}(X, Y)$, while the type I fixed points of φ are the roots of $XG_1(X, Y) - Y F_1(X, Y)$. $\square$

By combining the three identification lemmas we obtain a new fixed point theorem which concerns balls which φ maps onto $\mathbf{P}^1_K$. Previously known fixed point theorems (see [Baker and Rumely 2010, Theorems 10.83, 10.85, and 10.86]) concern domains $D \subset \mathbf{P}^1_K$ whose image $\varphi_*(D)$ is another domain, with $D \subset \varphi_*(D)$ or $\varphi_*(D) \subset D$, or closed sets X with $\varphi(X) \subseteq X$.

Theorem 4.6 (full image fixed point theorem). *Let $\varphi(z) \in K(z)$, with $\deg(\varphi) \geq 2$. Suppose $P \in \mathbf{P}^1_K$ and $\vec{v} \in T_P$. If $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$, then $B_P(\vec{v})^-$ contains either a (classical) fixed point of φ in $\mathbb{P}^1(K)$ or a repelling fixed point of φ in $\mathbf{H}^1_K$.*

Proof. Given points $P, Q \in \mathbb{P}^1_{\mathrm{Berk}}$ with $Q \neq P$, write $\mathrm{Ann}(P, Q)$ for the component of $\mathbf{P}^1_K \setminus \{P, Q\}$ containing (P, Q).

Assume $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$. If P is of type II, then $s_\varphi(P, \vec{v}) > 0$ and the result follows by combining Lemmas 2.1, 2.2, and 4.5. If P is of type III or IV, one reduces to the case of type II as follows: There is a point $Q \in B_P(\vec{v})^-$ such that $\varphi(\mathrm{Ann}(P, Q)) = \mathrm{Ann}(\varphi(P), \varphi(Q))$. Take any type II point $Z \in (P, Q)$ and let $\vec{w} \in T_Z$ be the direction towards Q. We claim that $\varphi(B_Z(\vec{w})^-) = \mathbf{P}^1_K$. Otherwise, $\varphi_*(B_Z(\vec{w})^-)$ would be the ball $B_{\varphi(Z)}(\varphi_*(\vec{w}))^-$. By the mapping properties of annuli $\varphi_*(\vec{w})$ is the direction in $T_{\varphi(Z)}$ containing $\varphi(Q)$, so

$$\varphi(B_P(\vec{v})^-) = \varphi(\mathrm{Ann}(P, Q)) \cup \varphi(B_Z(\vec{w})^-) = \mathrm{Ann}(\varphi(P), \varphi(Q)) \cup B_{\varphi(Z)}(\varphi_*(\vec{w}))^-$$

omits the point $\varphi(P)$, contradicting that $\varphi(B_P(\vec{v})^-) = \mathbf{P}^1_K$.

If P is of type I and $B_P(\vec{v})^- = \mathbf{P}^1_K \setminus \{P\}$ contains no type I fixed points of φ, then P is the only type I fixed point of φ. Hence it is a fixed point of multiplicity $d + 1 > 1$ and necessarily has multiplier 1. By a theorem of Rivera-Letelier (see [Rivera-Letelier 2003b, Theorem B] or [Baker and Rumely 2010, Theorem 10.82]) φ has at least one repelling fixed point in $\mathbf{P}^1_K$, so it has a repelling fixed point in $\mathbf{H}^1_K$, which clearly lies in $B_P(\vec{v})^-$. $\square$

The author does not know whether Theorem 4.6 can be generalized to domains D with $\varphi(D) = \mathbf{P}^1_K$, when D has more than one boundary point.

5. Slope formulas

In this section we establish formulas for the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at an arbitrary point $P \in \mathbf{H}^1_K$, in an arbitrary direction $\vec{v} \in T_P$. These formulas will be used in the proof of the weight formula in Section 6.

Let $f(z)$ be a function on $\mathbf{H}^1_K$ and write $\rho(P, Q)$ for the logarithmic path distance. Given a point $P \in \mathbf{H}^1_K$ and a direction $\vec{v} \in T_P$, the slope of f at P in the direction $\vec{v}$ is defined to be

$$\partial_{\vec{v}} f(P) = \lim_{\substack{Q \to P \\ Q \in B_P(\vec{v})^-}} \frac{F(Q) - F(P)}{\rho(Q, P)}, \tag{15}$$

provided the limit exists. For any two points $P_1, P_2 \in B_P(\vec{v})^-$, the paths $[P, P_1]$ and $[P, P_2]$ share a common initial segment, so the limit in (15) exists if and only if it exists for Q restricted to $[P, P_1]$. Since $\rho(P, Q)$ is invariant under the action of $\mathrm{GL}_2(K)$ on $\mathbf{H}^1_K$, $\partial_{\vec{v}} F(P)$ is independent of the choice of coordinates.

For the remainder of this section we will take $f(\,\cdot\,) = \mathrm{ordRes}_\varphi(\,\cdot\,)$.

Definition 6 (tangent space and valence in $\Gamma_{\mathrm{Fix,Repel}}$). For a point $P \in \Gamma_{\mathrm{Fix,Repel}}$, we write $T_{P,\mathrm{FR}}$ for the tangent space to P in $\Gamma_{\mathrm{Fix,Repel}}$, the set of directions $\vec{v} \in T_P$ such that there is an edge of $\Gamma_{\mathrm{Fix,Repel}}$ emanating from P in the direction $\vec{v}$. We write $v_{\mathrm{FR}}(P)$ for the valence of P in $\Gamma_{\mathrm{Fix,Repel}}$, the number of edges of $\Gamma_{\mathrm{Fix,Repel}}$ emanating from P.

Definition 7 (shearing directions). If P is a fixed point of φ in $\mathbf{P}^1_K$ (of any type I, II, III, or IV), a direction $\vec{v} \in T_P$ will be called a *shearing direction* if there is a type I fixed point in $B_P(\vec{v})^-$ but $\varphi_*(\vec{v}) \neq \vec{v}$. Let $N_{\mathrm{Shearing}}(P)$ be the number of shearing directions in T_P.

We will now give formulas for the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at each type II point P, in each direction $\vec{v} \in T_P$. The formulas break into several cases, depending on the nature of P. For the convenience of the reader, we first state all the formulas, then give the proofs.

Proposition 5.1. *Let $P \in \mathbf{H}^1_K$ be a type II fixed point of φ which is not* id*-indifferent. Then for each $\vec{v} \in T_P$, the slope of $f(\,\cdot\,) = \mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v}$ is*

$$\begin{aligned} \partial_{\vec{v}} f(P) &= (d^2 - d) - 2d \cdot \#F_\varphi(P, \vec{v}) + 2d \cdot \max(1, \#\widetilde{F}_\varphi(P, \vec{v})) \\ &= (d^2 - d) - 2d \cdot s_\varphi(P, \vec{v}) + 2d \cdot \begin{cases} 0 & \text{if } \varphi_*(\vec{v}) = \vec{v}, \\ 1 & \text{if } \varphi_*(\vec{v}) \neq \vec{v}. \end{cases} \end{aligned} \tag{16}$$

Furthermore, if $P \in \Gamma_{\mathrm{Fix,Repel}}$, then

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2-d)\cdot(v_{\mathrm{FR}}(P)-2)+2d\cdot(\deg_{\varphi}(P)-1+N_{\mathrm{Shearing}}(P)). \quad (17)$$

Proposition 5.2. *Let $P \in \mathbf{H}_K^1$ be a type II* id*-indifferent fixed point of φ. Then for each $\vec{v} \in T_P$, the slope of $f(\,\cdot\,) = \mathrm{ordRes}_{\varphi}(\,\cdot\,)$ at P in the direction $\vec{v}$ is*

$$\partial_{\vec{v}} f(P) = (d^2-d) - 2d\cdot s_{\varphi}(P,\vec{v}). \quad (18)$$

Furthermore, if $P \in \Gamma_{\mathrm{Fix,Repel}}$, then

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2-d)\cdot(v_{\mathrm{FR}}(P)-2). \quad (19)$$

Proposition 5.3. *Let $P \in \mathbf{H}_K^1$ be a type II point with $\varphi(P) \neq P$. Then for each $\vec{v} \in T_P$, the slope of $f(\,\cdot\,) = \mathrm{ordRes}_{\varphi}(\,\cdot\,)$ at P in the direction $\vec{v}$ is*

$$\partial_{\vec{v}} f(P) = d^2+d-2d\cdot \#F_{\varphi}(P,\vec{v}). \quad (20)$$

Furthermore, if $P \in \Gamma_{\mathrm{Fix,Repel}}$, then

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2-d)\cdot(v_{\mathrm{FR}}(P)-2)+2d\cdot(v_{\mathrm{FR}}(P)-2). \quad (21)$$

Finally, we consider the “slope of $\mathrm{ordRes}_{\varphi}(\,\cdot\,)$ in the direction away from Q”, at a type I point Q. Since $\mathrm{ordRes}_{\varphi}(Q) = \infty$, the definition of the slope given in (15) does not make sense. Instead, for a suitable Q_1 near Q, we show that $\mathrm{ordRes}_{\varphi}(\,\cdot\,)$ has constant slope in the direction of Q_1, for each $P \in (Q, Q_1)$.

Proposition 5.4. *Let Q be a type I point. Then there is a $Q_1 \in \mathbf{H}_K^1$ such that for each $P \in (Q, Q_1)$, the slope of $f(\,\cdot\,) = \mathrm{ordRes}_{\varphi}(\,\cdot\,)$ at P, in the direction $\vec{v}_1 \in T_P$ which points towards Q_1, is*

$$\partial_{\vec{v}_1} f(P) = \begin{cases} -(d^2-d) & \text{if } Q \text{ is fixed by } \varphi, \\ -(d^2+d) & \text{if } Q \text{ is not fixed by } \varphi. \end{cases}$$

We will now prove Propositions 5.1–5.4.

Proof of Proposition 5.1. After a change of coordinates, we can assume that $P = \zeta_G$ and $\vec{v} = \vec{v}_0$. Let (F, G) be a normalized representation of φ; write $F(X,Y) = a_d X^d + \cdots + a_0 Y^d$ and $G(X,Y) = b_d X^d + \cdots + b_0 Y^d$. For each $A \in K^{\times}$ we have, by [Rumely 2015, formula 13],

$$\begin{aligned} &\mathrm{ordRes}_{\varphi}(\zeta_{0,|A|}) - \mathrm{ordRes}_{\varphi}(\zeta_G) \\ &\qquad = (d^2+d)\mathrm{ord}(A) - 2d\cdot \min\Bigl(\min_{0\le i\le d} \mathrm{ord}(A^i a_i),\ \min_{0\le j\le d} \mathrm{ord}(A^{j+1} b_j)\Bigr). \end{aligned}$$

By hypothesis, the reductions $\widetilde{F}$ and $\widetilde{G}$ are nonzero; thus there are indices i and j such that $\tilde{a}_i \neq 0$ and $\tilde{b}_j \neq 0$ or equivalently that $\mathrm{ord}(a_i) = 0$ and $\mathrm{ord}(b_j) = 0$. Let ℓ_1 be the least index i for which $\mathrm{ord}(a_i) = 0$ and let ℓ_2 be the least index j for which $\mathrm{ord}(b_j) = 0$. If $\mathrm{ord}(A) > 0$ and $\mathrm{ord}(A)$ is sufficiently small, then

$$\begin{aligned}\min\Big(\min_{0\le i\le d}\mathrm{ord}(A^i a_i),\ \min_{0\le j\le d}\mathrm{ord}(A^{j+1} b_j)\Big) &= \min\big(\mathrm{ord}(A^{\ell_1} a_{\ell_1}),\ \mathrm{ord}(A^{\ell_2+1} b_{\ell_2})\big)\\ &= \min(\ell_1, \ell_2+1)\cdot \mathrm{ord}(A).\end{aligned}$$

It follows that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v} = \vec{v}_0$ is

$$d^2 + d - 2d\cdot\min(\ell_1, \ell_2+1). \tag{22}$$

We will now reformulate (22) using dynamical invariants. Letting $\widetilde{F}, \widetilde{G} \in \tilde{k}[X, Y]$ be the reductions of F and G, we can factor $\widetilde{F} = \widetilde{A}\cdot\widetilde{F}_0$ and $\widetilde{G} = \widetilde{A}\cdot\widetilde{G}_0$ where $\widetilde{A} = \mathrm{GCD}(\widetilde{F}, \widetilde{G})$. Write $\mathrm{ord}_X(\widetilde{F})$ and $\mathrm{ord}_X(\widetilde{G})$ for the power with which X occurs as a factor of $\widetilde{F}$ and $\widetilde{G}$. Using Faber's theorem [2013a, Lemma 3.17], it follows that

$$s_\varphi(P, \vec{v}_0) = \mathrm{ord}_X(\widetilde{A}) = \min\big(\mathrm{ord}_X(\widetilde{F}), \mathrm{ord}_X(\widetilde{G})\big) = \min(\ell_1, \ell_2).$$

On the other hand, since P is not id-indifferent for φ, by Lemma 2.1

$$s_\varphi(P, \vec{v}_0) = \#F_\varphi(P, \vec{v}_0) - \#\widetilde{F}_\varphi(P, \vec{v}_0).$$

Note that $\#\widetilde{F}_\varphi(P, \vec{v}_0) = 0$ holds if and only if $\mathrm{ord}_X(\widetilde{F}_0) = 0$, which in turn holds if and only if $\ell_1 \le \ell_2$. Thus, if $\#\widetilde{F}_\varphi(P, \vec{v}_0) = 0$, then $\ell_1 \le \ell_2$ and

$$\min(\ell_1, \ell_2+1) = \ell_1 = s_\varphi(P, \vec{v}_0) = \#F_\varphi(P, \vec{v}_0).$$

On the other hand, if $\#\widetilde{F}_\varphi(P, \vec{v}_0) > 0$ then $\ell_2 < \ell_1$ and $s_\varphi(P, \vec{v}_0) = \ell_2$, so

$$\min(\ell_1, \ell_2+1) = \ell_2 + 1 = s_\varphi(P, \vec{v}_0) + 1 = \#F_\varphi(P, \vec{v}_0) - \#\widetilde{F}_\varphi(P, \vec{v}_0) + 1.$$

It follows that

$$\begin{aligned}\min(\ell_1, \ell_2+1) &= \begin{cases}\#F_\varphi(P, \vec{v}_0) & \text{if } \#\widetilde{F}_\varphi(P, \vec{v}_0) = 0,\\ \#F_\varphi(P, \vec{v}_0) - \#\widetilde{F}_\varphi(P, \vec{v}_0) + 1 & \text{if } \#\widetilde{F}_\varphi(P, \vec{v}_0) > 0\end{cases}\\ &= \#F_\varphi(P, \vec{v}_0) - \max(1, \#\widetilde{F}_\varphi(P, \vec{v}_0)) + 1.\end{aligned} \tag{23}$$

Inserting (23) in (22) yields the first formula in (16). The second formula in (16) follows from $s_\varphi(P, \vec{v}) = \#F_\varphi(P, \vec{v}) - \#\widetilde{F}_\varphi(P, \vec{v})$, since $\varphi_*(\vec{v}) = \vec{v}$ if and only if $\#\widetilde{F}_\varphi(P, \vec{v}) > 0$.

If $P \in \Gamma_{\mathrm{Fix,Repel}}$, note that by Lemma 2.1, for a direction $\vec{v} \in T_{P,\mathrm{FR}}$, then $\varphi_*(\vec{v}) \neq \vec{v}$ if and only if $\vec{v}$ is a shearing direction. To obtain (17), sum the second formula in (16) over all $\vec{v} \in T_{P,\mathrm{FR}}$, getting

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d)\cdot v_{\mathrm{FR}}(P) - 2d\cdot \sum_{\vec{v}\in T_{P,\mathrm{FR}}} s_\varphi(P, \vec{v}) + 2d\cdot N_{\mathrm{Shearing}}(P). \tag{24}$$

By Lemma 2.1, $T_{P,\mathrm{FR}}$ contains all $\vec{v} \in T_P$ such that $s_\varphi(P, \vec{v}) > 0$. It follows that

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} s_\varphi(P, \vec{v}) = d - \deg_\varphi(P).$$

Inserting this in (24) yields (17). $\square$

Proof of Proposition 5.2. After a change of coordinates, we can assume that $P = \zeta_G$ and $\vec{v} = \vec{v}_0$. Let (F, G) be a normalized representation of φ; write $F(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G(X, Y) = b_d X^d + \cdots + b_0 Y^d$. Letting ℓ_1 and ℓ_2 be the least index such that $\mathrm{ord}(a_i) = 0$ and $\mathrm{ord}(b_j) = 0$, respectively, just as in Proposition 5.1 one sees that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v} = \vec{v}_0$ is

$$d^2 + d - 2d \cdot \min(\ell_1, \ell_2 + 1). \tag{25}$$

Since P is id-indifferent for φ, the reductions $\widetilde{F}$ and $\widetilde{G}$ are nonzero and if $\widetilde{A} = \mathrm{GCD}(\widetilde{F}, \widetilde{G})$ then $\widetilde{F}(X, Y) = X \cdot \widetilde{A}(X, Y)$ and $\widetilde{G}(X, Y) = Y \cdot \widetilde{A}(X, Y)$. Thus $\ell_1 = \mathrm{ord}_X(\widetilde{A}(X, Y)) + 1$ and $\ell_2 = \mathrm{ord}_X(\widetilde{A}(X, Y))$. By Faber's theorem [2013a, Lemma 3.17], we have $s_\varphi(P, \vec{v}_0) = \mathrm{ord}_X(\widetilde{A})$. Hence

$$\min(\ell_1, \ell_2 + 1) = \mathrm{ord}_X(\widetilde{A}) + 1 = s_\varphi(P, \vec{v}_0) + 1. \tag{26}$$

Inserting (26) in (25) yields (18).

When $P \in \Gamma_{\mathrm{Fix,Repel}}$, to obtain (19) sum (18) over all $\vec{v} \in T_{P,\mathrm{FR}}$, getting

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d) \cdot v_{\mathrm{FR}}(P) - 2d \cdot \sum_{\vec{v} \in T_{P,\mathrm{FR}}} s_\varphi(P, \vec{v}). \tag{27}$$

By Lemma 4.5, $T_{P,\mathrm{FR}}$ contains all $\vec{v} \in T_P$ such that $s_\varphi(P, \vec{v}) > 0$. Since $\deg_\varphi(P) = 1$, it follows that

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} s_\varphi(P, \vec{v}) = d - \deg_\varphi(P) = d - 1.$$

Inserting this in (27), and simplifying, yields (19). $\square$

Proof of Proposition 5.3. To prove (20) we use the machinery from Lemma 2.2. As in that lemma, we choose $\gamma \in \mathrm{GL}_2(K)$ such that $\gamma(\zeta_G) = P$ and $\gamma^{-1}(\varphi(P)) = \zeta_{0,r}$, for some $r \in |K^\times|$ with $0 < r < 1$. By replacing φ with φ^γ we can assume that $P = \zeta_G$ and $\varphi(P) = \zeta_{0,r}$.

Let $c \in K^\times$ be such that $|c| = r$; put $\Phi(z) = (1/c)\varphi(z)$. Then $\Phi(\zeta_G) = \zeta_G$, so Φ has nonconstant reduction and $\varphi(z) = c \cdot \Phi(z)$. Let (F, G) be a normalized representation of Φ; then (cF, G) is a normalized representation of φ. Using this representation, put $H(X, Y) = XG(X, Y) - Y \cdot cF(X, Y)$; this yields $\widetilde{H}(X, Y) = X \cdot \widetilde{G}(X, Y)$. On the other hand, if the type I fixed points of φ (listed with multiplicities) are $(a_i : b_i)$ for $i = 1, \ldots, d+1$, normalized so that $\max(|a_i|, |b_i|) = 1$ for each i, there is a constant C with $|C| = 1$ such that $H(X, Y) = C \cdot \prod_{i=1}^{d+1}(b_i X - a_i Y)$. Reducing this $\pmod{\mathfrak{m}}$ gives

$$X\widetilde{G}(X, Y) = \widetilde{H}(X, Y) = \widetilde{C} \cdot \prod_{i=1}^{d+1}(\tilde{b}_i X - \tilde{a}_i Y). \tag{28}$$

We will now consider what this means for the slope of $\mathrm{ordRes}_\varphi(\cdot)$ at P in a direction $\vec{v} \in T_P$. We parametrize the directions $\vec{v} \in T_P$ by points $a \in \mathbb{P}_1(\tilde{k})$. We will consider three cases, corresponding to the directions $\vec{v}_0$, $\vec{v}_a$ for $0 \neq a \in \tilde{k}$, and $\vec{v}_\infty$.

First consider the direction $\vec{v}_0 \in T_P$. We use the normalized representation (cF, G) for φ, expanding $cF(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G(X, Y) = b_d X^d + \cdots + b_0 Y^d$. As usual, for each $A \in K^\times$,

$$\begin{aligned}&\mathrm{ordRes}_\varphi(\zeta_{0,|A|}) - \mathrm{ordRes}_\varphi(\zeta_G)\\ &\qquad = (d^2 + d)\mathrm{ord}(A) - 2d \cdot \min\Big(\min_{0 \le i \le d} \mathrm{ord}(A^i a_i), \min_{0 \le j \le d} \mathrm{ord}(A^{j+1} b_j)\Big).\end{aligned} \tag{29}$$

Let $N = N_0 = \#F_\varphi(P, \vec{v}_0)$ be the number of fixed points of φ in $B_P(\vec{v}_0)^-$. By (28) we have $X^{N-1} \parallel \widetilde{G}(X, Y)$, so $\mathrm{ord}(b_\ell) > 0$ for $\ell = 0, \dots, N-2$ and $\mathrm{ord}(b_{N-1}) = 0$. Since $|c| < 1$ we see that $\mathrm{ord}(a_\ell) > 0$ for all ℓ. It follows that if $\mathrm{ord}(A) > 0$ and $\mathrm{ord}(A)$ is sufficiently small, then

$$\min\Big(\min_{0 \le i \le d} \mathrm{ord}(A^i a_i), \min_{0 \le j \le d} \mathrm{ord}(A^{j+1} b_j)\Big) = \mathrm{ord}(A^{(N-1)+1} b_{N-1}) = N \cdot \mathrm{ord}(A).$$

Inserting this in (29) shows that the slope of $\mathrm{ordRes}_\varphi(\cdot)$ at P in the direction $\vec{v}_0$ is

$$\partial_{\vec{v}_0} f(P) = d^2 + d - 2d \cdot N_0 = d^2 + d - 2d \cdot \#F_\varphi(P, \vec{v}_0). \tag{30}$$

Next consider a direction $\vec{v}_a \in T_P$, where $0 \neq a \in \tilde{k}$. Choose an $\alpha \in \mathcal{O}$ with $\tilde{\alpha} = a$ and conjugate φ by

$$\gamma_\alpha = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}.$$

A normalized representation for $\varphi_\alpha := (\varphi)^{\gamma_\alpha}$ is given by

$$\begin{pmatrix} F_\alpha(X, Y) \\ G_\alpha(X, Y) \end{pmatrix} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} cF \\ G \end{pmatrix} \cdot \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} cF(X + \alpha Y, Y) - \alpha G(X + \alpha Y, Y) \\ G(X + \alpha Y, Y) \end{pmatrix}.$$

Reducing this gives $\widetilde{F}_\alpha(X, Y) = -a\widetilde{G}(X + aY, Y)$ and $\widetilde{G}_\alpha(X, Y) = \widetilde{G}(X + aY, Y)$.

Let $N = N_a = \#F_\varphi(P, \vec{v}_a)$ be the number of fixed points of φ in $B_P(\vec{v}_a)^-$; from (28) it follows that $(X - aY)^N \parallel \widetilde{G}(X, Y)$. The direction $\vec{v}_\alpha$ for φ pulls back to $\vec{v}_0$ for φ_α, so $N = \#F_{\varphi_\alpha}(P, \vec{v}_0)$. Thus $X^N \parallel \widetilde{F}_\alpha(X, Y)$ and $X^N \parallel \widetilde{G}_\alpha(X, Y)$. Expanding $F_\alpha(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G_\alpha(X, Y) = b_d X^d + \cdots + b_0 Y^d$, we see that $\mathrm{ord}(a_\ell), \mathrm{ord}(b_\ell) > 0$ for $\ell = 0, \dots, N-1$, while $\mathrm{ord}(a_N) = \mathrm{ord}(b_N) = 0$.

If $\mathrm{ord}(A) > 0$ is sufficiently small, it follows that

$$\min\bigl(\min_{0\le i\le d} \mathrm{ord}(A^i a_i),\ \min_{0\le j\le d} \mathrm{ord}(A^{j+1} b_j)\bigr) = \mathrm{ord}(A^N a_N) = N \cdot \mathrm{ord}(A).$$

Inserting this in (29) shows that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v}_a$ is

$$\partial_{\vec{v}_a} f(P) = d^2 + d - 2d \cdot N_a = d^2 + d - 2d \cdot \#F_\varphi(P, \vec{v}_a). \tag{31}$$

Finally, consider the direction $\vec{v}_\infty \in T_P$. Conjugate φ by

$$\gamma_\infty = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

A normalized representation for $\varphi_\infty := (\varphi)^{\gamma_\infty}$ is given by

$$\begin{pmatrix} F_\infty(X,Y) \\ G_\infty(X,Y) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{pmatrix} cF \\ G \end{pmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{pmatrix} G(Y,X) \\ cF(Y,X) \end{pmatrix}.$$

Reducing this gives $\widetilde{F}_\infty(X,Y) = \widetilde{G}(Y,X)$ and $\widetilde{G}_\infty(X,Y) = 0$.

Let $N = N_\infty = \#F_\varphi(P, \vec{v}_\infty)$ be the number of fixed points of φ in $B_P(\vec{v}_\infty)^-$; by formula (28) we have $Y^N \parallel \widetilde{G}(X,Y)$. The direction $\vec{v}_\infty$ for φ pulls back to $\vec{v}_0$ for φ_∞, so $N = \#F_{\varphi_\infty}(P, \vec{v}_0)$. Thus $X^N \parallel \widetilde{F}_\infty(X,Y)$. Writing $F_\infty(X,Y) = a_d X^d + \cdots + a_0 Y^d$ and $G_\infty(X,Y) = b_d X^d + \cdots + b_0 Y^d$, we see that $\mathrm{ord}(a_\ell) > 0$ for $\ell = 0, \ldots, N-1$, while $\mathrm{ord}(a_N) = 0$; we have $\mathrm{ord}(b_\ell) > 0$ for all ℓ. Thus if $\mathrm{ord}(A) > 0$ is sufficiently small,

$$\min\bigl(\min_{0\le i\le d} \mathrm{ord}(A^i a_i),\ \min_{0\le j\le d} \mathrm{ord}(A^{j+1} b_j)\bigr) = \mathrm{ord}(A^N a_N) = N \cdot \mathrm{ord}(A).$$

Inserting this in (29) shows that the slope of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ at P in the direction $\vec{v}_\infty$ is

$$\partial_{\vec{v}_\infty} f(P) = d^2 + d - 2d \cdot N_\infty = d^2 + d - 2d \cdot \#F_\varphi(P, \vec{v}_\infty). \tag{32}$$

Combining (30), (31) and (32) yields (20).

If $P \in \Gamma_{\mathrm{Fix,Repel}}$, to obtain (19) sum (20) over all $\vec{v} \in T_{P,\mathrm{FR}}$, getting

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2+d) \cdot v_{\mathrm{FR}}(P) - 2d \cdot \sum_{\vec{v}\in T_{P,\mathrm{FR}}} \#F_\varphi(P, \vec{v}). \tag{33}$$

By Lemma 2.2, $T_{P,\mathrm{FR}}$ contains all $\vec{v} \in T_P$ such that $\#F_\varphi(P, \vec{v}) > 0$. Since φ has $d+1$ type I fixed points (counting multiplicities), it follows that

$$\sum_{\vec{v}\in T_{P,\mathrm{FR}}} \#F_\varphi(P, \vec{v}) = d + 1.$$

Inserting this in (33) yields (21). $\square$

Proof of Proposition 5.4. After a change of coordinates, we can assume that $Q = 0$. Since $\mathrm{ordRes}_\varphi(\cdot)$ is piecewise affine on paths in $\mathbf{H}^1_K$ (relative to the logarithmic path distance), it suffices to prove the result when $P = \zeta_{0,r}$ is a type II point with r sufficiently small and $\vec{v}_1 \in T_P$ is the direction $\vec{v}_\infty$.

Write $F(X,Y) = a_d X^d + \cdots + a_0 Y^d$, $G(X,Y) = b_d X^d + \cdots + b_0 Y^d$ and let (F,G) be a normalized representation of φ. Take $A \in K^\times$ and put $r = |A|$; by [Rumely 2015, formula (19)]

$$\begin{aligned}&\mathrm{ordRes}_\varphi(\zeta_{0,r})\\ &= \mathrm{ordRes}(F,G) + (d^2+d)\mathrm{ord}(A) - 2d\min\Bigl(\min_{0\le\ell\le d}\mathrm{ord}(A^\ell a_\ell),\ \min_{0\le\ell\le d}\mathrm{ord}(A^{\ell+1}b_\ell)\Bigr).\end{aligned}$$

First suppose that Q is fixed by φ, so $\varphi(0) = 0$. In this situation $a_0 = 0$ and $b_0 \neq 0$, so if $r = |A|$ is sufficiently small, then

$$\min\Bigl(\min_{0\le\ell\le d}\mathrm{ord}(A^\ell a_\ell),\ \min_{0\le\ell\le d}\mathrm{ord}(A^{\ell+1}b_\ell)\Bigr)$$

coincides with

$$\min\bigl(\mathrm{ord}(Aa_1), \mathrm{ord}(Ab_0)\bigr) = \mathrm{ord}(A) + \min\bigl(\mathrm{ord}(a_1), \mathrm{ord}(b_0)\bigr).$$

For such r we have

$$\mathrm{ordRes}_\varphi(\zeta_{0,r}) = \mathrm{ordRes}(F,G) - 2d\min\bigl(\mathrm{ord}(a_1), \mathrm{ord}(b_0)\bigr) + (d^2-d)\mathrm{ord}(A),$$

and since $\mathrm{ord}(A)$ measures the logarithmic path distance and increases as $r \to 0$, the slope of $\mathrm{ordRes}_\varphi(\cdot)$ at $\zeta_{0,r}$ in the direction $\vec{v}_\infty$ is $-(d^2-d)$.

Next suppose Q is not fixed by φ, so $\varphi(0) \neq 0$. In this case $a_0 \neq 0$. If $r = |A|$ is sufficiently small, then $\min\bigl(\min_{0\le\ell\le d}\mathrm{ord}(A^\ell a_\ell), \min_{0\le\ell\le d}\mathrm{ord}(A^{\ell+1}b_\ell)\bigr) = \mathrm{ord}(a_0)$. For such r we have

$$\mathrm{ordRes}_\varphi(\zeta_{0,r}) = \mathrm{ordRes}(F,G) - 2d\,\mathrm{ord}(a_0) + (d^2+d)\mathrm{ord}(A),$$

so the slope of $\mathrm{ordRes}_\varphi(\cdot)$ at $\zeta_{0,r}$ in the direction $\vec{v}_\infty$ is $-(d^2+d)$. □

6. The weight formula and the crucial set

In this section, we prove the weight formula (Theorem A) as Theorem 6.1. We then give an intrinsic formula for the crucial measure (Corollary 6.5) and note some of the functoriality properties of the crucial set and crucial measure.

We begin by defining weights $w_\varphi(P)$, motivated by Propositions 5.1, 5.2, and 5.3.

Definition 8 (weights). For each $P \in \mathbf{P}^1_K$, let $w_\varphi(P)$ be the following nonnegative integer:

(1) If $P \in \mathbf{H}^1_K$ and P is fixed by φ, put

$$w_\varphi(P) = \deg_\varphi(P) - 1 + N_{\mathrm{Shearing}}(P).$$

(2) If $P \in \mathbf{H}_K^1$ and P is not fixed by φ, let $v(P)$ be the number of directions $\vec{v} \in T_P$ such that $B_P(\vec{v})^-$ contains a type I fixed point of φ, and put
$$w_\varphi(P) = \max(0, v(P) - 2).$$

(3) If $P \in \mathbb{P}^1(K)$, put $w_\varphi(P) = 0$.

Our main theorem (Theorem A) is:

Theorem 6.1 (weight formula). *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then*
$$\sum_{P \in \mathbf{P}_K^1} w_\varphi(P) = d - 1. \tag{34}$$

Corollary 6.2. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then*
$$\sum_{\substack{\text{repelling fixed points} \\ P \in \mathbf{H}_K^1}} (\deg_\varphi(P) - 1) \leq d - 1. \tag{35}$$

In particular φ can have most $d-1$ repelling fixed points in $\mathbf{H}_K^1$.

In [Rumely 2014] it is shown that for each $d \geq 2$, there exist functions $\varphi \in K(z)$ of degree d which have $d-1$ repelling fixed points in $\mathbf{H}_K^1$, so Corollary 6.2 is sharp.

Although the weight formulas in Definition 8 are mysterious, they arise naturally, as shown by the proof of Theorem 6.1 and by Corollary 6.5 which gives an intrinsic formula for the crucial measure ν_φ.

The author's efforts to understand the weights lead to the following observations. It is not surprising that for a fixed point in $\mathbb{H}_{\text{Berk}}$, the degree should enter into the weight. Given that (35) holds, if there are infinitely many fixed points, most must have degree 1 and weight 0, so the contribution $\deg_\varphi(P) - 1$ to the weight is understandable. The contribution from the shearing directions is more unexpected. The presence of a shearing direction $\vec{v} \in T_P$ indicates a difference between the local and global behavior of φ: " near P" points in the direction $\vec{v}$ are moved to another direction $\vec{w}$ but "far away from P" there is a classical fixed point in the direction $\vec{v}$. There is a hidden similarity between the weight formulas for fixed points and points that are moved. If P is fixed by φ, there need be no shearing directions, and all the shearing directions count in the weight. For a point $P \in \mathbf{H}_K^1$ which is moved by φ, there are automatically some shearing directions: If P is outside $\Gamma_{\text{Fix,Repel}}$ all the classical fixed points lie in one direction and that direction is sheared. If P is in $\Gamma_{\text{Fix,Repel}}$ but is not a branch point, there are two directions containing classical fixed points and those directions are sheared. Such points must have weight 0. However, if P is a branch point of $\Gamma_{\text{Fix,Repel}}$, there are more than two shearing directions, and the weight counts the excess.

The following proposition makes the weights more explicit and describes which points have positive weight and which have weight zero:

Proposition 6.3 (properties of weights). *Let $P \in \mathbf{P}^1_K$. If P is a focused repelling fixed point, then $w_\varphi(P) = \deg_\varphi(P) - 1$. If P is an additively indifferent fixed point in Γ_{Fix} or is a multiplicatively indifferent fixed point which is a branch point of Γ_{Fix}, then $w_\varphi(P) = N_{\mathrm{Shearing}}(P)$. If P is an* id*-indifferent fixed point, then $w_\varphi(P) = 0$. If P is a nonfixed branch point of Γ_{Fix}, then $w_\varphi(P) = v(P) - 2$. For all P, $w_\varphi(P)$ is an integer with $0 \le w_\varphi(P) \le d - 1$ and*:

(A) $w_\varphi(P) > 0$ *if*

   (1) *P is a type II repelling fixed point of φ, or*
   (2) *P is an additively indifferent fixed point of φ which belongs to Γ_{Fix}, or*
   (3) *P is a multiplicatively indifferent fixed point of φ which is a branch point of Γ_{Fix}, or*
   (4) *P is a branch point of Γ_{Fix} which is moved by φ.*

(B) $w_\varphi(P) = 0$ *if*

   (1) *$P \notin \Gamma_{\mathrm{Fix,Repel}}$, or*
   (2) *P is not of type II, or*
   (3) *P is an additively indifferent fixed point which is not in Γ_{Fix}, or*
   (4) *P is a multiplicatively indifferent fixed point which is not a branch point of Γ_{Fix}, or*
   (5) *P is an* id*-indifferent fixed point, or*
   (6) *P is moved by φ and is not a branch point of Γ_{Fix}.*

Proof. If P is a focused repelling fixed point, the unique direction $\vec{v} \in T_P$ such that $B_P(\vec{v})^-$ contains type I fixed points is fixed by φ_*, so $N_{\mathrm{Shearing}}(P) = 0$ and $w_\varphi(P) = \deg_\varphi(P) - 1$. If P is an additively indifferent or multiplicatively indifferent fixed point, then $\deg_\varphi(P) = 1$ so $w_\varphi(P) = N_{\mathrm{Shearing}}(P)$. If P is id-indifferent, then $\deg_\varphi(P) = 1$ and each $\vec{v} \in T_P$ is fixed by φ_*, so $N_{\mathrm{Shearing}}(P) = 0$; thus $w_\varphi(P) = 0$. If P is a nonfixed branch point of Γ_{Fix}, then $v(P) > 2$ so $w_\varphi(P) = v(P) - 2$.

Clearly each type II repelling fixed point has $w_\varphi(P) > 0$. By results of Rivera-Letelier [2003b, Lemmas 5.3 and 5.4] (or [Baker and Rumely 2010, Lemma 10.80]), each fixed point of type III or IV has degree 1 and each of its tangent directions is fixed by φ_*, hence $w_\varphi(P) = 0$. By definition, each point of type I has $w_\varphi(P) = 0$.

Each additively indifferent fixed point $P \notin \Gamma_{\mathrm{Fix}}$ has type I fixed points in exactly one tangent direction and Lemma 2.1 shows that direction is fixed by φ_*; hence $w_\varphi(P) = 0$. On the other hand, each additively indifferent fixed point $P \in \Gamma_{\mathrm{Fix}}$ has type I fixed points in at least two tangent directions, one of which must be a shearing direction since an additively indifferent fixed point has exactly one fixed direction; hence $w_\varphi(P) > 0$.

Likewise, a multiplicatively indifferent fixed point P has two fixed directions and Lemma 2.1 shows each of them must contain type I fixed points; thus P belongs to Γ_{Fix}. If P is not a branch point of Γ_{Fix}, its two fixed directions are the only ones containing type I fixed points, so it has no shearing directions, and $w_\varphi(P) = 0$. If P is a branch point of Γ_{Fix}, there is at least one $\vec{v} \in T_P$ containing a type I fixed point besides the two fixed directions and that direction is a shearing direction, so $w_\varphi(P) > 0$.

If $P \in \mathbf{H}_K^1$ is not fixed by φ, there are three possibilities. Recall that $v(P)$ is the number of directions in T_P containing a type I fixed point. If $P \notin \Gamma_{\text{Fix}}$, there is only one direction $\vec{v} \in T_P$ containing type I fixed points, so $v(P) = 1$, giving $w_\varphi(P) = 0$. If $P \in \Gamma_{\text{Fix}}$ but P is not a branch point of Γ_{Fix}, then $v(P) = 2$, giving $w_\varphi(P) = 0$. If P is a branch point of Γ_{Fix}, then $v(P) \geq 3$, so $w_\varphi(P) > 0$.

The fact that $0 \leq w_\varphi(P) \leq d - 1$ follows from Theorem 6.1, whose proof only uses the formulas in Definition 8. $\square$

Using Proposition 6.3, the weight formula (34) in Theorem 6.1 can be made more explicit:

$$\sum_{\substack{\text{focused}\\\text{repelling f.p.s}}} (\deg_\varphi(P) - 1) + \sum_{\substack{\text{bifocused and}\\\text{multifocused f.p.s}}} (\deg_\varphi(P) - 1 + N_{\text{Shearing}}(P))$$
$$+ \sum_{\substack{\text{additively}\\\text{indifferent f.p.s in } \Gamma_{\text{Fix}}}} N_{\text{Shearing}}(P) + \sum_{\substack{\text{multiplicatively}\\\text{indifferent f.b.p.s of } \Gamma_{\text{Fix}}}} N_{\text{Shearing}}(P) + \sum_{\substack{\text{nonfixed}\\\text{b.p.s of } \Gamma_{\text{Fix}}}} (v(P) - 2)$$
$$= d - 1, \quad (36)$$

where f.p.s, b.p.s, and f.b.p.s stand for fixed points, branch points, and fixed branch points, respectively.

The idea for the proof of Theorem 6.1 is to take the Laplacian of $\text{ordRes}_\varphi(\,\cdot\,)$ on finite metrized subgraphs of $\Gamma_{\text{Fix,Repel}}$; by simplifying the Laplacians and taking a limit over a cofinal sequence of subgraphs, we obtain the result. By a finite metrized graph we mean a connected subgraph with a finite number of vertices and edges, such that each edge is equipped with a metric which makes it isometric to an interval of finite length in $\mathbb{R}$. For subgraphs of $\mathbf{H}_K^1$ we will always take the metric to be the one induced by the logarithmic path distance $\rho(x, y)$.

If $\Gamma \subset \mathbf{H}_K^1$ is a finite metrized subgraph, let $\text{CPA}(\Gamma)$ be the space of functions on Γ which are continuous and piecewise affine (with a finite number of pieces) with respect to the logarithmic path distance. For each $P \in \Gamma$, we write $T_{P,\Gamma}$ for the tangent space to P in Γ, namely the set of $\vec{v} \in T_P$ such that there is an edge of Γ emanating from P in the direction $\vec{v}$. If the valence of P in Γ is $v(P)$, then $T_{P,\Gamma}$ has $v(P)$ elements. If $f \in \text{CPA}(\Gamma)$, the Laplacian of F (as defined in [Baker and Rumely 2007]) is the measure

$$\Delta_\Gamma(f) = \sum_{P\in\Gamma} -\Bigg(\sum_{\vec{v}\in T_{P,\Gamma}} \partial_{\vec{v}}(f)(P)\Bigg)\delta_P(z), \tag{37}$$

where $\delta_P(z)$ is the Dirac measure at P. Here $\Delta_\Gamma(F)$ is a discrete measure, since the inner sum in (37) is 0 at any P which is not a branch point of Γ and where F does not change slope. It is well known (see for example [Baker and Rumely 2007]) that $\Delta_\Gamma(F)$ has total mass 0. The proof is as follows. Note that for any partition of Γ into finitely many segments $[P_i, Q_i]$ (disjoint except for their endpoints), such that the restriction of F to each $[P_i, Q_i]$ is affine, the slopes of F at P_i and Q_i, in the directions pointing into $[P_i, Q_i]$, are negatives of each other.

Before proving Theorem 6.1, we will need an identity relating the number of endpoints of a tree to the valences of its internal branch points. If Γ is a finite graph and $P \in \Gamma$, the valence $v_\Gamma(P)$ is the number of edges of Γ incident at P.

Lemma 6.4. *Let Γ be a finite tree with D endpoints. Then*

$$D - \sum_{\text{b.p.s } P\in\Gamma} (v_\Gamma(P) - 2) = 2, \tag{38}$$

where b.p.s stands for "branch points".

Proof. This is a reformulation of Euler's formula $V - E + F = 2$ for planar graphs.

If B is the number of branch points of Γ, then $V = D + B$. Each edge has two endpoints, so $E = \frac{1}{2}\big(\sum_{\text{vertices}} v(P)\big)$. Since Γ is a tree, if it is embedded as a planar graph then $F = 1$. Inserting these in Euler's formula yields

$$2D + 2B - \sum_{\text{e.p.s}} v(P) - \sum_{\text{b.p.s}} v(P) = 2, \tag{39}$$

where e.p.s stands for end points. At each endpoint we have $v(P) = 1$, so $\sum_{\text{e.p.s}} v(P) = D$. There are B branch points, so $2B = \sum_{\text{b.p.s}} 2$. Combining terms in (39) gives (38). □

Proof of Theorem 6.1. The idea is to restrict $\mathrm{ordRes}_\varphi(\cdot)$ to finite subgraphs of $\Gamma_{\text{Fix,Repel}}$, take its Laplacian and simplify. Since $\Gamma_{\text{Fix,Repel}}$ has branches of infinite length, to apply the theory of graph Laplacians for metrized graphs from [Baker and Rumely 2007], we cut off segments at the type I endpoints of $\Gamma_{\text{Fix,Repel}}$, obtaining a finite metrized tree Γ_{FR}. We then exhaust $\Gamma_{\text{Fix,Repel}}$ by a sequence of such trees.

For each type I fixed point α_i of φ, choose a type II point $Q_i \in \Gamma_{\text{Fix,Repel}}$ close enough to α_i that

(1) there are no branch points of $\Gamma_{\text{Fix,Repel}}$ in $[Q_i, \alpha_i]$, and

(2) at each $P \in [Q_i, \alpha_i)$, the slope of $\mathrm{ordRes}_\varphi(\cdot)$ at P in the (unique) direction $\vec{v} \in T_P$ pointing away from α_i in $\Gamma_{\text{Fix,Repel}}$ is $-(d^2 - d)$.

Since any two paths emanating from α_i share a common initial segment, such Q_i exist by Proposition 5.4.

Let Γ_{FR} be the subtree of $\Gamma_{\mathrm{Fix,Repel}}$ spanned by the focused repelling fixed points and the points Q_i. Suppose there are D_1 distinct type I fixed points (ignoring multiplicities) and D_2 focused repelling fixed points. Then the number of endpoints of Γ_{FR} is

$$D = D_1 + D_2.$$

Let $f(\,\cdot\,)$ be the restriction of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ to Γ_{FR}. Then f belongs to $\mathrm{CPA}(\Gamma_{\mathrm{FR}})$.

For each $P \in \Gamma_{\mathrm{FR}}$, write $T_{P,\mathrm{FR}}$ for its tangent space in Γ_{FR} (the set of $\vec{v} \in T_P$ such that there is an edge of Γ_{FR} emanating from P in the direction $\vec{v}$). If $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$, then $T_{P,\mathrm{FR}} = T_{P,\mathrm{FR}}$ and the valence of P in Γ_{FR} coincides with $v_{\mathrm{FR}}(P)$. If $P \in \{Q_1, \dots, Q_{D_1}\}$ then $T_{P,\mathrm{FR}}$ consists of the single direction $\vec{v}_{P,1} \in T_P$ pointing into Γ_{FR}.

For the Laplacian $\Delta_{\mathrm{FR}}(f)$ we have

$$-\Delta_{\mathrm{FR}}(f) = \sum_{P \in \{Q_1,\dots,Q_{D_1}\}} \partial_{\vec{v}_{P,1}} f(P)\delta_P(z) + \sum_{P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1,\dots,Q_{D_1}\}} \Bigg(\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) \Bigg) \delta_P(z). \tag{40}$$

By Proposition 5.4, if $P \in \{Q_1, \dots, Q_{D_1}\}$ then $\partial_{\vec{v}_{P,1}} f(P) = -(d^2 - d)$. We claim that if $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$, then

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2 - d) \cdot (v_{\mathrm{FR}}(P) - 2) + 2d \cdot w_\varphi(P). \tag{41}$$

If $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$ is of type II and is a repelling fixed point, or is a multiplicatively or additively indifferent fixed point, then (41) follows from Proposition 5.1 and the definition of $w_\varphi(P)$. If P is an id-indifferent fixed point, then (41) follows from Proposition 5.2 since $w_\varphi(P) = 0$ by Proposition 6.3.

If $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$ is a type II point with $\varphi(P) \neq P$, then by Propositions 3.1 and 3.5, P belongs to Γ_{Fix} and is not a point where a branch of $\Gamma_{\mathrm{Fix,Repel}} \setminus \Gamma_{\mathrm{Fix}}$ attaches to Γ_{Fix}. It follows that $v_{\mathrm{FR}}(P)$ (which is the valence of P in Γ_{FR} and $\Gamma_{\mathrm{Fix,Repel}}$) coincides with $v(P)$ (its valence in Γ_{Fix}). Hence (41) follows from Proposition 5.3 and the definition of $w_\varphi(P)$.

If $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$ is a type III point, then $v_{\mathrm{FR}}(P) = 2$ and $w_\varphi(P) = 0$, so the right side of (41) is 0. The left side of (41) is also 0, since $\mathrm{ordRes}_\varphi(\,\cdot\,)$ can change slope only at points of type II (see Theorem 1.1). Each $P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}$ is either of type II or type III, so this establishes (41) in all cases.

Since $\Delta_{\mathrm{FR}}(f)$ has total mass 0, it follows from (40) and (41) that

$$D_1 \cdot (-(d^2 - d)) + \sum_{P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1,\dots,Q_{D_1}\}} \Big((d^2 - d) \cdot (v_{\mathrm{FR}}(P) - 2) + 2d \cdot w_\varphi(P) \Big) = 0. \tag{42}$$

If P is a focused repelling fixed point, then $v_{\mathrm{FR}}(P) = 1$, so $v_{\mathrm{FR}}(P) - 2 = -1$. If P is not an endpoint or a branch point of Γ_{FR} then $v_{\mathrm{FR}}(P) - 2 = 0$. Moving the terms in (42) involving $(d^2 - d)$ to the right side, and noting that there are D_2 focused repelling fixed points, it follows that

$$\sum_{P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}} 2d \cdot w_\varphi(P) = (d^2 - d) \cdot \Bigg(D_1 + D_2 - \sum_{\text{b.p.s of } \Gamma_{\mathrm{FR}}} (v_{\mathrm{FR}}(P) - 2) \Bigg), \quad (43)$$

where b.p.s stands for branch points. By Lemma 6.4 the sum on the right side of (43) equals 2. Dividing through by $2d$ gives

$$\sum_{P \in \Gamma_{\mathrm{FR}} \setminus \{Q_1, \dots, Q_{D_1}\}} w_\varphi(P) = d - 1. \quad (44)$$

If we let the endpoints points Q_i approach the type I fixed points α_i, the corresponding graphs Γ_{FR} exhaust $\Gamma_{\mathrm{Fix,Repel}} \cap \mathbf{H}_K^1$. By Proposition 6.3, we have $w_\varphi(P) = 0$ for each type I fixed point and for each $P \in \mathbf{P}_K^1 \setminus \Gamma_{\mathrm{Fix,Repel}}$. Thus

$$\sum_{P \in \mathbf{P}_K^1} w_\varphi(P) = d - 1. \qquad \square$$

Definition 9 (the crucial set, crucial weights, and crucial measure). The set of points $P \in \mathbf{P}_K^1$ with weight $w_\varphi(P) > 0$ will be called the *crucial set* and their weights $w_\varphi(P)$ will be called the *crucial weights*. The probability measure

$$\nu_\varphi = \frac{1}{d-1} \sum_{P \in \mathbf{P}_K^1} w_\varphi(P) \delta_P(z) \quad (45)$$

will be called the *crucial measure* of φ.

The crucial set consists of the repelling fixed points in $\mathbf{H}_K^1$, the indifferent fixed points with a shearing direction, and the branch points of Γ_{Fix} which are moved by φ. It has at most $d-1$ elements. If φ has potential good reduction, it consists of the unique point where $\deg_\varphi(P) = d$. However, it can also consist of a single point in other ways: for each $k = 1, \dots, d-1$, it could be a fixed point P with $\deg_\varphi(P) = k$ and $d - k$ shearing directions or it could be a branch point of Γ_{Fix} with valence $d + 1$, which is moved. All of these possibilities actually occur; see [Rumely 2014].

If Γ is a finite metrized graph, there is a natural measure of total mass 1 attached to Γ, its "Branching Measure" $\mu_{\Gamma,\mathrm{Br}}$. This measure was introduced in [Chinburg and Rumely 1993], where it was called the canonical measure and used to define a pairing on the Arakelov divisor class group of a curve. It was studied in detail in [Baker and Rumely 2007] and was generalized by Shouwu Zhang [1993], who used it to construct the dualizing metric for the Serre duality theorem in nonarchimedean Arakelov theory.

When Γ is a tree, $\mu_{\Gamma,\mathrm{Br}}$ is supported on the branch points and endpoints of Γ and is given by the formula

$$\mu_{\Gamma,\mathrm{Br}} = \sum_{P\in\Gamma} \bigl(1 - \tfrac{1}{2} v_\Gamma(P)\bigr)\delta_P(z).$$

This formula makes sense even when Γ has edges of infinite length. Note that $\mu_{\Gamma,\mathrm{Br}}$ gives mass $\frac{1}{2}$ to each endpoint of Γ and negative mass $1-\frac{1}{2}v_\Gamma(P)$ to each branch point, so it is not in general a positive measure. The fact that $\mu_{\Gamma,\mathrm{Br}}$ has total mass 1 follows from Lemma 6.4. When $\Gamma = \Gamma_{\mathrm{Fix,Repel}}$, we will write μ_{Br} for its branching measure.

We can obtain an intrinsic formula for ν_φ by taking the Laplacian of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ on the tree $\Gamma_{\mathrm{Fix,Repel}}$: suitably normalized, the Laplacian decomposes as the difference of μ_{Br} and ν_φ. Note that μ_{Br} depends only on the topology of $\Gamma_{\mathrm{Fix,Repel}}$, while ν_φ encodes analytic properties of φ. Thus, ν_φ is the analytic part of a decomposition of the normalized Laplacian into topological and analytic parts.

Corollary 6.5. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Let $f(\,\cdot\,)$ be the restriction of* $\mathrm{ordRes}_\varphi(\,\cdot\,)$ *to $\Gamma_{\mathrm{Fix,Repel}}$ and let $\Delta_{\mathrm{Fix,Repel}}(f)$ be its Laplacian. Then*

$$\frac{1}{2d(d-1)}\Delta_{\mathrm{Fix,Repel}}(f) = \mu_{\mathrm{Br}} - \nu_\varphi.$$

Proof. Since $\Gamma_{\mathrm{Fix,Repel}}$ has edges of infinite length and $\mathrm{ordRes}_\varphi(\,\cdot\,)$ is unbounded, the Laplacian $\Delta_{\mathrm{Fix,Repel}}(f)$ must be defined in a more sophisticated way than formula (37). (See [Baker and Rumely 2010, §5] and [Jonsson 2015, §2] for general discussions of Laplacians on mildly infinite graphs.) Here is a distributional definition for it. Let $\mathcal{C}(\Gamma_{\mathrm{Fix,Repel}})$ be the space of continuous functions $\eta : \Gamma_{\mathrm{Fix,Repel}} \to \mathbb{R}$. Since $\Gamma_{\mathrm{Fix,Repel}}$ is compact, such functions are necessarily bounded. Let $\Lambda_f : \mathcal{C}(\Gamma_{\mathrm{Fix,Repel}}) \to \mathbb{R}$ be the continuous linear functional defined by

$$\Lambda_f(\eta) = \varinjlim_{\Gamma\subset\Gamma_{\mathrm{Fix,Repel}}} \int_\Gamma \eta\,\Delta_\Gamma(f),$$

where the limit is taken over the set of finite subgraphs $\Gamma \subset \Gamma_{\mathrm{Fix,Repel}}$, partially ordered by inclusion. Then $\Delta_{\mathrm{Fix,Repel}}(f)$ is the Radon measure representing Λ_f.

Let $\Gamma = \Gamma_{\mathrm{FR}}$ be a finite subgraph of $\Gamma_{\mathrm{Fix,Repel}}$ of the type considered in the proof of Theorem 6.1. Reformulating (40) using (41) and Proposition 5.4, one has

$$\Delta_\Gamma(f) = \sum_{P\in\{Q_1,\dots,Q_{D_1}\}} (d^2-d)\delta_P - \sum_{P\in\Gamma\setminus\{Q_1,\dots,Q_{D_1}\}} \bigl((d^2-d)(v_\Gamma(P)-2) + 2d\cdot w_\varphi(P)\bigr)\delta_P.$$

Using the definitions of $\mu_{\Gamma,\mathrm{Br}}$ and ν_φ, this can be rewritten as

$$\Delta_\Gamma(f) = 2d(d-1)\cdot(\mu_{\Gamma,\mathrm{Br}} - \nu_\varphi).$$

Letting $Q_1, \dots, Q_{D_1}$ approach the type I endpoints of $\Gamma_{\mathrm{Fix,Repel}}$, one sees that for any test function $\eta \in \mathcal{C}(\Gamma_{\mathrm{Fix,Repel}})$,

$$\int_\Gamma \eta\, \mu_{\Gamma,\mathrm{Br}} \to \int_{\Gamma_{\mathrm{Fix,Repel}}} \eta\, \mu_{\mathrm{Br}}.$$

Hence $\Delta_{\mathrm{Fix,Repel}}(f) = 2d(d-1)\cdot(\mu_{\mathrm{Br}} - \nu_\varphi)$. □

We close this section by noting some functoriality properties of the crucial set. First, the weights, crucial set, and crucial measure have the same equivariance properties under conjugation as fixed points:

Proposition 6.6. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let $\gamma \in \mathrm{GL}_2(K)$. Viewing γ as a linear fractional transformation, put $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$. Then the weights, crucial set, and crucial measure satisfy*

$$w_{\varphi^\gamma}(\gamma^{-1}(P)) = w_\varphi(P), \quad \mathrm{Cr}(\varphi^\gamma) = \gamma^{-1}(\mathrm{Cr}(\varphi)), \quad \textit{and} \quad \nu_{\varphi^\gamma} = \gamma^*(\nu_\varphi).$$

Proof. The group $\mathrm{GL}_2(K)$ acts transitively on type II points and, by the definition of $\mathrm{ordRes}_\varphi(\,\cdot\,)$ on type II points, for each $\tau \in \mathrm{GL}_2(K)$, if $P = \tau(\zeta_G)$ then $\mathrm{ordRes}_\varphi(\tau(\zeta_G)) = \mathrm{ord}(\mathrm{Res}(\varphi^\tau))$. It follows that

$$\mathrm{ordRes}_{\varphi^\gamma}(\tau(\zeta_G)) = \mathrm{ord}(\mathrm{Res}((\varphi^\gamma)^\tau)) = \mathrm{ordRes}_\varphi(\gamma(\tau(\zeta_G))).$$

Since $\mathrm{ordRes}_\varphi(\,\cdot\,)$ is determined by continuity from its values on type II points, we have $\mathrm{ordRes}_{\varphi^\gamma}(P) = \mathrm{ordRes}_\varphi(\gamma(P))$ for all $P \in \mathbf{P}^1_K$. Replacing P by $\gamma^{-1}(P)$ shows that

$$\mathrm{ordRes}_{\varphi^\gamma}(\gamma^{-1}(P)) = \mathrm{ordRes}_\varphi(P).$$

Furthermore, P is a fixed point of φ if and only if $\gamma^{-1}(P)$ is a fixed point of φ^γ and in that case the points have the same local degrees. This implies that $\Gamma_{\mathrm{Fix,Repel}}(\varphi^\gamma) = \gamma^{-1}(\Gamma_{\mathrm{Fix,Repel}}(\varphi))$. The assertions in the Proposition follow by taking Laplacians. □

Next, we show the crucial set and crucial measure are preserved under extension of the ground field. If $L \supset K$ is another complete, algebraically closed field, there is a canonical embedding $\mathbf{P}^1_K \hookrightarrow \mathbf{P}^1_L$. This is a special case of a general result about what Berkovich [1990] called *peaked points* and Poineau [2013] renamed *universal points*. Poineau[1] shows that whenever K is algebraically closed and L/K is any field extension, if X_K is any Berkovich space over K, then there is a canonical embedding $X_K \hookrightarrow X_L$.

[1]The author thanks one of the anonymous referees for bringing Poineau's result to his attention.

Proposition 6.7. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let $L \supseteq K$ be another complete algebraically closed valued field, whose valuation extends the one on K. Using subscripts L and K to denote objects computed over the corresponding ground fields, we have*

$$\mathrm{Cr}_L(\varphi) = \mathrm{Cr}_K(\varphi) \quad \textit{and} \quad \nu_{\varphi,L} = \nu_{\varphi,K}.$$

Proof. Faber [2013a, §4] constructs the embedding $\mathbf{P}^1_K \hookrightarrow \mathbf{P}^1_L$ in an elementary way and shows that it respects the action of φ on points and tangent spaces and preserves path distances and degrees of points. By the formulas in Definition 8, for each $P \in \mathbf{P}^1_K$ we have $w_{\varphi,L}(P) \geq w_{\varphi,K}(P)$. Applying Theorem 6.1 over L and over K we see that

$$\sum_{P \in \mathbf{P}^1_L} w_{\varphi,L}(P) = d - 1 = \sum_{P \in \mathbf{P}^1_K} w_{\varphi,K}(P).$$

Hence $w_{\varphi,L}(P) = w_{\varphi,K}(P)$ for each $P \in \mathbf{P}^1_K$ and $w_{\varphi,L}(P) = 0$ for each $P \in \mathbf{P}^1_L \backslash \mathbf{P}^1_K$. The assertions in the proposition follow from this. $\square$

Finally, the weights, crucial set, and crucial measures are Galois-equivariant. Recall that if σ is a continuous automorphism of K, then the action of σ on K extends to an action of σ on $\mathbf{P}^1_K$ which is continuous for both the weak and strong topologies, respects the logarithmic path distance, and is given on points of type II and type III by $\sigma(\zeta_{a,r}) = \zeta_{\sigma(a),r}$.

Proposition 6.8. *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let σ be a continuous automorphism of K. Then the weights, crucial set, and crucial measure satisfy*

$$w_{\sigma(\varphi)}(\sigma(P)) = w_\varphi(P), \quad \mathrm{Cr}(\sigma(\varphi)) = \sigma(\mathrm{Cr}(\varphi)), \quad \textit{and} \quad \nu_{\sigma(\varphi)} = \sigma_*(\nu_\varphi).$$

Proof. Since σ respects absolute value on K and since the resultant is defined by a determinant in the coefficients of φ, it follows that for each $\tau \in \mathrm{GL}_2(K)$,

$$\mathrm{ord}(\mathrm{Res}(\varphi^\tau)) = \mathrm{ord}(\mathrm{Res}(\sigma(\varphi^\tau))) = \mathrm{ord}(\mathrm{Res}(\sigma(\varphi)^{\sigma(\tau)})).$$

Since $\zeta_G = \zeta_{0,1}$, the description of the action of σ on type II points shows that $\sigma(\zeta_G) = \zeta_G$. This means that for the type II point $P = \tau(\zeta_G)$, one has $\sigma(P) = \sigma(\tau)(\zeta_G)$ and hence

$$\begin{aligned}\mathrm{ordRes}_{\sigma(\varphi)}(\sigma(P)) &= \mathrm{ordRes}_{\sigma(\varphi)}(\sigma(\tau)(\zeta_G)) = \mathrm{ord}(\mathrm{Res}(\sigma(\varphi)^{\sigma(\tau)})) \\ &= \mathrm{ord}(\mathrm{Res}(\varphi^\tau)) = \mathrm{ordRes}_\varphi(\tau(\zeta_G)) = \mathrm{ordRes}_\varphi(P).\end{aligned}$$

By continuity, this holds for all $P \in \mathbf{P}^1_K$.

Furthermore, P is a fixed point of φ if and only if $\sigma(P)$ is a fixed point of $\sigma(\varphi)$ and in that case the points have the same local degrees. This implies that

$\Gamma_{\mathrm{Fix,Repel}}(\sigma(\varphi)) = \sigma(\Gamma_{\mathrm{Fix,Repel}}(\varphi))$. The assertions in the proposition follow by taking Laplacians. □

7. Characterizations of the minimal resultant locus

In this section, we characterize MinResLoc(φ) from analytic and moduli-theoretic standpoints. We first give an analytic characterization. Before stating it we need two definitions.

Definition 10 (barycenter). The barycenter of a finite positive measure ν on $\mathbf{P}^1_K$ is the set of points $Q \in \mathbf{P}^1_K$ such that for each direction $\vec{v} \in T_Q$, at most half the mass of ν lies in $B_Q(\vec{v})^-$.

The notion of the barycenter of a measure on $\mathbf{P}^1_K$ is due to Benedetto and Rivera-Letelier, who used it in unpublished work on the "$s \log s$" bound for the number of k-rational preperiodic points of $\varphi(z) \in k(z)$, when k is a number field.

Definition 11 (the crucial tree). The crucial tree Γ_φ is the subtree of $\Gamma_{\mathrm{Fix,Repel}}$ spanned by the crucial set of φ. We define the *vertices* of Γ_φ to be the points of the crucial set (whether they are endpoints or interior points of Γ_φ) and the branch points of Γ_φ. The *edges* of Γ_φ are the closed segments between adjacent vertices.

Theorem 7.1 (dynamical characterization of MinResLoc(φ)). *Let $\varphi(z) \in K(z)$ have degree $d \geq 2$. Then* MinResLoc(φ) *is the barycenter of the crucial measure ν_φ. Equivalently, a point $Q \in \mathbf{P}^1_K$ belongs to* MinResLoc(φ) *if and only if for each $\vec{w} \in T_Q$*

$$\sum_{P \in B_Q(\vec{w})^-} w_\varphi(P) \leq \frac{d-1}{2}. \tag{46}$$

If d is even, then MinResLoc(φ) *is a vertex of the crucial tree Γ_φ. If d is odd, then* MinResLoc(φ) *is either a vertex or an edge of Γ_φ.*

Proof. To show that MinResLoc(φ) is the barycenter of ν_φ, we must show for each $Q \in \mathbf{P}^1_K$ that $Q \in$ MinResLoc(φ) if and only if for each $\vec{w} \in T_Q$,

$$\nu_\varphi(B_Q(\vec{w})^-) \leq \tfrac{1}{2}.$$

If $Q \notin \Gamma_{\mathrm{Fix,Repel}}$ this is trivial: $Q \notin$ MinResLoc(φ) since

$$\mathrm{MinResLoc}(\varphi) \subset \Gamma_{\mathrm{Fix,Repel}} \cap \mathbf{H}^1_K,$$

while if $\vec{w} \in T_Q$ is the direction towards $\Gamma_{\mathrm{Fix,Repel}}$ then $\nu_\varphi(B_Q(\vec{v})^-) = 1$. Similar reasoning applies when $Q \in \Gamma_{\mathrm{Fix,Repel}} \cap \mathbb{P}^1(K)$.

Let $Q \in \Gamma_{\mathrm{Fix,Repel}} \cap \mathbf{H}^1_K$ and write $f(\cdot) = \mathrm{ordRes}_\varphi(\cdot)$. Then $Q \in$ MinResLoc(φ) if and only if $\partial_{\vec{w}} f(Q) \geq 0$ for each $\vec{w} \in T_Q$. If $\vec{w} \in T_Q$ points away from $\Gamma_{\mathrm{Fix,Repel}}$ then $\partial_{\vec{w}} f(Q) > 0$ and $\nu_\varphi(B_Q(\vec{w})^-) = 0$. Hence it is enough to show that for each

$\vec{w} \in T_{Q,\mathrm{FR}}$, we have $\partial_{\vec{w}} f(Q) \geq 0$ if and only if $\nu_\varphi(B_Q(\vec{w})^-) \leq \frac{1}{2}$, or equivalently that (46) holds. For this, we use an argument like the one in the proof of Theorem 6.1.

Let Γ be the graph consisting of the part of $\Gamma_{\mathrm{Fix,Repel}}$ in $B_Q(\vec{w})^-$, together with Q. The endpoints of Γ are Q and the type I fixed points and focused repelling fixed points of φ in $B_Q(\vec{w})^-$. Let D_1' be the number of type I endpoints and let D_2' be the number of focused repelling endpoints, so Γ has $D' = 1 + D_1' + D_2'$ endpoints in all.

Suppose the type I fixed points of φ in $B_Q(\vec{w})^-$ are $\alpha_1, \ldots, \alpha_{D_1'}$. For each α_i, choose a type II point Q_i in $\Gamma_{\mathrm{Fix,Repel}}$ close enough to α_i that

(1) there are no branch points of Γ in $[Q_i, \alpha_i]$,

(2) at each $P \in [Q_i, \alpha_i)$, the slope of $\mathrm{ordRes}_\varphi(\cdot)$ at P in the (unique) direction $\vec{v}_i \in T_P$ pointing away from α_i in Γ is $-(d^2-d)$, and

(3) each $P \in [Q_i, \alpha_i]$ has weight $w_\varphi(P) = 0$.

Since only finitely many points have positive weight, Proposition 5.4 shows that such Q_i exist. Let $\widehat{\Gamma}$ be the finite metrized graph gotten by cutting the terminal segments $(Q_i, \alpha_i]$ off of Γ.

To simplify notation, write Q_0 for Q and let $\vec{v}_0 = \vec{w} \in T_Q$. Then the Laplacian $\Delta_{\widehat{\Gamma}}(f)$ satisfies

$$-\Delta_{\widehat{\Gamma}}(f) = \sum_{P \in \{Q_0, Q_1, \ldots, Q_{D_1'}\}} \partial_{\vec{v}_i} f(P) \delta_P(z) + \sum_{P \in \widehat{\Gamma} \setminus \{Q_0, Q_1, \ldots, Q_{D_1'}\}} \Bigg(\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) \Bigg) \delta_P(z). \tag{47}$$

Put $L = \partial_{\vec{w}} f(Q) = \partial_{\vec{v}_0} f(Q_0)$. By Proposition 5.4, if $P \in \{Q_1, \ldots, Q_{D_1'}\}$ then $\partial_{\vec{v}_i} f(P) = -(d^2-d)$. By formula (41), if $P \in \widehat{\Gamma} \setminus \{Q_0, Q_1, \ldots, Q_{D_1'}\}$, then

$$\sum_{\vec{v} \in T_{P,\mathrm{FR}}} \partial_{\vec{v}} f(P) = (d^2-d) \cdot (v_{\widehat{\Gamma}}(P) - 2) + 2d \cdot w_\varphi(P).$$

Inserting these values in (47) and using that $\Delta_{\widehat{\Gamma}}(f)$ has total mass 0, we see that

$$0 = L + D_1' \cdot (-(d^2-d)) + \sum_{P \in \widehat{\Gamma} \setminus \{Q_0, Q_1, \ldots, Q_{D_1'}\}} \big((d^2-d) \cdot (v_{\widehat{\Gamma}}(P) - 2) + 2d \cdot w_\varphi(P) \big).$$

If $P \in \widehat{\Gamma}$ is not an endpoint or a branch point, then $(d^2-d) \cdot (v_{\widehat{\Gamma}}(P) - 2) = 0$. There are D_2' focused repelling endpoints of $\widehat{\Gamma}$ in $B_Q(\vec{w})^-$ and for each of them we have $(d^2-d) \cdot (v_{\widehat{\Gamma}}(P) - 2) = -(d^2-d)$. Hence

$$L = \Bigg(D_1' + D_2' - \sum_{\text{b.p.s of } \widehat{\Gamma}} (v_{\widehat{\Gamma}}(P) - 2) \Bigg) \cdot (d^2-d) - 2d \cdot \sum_{P \in \widehat{\Gamma} \cap B_P(\vec{w})^-} w_\varphi(P). \tag{48}$$

Since $\widehat{\Gamma}$ has $D' = 1 + D_1' + D_2'$ endpoints, Lemma 6.4 gives

$$D_1' + D_2' - \sum_{\text{b.p.s of } \widehat{\Gamma}} (v_{\widehat{\Gamma}}(P) - 2) = 1. \tag{49}$$

It follows from (48) and (49) that $L \geq 0$ if and only if (46) holds. Thus Q belongs to $\mathrm{MinResLoc}(\varphi)$ if and only if Q is in the barycenter of ν_φ.

Clearly $\mathrm{MinResLoc}(\varphi) \subseteq \Gamma_\varphi$. To see that $\mathrm{MinResLoc}(\varphi)$ is either a vertex or an edge of Γ_φ, note that as a point P moves along Γ_φ, the distribution of ν_φ-mass in the various directions $\vec{v} \in T_P$ can change only when P passes through a vertex.

If $\mathrm{MinResLoc}(\varphi)$ consists of a vertex of Γ_φ, then we are done. Otherwise, $\mathrm{MinResLoc}(\varphi)$ contains a point Q in the interior of an edge e of Γ_φ. Since there are precisely two directions $\vec{v} \in T_Q$ for which the balls $B_Q(\vec{v})^-$ can contain ν_φ-mass and since each has mass at most $\frac{1}{2}$, each must have mass exactly $\frac{1}{2}$. This continues to hold for all P in the interior of e but it changes when P reaches an endpoint of e.

At an endpoint P_0 of e, for the direction $\vec{v}_0 \in T_{P_0}$ pointing into e we still have $\nu_\varphi(B_{P_0}(\vec{v}_0)^-) = \frac{1}{2}$, so for all $\vec{v} \in T_{P_0}$ with $\vec{v} \neq \vec{v}_0$ we necessarily have $\nu_\varphi(B_P(\vec{w})^-) \leq \frac{1}{2}$ and P_0 belongs to $\mathrm{MinResLoc}(\varphi)$. If P_0 is an endpoint of Γ_φ, this is all that needs to be said. If P_0 is a vertex of Γ_φ which is not a branch point, then P_0 belongs to the crucial set, so $\nu_\varphi(\{P_0\}) > 0$. Hence when P moves outside e, for the direction $\vec{v} \in T_P$ pointing towards e we will have $\nu_\varphi(B_P(\vec{v})^-) > \frac{1}{2}$, so $P \notin \mathrm{MinResLoc}(\varphi)$. Finally, if P_0 is a branch point of Γ_φ, there are at least two directions $\vec{v}_1, \vec{v}_2 \in T_{P_0}$ with $\vec{v}_1, \vec{v}_2 \neq \vec{v}_0$ such that $\nu_\varphi(B_{P_0}(\vec{v}_i)^-) > 0$. Hence when P moves outside e, for the direction $\vec{v} \in T_P$ pointing towards e, we will again have $\nu_\varphi(B_P(\vec{v})^-) > \frac{1}{2}$ and $P \notin \mathrm{MinResLoc}(\varphi)$. □

We next give a moduli-theoretic characterization of $\mathrm{MinResLoc}(\varphi)$.

The basic theorem in geometric invariant theory (GIT) concerning moduli spaces of rational functions is due to Silverman. Let R be a commutative ring with 1 and let $\varphi : \mathbb{P}^1_R \to \mathbb{P}^1_R$ be a morphism of degree $d \geq 2$. Fixing homogeneous coordinates on $\mathbb{P}^1_R$, the map φ corresponds to a pair of homogeneous functions $F(X,Y) = f_d X^d + \cdots + f_0 Y^d$ and $G(X,Y) = g_d X^d + \cdots + g_0 Y^d$ in $R[X,Y]$ such that $\mathrm{Res}(F,G)$ is a unit in R. Writing $f = (f_d, \ldots, f_0)$ and $g = (g_d, \ldots, g_0)$, let

$$Z_\varphi = Z_{f,g} = (f_d : \cdots : f_0 : g_d : \cdots : g_0) \in \mathbb{P}^{2d+1}(R)$$

be the point corresponding to φ. If $\mu = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$, the usual conjugation action of μ on φ defined by

$$\begin{aligned} \varphi^\mu &= \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \circ \begin{pmatrix} F \\ G \end{pmatrix} \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \begin{pmatrix} \delta F(\alpha X + \beta Y, \gamma X + \delta Y) - \beta G(\alpha X + \beta Y, \gamma X + \delta Y) \\ -\gamma F(\alpha X + \beta Y, \gamma X + \delta Y) + \alpha G(\alpha X + \beta Y, \gamma X + \delta Y) \end{pmatrix} \end{aligned}$$

induces an algebraic action of GL_2 on $\mathbb{P}^{2d+1}$. If $R = \Omega$ is an algebraically closed field, the groups $\mathrm{GL}_2(\Omega)$ and $\mathrm{SL}_2(\Omega)$ have the same orbits in $\mathbb{P}^{2d+1}(\Omega)$. For technical reasons, in GIT it is better to work with the action of SL_2; Silverman [1998, Theorems 1.1 and 1.3] proves:

Theorem 7.2 (Silverman). *There are open subschemes of* $\mathbb{P}^{2d+1}/\mathrm{Spec}(\mathbb{Z})$

$$\mathrm{Rat}_d \subseteq (\mathbb{P}^{2d+1})^s \subseteq (\mathbb{P}^{2d+1})^{ss}$$

(*the subschemes of rational morphisms of degree d*, *stable points*, *and semistable points*), *which are invariant under the conjugation action of* SL_2, *such that the quotients*

$$M_d = \mathrm{Rat}_d / \mathrm{SL}_2, \quad M_d^s = (\mathbb{P}^{2d+1})^s / \mathrm{SL}_2, \quad \textit{and} \quad M_d^{ss} = (\mathbb{P}^{2d+1})^{ss} / \mathrm{SL}_2$$

exist. M_d is a dense open subset of M_d^s and M_d^{ss}, M_d and M_d^s are geometric quotients, *and M_d^{ss} is a categorical quotient which is proper and of finite type over $\mathbb{Z}$.*

Geometrical and categorical quotients are quotients with certain desirable properties (see [Mumford et al. 1994] for the definitions). The fact that M_d is a geometric quotient includes the fact over any algebraically closed field Ω, the $\mathrm{SL}_2(\Omega)$ orbits of rational functions of degree d are in one-to-one correspondence with the points of $M_d(\Omega)$. The spaces M_d^s and M_d^{ss} are called the spaces of stable and semistable conjugacy classes of rational maps, respectively. Loosely, the stable locus $(\mathbb{P}^{2d+1})^s$ is the largest subscheme such that for any algebraically closed field Ω, the $\mathrm{SL}_2(\Omega)$-orbits of points in $(\mathbb{P}^{2d+1})^s(\Omega)$ are closed and are in one-to-one correspondence with the points of $M_d^s(\Omega)$. Loosely, the semistable locus $(\mathbb{P}^{2d+1})^{ss}$ is the largest subscheme for which a quotient makes sense: $\mathrm{SL}_2(\Omega)$-orbits of points in $(\mathbb{P}^{2d+1})^{ss}(\Omega)$ need not be closed but if one defines two orbits to be equivalent if their closures meet, points of $M_d^{ss}(\Omega)$ correspond to equivalence classes of $\mathrm{SL}_2(\Omega)$-orbits. Each equivalence class of orbits in $(\mathbb{P}^{2d+1})^{ss}(\Omega)$ contains a unique minimal closed orbit.

Recall that K is a complete, algebraically closed nonarchimedean valued field with ring of integers $\mathcal{O}$ and residue field $\tilde{k}$:

Definition 12 (semistable and stable reduction). Let $\varphi(z) \in K(z)$ have degree $d \geq 2$ and let (F, G) be a normalized representation of φ. Writing $F(X, Y) = f_d X^d + \cdots + f_0 Y^d$ and $G(X, Y) = g_d X^d + \cdots g_0 Y^d$, let

$$Z_\varphi = (f_d : \cdots : f_0 : g_d : \cdots : g_0) \in \mathbb{P}^{2d+1}(K)$$

be the point corresponding to φ. We will say that φ has *semistable reduction* if

$$\widetilde{Z}_\varphi = (\tilde{f}_d : \cdots : \tilde{f}_0 : \tilde{g}_d : \cdots : \tilde{g}_0) \in \mathbb{P}^{2d+1}(\tilde{k})$$

belongs to $(\mathbb{P}^{2d+1})^{ss}(\tilde{k})$ and that φ has *stable reduction* if $\widetilde{Z}_\varphi$ belongs to $(\mathbb{P}^{2d+1})^s(\tilde{k})$.

Making explicit the Hilbert–Mumford numerical criteria, Silverman [1998, Proposition 2.2] gives necessary and sufficient conditions for a point to be semistable or stable.

Proposition 7.3. *Let* SL_2 *act on* $\mathbb{P}^{2d+1}$ *as above and suppose* $\widetilde{Z} \in \mathbb{P}^{2d+1}(\tilde{k})$. *Then*:

(A) $\widetilde{Z}$ *belongs to* $(\mathbb{P}^{2d+1})^{ss}(\tilde{k})$ *if and only if for each* $\tilde{\tau} \in \mathrm{SL}_2(\tilde{k})$, *when* $\widetilde{Z}^{\tilde{\tau}}$ *is written as* $(\tilde{a}_d : \cdots : \tilde{a}_0 : \tilde{b}_d : \cdots : \tilde{b}_0)$, *either there is some* i *with* $(d+1)/2 \le i \le d$ *such that* $\tilde{a}_i \neq 0$ *or there is some* i *with* $(d-1)/2 \le i \le d$ *such that* $\tilde{b}_i \neq 0$.

(B) $\widetilde{Z}$ *belongs to* $(\mathbb{P}^{2d+1})^{s}(\tilde{k})$ *if and only if for each* $\tilde{\tau} \in \mathrm{SL}_2(\tilde{k})$, *when* $\widetilde{Z}^{\tilde{\tau}}$ *is written as* $(\tilde{a}_d : \cdots : \tilde{a}_0 : \tilde{b}_d : \cdots : \tilde{b}_0)$, *either there is some* i *with* $(d+1)/2 < i \le d$ *such that* $\tilde{a}_i \neq 0$ *or there is some* i *with* $(d-1)/2 < i \le d$ *such that* $\tilde{b}_i \neq 0$.

Proof. This is [Silverman 1998, Proposition 2.2] in the special case $\Omega = \tilde{k}$, with two modifications. Silverman formulates conditions (A) and (B) as characterizing the "unstable" and "not stable" points of $\mathbb{P}^{2d+1}(\tilde{k})$. Our assertions are the contrapositives of his: by definition, "semistable" is "not unstable" and "stable" is "not *not stable*". Second, Silverman writes $\varphi(z) = (a_0 z^d + \cdots + a_d)/(b_0 z^d + \cdots + b_d)$, indexing coefficients of rational function in the opposite order. We have adjusted his coefficient ranges to account for this. □

The connection between semistability and having minimal resultant is due to Szpiro, Tepper, and Williams, who proved the implication "semistable reduction implies minimal resultant" in part A of the following theorem using a moduli-theoretic argument (see [Szpiro et al. 2014, Theorem 3.3]). Here we establish the converse as well; however, the theorem of Szpiro, Tepper, and Williams holds in arbitrary dimension n, whereas ours applies only when $n = 1$.

Theorem 7.4 (moduli-theoretic characterization of $\mathrm{MinResLoc}(\varphi)$). *Let* $\varphi(z) \in K(z)$ *have degree* $d \ge 2$. *Suppose* $\gamma \in \mathrm{GL}_2(K)$ *and put* $P = \gamma(\zeta_G)$, *then*:

(A) *P belongs to* $\mathrm{MinResLoc}(\varphi)$ *if and only if* φ^γ *has semistable reduction in the sense of* GIT.

(B) $\mathrm{MinResLoc}(\varphi)$ *consists of the single point* P *if and only if* φ^γ *has stable reduction in the sense of* GIT.

Proof. We first show (A). Fixing γ, put $P = \gamma(\zeta_G)$. Note that $P \in \mathrm{MinResLoc}(\varphi)$ if and only if $\partial_{\vec{v}} \mathrm{ordRes}_\varphi(P) \ge 0$ for each $\vec{v} \in T_P$. After replacing φ with φ^γ, we can assume that $P = \zeta_G$. Let $Z_\varphi \in \mathbb{P}^{2d+1}(K)$ be the point corresponding to φ and let $\widetilde{Z}_\varphi \in \mathbb{P}^{2d+1}(\tilde{k})$ be its reduction. We will index directions $\vec{v} \in T_{\zeta_G}$ by points $a \in \mathbb{P}^1(\tilde{k})$.

Fix a direction $\vec{v}_a \in T_{\zeta_G}$ and choose $\tilde{\tau} \in \mathrm{SL}_2(\tilde{k})$ so that $\tilde{\tau}(\infty) = a$. We will show that $\partial_{\vec{v}_a} \mathrm{ordRes}_\varphi(\zeta_G) \ge 0$ if and only if the condition of Proposition 7.3(A) holds for $(\widetilde{Z}_\varphi)^{\tilde{\tau}}$.

Lift $\tilde{\tau}$ to $\tau \in \mathrm{SL}_2(\mathcal{O})$. Let (F, G) be a normalized representation of φ^τ and write $F(X, Y) = a_d X^d + \cdots + a_0 Y^d$ and $G(X, Y) = b_d X^d + \cdots + b_0 Y^d$. Since the action of SL_2 commutes with reduction, it follows that

$$(\widetilde{Z}_\varphi)^{\tilde{\tau}} = \widetilde{(Z_{\varphi^\tau})} = (\tilde{a}_d : \cdots \tilde{a}_0 : \tilde{b}_d : \cdots : \tilde{b}_0).$$

Since $\tau_*(\vec{v}_\infty) = \vec{v}_a$, the function $\mathrm{ordRes}_\varphi(\,\cdot\,)$ is nondecreasing in the direction $\vec{v}_a$ at ζ_G if and only if $\mathrm{ordRes}_{\varphi^\tau}(\,\cdot\,)$ is nondecreasing in the direction $\vec{v}_\infty$. Take $A \in K^\times$ with $|A| \geq 1$. By [Rumely 2015, formula (13)], we have

$$\begin{aligned}
&\mathrm{ordRes}_{\varphi^\tau}(\zeta_{0,|A|}) - \mathrm{ordRes}_{\varphi^\tau}(\zeta_G)\\
&\qquad = (d^2+d)\mathrm{ord}(A) - 2d \cdot \min_{0 \le i \le d} \big(\mathrm{ord}(A^i a_i), \mathrm{ord}(A^{i+1} b_i)\big)\\
&\qquad = \max_{0 \le i \le d} \big((d^2+d-2di)\mathrm{ord}(A) - 2d\mathrm{ord}(a_i),\\
&\qquad\qquad\qquad (d^2+d-2d(i+1))\mathrm{ord}(A) - 2d\mathrm{ord}(b_i)\big). \qquad (50)
\end{aligned}$$

The terms in (50) which are nondecreasing as $|A|$ increases are the ones involving $\mathrm{ord}(a_i)$ for $(d+1)/2 \le i \le d$ and the ones involving $\mathrm{ord}(b_i)$ for $(d-1)/2 \le i \le d$.

Since (F, G) is normalized, we have $\mathrm{ord}(a_i) \ge 0$ and $\mathrm{ord}(b_i) \ge 0$ for all i and there is at least one i for which $\mathrm{ord}(a_i) = 0$ or $\mathrm{ord}(b_i) = 0$. Hence the terms with $\mathrm{ord}(a_i) > 0$ or $\mathrm{ord}(b_i) > 0$ cannot be the maximal ones in (50) when $\mathrm{ord}(A)$ is near 0. It follows that $\partial_{\vec{v}_\infty} \mathrm{ordRes}_{\varphi^\tau}(\zeta_G) \ge 0$ if and only if

$$\begin{aligned}
\mathrm{ord}(a_i) &= 0 \quad \text{for some } i \text{ with } (d+1)/2 \le i \le d, \text{ or}\\
\mathrm{ord}(b_i) &= 0 \quad \text{for some } i \text{ with } (d-1)/2 \le i \le d.
\end{aligned}$$

Since $\mathrm{ord}(a_i) = 0$ if and only if $\tilde{a}_i \neq 0$ and $\mathrm{ord}(b_i) = 0$ if and only if $\tilde{b}_i \neq 0$, these are precisely the conditions on $(\widetilde{Z}_\varphi)^{\tilde{\tau}}$ from Proposition 7.3(A).

Since $\tau_*(\vec{v}_\infty)$ runs over all $\vec{v}_a \in T_P$ as $\tilde{\tau}$ runs over $\mathrm{SL}_2(\tilde{k})$, it follows that P belongs to $\mathrm{MinResLoc}(\varphi)$ if and only if φ^γ has semistable reduction.

Part (B) is proved similarly, using that $P = \gamma(\zeta_G)$ is the unique point in $\mathrm{MinResLoc}(\varphi)$ if and only if $\partial_{\vec{v}} \mathrm{ordRes}_\varphi(P) > 0$ for each $\vec{v} \in T_P$ and that the terms in (50) which are increasing as $|A|$ increases are the ones involving $\mathrm{ord}(a_i)$ for $(d+1)/2 < i \le d$ and the ones involving $\mathrm{ord}(b_i)$ for $(d-1)/2 < i \le d$. $\square$

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