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**A uniform classification of discrete series  
representations of affine Hecke algebras**

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# A uniform classification of discrete series representations of affine Hecke algebras

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We give a new and independent parametrization of the set of discrete series characters of an affine Hecke algebra  $\mathcal{H}_v$ , in terms of a canonically defined basis  $\mathcal{B}_{\text{gm}}$  of a certain lattice of virtual elliptic characters of the underlying (extended) affine Weyl group. This classification applies to all semisimple affine Hecke algebras  $\mathcal{H}$ , and to all  $v \in \mathcal{Q}$ , where  $\mathcal{Q}$  denotes the vector group of positive real (possibly unequal) Hecke parameters for  $\mathcal{H}$ . By analytic Dirac induction we define for each  $b \in \mathcal{B}_{\text{gm}}$  a continuous (in the sense of Opdam and Solleveld (2010)) family  $\mathcal{Q}_b^{\text{reg}} := \mathcal{Q}_b \setminus \mathcal{Q}_b^{\text{sing}} \ni v \rightarrow \text{Ind}_D(b; v)$ , such that  $\epsilon(b; v)\text{Ind}_D(b; v)$  (for some  $\epsilon(b; v) \in \{\pm 1\}$ ) is an irreducible discrete series character of  $\mathcal{H}_v$ . Here  $\mathcal{Q}_b^{\text{sing}} \subset \mathcal{Q}$  is a finite union of hyperplanes in  $\mathcal{Q}$ .

In the nonsimply laced cases we show that the families of virtual discrete series characters  $\text{Ind}_D(b; v)$  are piecewise rational in the parameters  $v$ . Remarkably, the formal degree of  $\text{Ind}_D(b; v)$  in such piecewise rational family turns out to be rational. This implies that for each  $b \in \mathcal{B}_{\text{gm}}$  there exists a universal rational constant  $d_b$  determining the formal degree in the family of discrete series characters  $\epsilon(b; v)\text{Ind}_D(b; v)$ . We will compute the canonical constants  $d_b$ , and the signs  $\epsilon(b; v)$ . For certain geometric parameters we will provide the comparison with the Kazhdan–Lusztig–Langlands classification.

1. Motivation and goals	1090
2. Massive pure elliptic virtual characters	1094
3. One-dimensional algebraic families of discrete series representations, and their limits	1103
4. Explicit results; comparison to the Kazhdan–Lusztig–Langlands classification	1113
References	1132

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## 1. Motivation and goals

Let  $\mathcal{R} = (X, R_0, Y, R_0^\vee, F_0)$  be a based root datum. In particular,  $X, Y$  are  $\mathbb{Z}$ -lattices of finite rank in perfect duality,  $R_0 \subset X$  is the set of roots,  $R_0^\vee \subset Y$  is the set of coroots, and  $F_0 \subset R_0$  is the set of simple roots. Define the (extended) affine Weyl group  $W := W_0 \ltimes X$ , where  $W_0 = W(R_0)$  is the finite Weyl group associated with the root system  $R_0$ . Let  $R_{0,+} \supset F_0$  denote the positive roots. We denote by  $S_0$  the set of simple reflections of  $W_0$ . Associated to  $\mathcal{R}$  one has a canonical Laurent polynomial algebra  $\Lambda$  generated by invertible ‘‘Hecke parameters’’, and the generic extended affine Hecke algebra  $\mathcal{H}_\Lambda$  over  $\Lambda$  (see, e.g., [Opdam and Solleveld 2010]). Let  $\mathcal{Q}$  be the real vector group of the algebraic torus associated with  $\Lambda$ . If  $v \in \mathcal{Q}$  then we denote by  $\mathcal{H}_v$  the corresponding specialization of  $\mathcal{H}_\Lambda$ .

The Hecke algebra  $\mathcal{H}_v$  has a natural structure of a normalized Hilbert algebra and an abstract Plancherel formula [Opdam 2004]. It is a fundamental question to classify explicitly the irreducible discrete series characters (the simple summands in the Plancherel decomposition) and compute their formal degrees, i.e., the Plancherel mass of the discrete series characters.

For the Hecke algebras  $\mathcal{H}_v$  with equal parameters of a simply connected root datum, the classification of the irreducible discrete series modules in terms of the Kazhdan–Lusztig–Langlands parameters was obtained in the seminal paper of Kazhdan and Lusztig [1987] and also by Ginzburg [Chriss and Ginzburg 1997]. This classification was generalized later by Lusztig [2002] for the Hecke algebras that occur in relation with the unipotent representations of quasisimple  $p$ -adic groups with connected center. In the case when the  $p$ -adic group is the split form of  $\mathrm{SO}(2n + 1)$ , the parametrization of irreducible discrete series modules was also determined by Waldspurger [2004]. By applying Clifford theory to Kazhdan–Lusztig theory, Reeder [2002] extended the Kazhdan–Lusztig classification to root data of arbitrary isogeny type in the case when the parameters are equal. For unequal parameters unipotent Hecke algebras, this method was carried out recently by Aubert, Baum, Plymen and Solleveld [Aubert et al. 2017]. In the case of the Hecke algebra of affine type  $C_n$ , a different classification in terms of Kato’s ‘‘exotic geometry’’ was offered in [Ciubotaru and Kato 2011].

Using a different, analytic approach, a complete explicit Plancherel decomposition for affine Hecke algebras of arbitrary isogeny and with arbitrary positive parameters was obtained in [Opdam 2004; 2007]. The program was continued in [Opdam and Solleveld 2010], where a classification of irreducible discrete series in this generality is obtained, except that for root data of type  $E$ , the authors had to also rely on certain results from Kazhdan and Lusztig [1987].

In this paper, we give a new and uniform classification of the set of discrete series modules of  $\mathcal{H}_v$  in terms of certain canonical orthonormal subset  $\mathcal{B}_{\mathrm{gm}}$  of the elliptic

character lattice of  $W$  for arbitrary positive parameters and arbitrary root data, which is independent of the previous classifications, including the Kazhdan–Lusztig classification. We also give algebraic models for the discrete series modules and we study their formal degrees from the perspective of parameter deformations, in particular, we obtain complete and explicit closed formulas for the formal degrees of all irreducible discrete series modules. At several places in our proofs, we rely on [Opdam 2007] and [Opdam and Solleveld 2010] for the classification of the central characters of irreducible discrete series in terms of residual points, on [Opdam 2004] and [Opdam and Solleveld 2009] for the elements of elliptic theory for affine Hecke algebras, and on [Ciubotaru et al. 2014] for several facts about Dirac induction for graded affine Hecke algebras.

**1A. Uniform classification of the discrete series.** Let  $\bar{\mathcal{R}}_{\mathbb{Z}}(W)$  be the lattice of elliptic virtual characters of  $W$ , equipped with the Euler–Poincaré pairing, and let  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)$  denote the lattice of elliptic virtual characters of  $\mathcal{H}_v$ . If  $\pi$  is an element of  $\mathcal{R}_{\mathbb{Z}}(\mathcal{H}_v)$ , let us denote by  $\bar{\pi}$  its image in  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)$ .

In this paper we will use a basic tool, the so-called “scaling map”:

$$\begin{aligned} \lim_{v \rightarrow 1} : \bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v) &\rightarrow \bar{\mathcal{R}}_{\mathbb{Z}}(W) \\ [\pi] &\rightarrow \left[ \lim_{\epsilon \rightarrow 0} \pi_{v^\epsilon} \right], \end{aligned} \tag{1}$$

where  $\pi_{v^\epsilon} := \pi \circ j_\epsilon^{-1}$ . Here  $j_\epsilon : \mathcal{H}_v^{\text{an}}(U) \rightarrow \mathcal{H}_{v^\epsilon}^{\text{an}}(\sigma_\epsilon(U))$  ( $\epsilon > 0$ ) is the isomorphism between the analytic localizations  $\mathcal{H}_v^{\text{an}}(U)$  and  $\mathcal{H}_{v^\epsilon}^{\text{an}}(\sigma_\epsilon(U))$  of the affine Hecke algebra as introduced in [Opdam 2004, Theorem 5.3]. It is easy to see that the family of isomorphisms  $\{j_\epsilon^{-1}\}_{\epsilon > 0}$  has a well defined limit at  $\epsilon = 0$ , defining a homomorphism  $i_0 : \mathbb{C}[W] \rightarrow \mathcal{H}_v^{\text{an}}(U)$  (see [Solleveld 2012, Proposition 4.1.2]). This explains the existence of the desired “scaling map”  $\lim_{v \rightarrow 1} : \bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v) \rightarrow \bar{\mathcal{R}}_{\mathbb{Z}}(W)$  as in (1). The isomorphisms  $j_\epsilon$  ( $\epsilon > 0$ ) induce isometric isomorphisms [Opdam and Solleveld 2009, Theorem 3.5(b)]:

$$(j_\epsilon^{-1})^* : \bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v) \rightarrow \bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_{v^\epsilon}). \tag{2}$$

Consequently, the limit  $\lim_{v \rightarrow 1}$  of (1) is an isometry too [Opdam and Solleveld 2009, Theorem 3.5(b)].

Let us denote by  $\mathcal{Y}_v \subset \bar{\mathcal{R}}_{\mathbb{Z}}(W)$  the image of this map, and by  $\mathcal{Y}_{v-m} \subset \mathcal{Y}_v$  the image of the sublattice of  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)$  of the virtual discrete series characters of  $\mathcal{H}_v$ . Let  $\mathcal{Y}_{\text{gm}} \subset \bar{\mathcal{R}}_{\mathbb{Z}}(W)$  be the smallest sublattice which contains all lattices  $\mathcal{Y}_{v-m}$  (see Definition 2.5). We call  $\mathcal{Y}_{\text{gm}}$  the lattice of *generically massive* elliptic characters of  $W$ , and  $\mathcal{Y}_{v-m}$  the sublattice of *v-massive* elliptic characters of  $W$ .

The lattice  $\mathcal{Y}_{\text{gm}}$  possesses a distinguished orthonormal basis  $\mathcal{B}_{\text{gm}}$  characterized by a positivity property to be explained below. To each  $b \in \mathcal{B}_{\text{gm}}$  we will assign a

subset (nonempty by definition)  $\mathcal{Q}_b^{\text{reg}} \subset \mathcal{Q}$  of the space of parameters by:

$$\mathcal{Q}_b^{\text{reg}} := \{ \mathbf{v} \in \mathcal{Q} \mid \exists \text{ an irreducible discrete series } \pi \text{ of } \mathcal{H}_{\mathbf{v}} \text{ such that } \lim_{\epsilon \rightarrow 0} \overline{\pi_{\mathbf{v}^\epsilon}} = b \}.$$

According to [Opdam and Solleveld 2010], the complement  $\mathcal{Q}_b^{\text{sing}}$  of  $\mathcal{Q}_b^{\text{reg}}$  is a union of finitely many hyperplanes (depending on  $b$ ). For  $y \in \mathcal{Y}_{\text{gm}}$  in general we put  $\mathcal{Q}_y^{\text{reg}} := \bigcap_{b \in \text{Supp}(y)} \mathcal{Q}_b^{\text{reg}}$ . Combining with the technique of analytic Dirac induction (introduced in [Ciubotaru et al. 2014] in the context of graded affine Hecke algebras) we can associate a family of virtual discrete series characters

$$\mathcal{Q}_y^{\text{reg}} \ni \mathbf{v} \rightarrow \text{Ind}_D(y; \mathbf{v}) \in \mathcal{R}_{\mathbb{Z}}(\mathcal{H}_{\mathbf{v}})$$

(called the Dirac induction of  $y$  at  $\mathbf{v}$ , see Definition 2.7), which depends linearly on  $y$ , for each fixed  $\mathbf{v} \in \mathcal{Q}_y^{\text{reg}}$ . If  $b \in \mathcal{B}_{\text{gm}}$  and  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$  then  $\text{Ind}_D(b; \mathbf{v})$  is an irreducible character up to a sign, characterized by the property  $\lim_{\epsilon \rightarrow 0} \overline{(\text{Ind}_D(b; \mathbf{v}^\epsilon))} = b$ . We will prove (see Proposition 3.8) that the family  $\text{Ind}_D(y; \mathbf{v})$  depends continuously on  $\mathbf{v} \in \mathcal{Q}_y^{\text{reg}}$  in the sense of [Opdam and Solleveld 2010]. The ‘‘Vogan conjecture’’ (see [Ciubotaru et al. 2014]) allows one to compute the generic central character  $W_0 r_b$  of  $\text{Ind}_D(b)$  explicitly, where  $r_b \in T_\Lambda := \text{Hom}(X, \Lambda^\times)$  is a *generic residual point* in the sense of [Opdam and Solleveld 2010].

To such an orbit of generic residual points  $W_0 r_b$  we associated in [Opdam and Solleveld 2010] an explicit rational function  $m_b^{\mathcal{Q}} := m_{W_0 r_b}$  on  $\mathcal{Q}$  which is regular on  $\mathcal{Q}$ , and with the property that  $\mathcal{Q}_b^{\text{reg}} = \{ \mathbf{v} \in \mathcal{Q} \mid m_b(\mathbf{v}) \neq 0 \}$ .

Let  $\mathcal{R}_{\mathbb{Z}, \text{temp}}(\mathcal{H}_{\mathbf{v}})$  denote the Grothendieck group of finite-dimensional tempered  $\mathcal{H}_{\mathbf{v}}$ -representations and let  $\overline{\mathcal{R}}_{\mathbb{Z}, \text{temp}}(\mathcal{H}_{\mathbf{v}})$  denote the image of  $\mathcal{R}_{\mathbb{Z}, \text{temp}}(\mathcal{H}_{\mathbf{v}})$  in  $\overline{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_{\mathbf{v}})$ . As a consequence of the parabolic Langlands classification, it is easy to see that  $\overline{\mathcal{R}}_{\mathbb{Z}, \text{temp}}(\mathcal{H}_{\mathbf{v}}) = \overline{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_{\mathbf{v}})$ . Extend the notion of formal degree  $\text{fdeg}$  linearly to  $\mathcal{R}_{\mathbb{Z}, \text{temp}}(\mathcal{H}_{\mathbf{v}})$ . Observe that in this way, the function  $\text{fdeg}$  naturally descends to  $\overline{\mathcal{R}}_{\mathbb{Z}, \text{temp}}(\mathcal{H}_{\mathbf{v}})$ .

**Theorem 1.1.** *Retain the previous notation.*

- (a)  $\mathcal{Y}_{\text{gm}}$  has a **unique orthonormal basis**  $\mathcal{B}_{\text{gm}}$  such that for all  $b \in \mathcal{B}_{\text{gm}}$ , and for all  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$ ,  $d_b(\mathbf{v}) := m_b(\mathbf{v})^{-1} \text{fdeg}(\overline{\text{Ind}_D(b; \mathbf{v})}) > 0$ .
- (b)  $\overline{\text{Ind}_D(b; \mathbf{v})}$  is represented by a virtual character  $\text{Ind}_D(b; \mathbf{v})$  of  $\mathcal{H}_{\mathbf{v}}$  which is plus or minus an irreducible discrete series.
- (c) The central character of  $\text{Ind}_D(b; \mathbf{v})$  is the specialization at  $\mathbf{v}$  of a  $W_0$ -orbit of generic residual points  $W_0 r_b$ , with  $r_b \in T_\Lambda := \text{Hom}(X, \Lambda^\times)$ .
- (d) The family  $\text{Ind}_D(b; \mathbf{v})$  depends continuously on  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$  (in the sense of [Opdam and Solleveld 2010]).
- (e) For all  $b \in \mathcal{B}_{\text{gm}}$ , the signature function  $\mathcal{Q}_b^{\text{reg}} \ni \mathbf{v} \rightarrow \epsilon(b; \mathbf{v}) \in \{\pm 1\}$  such that  $\epsilon(b; \mathbf{v}) \text{Ind}_D(b; \mathbf{v})$  is an irreducible discrete series character, is locally constant.

(f) For each  $\mathbf{v} \in \mathcal{Q}$ , define  $\mathcal{B}_{\mathbf{v}-m} = \{b \in \mathcal{B}_{\text{gm}} \mid m_b(\mathbf{v}) \neq 0\}$ . The assignment

$$\mathcal{B}_{\mathbf{v}-m} \ni b \rightarrow \epsilon(b; \mathbf{v}) \text{Ind}_D(b; \mathbf{v})$$

yields a canonical bijection between  $\mathcal{B}_{\mathbf{v}-m}$  and the set of irreducible discrete series characters of  $\mathcal{H}_{\mathbf{v}}$ .

(g) For all  $b \in \mathcal{B}_{\text{gm}}$ :  $d_b(\mathbf{v}) = d_b$  is **independent** of  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$  (where  $d_b(\mathbf{v})$  is the positive function defined in (a)), and  $d_b \in \mathbb{Q}_+$ . In other words, for all  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$ ,

$$\text{fdeg}(\text{Ind}_D(b; \mathbf{v})) = d_b m_b(\mathbf{v})$$

(a rational function of  $\mathbf{v}$ , regular in all points of  $\mathcal{Q}$ , with zero locus  $\mathcal{Q}_b^{\text{sing}}$ ).

This represents first of all a new classification of the discrete series of  $\mathcal{H}_{\mathbf{v}}$ , which is uniform in the sense that it applies to all irreducible root data and all parameters  $\mathbf{v} \in \mathcal{Q}$ . It is explicit in the sense that for cases where a classification of discrete series has been given in other terms in the literature (e.g., in terms of Kazhdan–Lusztig–Langlands parameters) the comparison can be explicitly given. The main tool for making this uniform classification explicit in specific cases is the Hecke algebra version of the “Vogan conjecture” established first in [Barbasch et al. 2012] and sharpened to the version that we need here in [Ciubotaru et al. 2014]. This enables the explicit computation in terms of  $b \in \mathcal{B}_{\text{gm}}$  of the central character  $W_0 r_b$ . More precisely, motivated by the ideas of Vogan and Huang and Pandžić for  $(\mathfrak{g}, K)$ -modules of real reductive groups, [Barbasch et al. 2012] introduced the notion of Dirac cohomology of a finite dimensional graded Hecke algebra module  $M$ . The Dirac cohomology is a finite dimensional representation of the pin double cover of the finite Weyl group and the main idea is that if  $M$  has a central character (in particular, if it is a simple module), then the Dirac cohomology, if nonzero, determines its central character. In the present setting, to every  $b \in \mathcal{B}_{\text{gm}}$ , one can attach canonically an irreducible representation of the pin double cover of an “endoscopic” subgroup of  $W_0$  in such a way that this representation occurs in the Dirac cohomology of the graded Hecke algebra module supported on  $\text{Ind}_D(b; \mathbf{v})$ , and this in turn, using the idea just explained, determines the central character of  $\text{Ind}_D(b; \mathbf{v})$ .

Let  $b \in \mathcal{B}_{\text{gm}}$ , let  $C \subset \mathcal{Q}_b$  be a connected component (an open cone in  $\mathcal{Q}$ ), and let  $\mathbf{v}_0 \in \partial(C) \subset \mathcal{Q}_b^{\text{sing}}$ . An underlying issue is the behavior of the families  $\text{Ind}_D(b; \mathbf{v})$  near  $\mathbf{v}_0 \in \mathcal{Q}_b^{\text{sing}}$ . These questions play a technical role in the proof of the above Theorem, and are of independent interest. To be sure, the family  $\text{Ind}_D(b; \mathbf{v})$  is *not* continuous in any neighborhood of  $\mathbf{v}_0$ , which is one of the reasons that Theorem 1.1(g) is surprising and noteworthy. We will show that there is a sense in which the family  $\text{Ind}_D(b; \mathbf{v})$  can be extended along smooth curves in  $\mathcal{Q}_{\mathbb{C}}$  as an algebraic family of genuine characters, provided one lifts the condition that the

characters in the family are discrete series characters. More precisely, consider an affine smooth curve  $\mathcal{C} \subset \mathcal{Q}_{\mathbb{C}}$  which intersects  $\bar{C}$  in a real curve containing  $\mathbf{v}_0 \in \bar{C}$ . Then  $\mathcal{C} \cap C \ni \mathbf{v} \rightarrow \text{Ind}_D(b; \mathbf{v})$  extends as a rational family of generically irreducible genuine characters to a finite branched covering  $\tilde{\mathcal{C}}$  of  $\mathcal{C}$ . Importantly, this family is unramified at points of  $\mathcal{C} \cap \bar{C}$ , and regular at the points of  $\tilde{\mathcal{C}}$  lying above  $\mathbf{v}_0$ . In particular one can define a limit  $\text{Ind}_D(b, C; \mathbf{v}_0)$  at  $\mathbf{v}_0$  of the family of discrete series along  $\mathcal{C}$  “from the direction of  $C$ ”. This “limit of discrete series” is a tempered character, which depends on  $(C, \mathbf{v}_0)$ . We expect that it does not depend on the choice of  $\mathcal{C}$ . Notice that if  $\mathcal{C}$  intersects the boundary of  $C$  transversally at  $\mathbf{v}_0$  then  $\mathcal{C}$  will also intersect the chamber  $C_-$  opposite to  $C$  with respect to  $\mathbf{v}_0$ , and thus there exists also a limit  $\text{Ind}_D(b, C_-, \mathcal{C}; \mathbf{v})$ . It would be interesting to investigate how these two limits are related to each other, in terms of the relevant analytic R-group.

## 2. Massive pure elliptic virtual characters

### 2A. Elliptic virtual characters of affine Weyl groups.

**2A1.** *Elliptic virtual characters of affine Weyl groups.* We identify  $X$  with the normal subgroup  $\{e\} \times X \subset W$ , and  $W_0$  with the subgroup  $W_0 \times \{0\}$ . Let  $E = \mathbb{R} \otimes X$ ; then  $W$  acts naturally on the Euclidean space  $E$  as a group of affine isometries.

The lattice  $X \subset W$  is the normal subgroup of elements whose conjugacy class is finite. A centralizer of an element  $x \in X$  is called a *Levi subgroup* of  $W$ . There are finitely many Levi subgroups of  $W$ , and this collection is conjugation invariant. Each Levi subgroup  $L \subset W$  is itself an affine Weyl group  $L = W_L \ltimes X$ , where  $W_L$  is a Levi subgroup of  $W_0$  (the isotropy group of  $x$  in  $W_0$ ). Then  $W_L$  is a Coxeter group, and has a unique set of simple reflections  $S_L$  consisting of reflection  $r_\alpha \in W_0$  with  $\alpha \in R_{0,+}$ . Every Levi subgroup is conjugate to a standard Levi subgroup. We call  $L$  *standard* if  $S_L \subset S_0$ .

An element  $w \in W$  is called *elliptic* if  $w$  does not belong to any proper Levi subgroup  $L \subset W$  (see [Opdam and Solleveld 2009]). The following are easily seen to be equivalent:

- (a)  $w \in W$  is elliptic.
- (b) The canonical image of  $w$  in  $W_0$  is elliptic (with respect to the action of  $W_0$  on  $E$ ).
- (c) The centralizer of  $w$  in  $W$  is finite.
- (d) The conjugacy class of  $w$  is a union of left (or equivalently right) cosets of a sublattice of  $X$  of maximal rank.
- (e)  $w$  has isolated fixed points in  $E$ .
- (f)  $w$  has a unique fixed point in  $E$ .



The set of elliptic elements is a finite union of conjugacy classes.

An (extended) *parahoric subgroup* of  $W$  is the pointwise stabilizer of an affine subspace in  $E$ . Given  $e \in E$  consider the isotropy group  $W_e \subset W$ . Then  $W_e$  has a natural faithful linear action on  $T_e(T) \simeq E$ , and an element  $w \in W_e$  is said to be elliptic if  $w$  is elliptic in  $W_e$  with respect to the action on  $T_e(T)$ , in the sense of Reeder [2001].

We denote by  $\mathcal{R}_{\mathbb{Z}}(W)$  the Grothendieck group of the category of  $\mathbb{C}[W]$ -modules of finite length, and by  $\mathcal{R}_{\mathbb{C}}(W)$  its complexification. The character map defines an embedding of  $\mathcal{R}_{\mathbb{C}}(W)$  into the space of complex class functions on  $W$ . Let  $\bar{\mathcal{R}}_{\mathbb{C}}(W)$  denote the complex valued class functions on  $W$  supported on the set of elliptic conjugacy classes. When we compose the character map with the restriction map we obtain a surjective map from  $\mathcal{R}_{\mathbb{C}}(W)$  to  $\bar{\mathcal{R}}_{\mathbb{C}}(W)$ . In [Opdam and Solleveld 2009] it was shown that the kernel of this map is spanned by the set of characters which are induced from proper Levi subgroups. We will identify  $\bar{\mathcal{R}}_{\mathbb{C}}(W)$  with this quotient of  $\mathcal{R}_{\mathbb{C}}(W)$ . We denote by  $\bar{\mathcal{R}}_{\mathbb{Z}}(W) \subset \mathcal{R}_{\mathbb{C}}(W)$  the image of  $\mathcal{R}_{\mathbb{Z}}(W)$ , and refer to this lattice as the group of elliptic virtual characters.

There exists a unique conjugation invariant measure [Opdam and Solleveld 2009, Theorem 3.3(c)]  $\mu_{\text{ell}}$  on  $W$ , which is supported on the elliptic conjugacy classes, and which is defined by  $\mu_{\text{ell}}((1-w)(X)) = |W_0|^{-1}$  if  $(1-w)(X)$  has maximal rank, and  $\mu_{\text{ell}}((1-w)(X)) = 0$  otherwise. This defines an integral positive semidefinite Hermitian pairing, the *elliptic pairing*  $\text{EP}_W$  on  $\mathcal{R}_{\mathbb{C}}(W)$  by integrating  $f\bar{g}$  over  $W$  with respect to the measure  $\mu_{\text{ell}}$ . The Euler–Poincaré pairing on  $\mathcal{R}_{\mathbb{Z}}(W)$  is expressed by  $\text{EP}_W$ . More precisely [Opdam and Solleveld 2009], given virtual representations  $U$  and  $V$  of  $W$ , with characters  $\chi_U$  and  $\chi_V$  respectively, one has

$$\text{EP}_W(U, V) = \int_{w \in W} \chi_U(w) \overline{\chi_V(w)} d\mu_{\text{ell}}(w) = \sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}_W^i(U, V). \quad (3)$$

In particular  $\text{EP}_W$  is integral on  $\mathcal{R}_{\mathbb{Z}}(W)$ . By [Opdam and Solleveld 2009] the radical of  $\text{EP}_W$  is exactly the kernel of the quotient map  $\mathcal{R}_{\mathbb{C}}(W) \rightarrow \bar{\mathcal{R}}_{\mathbb{C}}(W)$ , hence in particular  $\text{EP}_W$  descends to a positive definite integral inner product on  $\bar{\mathcal{R}}_{\mathbb{Z}}(W)$ .

The Weyl group  $W_0$  acts naturally on the algebraic torus  $T = \text{Hom}(X, \mathbb{C}^\times)$ . Clearly  $w \in W_0$  is elliptic if and only if  $w$  has finitely many fixed points on  $T$ . It was shown in [Opdam and Solleveld 2009] that the set of elliptic conjugacy classes of  $W$  and the set of  $W_0$ -orbits of pairs  $(C, t)$  with  $C \in W_t$  an elliptic conjugacy class (with respect to the faithful action of  $W_t$  on  $T_t(T)$ ) and  $t \in T$  have the same cardinality. Here the action of  $W_0$  is defined by  $w(C, t) = (wCw^{-1}, wt)$ .

Elements  $f \in \bar{\mathcal{R}}_{\mathbb{C}}(W)$  can be viewed as tracial functionals  $f \in \mathbb{C}[W]^*$  supported on the set of elliptic conjugacy classes. Hence the center

$$Z(\mathbb{C}[W]) = \mathbb{C}[X]^{W_0} \subset \mathbb{C}[W]$$

acts on  $\bar{\mathcal{R}}_{\mathbb{C}}(W)$  by multiplication, i.e.,  $z.f(a) := f(za)$  for all  $z \in Z(\mathbb{C}[W])$ ,  $f \in \bar{\mathcal{R}}(W)$ , and  $a \in \mathbb{C}[W]$ . Using Mackey theory we showed in [Opdam and Solleveld 2009] that there exists an isometric isomorphism

$$\text{Ind} := \bigoplus_{s \in W_0 \setminus T} \text{Ind}_s : \bigoplus_{s \in W_0 \setminus T} \bar{\mathcal{R}}_{\mathbb{C}}(W_s) \rightarrow \bar{\mathcal{R}}_{\mathbb{C}}(W). \tag{4}$$

Here  $\bar{\mathcal{R}}_{\mathbb{C}}(W_s)$  is equipped with the elliptic inner product [Reeder 2001] for the isotropy group  $W_s$  with respect to its natural faithful representation on the tangent space  $T_s(T)$  of  $T$  at  $s$ , and the direct sum is an orthogonal direct sum. Furthermore  $\text{Ind}_s$  is the linear map on  $\bar{\mathcal{R}}_{\mathbb{C}}(W_s)$  realized by the Mackey induction functor. The image of  $\text{Ind}_s$  equals

$$\text{Im}(\text{Ind}_s) = \bar{\mathcal{R}}_{\mathbb{C}}(W)_{W_0s}, \tag{5}$$

the  $Z(\mathbb{C}[W])$ -eigenspace in  $\bar{\mathcal{R}}_{\mathbb{C}}(W)$  with eigenvalue  $W_0s$ , hence (4) gives the orthogonal decomposition of  $\bar{\mathcal{R}}_{\mathbb{C}}(W)$  as a direct sum of  $Z(\mathbb{C}[W])$ -eigenspaces. By Mackey theory for  $W = W_0 \times X$  this decomposition is compatible with the integral structure. It follows that we also have an orthogonal direct sum decomposition of lattices:

$$\text{Ind} := \bigoplus_{s \in W_0 \setminus T} \text{Ind}_s : \bigoplus_{s \in W_0 \setminus T} \bar{\mathcal{R}}_{\mathbb{Z}}(W_s) \rightarrow \bar{\mathcal{R}}_{\mathbb{Z}}(W). \tag{6}$$

**Definition 2.1** ([Ciubotaru et al. 2014]). Let  $\mathcal{X}$  be a  $\mathbb{Z}$ -lattice equipped with an integral positive definite bilinear form. An element  $x \in \mathcal{X}$  is called *pure* if  $x$  is not a nontrivial orthogonal sum in  $\mathcal{X}$ .

**2B. Affine Hecke algebras and Dirac induction.** Unfortunately we do not know how to define a Dirac-type operator for affine Hecke algebras. Using appropriate versions of Lusztig’s reduction theorems and results of [Ciubotaru et al. 2014; Opdam and Solleveld 2009; 2010], we can nevertheless define Dirac-type induction from a well-defined subspace of the space of elliptic characters of the affine Weyl group to the space of virtual discrete series characters of  $\mathcal{H}_v$ .

For affine Hecke algebras we use the setup and notation of [Opdam and Solleveld 2010, Section 2]. Thus given a based root datum  $\mathcal{R}$  let  $\Lambda$  denote the canonically associated Laurent polynomial ring of Hecke parameters  $v(s)$ , and let  $\mathcal{H}_{\Lambda} = \mathcal{H}_{\Lambda}(\mathcal{R})$  denote the associated affine Hecke algebra defined over  $\Lambda$ . Let  $\mathcal{Q}_c = \text{Hom}(\Lambda, \mathbb{C})$ , the group of complex points of an algebraic torus. Let  $\mathcal{Q} \subset \mathcal{Q}_c$  be the real vector group, the identity component of the group of real points.

We denote the canonical  $\Lambda$ -basis of  $\mathcal{H}_{\Lambda}$  by  $N_w$  (with  $w \in W$ ), where the normalization is such that for affine simple reflections  $s \in S$  we have

$$(N_s - v(s))(N_s + v(s)^{-1}) = 0. \tag{7}$$

An element  $v \in \mathcal{Q}$  is determined by its coordinates  $v(s) := v(s)(v)$  with  $s \in S$ . We denote by  $\mathcal{H}_v = \mathcal{H}_\Lambda(\mathcal{R}) \otimes_{\mathbb{C}_v}$  the corresponding specialized affine Hecke algebra, specialized at  $v$ .

According to the Bernstein–Lusztig–Zelevinski presentation of  $\mathcal{H}_\Lambda$  we have a unique abelian subalgebra  $\mathcal{A} = \mathbb{C}[\theta_x \mid x \in X] \subset \mathcal{H}$  such that  $\theta_x = N_x$  if  $x \in X$  is dominant. Then  $\mathcal{A} \simeq \mathbb{C}[X]$ , and the center  $Z(\mathcal{H}_\Lambda)$  is equal to  $A^{W_0} \simeq \Lambda[X]^{W_0}$ .

Given  $v \in \mathcal{Q}$ , consider the quotient  $\bar{\mathcal{R}}_{\mathbb{C}}(\mathcal{H}_v)$  of the complexified Grothendieck ring  $\mathcal{R}_{\mathbb{C}}(\mathcal{H}_v)$  of finite length representations of  $\mathcal{H}_v$  by the subspace generated by the properly parabolically induced representations. By [Ciubotaru and He 2014] this is a finite dimensional complex vector space for all  $v \in \mathcal{Q}$ . Notice also that  $\bar{\mathcal{R}}_{\mathbb{C}}(\mathcal{H}_{v=1}) = \bar{\mathcal{R}}_{\mathbb{C}}(W)$ . The image of the lattice of virtual characters is denoted by  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)$ . The center  $Z(\mathcal{H}_v)$  acts on  $\bar{\mathcal{R}}_{\mathbb{C}}(\mathcal{H}_v)$ , and by Schur’s lemma we have a decomposition

$$\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v) = \bigoplus_{t \in W_0 \backslash T} \bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)_{W_0 t}. \tag{8}$$

If  $v \in \mathcal{Q}$  is a positive parameter then [Opdam and Solleveld 2009] asserts that  $\mathcal{H}_v$  has finite global dimension, and we define an integral bilinear form  $\text{EP}_{\mathcal{H}}$  on  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)$  by

$$\text{EP}_{\mathcal{H}}(U, V) = \sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}_{\mathcal{H}_v}^i(U, V). \tag{9}$$

As mentioned above, there exists [Opdam and Solleveld 2009] a “scaling map”  $\lim_{v \rightarrow 1} : \bar{\mathcal{R}}_{\mathbb{C}}(\mathcal{H}_v) \rightarrow \bar{\mathcal{R}}_{\mathbb{C}}(W)$  which is an isometry. In particular  $\text{EP}_{\mathcal{H}}$  is itself symmetric and positive semidefinite. One way to understand  $\lim_{v \rightarrow 1}$  is via Lusztig’s reduction results to graded affine Hecke algebras, combined with Clifford theory, and the restriction map from graded affine Hecke algebra representations to representations of the corresponding Weyl group. This is what we will look into in the next paragraph.

**2B1. Clifford theory for extensions of graded affine Hecke algebras.** Consider  $v \in \mathcal{Q}$ , and let  $V$  be an irreducible representation of  $\mathcal{H}_v$ . Recall the polar decomposition  $T = T_u T_v$ , where  $T_u = \text{Hom}(X, S^1)$  and  $T_v = \text{Hom}(X, \mathbb{R}_{>0})$ . Let the central character of  $V$  be  $W_0 t$  with  $t = sc$  where  $s \in T_u$  is a unitary element, and  $c \in T_v$ . Let  $F_{s,1}$ ,  $R_{s,1}$  and  $\Gamma_s$  be as in [Opdam and Solleveld 2010, Definition 2.5], so that the isotropy group  $W_s$  of  $s$  in  $W_0$  equals  $W_s = W(R_{s,1}) \rtimes \Gamma_s$ , and  $\alpha(t) > 0$  for all  $\alpha \in R_{s,1}$ . We recall that  $\Gamma_s$  is a finite abelian group, acting on  $F_{s,1}$  by diagram automorphisms preserving  $k_{s,1}$ . Lusztig’s reduction theorems [1989a] in the version discussed in [Opdam and Solleveld 2010, Theorems 2.6 and 2.8, Corollary 2.10] imply that the category of finite dimensional representations of  $\mathcal{H}_v$  with central character  $W_0 t$  is equivalent to the category of finite dimensional representations

of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma(t)$  with real central character  $W(R_{s,1})\xi$ . Here  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$  is the graded affine Hecke algebra as defined in [Opdam and Solleveld 2010, Section 2],  $\xi \in T_s(T)$  is the unique vector in the real span of  $R_{s,1}^\vee$  such that  $\alpha(t) = e^{\alpha(\xi)}$  for all  $\alpha \in R_{s,1}$  and  $\Gamma(t) \subset \Gamma_s$  is the isotropy group of the central character  $W(R_{s,1})\xi$  of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$ .

Clifford theory [Ram and Ramage 2003] for the crossed product

$$\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s$$

says that the irreducible characters of this algebra are obtained as follows. Let  $U$  be an irreducible representation of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$ . Let  $\Gamma_U \subset \Gamma_s$  be the isotropy subgroup for the equivalence class  $[U]$  of irreducible representations of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$ . Then twisting  $U$  by elements of  $\Gamma_U$  equips  $U$  with a representation of a twisted group algebra  $\mathbb{C}[\Gamma_U; \eta_U]$  with respect to a 2-cocycle  $\eta_U$  of  $\Gamma_U$  with values in  $\mathbb{C}^\times$ . Consider a simple module  $M$  of  $\mathbb{C}[\Gamma_U; \eta_U^{-1}]$ , then

$$N_{\mathbb{H}}(U, M) := \text{Ind}_{\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_U}^{\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s} (U \otimes M)$$

is an irreducible  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s$ -module, and all its irreducible modules are equivalent to such a module. Moreover,  $N_{\mathbb{H}}(U, M) \simeq N_{\mathbb{H}}(U', M')$  if and only if  $U' \simeq U \circ \gamma^{-1}$  for some  $\gamma \in \Gamma_s$ , and  $M' \simeq M \circ \gamma^{-1}$ .

Observe that when  $U$  has central character  $W(R_{s,1})\xi$  then  $\Gamma_U \subset \Gamma(t)$ . Thus Clifford theory implies that the set of irreducible modules of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s$  with central character  $W_s\xi$  is in natural bijection with the set of irreducible modules of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma(t)$  with central character  $W(R_{s,1})\xi$ . In fact it follows from the proof of Lusztig’s reduction theorem that this bijection between the respective sets of irreducibles arises from a Morita equivalence of the two algebras, formally completed at the appropriate central characters  $W_s\xi$  and  $W(R_{s,1})\xi$  respectively. Therefore, by the above, the category of finite dimensional representations of  $\mathcal{H}_v$  with central character  $W_{0t}$  is naturally equivalent with the category of finite dimensional representations of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s$  with real central character  $W_s\xi$ . In particular we have a natural isomorphism

$$\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)_{W_{0t}} \simeq \bar{\mathcal{R}}_{\mathbb{Z}}(\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s)_{W_s\xi}. \tag{10}$$

It is an interesting question what the central support of  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)$  is. Clearly, if  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)_{W_{0t}} \neq 0$  then, by the above, one has  $T_s(E)^{W_s} = 0$ .

Since  $\mathbb{C}[W_s] = \mathbb{C}[W(R_{s,1})] \rtimes \Gamma_s$ , we have a similar description of the set of irreducibles of  $\mathbb{C}[W_s]$  as modules of the form  $N_{W_s}(X, M)$  where  $X$  is an irreducible for  $W(R_{s,1})$ .

The restriction functor

$$\text{Res}_{W_s}: \mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s\text{-modules} \rightarrow \mathbb{C}[W_s] = \mathbb{C}[W(R_{s,1})] \rtimes \Gamma_s\text{-modules},$$

induces a homomorphism on the level of the Grothendieck groups of representations of finite length. Via the above correspondences, the “scaling map”  $\lim_{v \rightarrow 1}$  (more precisely,  $\lim_{\epsilon \rightarrow 0} \pi_{v^\epsilon}$ ) corresponds to taking the limit  $\epsilon \rightarrow 0$  of the family of twists by linear scaling isomorphisms,

$$\psi_\epsilon : \mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s \rightarrow \mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; \epsilon k_s) \rtimes \Gamma_s,$$

defined by  $\phi_\epsilon(\xi) = \epsilon^{-1}\xi$ . This is the restriction map  $\text{Res}_{W_s}$ . In particular we see:

**Corollary 2.2.** *The map  $[\pi] \rightarrow \lim_{\epsilon \rightarrow 0} [\pi_{v^\epsilon}]$  respects the lattices of virtual characters, and defines an isometric map  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v) \rightarrow \bar{\mathcal{R}}_{\mathbb{Z}}(W)$  sending  $\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)_{W_0 t}$  to  $\bar{\mathcal{R}}_{\mathbb{Z}}(W)_{W_0 s}$ , where  $t = sc \in T_u T_v$  as before. More precisely, if  $\pi$  corresponds to the module  $U$  of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s$  via (10), and  $b_s = [U|_{W_s}]$ , then  $\lim_{\epsilon \rightarrow 0} [\pi_{v^\epsilon}] = \text{Ind}_s(b_s)$ .*

**2B2. Residual points.** Let  $R_1$  be the reduced root subsystem of the inmultiplicable roots of the possibly nonreduced root system

$$R_{\text{nr}} = R_0 \cup \{2\alpha \mid \alpha^\vee \in 2Y \cap R_0^\vee\}. \tag{11}$$

For  $\beta \in R_{\text{nr}}$ , define the parameters  $v_{\beta^\vee}$  in terms of the  $v(s)$  as in [Opdam and Solleveld 2010, (7), (8)]. For every  $v \in \mathcal{Q}$ , recall the Macdonald  $c$ -function,  $c = \prod_{\alpha \in R_{1,+}} c_\alpha$ , a rational function on  $T$ , where

$$c_\alpha(t, v) = \frac{(1 + v_{\alpha^\vee}^{-1} \alpha(t)^{-\frac{1}{2}})(1 - v_{\alpha^\vee}^{-1} v_{2\alpha^\vee}^{-2} \alpha(t)^{-\frac{1}{2}})}{1 - \alpha(t)^{-1}}. \tag{12}$$

If  $\alpha \in R_1 \setminus R_0$  then  $\alpha/2$  is a character of  $T$ ; however, if  $\alpha \in R_0 \cap R_1$  then  $v_{2\alpha^\vee} = 1$ , and we interpret the numerator of  $c_\alpha$  as  $(1 - v_{\alpha^\vee}^{-2} \alpha(t)^{-1})$ . Thus for all  $\alpha \in R_1$ , the expression for  $c_\alpha$  defines a rational function on  $T$  indeed. Set  $\eta(t) = (c(t)c(t^{-1}))^{-1}$ . Define the pole order  $i_{\{r\}}$  of  $\eta$  at  $r \in T$  as in [Opdam and Solleveld 2010, (34)]. By [Opdam 2007, Theorem 6.1],  $i_{\{r\}} \geq \text{rk}(R_0)$  for all  $r \in T$ .

**Definition 2.3** ([Opdam and Solleveld 2010, Definitions 2.39 and 2.40]). An element  $r \in T$  is called a *residual point* of  $(\mathcal{R}, v)$  if  $i_{\{r\}} = \text{rk}(R_0)$ . The set of  $(\mathcal{R}, v)$ -residual points is denoted by  $\text{Res}(\mathcal{R}, v)$ .

A  $\mathcal{Q}_c$ -valued point  $r \in T_\Lambda$  is called a  $(\mathcal{Q})$ -generic residual point of  $\mathcal{R}$  if there exists an open dense subset  $U \subset \mathcal{Q}$  such that the points  $r(v) \in \text{Res}(\mathcal{R}, v)$  for all  $v \in U$ . The set of generic residual points of  $\mathcal{R}$  is denoted by  $\text{Res}(\mathcal{R})$  (or  $\text{Res}(\mathcal{R}, \mathcal{Q})$  if confusion is possible).

Let  $r$  be a  $\mathcal{Q}$ -generic residual point. As in [Opdam and Solleveld 2010, (40)], we define the mass function  $m_{W_0 r}$  as the rational function on  $\mathcal{Q}$  defined by

$$m_{W_0 r} = m_{W_0 r}^{\mathcal{Q}} = \frac{\prod'_{\alpha \in R_1} (\alpha(r)^{-1} - 1)}{\prod'_{\alpha \in R_1} (v_{\alpha^\vee}^{-1} \alpha(r)^{-\frac{1}{2}} + 1) \prod'_{\alpha \in R_1} (v_{\alpha^\vee}^{-1} v_{2\alpha^\vee}^{-2} \alpha(r)^{-\frac{1}{2}} - 1)}. \tag{13}$$

Here  $\prod'$  means that the factors which are identically zero as functions on  $\mathcal{Q}$  are omitted. The function  $m_{W_0 r}$  is regular on  $\mathcal{Q}$  (see [Opdam and Solleveld 2010, Theorem 2.60, Corollary 4.4, Theorem 4.6]), hence, in particular, continuous. Comments similar to those above concerning the interpretation of the roots  $\alpha(r)^{-\frac{1}{2}}$  apply. Observe that the expression in (13) is independent of the choice of representative  $r$  in its  $W_0$ -orbit.

We will also need the related notions for the graded affine Hecke algebra. Let  $R_1 \subset V^*$  be a semisimple reduced root system and let  $\mathcal{K}$  be the space of  $W_0$ -invariant real valued functions on  $R_1$ .

**Definition 2.4** ([Heckman and Opdam 1997; Opdam and Solleveld 2010, Definition 2.55]). For  $k \in \mathcal{K}$ , a point  $h \in V$  is called  $(R_1, k)$ -residual if

$$|\{\alpha \in R_1 \mid \alpha(h) = k_\alpha\}| = |\{\alpha \in R_1 \mid \alpha(h) = 0\}| + \dim(V). \tag{14}$$

We denote by  $\text{Res}^{\text{lin}}(R_1)$  the set of linear maps  $\xi : \mathcal{K} \rightarrow V$  such that for almost all  $k$ , the point  $\xi(k) \in V$  is  $(R_1, k)$ -residual. The elements  $\xi \in \text{Res}^{\text{lin}}(R_1)$  are called the *generic linear residual points*.

**2B3. Massive elliptic representations.**

**Definition 2.5.** We define

$$\mathcal{Y}_v := \lim_{v \rightarrow 1} (\bar{\mathcal{R}}_{\mathbb{Z}}(\mathcal{H}_v)) \subset \bar{\mathcal{R}}_{\mathbb{Z}}(W), \tag{15}$$

and let  $\mathcal{Y}_{v-m} \subset \mathcal{Y}_v$  be the sublattice generated by the limits  $\lim_{\epsilon \rightarrow 0} \pi_{v^\epsilon}$  of *discrete series representations* of  $\mathcal{H}_v$ . We call  $\mathcal{Y}_{v-m}$  the lattice of  $v$ -massive elliptic representations of  $W$ . Finally let  $\mathcal{Y}_{\text{gm}} \subset \bar{\mathcal{R}}_{\mathbb{Z}}(W)$  be the sublattice generated by  $\bigcup_{v \in \mathcal{Q}} \mathcal{Y}_{v-m}$ , the lattice of *generically massive* elliptic representations of  $W$ . In general,  $\mathcal{Y}_{\text{gm}} \neq \bar{\mathcal{R}}_{\mathbb{Z}}(W)$ .

**Proposition 2.6.** *The lattice  $\mathcal{Y}_{\text{gm}}$  admits an orthonormal basis  $\mathcal{B}_{\text{gm}} \subset \mathcal{Y}_{\text{gm}}$ . If  $b \in \mathcal{B}_{\text{gm}}$  then  $b \in \bar{\mathcal{R}}_{\mathbb{Z}}(W)_{W_0 s}$  for some  $s \in T_u$  such that  $\text{rk}(R_{s,1}) = \dim(T_u)$ .*

*Proof.* It follows from the classification [Opdam and Solleveld 2010, Section 5] that an irreducible discrete series  $(V, \pi_v)$  of  $\mathcal{H}_v$  has central character  $W_0 r(v)$  for some generic residual point  $r$ , and we can write  $r = sc$  with  $s \in T_u$  and  $c = \exp(\xi)$ , where  $\xi$  is a generic linear residual point for  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$  whose evaluation at  $k_s$  is residual. In particular, the rank of  $R_{s,1}$  is equal to the dimension of  $T_u$ .

By [Opdam and Solleveld 2010] it also follows that  $\pi$  corresponds via (10) to the representation of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s$  induced by the irreducible representation  $U \otimes M$  of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_U$ , where  $U$  is an irreducible discrete series character of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$  (here we also use the discussion in the text above). In Corollary 2.2 we have seen that the limit  $b := \lim_{\epsilon \rightarrow 0} \bar{\pi}_{v^\epsilon}$  equals  $b = \text{Ind}_s(b_s)$  with  $b_s = \overline{U|_{W_s}}$ . By the results of [Ciubotaru et al. 2014]

and of [Opdam and Solleveld 2010] we see that the algebraic Dirac induction for  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$  of  $b_s$  yields an elliptic representation supported by the central character  $W(R_{s,1})\xi(k_s)$ , which is (by [Opdam and Solleveld 2010]) residual for all  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$ , a complement in  $\mathcal{Q}$  of finitely many hyperplanes. Recall that, by [Opdam 2007], a central character  $W(R_{s,1})\xi(k_s)$  for  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$  is residual if and only if  $W_s\xi(k_s)$  is residual for  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_s$ . (This is because of the invariance of residual central characters under Dynkin diagram automorphisms.)

Let  $\mathbf{v}' \in \mathcal{Q}_b^{\text{reg}}$ . By the main result of [Ciubotaru et al. 2014], since  $W_s\xi(k'_s)$  is residual there exists a virtual discrete series character  $U'$  of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k'_s)$  with  $\overline{U'|_{W_s}} = b_s$  ( $U'$  is the analytic Dirac induction of  $b_s$ ). Again using the classification of the discrete series of [Opdam and Solleveld 2010], and (10), there exists a virtual discrete series character  $\pi'_{\mathbf{v}'}$  of  $\mathcal{H}_{\mathbf{v}'}$  with  $b = \text{Ind}_s(b_s) = \lim_{\epsilon \rightarrow 0} \overline{\pi_{\mathbf{v}'\epsilon}}$ . Since  $\mathcal{Y}_{\text{gm}} \subset \overline{\mathcal{R}_{\mathbb{Z}}(W)}$  it is clear that  $\mathcal{Y}_{\text{gm}}$  is finitely generated. Choose a finite collection of (irreducible) discrete series  $\pi_i$  of  $\mathcal{H}_{\mathbf{v}_i}$  such that the corresponding limits  $b_i$  are linearly independent and generate  $\mathcal{Y}_{\text{gm}}$ . By the arguments above, if  $\mathbf{v} \in \bigcap_i \mathcal{Q}_{b_i}^{\text{reg}}$  then there exist virtual discrete series characters  $\pi'_i$  of  $\mathcal{H}_{\mathbf{v}}$  with

$$b_i := \lim_{\epsilon \rightarrow 0} \overline{\pi'_{i,\mathbf{v}\epsilon}}.$$

Consequently,  $\mathcal{Y}_{\text{gm}} = \mathcal{Y}_{\mathbf{v}-m}$ . Since the limit map is an isometry and since (by [Opdam and Solleveld 2009]) the irreducible discrete series form an orthonormal set with respect to  $\text{EP}_{\mathcal{H}}$ ,  $\mathcal{Y}_{\mathbf{v}-m}$  (and thus  $\mathcal{Y}_{\text{gm}}$ ) admits an orthonormal basis.  $\square$

At this point, the basis  $\mathcal{B}_{\text{gm}}$  from Proposition 2.6 is not unique. The canonical choice for the basis  $\mathcal{B}_{\text{gm}}$  is obtained in Corollary 3.19.

**2B4. Dirac induction for affine Hecke algebras.**

**Definition 2.7.** Given  $b \in \mathcal{B}_{\text{gm}}$  and  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$ , we define  $\text{Ind}_D(b; \mathbf{v})$  (the “Dirac induction of  $b$ ”) as the unique virtual discrete series character of  $\mathcal{H}_{\mathbf{v}}$  whose scaling limit satisfies  $\lim_{\epsilon \rightarrow 0} \text{Ind}_D(b; \mathbf{v}\epsilon) = b$ . Up to a sign  $\epsilon(b; \mathbf{v}) \in \{\pm 1\}$ ,  $\text{Ind}_D(b; \mathbf{v})$  is an irreducible discrete series character. For all  $\mathbf{v} \in \mathcal{Q}$  this defines a bijection

$$\mathcal{B}_{\mathbf{v}-m} \ni b \rightarrow \epsilon(b; \mathbf{v}) \text{Ind}(b; \mathbf{v}) \in \Delta_{\mathbf{v}},$$

where  $\Delta_{\mathbf{v}} := \Delta(\mathcal{H}_{\mathbf{v}})$  denotes the set of isomorphism classes of irreducible discrete series representations of  $\mathcal{H}_{\mathbf{v}}$ .

In the proof of Proposition 2.6 we have seen that  $\text{Ind}_D(b; \mathbf{v})$  is not directly constructed as the index of a Dirac-type operator but rather, it corresponds via (10) and a Morita equivalence to the discrete series  $U \otimes M$  of  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s) \rtimes \Gamma_U$ , with  $U = \text{Ind}_D^{\text{an}}(b_s, k_s)$  the analytic Dirac induction (defined in [Ciubotaru et al. 2014]) of  $b_s$  for  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$ . The existence of  $\text{Ind}_D(b; \mathbf{v})$  follows

from this construction. It also follows from this construction that  $\pm \text{Ind}_D(b; \mathbf{v})$  is irreducible.

**2B5.** *The generic Vogan central character map.* Let  $\mathcal{B}_{\text{gm}}$  be an orthonormal basis of  $\mathcal{Y}_{\text{gm}}$ . As we have seen, given  $b \in \mathcal{B}_{\text{gm}}$  the set  $\mathcal{Q}_b^{\text{reg}} = \{\mathbf{v} \mid b \in \mathcal{B}_{\mathbf{v}-m}\}$  is the complement of finitely many hyperplanes in  $\mathcal{Q}$ . As in the proof of Proposition 2.6, to each  $b \in \mathcal{B}_{\text{gm}}$  we have a canonically associated orbit of generic residual points  $W_0 r_b$ , with  $r_b = s \exp \xi_s$ , and  $\xi_s$  the generic linear residual point associated to  $b_s \in \overline{R_{\mathbb{Z}}(W_s)}$  by the generic version of ‘‘Vogan’s conjecture’’ (see [Ciubotaru et al. 2014, Theorem 3.2]). Strictly speaking, the results of [Ciubotaru et al. 2014] apply to give a residual central character of a nonextended graded affine Hecke algebra, but as already noted in the proof of Proposition 2.6, one may use the invariance [Opdam 2007] of these central characters under diagram automorphisms to obtain the desired central character in our more general setting.

Let us denote the resulting generic central character map (see [Opdam and Solleveld 2010]) by:

$$\begin{aligned} \text{gcc}_B : \mathcal{B}_{\text{gm}} &\rightarrow W_0 \setminus \text{Hom}(X, \Lambda^\times) = W_0 \setminus T_\Lambda, \\ b &\rightarrow W_0(s \exp \xi_s) \end{aligned}$$

(the generic Vogan central character map). From the proof of Proposition 2.6 and Definition 2.7 we obtain:

**Corollary 2.8.** *For all  $b \in \mathcal{B}_{\text{gm}}$  and  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$  we have*

$$\text{gcc}_{\mathbf{v}}(\text{Ind}_D(b; \mathbf{v})) = \text{gcc}_B(b) = W_0 r_b.$$

Moreover,  $W_0 r_b(1) = W_0 s$  if and only if  $b$  can be written as  $b = \text{Ind}_s(b_s)$ .

**Definition 2.9.** We put  $m_b := m_{W_0 r_b}$ , where  $m_{W_0 r_b}$  denotes the mass function associated to the orbit of  $\mathcal{Q}$ -generic residual points  $\text{gcc}_B(b) = W_0 r_b$  in (13). By [Opdam and Solleveld 2010], and we have for all  $b \in \mathcal{B}$  that

$$\mathcal{Q}_b^{\text{reg}} = \{\mathbf{v} \in \mathcal{Q} \mid m_b(\mathbf{v}) \neq 0\}.$$

When  $\mathcal{Q}'$  is a subtorus of  $\mathcal{Q}$ , we will write  $m_b^{\mathcal{Q}'}$  for the function  $m_{W_0 r'_b}$  defined with respect to the  $\mathcal{Q}'$ -generic residual point  $r'_b$  obtained from  $r_b$  by base change to  $\mathcal{Q}'$ . Notice that  $m_b^{\mathcal{Q}'} = d' m_b|_{\mathcal{Q}'}$  for some  $d' \in \mathbb{Q}$ , with  $d' \neq 0$  if and only if  $\mathcal{Q}_b^{\text{reg}} \cap \mathcal{Q}' \neq \emptyset$ .

**Theorem 2.10.** *The generic Vogan central character map  $\text{gcc}_B$  is a surjection  $\text{gcc}_B : \mathcal{B}_{\text{gm}} \rightarrow W_0 \setminus \text{Res}(\mathcal{R})$ .*

*Proof.* In [Opdam 2004] it was shown (also see [Opdam and Solleveld 2010]) that for any  $\mathbf{v} \in \mathcal{Q}$ , every orbit of residual points of  $\mathcal{H}_{\mathbf{v}}$  is the specialization at  $\mathbf{v}$  of a generic orbit of residual points  $W_0 r$  at  $\mathbf{v}$ . The main theorem of [Opdam 2004] states that for any  $\mathbf{v} \in \mathcal{Q}$ , and any orbit of residual cosets  $W_0 r(\mathbf{v})$ , there exists a discrete



series character  $\pi$  of  $\mathcal{H}_v$  such that  $W_0r(v)$  is the central character of  $\pi$ . The set  $W_0 \setminus \text{Res}(\mathcal{R})$  is finite, hence

$$\mathcal{Q}^{\text{gen}} := \left\{ v \in \bigcap_{W_0r \in W_0 \setminus \text{Res}(\mathcal{R})} \mathcal{Q}_{W_0r}^{\text{reg}} \mid \text{For all } r, r' \in \text{Res}(\mathcal{R}): W_0r(v) = W_0r'(v) \text{ only if } Wr = W_0r' \right\}$$

is open and dense in  $\mathcal{Q}$ . Let  $v \in \mathcal{Q}^{\text{gen}}$  and  $W_0r \in W_0 \setminus \text{Res}(\mathcal{R})$ . By the above there exists a  $\pi \in \Delta_v$  such that  $cc(\pi) = W_0r(v)$ . Since  $v \in \mathcal{Q}^{\text{gen}}$  this implies that  $\text{gcc}_v(\pi) = W_0r$ . (For the definition of the generic central character map  $\text{gcc}_v$ , see [Opdam and Solleveld 2010, Definition 5.4].) Put  $b := \lim_{\epsilon \rightarrow 0} \pi_{v^\epsilon}$ . By Corollary 2.8 we have  $\text{gcc}_B(b) = W_0r$ .  $\square$

Denote by  $\mathcal{B}_{W_0r} \subset \mathcal{B}_{\text{gm}}$  the fiber  $\text{gcc}_B^{-1}(W_0r)$  of the map from Theorem 2.10.

### 3. One-dimensional algebraic families of discrete series representations, and their limits

**3A. One-dimensional algebraic families of discrete series.** Given a generic residual point  $r \in \text{Hom}(X, \Lambda^\times)$  we know that  $r(v)$  is a residual point for all  $v \in \mathcal{Q}_{W_0r}^{\text{reg}}$ , which is the complement of finitely many hyperplanes in  $\mathcal{Q}$ . By [Opdam and Solleveld 2010, Corollary 5.9]) we have a nonempty set of irreducible discrete series characters of  $\mathcal{H}_v$  with *generic central character*  $W_0r$ . Denote this nonempty set by:

$$\text{DS}_{W_0r, v} := \{ \pi \in \Delta(\mathcal{H}_v) \mid \text{gcc}(\pi) = b \}.$$

Hence  $W_0r(v)$  is the central character of the following nonempty union:

$$\text{DS}_{W_0r(v)} := \bigcup_{\{W_0r' \mid W_0r'(v) = W_0r(v)\}} \text{DS}_{W_0r', v}.$$

Let  $v \in \mathcal{Q}_{\mathbb{C}}$ . For each  $r'(v) \in W_0r(v)$ , choose a convex open neighborhood  $U_{r'(v)}$  of  $r'(v) \in T$  such that the only residual cosets of the  $\mu$ -function  $\mu_{\mathcal{R}, v}$  of  $\mathcal{H}_v$  which intersect  $U_{r'(v)}$  in fact contain  $r'(v)$ . Let  $U_{r'(v)}^{\text{reg}} = U_{r'(v)} \cap T^{\text{reg}}$  be the complement in  $U_{r'(v)}$  of the union of the set of residual cosets of  $\mu_{\mathcal{R}, v}$  in  $T$ . The choice of  $U_{r'(v)}$  as above implies that  $U_{r'(v)}^{\text{reg}}$  is homeomorphic to the complement of a central hyperplane arrangement in a complex vector space with origin  $r'(v)$ . In particular, the homology group  $H_n(U_{r'(v)}^{\text{reg}}, \mathbb{Z})$  only depends on the local structure of the pole hyperplane arrangement at  $r'(v)$ , and if we would shrink  $U_{r'(v)}$  to a smaller convex open neighborhood  $U_{r'(v)}^{\prime, \text{reg}} \subset U_{r'(v)}^{\text{reg}}$  of  $r'(v)$  this would induce a canonical isomorphism  $H_n(U_{r'(v)}^{\prime, \text{reg}}, \mathbb{Z}) = H_n(U_{r'(v)}^{\text{reg}}, \mathbb{Z})$ . Let us denote the direct limit  $\varinjlim H_n(U_{r'(v)}^{\prime, \text{reg}}, \mathbb{Z})$  by  $H_{n, r'(v)}(\mathbb{Z})$ .

In general, if  $U \subset T$  is homeomorphic to a complex ball  $B \subset \mathfrak{t}$ , via the exponential mapping of the complex algebraic torus  $T$ , then  $U^{\text{reg}} := U \cap T^{\text{reg}}$  is homeomorphic to the intersection of  $B$  with the complement of an affine hyperplane arrangement. In this section we will need some basic facts about the topology of hyperplane arrangements, see [Orlik and Terao 1992; Schechtman and Varchenko 1991]. By [Schechtman and Varchenko 1991, Paragraph 4.4] it easily follows that  $H_n(U^{\text{reg}}, \mathbb{Z})$  is in a canonical way a *direct sum* of the  $H_{n,p}(\mathbb{Z})$  where  $p$  runs over the set of points of  $U$  which lie in the intersection lattice generated by the codimension one residual cosets of  $\mu_{\mathcal{R},v}$ . Let  $\pi_p : H_n(U^{\text{reg}}, \mathbb{Z}) \rightarrow H_{n,p}(\mathbb{Z})$  denote the corresponding projection.

Let us denote the collection of open sets  $\{U_{r'(\mathbf{v})}\}_{r' \in W_0r}$  by  $\mathcal{U}$ . Let  $\mathcal{O}_{\mathcal{U},\mathbf{v}} \subset \mathcal{Q}_{\mathbb{C}}$  be an open ball with center  $\mathbf{v}$  with the property that for all  $\mathbf{v}' \in \mathcal{O}_{\mathcal{U},\mathbf{v}}$  and all  $r' \in W_0r$  we have  $r'(\mathbf{v}') \in U_{r'(\mathbf{v})}$ . Given  $\mathbf{v} \in \mathcal{Q}_{\mathbb{C}}$  and a homology class  $[\xi_{r'(\mathbf{v})}] \in H_n(U_{r'(\mathbf{v})}, \mathbb{Z})$  for each  $r'(\mathbf{v}) \in W_0r(\mathbf{v})$ , this defines for all  $\mathbf{v}' \in \mathcal{O}_{\mathcal{U},\mathbf{v}}$  a unique class  $\pi_{r'(\mathbf{v}')}([\xi_{r'(\mathbf{v}')}] \in H_{n,r'(\mathbf{v}')}(Z)$ . It is easy to see that this procedure defines a topology basis of the étale space of a sheaf  $\mathcal{F}_{n,r'}^H$  of abelian groups over  $\mathcal{Q}_{\mathbb{C}}$ , with stalks  $H_{n,r'(\mathbf{v})}(Z)$ . Let  $\mathcal{Q}_{W_0r,\mathbb{C}}^{\text{gen}} \subset \mathcal{Q}_{\mathbb{C}}$  be the Zariski-open set of  $\mathbf{v} \in \mathcal{Q}_{\mathbb{C}}$  such that  $|W_0r(\mathbf{v}')|$  is locally constant at  $\mathbf{v}$ , and such that if  $r'$  is a generic residual point such that  $W_0r(\mathbf{v}) = W_0r'(\mathbf{v})$  then  $W_0r = W_0r'$ .<sup>1</sup> Clearly, the sheaf  $\mathcal{F}_{n,r}^H$  is locally trivial in the analytic topology on  $\mathbf{v} \in \mathcal{Q}_{W_0r,\mathbb{C}}^{\text{gen}}$ . We have shown:

**Lemma 3.1.** *For each generic residual point  $r$ , the homology groups  $H_{n,r(\mathbf{v})}(Z)$  ( $\mathbf{v} \in \mathcal{Q}_{\mathbb{C}}$ ) are the stalks of a sheaf  $\mathcal{F}_{n,r}^H$  (in the analytic topology) of finitely generated abelian groups on  $\mathcal{Q}_{\mathbb{C}}$ , which is locally trivial on the Zariski-open set  $\mathcal{Q}_{W_0r,\mathbb{C}}^{\text{gen}}$ .*

The main results of [Opdam 2004] and of [Opdam and Solleveld 2010] show that for each  $\mathbf{v} \in \mathcal{Q}_{W_0r}^{\text{reg}}$  we can choose, for each  $r'(\mathbf{v}) \in W_0r(\mathbf{v})$ , classes  $\xi_{r'(\mathbf{v})} \in H_{n,r'(\mathbf{v})}(Z)$  and, for each  $\chi \in \text{DS}_{W_0r(\mathbf{v})}$ , constants  $c_{\chi,C} \in \mathbb{Q}_+$  depending only on the connected component  $C = C_{\chi,\mathbf{v}} \subset \mathcal{Q}_{\text{gcc}_v}^{\text{reg}}$  to which  $\mathbf{v}$  belongs (where  $\text{gcc}_v(\chi) = W_0r''$  denotes the generic central character [Opdam 2004, Definition 5.4] of  $\chi$ ), such that for all  $h \in \mathcal{H}_{\mathbf{v}}$ ,

$$\sum_{\chi \in \text{DS}_{W_0r(\mathbf{v})}} c_{\chi,C} \chi_{\mathbf{v}}(h) = m_{W_0r(\mathbf{v})}^{-1} \sum_{r'(\mathbf{v}) \in W_0r(\mathbf{v})} \int_{t \in \xi_{r'(\mathbf{v})}} K_{\mathbf{v}}(h, t), \tag{16}$$

where  $K_{\mathbf{v}}(h, t) = E_t(\mathbf{v}; h) \Delta(t)^{-1} \mu_{\mathcal{R},\mathbf{v}} dt$  is a rational  $(n, 0)$ -form in  $t \in T$ , with  $\Delta(t) := \prod_{\alpha > 0} (1 - \alpha(t)^{-1})$  and with  $E_t(\mathbf{v}; h)$  a linear functional in  $h \in \mathcal{H}_{\Lambda}$ , which is regular on  $(t, \mathbf{v}) \in T \times \mathcal{Q}_{\mathbb{C}}$ , and such that  $E_t(\mathbf{v}; a) = a(t) \Delta(t)$  for all  $a \in \mathcal{A}$ .

We now prove a fundamental continuity property of (16) (or even the full Plancherel decomposition of the trace of  $\mathcal{H}_{\mathbf{v}}$ ) with respect to the topology of the sheaves  $\mathcal{F}_{n,r}^H$ .

<sup>1</sup>Using the results of [Opdam 2007] it is easy to see that  $\mathcal{Q}_{W_0r,\mathbb{C}}^{\text{gen}} \cap \mathcal{Q} = \mathcal{Q}_{W_0r}^{\text{gen}}$ , in the notation of [Opdam and Solleveld 2010].

**Lemma 3.2.** *Let  $r$  be a generic residual point and let  $\mathbf{v} \in \mathcal{Q}_{W_0 r}^{\text{reg}}$ . For  $r' \in W_0 r$  and for  $\mathbf{v}'$  in a sufficiently small neighborhood of  $\mathbf{v}$ , the local homology classes  $[\xi_{r'(\mathbf{v}')}] \in H_{n,r'(\mathbf{v}')}(\mathbb{Z})$  in (16) form a local section of  $\mathcal{F}_{n,r'}^H$ .*

*Proof.* We use the analogous construction to the above of sheaves  $\mathcal{F}_{n,r_L}^H(\mathbb{Z})$  for all  $r_L \in T_L$  a generic residual point. Recall the residue lemma [Opdam 2004, Lemma 3.4]. By [Opdam 2004, Proposition 3.7] we can realize the local traces  $\mathfrak{X}_c$  defined in [Opdam 2004, Lemma 3.4] “explicitly” by a system of local cycles  $\xi_{L(\mathbf{v})} \subset \mathcal{B}_{L(\mathbf{v})}(r_L(\mathbf{v}), \delta) \subset T_{L(\mathbf{v})}$  (for all quaresidual cosets  $L(\mathbf{v}) \subset T$ ) such that for  $t_0 \in T_v$  deep in the negative Weyl chamber, the cycles  $t_0 T_u$  and  $\bigcup_L \xi_{L(\mathbf{v})} \times T_u^L$  are homologous. Moreover, the  $\xi_{L(\mathbf{v})}$  satisfy certain local properties (see [Opdam 2004, Proposition 3.7], items (i), (ii) and (iii)) which guarantee that for all  $h \in \mathcal{H}_v$ , the functional

$$\mathbb{C}[T] \ni f \rightarrow \int_{\xi_{L(\mathbf{v})} \times T_u^L} f(t) K_v(h, t)$$

can be written as a distribution on  $r_L(\mathbf{v})T_u$  with support in  $L(\mathbf{v})^{\text{temp}} = r_L(\mathbf{v})T_u^L$ , applied to  $f$  (which is zero if  $L_v$  is not a residual coset).

The definition of the local sections of the sheaves  $\mathcal{F}_{n,r_{L'}}^H$  is such that the local properties (i), (ii) and (iii) mentioned in [Opdam 2004, Proposition 3.7]) can be satisfied by cycles

$$\xi_{L'(\mathbf{v}')} \subset \mathcal{B}_{L'(\mathbf{v}')}(r_{L'}(\mathbf{v}'), \delta) \subset T_{L'(\mathbf{v}')}$$

representing the classes  $[\xi_{L'(\mathbf{v}')}] \in H_{n,r_{L'}(\mathbf{v}')}(\mathbb{Z})$  for  $\mathbf{v}'$  in a small neighborhood of  $\mathbf{v}$ . Moreover, it is automatic that  $t_0 T_u$  is homologous to the union of  $\bigcup_L \xi_{L(\mathbf{v}')} \times T_u^L$  (union over the generic residual cosets  $L$ ) with a collection of cycles of the form  $\xi_{L'(\mathbf{v}')} \times T_u^{L'}$ , where  $L'(\mathbf{v}')$  is quaresidual (which are irrelevant). This implies the result. □

**Corollary 3.3.** *For  $\mathbf{v} \in \mathcal{Q}_{W_0 r}^{\text{reg}}$ , the left-hand side of (16) extends to a central functional on  $\mathcal{H}_{\mathbf{v}'}$  for all  $\mathbf{v}'$  in a Zariski-open neighborhood of  $\mathbf{v}$  in  $\mathcal{Q}_{\mathbb{C}}$ . For  $h \in \mathcal{H}_{\Lambda}$  this functional takes values in the quotient field  $Q_{\Lambda}$  of  $\Lambda$ . The values of this functional restrict to continuous functions on the connected component  $C$  of  $\mathbf{v}$  in  $\bigcap_{\chi \in \text{DS}_{W_0 r(\mathbf{v})}} \mathcal{Q}_{\text{gcc}_v(\chi)}^{\text{reg}}$ .*

*Proof.* We extend the left-hand side in a neighborhood of  $\mathbf{v}$  by taking local sections of classes  $[\xi_{r'(\mathbf{v})}]$  in  $\mathcal{F}_{n,r'}^H$ , and use the right-hand side of (16) to define the left-hand side. We know from [Opdam and Solleveld 2010] that the left-hand side extends, for all  $h \in \mathcal{H}_{\Lambda}$ , continuously on  $\bigcap_{\chi \in \text{DS}_{W_0 r(\mathbf{v})}} \mathcal{Q}_{\text{gcc}_v(\chi)}^{\text{reg}}$ . Let  $\chi \in \text{DS}_{W_0 r(\mathbf{v})}$  and let  $\text{gcc}_v(\chi) := W_0 r'$ . Let

$$\mathbf{v}' \in \bigcap_{\chi \in \text{DS}_{W_0 r(\mathbf{v})}} C_{\chi, \mathbf{v}} \cap \mathcal{Q}_{\text{gcc}_v(\chi)}^{\text{gen}}$$

Then the left-hand side of (16) can be written as  $\sum_{\{W_{0r'}|W_{0r'}(v)=W_{0r}(v)\}} \Sigma_{W_{0r'}(h; v')$ , with

$$\Sigma_{W_{0r'}(h; v') := \sum_{\chi \in \text{DS}_{W_{0r'}, v'}} c_{\chi, C} \chi_{v'}(h). \tag{17}$$

We have

$$\Sigma_{W_{0r'}(h; v') = m_{W_{0r'}(v')}^{-1} \sum_{r''(v') \in W_{0r'}(v')} \int_{t \in \xi_{r''(v')}} K_{v'}(h, t),$$

where, for  $v'$  in a sufficiently small neighborhood of  $v$ , the class  $[\xi_{r''(v')}]$  is equal to the image  $\pi_{r''(v')}(\xi_{r''(v)}) \in H_{n, r''(v')}(\mathbb{Z})$  of the constant cycle  $\xi_{r''(v)}$ , by Lemmas 3.1 and 3.2. It follows that in a small analytic neighborhood of  $v$  the expression  $\Sigma_{W_{0r'}(h; v')$  is given by an element of  $\mathcal{Q}_\Lambda$ . This extends  $v' \rightarrow \Sigma_{W_{0r'}(h; v')$  to a rational function on a Zariski-open set of  $\mathcal{Q}_\mathbb{C}$ .

The continuity of the values of  $\Sigma_{W_{0r'}(h; v')$  in the connected component  $C \subset \mathcal{Q}_{W_{0r'}}^{\text{reg}}$  of  $v'$  follows from [Opdam and Solleveld 2010, Theorem 3.4, Proposition 3.7], and this implies the continuity assertion in the theorem.  $\square$

**Remark 3.4.** We note that  $\Sigma_{W_{0r}}(h; \cdot)$  is not rational on  $\bigcap_{\chi \in \text{DS}_{W_{0r}(v)}} \mathcal{Q}_{\text{gcc}_v}^{\text{reg}}(\chi)$ . The rational functions of Corollary 3.3 depend on the connected component of  $v$  in  $\bigcap_{\chi \in \text{DS}_{W_{0r}(v)}} \mathcal{Q}_{\text{gcc}_v}^{\text{reg}}(\chi)$ .

Let  $\mathcal{C} \subset \mathcal{Q}$  be (the set of complex points of) an irreducible real algebraic curve which is not contained in  $\bigcup_{W_{0r}} \mathcal{Q}_{W_{0r}}^{\text{sing}}$ . Let  $\mathfrak{R}$  be the ring of regular functions of  $\mathcal{C}$ , with quotient field  $\mathfrak{K}$ . Thus  $\mathfrak{R}$  is a quotient of  $\Lambda$ , and we can form  $\mathcal{H}_{\mathfrak{R}} := \mathcal{H}_\Lambda \otimes_\Lambda \mathfrak{R}$ . Let  $P = v_0 \in \mathcal{C}$  be a smooth point of  $\mathcal{C}$ , and let  $C \subset \mathcal{Q}_{W_{0r}}^{\text{reg}}$  be a connected component such that  $v_0 \in \bar{C}$ . Let  $v' \in C \cap \mathcal{Q}_{W_{0r}}^{\text{gen}}$ , and consider  $\Sigma_{W_{0r}}(h; v')$ . According to Corollary 3.3, this positive rational linear combination of the discrete series characters with central character  $W_{0r}(v')$  extends as a rational function in  $v'$  which is continuous on  $C$ . By Corollary 3.3 we can restrict the values of  $\Sigma_{W_{0r}}(h; v')$  to  $v' \in \mathcal{C} \cap C$ . By the argument of [Kollár and Nowak 2015, Proposition 7] it is clear that this restriction defines a rational function on  $\mathcal{C}$  which we will denote by  $\Sigma_{W_{0r}, C}^{\mathcal{C}}(h) \in \mathfrak{R}$ , depending linearly on  $h \in \mathcal{H}_{\mathfrak{R}}$ . These rational functions are continuous on  $\mathcal{C} \cap C$ .

**Lemma 3.5.** *Assume that  $\mathcal{C} \cap \bar{C}$  consists of smooth points and let  $P = v_0 \in \mathcal{C} \cap \bar{C}$ . For all  $h \in \mathcal{H}_{\mathfrak{R}}$  we have: The rational function  $\Sigma_{W_{0r}, C}^{\mathcal{C}}(h)$  is regular in  $P = v_0$ . There exists a Zariski-open neighborhood  $\mathfrak{U} \subset \mathcal{C}$  of  $\mathcal{C} \cap \bar{C}$  such that  $\Sigma_{W_{0r}, C}^{\mathcal{C}}(h) \in \mathfrak{R}(\mathfrak{U})$  for all  $h \in \mathcal{H}_{\mathfrak{R}(\mathfrak{U})}$ .*

*Proof.* Since  $\Sigma_{W_{0r}, C}^{\mathcal{C}}(h)$  is rational and  $P$  is a smooth point of the curve  $\mathcal{C}$ , it is enough to show that  $v \rightarrow \Sigma_{W_{0r}, C}^{\mathcal{C}}(h; v)$  is bounded in a neighborhood of  $v_0 = P \in \mathcal{C}$  for  $v \in \mathcal{C} \cap C$ . But  $\Sigma_{W_{0r}}(\cdot; v)$  is a positive functional on  $\mathcal{H}_v$  for all  $v \in C$ , and if  $v \in \mathcal{C} \cap C$  then  $\Sigma_{W_{0r}}(T_e; v) = \sum_{\chi \in \text{DS}_{W_{0r}, v}} c_{\chi, C} \deg(\chi) = C \in \mathbb{Q}_+$  is constant.

By [Opdam 2004, Corollary 2.17(i)] we see that  $|\Sigma_{W_0r,C}^{\mathfrak{C}}(h; \mathbf{v})| \leq C \|h\|_o$ . This implies the regularity at  $P$  of the rational function  $\mathbf{v} \rightarrow \Sigma_{W_0r,C}^{\mathfrak{C}}(h; \mathbf{v})$  for all  $h \in \mathcal{H}_{\mathfrak{R}}$ . Note that for  $z \in \mathcal{Z} \subset \mathcal{H}_{\mathfrak{R}}$  we have  $\Sigma_{W_0r,C}^{\mathfrak{C}}(zh; \mathbf{v}) = z(W_0r(\mathbf{v}))\Sigma_{W_0r,C}^{\mathfrak{C}}(h; \mathbf{v})$ , while  $\mathbf{v} \rightarrow z(W_0r(\mathbf{v})) \in \mathfrak{R}$  is regular on  $\mathfrak{C}$ . Since  $\mathcal{H}_{\mathfrak{R}}$  is finitely generated over  $\mathcal{Z}$  it follows that we can find an open neighborhood  $\mathfrak{U}$  of  $\mathfrak{C} \cap \bar{C}$  on which  $\mathbf{v} \rightarrow \Sigma_{W_0r,C}^{\mathfrak{C}}(h; \mathbf{v})$  is regular for all  $h \in \mathcal{H}_{\mathfrak{R}}$ .  $\square$

From now on we will assume that  $\mathfrak{C} \cap \bar{C}$  consists of smooth points of  $\mathfrak{C}$ , and that  $\mathfrak{U} \subset \mathfrak{C}$  is as in Lemma 3.5. We have shown that

$$\Sigma_{W_0r,C}^{\mathfrak{C}} \in \mathcal{H}_{\mathfrak{R}(\mathfrak{U})}^* := \text{Hom}_{\mathfrak{R}(\mathfrak{U})}(\mathcal{H}_{\mathfrak{R}(\mathfrak{U})}, \mathfrak{R}(\mathfrak{U}))$$

is a central functional in the  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})} \otimes_{\mathfrak{R}(\mathfrak{U})} \mathcal{H}_{\mathfrak{R}(\mathfrak{U})}^{\text{op}}$ -module  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})}^*$ , supported by the central character  $W_0r$ .

**Definition 3.6.**  $M_{W_0r,C}^{\mathfrak{U}} \subset \mathcal{H}_{\mathfrak{R}(\mathfrak{U})}^*$  denotes the  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})} \otimes_{\mathfrak{R}(\mathfrak{U})} \mathcal{H}_{\mathfrak{R}(\mathfrak{U})}^{\text{op}}$ -submodule generated by  $\Sigma_{W_0r,C}^{\mathfrak{C}}$ .

We will now show that a generic family  $\chi$  of discrete series characters as discussed in [Opdam and Solleveld 2010] always admits an algebraic model when we restrict the parameter space to a curve  $\mathfrak{C}$  in  $\mathcal{Q}$ . Moreover, we can find such models that are regular at the intersection of the curve with the boundary of the connected component  $C \subset \mathcal{Q}_{\text{gcc}(\chi)}^{\text{reg}}$  (an open cone) on which  $\chi$  lives. More precisely:

**Proposition 3.7.** (i)  $M_{W_0r,C}^{\mathfrak{U}}$  is an  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})} \otimes_{\mathfrak{R}(\mathfrak{U})} \mathcal{H}_{\mathfrak{R}(\mathfrak{U})}^{\text{op}}$ -module. The central subalgebra  $\mathcal{Z}_{\mathfrak{R}(\mathfrak{U})} \otimes_{\mathfrak{R}(\mathfrak{U})} \mathcal{Z}_{\mathfrak{R}(\mathfrak{U})}$  acts via scalar multiplication via the character  $(z_1 \otimes z_2) \rightarrow (z_1 z_2)(W_0r) \in \mathfrak{R}(\mathfrak{U})$ .

(ii)  $M_{W_0r,C}^{\mathfrak{U}}$  is a locally free  $\mathfrak{R}(\mathfrak{U})$ -module of finite rank.

(iii)  $M_{W_0r,C}^{\mathfrak{U}}$  has a canonical  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})}$ -algebra structure such that, if we put  $e = m_{W_0r} \Sigma_{W_0r,C}^{\mathfrak{C}} \in M_{W_0r,C}^{\mathfrak{U}} \subset \mathcal{H}_{\mathfrak{R}(\mathfrak{U})}^*$ , the map  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})} \ni h \rightarrow he$  is an algebra homomorphism.

(iv) If  $\mathbf{v} \in \mathfrak{U} \cap C$  then

$$M_{W_0r,C}^{\mathfrak{U}} \otimes \mathbb{C}_{\mathbf{v}} \simeq \bigoplus_{\chi \in \text{DS}_{W_0r,\mathbf{v}}} \text{End}_{\mathbb{C}}(V_{\chi_{\mathbf{v}}}),$$

where  $V_{\chi_{\mathbf{v}}}$  is a vector space on which the discrete series character  $\chi_{\mathbf{v}}$  is realized.

(v) We may assume without loss of generality that  $\mathfrak{U}$  is smooth. There exists a Zariski-open subset  $\mathfrak{U}^* \subset \mathfrak{U}$  with  $\mathfrak{C} \cap C \subset \mathfrak{U}^*$  such that

$$M_{W_0r,C}^{\mathfrak{U}^*} := M_{W_0r,C}^{\mathfrak{U}} \otimes_{\mathfrak{R}(\mathfrak{U})} \mathfrak{R}(\mathfrak{U}^*)$$

is a locally free separable  $\mathfrak{R}(\mathfrak{U}^*)$ -algebra of finite rank.

(vi) There exists a branched covering  $\phi : \tilde{\mathfrak{U}} \rightarrow \mathfrak{U}$  such that the following holds true. Let  $\mathfrak{L} \supset \mathfrak{K}$  denote the function field of  $\tilde{\mathfrak{U}}$ , and let  $\tilde{\mathfrak{R}}(\mathfrak{U}) := \mathfrak{R}(\tilde{\mathfrak{U}})$  be the integral

closure of  $\mathfrak{R}(\mathfrak{U})$  in  $\mathfrak{L}$ . Put  $\tilde{M}_{W_0r,C}^{\mathfrak{U}} := M_{W_0r,C}^{\mathfrak{U}} \otimes \tilde{\mathfrak{R}}(\mathfrak{U})$ . There exists a finite set of projective  $\tilde{\mathfrak{R}}(\mathfrak{U})$ -modules of finite rank  $\{L_i\}_{i \in \mathcal{I}}$ , such that

$$\tilde{M}_{W_0r,C}^{\mathfrak{U}} \subset \tilde{M}_{W_0r,C}^{\mathfrak{U},\max} \simeq \bigoplus_{i \in \mathcal{I}} \text{End}_{\tilde{\mathfrak{R}}(\mathfrak{U})}(L_i)$$

is an  $\tilde{\mathfrak{R}}(\mathfrak{U})$ -order. This turns  $L_i$  into an  $\mathcal{H}_{\tilde{\mathfrak{R}}(\mathfrak{U})}$ -module, for each  $i \in \mathcal{I}$ .

(vii) With  $\mathfrak{U}^*$  as in (v), define  $\tilde{\mathfrak{U}}^* := \phi^{-1}(\mathfrak{U}^*)$  and let  $\mathfrak{R}(\tilde{\mathfrak{U}}^*) \subset \mathfrak{L}$  be the integral closure of  $\mathfrak{R}(\mathfrak{U}^*)$  (which is the localization of  $\mathfrak{R}(\mathfrak{U}^*)$  at  $\tilde{\mathfrak{U}}^*$ ). Put  $\tilde{M}_{W_0r,C}^{\mathfrak{U}^*} := M_{W_0r,C}^{\mathfrak{U}^*} \otimes_{\mathfrak{R}(\mathfrak{U}^*)} \mathfrak{R}(\tilde{\mathfrak{U}}^*)$ . Then we have an isomorphism

$$\tilde{M}_{W_0r,C}^{\mathfrak{U}^*} = \tilde{M}_{W_0r,C}^{\mathfrak{U}^*,\max} \simeq \bigoplus_{i \in \mathcal{I}} \text{End}_{\tilde{\mathfrak{R}}(\mathfrak{U})}(L_i^*),$$

with  $L_i^* := L_i \otimes_{\tilde{\mathfrak{R}}(\mathfrak{U})} \mathfrak{R}(\tilde{\mathfrak{U}}^*)$ .

(viii) The covering  $\phi$  (as in (vi)) is not branched at points of  $\mathfrak{C} \cap C$ . Let  $\mathbf{v} \in \mathfrak{C} \cap C$  and let  $\tilde{\mathbf{v}}$  be a closed point lying above  $\mathbf{v} \in \mathfrak{C} \cap C$ . There exists a unique bijection  $i \rightarrow \chi_i$  from  $\mathcal{I}$  onto  $\text{DS}_{W_0r,\mathbf{v}}$  such that  $L_{i,\tilde{\mathbf{v}}} := L_i \otimes \mathbb{C}_{\tilde{\mathbf{v}}} \simeq V_{\chi_i,\mathbf{v}}$  as representation of  $\mathcal{H}_{\mathbf{v}}$ , via the canonical isomorphism  $\mathcal{H}_{\mathbf{v}} \simeq \mathcal{H}_{\tilde{\mathfrak{R}}(\mathfrak{U}),\tilde{\mathbf{v}}}$ .

*Proof.* Assertion (i): This is immediate from the definition.

Assertion (ii): Since  $\Sigma_{W_0r,C}^{\mathfrak{C}}$  is central and is a  $\mathcal{Z}^{\mathfrak{U}}$ -eigenfunction with eigenvalue  $W_0r$ , it is clear that  $M_{W_0r,C}^{\mathfrak{U}}$  is finitely generated over  $\mathfrak{R}(\mathfrak{U})$ . It is also obviously torsion free over  $\mathfrak{R}(\mathfrak{U})$ , implying that the localization  $M_{W_0r,C,Q}^{\mathfrak{U}}$  is free over  $\mathfrak{R}_Q$  for all  $Q \in \mathfrak{U}$  (since the local rings are principal ideal domains). Since  $M_{W_0r,C}^{\mathfrak{U}}$  is finitely generated, this implies that it is a locally free module.

Assertion (iii): We need to verify that the kernel of the surjective  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})}$ -module homomorphism  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})} \ni h \rightarrow he \in M_{W_0r,C}^{\mathfrak{R}(\mathfrak{U})}$  is a two-sided ideal. This is a trivial consequence of the centrality of  $e$ . This turns  $M_{W_0r,C}^{\mathfrak{R}(\mathfrak{U})}$  into an  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})}$ -algebra.

Assertions (iv) and (v): We have already established that  $M_{W_0r,C}^{\mathfrak{R}(\mathfrak{U})}$  is a locally free  $\mathcal{H}_{\mathfrak{R}(\mathfrak{U})}$ -algebra of finite rank. When we specialize at  $\mathbf{v} \in C \cap \mathfrak{U}$ , we may use the fact [Delorme and Opdam 2008, Corollary 5.8] that  $e_{\mathbf{v}} \in \mathcal{S}_{\mathbf{v}} \subset \mathcal{H}_{\mathbf{v}}$  is a central idempotent of the Schwartz algebra  $\mathcal{S}_{\mathbf{v}}$  to see that the specialization  $M_{W_0r,C,\mathbf{v}} = M_{W_0r,C}^{\mathfrak{U}} \otimes \mathbb{C}_{\mathbf{v}}$  equals the finite dimensional semisimple algebra as stated in (iv). Hence the trace form of  $M_{W_0r,C}^{\mathfrak{U}}$  is nondegenerate over  $\mathfrak{R}$ , and by shrinking  $\mathfrak{U}^*$  sufficiently we may assume that the discriminant of the trace form of  $M_{W_0r,C}^{\mathfrak{U}}$  is nonvanishing on  $\mathfrak{U}^*$ . We arrive at the conclusion of (v) (see, e.g., [Artin 1999, Section IV.3]).

Assertion (vi): By (v) it follows that  $M_{W_0r,C}^{\mathfrak{R}} := M_{W_0r,C}^{\mathfrak{U}} \otimes_{\mathfrak{R}(\mathfrak{U})} \mathfrak{R}$  is a finite separable  $\mathfrak{R}$ -algebra. Hence  $M_{W_0r,C}^{\mathfrak{R}}$  is isomorphic to a finite direct sum of central simple algebras over finite separable field extensions  $\mathfrak{R}_i$  of  $\mathfrak{R}$  (see [Artin 1999, Section IV.3]). Tsen's Theorem [Artin 1999, Section III.15] implies that in fact  $M_{W_0r,C}^{\mathfrak{R}}$  is a direct sum

of full matrix algebras  $\text{Mat}_{d_i \times d_i}(\mathfrak{K}_i)$ . Consider the compositum  $\mathfrak{L}$  of the  $\mathfrak{K}_i$ . It is easy to see that for all  $i$ , the tensor product  $\mathfrak{K}_i \otimes_{\mathfrak{K}} \mathfrak{L}$  is isomorphic to a finite direct sum of copies of  $\mathfrak{L}$ . It follows that  $\mathfrak{L}$  is a finite separable extension of  $\mathfrak{K}$  such that  $M_{W_0r,C}^{\mathfrak{L}} := M_{W_0r,C}^{\mathfrak{L}} \otimes_{\mathfrak{R}(\mathfrak{L})} \mathfrak{L} = M_{W_0r,C}^{\mathfrak{K}} \otimes_{\mathfrak{K}} \mathfrak{L}$  is isomorphic to a finite direct sum of full matrix algebras  $M_{W_0r,C,i}^{\mathfrak{L}} \simeq \text{End}_{\mathfrak{L}}(\mathfrak{L}^{d_i})$  over  $\mathfrak{L}$ . Let  $\Pi_i$  denote the projection homomorphism  $M_{W_0r,C}^{\mathfrak{L}} \rightarrow M_{W_0r,C,i}^{\mathfrak{L}}$ . Let  $\tilde{\mathfrak{R}}(\mathfrak{L})$  be the integral closure of  $\mathfrak{R}(\mathfrak{L})$  in  $\mathfrak{L}$ , which is a Dedekind domain with fraction field  $\mathfrak{L}$ . The inclusion  $\mathfrak{R}(\mathfrak{L}) \subset \tilde{\mathfrak{R}}(\mathfrak{L})$  gives rise to a ramified covering  $\phi : \tilde{\mathfrak{U}} \rightarrow \mathfrak{U}$ . We may assume, by shrinking  $\mathfrak{U}^*$  (as in (v)) if necessary, that  $\phi : \tilde{\mathfrak{U}}^* := \phi^{-1}(\mathfrak{U}^*) \rightarrow \mathfrak{U}^*$  is finite étale.

Now  $\tilde{M}_{W_0r,C}^{\mathfrak{L}} := M_{W_0r,C}^{\mathfrak{L}} \otimes_{\mathfrak{R}(\mathfrak{L})} \tilde{\mathfrak{R}}(\mathfrak{L}) \subset M_{W_0r,C}^{\mathfrak{L}}$  is clearly an  $\tilde{\mathfrak{R}}(\mathfrak{L})$ -order in  $M_{W_0r,C}^{\mathfrak{L}}$ . Hence the projection  $\Pi_i(\tilde{M}_{W_0r,C}^{\mathfrak{L}})$  is an  $\tilde{\mathfrak{R}}(\mathfrak{L})$ -order in  $M_{W_0r,C,i}^{\mathfrak{L}} \simeq \text{End}_{\mathfrak{L}}(\mathfrak{L}^{d_i})$ . By [Artin 1999, Theorem IV.4.1, Proposition IV.4.7] this is contained in a maximal  $\tilde{\mathfrak{R}}(\mathfrak{L})$ -order in  $M_{W_0r,C,i}^{\mathfrak{L}}$  which must be a trivial Azumaya algebra of the form  $\text{End}_{\tilde{\mathfrak{R}}(\mathfrak{L})}(L_i)$  where  $L_i \subset \mathfrak{L}^{d_i}$  denotes an  $\tilde{\mathfrak{R}}(\mathfrak{L})$ -lattice. Hence we have

$$\tilde{M}_{W_0r,C}^{\mathfrak{L}} \subset \bigoplus_i \Pi_i(\tilde{M}_{W_0r,C}^{\mathfrak{L}}) \subset \bigoplus_i \text{End}_{\tilde{\mathfrak{R}}(\mathfrak{L})}(L_i) =: \tilde{M}_{W_0r,C}^{\mathfrak{L},\max}.$$

This turns  $L_i$  into an  $\mathcal{H}_{\tilde{\mathfrak{R}}(\mathfrak{L})}$ -module, and proves (vi).

Assertions (vii) and (viii): Localizing at  $\tilde{\mathfrak{U}}^*$  we obtain  $\tilde{M}_{W_0r,C}^{\mathfrak{L}*} \subset \bigoplus_i \text{End}_{\tilde{\mathfrak{R}}(\mathfrak{L})}(L_i^*)$ , where  $L_i^* := L_i \otimes_{\tilde{\mathfrak{R}}(\mathfrak{L})} \tilde{\mathfrak{R}}(\tilde{\mathfrak{U}}^*)$  is an  $\tilde{\mathfrak{R}}(\tilde{\mathfrak{U}}^*)$ -lattice in  $\mathfrak{L}^{d_i}$ . Since  $\tilde{M}_{W_0r,C}^{\mathfrak{L}*}$  is separable, by (v), we have  $\tilde{M}_{W_0r,C}^{\mathfrak{L}*} = (\tilde{M}_{W_0r,C}^{\mathfrak{L}})^* := \text{Hom}_{\tilde{\mathfrak{R}}(\tilde{\mathfrak{U}}^*)}(\tilde{M}_{W_0r,C}^{\mathfrak{L}}, \tilde{\mathfrak{R}}(\tilde{\mathfrak{U}}^*))$ . As in the proof of [Artin 1999, Corollary IV.4.5] this implies that  $\tilde{M}_{W_0r,C}^{\mathfrak{L}*}$  is a maximal  $\tilde{\mathfrak{R}}(\tilde{\mathfrak{U}}^*)$ -order, hence we actually have  $\tilde{M}_{W_0r,C}^{\mathfrak{L}*} = \bigoplus_i \text{End}_{\tilde{\mathfrak{R}}(\mathfrak{L})}(L_i^*)$ , proving (vii). When we take  $\tilde{v}$  lying above a  $v \in \mathfrak{U}^* \cap C$ , then specializing at  $\tilde{v}$  and comparing (iv) and (vii) we obtain (viii). Observe that the structural result (vi) implies that  $\phi$  can not be ramified at  $v$ . Indeed, when assuming the contrary then (vi) would imply that the number of discrete series characters of  $\mathcal{H}_v$  with generic central character  $W_0r$  would be strictly smaller than the number of such discrete series characters of  $\mathcal{H}_{v'}$  with  $v' \in C$  in a small neighborhood of  $v$ . This contradicts [Opdam and Solleveld 2010, Corollary 5.11].  $\square$

**Proposition 3.8.** *Let  $b \in \mathcal{B}_{\text{gm}}$ . The family  $\mathcal{Q}_b^{\text{reg}} \ni v \rightarrow \epsilon(b; v) \text{Ind}_D(b; v)$  is continuous (in the sense of [Opdam and Solleveld 2010, Section 3]).*

*Proof.* The discrete series character  $\epsilon(b; v) \text{Ind}_D(b; v)$  of  $\mathcal{H}_v$  has generic central character  $\text{gcc}_v(\pi_v) = W_0r$  with  $r = r_b$  by Corollary 2.8. By [Opdam and Solleveld 2010, Definition 5.10] we know therefore that  $\epsilon(b; v) \text{Ind}_D(b; v)$  is also the specialization at  $v$  of a unique continuous family of generic irreducible discrete series characters  $\chi \in \Delta_{W_0r_b}^{\text{gen}}(\mathcal{R})$  with generic central character  $W_0r_b$  on  $\mathcal{Q}_b^{\text{gen}}$ . By [Opdam and Solleveld 2010, Corollary 5.8]  $\Delta_{W_0r_b}(\mathcal{R})$ , is a constant sheaf with finite fiber. Let  $C$  be a connected component of  $\mathcal{Q}_b^{\text{reg}}$ , and let  $\mathcal{C}$  be any irreducible real curve in

$\mathcal{Q}$  which intersects  $C$ . Let  $I_{\mathcal{C}} \subset \mathcal{C} \cap C$  be connected. By the above it is enough to show that  $\epsilon(b; \mathbf{v}) \text{Ind}_D(b; \mathbf{v})$  and  $\chi_{\mathbf{v}}$  are equivalent when  $\mathbf{v} \in I_{\mathcal{C}}$ . In other words, it is enough to show that for all  $\mathbf{v} \in I_{\mathcal{C}}$  we have  $\lim_{\epsilon \rightarrow 0} [\chi_{\mathbf{v}^\epsilon}] = b$ . By Proposition 3.7, the restriction of  $\chi$  to  $\mathcal{C} \cap C$  is realized by an algebraic model  $L$  defined over some Zariski-open set  $\tilde{\mathcal{U}}$  of a ramified cover  $\tilde{\mathcal{C}}$  of  $\mathcal{C}$ , such that  $I_{\mathcal{C}} \subset \mathcal{U}$  and there is no ramification at the points of  $I_{\mathcal{C}}$ . Choose a lift  $\tilde{I}_{\mathcal{C}}$  of  $I_{\mathcal{C}}$ .

By Corollary 2.8 we know that  $b = \text{Ind}_s(b_s)$ , where  $\text{gcc}_B(b) = W_0 r$  for some generic residual point  $r = s c$  with  $s \in T_u$  a fixed special point, and  $c = \exp(\xi)$  with  $\xi$  a generic linear residual point for  $\mathbb{H}(R_{s,1}, T_s(E), F_{s,1}; k_s)$ . By Proposition 3.7 we know that  $\epsilon(b, C) \text{Ind}_D(b, C; \mathbf{v})$  is realized by an algebraic family  $L$  defined over  $\mathfrak{R}(\tilde{\mathcal{U}}^*)$ . By shrinking  $\mathcal{U}^*$  if necessary, we may assume that for all  $\tilde{\mathbf{v}} \in \tilde{\mathcal{U}}^*$  we have, for all  $r, r' \in W_0 r$ , that  $s W_s c \neq s' W_{s'} c'$  (with  $r' = s' c'$ ) if and only if  $s W_s c(\mathbf{v}) \cap s' W_{s'} c'(\mathbf{v}) = \emptyset$  with  $\mathbf{v} = \phi(\tilde{\mathbf{v}})$ . Let  $m_s$  be the ideal in  $\mathfrak{R}(\tilde{\mathcal{U}}^*)[X]$  corresponding to the finite set  $s W_s c$  of  $\mathfrak{R}(\tilde{\mathcal{U}}^*)$ -points of  $T$ . By our choice of  $\mathfrak{R}(\tilde{\mathcal{U}}^*)$ , we have that  $m_s + m'_s = \mathfrak{R}(\tilde{\mathcal{U}}^*)[X]$  whenever  $s W_s c \neq s' W_{s'} c'$ . Hence by the Chinese remainder theorem we have

$$(\mathfrak{R}(\tilde{\mathcal{U}}^*)[X])_{\widehat{W_0 r}} \simeq \bigoplus_{s' \in W_0 s} (\mathfrak{R}(\tilde{\mathcal{U}}^*)[X])_{\widehat{s' W_{s'} c'}}$$

Let  $1 = \sum_{s' \in W_0 s} e_{s'}$  be the corresponding decomposition into orthogonal idempotents of  $1 \in (\mathfrak{R}(\tilde{\mathcal{U}}^*)[X])_{\widehat{W_0 r}}$ . Let  $\mathcal{H}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)} = \mathcal{H}_\Lambda \otimes_\Lambda \mathfrak{R}(\tilde{\mathcal{U}}^*)$ . Since  $\mathcal{H}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)}$  contains the Bernstein subalgebra  $\mathcal{A}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)} \simeq \mathfrak{R}(\tilde{\mathcal{U}}^*)[X]$ , we can define the idempotents  $e_s \in (\mathcal{H}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)})_{\widehat{W_0 r}}$ . Since  $\text{gcc}(L_b) = W_0 r$ , we can consider  $L$  as a module over  $(\mathcal{H}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)})_{\widehat{W_0 r}}$ . Hence the  $e_s$  act on  $L$ , and are mapped to idempotent elements in  $\text{End}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)}(L)$  such that  $\text{Id} = \sum_{s' \in W_0 s} e_{s'}$ .

Following Lusztig [1989a, Lemmas 8.14, 8.15] we know that

$$\mathcal{H}_{s, \mathfrak{R}(\tilde{\mathcal{U}}^*)} := (\mathcal{H}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)}(\mathcal{R}_s))_{\widehat{W_s c}} \rtimes \Gamma_s$$

is isomorphic to the algebra  $(\mathcal{H}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)})_{s, W_0 r} := e_s (\mathcal{H}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)})_{\widehat{W_0 r}} e_s$ , via the map (in the notation of [loc. cit.])  $f.T_{w,s}.\gamma \rightarrow f.e_s.T_w.e_s.T_\gamma.e_s$ . Hence the finite, locally free  $\mathfrak{R}(\tilde{\mathcal{U}}^*)$  module  $L_s := e_s L$  is a module over  $\mathcal{H}_{s, \mathfrak{R}(\tilde{\mathcal{U}}^*)}$ , and in particular over the finite-type Hecke subalgebra

$$\mathcal{H}_{s, \mathfrak{R}(\tilde{\mathcal{U}}^*), 0} := \mathcal{H}_{\mathfrak{R}(\tilde{\mathcal{U}}^*)}(W(R_{1,s})) \rtimes \Gamma_s$$

in the obvious way. By shrinking  $\mathfrak{R}(\tilde{\mathcal{U}}^*)$  if necessary we may assume that  $L_s$  is actually a free  $\mathfrak{R}(\tilde{\mathcal{U}}^*)$ -module. Let  $\chi_s$  denote the character of  $L_s$  (considered as a function of  $\mathbf{v} \in I_{\mathcal{C}}$ ). By Corollary 2.2 we see that for each  $\mathbf{v} \in I_{\mathcal{C}}$  we have  $\lim_{\epsilon \rightarrow 0} [\chi_{\mathbf{v}^\epsilon}] = \text{Ind}_s(b_{\mathbf{v},s})$ , where  $b_{\mathbf{v},s} = \lim_{\epsilon \rightarrow 0} [\chi_{s, \mathbf{v}^\epsilon}]$ . This limit clearly only depends on the restriction of  $L_s$  to  $\mathcal{H}_{s, \mathfrak{R}(\tilde{\mathcal{U}}^*), 0}$ . Now it is well known that (see, e.g.,



[Opdam 1995, Theorem 5])<sup>2</sup> the irreducible characters of  $\mathcal{H}_{s, \mathfrak{R}(\tilde{\mathfrak{U}}^*), 0}$  take values in  $\Lambda$ . This implies that  $b_{v,s}$  is independent of  $v \in I_{\mathfrak{C}}$ , as desired.  $\square$

**Corollary 3.9.**  $\epsilon(b, \cdot)$  is locally constant on  $\mathcal{Q}_b^{\text{reg}}$ .

*Proof.* The continuity of  $\mathcal{Q}_b^{\text{reg}} \ni v \rightarrow \epsilon(b; v) \text{Ind}_D(b; v)$  implies that on each connected component  $C$  of  $\mathcal{Q}_b^{\text{reg}}$  the sign  $C \ni v \rightarrow \epsilon(b, v)$  is continuous, hence constant.  $\square$

**Definition 3.10.** If  $C \subset \mathcal{Q}_b^{\text{reg}}$  is a connected component, we define  $\epsilon(b, C) := \epsilon(b, v) \in \{\pm 1\}$  for any choice of  $v \in C$ .

**3B. Limits of discrete series.** Let  $b \in \mathcal{B}_{\text{gm}}$  let  $C \subset \mathcal{Q}_b^{\text{reg}}$  be a connected component, and let  $v_0 \in \partial C$ . Choose a connected real algebraic curve  $\mathfrak{C}_r = \mathfrak{C} \cap \mathcal{Q} \subset \mathcal{Q}$  which meets  $C$ , and contains  $v_0 \in \mathfrak{C}_r$  as a smooth point of  $\mathfrak{C}_r$ . Extend  $\mathfrak{C}_r$  to a complex affine algebraic curve  $\mathfrak{C} \subset \mathcal{Q}_{\mathbb{C}}$ . Assume we have chosen the structures as in Proposition 3.7 with respect to  $W_0r = \text{gcc}_B(b)$ ,  $v_0$ ,  $C$  and  $\mathfrak{C}$ . Let  $v$  be a point in  $\mathfrak{C} \cap C$  such that  $v_0$  and  $v$  are in the same connected component of  $\mathfrak{C} \cap \bar{C}$ . Given a  $\tilde{v} \in \tilde{\mathfrak{U}}^*$  above  $v \in \mathfrak{C} \cap C$ , let us denote by  $L_b$  the unique  $\mathfrak{R}(\mathfrak{U})$ -lattice as in Proposition 3.7(vi) such that  $L_{i, \tilde{v}} \simeq V_{\chi_{b,v}}$  (according to Proposition 3.7(viii)), where  $\chi_{b,v} = \epsilon(b; v) \text{Ind}_D(b; v)$ . The interval  $I_{\mathfrak{C}} \subset \mathfrak{C}_r$  connecting  $v$  to  $v_0$  has a unique lift  $\tilde{I}_{\mathfrak{C}}$  starting at  $\tilde{v} \in \tilde{\mathfrak{U}}^*$ . Let  $\tilde{v}_0 \in \tilde{\mathfrak{U}}$  be the unique endpoint of this lifted interval lying above  $v_0$ .

**Definition 3.11.** We denote by  $\text{Ind}_D(b, C; v_0)$  the unique virtual character of  $\mathcal{H}_{v_0}$  such that  $\epsilon(b, C) \text{Ind}_D(b, C; v_0)$  is realized by  $L_{b, \tilde{v}_0}$ . We call this representation the *limit of the family of discrete series*  $\chi_b = \epsilon(b; v) \text{Ind}_D(b; v)$  along  $\mathfrak{C}$  from  $C$  at  $v_0$ .

**Remark 3.12.** A priori  $\text{Ind}_D(b, C; v_0)$  may depend on the chosen curve and the model  $L_b$  of  $\epsilon(b; v) \text{Ind}_D(b)$  defined over  $\mathfrak{R}(\mathfrak{U})$ , but we suppress this from the notation.

**Corollary 3.13.**  $\epsilon(b, C) \text{Ind}_D(b, C; v_0)$  is a genuine tempered character.

*Proof.* Since the limit  $\epsilon(b, C) \text{Ind}_D(b, C; v_0)$  was defined by specializing the  $\mathfrak{R}(\mathfrak{U})$ -module  $L_b$  at  $\tilde{v} = \tilde{v}_0$ , it is by definition a genuine character.

The central character of  $L_b(\tilde{v})$  is  $W_0r(v)$  (with  $v = \phi(\tilde{v})$ ), hence for all  $z \in \mathcal{Z}_{\mathfrak{R}(\mathfrak{U})}$  we have  $z(W_0r) \in \mathfrak{R}(\mathfrak{U})$ . Recall that  $\mathcal{Z}_{\mathfrak{R}(\mathfrak{U})} = \mathcal{A}_{\mathfrak{R}(\mathfrak{U})}^{W_0}$ , hence the generalized  $\mathcal{A}_{\mathfrak{R}(\mathfrak{U})}$  weight spaces belongs to the set  $W_0r \subset T(\mathfrak{R}(\mathfrak{U}))$ .

Let  $r'(v_0) \in W_0r(v_0)$ , and let  $\tilde{v} \in \tilde{I}_{\mathfrak{C}}$  be close to  $\tilde{v}_0$ . Choose a basis  $\{x_1, \dots, x_n\}$  of  $X$ , and consider the corresponding commuting elements  $\theta_i \in \mathcal{A}$  acting in  $L_b(\tilde{v})$ . For each  $i$  consider the set  $S_{i, r'(v_0)} := \{r'' \in W_0r \mid x_i(r''(v_0)) = x_i(r'(v_0))\}$  of

<sup>2</sup>For a discussion of the rationality properties of characters for finite Hecke algebras, see also [Geck and Pfeiffer 2000, Section 9.3], and the references therein.

generalized eigenvalues of  $\theta_i$  (acting in  $L_b(\tilde{\mathbf{v}})$ ) which coalesce to  $x_i(r'(\mathbf{v}_0))$  at  $\mathbf{v} = \mathbf{v}_0$ . Then the projection  $\Pi_{r', \tilde{\mathbf{v}}}$  onto the direct sum of the generalized  $\mathcal{A}$ -weight spaces in  $L_b(\tilde{\mathbf{v}})$  which coalesce to  $r'(\mathbf{v}_0)$  is the composition of the commuting projection operators  $\Pi_{i, r', \tilde{\mathbf{v}}}$  onto the direct sum of the generalized eigenspaces  $r''(\mathbf{v})$  of  $\theta_i(\tilde{\mathbf{v}})$  with  $r'' \in S_{i, r'(\mathbf{v}_0)}$ . By holomorphic functional calculus we have

$$\Pi_{i, r', \tilde{\mathbf{v}}} = \frac{1}{2\pi i} \int_{\partial D_i} (z \text{Id} - \theta_i(\tilde{\mathbf{v}}))^{-1} dz,$$

where  $D_i$  is a fixed disk around  $x_i(r'(\mathbf{v}_0))$  such that  $x_i(r''(\mathbf{v})) \in D_i$  if and only if  $r'' \in S_{i, r'(\mathbf{v}_0)}$  (such  $D_i$  clearly exist, provided  $\mathbf{v}$  is sufficiently close to  $\mathbf{v}_0$ ). Hence the  $\Pi_{i, r', \tilde{\mathbf{v}}}$  are continuous in  $\tilde{\mathbf{v}}$ . In particular  $\Pi_{r', \tilde{\mathbf{v}}}$  is continuous in  $\tilde{\mathbf{v}}$ , implying that the dimension of its image is independent of  $\mathbf{v}$ . Therefore  $r'(\mathbf{v}_0)$  is a generalized  $\mathcal{A}$ -weight of  $L_b(\tilde{\mathbf{v}}_0)$  if and only if for some  $\mathbf{v} \in I_{\mathcal{C}}$  sufficiently close to  $\mathbf{v}_0$ , there exists a  $r''(\mathbf{v})$  with  $r''(\mathbf{v}_0) = r'(\mathbf{v}_0)$  (i.e.,  $r'' \in \bigcap_i S_{i, r'(\mathbf{v}_0)}$ ) such that  $r''(\mathbf{v})$  is a generalized  $\mathcal{A}$ -weight of the discrete series representation  $L_b(\tilde{\mathbf{v}})$ . In particular,  $r'(\mathbf{v})$  is a limit of  $\mathcal{A}$ -weights which meet the Casselman condition for temperedness [Opdam 2004, Lemma 2.20]. This is a closed condition, hence the generalized  $\mathcal{A}$ -weights of  $L_b(\tilde{\mathbf{v}}_0)$  satisfy the Casselman conditions themselves. The mentioned lemma implies that  $L_b(\tilde{\mathbf{v}}_0) = \epsilon(b, C) \text{Ind}_D(b, C; \mathbf{v}_0)$  is tempered.  $\square$

**Corollary 3.14.** *The tempered character  $\epsilon(b, C) \text{Ind}_D(b, C; \mathbf{v}_0)$  is a member of an algebraic family of characters of  $\mathcal{H}_{\tilde{\mathfrak{X}}(\mathfrak{A})}$ -characters with values in  $\tilde{\mathfrak{X}}(\mathfrak{A})$ . For generic element  $\tilde{\mathbf{v}} \in \tilde{\mathfrak{X}}$  this character is irreducible, and for  $\tilde{\mathbf{v}} \in \tilde{\mathfrak{X}}^* \cap C$  it is an irreducible discrete series.*

**Remark 3.15.** The limit  $\epsilon(b, C) \text{Ind}_D(b, C; \mathbf{v}_0)$  may be irreducible or not. In the case of the Hecke algebras of type  $C_n^{(1)}$  the limits of discrete series were constructed using the geometric model [Kato 2009] of characters of the generic affine Hecke algebra of this type. In this situation it is known that the limits of the discrete series at nontrivial singular parameters are always irreducible [Ciubotaru et al. 2012]. In general the limit to the trivial parameter  $\mathbf{v} = 1$  will be a reducible character of the affine Weyl group (the only exceptions being the “one  $W$ -type” discrete series characters).

**Proposition 3.16.**  $\lim_{\epsilon \rightarrow 0} [\text{Ind}_D(b, C; \mathbf{v}_0^\epsilon)] = b \in \mathcal{B}_{\text{gm}} \subset \bar{\mathcal{R}}_{\mathbb{Z}}(W).$

*Proof.* The proof is exactly the same as the proof of Proposition 3.8.  $\square$

**Corollary 3.17.** *The covering  $\phi$  is not ramified at the points of  $\mathcal{C} \cap \bar{C}$ .*

*Proof.* The argument of Proposition 3.8 proves that  $b_s = \lim_{\epsilon \rightarrow 0} [\chi_{s, \mathbf{v}^\epsilon}]$  for any  $\mathbf{v}$  in a neighborhood of  $\mathbf{v}_0$ . On the other hand, the discrete series of  $\mathcal{H}_{\mathbf{v}}$  are parametrized by the  $b \in \mathcal{B}_{\mathbf{v}-\text{gm}} = \mathcal{B}_{\text{gm}} \cap \mathcal{Y}_{\mathbf{v}-\text{gm}}$  by Corollary 2.2, Proposition 2.6 and Section 2B4. Moreover, for  $b$  in the image of  $\text{Ind}_s$ , the limit map  $b \rightarrow b_s$  is an isometry by

Corollary 2.2. Hence the generic discrete series with generic central character  $W_0r$  are also parametrized by the corresponding elliptic class  $b_s$  of  $W_s$ . Therefore the algebraic continuation of the discrete series characters  $\epsilon(b, C) \text{Ind}_D(b; \mathbf{v})$  (realized by  $L_b$ ) on  $\widetilde{\mathfrak{U}}^*$  can not have monodromy around  $\mathbf{v}_0$ , by the above. We conclude that the character values are regular functions in a neighborhood of  $\mathbf{v}_0 \in \mathfrak{U}$ .  $\square$

**3C. The rationality of the generic formal degree.**

**3C1.** *The universality of the rational factors of the generic formal degree.* The following result follows now simply from [Ciubotaru and Opdam 2015, Corollary 5.7; Ciubotaru et al. 2014; Opdam and Solleveld 2010]:

**Theorem 3.18.** *Fix  $b \in \mathcal{B}_{\text{gm}}$ . The formal degree of a continuous family of virtual discrete series characters  $\mathcal{Q}_b^{\text{reg}} \ni \mathbf{v} \rightarrow \text{Ind}_D(b; \mathbf{v})$  as a function of  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$  is a rational function of the form  $d_b m_b$ , with  $d_b \in \mathbb{Q}^\times$ .*

*Proof.* By [Ciubotaru and Opdam 2015, Corollary 5.7] the formal degree of  $\text{Ind}_D(b; \mathbf{v})$  is a linear combination of rational functions of the  $\mathbf{v}_s$ , which only depends on the elliptic class  $\lim_{\epsilon \rightarrow 0} \text{Ind}_D(b; \mathbf{v}^\epsilon)$ . This is, by definition,  $b$  (and therefore independent of  $\mathbf{v}$ ). On the other hand we have shown that this family is continuous and (by [Ciubotaru et al. 2014]) has generic central character  $W_0r_b$ . By [Opdam and Solleveld 2010, Theorem 4.6] we conclude that the formal degree has the form  $d_b(C) m_b$  for some constants  $d_b(C)$  depending on the connected component  $C$  of  $\mathcal{Q}_b^{\text{reg}}$  in which  $\mathbf{v}$  lies. But the rationality implies that these constants need to be equal on all chambers.  $\square$

**Corollary 3.19.** *We can choose the basis vector  $b \in \mathcal{B}_{\text{gm}}$  uniquely such that  $d_b > 0$ . With this choice,  $\epsilon(b, C)$  will be equal to the sign of  $m_b(\mathbf{v})$  for  $\mathbf{v} \in C$ .*

**3D. Proof of Theorem 1.1.** The proof of Theorem 1.1 is all completed now, and we point out where the various parts of it have been proved. Part (a) is in Proposition 2.6 and Corollary 3.19. Part (b) is in Section 2B4. Part (c) is Corollary 2.8. Part (d) is in Proposition 3.8. Part (e) is Corollary 3.9. Part (f) follows from Corollary 2.2, Proposition 2.6, and Section 2B4. Part (g) is Theorem 3.18.

**4. Explicit results; comparison with the Kazhdan–Lusztig–Langlands classification**

In this section we will compare the uniform classification of the discrete series with the Kazhdan–Lusztig–Langlands classification when an affine Hecke algebra arises in the context of a unipotent type of an unramified simple group defined over a nonarchimedean local field. We will also compute, for all affine Hecke algebras of simple type, the canonically positive basis  $\mathcal{B}_{\text{gm}}$  and the fundamental rational constants  $d_b$  appearing in the generic Plancherel measure. We will moreover

explain how the general semisimple case can be reduced to the case of simple type. In addition, we will show that in each connected component of  $\mathcal{Q}_b^{\text{reg}}$  the generic discrete series character  $\epsilon(b; \mathbf{v}) \text{Ind}_D(b; \mathbf{v})$  takes values in  $\mathcal{Q}_\Lambda$ , the quotient field of  $\Lambda$ . Needless to say, the results in this section are based on case-by-case methods.

**4A. Determination of the canonically positive basis  $\mathcal{B}_{\text{gm}}$ .**

**4A1. The sign of  $m_b(\mathbf{v})$ .** In light of Corollary 3.19, we need to analyze the sign of  $m_b(\mathbf{v})$ . Specialize

$$\mathbf{v}(s) = \mathbf{v}^{f_s}, \quad \text{with } \mathbf{v} > 1, f_s \in \mathbb{R}.$$

Write  $r = sc$  as before and let  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$  denote the corresponding graded affine Hecke algebra. Recall that the parameter function  $k_s$  is given by [Opdam and Solleveld 2010, (26)]

$$k_s(\alpha) = \begin{cases} \log_{\mathbf{v}}(v_{\alpha^\vee}^2) & \text{if } \alpha \in R_0 \cap R_1, \text{ or if } \alpha = 2\beta, \beta \in R_0 \text{ and } \beta(s) = 1, \\ \log_{\mathbf{v}}(v_{\alpha^\vee}^2 v_{2\alpha^\vee}^4) & \text{if } \alpha = 2\beta, \beta \in R_0, \beta(s) = -1. \end{cases} \tag{18}$$

Write  $c = v^{\bar{c}}$ . The following proposition says that the sign of the function  $m_{W_0,r}$  can in fact be detected at the graded Hecke algebra level.

**Proposition 4.1.** *The sign of  $m_{W_0,r}$  from (13) equals the sign of the expression*

$$\bar{m}_{W_0\bar{c}} = \frac{\prod'_{\alpha \in R_{s,1}} \alpha(\bar{c})}{\prod'_{\alpha \in R_{s,1}} (\alpha(\bar{c}) - k_s(\alpha))}. \tag{19}$$

*This expression is a polynomial which equals the product of  $n$  rational linear forms in the parameters  $k_s(\alpha)$  (where  $n$  is the rank of  $R_0$ ).*

*Proof.* For every  $\alpha \in R_1$ ,  $\alpha(s)$  is a root of unity. Write  $R_1 = R_{s,1} \sqcup R_1^{s,-1} \sqcup R_1^{s,z}$ , where  $R_1^{s,-1} = \{\alpha \in R_1 \mid \alpha(s) = -1\}$  and  $R_1^{s,z} = \{\alpha \in R_1 \mid \alpha(s)^2 \neq 1\}$ . If  $\alpha$  is a root in  $R_1 \setminus R_0$ , then  $\alpha$  is a root of type  $A_1$  in  $B_n$  and from the classification of isolated semisimple elements  $s$  in this situation, we see that  $\alpha \in R_{s,1}$ .

We break up the product expression for  $m_{W_0,r}$  according to the three types of roots:  $R_{s,1}, R_1^{s,-1}, R_1^{s,z}$ . Take  $R_1^{s,-1}$  first and we analyze the contribution of its numerator and denominator. The numerator is a product of expressions  $(-\alpha(c) - 1)(-\alpha(c)^{-1} - 1) > 0$ , since  $\alpha(c) > 0$ , one factor for each positive root  $\alpha \in R_1^{s,-1}$ . For the denominator, for each positive root  $\alpha \in R_1^{s,-1}$  (which by the remark above it is in  $R_0$ ), we have a factor  $(-v_{\alpha^\vee}^{-2}\alpha(c)^{-1} - 1)(-v_{\alpha^\vee}^{-2}\alpha(c) - 1) > 0$ . Therefore, the part of the expression corresponding to  $R_1^{s,-1}$  is positive.

Consider now  $R_1^{s,z}$ . Its contribution equals

$$\frac{\prod'_{\alpha \in R_1^{s,z}} (\alpha(s)\alpha(c) - 1)}{\prod'_{\alpha \in R_1^{s,z}} (\alpha(s)v_{\alpha^\vee}^{-2}\alpha(c) - 1)}.$$

The argument in [Opdam 2004, Theorem 3.27(v)] shows that both the numerator and the denominator are polynomial expressions in  $\mathbf{v}$  with rational (in fact integer) coefficients. Moreover, since  $\alpha(s)$  is not real, these polynomials do not afford real roots in  $\mathbf{v}$ . It follows that they are always positive or always negative for real  $\mathbf{v}$ . But it is clear that for  $\mathbf{v} = 0$  the fraction above equals  $1 > 0$ , and therefore it is always positive.

In conclusion, the sign of  $m_{W_0r}$  equals the sign of the expression

$$m_{W_0r} = \frac{\prod'_{\alpha \in R_{s,1}} (\alpha(c)^{-1} - 1)}{\prod'_{\alpha \in R_{s,1}} (v_{\alpha^\vee}^{-1} \alpha(c)^{-1/2} + 1) \prod'_{\alpha \in R_{s,1}} (v_{\alpha^\vee}^{-1} v_{2\alpha^\vee}^{-2} \alpha(c)^{-1/2} - 1)}.$$

The numerator can be rewritten as  $\prod_{\alpha \in R_{s,1}^+} \alpha(c)^{-1} (-1)(\alpha(c) - 1)^2$  and therefore its sign is  $(-1)^{n_s}$ , where  $n_s = \#\{\alpha \in R_{s,1}^+ \mid \alpha(c) \neq 1\} = \#\{\alpha \in R_{s,1}^+ \mid \alpha(\bar{c}) \neq 0\}$ . But this is also the sign of the numerator in (19).

Let  $\alpha \in R_0^+ \cap R_{s,1}$  be given. Then  $\alpha$  and  $-\alpha$  give a contribution in the denominator of  $v_{\alpha^\vee}^{-4} (\alpha(c)^{-1} - v_{\alpha^\vee}^2) (\alpha(c) - v_{\alpha^\vee}^2) = v_{\alpha^\vee}^{-4} (\mathbf{v}^{-\alpha(\bar{c})} - \mathbf{v}^{k_s(\alpha)}) (\mathbf{v}^{\alpha(\bar{c})} - \mathbf{v}^{k_s(\alpha)})$ . Clearly, the sign of this expression is the same as the sign of  $(-\alpha(\bar{c}) - k_s(\alpha)) (\alpha(\bar{c}) - k_s(\alpha))$ .

Now suppose that  $\alpha = 2\beta$  with  $\beta \in R_0$ . If  $\beta(s) = 1$ , the factors of the form  $(v_{\alpha^\vee}^{-1} \alpha(r)^{-1/2} + 1) = (v_{\alpha^\vee} \beta(c)^{-1} + 1)$  are all positive and can be ignored. So the contribution in the denominator comes from the factors  $(v_{\alpha^\vee}^{-1} v_{2\alpha^\vee}^{-2} \alpha(r)^{-1/2} - 1) = (\mathbf{v}^{-k_s(\alpha)/2} \beta(c)^{-1} - 1)$ . Grouping together the factors corresponding to  $\alpha$  and  $-\alpha$  as before and multiplying by powers of  $\mathbf{v}$ , we see that the contribution to the sign is given by  $(\mathbf{v}^{\alpha(\bar{c})/2} - \mathbf{v}^{k_s(\alpha)/2}) (\mathbf{v}^{-\alpha(\bar{c})/2} - \mathbf{v}^{k_s(\alpha)/2})$ . But this has the same sign as  $(-\alpha(\bar{c}) - k_s(\alpha)) (\alpha(\bar{c}) - k_s(\alpha))$ .

Finally, if  $\beta(s) = -1$ , then the factor

$$(v_{\alpha^\vee}^{-1} v_{2\alpha^\vee}^{-2} \alpha(r)^{-1/2} - 1) = -(v_{\alpha^\vee}^{-1} v_{2\alpha^\vee}^{-2} \beta(c)^{-1} + 1)$$

is negative and the contributions of these factors for  $\alpha$  and  $-\alpha$  cancel out. Thus we remain with the factor  $(v_{\alpha^\vee}^{-1} \alpha(r)^{-1/2} + 1) = (-\mathbf{v}^{-k_s(\alpha)/2} \beta(c) + 1)$  and the same analysis as in the previous case applies.

The regularity of  $m_{W_0r}$  on  $\mathcal{Q}$  implies that  $\bar{m}_{W_0\bar{c}}$  is bounded if the  $k_s(\alpha)$  take arbitrary real values. At the same time,  $\bar{m}_{W_0\bar{c}}$  is a rational function whose numerator and denominator are products of *rational* linear expressions in the  $k_s(\alpha)$ , with (as follows from Definition 2.3) exactly  $n$  more factors in the numerator than in the denominator. It follows that the denominator divides the numerator, leaving a polynomial in the  $k_s(\alpha)$  of degree  $n$  of the desired form.  $\square$

**Example 4.2.** Consider the affine Hecke algebra of type  $G_2$  with parameters  $k_1$  and  $k_2$  as in Section 4B3 below and in Table 1. In particular, let

$$\bar{c}_1 = [k_1, k_2], \quad \bar{c}_2 = [k_1, -k_1 + k_2], \quad \text{and} \quad \bar{c}_3 = \left[ k_1, \frac{1}{2}(-k_1 + k_2) \right],$$

be the three generic linear residual points in the graded affine Hecke algebra of type  $G_2$ . The corresponding polynomials in Proposition 4.1 are:

$$\begin{aligned} \bar{m}_{W_0\bar{c}_1} &= \frac{1}{36}(k_1 + 2k_2)(2k_1 + 3k_2), \\ \bar{m}_{W_0\bar{c}_2} &= -\frac{1}{36}(2k_1 - 3k_2)(k_1 - 2k_2), \\ \bar{m}_{W_0\bar{c}_3} &= \frac{1}{36}k_1k_2. \end{aligned} \tag{20}$$

Notice that the zeros of these polynomials give precisely the sets  $Q_{b_i}^{\text{sing}}$ ,  $1 \leq i \leq 3$ , see [Heckman and Opdam 1997, Proposition 4.15; Opdam and Solleveld 2010, Table 4].

**Remark 4.3.** Fix  $b \in \mathcal{B}_{\text{gm}}$  and suppose  $v_0 \in Q_b^{\text{reg}} \setminus Q_b^{\text{gen}}$  is a nongeneric, but regular parameter. By definition,  $b$  is in  $\mathcal{B}_{v_0-m}$  and let  $r_0 = \lim_{v \rightarrow v_0} r_b(v)$ . The irreducible discrete series at  $v_0$  in the family defined by  $b$  is  $\text{ds}(b, v_0) = \epsilon(b; v_0) \text{Ind}_D(b; v_0)$ . By Theorem 1.1, the sign  $\epsilon(b; v)$  is locally constant and therefore taking  $\lim_{v \rightarrow v_0}$  in the formula of Theorem 1.1(g), we see that  $\epsilon(b; v_0)$  equals the sign of  $m_{W_0r}(v_0) = \lim_{v \rightarrow v_0} m_{W_0r_b}(v)$ , where  $m_{W_0r_b}$  is defined by (13). This equals the sign of  $\bar{m}_{W_0\bar{c}_b}$ .

**4B. Tables.** We present the explicit form of the canonically positive basis  $\mathcal{B}_{\text{gm}}$ , as well as the generic residual central characters  $W_0r_b$ , the constants  $d_b$  in the case when  $\mathcal{R}$  is a simple root datum of type  $C_n^{(1)}$  (with three parameters),  $B_n^{(1)}$ ,  $C_n$  adjoint,  $G_2$ , and  $F_4$  (with two parameters). For the affine Hecke algebra of a simple, nonsimply laced root datum of arbitrary isogeny, one may deduce the relevant information from the cases listed above, by specializing the parameters appropriately and by the use of induction and restriction (see Section 4E).

**4B1.**  $C_n^{(1)}$ . Let

$$X = \mathbb{Z}^n = \langle \epsilon_1, \dots, \epsilon_n \rangle, \quad R_0 = \{\pm\epsilon_i \pm \epsilon_j, i \neq j, \pm\epsilon_i\}, \quad F_0 = \{\epsilon_i - \epsilon_{i+1}, \epsilon_n\}.$$

For simplicity of notation, we use the coordinates  $\epsilon_i$  for the basis of  $Y$  as well. We have  $W = X \rtimes W_0 = \mathbb{Z}R_0 \rtimes W_0$ . The affine simple roots are

$$F = \{(\epsilon_i - \epsilon_{i+1}, 0) \cup \{(2\epsilon_n, 0)\} \cup (-2\epsilon_1, 1)$$

and the affine Dynkin diagram (with parameters) is

$$v_0 \implies v_1 \text{ --- } v_1 \text{ --- } \dots \text{ --- } v_1 \longleftarrow v_2. \tag{21}$$

Here the affine simple root is  $\alpha_0 = (-2\epsilon_1, 1)$  and it gets the label  $v_0$ . Let  $\mathcal{H}(v_0, v_1, v_2)$  be the affine Hecke algebra with generators  $N_0, N_1, \dots, N_n$ . We need to switch to the Bernstein presentation. The algebra  $\mathcal{H}(v_0, v_1, v_2)$  is generated by  $N_1, \dots, N_n$

and  $\theta_1^{\pm 1}, \dots, \theta_n^{\pm 1}$  (here  $\theta_i = \theta_{\epsilon_i}$ ) subject to the relations

$$\begin{aligned} \theta_i N_i - N_i \theta_{i+1} &= (v_1 - v_1^{-1})\theta_i, & 1 \leq i \leq n - 1, \\ \theta_i N_j &= N_j \theta_i, & |i - j| \geq 2, \\ \theta_n N_n - N_n \theta_n^{-1} &= (v_2 - v_2^{-1})\theta_n + (v_0 - v_0^{-1}), \end{aligned} \tag{22}$$

the usual Hecke relations and the commutation of the  $\theta_i$ 's. In the change of presentation, we set  $\theta_i = N_0 N_{s_{\epsilon_i}}$ , where  $s_{\epsilon_i} \in W_0$  is the reflection corresponding to the root  $\epsilon_i$ .

It is immediate that the assignment  $N_0 \mapsto -v_0^{-1}$  gives a surjective algebra homomorphism onto the finite Hecke algebra  $\mathcal{H}_f(C_n, v_1, v_2)$  of type  $C_n$  with parameters  $v_1, v_2$ . Translating to the Bernstein presentation, we find that the assignment

$$N_i \mapsto N_i, \quad \theta_i \mapsto -v_0^{-1} N_{s_{\epsilon_i}}, \quad 1 \leq i \leq n, \tag{23}$$

extends to a surjective algebra homomorphism onto the Hecke algebra  $\mathcal{H}_f(C_n, v_1, v_2)$  of finite type. This allows us to lift every simple  $\mathcal{H}_f(C_n, v_1, v_2)$ -module to a simple module of the affine Hecke algebra. The simple  $\mathcal{H}_f(C_n, v_1, v_2)$ -modules are parametrized by bipartitions  $(\lambda, \mu)$  of  $n$ . It is particularly useful to use Hoefsmit's construction of such modules, see [Geck and Pfeiffer 2000, pages 322–325], since in that realization the  $N_{s_{\epsilon_i}}$  act diagonally.

We recall the construction. Denote by  $V_{(\lambda, \mu)}(v)$  the simple module of  $\mathcal{H}_f(C_n)$  parametrized by  $(\lambda, \mu)$ . Its basis is indexed by left-justified decreasing standard tableaux of shape  $(\lambda, \mu)$ . Let  $\dagger$  denote such a Young tableau. Then

$$N_{s_{\epsilon_j}} \cdot \dagger = \text{ct}(\dagger, n - j + 1)\dagger, \quad \text{where } \text{ct}(\dagger, k) = \begin{cases} v_1^{2(y-x)} v_2 & \text{if } k \text{ occurs in } \lambda, \\ -v_1^{2(y-x)} v_2^{-1} & \text{if } k \text{ occurs in } \mu, \end{cases}$$

where  $(x, y)$  are the coordinates of the box in which  $k$  occurs. (The coordinates  $(x, y)$  of a box in the Young tableau increase to the right in  $y$  and down in  $x$ .)

This means that in  $\tilde{V}_{(\lambda, \mu)}(v)$ , the lift of  $V_{(\lambda, \mu)}$  to the affine Hecke algebra, we have

$$\theta_j \cdot \dagger = -v_0^{-1} \text{ct}(\dagger, n - j + 1)\dagger. \tag{24}$$

The condition that a simple module is a discrete series module is that the eigenvalues of the product  $\theta_1 \cdot \theta_2 \cdots \theta_j$  are all smaller than 1 in absolute value, for all  $j = 1, \dots, n$ . From (24), we see that when the absolute value of the specialization of  $v_0$  is much larger than that of  $v_1$  and  $v_2$ , every  $V_{(\lambda, \mu)}(v)$  is a discrete series module. Moreover, the central characters of these modules are all distinct at generic values of the parameters, and since, by [Opdam and Solleveld 2010], the dimension of the space  $\bar{\mathcal{R}}_{\mathbb{Z}}(W)$  equals the number of bipartitions of  $n$ , it follows that the set  $\{\lim_{v \rightarrow 1} \tilde{V}_{(\lambda, \mu)}(v)\}$  is an orthonormal basis of  $\mathcal{Y}_{\text{gm}} = \bar{\mathcal{R}}_{\mathbb{Z}}(W)$ . In order to determine

the canonically positive basis  $\mathcal{B}_{\text{gm}}$ , it remains therefore to determine the signs  $\epsilon(b, \mathbf{v})$  for each  $b = \lim_{v \rightarrow 1} \tilde{V}_{(\lambda, \mu)}(v)$ .

To this end, we need to examine the formal degrees of these modules. Denote by  $R_0^{\text{sh}} = \{\pm\epsilon_i\}$ , the short roots, and by  $R_0^{\text{lo}} = \{\pm\epsilon_i \pm \epsilon_j\}$ , the long roots. Specialize

$$v_1 = v, \quad v_2 = v^{k_+ + k_-}, \quad v_0 = v^{k_+ - k_-},$$

where  $v > 1$  and  $k_-, k_+ \in \mathbb{R}$ . Notice that  $R_1 \setminus R_0 = \{\pm 2\epsilon_i\}$ . The formula for the generic formal degree [Opdam and Solleveld 2010, Theorem 4.6 and (40)] gives in our particular case that the formal degree of a discrete series  $\pi(b, \mathbf{v})$  with central character  $W_0 r_b(\mathbf{v})$  is

$$\text{fdeg}(\pi(b, \mathbf{v})) = \frac{d_b \epsilon(b, k_+, k_-) \prod'_{\alpha \in R_1} (\alpha(r_b(v))^{-1} - 1)}{\prod'_{\alpha \in R_0^{\text{lo}}} (v^{-2}\alpha(r_b(v))^{-1} - 1)} \cdot \frac{1}{\prod'_{\alpha \in R_0^{\text{sh}}} (v^{-2k_+}\alpha(r_b(v))^{-1} - 1) \prod'_{\alpha \in R_0^{\text{sh}}} (v^{-2k_-}\alpha(r_b(v))^{-1} + 1)},$$

where  $\epsilon(b, k_+, k_-)$  is a sign to be made explicit (see (28)). From [Ciubotaru et al. 2012, Theorem 4.7] (see the remark in the proof of [Opdam 2016, Theorem 4.12]), we know that

$$d_b = 1. \tag{25}$$

Fix a bipartition  $(\lambda, \mu)$ . The corresponding central character  $W_0 r_b(\mathbf{v})$  is the  $W_0$ -orbit of the string

$$r_{(\lambda, \mu)} = ((-v^{2(y-x)}v^{2k_-} \mid (x, y) \in \lambda); (v^{2(y'-x')}v^{-2k_+} \mid (x', y') \in \mu)). \tag{26}$$

and therefore

$$\bar{c}_{(\lambda, \mu)} = (\bar{c}_\lambda(k_-), \bar{c}_\mu(-k_+)), \quad \text{where } \bar{c}_\lambda(k) = ((y-x) + k \mid (x, y) \in \lambda).$$

We will use Proposition 4.1 to compute the signs. Denote by  $B_{|\lambda|}$  the root subsystem of  $R_0$  given by the roots that involve only coordinates  $\epsilon_i$ ,  $1 \leq i \leq |\lambda|$ . Similarly, let  $B_{|\mu|}$  be the root subsystem in coordinates  $\epsilon_j$ ,  $|\lambda| < j \leq n$ . Let  $D_{|\lambda|} = B_{|\lambda|} \cap R_0^{\text{lo}}$  and  $A_1^{|\lambda|} = B_{|\lambda|} \cap R_0^{\text{sh}}$ , and similarly for  $\mu$ . Notice that  $\mathbb{H}(R_{s,1}, T_s(T), F_{s,1}; k_s)$  is isomorphic to the product of graded affine Hecke algebras  $\mathbb{H}(B_{|\lambda|}, k_-) \times \mathbb{H}(B_{|\mu|}, k_+)$ , where by  $\mathbb{H}(B_\ell, k)$  we denote the graded affine Hecke algebra of type  $B_\ell$  with parameters 1 on the long roots and  $m$  on the short roots. Define for  $\lambda$  (and similarly for  $\mu$ ):

$$\bar{m}_\lambda(k) = \frac{\prod'_{\alpha \in B_{|\lambda|}} \alpha(\bar{c}_\lambda(k))}{\prod'_{\alpha \in D_{|\lambda|}} (\alpha(\bar{c}_\lambda(k)) - 1) \prod'_{\alpha \in A_1^{|\lambda|}} (\alpha(\bar{c}_\lambda(k)) - k)}, \tag{27}$$

$\epsilon(\lambda, k)$  = the sign of  $\bar{m}_\lambda(k)$ .



Then from Proposition 4.1,  $\bar{m}_\lambda(k)$  is a degree  $n$  polynomial and

$$\epsilon(b, k_+, k_-) = \epsilon(\lambda, k_-)\epsilon(\mu, -k_+) = \epsilon(\lambda, k_-)\epsilon(\mu^T, k_+), \tag{28}$$

where  $\mu^T$  denotes the transpose partition to  $\mu$ .

As noticed before, in the chamber where  $k_- \gg k_+ \gg 0$ , the discrete series are  $V_{(\lambda, \mu)}(\mathbf{v})$ . Let  $\epsilon(\lambda, \mu) = \lim_{k_\pm \rightarrow \infty} \epsilon(b, k_+, k_-)$ . In conclusion, we have proved:

**Proposition 4.4.** *For the affine Hecke algebra of type  $C_n^{(1)}$ , the canonically positive basis  $\mathcal{B}_{\text{gm}, C}$  of  $\mathcal{Y}_{\text{gm}} = \bar{\mathcal{R}}_{\mathbb{Z}}(W)$  is*

$$\mathcal{B}_{\text{gm}, C} = \{b_{(\lambda, \mu)} := \epsilon(\lambda, \mu)\tilde{V}_{(\lambda, \mu)}(1) \mid (\lambda, \mu) \text{ bipartition of } n\},$$

where  $\tilde{V}_{(\lambda, \mu)}(1)$  is the irreducible  $W$ -representation with central character  $W_0^{S(\lambda, \mu)}$ ,

$$s_{(\lambda, \mu)} = (\underbrace{-1, \dots, -1}_{|\lambda|}, \underbrace{1, \dots, 1}_{|\mu|}),$$

and whose restriction to  $W_0$  is the irreducible  $W_0$ -representation labeled by the bipartition  $(\lambda, \mu)$ .

**4B2.**  $B_n^{(1)}, C_n$  *adjoint.* The Iwahori–Hecke algebra of  $\text{Spin}_{2n+1}(F)$  is associated with the root datum of type  $B_n^{(1)}$ , for which the parameter space  $\mathcal{Q}$  is two dimensional. This two parameter family is isogenous to a specialization  $\mathcal{H}_B$  of the generic affine Hecke algebra  $\mathcal{H}_C$  of type  $C_n^{(1)}$  discussed in the previous section, namely  $\mathcal{H}_B = \mathcal{H}_C|_{v_0=1}$  (or equivalently  $k_- = k_+$ ). Observe that  $\mathcal{H}_B$  is associated with the extended diagram of type  $B_n^{(1)}$ , i.e.,  $R_0$  has type  $C_n$ , and  $X$  is its weight lattice. The Iwahori–Hecke algebra of  $\text{PGSp}_{2n}(F)$  is associated with the extended diagram of type  $C_n^{(1)}$ , and has a root datum with  $R_0$  of type  $B_n$  with  $X$  its weight lattice. The generic affine Hecke algebra  $\mathcal{H}_{C, \text{ad}}$  of this type has a two dimensional parameter space  $\mathcal{Q}_{\text{ad}} \subset \mathcal{Q}$  given by  $v_0 = v_2$ . Clearly,  $\mathcal{H}_{C, \text{ad}}$  is isogenous to the specialization  $\mathcal{H}_{C, \mathcal{Q}_{\text{ad}}} = \mathcal{H}_C|_{v_0=v_2}$  (or equivalently  $k_- = 0$ ) of the generic affine Hecke algebra  $\mathcal{H}_C$  of type  $C_n^{(1)}$ . To find the bases  $\mathcal{B}_{\text{gm}}$  of these two parameter generic Hecke algebras, we use some general results on isogeny (see Section 4E) in conjunction with the following remark:

**Remark 4.5.** Consider an arbitrary generic affine Hecke algebra  $\mathcal{H}_\Lambda$  over the parameter ring  $\Lambda$  as in Section 2B, with parameter space  $\mathcal{Q}$  and canonically positive basis  $\mathcal{B}_{\text{gm}}$ . Suppose that we have a quotient  $\Lambda'$  of  $\Lambda$  corresponding to a subtorus  $\mathcal{Q}' \subset \mathcal{Q}$ , and let  $\mathcal{B}'_{\text{gm}}$  be the corresponding basis for  $\mathcal{H}_{\Lambda'}$ . For any  $b \in \mathcal{B}_{\text{gm}}$  we have  $m_b^{\mathcal{Q}'} = d'_b m_b|_{\mathcal{Q}'}$  (see Definition 2.9) with

$$\pm b \in \mathcal{B}'_{\text{gm}} \iff \mathcal{Q}_b^{\text{reg}} \cap \mathcal{Q}' \neq \emptyset \iff d'_b \neq 0,$$

and

$$\mathcal{B}'_{\text{gm}} = \{\text{sign}(d'_b)b \mid b \in \mathcal{B}_{\text{gm}} \text{ and } \mathcal{Q}_b^{\text{reg}} \cap \mathcal{Q}' \neq \emptyset\}. \tag{29}$$

**Proposition 4.6.** (1) *The canonically positive basis of  $\mathcal{H}_B$  is  $\mathcal{B}_{\text{gm},B} = \mathcal{B}_{\text{gm},C}$ .*  
 (2) *The canonically positive basis of  $\mathcal{H}_{C,\mathcal{Q}_{\text{ad}}}$  is (see [Opdam and Solleveld 2010, Proposition 6.4] for the notion of  $\lambda$ -regular)*

$$\mathcal{B}_{\text{gm},C,\mathcal{Q}_{\text{ad}}} = \{ \epsilon_{C,\mathcal{Q}_{\text{ad}}}(\lambda) b_{(\lambda,\mu)} \mid 0 \text{ is } \lambda\text{-regular} \},$$

where  $\epsilon_{C,\mathcal{Q}_{\text{ad}}}(\lambda)$  is the sign of  $\bar{m}_\lambda^{C,\mathcal{Q}_{\text{ad}}}(0) \cdot \bar{m}_\lambda(0)$ .

*Proof.* By Remark 4.5 and formula (28) we need to compute the sign

$$\text{sign} \left( \lim_{(k_+,k_-) \rightarrow (k_{0,+},k_{0,-})} \bar{m}'_\lambda(k_-) \bar{m}'_{\mu^T}(k_+) \right) \cdot \epsilon(\lambda, k_{0,-}) \epsilon(\mu^T, k_{0,+}),$$

where  $k_{0,+} = k_{0,-} = k$  in the first case, and  $k_{0,+} = k, k_{0,-} = 0$  in the second case (here  $k$  is a generic real number), and  $\bar{m}'$  denotes the polynomial  $\bar{m}_\lambda^B$  (in the first case) or  $\bar{m}_\lambda^{C,\mathcal{Q}_{\text{ad}}}$  (in the second case) defined as  $\bar{m}$ , but with  $\bar{c}$  generic on the relevant 2-dimensional parameter space instead of the 3-dimensional parameter space of  $\mathcal{H}_C$ . The combinatorial condition that 0 is  $\lambda$ -regular is precisely designed so that  $\bar{m}_\lambda(0) \neq 0$ .

It is obvious that in the first case the sign is +1. Since in the second case the contribution of  $k_+$  is +1 again, the claim follows. □

**Remark 4.7.** Let  $\lambda$  be an arbitrary partition. The signs  $\epsilon_{C,\mathcal{Q}_{\text{ad}}}(\lambda)$  that appear in Proposition 4.6 can be computed in the terms of the diagram of the partition. We need to count the signs contributed by the nonzero linear factors that appear in  $\bar{m}_\lambda(k)$ ,  $k$  generic but which become zero at  $k = 0$ . These are all factors of the form  $\pm k, \pm 2k$ . One combinatorial answer is that the resulting sign is  $(-1)^\ell$ , where  $\ell$  is the number of pairs of distinct boxes  $\{x_1, x_2\}$  in the Young diagram of  $\lambda$  that are “almost symmetric” with respect to the main diagonal. I.e., suppose  $x_1$  is the box on or below the diagonal and  $x'_1$  is the flip of  $x_1$  with respect to the diagonal, then  $x_2$  is one unit away from  $x'_1$  in one of the four directions.

**4B3.**  $G_2, F_4$ . For the affine Hecke algebras of types  $G_2$  and  $F_4$ , we work with the following coordinates. For  $G_2$ , the affine diagram is

$$\alpha_0 \text{ --- } \alpha_1 \implies \alpha_2,$$

with parameters  $v(s_i) = v^{2k_i}, i = 1, 2$ , while for  $F_4$ , it is

$$\alpha_0 \text{ --- } \alpha_1 \text{ --- } \alpha_2 \implies \alpha_3 \text{ --- } \alpha_4,$$

with parameters  $v(s_1) = v^{2k_1}, v(s_3) = v^{2k_2}$ . In both cases,  $v > 1$  and  $k_i \in \mathbb{R}$ .

Let  $\omega_i^\vee$  denote the fundamental coweights. A central character is of the form  $r_b = sc$ , where  $c = v^{\sum_i a_i \omega_i^\vee}$ , where  $s$  is a compact element of the torus. The generic residual central characters are listed in Tables 1 and 2 (Parts 1 and 2). In these

$b$	$s$	$W_0c$	$d_b$	$G_2$	$\epsilon$
$b_1$	1	$[k_1, k_2]$	1	$[G_2, 1]$	1
$b_2$	1	$[k_1, -k_1 + k_2]$	1	$[G_2(a_1), (21)]$	-1
$b_3$	1	$[k_1, \frac{1}{2}(-k_1 + k_2)]$	1/2	$[G_2(a_1), (3)]$	1
$b_4$	$2A_1$	$[-\frac{1}{2}k_1 - \frac{3}{2}k_2, k_2]$	1/2	$[2A_1, 1]$	1
$b_5$	$A_2$	$[k_1, -k_1]$	1/3	$[A_2, 1]$	1

$b$	$E_6 \subset E_8$	$\epsilon$	${}^3E_6$	$\epsilon$
$b_1$	$[A_2E_6, \theta]$	1	$E_6$	1
$b_2$	$[A_2E_6(a_1), \theta]$	-1	$E_6(a_1)$	-1
$b_3$	$[A_2E_6(a_3), \theta]$	1	$E_6(a_3)$	1
$b_4$	$[A_1A_2A_5, \theta]$	1	$A_1A_5$	1
$b_5$	$[A_8, \theta]$	1	$A_2^3$	1

**Table 1.**  $G_2$ .

tables, if  $s \neq 1$ , we specify it by the type of its centralizer in the second column. In the third column, we give  $c$  in the form  $[a_i]$ .

For each generic residual central character  $W_0r_b$ , we compute the function  $m_b(\mathbf{v})$  using formula (40) from [Opdam and Solleveld 2010]. To obtain the constant  $d_b$ , we compute the limits of  $m_b(\mathbf{v})$  in equal parameter case, e.g.,  $k_1 \rightarrow 1, k_2 \rightarrow 1$  and also in the unequal parameter cases that appear in  $E_8$ . Then we compare the results with the formulas for formal degrees in [Reeder 2000; 1994]. To complete the determination of the Langlands parameter, i.e., the representation of the component group, we computed, when needed, the  $W_0$ -structure of the specialization of the family of discrete series and compared it with the  $K$ -structure of the representations in Reeder’s tables. The relevant unequal parameter cases that appear in  $E_8$  are: the affine Hecke algebra of type  $F_4$  that controls the subcategory of unipotent representations where the parahoric subgroup is  $D_4$  in  $E_8$ , and the affine Hecke algebra of type  $G_2$  for the subcategories of unipotent representations where the parahoric subgroup is  $E_6$  in  $E_8$ . For this comparison, we need to multiply the specialization of the formal degree in the unequal parameters Hecke algebra by the factor  $\rho(1)/P_J(v^2)$ , where  $\rho$  is the appropriate cuspidal unipotent representation (whose dimension is given in the tables of [Carter 1985]) and  $P_J(v^2)$  is the Poincaré polynomial of the finite Hecke algebra corresponding to the parahoric  $J$ .

In addition, the Iwahori–Hecke algebra for the quasisplit exceptional  $p$ -adic group  ${}^3E_6$  (respectively,  ${}^2E_7$ ) is isomorphic to a direct sum of three (respectively, two) copies of an affine Hecke algebra with unequal parameters of type  $G_2$  (respectively,  $F_4$ ). To compare the formal degrees in these cases against the results of [Reeder 1994], we need to multiply the specialization of formal degrees for the affine Hecke

$b$	$s$	$W_0c$	$d_b$	$F_4$	$\epsilon$
$b_1$	1	$[k_1, k_1, k_2, k_2]$	1	$[F_4, 1]$	1
$b_2$	1	$[k_1, k_1, k_2 - k_1, k_2]$	1	$[F_4(a_1), -]$	-1
$b_3$	1	$[k_1, k_1, k_2 - k_1, k_1]$	1	$[F_4(a_1), +]$	1
$b_4$	1	$[k_1, k_1, k_2 - 2k_1, k_2]$	1	$[F_4(a_3), (211)]$	1
$b_5$	1	$[k_1, k_1, k_2 - 2k_1, 2k_1]$	1	$[F_4(a_2), +]$	1
$b_6$	1	$[k_1, k_1, k_2 - 2k_1, k_1]$	1	$[F_4(a_3), (31)]$	-1
$b_7$	1	$[k_1, k_1, k_2 - 2k_1, -2k_2]$	1	$[F_4(a_2), -]$	-1
$b_8$	1	$[0, k_1, 0, k_2 - k_1]$	1/6	$[F_4(a_3), (4)]$	1
$b_9$	1	$[0, k_1, 0, k_2 - k_1]$	1/3	$[F_4(a_3), (22)]$	1
$b_{10}$	$B_4$	$[k_1/2, k_1, k_2, -3k_1 - 2k_2]$	1/2	$[B_4, +]$	1
$b_{11}$	$B_4$	$[2k_1, -k_1, k_2, -k_1 - 2k_2]$	1/2	non-ds	
$b_{12}$	$B_4$	$[0, k_1, -k_1 + k_2, -2k_2]$	1/2	$[B_4(531), \epsilon'']$	-1
$b_{13}$	$B_4$	$[k_1, k_1, -2k_1 + k_2, k_1 - 2k_2]$	1/2	$[B_4(531), 1]$	1
$b_{14}$	$B_4$	$[k_1, k_1, -3k_1 + k_2, 3k_1 - 2k_2]$	1/2	$[B_4(531), \epsilon']$	-1
$b_{15}$	$C_3A_1$	$[-2k_1 - 3k_2, k_1, k_2, k_2]$	1/2	$[C_3 \times A_1, +]$	1
$b_{16}$	$C_3A_1$	$[-2k_1, k_1, -k_2, 2k_2]$	1/2	$[C_3(42) \times A_1, ++]$	1
$b_{17}$	$C_3A_1$	$[-2k_1 + 3k_2, k_1, -k_2, -k_2]$	1/2	$[C_3(42) \times A_1, +-]$	-1
$b_{18}$	$2A_2$	$[k_1, -k_1 - 2k_2, k_2, k_2]$	1/3	$[2A_2, 1]$	1
$b_{19}$	$A_3A_1$	$[k_1, k_1, -3k_1/2 - k_2/2, k_2]$	1/4	$[A_1A_3, 1]$	1

**Table 2.** (Part 1)  $F_4$ .

algebra by the factor

$$\frac{(q^{1/2} - q^{-1/2})^{n+1}}{|\Omega| \prod_O (q^{|O|/2} - q^{-|O|/2})},$$

where  $n + 1$  is the number of nodes in the affine Dynkin diagram, and  $O$  ranges over the Galois orbits in the affine Dynkin diagram. See [Opdam 2016, (25)] for more details. For the nonsplit inner forms, this procedure allows us to find the L-packet to which the representation should belong, but it is not sufficient to enable us to attach the representation of the component group.<sup>3</sup>

**4C. Relation with Kazhdan–Lusztig parameters.** Let  $\mathcal{R}$  be a semisimple root datum. Let  $G$  be the connected complex semisimple group with root datum  $\mathcal{R}$ . Consider the generic affine Hecke algebra  $\mathcal{H}$  with root datum  $\mathcal{R}$  and equal parameters, i.e.,  $v(s) = v(s') = v$  for all  $s, s' \in S$ . If  $G$  is simply connected, the Kazhdan–Lusztig classification [Kazhdan and Lusztig 1987] applies to give a

<sup>3</sup>This is a subtle question, see [Opdam 2016] for results in this sense. We plan to return to this question in future work.

$b$	$D_4 \subset E_8$	$\epsilon$	${}^2E_7$	$\epsilon$
$b_1$	$[A_1 E_7, -]$	1	$E_7$	1
$b_2$	$[A_1 E_7(a_4), --]$	-1	$E_7(a_1)$	-1
$b_3$	$[A_1 E_7(a_2), -]$	1	$E_7(a_2)$	1
$b_4$	$[A_1 E_7(a_3), +-]$	1	$E_7(a_3)$	1
$b_5$	$[A_1 E_7(a_3), -+]$	-1	$E_7(a_3)$	-1
$b_6$	$[A_1 E_7(a_4), +-]$	1	$E_7(a_4)$	1
$b_7$	$[A_1 E_7(a_1), -]$	-1	$E_7(a_4)$	-1
$b_8$	$[A_1 E_7(a_5), -3]$	1	$E_7(a_5)$	1
$b_9$	$[A_1 E_7(a_5), -21]$	1	$E_7(a_5)$	1
$b_{10}$	$[D_8, -]$	1	$A_1 D_6$	1
$b_{11}$	$[D_8(5, 11), -]$	1	$A_1 D_6(3, 9)$	-1
$b_{12}$	$[D_8(1, 3, 5, 7), r]$	1	$A_1 D_6(5, 7)$	-1
$b_{13}$	$[D_8(7, 9), -]$	-1	non-ds	
$b_{14}$	$[D_8(3, 13), -]$	-1	non-ds	
$b_{15}$	$[A_3 D_5, -1]$	1	$A_1 D_6$	1
$b_{16}$	$[A_3 D_5(3, 7), -1]$	-1	$A_1 D_6(5, 7)$	1
$b_{17}$	non-ds		$A_1 D_6(3, 9)$	-1
$b_{18}$	$[A_1 A_2 A_5, -1]$	1	$A_2 A_5$	1
$b_{19}$	$[A_1 A_7, -]$	1	$A_1 A_3^2$	1

**Table 2.** (Part 2)  $F_4$ .

parametrization of the simple discrete series  $\mathcal{H}_{v_0}$ -modules,  $v_0 > 1$ , in terms of

$$DS_{KL}(\mathcal{R}) = G \setminus \{(x, \phi) \mid x \in G \text{ elliptic, } \phi \in \widehat{A(x)} \text{ such that } H^{\text{top}}(\mathcal{B}_x)^\phi \neq 0\}. \quad (30)$$

This result has been extended by Reeder [2002] (also see [Aubert et al. 2017]) to the case where  $\mathcal{R}$  has arbitrary isogeny type. Recall that we say that  $x$  is elliptic in  $G$  if the conjugacy class of  $x$  does not meet any proper Levi subgroup of  $G$ . Here we denote by  $A(x) = Z_G(x)/Z_G(x)^0 Z(G)$  the component group of the centralizer of  $x$  in  $G$  (mod the center of  $G$ ), by  $\mathcal{B}_x$  the Springer fiber of  $x$  in  $G$ , and by  $H^{\text{top}}(\mathcal{B}_x)^\phi$  the  $\phi$ -isotypic component of  $A(x)$  in the top cohomology of  $\mathcal{B}_x$ .

As is well known, by the construction of Springer extended by Lusztig and Kato in the simply connected case (see, for example, [Reeder 2000, Section 8]), and further extended by Reeder [2002, Section 3] to the arbitrary semisimple case, the full cohomology groups  $H^\bullet(\mathcal{B}_x)^\phi$  carry an action of  $W$ . We define

$$\mathcal{B}_{DS}^{KL} = \{h_{x,\phi} := H^\bullet(\mathcal{B}_x)^\phi \otimes \varepsilon \mid (x, \phi) \in DS_{KL}(\mathcal{R})\}, \quad (31)$$

where  $\varepsilon$  is the sign character of  $W$ .

Let  $x$  be an elliptic parameter as above and write  $x = su$  for the Jordan decomposition, with  $s \in T_u$ . Let  $\psi : \mathrm{SL}(2, \mathbb{C}) \rightarrow G$  be the Lie homomorphism such that  $\phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = u$ . Set

$$\tau = s\phi\left(\begin{pmatrix} v^{-1} & 0 \\ 0 & v \end{pmatrix}\right) \in T_u T_v \quad \text{and} \quad q = v^2.$$

The following Proposition follows easily from the work of Kazhdan and Lusztig [Kazhdan and Lusztig 1987; Lusztig 1995; 1989b] and the definitions.

**Proposition 4.8.** *Let  $\pi_{v, h_x, \phi}$  be the Kazhdan–Lusztig representation of  $\mathcal{H}$  associated to the elliptic Kazhdan–Lusztig parameter  $(x, \phi)$ . Then*

$$\pi_{v, h_x, \phi} = \mathrm{Ind}_D(h_x, \phi, v). \tag{32}$$

*In particular,  $\pi_{v, h_x, \phi}$  is a discrete series character with central character  $W_0\tau$ .*

*Proof.* By [Kazhdan and Lusztig 1987], the representation  $\pi_{v, h_x, \phi}$  is an irreducible discrete series with central character  $W_0\tau$ . We know by [Reeder 2001, Theorem 5.11.1] that  $\lim_{v \rightarrow 1} \pi_{v, h_x, \phi} = h_x, \phi$ , hence (32) follows directly from Definition 2.7.  $\square$

If  $\mathcal{R}$  is of simply connected type then the formal degree is given by the formula [Opdam 2016; Reeder 2000]:

$$\mathrm{fdeg}(\pi_{v, h_x, \phi}) = \frac{\phi(1)}{|A(x)||Z(G)|} m_v^{\mathrm{KL}}(\tau), \tag{33}$$

where

$$m_v^{\mathrm{KL}}(\tau) = q^{|R|/2} \frac{\prod'_{\alpha \in R} (\alpha(\tau) - 1)}{\prod'_{\alpha \in R} (q\alpha(\tau) - 1)}.$$

**Proposition 4.9.** *Equation (33) holds for all semisimple root data  $\mathcal{R}$ .*

*Proof.* Let  $G'$  be an arbitrary connected complex semisimple group with root datum  $\mathcal{R}'$ , and let  $1 \rightarrow C \rightarrow G \rightarrow G' \rightarrow 1$  be the universal covering of  $G'$ . Consider  $x' = s'u' \in G'$  elliptic, and let  $x = su \in G$  be a lifting of  $x'$ . Let  $A(x), A(x') \subset G_{\mathrm{ad}}$  be the centralizers of the unramified Langlands parameters associated to  $x$  and  $x'$  as in the text above Proposition 4.9 (these are finite subgroups, since  $x$  and  $x'$  are elliptic). We define a homomorphism  $A(x') \rightarrow C$  by  $a \rightarrow asa^{-1}s^{-1} \in C$ , whose image we denote by  $C_x$ . Similar to [Reeder 2002, Section 3.3], this give rise to an exact sequence:

$$1 \rightarrow A(x) \rightarrow A(x') \rightarrow C_x \rightarrow 1$$

(more precisely,  $A(x)$  and  $A(x')$  are the images in  $G_{\mathrm{ad}}$  of Reeder’s  $A_{\tau, u}$  and  $A_{\tau, u}^+$  respectively). Let  $(x, \phi) \in \mathrm{DS}_{\mathrm{KL}}(\mathcal{R})$ . Let  $C_{x, \phi} \subset C_x$  be the isotropy group of  $\phi$  with respect to the natural action of  $C_x$  on the set of equivalence classes of irreducible characters of  $A(x)$ . Let  $\mu$  denote the complex 2-cocycle of  $C_{x, \phi}$  associated to  $\phi$ ,

and  $E_{x,\phi} = \mathbb{C}[C_{x,\phi}, \mu]$  the corresponding twisted group algebra. By Mackey theory (see [Reeder 2002, Section 3.3]) we have

$$E_{x,\phi} \simeq \text{End}_{\mathbb{C}[A(x')]}(\text{Ind}_{A(x)}^{A(x')} \phi),$$

and thus

$$\text{Ind}_{A(x)}^{A(x')} \phi = \bigoplus_{\psi \in \text{Irr}(E_{x,\phi})} \psi \otimes \rho_\phi^\psi, \tag{34}$$

with  $\rho_\phi^\psi \in \text{Irr}(A(x'))$  as  $E_{x,\phi} \otimes \mathbb{C}[A(x')]$ -module. Moreover, all irreducible characters  $\rho$  of  $A(x')$  appear as some  $\rho \simeq \rho_\phi^\psi$ , and  $\rho_\phi^\psi \simeq \rho_{\phi'}^{\psi'}$  if and only if  $\phi'$  is a twist of  $\phi$  by an element of  $C_x$ , and  $\psi'$  and  $\psi$  correspond accordingly via this twist. By counting the multiplicity of  $\rho_\phi^\psi$  in the regular representation of  $A(x')$  using (34) we see:

$$\dim(\rho_\phi^\psi) = \frac{|C_x|}{|C_{x,\phi}|} \dim(\phi) \dim(\psi). \tag{35}$$

On the other hand we consider the affine Hecke algebras  $\mathcal{H}_v$  and  $\mathcal{H}'_v$ . Reeder [Reeder 2002, Section 1.5, Lemma 3.5.2] showed  $\mathcal{H}'_v = (\mathcal{H}_v)^C$ , and

$$\pi_{v,h_{x,\phi}}|_{\mathcal{H}'_v} = \bigoplus_{\psi \in \text{Irr}(E_{x,\phi})} \psi \otimes \pi_{v,h_{x',\rho_\phi^\psi}}. \tag{36}$$

Now  $\pi_{v,h_{x',\rho_\phi^\psi}}$  will arise as a summand of  $\pi_{v,h_{\tilde{x},\tilde{\phi}}}|_{\mathcal{H}'_v}$  if and only if  $(\tilde{x}, \tilde{\phi})$  is a twist of  $(x, \phi)$  by an element of  $C$ . The Plancherel decomposition of the trace  $\tau'$  of the normalized Hecke algebra  $\mathcal{H}'_v$  is obtained by restricting the Plancherel decomposition of the trace  $\tau$  of  $\mathcal{H}_v$ , since (see [Opdam 2016, Paragraph 2.4.1]) we have  $\tau' = \tau|_{\mathcal{H}'_v}$ . The above shows that in this restriction, the character of  $\pi_{v,h_{x',\rho_\phi^\psi}}$  will appear with formal degree equal to the formal degree of  $\pi_{v,h_{x,\phi}}$  with respect to  $\mathcal{H}_v$ , multiplied by the multiplicity

$$\frac{|C|}{|C_{x,\phi}|} \dim(\psi). \tag{37}$$

Considering that clearly  $m_v^{\text{KL}}(\tau) = m_v^{\text{KL}}(\tau')$ , and using (33), (34), (35), and (37), we see that

$$\text{fdeg}(\pi_{v,h_{x',\rho_\phi^\psi}}) = \frac{\rho_\phi^\psi(1)}{|A(x')||Z(G')|} m_v^{\text{KL}}(\tau'),$$

as was to be proved. □

By [Reeder 2000, Proposition 7.2],  $m_v^{\text{KL}}(\tau) = q^{\dim \mathcal{B}_u} (q-1)^\ell |M_0^s| R(q)$ , where  $\ell = \dim T$ ,  $M_0^s$  is as in [Reeder 2000, Lemma 7.1], and  $R(q)$  is a rational function in  $q$  that has the property that  $R(0) = 1$  and  $R(1) \neq 0$ . On the other hand, we know by [Opdam 2004, Proposition 3.27(v)] that  $m_v^{\text{KL}}(\tau) q^{-\dim \mathcal{B}_u}$  must equal a scalar times a rational function in cyclotomic polynomials in  $q$ . This means that  $R(q)$  is a scalar times a rational function where both the numerator and the denominator are products of cyclotomic polynomials  $\Phi_n(q)$  with  $n \geq 2$ . Since  $R(0) = 1$ , the scalar

must be in fact 1. But then  $R(q) > 0$  whenever  $q$  is specialized to any real number greater than  $-1$ , which implies that  $m_v^{\text{KL}}(\tau) > 0$  whenever  $q > 1$ .

Notice that the same conclusion follows from our sign formula (19). In the equal parameter case, we have  $k_s \equiv 1$ , and therefore the number of contributions of  $(-1)$  in (19) equals the number of roots  $\alpha(\bar{c}) < 0$  plus the number of roots  $\alpha(\bar{c}) < 1$ . Because of integrality of the central character  $\bar{c}$  and the fact that the roots of  $\bar{c}$  that vanish come in pairs, it follows, that this is an even number, thus the sign of  $\bar{m}_{W_0\bar{c}}$  is positive.

**Proposition 4.10.** *Suppose  $\mathcal{R}$  is a semisimple root datum and let  $\mathbf{v}_0$  denote the specialization in the equal parameters case.*

- (a)  $\mathcal{B}_{\mathbf{v}_0-m} = \{b_{x,\phi} := \epsilon(x, \phi)h_{x,\phi} \mid h_{x,\phi} \in \mathcal{B}_{\text{DS}}^{\text{KL}}\}$ , where  $\epsilon(x, \phi) = \epsilon(b_{x,\phi}; \mathbf{v}_0)$  is the sign of  $m_{W_0r_b}^{\text{KL}}(\mathbf{v}_0)$  (see Remark 4.3).
- (b) If  $\mathcal{R}$  is simply laced, then  $\mathcal{B}_{\text{gm}} = \mathcal{B}_{\text{DS}}^{\text{KL}}$  and

$$d_{b_{x,\phi}} = \frac{\phi(1)}{|A(x)||Z(G)|} \quad \text{for all } b_{x,\phi} \in \mathcal{B}_{\text{gm}}.$$

- (c) Suppose  $\mathcal{R}$  is a root datum of type  $G_2$  or  $F_4$ . Then

$$\mathcal{B}_{\text{gm}}(G_2) = \{\epsilon(x, \phi)h_{x,\phi} \mid h_{x,\phi} \in \mathcal{B}_{\text{DS}}^{\text{KL}}\}$$

and

$$\mathcal{B}_{\text{gm}}(F_4) = \{\epsilon(x, \phi)h_{x,\phi} \mid h_{x,\phi} \in \mathcal{B}_{\text{DS}}^{\text{KL}}\} \cup \{b_{11} = H^\bullet(\mathcal{B}_{x_{B_4}})^{\text{triv}} \otimes \varepsilon\},$$

with  $x_{B_4} = s_{B_4}u_{(711)}$ , where  $Z_G(s_{B_4})$  is of type  $B_4$  and  $u_{(711)}$  is a representative of the subregular unipotent class in  $B_4$ . The explicit signs  $\epsilon(x, \phi)$  are given in the sixth column in Tables 1 and 2 (Part 1), while the constants  $d_{b_{x,\phi}}$  are listed in the fourth column of the tables.

*Proof.* As in Remark 4.3, for every  $b \in \mathcal{B}_{\mathbf{v}_0-m}$ ,  $\text{ds}(b; \mathbf{v}_0) = \epsilon(b; \mathbf{v}_0) \text{Ind}(b; \mathbf{v}_0)$  and we know that the sign  $\epsilon(b, \mathbf{v}_0)$  is given by the sign of the generic  $m$ -function. Applying the restriction map, we get that  $\text{Res}(\text{ds}(b; \mathbf{v}_0)) = \epsilon(b; \mathbf{v}_0) \text{Ind}(b; \mathbf{v}_0)$ . On the other hand, as explained above  $\text{Res}(\text{ds}(b; \mathbf{v}_0))$  equals an  $h_{x,\phi}$  for some  $(x, \phi) \in \text{DS}_{\text{KL}}(\mathcal{R})$ . This is the claim in part (a).

Part (b) follows immediately from (a) by Proposition 4.9 and the discussion preceding the present proposition: in the simply laced case, the generic  $m$ -function equals  $m_v^{\text{KL}}(\tau)$ , which is positive.

Part (c) is contained in Section 4B3, where Tables 1 and 2 were computed.  $\square$

**Remark 4.11.** If  $\mathcal{R}$  is a simply connected root datum of type  $B$  or  $C$ , then one can relate the elements of  $\mathcal{B}_{\text{DS}}^{\text{KL}}$  to the bipartitions in the basis  $\mathcal{B}_{\text{gm}}$  from Proposition 4.4 using the combinatorial algorithms of Slooten [2008], see also [Opdam 2016]. If



$(x, \phi)$  is a discrete Kazhdan–Lusztig parameter as above, let  $(\lambda(x, \phi), \mu(x, \phi))$  be the bipartition associated by the algorithms in [loc. cit.]. Then

$$\mathcal{B}_{\text{DS}}^{\text{KL}} = \{ \epsilon(\lambda(x, \phi), \mu(x, \phi)) \cdot b_{(\lambda(x, \phi), \mu(x, \phi))} \mid (x, \phi) \in \text{DS}_{\text{KL}}(\mathcal{R}) \},$$

where  $\epsilon(\lambda(x, \phi), \mu(x, \phi))$  equals the sign of  $\lim_{k_s(\alpha) \rightarrow 1} \bar{m}_{W_0 \bar{c}}$  from (19).

**4D. Pin cover of the Weyl group.** Suppose  $\mathcal{W}$  is a finite Weyl group with its reflection representation  $\mathcal{E}$ . Fix a positive definite  $\mathcal{W}$ -invariant symmetric bilinear form on  $\mathcal{E}$ . Define the Clifford algebra  $C(\mathcal{E})$  and the pin cover  $p : \tilde{\mathcal{W}} \rightarrow \mathcal{W}$  as in [Ciubotaru et al. 2014, Section 3.1]. Let  $\det$  be the determinant character of  $\tilde{\mathcal{W}}$  acting on  $\mathcal{E}$ . As in [Ciubotaru et al. 2014, Section 4.1], define  $\tilde{\mathcal{W}}'$  to be equal to  $\tilde{\mathcal{W}}$ , when  $\dim \mathcal{E}$  is odd, and to equal  $\ker \det$  (an index two subgroup), when  $\dim \mathcal{E}$  is even.

If  $\dim \mathcal{E}$  is odd, then  $C(\mathcal{E})$  has two nonisomorphic simple complex modules; we denote them  $S^+$  and  $S^-$ . When  $\dim \mathcal{E}$  is even,  $C(\mathcal{E})$  has a unique simple complex module whose restriction to the even part  $C_0(\mathcal{E})$  splits into a direct sum of two nonisomorphic modules, denoted again  $S^+$  and  $S^-$ . We fix the choice of  $S^+$  (and  $S^-$ ) in both cases once and for all.

An irreducible  $\tilde{\mathcal{W}}'$ -representation is said to be genuine if it does not factor through  $p(\tilde{\mathcal{W}}')$ . Two nonisomorphic irreducible  $\tilde{\mathcal{W}}'$ -representations are said to be associate if one is the det-dual of the other, when  $\dim \mathcal{E}$  is odd, and if they both occur in the restriction to  $\tilde{\mathcal{W}}'$  of an irreducible  $\tilde{\mathcal{W}}$ -module, when  $\dim \mathcal{E}$  is even. For example  $\{S^+, S^-\}$  is a pair of associate  $\tilde{\mathcal{W}}'$ -representations. Denote by  $\text{Irr}^2 \tilde{\mathcal{W}}'$  the set of associate genuine pairs.

We apply these constructions in the case when  $\mathcal{R}$  is a semisimple, simply connected root datum,  $\mathcal{W} = W_s$ , with  $s$  an isolated element of  $T_u$  and  $\mathcal{E} = T_s(T) \cong E$ . For the connections with elliptic theory, see [Ciubotaru et al. 2014, Section 4] for example. Given  $\{\tilde{\sigma}^+, \tilde{\sigma}^-\}$ , let  $\xi_s(\tilde{\sigma}^+) = \xi_s(\tilde{\sigma}^-)$  be a representative of the generic central character defined by [Ciubotaru et al. 2014, Theorem 3.2]. Let

$$\text{Irr}_{\text{gm}}^2 \tilde{W}'_s = \{ \{\tilde{\sigma}^+, \tilde{\sigma}^-\} \in \text{Irr}^2 \tilde{W}'_s \mid \xi_s(\tilde{\sigma}^+) \text{ is generically residual for } (R_{s,1}, k_s) \}$$

and

$$\text{Irr}_{\text{KL}}^2 \tilde{W}'_s = \{ \{\tilde{\sigma}^+, \tilde{\sigma}^-\} \in \text{Irr}_{\text{gm}}^2 \tilde{W}'_s \mid \xi_s(\tilde{\sigma}^+)(1) \text{ is residual for } (R_{s,1}, k_s = 1) \},$$

where  $\xi_s(\tilde{\sigma}^+)(1)$  denotes the specialization at the equal parameter case  $k_s = 1$ .

Write  $\mathcal{B}_{\text{gm}} = \bigsqcup_s \text{Ind}_s(\mathcal{B}_{\text{gm},s})$  for the canonically positive basis, where  $\mathcal{B}_{\text{gm},s}$  is a certain orthonormal subset of  $\bar{R}_{\mathbb{Z}}(W_s)$ . By [Ciubotaru et al. 2014, Theorem 4.2], for each  $b_s \in \mathcal{B}_{\text{gm},s}$  there exists a pair  $(\tilde{\sigma}_{b_s}^+, \tilde{\sigma}_{b_s}^-)$  of associate genuine irreducible  $\tilde{W}'_s$ -representations such that

$$b_s = \Delta(\tilde{\sigma}_{b_s}^+, \tilde{\sigma}_{b_s}^-) := \frac{\tilde{\sigma}_{b_s}^+ - \tilde{\sigma}_{b_s}^-}{S^+ - S^-} \quad \text{in } \bar{R}_{\mathbb{Z}}(W_s).$$

Moreover, as explained in Section 2B4,  $\xi_s(\tilde{\sigma}_{b_s}^+) = r_{b_s}$ ; in particular,

$$\{\tilde{\sigma}_{b_s}^+, \tilde{\sigma}_{b_s}^-\} \in \text{Irr}_{\text{gm}}^2 \widetilde{W}_s'.$$

By the same results in [loc. cit.], the assignment  $b_s \mapsto \{\tilde{\sigma}_{b_s}^+, \tilde{\sigma}_{b_s}^-\}$  is injective. Thus if we define  $\text{Irr}_{\text{gm}}^2 \widetilde{W}_s' = \{(\tilde{\sigma}_{b_s}^+, \tilde{\sigma}_{b_s}^-) \mid b_s \in \mathcal{B}_{\text{gm},s}\}$ , then the map  $\Delta$  defines a bijection

$$\Delta : \text{Irr}_{\text{gm}}^2 \widetilde{W}_s' \rightarrow \mathcal{B}_{\text{gm},s}.$$

In fact,  $\text{Irr}_{\text{gm}}^2 \widetilde{W}_s'$  is just  $\text{Irr}_{\text{gm}}^2 \widetilde{W}_s'$  with a particular choice of  $\tilde{\sigma}^+$  versus  $\tilde{\sigma}^-$ .

We can explain this choice independently in the case of  $\text{Irr}_{\text{KL}}^2 \widetilde{W}_s'$ . Since  $\xi_s(\tilde{\sigma}^+)(1)$  is a residual point, we know by [Opdam 2004] that  $\xi_s(\tilde{\sigma}^+)(1)$  satisfies the same combinatorial condition as the Bala–Carter condition for half of the middle element of a distinguished Lie triple in  $Z_G(s)$ . Thus, the pair  $\{\tilde{\sigma}^+, \tilde{\sigma}^-\} \in \text{Irr}_{\text{KL}}^2 \widetilde{W}_s'$  determines the (conjugacy class of an) element  $u$ , and therefore the elliptic element  $x = su$  in the Kazhdan–Lusztig parametrization. To formalize this, denote by  $u(\{\tilde{\sigma}^+, \tilde{\sigma}^-\})$  a representative of the unipotent class attached in this way.

Let  $\Omega_{\widetilde{W}_s}$  be the Casimir element of  $\widetilde{W}_s$ , e.g., [Ciubotaru et al. 2014, (3.2.5)]. The scalar  $\tilde{\sigma}^+(\Omega_{\widetilde{W}_s})$  by which  $\Omega_{\widetilde{W}_s}$  acts in  $\tilde{\sigma}^+$  (which does not depend on the choice of  $\tilde{\sigma}^+$  versus  $\tilde{\sigma}^-$ ) equals the squared norm of  $\xi_s(\tilde{\sigma}^+)$ .

It remains to discuss how to associate a representation  $\phi$  of the component group  $A(x)$ . This will also lead to the desired canonical choice for  $\tilde{\sigma}^\pm$ . Let  $x = su$  be an elliptic element of  $G$ . Recall that this means that  $u$  is distinguished in  $Z_G(s)$ . Let  $X_s(u, \phi)$  and  $\sigma_s(u, \phi)$  denote the standard and irreducible  $W_s$ -representations afforded via Springer theory by  $H^\bullet(\mathcal{B}_u^s)^\phi \otimes \varepsilon$  and  $H^{2d_u^s}(\mathcal{B}_u^s)^\phi \otimes \varepsilon$  respectively, where  $\mathcal{B}_u^s$  is the Springer fiber of  $u$  in  $Z_G(s)$  and  $d_u^s = \dim \mathcal{B}_u^s$ . As above, there exist associate irreducible representations  $\tilde{\sigma}_s(u, \phi)^\pm$  such that

$$X_s(u, \phi) \otimes (S^+ - S^-) = \tilde{\sigma}_s(u, \phi)^+ - \tilde{\sigma}_s(u, \phi)^-.$$

As noticed in [Ciubotaru and Trapa 2013; Ciubotaru and He 2015], if we write

$$X_s(u, \phi) = \sigma_s(u, \phi) + \sum_{u' > u} a_{(u', \phi')} X_s(u', \phi')$$

(by the results of Borho and Macpherson), then one can deduce that  $\tilde{\sigma}_s(u, \phi)^\pm$  can only appear in  $\sigma_s(u, \phi) \otimes S^\pm$  and in no other  $X_s(u', \phi') \otimes S^\pm$ . Moreover, by the Dirac theory, the scalar  $\tilde{\sigma}_s(u, \phi)^\pm(\Omega_{\widetilde{W}_s})$  equals the squared norm of half a middle element for a Lie triple of  $u$ , and this is the *minimal* such scalar that may appear for an irreducible constituent of  $\sigma_s(u, \phi) \otimes S^\pm$ . By [Ciubotaru 2012] (case-by-case) or [Ciubotaru and He 2015] (uniformly),  $\tilde{\sigma}_s(u, \phi)^+$  appears with multiplicity one in  $\sigma_s(u, \phi) \otimes S^+$  (and similarly for the  $-$  case).

The conclusion is that  $\tilde{\sigma}_s(u, \phi)^+$  is characterized by the property that it is the unique irreducible constituent of  $\sigma_s(u, \phi) \otimes S^+$  for which the scalar by which  $\Omega_{\widetilde{W}_s}$

acts in it is minimal among all the constituents of the tensor product. Moreover,  $\tilde{\sigma}_s(u, \phi)^+$  does not appear as such a “minimal representation” in any other tensor product of this form. This gives a one-to-one correspondence between the sets  $\{\sigma_s(u, \phi)\}$  and  $\{\tilde{\sigma}_s(u, \phi)^+\}$ . Given a  $\tilde{\sigma}^+$  in the latter set, denote by  $\phi(\tilde{\sigma}^+)$  the local system  $\phi$  for the Springer representation in the former set. In particular, we have a preferred representation  $\tilde{\sigma}^+$  in any pair belonging to  $\text{Irr}_{\text{KL}}^2(\widetilde{W}_s')$ . Thus we get a set  $\underline{\text{Irr}}_{\text{KL}}^2(\widetilde{W}_s')$  of ordered pairs. In summary we have:

**Proposition 4.12.** *Suppose  $\mathcal{R}$  is a simply connected semisimple root datum. Then, with the canonical choice of representations  $\tilde{\sigma}^+$  as above,*

$$\bigsqcup_s \text{Ind}_s \circ \Delta(\underline{\text{Irr}}_{\text{KL}}^2(\widetilde{W}_s')) = \mathcal{B}_{\text{DS}}^{\text{KL}}.$$

*Given a pair  $(\tilde{\sigma}^+, \tilde{\sigma}^-) \in \underline{\text{Irr}}_{\text{KL}}^2(\widetilde{W}_s')$ , the Kazhdan–Lusztig parameter  $(x = su, \phi)$  is given by  $u = u(\{\tilde{\sigma}^+, \tilde{\sigma}^-\})$  and  $\phi = \phi(\tilde{\sigma}^+)$  with the notation as in the previous paragraph.*

**Remark 4.13.** If, in addition  $\mathcal{R}$  is simply laced, this also implies that

$$\mathcal{B}_{\text{gm}} = \bigsqcup_s \text{Ind}_s \circ \Delta(\underline{\text{Irr}}_{\text{KL}}^2(\widetilde{W}_s')).$$

When  $\mathcal{R}$  is not simply laced, the relation between  $\mathcal{B}_{v_0-m}$  and  $\mathcal{B}_{\text{DS}}^{\text{KL}}$  (and therefore  $\bigsqcup_s \text{Ind}_s \circ \Delta(\underline{\text{Irr}}_{\text{KL}}^2(\widetilde{W}_s'))$ ) was given in Section 4C.

**Remark 4.14.** The case of root data of type  $E$  is particularly interesting. Suppose  $\mathcal{R}$  is a simply connected, type  $E$ , root datum. Let  $\{\tilde{\sigma}^+, \tilde{\sigma}^-\} \in \underline{\text{Irr}}^2(\widetilde{W}_s')$  be given, with the notation as above. We know a priori that  $\xi_s(\tilde{\sigma}^+)$  equals  $k/2$  times the middle element of a Lie triple of a quasidistinguished unipotent element  $u$  of  $Z_G(s)$  (see [Ciubotaru et al. 2014, Section 5]). By [Opdam 2004, Appendix] (also see [Opdam and Solleveld 2010]), this is a residual point for  $(R_s, k)$  if and only if  $u$  is distinguished in  $Z_G(s)$ . The key, empirical, observation is that for the root systems  $R_s$  occurring in a type  $E$  root system, the squared norms of the middle element of a quasidistinguished Lie triple uniquely identifies the unipotent class. In particular, the scalar by which  $\Omega_{\widetilde{W}_s}$  acts in  $\tilde{\sigma}^+$  already identifies uniquely the unipotent class of  $u(\{\tilde{\sigma}^+, \tilde{\sigma}^-\})$ , and in particular if  $\{\tilde{\sigma}^+, \tilde{\sigma}^-\} \in \underline{\text{Irr}}_{\text{gm}}^2(\widetilde{W}_s')$ . In this way we can effectively determine the base  $\mathcal{B}_{\text{DS}}^{\text{KL}} = \mathcal{B}_{\text{gm}}$  of Proposition 4.12, and for each  $b \in \mathcal{B}_{\text{gm}}$ , the central character  $W_0\tau$  (with  $\tau = s \exp(\xi_s)$ ) of  $\text{Ind}_D(b; v)$ . This recovers the classification of the discrete series characters of  $\mathcal{H}$  starting from the Springer theory of  $W_s$ .

**4E. The general semisimple case.** Let us consider the determination of the basis  $\mathcal{B}_{\text{gm}}$  for an arbitrary semisimple root datum  $\mathcal{R}$ . Recall that if  $\mathcal{H} \subset \mathcal{H}'$  are isogenous extended affine Hecke algebras such that  $R_{\text{nr}} = R'_{\text{nr}}$  (see (11)), so that in particular

we have a canonical identification  $\mathcal{Q} = \mathcal{Q}'$ , then the functors induction  $\text{Ind}_{\mathcal{H}}^{\mathcal{H}'}$  and restriction  $\text{Res}_{\mathcal{H}}^{\mathcal{H}'}$  respect the discrete series [Delorme and Opdam 2011, Lemma 6.3] in the sense that an irreducible discrete series is mapped to a finite direct sum of irreducible discrete series. From this we easily obtain:

**Proposition 4.15.** *Let  $b \in \mathcal{B}_{\text{gm}}$  (respectively  $b' \in \mathcal{B}'_{\text{gm}}$ ), and let  $\mathbf{v} \in \mathcal{Q}_b^{\text{reg}}$  (respectively  $\mathbf{v} \in \mathcal{Q}_{b'}^{\text{reg}}$ ). Let  $\delta'$  (respectively  $\delta$ ) be an irreducible summand of  $\text{Ind}_{\mathcal{H}}^{\mathcal{H}'}(\text{Ind}_D(b; \mathbf{v}))$  (respectively  $\text{Res}_{\mathcal{H}}^{\mathcal{H}'}(\text{Ind}_D(b'; \mathbf{v}))$ ). Then  $\lim_{\mathbf{v} \rightarrow 1} \delta'$  is an element of  $\mathcal{B}'_{\text{gm}}$  and all elements of  $\mathcal{B}'_{\text{gm}}$  are of this form, and similarly,  $\lim_{\mathbf{v} \rightarrow 1} \delta$  is an element of  $\mathcal{B}_{\text{gm}}$  and all elements of  $\mathcal{B}_{\text{gm}}$  are of this form.*

The following result reduces the explicit computation of the restrictions and inductions of discrete series to the case of restriction and induction of representations of the extended affine Weyl groups  $W \subset W'$ .

**Proposition 4.16.** *We have  $\text{Ind}_{\mathcal{H}}^{\mathcal{H}'}(\text{Ind}_D(b; \mathbf{v})) = \text{Ind}_D(\text{Ind}_W^{W'}(b); \mathbf{v})$  and similarly  $\text{Res}_{\mathcal{H}}^{\mathcal{H}'}(\text{Ind}_D(b; \mathbf{v})) = \text{Ind}_D(\text{Res}_W^{W'}(b); \mathbf{v})$ .*

*Proof.* This follows easily from Definition 2.7 and the proof of [Delorme and Opdam 2011, Lemma 6.3], by taking the limit  $\lim_{\mathbf{v} \rightarrow 1}$  on both sides.  $\square$

**Remark 4.17.** From Propositions 4.15 and 4.16, it follows that for every  $b \in \mathcal{B}_{\text{gm}}$ , the induced character  $\text{Ind}_W^{W'}(b)$  is a positive integral linear combination of elements of  $\mathcal{B}'_{\text{gm}}$ . Therefore, in practice, in order to obtain the elements of  $\mathcal{B}'_{\text{gm}}$  that lie in the support of  $\text{Ind}_W^{W'}(b)$ , it is sufficient to decompose  $\text{Ind}_W^{W'}(b)$  as a positive integral linear combination of unit vectors in  $\bar{R}(W')$ . The resulting unit vectors are all in  $\mathcal{B}'_{\text{gm}}$  automatically by the canonical positivity property of  $\mathcal{B}'_{\text{gm}}$ .

The same analysis holds for the restrictions  $\text{Res}_W^{W'}(b)$ .

By the results of Sections 4B, 4C, and 4D we have determined the basis  $\mathcal{B}_{\text{gm}}$  for at least one representative of each isogeny class of irreducible root data. Using Remark 4.17 it is now easy to determine  $\mathcal{B}_{\text{gm}}$  for an arbitrary irreducible root datum.

Now suppose that  $\mathcal{R}$  is an arbitrary semisimple based root datum. The finite abelian group  $\Omega$  acts faithfully on the affine Coxeter diagram of  $\mathcal{R}$  by special affine diagram automorphisms, and it is easy to see that the special affine diagram automorphisms respect the irreducible components of the diagram of  $\mathcal{R}$ . Let  $D_i$  with  $i = 1, 2, \dots, e$  denote the irreducible components of the affine Coxeter diagram of  $\mathcal{R}$ , and let  $\Omega_i$  denote the image of  $\Omega$  in the group of special affine diagram automorphisms of  $D_i$ . Then  $\Omega \subset \Omega_1 \times \dots \times \Omega_e$ . Let  $\mathcal{R}' = \mathcal{R}_1 \times \mathcal{R}_2 \times \dots \times \mathcal{R}_e$  be the root datum which is isogenous to  $\mathcal{R}$ , such that  $\Omega' = \Omega_1 \times \dots \times \Omega_e$ .

**Proposition 4.18.** *We have  $\mathcal{Q} = \mathcal{Q}' = \mathcal{Q}_1 \times \dots \times \mathcal{Q}_e$ . In particular, Propositions 4.15 and 4.16 apply to the isogeny  $\mathcal{H} \subset \mathcal{H}'$ .*

*Proof.* We need to verify that  $R_{nr} = R'_{nr}$ . But this reduces to the actions of the groups  $\Omega$  and  $\Omega'$  on the irreducible components. The result follows easily from the definition of  $R_{nr}$  (see, e.g., [Opdam and Solleveld 2010, Section 2]).  $\square$

In the notation of Proposition 4.18, we have  $\mathcal{B}' = \mathcal{B}'_1 \times \cdots \times \mathcal{B}'_e$ , which can be determined explicitly since the factors are of irreducible type. Hence we can determine  $\mathcal{B}$  using Propositions 4.15 and 4.16.

**4F. Rationality of characters of discrete series.** In this subsection, we discuss the character value ring for the families of generic discrete series considered before. In light of Corollary 3.3, we know that for every generic residual central character  $W_{0r}$  and for every  $\mathbf{v} \in \mathcal{Q}_{W_{0r}}^{\text{reg}}$ , the central functional given by the “stable combination” of generic characters (16) takes values in  $\mathcal{Q}_\Lambda$ , the quotient field of  $\Lambda$ , i.e.,

$$\sum_{\chi \in \text{DS}_{W_{0r}(\mathbf{v})}} c_{\chi,C} \chi_{\mathbf{v}}(h) \in \mathcal{Q}_\Lambda, \quad \text{for all } h \in \mathcal{H}_\Lambda. \tag{38}$$

**Proposition 4.19.** *Suppose the root datum  $\mathcal{R}$  of the generic affine Hecke algebra  $\mathcal{H}_\Lambda$  is simple and not simply laced. Let  $W_{0r}$  be a generic residual central character and let  $\mathbf{v} \in \mathcal{Q}_{W_{0r}}^{\text{reg}}$  be given. Then for every discrete series character  $\chi \in \text{DS}_{W_{0r}(\mathbf{v})}$ , we have  $\chi_{\mathbf{v}}(h) \in \mathcal{Q}_\Lambda$ , for all  $h \in \mathcal{H}_\Lambda$ .*

*Proof.* By the classification of generic discrete series families [Opdam 2004; Opdam and Solleveld 2010], we know that, when  $\mathcal{R}$  is not simply laced, the cardinality of  $\text{DS}_{W_{0r}(\mathbf{v})}$  is always 1, except if  $\mathcal{R} = F_4$  and  $W_{0r}$  is the central character  $[0, k_1, 0, k_2 - k_1]$  from Table 2. Therefore, with this exception, the claim immediately follows from (38).

Let’s consider now this exceptional residual central character in  $F_4$ . In the same coordinates as in Table 2, we have  $r = v^{2[0, k_1, 0, k_2 - k_1]}$  and

$$\mathcal{Q}_{W_{0r}}^{\text{reg}} = \{k_1, k_2 \in \mathbb{R} \mid k_1 k_2 \neq 0\}.$$

There are always two families of discrete series  $\pi_{\mathbf{v},\text{I}} = \epsilon(b_8; \mathbf{v}) \text{Ind}_D(b_8; \mathbf{v})$  and  $\pi_{\mathbf{v},\text{II}} = \epsilon(b_9; \mathbf{v}) \text{Ind}_D(b_9; \mathbf{v})$  for  $\mathbf{v} \in \mathcal{Q}_{W_{0r}}^{\text{reg}}$ , where  $b_8, b_9$  are as in Table 2. From (38), we know that  $\pi_{\mathbf{v},\text{I}} + \pi_{\mathbf{v},\text{II}}$  takes values in  $\mathcal{Q}_\Lambda$ , so it suffices to show that  $\pi_{\mathbf{v},\text{II}}$  has the same property. Assume first that  $k_1 > 0$  and  $k_2 > 0$ . Then  $\pi_{\mathbf{v},\text{II}}$  is a 10-dimensional module that has the property that it is  $\mathcal{A}$ -semisimple, i.e., each generalized weight space under the action of Bernstein’s abelian subalgebra  $\mathcal{A}$  is one-dimensional. This claim follows from [Reeder 2000, page 80], where the weight diagram of this module is given under the label  $[A_1 E_7(a_5), -21]$ . Let  $\text{Wt}(\pi_{\mathbf{v},\text{II}})$  denote the set of weights. As it is well known, we may choose a basis of the module consisting of weight vectors  $\dagger_\lambda, \lambda \in \text{Wt}(\pi_{\mathbf{v},\text{II}})$  such that for every  $s \in S_0$ , the action of  $N_s$  is given

by

$$N_s \cdot \dagger_\lambda = \begin{cases} \frac{v(s) - v(s)^{-1}}{1 - \theta_{-\alpha}(\lambda)} \dagger_\lambda, & \text{if } s(\lambda) \notin \text{Wt}(\pi_{v,\text{II}}), \\ \frac{v(s) - v(s)^{-1}}{1 - \theta_{-\alpha}(\lambda)} \dagger_\lambda + \left( v(s)^{-1} + \frac{v(s) - v(s)^{-1}}{1 - \theta_{-\alpha}(\lambda)} \right) \dagger_{s(\lambda)} & \text{if } s(\lambda) \in \text{Wt}(\pi_{v,\text{II}}), \end{cases}$$

where  $s = s_\alpha$ . This means that both types of generators,  $\theta_x$  and  $N_s$  act in this basis via matrices with entries in  $Q_\Lambda$ , and the claim follows.

To treat the other three quadrants in  $k_1, k_2$ , notice that the affine Hecke algebra of type  $F_4$  has three involutions of the form  $N_s \mapsto -N_s^{-1}$ , where  $s$  ranges over the long simple reflections, the short simple reflections, and all the simple reflections, respectively. (The last one is the Iwahori–Matsumoto involution.) Applying these involutions to the modules  $\pi_{v,\text{I}}$  and  $\pi_{v,\text{II}}$  from the  $k_1 > 0, k_2 > 0$  case, we obtain the discrete series in the other three cases.  $\square$

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Dan.Ciubotaru@maths.ox.ac.uk

*Mathematical Institute, University of Oxford,  
Andrew Wiles Building, Oxford, OX2 6GG, United Kingdom*

e.m.opdam@uva.nl

*Korteweg-de Vries Institute for Mathematics,  
Universiteit van Amsterdam, Science Park 105-107,  
1098 XG Amsterdam, Netherlands*



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# Algebra & Number Theory

Volume 11 No. 5 2017

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Hybrid sup-norm bounds for Maass newforms of powerful level ABHISHEK SAHA	1009
Collinear CM-points YURI BILU, FLORIAN LUCA and DAVID MASSER	1047
A uniform classification of discrete series representations of affine Hecke algebras DAN CIUBOTARU and ERIC OPDAM	1089
An explicit bound for the least prime ideal in the Chebotarev density theorem JESSE THORNER and ASIF ZAMAN	1135
Modular curves of prime-power level with infinitely many rational points ANDREW V. SUTHERLAND and DAVID ZYWINA	1199
Some sums over irreducible polynomials DAVID E. SPEYER	1231



1937-0652(2017)11:5;1-G