The equations defining blowup algebras of height
three Gorenstein ideals

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We find the defining equations of Rees rings of linearly presented height three Gorenstein ideals. To prove our main theorem we use local cohomology techniques to bound the maximum generator degree of the torsion submodule of symmetric powers in order to conclude that the defining equations of the Rees algebra and of the special fiber ring generate the same ideal in the symmetric algebra. We show that the ideal defining the special fiber ring is the unmixed part of the ideal generated by the maximal minors of a matrix of linear forms which is annihilated by a vector of indeterminates, and otherwise has maximal possible height. An important step in the proof is the calculation of the degree of the variety parametrized by the forms generating the height three Gorenstein ideal.

1. Introduction

This paper deals with the algebraic study of rings that arise in the process of blowing up a variety along a subvariety. These rings are the Rees ring and the special fiber ring of an ideal. Rees rings are also the bihomogeneous coordinate rings of graphs of rational maps between projective spaces, whereas the special fiber rings are the homogeneous coordinate rings of the images of such maps. More precisely, let R be a standard graded polynomial ring over a field and I be an ideal of R minimally generated by forms $g_1, \ldots, g_n$ of the same degree. These forms define a rational map Ψ whose image is a variety X. The bihomogeneous coordinate ring of the graph of Ψ is the Rees algebra of the ideal I, which is defined as the graded subalgebra $\mathcal{R}(I) = R[It]$ of the polynomial ring $R[t]$. This algebra contains, as direct summands, the ideal I as well as the homogeneous coordinate ring of the variety X.

It is a fundamental problem to find the implicit defining equations of the Rees ring and thereby of the variety X. This problem has been studied extensively

by commutative algebraists, algebraic geometers, and, most recently, by applied mathematicians in geometric modeling. A complete solution requires that one begins with a knowledge of the structure of the ideal I. There is a large body of work dealing with perfect ideals of height two, which are known to be ideals of maximal minors of an almost square matrix; see for instance [Herzog et al. 1983; Morey 1996; Morey and Ulrich 1996; Cox et al. 2008; Hong et al. 2008; Busé 2009; Kustin et al. 2011; Cortadellas Benítez and D'Andrea 2013; 2014; Nguyen 2014; Madsen 2015; Boswell and Mukundan 2016; Kustin et al. 2017a]. Much less is known for the "next cases", determinantal ideals of arbitrary size [Bruns et al. 2013; 2015] and Gorenstein ideals of height three [Morey 1996; Johnson 1997], the class of ideals that satisfy the Buchsbaum–Eisenbud structure theorem [Buchsbaum and Eisenbud 1977]. In the present paper we treat the case where I is a height three Gorenstein ideal and the entries of a homogeneous presentation matrix of I are linear forms or, more generally, generate a complete intersection ideal; see Theorem 9.1 and Remark 9.2. Height three Gorenstein ideals have been studied extensively, and there is a vast literature dealing with the various aspects of the subject; see for instance [Watanabe 1973; Buchsbaum and Eisenbud 1977; Elias and Iarrobino 1987; Boffi and Sánchez 1992; Kustin and Ulrich 1992; Iarrobino 1994; Aberbach et al. 1995; De Negri and Valla 1995; Diesel 1996; Geramita and Migliore 1997; Migliore and Peterson 1997; Kleppe and Miró-Roig 1998; Iarrobino and Kanev 1999; Polini and Ulrich 1999; Conca and Valla 2000; Hartshorne 2004; Boij et al. 2014; Kimura and Terai 2015].

Though it may be impossible to determine the implicit equations of Rees rings for general classes of ideals, one can still hope to bound the degrees of these equations, which is in turn an important step towards finding them. In the present paper we address this broader issue as well.

Traditionally one views the Rees algebra as a natural epimorphic image of the symmetric algebra $\mathrm{Sym}(I)$ of I and one studies the kernel of this map,

$$0 \to \mathcal{A} \to \mathrm{Sym}(I) \to \mathcal{R}(I) \to 0.$$

The kernel $\mathcal{A}$ is the R-torsion submodule of $\mathrm{Sym}(I)$. The defining equations of the symmetric algebra can be easily read from the presentation matrix of the ideal I. Hence the simplest situation is when the symmetric algebra and the Rees algebra are isomorphic, in which case the ideal I is said to be of *linear type*.

When the ideal I is perfect of height two, or is Gorenstein of height three or, more generally, is in the linkage class of a complete intersection, then I is of linear type if and only if the minimal number of generators of the ideal $I_{\mathfrak{p}}$, $\mu(I_{\mathfrak{p}})$, is at most $\dim R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \in V(I)$; see [Huneke 1982; Herzog et al. 1983]. If the condition $\mu(I_{\mathfrak{p}}) \le \dim R_{\mathfrak{p}}$ is only required for every nonmaximal prime ideal $\mathfrak{p} \in V(I)$, then we say that I satisfies G_d, where d is the dimension of

the ring R. The condition G_d can be interpreted in terms of the height of Fitting ideals, $i < \mathrm{ht}\,\mathrm{Fitt}_i(I)$, whenever $i < d$. The property G_d is always satisfied if R/I is 0-dimensional or if I is a generic perfect ideal of height two or a generic Gorenstein ideal of height three. For ideals in the linkage class of a complete intersection the assumption G_d means that $\mathcal{A}$ is annihilated by a power of the maximal homogeneous ideal $\mathfrak{m}$ of R, in other words $\mathcal{A}$ is the zeroth local cohomology $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I))$. One often requires the condition G_d when studying Rees rings of ideals that are not necessarily of linear type.

To identify further implicit equations of the Rees ring, in addition to the equations defining the symmetric algebra, one uses the technique of *Jacobian dual*; see for instance [Vasconcelos 1991]. Let $x_1, \ldots, x_d$ be the variables of the polynomial ring R and $T_1, \ldots, T_n$ be new variables. Let φ be a minimal homogeneous presentation matrix of the forms $g_1, \ldots, g_n$ generating I. There exits a matrix B with d rows and entries in the polynomial ring $S = R[T_1, \ldots, T_n]$ such that the equality of row vectors

$$[T_1, \ldots, T_n] \cdot \varphi = [x_1, \ldots, x_d] \cdot B$$

holds. Choose B so that its entries are linear in $T_1, \ldots, T_n$. The matrix B is called a *Jacobian dual* of φ. If φ is a matrix of linear forms, then the entries of B can be taken from the ring $T = k[T_1, \ldots, T_n]$; this B is uniquely determined by φ and is called *the* Jacobian dual of φ. The choice of generators of I gives rise to homogeneous epimorphisms of graded R-algebras

$$S \twoheadrightarrow \mathrm{Sym}(I) \twoheadrightarrow \mathcal{R}(I).$$

We denote by $\mathcal{L}$ and $\mathcal{J}$ the ideals of S defining $\mathrm{Sym}(I)$ and $\mathcal{R}(I)$, respectively. The ideal $\mathcal{L}$ is generated by the entries of the row vector $[T_1, \ldots, T_n] \cdot \varphi = [x_1, \ldots, x_d] \cdot B$. Cramer's rule shows that

$$\mathcal{L} + I_d(B) \subset \mathcal{J},$$

and if equality holds, then $\mathcal{J}$ or $\mathcal{R}(I)$ is said to have the *expected form*. The expected form is the next best possibility for the defining ideal of the Rees ring if I is not of linear type. Linearly presented ideals whose Rees ring has the expected form are special cases of ideals of fiber type. An ideal I is of *fiber type* if the defining ideal $\mathcal{J}$ of $\mathcal{R}(I)$ can be reconstructed from the defining ideal $I(X)$ of the image variety X or, more precisely,

$$\mathcal{J} = \mathcal{L} + I(X)S.$$

It has been shown in [Morey and Ulrich 1996] that the Rees ring has the expected form if I is a linearly presented height two perfect ideal satisfying G_d. In this situation the Rees algebra is Cohen–Macaulay. There has been a great deal of work investigating the defining ideal of Rees rings when I is a height two perfect ideal

that either fails to satisfy G_d or is not linearly presented. In this case $\mathcal{R}(I)$ usually does not have the expected form and is not Cohen–Macaulay; see for instance [Cox et al. 2008; Hong et al. 2008; Busé 2009; Kustin et al. 2011; Cortadellas Benítez and D'Andrea 2013; 2014; Nguyen 2014; Boswell and Mukundan 2016; Madsen 2015; Kustin et al. 2017a].

Much less is known for height three Gorenstein ideals. In this case n is odd and the presentation matrix φ can be chosen to be alternating, according to the Buchsbaum–Eisenbud structure theorem [Buchsbaum and Eisenbud 1977]. The defining ideal $\mathcal{J}$ of the Rees ring has been determined, provided that I satisfies G_d, $n = d + 1$, and moreover φ has linear entries [Morey 1996] or, more generally, $I_1(\varphi)$ is generated by the entries of a single row of φ [Johnson 1997]. It turns out that $\mathcal{R}(I)$ does not have the expected form, but is Cohen–Macaulay under these hypotheses. On the other hand, this is the only case when the Rees ring is Cohen–Macaulay. In fact, the Rees ring of a height three Gorenstein ideal satisfying G_d is Cohen–Macaulay if and only if either $n \leq d$ or else $n = d + 1$ and $I_1(\varphi)$ is generated by the entries of a single generalized row of φ [Polini and Ulrich 1999]. Observe that if $n = d + 1$ then d is necessarily even. The "approximate resolutions" of the symmetric powers of I worked out in [Kustin and Ulrich 1992] show that if $n \geq d + 1$ and d is even, then $\mathrm{Sym}(I)$ has R-torsion in too low a degree for $\mathcal{J}$ to have the expected form; see also [Morey 1996]. As it turns out, the alternating structure of the presentation matrix φ is responsible for "unexpected" elements in $\mathcal{J}$. Let φ be an alternating matrix of linear forms presenting a height three Gorenstein ideal, and let B be its Jacobian dual. As in [Kustin and Ulrich 1992] one builds an alternating matrix from φ and B,

$$\mathfrak{B} = \begin{bmatrix} \varphi & -B^{\mathrm{t}} \\ B & 0 \end{bmatrix}.$$

The submaximal Pfaffians of this matrix are easily seen to belong to $\mathcal{J}$. We regard the last of these Pfaffians as a polynomial in $T[x_1, \ldots, x_d]$ and consider its content ideal $C(\varphi)$ in T. The main theorem of the present paper says that $C(\varphi)$, together with the expected equations, generates the ideal $\mathcal{J}$.

Theorem 9.1. *Let I be a linearly presented height three Gorenstein ideal that satisfies G_d. Then the defining ideal of the Rees ring of I is*

$$\mathcal{J} = \mathcal{L} + I_d(B)S + C(\varphi)S,$$

and the defining ideal of the variety X is

$$I(X) = I_d(B) + C(\varphi),$$

where φ is a minimal homogeneous alternating presentation matrix for I. In particular, I is of fiber type and, if d is odd, then $C(\varphi) = 0$ and $\mathcal{R}(I)$ has the expected form.

We now outline the method of proof of our main theorem. The proof has essentially four ingredients, each of which is of independent interest and applies in much greater generality than needed here. Some of these tools are developed in other articles [Kustin and Ulrich 1992; Kustin et al. 2015; 2016b; 2017b]. We describe the content of these articles to the extent necessary for the present paper.

The first step is to prove that any ideal I satisfying the assumptions of Theorem 9.1 is of fiber type (Corollary 6.4). Once this is done, it suffices to determine $I(X)$. A crucial tool in this context is the standard bigrading of the symmetric algebra. If δ is the degree of the generators of I, then the R-module $I(\delta)$ is generated in degree zero and its symmetric algebra $\text{Sym}(I(\delta))$ is naturally standard bigraded. As it turns out, the ideal I is of fiber type if and only if the bihomogeneous ideal $\mathscr{A} = \text{H}^0_\mathfrak{m}(\text{Sym}(I(\delta)))$ of the symmetric algebra is generated in degrees $(0, \star)$ or, equivalently, the R-modules $\text{H}^0_\mathfrak{m}(\text{Sym}_q(I(\delta)))$ are generated in degree zero for all nonnegative integers q. This suggests the following general question: If M is a finitely generated graded R-module, what are bounds for the generator degrees of the local cohomology modules $\text{H}^i_\mathfrak{m}(M)$? This question is addressed in [Kustin et al. 2015]. There we consider *approximate resolutions*, which are homogeneous complexes of finitely generated graded R-modules C_j of sufficiently high depth,

$$C_\bullet : \quad \cdots \to C_1 \to C_0 \to 0,$$

such that $\text{H}_0(C_\bullet) \cong M$ and $\text{H}_j(C_\bullet)$ have sufficiently small dimension for all positive integers j. We prove, for instance, that $\text{H}^0_\mathfrak{m}(M)$ is concentrated in degrees $\leq b_0(C_d) - d$ and is generated in degrees $\leq b_0(C_{d-1}) - d + 1$, where $b_0(C_j)$ denotes the largest generator degree of C_j. These theorems from [Kustin et al. 2015] are reproduced in the present paper as Corollary 5.4 and Theorem 5.6. Whereas similar results for the concentration degree of local cohomology modules have been established before to study Castelnuovo–Mumford regularity (see, e.g., [Gruson et al. 1983]), it appears that the sharper bound on the generation degree is new and more relevant for our purpose. Under the hypotheses of Theorem 9.1 the complexes $\mathscr{D}^q_\bullet$ from [Kustin and Ulrich 1992] are approximate resolutions of the symmetric powers $\text{Sym}_q(I(\delta))$, and we deduce that indeed the R-modules $\text{H}^0_\mathfrak{m}(\text{Sym}_q(I(\delta)))$ are generated in degree zero. This is done in Theorem 6.1 and Corollary 6.4.

In fact, Theorem 6.1 is considerably more general; it deals with height two perfect ideals and height three Gorenstein ideals that are not necessarily linearly presented. We record an explicit exponent N so that $\mathfrak{m}^N \cdot \mathscr{A} = 0$ and we provide an explicit bound for the largest x-degree p so that $\mathscr{A}_{(p,*)}$ contains a minimal bihomogeneous generator of $\mathscr{A}$. Theorem 6.1 is a significant generalization of part of [Kustin et al. 2017a], where the same results are established under fairly restrictive hypotheses.

With the hypotheses of Theorem 9.1 we know that $I_d(B) + C(\varphi) \subset I(X)$; see [Kustin et al. 2017b] or Corollary 8.8. To show that this inclusion is an equality,

we first prove that the two ideals have the same height, which is $n - d$ if $d \leq n$. In fact, we show much more under weaker hypotheses: Assume I is any ideal of R generated by forms of degree δ, I is of linear type on the punctured spectrum, and a sufficiently high symmetric power $\mathrm{Sym}_q(I(\delta))$ has an approximate free resolution that is linear for the first d steps, then up to radical, $\mathcal{J}$ has the expected form, in symbols,

$$\mathcal{J} = \sqrt{(\mathcal{L}, I_d(B))}$$

and, as a consequence,

$$I(X) = \sqrt{I_d(B')},$$

where B' is the Jacobian dual of the largest submatrix of φ that has linear entries; this is Theorem 7.2. We point out that even though our primary interest is the defining ideal $I(X)$ of the variety X, we need to consider the entire bigraded symmetric algebra of I since our main assumption uses the first component of the bigrading, which is invisible to the homogeneous coordinate ring of X.

The third step in the proof of Theorem 9.1 is to show that the ideal $I_d(B) + C(\varphi)$ is unmixed, in the relevant case when $d < n$. In [Kustin et al. 2017b] we construct, more generally, complexes associated to any d by n matrix B with linear entries in T that is annihilated by a vector of indeterminates, $B \cdot \underline{T}^{\mathrm{t}} = 0$. The ideal of d by d minors of such a matrix cannot have generic height $n - d + 1$. However, if the height is $n - d$, our complexes give resolutions of $I_d(B)$ and of $I_d(B) + C(\varphi)$, although these ideals fail to be perfect in general. Thus, we obtain resolutions in the context of Theorem 9.1 because we have seen in the previous step that $I_d(B)$ has height $n - d$. Using these resolutions we prove that the ideal $I_d(B) + C(\varphi)$ is unmixed and hence is the unmixed part of $I_d(B)$. In [Kustin et al. 2017b] we also compute the multiplicity of the rings defined by these ideals.

We have seen that the two ideals $I_d(B) + C(\varphi)$ and $I(X)$ are unmixed and have the same height. Thus, to conclude that the inclusion $I_d(B) + C(\varphi) \subset I(X)$ is an equality it suffices to prove that the rings defined by these ideals have the same multiplicity. Since the multiplicity of the first ring has been computed in [Kustin et al. 2017b], it remains to determine the multiplicity of the coordinate ring $A = T/I(X)$ or, equivalently, the degree of the variety X. To do so we observe that the rational map Ψ is birational onto its image because the presentation matrix φ is linear (Proposition 4.1). Therefore the degree of X can be expressed in terms of the multiplicity of a ring defined by a certain residual intersection of I (Proposition 4.2). Finally, the multiplicity of such residual intersections can be obtained from the resolutions in [Kustin and Ulrich 1992]; see [Kustin et al. 2016b] and Theorem 4.4. Once we prove in Theorem 9.1 that $I(X) = I_d(B) + C(\varphi)$, then in Corollary 9.3 we are able to harvest much information from the results of [Kustin et al. 2017b]. In

particular, we learn the depth of the homogeneous coordinate ring A of X, the entire Hilbert series of A, and the fact that $I(X)$ has a linear resolution when d is odd.

2. Conventions and notation

2.1. If ψ is a matrix, then ψ^{t} is the transpose of ψ. If ψ is a matrix (or a homomorphism of finitely generated free R-modules), then $I_r(\psi)$ is the ideal generated by the $r \times r$ minors of ψ (or any matrix representation of ψ).

2.2. We collect names for some of the invariants associated to a graded module. Let R be a graded Noetherian ring and M be a graded R-module. Define

$$\operatorname{topdeg} M = \sup\{ j \mid M_j \neq 0\},$$
$$\operatorname{indeg} M = \inf\{ j \mid M_j \neq 0\}, \text{ and}$$
$$b_0(M) = \inf\left\{ p \mid R\left(\bigoplus_{j \leq p} M_j\right) = M\right\}.$$

If R is nonnegatively graded, R_0 is local, $\mathfrak{m}$ is the maximal homogeneous ideal of R, M is finitely generated, and i is a nonnegative integer, then also define

$$a_i(M) = \operatorname{topdeg} \mathrm{H}^i_{\mathfrak{m}}(M) \quad \text{and} \quad b_i(M) = \operatorname{topdeg} \operatorname{Tor}^R_i(M, R/\mathfrak{m}).$$

Observe that both definitions of the maximal generator degree $b_0(M)$ give the same value. The expressions "topdeg", "indeg", and "b_i" are read "top degree", "initial degree", and "maximal i-th shift in a minimal homogeneous resolution", respectively. If M is the zero module, then

$$\operatorname{topdeg}(M) = b_0(M) = -\infty \quad \text{and} \quad \operatorname{indeg} M = \infty.$$

In general one has

$$a_i(M) < \infty \quad \text{and} \quad b_i(M) < \infty.$$

2.3. Recall the numerical functions of 2.2. Let R be a nonnegatively graded Noetherian ring with R_0 local and $\mathfrak{m}$ be the maximal homogeneous ideal of R. The a-invariant of R is defined to be

$$a(R) = a_{\dim R}(R).$$

Furthermore, if ω_R is the graded canonical module of R, then

$$a(R) = - \operatorname{indeg} \omega_R.$$

2.4. Let R be a standard graded polynomial ring over a field k, $\mathfrak{m}$ be the maximal homogeneous ideal of R, and M be a finitely generated graded R-module. Recall the numerical functions of 2.2. The Castelnuovo–Mumford *regularity* of M is

$$\operatorname{reg} M = \sup\{a_i(M) + i\} = \sup\{b_i(M) - i\}.$$

2.5. Let I be an ideal in a Noetherian ring. The *unmixed part* I^{unm} of I is the intersection of the primary components of I that have height equal to the height of I.

2.6. Let R be a Noetherian ring, I be a proper ideal of R, and M be a finitely generated nonzero R-module. We denote the projective dimension of the R-module M by $\mathrm{pd}_R M$. The *grade* of I is the length of a maximal regular sequence on R which is contained in I. (If R is Cohen–Macaulay, then the grade of I is equal to the height of I.) The R-module M is called *perfect* if the grade of the annihilator of M (denoted $\mathrm{ann}\, M$) is equal to the projective dimension of M. (The inequality grade $\mathrm{ann}\, M \le \mathrm{pd}_R M$ holds automatically.) The ideal I in R is called a *perfect ideal* if R/I is a perfect R-module. A perfect ideal I of grade g is a *Gorenstein ideal* if $\mathrm{Ext}_R^g(R/I, R)$ is a cyclic R-module.

2.7. Denote by $\mu(M)$ the minimal number of generators of a finitely generated module M over a local ring R. Recall from [Artin and Nagata 1972] that an ideal I in a Noetherian ring R satisfies the condition G_s if $\mu(I_{\mathfrak{p}}) \le \dim R_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \le s - 1$. The condition G_s can be interpreted in terms of the height of Fitting ideals. An ideal I of positive height satisfies G_s if and only if $i < \mathrm{ht}\, \mathrm{Fitt}_i(I)$ for $0 < i < s$.

The property G_d, for $d = \dim R$, is always satisfied if R/I is 0-dimensional or if I is a generic perfect ideal of grade two or a generic Gorenstein ideal of grade three.

2.8. Let R be a Noetherian ring, I be an ideal of height g, K be a proper ideal, and s be an integer with $g \le s$.

(a) The ideal K is called an *s-residual intersection* of I if there exists an s-generated ideal $\mathfrak{a} \subset I$ such that $K = \mathfrak{a} : I$ and $s \le \mathrm{ht}\, K$.

(b) The ideal K is called a *geometric s-residual intersection* of I if K is an s-residual intersection of I and if, in addition, $s + 1 \le \mathrm{ht}(I + K)$.

(c) The ideal I is said to be *weakly s-residually S_2* if for every i with $g \le i \le s$ and every geometric i-residual intersection K of I, R/K is S_2.

2.9. If I is an ideal of a ring R, then the *Rees ring* of I, denoted $\mathcal{R}(I)$, is the subring $R[It]$ of the polynomial ring $R[t]$. There is a natural epimorphism of R-algebras from the symmetric algebra of I, denoted $\mathrm{Sym}(I)$, onto $\mathcal{R}(I)$. If this map is an isomorphism, we say that the ideal I is of *linear type*.

3. Defining equations of graphs and images of rational maps

Data 3.1. Let k be a field and let $R = k[x_1, \ldots, x_d]$, $T = k[T_1, \ldots, T_n]$, and $S = R[T_1, \ldots, T_n]$ be polynomial rings. Let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the maximal homogeneous ideal of R, and I a homogeneous ideal of R minimally generated by forms $g_1, \ldots, g_n$ of the same positive degree δ.

Such forms define a rational map

$$\Psi = [g_1 : \cdots : g_n] : \mathbb{P}_k^{d-1} \to \mathbb{P}_k^{n-1}$$

with base locus $V(I)$. We write X for the closed image of Ψ, the variety parametrized by Ψ, and $A = T/I(X)$ for the homogeneous coordinate ring of this variety.

In this section we collect some general facts about rational maps and Rees rings. Data 3.1 is in effect throughout.

3.2. Recall the definition of the Rees ring $\mathcal{R}(I) = R[It] \subset R[t]$ from 2.9. We consider the epimorphism of R-algebras

$$\pi : S \twoheadrightarrow \mathcal{R}(I)$$

that sends T_i to $g_i t$. Let $\mathcal{J}$ be the kernel of this map, which is the ideal defining the Rees ring.

We set $\deg x_i = (1, 0)$, $\deg T_i = (0, 1)$, and $\deg t = (-\delta, 1)$. Thus S and the subring $\mathcal{R}(I)$ of $R[t]$ become standard bigraded k-algebras, π is a bihomogeneous epimorphism, and $\mathcal{J}$ is a bihomogeneous ideal of S. If M is a bigraded S-module and p a fixed integer, we write

$$M_{(p,\star)} = \bigoplus_j M_{(p,j)} \quad \text{and} \quad M_{(>p,\star)} = \bigoplus_{i>p} M_{(i,\star)}.$$

3.3. The Rees algebra $\mathcal{R}(I)$ is the bihomogeneous coordinate ring of the graph of Ψ. In fact, the natural morphisms of projective varieties

$$
\begin{array}{ccc}
\mathbb{P}_k^{d-1} \times \mathbb{P}_k^{n-1} & \longleftarrow & \text{graph } \Psi \\
\downarrow & & \downarrow \\
\mathbb{P}_k^{n-1} & \longleftarrow & X = \text{im } \Psi
\end{array}
$$

correspond to bihomogeneous homomorphisms of k-algebras,

$$
\begin{array}{ccc}
S & \xrightarrow{\pi} & \mathcal{R}(I) \\
\uparrow & & \uparrow \\
T & \xrightarrow{\pi_{|T}} & A.
\end{array}
$$

Since $T = S_{(0,\star)}$, we see that

$$A = \mathcal{R}(I)_{(0,\star)} = k[g_1 t, \ldots, g_n t]$$

and

$$I(X) = \mathcal{J}_{(0,\star)} = \mathcal{J} \cap T. \tag{3.3.1}$$

We also observe that there is a not necessarily homogeneous isomorphism of k-algebras

$$k[g_1 t, \ldots, g_n t] \cong k[g_1, \ldots, g_n] \subset R.$$

Moreover, the identification $\mathcal{R}(I)_{(0,\star)} \cong \mathcal{R}(I)/\mathcal{R}(I)_{(>0,\star)}$ gives a homogeneous isomorphism

$$A \cong \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I).$$

The latter ring is called the *special fiber ring* of I and its dimension is the *analytic spread* of I, written $\ell(I)$. The analytic spread plays an important role in the study of reductions of ideals and satisfies the inequality

$$\ell(I) \leq \min\{d, n\}. \tag{3.3.2}$$

Notice that $\ell(I) = \dim A = \dim X + 1$.

3.4. The graded R-module $I(\delta)$ is generated in degree zero and hence its symmetric algebra $\mathrm{Sym}(I(\delta))$ is a standard bigraded k-algebra. There is a natural bihomogeneous epimorphism of k-algebras

$$\mathrm{Sym}(I(\delta)) \twoheadrightarrow \mathcal{R}(I),$$

whose kernel we denote by $\mathcal{A}$. The epimorphism π from 3.2 factors through this map and gives a bihomogeneous epimorphism of k-algebras

$$S \twoheadrightarrow \mathrm{Sym}(I(\delta)).$$

We write $\mathcal{L}$ for the kernel of this epimorphism. The advantage of $\mathcal{L}$ over $\mathcal{J}$ is that $\mathcal{L}$ can be easily described in terms of a presentation matrix of I; see 3.6. Notice that

$$\mathcal{A} = \mathcal{J}/\mathcal{L}.$$

The ideal I is of linear type if any of the following equivalent conditions hold:

- $\mathcal{A} = 0$.
- $\mathcal{J} = \mathcal{L}$.
- $\mathcal{J}$ is generated in bidegrees $(\star, 1)$.

Notice that if I is of linear type then necessarily $I(X) = 0$.

The ideal I is of *fiber type* if any of the following equivalent conditions hold:

- $\mathcal{A} = I(X) \cdot \mathrm{Sym}(I(\delta))$.
- $\mathcal{A}$ is generated in bidegrees $(0, \star)$.
- $\mathcal{J} = \mathcal{L} + I(X) \cdot S$.
- $\mathcal{J}$ is generated in bidegrees $(\star, 1)$ and $(0, \star)$.

See (3.3.1) for the equivalence of these conditions.

3.5. The ideal $\mathscr{A}$ in the exact sequence

$$0 \to \mathscr{A} \to \mathrm{Sym}(I(\delta)) \to \mathscr{R}(I) \to 0$$

is the R-torsion of $\mathrm{Sym}(I(\delta))$. Hence $\mathscr{A}$ contains the zeroth local cohomology of $\mathrm{Sym}(I(\delta))$ as an R-module, which in turn contains the socle of $\mathrm{Sym}(I(\delta))$ as an R-module,

$$\mathscr{A} \supset \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta))) = 0 :_{\mathrm{Sym}(I(\delta))} \mathfrak{m}^{\infty} \supset 0 :_{\mathrm{Sym}(I(\delta))} \mathfrak{m}. \qquad (3.5.1)$$

3.6. Consider the row vectors $\underline{x} = [x_1, \ldots, x_d]$, $\underline{T} = [T_1, \ldots, T_n]$, $\underline{g} = [g_1, \ldots, g_n]$, and let φ be a minimal homogeneous presentation matrix of $\underline{g}$. There exists a matrix B with d rows and entries in the polynomial ring S such that the equality of row vectors

$$\underline{T} \cdot \varphi = \underline{x} \cdot B \qquad (3.6.1)$$

holds. If B is chosen so that its entries are linear in $T_1, \ldots, T_n$, then B is called a *Jacobian dual* of φ. We say that a Jacobian dual is *homogeneous* if the entries of any fixed column of B are homogeneous of the same bidegree. Whenever φ is a matrix of linear forms, then the entries of B can be taken from the ring $k[T_1, \ldots, T_n]$; this B is uniquely determined by φ and is called *the* Jacobian dual of φ; notice that the entries of B are linear forms and that B is indeed a Jacobian matrix of the T-algebra $\mathrm{Sym}(I(\delta))$.

The ideal $\mathscr{L}$ defining the symmetric algebra is equal to $I_1(\underline{T} \cdot \varphi)$, which coincides with $I_1(\underline{x} \cdot B)$. For the Rees algebra, the inclusions

$$\mathscr{L} + I_d(B) \subset \mathscr{L} :_S \mathfrak{m} \subset \mathscr{J} \qquad (3.6.2)$$

obtain; see also [Vasconcelos 1991]. If the equality $\mathscr{L} + I_d(B) = \mathscr{J}$ holds, then $\mathscr{J}$ or $\mathscr{R}(I)$ is said to have the *expected form*. Notice that in this case the inclusions of (3.5.1) are equalities, in other words, $\mathscr{A}$ is the socle of $\mathrm{Sym}(I(\delta))$ as an R-module.

3.7. It is easier to determine when the first inclusion of (3.5.1) is an equality. Namely,

$$\mathscr{A} = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta))) \iff I \text{ is of linear type on the punctured spectrum of } R.$$
$$(3.7.1)$$

The right hand side means that $\mathscr{A}_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ in $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$.

This leads to the question of when an ideal is of linear type. Thus let J be an ideal of a local Cohen–Macaulay ring of dimension d. If J is perfect of height two or Gorenstein of height three, then J is in the linkage class of a complete intersection [Gaeta 1953; Watanabe 1973]. If J is in the linkage class of a complete intersection, then J is strongly Cohen–Macaulay [Huneke 1982, 1.11] and hence satisfies the sliding depth condition of [Herzog et al. 1985]. If J satisfies the sliding depth condition and is G_{d-2}, then J is weakly $(d-2)$-residually S_2 in the sense of

2.8.(c) [Herzog et al. 1985, 3.3]. In the presence of the weak $(d-2)$-residually S_2 condition it follows from [Chardin et al. 2001, 3.6(b)] that

$$J \text{ is of linear type} \iff J \text{ satisfies } G_{d+1};$$

in particular, since the residually S_2 condition localizes,

$$J \text{ is of linear type on the punctured spectrum} \iff J \text{ satisfies } G_d.$$

(Alternatively, one uses [Herzog et al. 1983].)

Combining these facts we see that if the ideal I of Data 3.1 is perfect of height two or Gorenstein of height three then

$$I \text{ is of linear type} \iff I \text{ satisfies } G_d \text{ and } n \leq d \qquad (3.7.2)$$

and

$$\mathcal{A} = H^0_{\mathfrak{m}}(\text{Sym}(I(\delta))) \iff I \text{ satisfies } G_d. \qquad (3.7.3)$$

4. The degree of the image of a rational map

This section deals with the dimension and the degree of the image of rational maps as in Data 3.1. Applied to the special case of linearly presented Gorenstein ideals of height three, this information will be an important ingredient in the proof of Theorem 9.1.

We first treat the question of when the rational map Ψ is birational onto its image X (Proposition 4.1). We then express the degree of X in terms of the multiplicity of a ring defined by a residual intersection of the ideal I defining the base locus of Ψ (Proposition 4.2). The multiplicity of rings defined by residual intersections of linearly presented height three Gorenstein ideals was computed in [Kustin et al. 2016b], using the resolutions worked out in [Kustin and Ulrich 1992]. For completeness we state this result in Theorem 4.3. Finally, we combine Propositions 4.1, 4.2, and Theorem 4.3 in Theorem 4.4 to compute the degree of X under the hypotheses of Theorem 9.1.

Proposition 4.1. *Adopt Data 3.1 and assume further that the ideal I is linearly presented.*

(a) *If the ring R/I has dimension zero, then the map Ψ is biregular onto its image. In particular, A has multiplicity δ^{d-1}.*

(b) *If A has dimension d, then the rational map Ψ is birational onto its image.*

Proof. We may assume that the field k is algebraically closed. Let φ be a minimal presentation matrix of $\underline{g} = g_1, \ldots, g_n$ with homogeneous linear entries in R and let $P \in \mathbb{P}^{n-1}_k$ be a point in X. According to [Eisenbud and Ulrich 2008] the

ideal $I_1(P \cdot \varphi) : I^\infty$ defines the fiber $\Psi^{-1}(P)$ scheme theoretically. This ideal cannot be the unit ideal because the fiber is not empty. On the other hand, the ideal $I_1(P \cdot \varphi)$ is generated by linear forms, hence it is a prime ideal of R. Thus $I_1(P \cdot \varphi) : I^\infty = I_1(P \cdot \varphi)$ is generated by linear forms.

(a) We consider the embedding $A \cong k[\underline{g}] \subset R$ which corresponds to the rational map Ψ onto its image. If the ring $R/I = R/(\underline{g})$ has dimension zero, then R is finitely generated as an A-module. It follows that the fiber $\Psi^{-1}(P)$ consists of finitely many points. Since this fiber is defined by a linear ideal, it consists of a single reduced point. This shows that Ψ is biregular onto its image.

The claim about the multiplicity follows because, in particular, Ψ is birational onto its image; see, for instance, [Kustin et al. 2016a, 5.8].

(b) Let Q be a general point in $\mathbb{P}_k^{d-1}$ and P be its image $\Psi(Q)$ in X. Since $A \cong k[\underline{g}]$ and R have the same dimension, the fiber $\Psi^{-1}(P)$ consists of finitely many points. Therefore, again, $\Psi^{-1}(P)$ consists of a single reduced point, which means that Ψ is birational onto its image. $\qquad\square$

Proposition 4.2. *Adopt Data 3.1. Further assume that the field k is infinite, that $d \leq n$, and that the ideal I is weakly $(d-2)$-residually S_2 and satisfies G_d. Let $f_1, \ldots, f_{d-1}$ be general k-linear combinations of the generators $g_1, \ldots, g_n$ of I. Then the following statements hold:*

(a) *The ring $R/((f_1, \ldots, f_{d-1}) : I)$ is Cohen–Macaulay of dimension one.*

(b) *The ring A has dimension d.*

(c) *The ring A has multiplicity*

$$e(A) = \frac{1}{r} \cdot e\left(\frac{R}{(f_1, \ldots, f_{d-1}) : I} \right),$$

where r is the degree of the rational map Ψ.

Proof. Let $f_1, \ldots, f_d$ be general k-linear combinations of the generators $g_1, \ldots, g_n$ of I. The G_d property of the ideal I implies that

$$d - 1 \leq \mathrm{ht}((f_1, \ldots, f_{d-1}) : I) \quad \text{and} \quad d \leq \mathrm{ht}((f_1, \ldots, f_{d-1}) : I, f_d), \quad (4.2.1)$$

according to [Artin and Nagata 1972, 2.3] or [Ulrich 1994, 1.6(a)]. Furthermore the ideal $(f_1, \ldots, f_{d-1}) : I$ is proper because I requires at least d generators. Now 3.1 and 3.3(a) of [Chardin et al. 2001] imply that $(f_1, \ldots, f_{d-1}) : I$ is unmixed of height $d - 1$, which proves (a).

To show (b) we suppose that $\dim A < d$. In this case the ring $k[g_1, \ldots, g_n] \cong A$ is integral over the subring $k[f_1, \ldots, f_{d-1}]$, and therefore the ideal I is integral over the ideal $(f_1, \ldots, f_{d-1})$. In 3.7 we have seen that, locally on the punctured spectrum, I is of linear type and hence cannot be integral over a proper subideal. Thus $d \leq \mathrm{ht}((f_1, \ldots, f_{d-1}) : I)$, which contradicts (a).

We now prove (c). Item (b) allows us to apply [Kustin et al. 2016a, 3.7], which gives

$$e(A) = \frac{1}{r} \cdot e\left(\frac{R}{(f_1, \ldots, f_{d-1}) : I^\infty}\right).$$

On the other hand, item (a) and the inequality $d \le \mathrm{ht}((f_1, \ldots, f_{d-1}) : I, f_d)$ imply that the element f_d of I is regular on $R/((f_1, \ldots, f_{d-1}) : I)$. It follows that $(f_1, \ldots, f_{d-1}) : I^\infty = (f_1, \ldots, f_{d-1}) : I$. $\qquad\square$

The following statement, which is used in the proof of Theorem 4.4, is part of [Kustin et al. 2016b, 1.2]. We do not need the entire h-vector of $\overline{R}$ in the present paper; but the entire h-vector is recorded in [Kustin et al. 2016b, 1.2].

Theorem 4.3. *Adopt Data 3.1. Further assume that the ideal I is a linearly presented Gorenstein ideal of height three. Let $\mathfrak{a}$ be a subideal of I, minimally generated by s homogeneous elements for some s with $3 \le s$, let α be the minimal number of generators of $I/\mathfrak{a}$, J be the ideal $\mathfrak{a} : I$, and $\overline{R} = R/J$. Assume that*

(1) *the homogeneous minimal generators of $\mathfrak{a}$ live in two degrees:* $\mathrm{indeg}(I)$ *and* $\mathrm{indeg}(I) + 1$,

(2) *the ideal J is an s-residual intersection of I (that is, $J \ne R$ and $s \le \mathrm{ht}\, J$).*

Then the ring $\overline{R}$ has multiplicity

$$e(\overline{R}) = \sum_{i=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \binom{s+\alpha-2-2i}{s-1}.$$

The latter is also equal to the number of monomials $\boldsymbol{m}$ of degree at most $\alpha - 1$ in s variables with $(\deg \boldsymbol{m}) + \alpha$ odd.

Theorem 4.4, the main result of this section, is where the dimension and the degree of the variety X are computed.

Theorem 4.4. *Adopt Data 3.1. Further assume that $d \le n$, that I is a linearly presented Gorenstein ideal of height three, and that I satisfies G_d. Then A has dimension d and multiplicity*

$$e(A) = \sum_{i=0}^{\lfloor \frac{n-d}{2} \rfloor} \binom{n-2-2i}{d-2},$$

which is equal to

- *the number of monomials of even degree at most $n-d$ in $d-1$ variables, if d is odd,*
- *the number of monomials of odd degree at most $n-d$ in $d-1$ variables, if d is even.*

Proof. We may assume that the field k is infinite. We first observe that the ideal I satisfies the hypotheses of Proposition 4.2; see 3.7. According to Propositions 4.2.(b), 4.1.(b), and 4.2.(c), the dimension of A is d and the multiplicity of A is

$$e(A) = e\left(\frac{R}{(f_1, \ldots, f_{d-1}) : I}\right),$$

where $f_1, \ldots, f_{d-1}$ are general k-linear combinations of homogeneous minimal generators of I.

Write $J = (f_1, \ldots, f_{d-1}) : I$. This ideal is a $(d-1)$-residual intersection of I by (4.2.1). Apply Theorem 4.3 with s replaced by $d-1$ and α replaced by $n-d+1$ in order to obtain

$$e(A) = e(R/J) = \sum_{i=0}^{\lfloor \frac{n-d}{2} \rfloor} \binom{n-2-2i}{d-2}. \qquad \square$$

5. Degree bounds for local cohomology

An important step in the proof of Theorem 9.1 is to show that an ideal is of fiber type (see 3.4 for a definition) and to identify a power of the maximal ideal that annihilates the torsion of the symmetric algebra. In (3.7.3) and 3.4 we have seen that this amounts to finding degree bounds for local cohomology modules. Such degree bounds are given in [Kustin et al. 2015], we recall them here. These results are applied to the study of blowup algebras in Sections 6 and 7.

Data 5.1. Let R be a nonnegatively graded Noetherian algebra over a local ring R_0 with $\dim R = d$, let $\mathfrak{m}$ be the maximal homogeneous ideal of R, let M be a graded R-module, and let

$$C_\bullet : \quad \cdots \to C_1 \to C_0 \to 0$$

be a homogeneous complex of finitely generated graded R-modules with $H_0(C_\bullet) \cong M$.

In [Kustin et al. 2015] we find bounds on the degrees of interesting elements of the local cohomology modules $H_\mathfrak{m}^i(M)$ in terms of information about the ring R and information that can be read from the complex $C_\bullet$. The ring R need not be a polynomial ring, the complex $C_\bullet$ need not be finite, need not be acyclic, and need not consist of free modules, and the parameter i need not be zero. Instead, we impose hypotheses on the Krull dimension of $H_j(C_\bullet)$ and the depth of C_j in order to make various local cohomology modules $H_\mathfrak{m}^\ell(H_j(C_\bullet))$ and $H_\mathfrak{m}^\ell(C_j)$ vanish.

Recall that if M and N are modules over a ring, then N is a *subquotient* of M if N is isomorphic to a submodule of a homomorphic image of M or, equivalently, if N is isomorphic to a homomorphic image of a submodule of M. Also recall the numerical functions of 2.2.

The next proposition is [Kustin et al. 2015, 3.6].

Proposition 5.2. *Adopt Data 5.1. Fix an integer i with $0 \leq i \leq d$. Assume that*

(1) $j + i + 1 \leq \operatorname{depth} C_j$ *for all j with $0 \leq j \leq d - i - 1$, and*

(2) $\dim H_j(C_\bullet) \leq j + i$ *for all j with $1 \leq j$.*

Then

(a) $H_\mathfrak{m}^i(M)$ *is a graded subquotient of* $H_\mathfrak{m}^d(C_{d-i})$, *and*

(b) $a_i(M) \leq b_0(C_{d-i}) + a(R)$.

Remark 5.3. Typically, one applies Proposition 5.2 when the modules C_j are maximal Cohen–Macaulay modules (for example, free modules over a Cohen–Macaulay ring) because, in this case, hypothesis (1) about depth C_j is automatically satisfied. Similarly, if

$$H_j(C_{\bullet\mathfrak{p}}) = 0 \text{ for all } j \text{ and } p \text{ with } 1 \leq j \leq d-i-1,$$

$$\mathfrak{p} \in \operatorname{Spec}(R), \text{ and } i+2 \leq \dim R/\mathfrak{p}, \tag{5.3.1}$$

then hypothesis (2) is satisfied.

We record an immediate consequence of Proposition 5.2; this is [Kustin et al. 2015, 3.8].

Corollary 5.4. *Adopt the hypotheses of Proposition 5.2, with $i = 0$. Then the following statements hold*:

(a) *If $C_d = 0$, then $H_\mathfrak{m}^0(M) = 0$.*

(b) *If $C_d \neq 0$, then $[H_\mathfrak{m}^0(M)]_p = 0$ for all p with $b_0(C_d) + a(R) < p$.*

The next result, an obvious consequence of Corollary 5.4, is [Kustin et al. 2015, 3.9].

Corollary 5.5. *Adopt the hypotheses of Proposition 5.2, with $i = 0$. Assume further that $R = k[x_1, \ldots, x_d]$ is a standard graded polynomial ring over a field and that the subcomplex*

$$C_d \to \cdots \to C_1 \to C_0 \to 0$$

of $C_\bullet$ is a q-linear complex of free R-modules, for some integer q (that is, $C_j \cong R(-j-q)^{\beta_j}$ for $0 \leq j \leq d$). Then $H_\mathfrak{m}^0(M)$ is concentrated in degree q; that is, $[H_\mathfrak{m}^0(M)]_p = 0$ for all p with $p \neq q$.

The next theorem is [Kustin et al. 2015, 4.8] and the main result of this section. It provides an upper bound for the generation degree of the zeroth local cohomology.

Theorem 5.6. *Let $R = k[x_1, \ldots, x_d]$ be a standard graded polynomial ring over a field, with maximal homogeneous ideal $\mathfrak{m}$, let*

$$C_\bullet: \cdots \to C_2 \to C_1 \to C_0 \to 0$$

be a homogeneous complex of finitely generated graded R-modules, and $M = H_0(C_\bullet)$. Assume that

- $\dim H_j(C_\bullet) \leq j$ *whenever* $1 \leq j \leq d-1$, *and*
- $\min\{d, j+2\} \leq \operatorname{depth} C_j$ *whenever* $0 \leq j \leq d-1$.

Then $b_0(H_{\mathfrak{m}}^0(M)) \leq b_0(C_{d-1}) - d + 1$.

Remark 5.7. The hypotheses of Theorem 5.6 about depth and dimension are satisfied if the modules C_j are free and condition (5.3.1) holds.

6. Bounding the degrees of defining equations of Rees rings

In this section we apply the local cohomology techniques of [Kustin et al. 2015] to bound the degrees of the defining equations of Rees rings. This style of argument was inspired by work of Eisenbud, Huneke, and Ulrich; see [Eisenbud et al. 2006].

To apply the results of [Kustin et al. 2015], most notably Corollary 5.4 and Theorem 5.6 of the present paper, we need "approximate resolutions" $C_\bullet$ of the symmetric powers of $I(\delta)$, for an ideal I as in Data 3.1 that satisfies G_d. If I is perfect of height two, we take $C_\bullet$ to be a homogeneous strand of a Koszul complex. The fact that this choice of $C_\bullet$ satisfies the hypotheses of Corollary 5.4 and Theorem 5.6 is shown in the proof of Theorem 6.1.(a). If I is Gorenstein of height three, we take $C_\bullet$ to be one of the complexes $\mathscr{D}_\bullet^q(\varphi)$ from [Kustin and Ulrich 1992]. In this paper these complexes are introduced in (6.1.4); they are shown to satisfy the hypotheses of Corollary 5.4 and Theorem 5.6 in Lemma 6.3.

In Theorem 6.1 we carefully record the degrees of the entries in a minimal homogeneous presentation matrix for I; but we do not insist that these entries be linear. For the ideal $\mathscr{A}$ of the symmetric algebra defined in 3.4, we give an explicit exponent N so that $\mathfrak{m}^N \cdot \mathscr{A} = 0$ and we obtain an explicit bound for the largest x-degree p so that $\mathscr{A}_{(p,*)} = \bigoplus_q \mathscr{A}_{(p,q)}$ contains a minimal bihomogeneous generator of $\mathscr{A}$.

Assertions (ai) and (aii) of Theorem 6.1, about perfect height two ideals, are significant generalizations of Corollaries 2.5(2) and 2.16(1), respectively, in [Kustin et al. 2017a], where the same results are obtained for $d = 2$ and $n = 3$. The starting point for [Kustin et al. 2017a] is the perfect pairing of Jouanolou [1996; 1997], which exists because the symmetric algebra of a three generated height two perfect ideal is a complete intersection. With the present hypotheses, the symmetric algebra is only a complete intersection on the punctured spectrum of R and there is no perfect pairing analogous to the pairing of Jouanolou.

Theorem 6.1 is particularly interesting when the ideal I is linearly presented. Indeed, assertions (aii) and (bii) say that I is of fiber type. For height two perfect ideals this was already observed in [Morey and Ulrich 1996]. For height

three Gorenstein ideals, this is a new result and a main ingredient in the proof of
Theorem 9.1. The application of Theorem 6.1 to the situation where I is a linearly
presented height three Gorenstein ideal is recorded as Corollary 6.4.

Theorem 6.1. *Adopt Data 3.1 and recall the bihomogeneous ideal $\mathcal{A}$ of the symmetric algebra defined in 3.4. Assume further that I satisfies the condition G_d.*

(a) *Assume that the ideal I is perfect of height two and the column degrees of a homogeneous Hilbert–Burch matrix minimally presenting I are $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{n-1}$. Then the following statements hold when $d \leq n-1$:*

  (i) *If $\sum_{i=1}^{d}(\epsilon_i - 1) < p$, then $\mathcal{A}_{(p,*)} = 0$. In particular,*

$$\mathfrak{m}^{1+\sum_{i=1}^{d}(\epsilon_i - 1)}\mathcal{A} = 0.$$

  (ii) *If $\mathcal{A}_{(p,*)}$ contains a minimal bihomogeneous generator of $\mathcal{A}$, then*

$$p \leq \sum_{i=1}^{d-1}(\epsilon_i - 1).$$

(b) *Assume that the ideal I is Gorenstein of height three and every entry of a homogeneous alternating matrix minimally presenting I has degree D. Then the following statements hold:*

  (i) *If*

$$\begin{cases} d(D-1) < p & \text{when } d \text{ is odd,} \\ (d-1)(D-1) + \frac{1}{2}(n-d+1)D - 1 < p & \text{when } d \text{ is even,} \end{cases}$$

  then $\mathcal{A}_{(p,*)} = 0$. In particular,

$$\begin{cases} \mathfrak{m}^{d(D-1)+1}\mathcal{A} = 0 & \text{if } d \text{ is odd,} \\ \mathfrak{m}^{(d-1)(D-1)+(n-d+1)D/2}\mathcal{A} = 0 & \text{if } d \text{ is even.} \end{cases}$$

  (ii) *If $\mathcal{A}_{(p,*)}$ contains a minimal bihomogeneous generator of $\mathcal{A}$, then*

$$p \leq (d-1)(D-1).$$

Remarks. (a) We may safely assume that $d \leq n-1$ in Theorem 6.1 because otherwise $\mathcal{A} = 0$; see (3.7.2).

(b) The assumptions on the degrees of the entries in a presentation matrix do not impose any restriction: In (a), the inequalities $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{n-1}$ can be achieved by a permutation of the columns. In (b), all entries necessarily have the same degree, as can be seen from the symmetry of the minimal homogeneous R-resolution of R/I.

(c) Results similar to, but more complicated than, Theorem 6.1 can be obtained without the assumption that the ideal I is generated by forms of the same degree.

(d) The degree δ of the forms $g_1, \ldots, g_n$ in Data 3.1 is $\sum_{i=1}^{n-1} \epsilon_i$ in the setting of Theorem 6.1.(a) and is $\frac{1}{2}D(n-1)$ in Theorem 6.1.(b); see [Burch 1968] and [Buchsbaum and Eisenbud 1977], respectively.

Proof. We have seen in (3.7.3) that the ideal $\mathcal{A}$ of $\mathrm{Sym}(I(\delta))$ is equal to the local cohomology module $\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta)))$. We apply Corollary 5.4 and Theorem 5.6 to learn about the R-modules

$$\mathrm{H}^0_{\mathfrak{m}}(\mathrm{Sym}_q(I(\delta))) = \bigoplus_p \mathcal{A}_{(p,q)} = \mathcal{A}_{(*,q)}. \tag{6.1.1}$$

(a) Let φ be a minimal homogeneous Hilbert–Burch matrix for I. In other words, φ is an $n \times (n-1)$ matrix with homogeneous entries from R such that

$$0 \to F_1 \xrightarrow{\varphi} F_0 \xrightarrow{g} I(\delta) \to 0$$

is a homogeneous exact sequence of R-modules, where $\underline{g} = [g_1, \ldots, g_n]$ is the row vector of 3.6,

$$F_0 = R^n \quad \text{and} \quad F_1 = \bigoplus_{i=1}^{n-1} R(-\epsilon_i).$$

It induces a bihomogeneous presentation of the symmetric algebra

$$(\mathrm{Sym}(F_0) \otimes_R F_1)(0, -1) \xrightarrow{\ell} \mathrm{Sym}(F_0) \to \mathrm{Sym}(I(\delta)) \to 0,$$

where $\mathrm{Sym}(F_0)$ is the standard bigraded polynomial ring S of 3.2 and $\underline{\ell}$ is the row vector $[T_1, \ldots, T_n] \cdot \varphi$ of 3.6. We consider the Koszul complex of $\underline{\ell}$,

$$\mathbb{K}_\bullet = \bigwedge_S^\bullet ((S \otimes_R F_1)(0, -1)) = S \otimes_R \bigwedge_R^\bullet (F_1(0, -1)).$$

It is a bihomogeneous complex of free S-modules with $\mathrm{H}_0(\mathbb{K}_\bullet)$ equal to $\mathrm{Sym}(I(\delta))$.

The ideal I is perfect of height two and satisfies G_d (see 2.7). It follows that the sequence $\underline{\ell} = \ell_1, \ldots, \ell_{n-1}$ is a regular sequence locally on the punctured spectrum of R (see Observation 6.2); and therefore the Koszul complex $\mathbb{K}_\bullet$ is acyclic on the punctured spectrum of R. In particular, for each nonnegative integer q, the component of degree $(\star, q)$ of $\mathbb{K}_\bullet$ is a homogeneous complex of graded free R-modules, has zeroth homology equal to $\mathrm{Sym}_q(I(\delta))$, and is exact on the punctured spectrum of R. This component looks like

$$C^q_\bullet : 0 \to C^q_{n-1} \to C^q_{n-2} \to \cdots \to C^q_1 \to C^q_0 \to 0,$$

with

$$C^q_r = \mathrm{Sym}_{q-r}(F_0) \otimes_R \bigwedge^r F_1 \cong \bigoplus_{1 \le i_1 < \cdots < i_r \le n-1} R^{b_r}\left(-\sum_{j=1}^r \epsilon_{i_j}\right)$$

for

$$b_r = \text{rank}(\text{Sym}_{q-r}(F_0)) = \binom{q-r+n-1}{n-1}.$$

The assumption $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{n-1}$ yields that if $C_r^q \neq 0$ then

$$b_0(C_r^q) = \sum_{i=1}^{r} \epsilon_i. \tag{6.1.2}$$

Notice that if $C_r^q = 0$ then $b_0(C_r^q) = -\infty$.

Apply Corollary 5.4 (see also Remark 5.3) to the complex $C_\bullet^q$ to conclude that

$$[\text{H}_\mathfrak{m}^0(\text{Sym}_q(I(\delta)))]_p = 0 \quad \text{for all } p \text{ with } b_0(C_d^q) + a(R) < p. \tag{6.1.3}$$

Recall that the index q in (6.1.3) is arbitrary. Apply (6.1.1), (6.1.3), and (6.1.2) to conclude that

$$\mathcal{A}_{(p,*)} = 0 \quad \text{for all } p \text{ with } \sum_{i=1}^{d} \epsilon_i - d < p.$$

Assertion (ai) has been established.

Apply Theorem 5.6 (see also Remark 5.7) to the complex $C_\bullet^q$ to conclude that every minimal homogeneous generator α of $\text{H}_\mathfrak{m}^0(\text{Sym}_q(I(\delta)))$ satisfies

$$\deg \alpha \leq b_0(C_{d-1}^q) - d + 1.$$

Once again, (6.1.1) and (6.1.2) yield that

$$b_0(\mathcal{A}_{(*,q)}) \leq \sum_{i=1}^{d-1} \epsilon_i - d + 1.$$

The parameter q remains arbitrary. Every minimal bihomogeneous generator of $\mathcal{A}$ is a minimal homogeneous generator of $\mathcal{A}_{(*,q)}$, for some q. We conclude that if $\mathcal{A}_{(p,q)}$ contains a minimal bihomogeneous generator of $\mathcal{A}$, then $p \leq \sum_{i=1}^{d-1} \epsilon_i - d + 1$, and this establishes assertion (aii).

(b) Let φ be an $n \times n$ homogeneous alternating matrix which minimally presents I, and, for each nonnegative integer q, let

$$\mathcal{D}_\bullet^q(\varphi) : 0 \to \mathcal{D}_{n-1}^q \to \mathcal{D}_{n-2}^q \to \cdots \to \mathcal{D}_1^q \to \mathcal{D}_0^q \to 0$$

be the complex from [Kustin and Ulrich 1992, Definition 2.15 and Figure 4.7] that is associated to the alternating matrix φ. The zeroth homology of $\mathcal{D}_\bullet^q(\varphi)$ is $\text{Sym}_q(I(\delta))$; see [Kustin and Ulrich 1992, 4.13.b]. The hypothesis that every entry

of φ has degree D yields that

$$
\mathscr{D}_r^q = \begin{cases}
K_{q-r,r} = R(-rD)^{\beta_r^q} & \text{if } r \leq \min\{q, n-1\}, \\
Q_q = R\big(-(r-1)D - \frac{1}{2}(n-r+1)D\big) & \text{if } r = q+1 \leq n-1, \ q \text{ odd}, \\
0 & \text{if } r = q+1, \ q \text{ even}, \\
0 & \text{if } \min\{q+2, n\} \leq r,
\end{cases}
\tag{6.1.4}
$$

for some nonzero Betti numbers β_r^q. Indeed, the graded R-module $\mathrm{Sym}_q(I(\delta))$ is generated in degree zero; moreover, each map $K_{a,b} \to K_{a+1,b-1}$ is linear in the entries of φ (see [Kustin and Ulrich 1992, 2.15.d]) and the map $Q_q \to K_{0,q}$ may be represented by a column vector whose entries consist of the Pfaffians of the principal $(n-r+1) \times (n-r+1)$ alternating submatrices of φ (see [Kustin and Ulrich 1992, 2.15.c and 2.15.f]).

Notice from (6.1.4) that

$$
\text{if } \mathscr{D}_r^q = Q_q, \text{ then } \mathscr{D}_{r+1}^q = 0.
\tag{6.1.5}
$$

Notice also that

$$
\max_{\{q\,|\,\mathscr{D}_d^q \neq 0\}} b_0(\mathscr{D}_d^q) \leq \begin{cases}
dD & \text{if } d \text{ is odd}, \\
(d-1)D + \frac{1}{2}(n-d+1)D & \text{if } d \text{ is even},
\end{cases}
\tag{6.1.6}
$$

$$
\max_{\{q\,|\,\mathscr{D}_d^q \neq 0\}} b_0(\mathscr{D}_{d-1}^q) \leq (d-1)D;
\tag{6.1.7}
$$

one uses (6.1.5) in order to see (6.1.7) because the condition $\mathscr{D}_d^q \neq 0$ forces $\mathscr{D}_{d-1}^q = K_{q-d+1,d-1}$.

By hypothesis the ideal I satisfies the condition G_d; so I satisfies the hypothesis (6.3.1) of Lemma 6.3 for $s = d-1$; and therefore, the complex $(\mathscr{D}_\bullet^q(\varphi))_\mathfrak{p}$ is acyclic for every $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R_\mathfrak{p} \leq d-1$ and every nonnegative integer q. The hypotheses of Corollary 5.4 and Theorem 5.6 are satisfied by the complexes $\mathscr{D}_\bullet^q(\varphi)$. We deduce that either $\mathscr{D}_d^q$ and $\mathrm{H}_\mathfrak{m}^0(\mathrm{Sym}_q(I(\delta)))$ are both zero, or else $\mathscr{D}_d^q \neq 0$, $[\mathrm{H}_\mathfrak{m}^0(\mathrm{Sym}_q(I(\delta)))]_p = 0$ for all p with $b_0(\mathscr{D}_d^q) + a(R) < p$, and every minimal homogeneous generator α of $\mathrm{H}_\mathfrak{m}^0(\mathrm{Sym}_q(I(\delta)))$ satisfies

$$
\deg \alpha \leq b_0(\mathscr{D}_{d-1}^q) - d + 1.
$$

In the case $\mathscr{D}_d^q \neq 0$ we apply (6.1.1), (6.1.6), and (6.1.7) to conclude that

$$
\mathscr{A}_{(p,q)} = 0 \quad \text{for} \quad \begin{cases}
dD - d < p & \text{if } d \text{ is odd}, \\
(d-1)D + \frac{1}{2}(n-d+1)D - d < p & \text{if } d \text{ is even},
\end{cases}
$$

and every minimal homogeneous generator α of $\mathscr{A}_{(*,q)}$ satisfies

$$
\deg \alpha \leq (d-1)D - d + 1.
$$

Keep in mind that if $\alpha \in \mathcal{A}_{(p,q)}$ is a minimal bihomogeneous generator of $\mathcal{A}$, then α is also a minimal homogeneous generator of $\mathcal{A}_{(*,q)}$. Assertions (bi) and (bii) have been established. $\qquad\square$

Two small facts were used in the proof of Theorem 6.1 to guarantee that the complexes $C_\bullet^q$ and $\mathcal{D}_\bullet^q(\varphi)$ satisfy the hypotheses of Corollary 5.4 and Theorem 5.6. We prove these facts now.

Observation 6.2. Let R be a Cohen–Macaulay local ring and I be an ideal of positive height. Assume that $\mu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p} + 1$ for each prime ideal $\mathfrak{p} \in V(I)$. If $\mathrm{Sym}(I) \cong P/J$ is any homogeneous presentation, where P is a standard graded polynomial ring over R in n variables and J is a homogeneous ideal of P, then J has height $n - 1$.

Proof. One has

$$\mathrm{ht}\, J = \dim P - \dim P/J = \dim R + n - \dim \mathrm{Sym}(I) = n - 1,$$

where the first equality holds because R is Cohen–Macaulay local and J is homogeneous and the last equality follows from the dimension formula for symmetric algebras; see [Huneke and Rossi 1986, 2.6]. $\qquad\square$

Lemma 6.3. *Let R be a Cohen–Macaulay ring and I be a perfect Gorenstein ideal of height three. Let φ be an alternating presentation matrix for I, $\{\mathcal{D}_\bullet^q(\varphi)\}$ be the family of complexes in [Kustin and Ulrich 1992] which is associated to φ, and s be an integer. If*

$$\mu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p} + 1 \quad \textit{for each prime ideal } \mathfrak{p} \in V(I) \textit{ with } \dim R_\mathfrak{p} \leq s, \quad (6.3.1)$$

then the complex $(\mathcal{D}_\bullet^q(\varphi))_\mathfrak{p}$ is acyclic for every $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R_\mathfrak{p} \leq s$ and every nonnegative integer q.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R_\mathfrak{p} \leq s$. Notice that $(\mathcal{D}_\bullet^q(\varphi))_\mathfrak{p} = \mathcal{D}_\bullet^q(\varphi_\mathfrak{p})$. This complex has length at most $n - 1$, where n is the size of the matrix $\varphi_\mathfrak{p}$ according to [Kustin and Ulrich 1992, 4.3.d]; see also (6.1.4). Moreover, by assumption (6.3.1), the matrix $\varphi_\mathfrak{p}$ satisfies condition WMC_1, and therefore WPC_1, see [Kustin and Ulrich 1992, 5.8, 5.9, 5.11.a]. Now [Kustin and Ulrich 1992, 6.2] shows that the complex $\mathcal{D}_\bullet^q(\varphi_\mathfrak{p})$ is acyclic. $\qquad\square$

Theorem 6.1 is particularly interesting when I is a linearly presented height three Gorenstein ideal. One quickly deduces that such ideals are of fiber type; see 3.4 for the definition. This is a new result and a main ingredient in the proof of Theorem 9.1.

Corollary 6.4. *Adopt the setting of Theorem 6.1.(b) with $D = 1$. Then the ideal I is of fiber type. Furthermore, $\mathcal{A}$ is annihilated by $\mathfrak{m}$ when d is odd and by $\mathfrak{m}^{(n-d+1)/2}$ when d is even.*

Proof. Theorem 6.1.(bii) shows that the ideal $\mathscr{A}$ of $\mathrm{Sym}(I(\delta))$ is generated by $\mathscr{A}_{(0,\star)}$, which means that I is of fiber type; see 3.4. The assertion about which power of $\mathfrak{m}$ annihilates $\mathscr{A}$ is explicitly stated in Theorem 6.1.(bi).   □

7. The defining ideal of Rees rings up to radical

In this section we mainly work under the assumption that the ideal I of Data 3.1 is of linear type on the punctured spectrum of R and that a sufficiently high symmetric power $\mathrm{Sym}_t(I(\delta))$ has an approximate free resolution that is linear for the first d steps. With these hypotheses alone we prove, somewhat surprisingly, that the defining ideal $\mathscr{J}$ of the Rees ring has the expected form up to radical; see 3.6 for the definition of expected form. We also describe the defining ideal $I(X)$ of the variety X up to radical and we deduce that $I_d(B')$ has maximal possible height, where B' is the Jacobian dual of the linear part of a minimal homogeneous presentation matrix of I; see 3.6 for the definition of Jacobian dual. This is done in Theorem 7.2. The computation of the height of $I_d(B')$ is relevant for the proof of Theorem 9.1 because it ensures that the complexes constructed in [Kustin et al. 2017b] are free resolutions of $I_d(B)$ and of $I_d(B) + C(\varphi)$, where $B = B'$ is the full Jacobian dual in this case and $C(\varphi)$ is the content ideal of Definition 8.1. A step towards the proof of Theorem 7.2 is Lemma 7.1, where we show that, up to radical, $\mathscr{A}$ is the socle of the symmetric algebra as an R-module and I is of fiber type; see 3.4 for the definition of fiber type. We deduce this statement from the other parts of the same lemma, which say that even locally on the punctured spectrum of the polynomial ring S, the ideal $\mathscr{A}$ belongs to the socle and I is of fiber type. These facts are proved using the methods of [Kustin et al. 2015], most notably Corollary 5.5 of the present paper.

The assumption that I is of linear type on the punctured spectrum of R is rather natural; see 3.7. It is satisfied for instance if I is perfect of height two or Gorenstein of height three and I satisfies G_d. If such an ideal is, in addition, linearly presented, then the hypothesis about an approximate resolution $C_\bullet$ of $\mathrm{Sym}_t(I(\delta))$ holds as well; see Remark 7.4.

The index t of the symmetric power $\mathrm{Sym}_t(I(\delta))$ in Lemma 7.1 and Theorem 7.2 can be any integer greater than or equal to the relation type of I. The *relation type* of an ideal I as in Data 3.1 is the largest integer q such that $\mathscr{A}_{(*,q)}$ contains a minimal bihomogeneous generator of $\mathscr{A}$. We recall the bihomogeneous ideal $\mathscr{A} = \mathscr{J}/\mathscr{L}$ of $\mathrm{Sym}(I(\delta))$ as well as the bihomogeneous ideals $\mathscr{L}$ and $\mathscr{J}$ of S that define the symmetric algebra and the Rees ring, respectively; see 3.2 and 3.4.

Lemma 7.1. *Adopt* Data 3.1. *Suppose further that I is of linear type on the punctured spectrum of R and that for some index t greater than or equal to the relation type of I there exists a homogeneous complex*

$$C_\bullet : \cdots \to C_1 \to C_0 \to 0$$

of finitely generated graded free R-modules such that

(1) $H_0(C_\bullet) = \mathrm{Sym}_t(I(\delta))$,

(2) *the subcomplex $C_d \to \cdots \to C_0 \to 0$ of $C_\bullet$ is linear (that is $C_j = R(-j)^{\beta_j}$ for $0 \le j \le d$), and*

(3) $\dim H_j(C_\bullet) \le j$ *for* $1 \le j \le d-1$.

Let $\mathfrak{M}$ be the ideal $(x_1, \ldots, x_d, T_1, \ldots, T_n)$ of S. Then the following statements hold:

(a) *On $\mathrm{Spec}(S) \setminus \{\mathfrak{M}\}$, the ideal $\mathscr{A}$ of $\mathrm{Sym}(I(\delta))$ is annihilated by $\mathfrak{m}$; that is,*

$$(\mathfrak{m}\mathscr{A})_{\mathfrak{P}} = 0 \quad \text{for every } \mathfrak{P} \in \mathrm{Spec}(S) \setminus \{\mathfrak{M}\}.$$

(b) *On $\mathrm{Spec}(S) \setminus \{\mathfrak{M}\}$, I is of fiber type; that is,*

$$\mathscr{A}_{\mathfrak{P}} = I(X) \cdot \mathrm{Sym}(I(\delta))_{\mathfrak{P}} \quad \text{for every } \mathfrak{P} \in \mathrm{Spec}(S) \setminus \{\mathfrak{M}\}.$$

(c) *The ideal $\mathscr{J}$ of S satisfies*

$$\mathscr{J} = \sqrt{\mathscr{L} :_S \mathfrak{m}} = \sqrt{(\mathscr{L}, I(X))}.$$

Proof. Recall that $\mathscr{A} = H^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta)))$ since I is of linear type on the punctured spectrum of R; see (3.7.1). In particular,

$$\mathscr{A}_{(\star, t)} = H^0_{\mathfrak{m}}(\mathrm{Sym}_t(I(\delta))).$$

Apply Corollary 5.5 to the R-module $\mathrm{Sym}_t(I(\delta))$ to conclude that

$$\mathscr{A}_{(\star, t)} = \mathscr{A}_{(0, t)}.$$

Therefore

$$\mathscr{A}_{(\star, \ge t)} = \mathscr{A}_{(0, \ge t)}$$

because, by the definition of relation type, $\mathscr{A}_{(\star, t)}$ generates $\mathscr{A}_{(\star, \ge t)}$ as a module over $\mathrm{Sym}(I(\delta))$, hence over $\mathrm{Sym}(I(\delta))_{(0, \star)}$. It follows that $(T_1, \ldots, T_n)^t \mathscr{A} \subset \mathscr{A}_{(0, \ge t)}$, which gives

$$(T_1, \ldots, T_n)^t \mathscr{A}_{(>0, \star)} = 0.$$

Since, moreover, $\mathscr{A}_{(>0, \star)}$ vanishes locally on $\mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$, we deduce that

$$\mathscr{A}_{(>0, \star)} = 0 \quad \text{locally on } \mathrm{Spec}(S) \setminus \{\mathfrak{M}\}.$$

This completes the proof of (a) because $\mathfrak{m}\mathscr{A} \subset \mathscr{A}_{(>0, \star)}$, and of (b) because $\mathscr{A} = I(X) + \mathscr{A}_{(>0, \star)}$. As for (c), we recall the obvious inclusion $\mathscr{L} :_S \mathfrak{m} \subset \mathscr{J}$. Part (a) shows that if $\mathfrak{P} \in \mathrm{Spec}(S) \setminus \{\mathfrak{M}\}$ contains $\mathscr{L} :_S \mathfrak{m}$ then $\mathfrak{P}$ contains $\mathscr{J}$. Since $\mathfrak{M}$ contains $\mathscr{J}$ anyway, we deduce that $\mathscr{J} \subset \sqrt{\mathscr{L} :_S \mathfrak{m}}$. The same argument, with part (a) replaced by (b), shows that $\mathscr{J} = \sqrt{(\mathscr{L}, I(X))}$. $\qquad\square$

Theorem 7.2. *Retain all of the notation and hypotheses of Lemma 7.1. Let φ be a minimal homogeneous presentation matrix of $g_1, \ldots, g_n$, φ' be the submatrix of φ that consists of the linear columns of φ, B be a homogeneous Jacobian dual of φ, and B' be the Jacobian dual of φ' in the sense of 3.6. Then the following statements hold*:

(a) *The ideal $I(X)$ of T which defines the variety X satisfies*

$$I(X) = \sqrt{I_d(B')}.$$

(b) $\mathrm{ht}(I_d(B')) = n - \dim A$.

(c) *The ideal $\mathcal{J}$ of S which defines the Rees ring $\mathcal{R}(I)$ satisfies*

$$\mathcal{J} = \sqrt{(\mathcal{L}, I_d(B'))} = \sqrt{(\mathcal{L}, I_d(B))}.$$

Proof. Part (b) is an immediate consequence of (a). We prove items (a) and (c) simultaneously. We first notice that B' is a submatrix of B, the ideal $I_d(B)$ of S is bihomogeneous, the two ideals $(\mathfrak{m}, I_d(B))$ and $(\mathfrak{m}, I_d(B'))$ of S are equal, and the two ideals $I_d(B) \cap T$ and $I_d(B')$ of T are equal.

From Lemma 7.1.(c) we know that

$$\mathcal{J} = \sqrt{\mathcal{L} :_S \mathfrak{m}}.$$

The ideal $\mathcal{L} :_S \mathfrak{m}$ is the annihilator of the S-module $M = \mathfrak{m}S/\mathcal{L}$. The ideal $\mathfrak{m}S$ is generated by the entries of $\underline{x}$ and the ideal $\mathcal{L}$ is generated by the entries of $\underline{x} \cdot B$. It follows that M is presented by $[\Pi \mid B]$, where Π is a presentation matrix of $\underline{x}$ with entries in $\mathfrak{m}$. Therefore

$$\mathcal{L} :_S \mathfrak{m} = \mathrm{ann}_S M \subset \sqrt{\mathrm{Fitt}_0(M)} \subset \sqrt{(\mathfrak{m}, I_d(B))}.$$

It follows that

$$\mathcal{J} \subset \sqrt{(\mathfrak{m}, I_d(B))} = \sqrt{(\mathfrak{m}, I_d(B'))}.$$

Intersecting with T we obtain

$$I(X) = \mathcal{J} \cap T \subset \sqrt{(\mathfrak{m}, I_d(B'))} \cap T = \sqrt{(\mathfrak{m}, I_d(B')) \cap T} = \sqrt{I_d(B')},$$

where the last two radicals are taken in the ring T. This proves part (a).

From Lemma 7.1.(c) we also know that

$$\mathcal{J} = \sqrt{(\mathcal{L}, I(X))},$$

and then by part (a)

$$\mathcal{J} = \sqrt{(\mathcal{L}, I_d(B'))}.$$

Finally, recall the inclusions $I_d(B') \subset I_d(B) \subset \mathcal{J}$, and the proof of part (c) is complete. $\qquad\square$

Remark 7.3. Retain the notation and the hypotheses of Theorem 7.2, and recall the definition of analytic spread from 3.3. If $d < n$ then $\ell(I) < n$; and if $\ell(I) < n$ then φ' has at least $n - 1$ columns.

Proof. Write s for the number of columns of φ', which is also the number of columns of B'. One has

$$n - d \le n - \ell(I) = n - \dim A = \operatorname{ht} I_d(B') \le \max\{s - d + 1, 0\},$$

where the first inequality holds by (3.3.2), the first equality follows from 3.3, the second equality is Theorem 7.2.(b), and the last inequality holds by the Eagon–Northcott bound on the height of determinantal ideals. These inequalities show that if $0 < n - d$ then $0 < n - \ell(I)$ and if $0 < n - \ell(I)$ then $n - d \le s - d + 1$. □

Remark 7.4. Adopt Data 3.1. Further assume that I is perfect of height two or Gorenstein of height three, I is linearly presented, and I satisfies G_d. Then the hypotheses of Lemma 7.1 and Theorem 7.2 are satisfied.

Proof. We know from (3.7.1) and (3.7.3) that the ideal I is of linear type on the punctured spectrum.

We now establish the existence of a complex $C_\bullet$. In the first case, when I is perfect of height two, we can take $C_\bullet$ to be the complex $C_\bullet^t = \mathbb{K}_{\bullet(\star,t)}$ defined in the proof of Theorem 6.1.(a). This is a linear complex of finitely generated graded free R-modules which is acyclic on the punctured spectrum and has $\operatorname{Sym}_t(I(\delta))$ as zeroth homology. These facts are shown in the proof of Theorem 6.1.(a), most notably in (6.1.2).

In the second case, when I is Gorenstein of height three, we take $C_\bullet$ to be the complex $\mathscr{D}_\bullet^t(\varphi)$ used in the proof of Theorem 6.1.(b), with the additional restriction that $d \le t$. This is a complex of finitely generated graded free R-modules which has $\operatorname{Sym}_t(I(\delta))$ as zeroth homology. Moreover, this complex is acyclic on the punctured spectrum, see Lemma 6.3, and it is linear for the first d steps, see (6.1.4). □

We end this section by recording a fact contained in the proof of Theorem 6.1.(b), that may well be of independent interest. Recall the functions a_i and reg from 2.2 and 2.4.

Proposition 7.5. *Adopt Data 3.1. Further assume that I is a linearly presented Gorenstein ideal of height three. Let q be a nonnegative integer. If $\mu(I_\mathfrak{p}) \le \dim R_\mathfrak{p} + 1$ for each prime ideal $\mathfrak{p} \in V(I)$ with $\dim R_\mathfrak{p} \le d - 2$, then*

$$a_i(\operatorname{Sym}_q(I)) \le \begin{cases} q\delta - i & \text{if } q \text{ is even or } q \ne d - i - 1, \\ q\delta + \frac{1}{2}(n - q) - 1 - i & \text{if } q \text{ is odd and } q = d - i - 1. \end{cases}$$

In particular,

$$\operatorname{reg} \operatorname{Sym}_q(I) = q\delta \qquad \text{if } q \text{ is even or } d \le q,$$
$$\operatorname{reg} \operatorname{Sym}_q(I) \le q\delta + \frac{1}{2}(n - q) - 1 \quad \text{if } q \text{ is odd and } q \le d - 1.$$

Proof. Apply Proposition 5.2 with $M = \mathrm{Sym}_q(I(\delta))$ and $C_\bullet$ the complex $\mathscr{D}_\bullet^q(\varphi)$ of (6.1.4). Lemma 6.3 guarantees that $\dim H_j(\mathscr{D}_\bullet^q(\varphi)) \leq 1$ for every j with $1 \leq j$. Thus Proposition 5.2.(a) (see also Remark 5.3) gives that $H_\mathfrak{m}^i(\mathrm{Sym}_q(I(\delta)))$ is a graded subquotient of $H_\mathfrak{m}^d(\mathscr{D}_{d-i}^q)$ for $0 \leq i \leq d$. From (6.1.4) we see that either $\mathscr{D}_{d-i}^q = 0$ or else

$$\mathscr{D}_{d-i}^q \cong \begin{cases} R(-d+i)^{\beta_{d-i}^q} & \text{if } q \text{ is even or } q \neq d-i-1, \\ R\left(-d+i+1-\tfrac{1}{2}(n-q)\right) & \text{if } q \text{ is odd and } q = d-i-1. \end{cases}$$

Therefore the top degree of $H_\mathfrak{m}^d(\mathscr{D}_{d-i}^q)$ is at most

$$d-i-d = -i \qquad \text{if } q \text{ is even or } q \neq d-i-1,$$
$$d-i-1+\tfrac{1}{2}(n-q)-d = -i-1+\tfrac{1}{2}(n-q) \quad \text{if } q \text{ is odd and } q = d-i-1.$$

Notice the second case cannot occur if q is even or $d \leq q$.

On the other hand, the fact that the R-module $\mathrm{Sym}_q(I(\delta))$ is generated in degree 0 implies that the regularity of $\mathrm{Sym}_q(I(\delta))$ is nonnegative. Now the assertions follow. $\qquad\square$

8. The candidate ideal $C(\varphi)$

In the present section we exhibit the candidate for the defining ideal of the variety X. Most of this material is taken from [Kustin et al. 2017b].

Definition 8.1. Let k be a field and n and d be positive integers. For each $n \times n$ alternating matrix φ with linear entries from the polynomial ring $R = k[x_1, \ldots, x_d]$, define an ideal $C(\varphi)$ in the polynomial ring $T = k[T_1, \ldots, T_n]$ as follows: Let B be the $d \times n$ matrix with linear entries from T such that the matrix equation

$$[T_1, \ldots, T_n] \cdot \varphi = [x_1, \ldots, x_d] \cdot B \tag{8.1.1}$$

holds. Consider the $(n+d) \times (n+d)$ alternating matrix

$$\mathfrak{B} = \begin{bmatrix} \varphi & -B^{\mathrm{t}} \\ B & 0 \end{bmatrix},$$

with entries in the polynomial ring $S = k[x_1, \ldots, x_d, T_1, \ldots, T_n]$. Let F_i be $(-1)^{n+d-i}$ times the Pfaffian of $\mathfrak{B}$ with row and column i removed. View each F_i as a polynomial in $T[x_1, \ldots, x_d]$ and let

$$C(\varphi) = c_T(F_{n+d})$$

be the content ideal of F_{n+d} in T.

A description of $C(\varphi)$ which is almost coordinate-free is given in [Kustin et al. 2017b]. The description depends on the choice of a direct sum decomposition of the degree one component of the ring T into $V_1 \oplus V_2$ where V_1 has dimension one.

(The almost coordinate-free description of $C(\varphi)$ depends on the choice of direct sum decomposition; however every decomposition gives rise to the same ideal $C(\varphi)$; see also Proposition 8.4.(b) below).

For the convenience of the reader, we provide direct proofs of the more immediate properties of the ideal $C(\varphi)$; most notably we describe a generating set of this ideal in Proposition 8.5. The more difficult results about depth, unmixedness, Hilbert series, and resolutions, however, are merely restated from [Kustin et al. 2017b]; see Theorem 8.7.

Lemma 8.2. *Retain the setting of Definition 8.1. There exists a polynomial h in S so that the equality of row vectors*

$$[F_1, \ldots, F_{n+d}] = h \cdot [T_1, \ldots, T_n, -x_1, \ldots, -x_d]$$

obtains.

Proof. We may assume that the row vector $\underline{F} = [F_1, \ldots, F_{n+d}]$ is not zero. Then $\mathfrak{B}$ has rank at least $n+d-1$. The alternating property of φ and equality (8.1.1) imply

$$0 = \underline{T} \cdot \varphi \cdot \underline{T}^{\mathrm{t}} = \underline{x} \cdot B \cdot \underline{T}^{\mathrm{t}}.$$

Since the matrix B has only entries in the polynomial ring T, we conclude that

$$B \cdot \underline{T}^{\mathrm{t}} = 0 \quad \text{and} \quad \underline{T} \cdot B^{\mathrm{t}} = 0.$$

Hence

$$[T_1, \ldots, T_n, -x_1, \ldots, -x_d] \cdot \mathfrak{B} = 0.$$

Since

$$[F_1, \ldots, F_{n+d}] \cdot \mathfrak{B} = 0$$

and $\mathfrak{B}$ has almost maximal rank, there exists an element h in the quotient field of S such that

$$[F_1, \ldots, F_{n+d}] = h \cdot [T_1, \ldots, T_n, -x_1, \ldots, -x_d].$$

The element h is necessarily in S because the ideal $(T_1, \ldots, T_n, x_1, \ldots, x_d)$ of S has grade at least two. □

Lemma 8.2 shows that either h and the submaximal Pfaffians of $\mathfrak{B}$ all vanish, or else these elements are all nonzero. In the latter case, $n+d$ has to be odd. Moreover, regarding S as standard bigraded with $\deg x_j = (1, 0)$ and $\deg T_i = (0, 1)$, we see that $\mathfrak{B}$ is the matrix of a bihomogeneous linear map. Thus the Pfaffians of this matrix are bihomogeneous, hence h is bihomogeneous, and computing bidegrees one sees that

$$\deg h = \left(\frac{n-d-1}{2}, d-1 \right).$$

In particular, if $h \neq 0$ then $d < n$. The next remark is now immediate.

Remark 8.3. Retain the setting of Definition 8.1.

(a) The ideal $C(\varphi)$ of T is generated by homogeneous forms of degree $d - 1$.

(b) If $n \leq d$ or $n + d$ is even, then $C(\varphi) = 0$.

Proposition 8.4 shows that the ideal $C(\varphi)$ can be defined using any submaximal Pfaffian of $\mathfrak{B}$; it also relates $C(\varphi)$ to the socle modulo the ideal $I_d(B)$ of T.

Proposition 8.4. *Retain the setting of Definition 8.1.*

(a) $c_T(F_i) = T_i \cdot c_T(F_{n+j}) = T_i \cdot C(\varphi)$ *for every* $1 \leq i \leq n$ *and* $1 \leq j \leq d$.

(b) $c_T(F_{n+j}) = C(\varphi)$ *for every* $1 \leq j \leq d$.

(c) $C(\varphi) \subset I_d(B) :_T (T_1, \ldots, T_n)$; *in other words, the image of* $C(\varphi)$ *is contained in the socle of the standard graded k-algebra* $T/I_d(B)$.

Proof. Lemma 8.2 shows that

$$c_T(F_i) = c_T(h \cdot T_i) = T_i \cdot c_T(h) \quad \text{for } 1 \leq i \leq n,$$
$$c_T(F_{n+j}) = c_T(h \cdot x_j) = c_T(h) \qquad \text{for } 1 \leq j \leq d.$$

This proves (a). Part (b) is an immediate consequence of (a).

Finally, the definition of F_i shows that in the range $1 \leq i \leq n$,

$$F_i \in I_d(B) \cdot S, \quad \text{and therefore} \quad c_T(F_i) \subset I_d(B).$$

Now apply (a) to deduce (c). $\qquad\qquad\square$

In light of Remark 8.3.(b) we now assume that $d < n$ and $n + d$ is odd. Fix an integer i with $1 \leq i \leq n$. Let $J = \{j_1, \ldots, j_d\}$ be a subset of $\{1, \ldots, n\} \setminus \{i\}$ with $j_1 < \cdots < j_s < i < j_{s+1} < \cdots < j_d$. Write $|J| = j_1 + \cdots + j_d - d + s$. Let $\mathrm{Pf}_J(\varphi_i)$ be the Pfaffian of the matrix obtained from φ by deleting rows and columns $i, j_1, \ldots, j_d$ and denote by $\Delta_J(B)$ the determinant of the $d \times d$ matrix consisting of columns $j_1, \ldots, j_d$ of B.

Let $k[y_1, \ldots, y_d]$ be another polynomial ring in d variables that acts on the polynomial ring $T[x_1, \ldots, x_d]$ by contraction and let $\circ$ denote the contraction operation.

For M a monomial in $k[y_1, \ldots, y_d]$ of degree $\frac{1}{2}(n - d - 1)$ we define

$$f_M = \frac{1}{T_i} \sum_J (-1)^{|J|} (M \circ \mathrm{Pf}_J(\varphi_i)) \cdot \Delta_J(B), \qquad (8.4.1)$$

where J ranges over all subsets of $\{1, \ldots, n\} \setminus \{i\}$ of cardinality d. Notice that, since $\mathrm{Pf}_J(\varphi_i) \in k[x_1, \ldots, x_d]$ is a homogeneous polynomial of degree $\frac{1}{2}(n - d - 1)$,

the element $M \circ \mathrm{Pf}_J(\varphi_i) \in k$ is simply the coefficient in this polynomial of the monomial in $k[x_1, \ldots, x_d]$ corresponding to M.

Similarly, fix an integer j with $1 \le j \le d$. Let $J = \{j_1, \ldots, j_{d-1}\}$ be a subset of $\{1, \ldots, n\}$ and set $\|J\| = j_1 + \cdots + j_{d-1}$. Write $\mathrm{Pf}_J(\varphi)$ for the Pfaffian of the matrix obtained from φ by deleting rows and columns $j_1, \ldots, j_{d-1}$ and denote by $\Delta_J(B_j)$ the determinant of the matrix consisting of columns $j_1, \ldots, j_{d-1}$ of B with row j removed. Let $M \in k[y_1, \ldots, y_d]$ be a monomial of degree $\frac{1}{2}(n - d + 1)$ that is divisible by y_j. We define

$$h_M = \sum_J (-1)^{\|J\|}(M \circ \mathrm{Pf}_J(\varphi)) \cdot \Delta_J(B_j), \qquad (8.4.2)$$

where J ranges over all subsets of $\{1, \ldots, n\}$ of cardinality $d - 1$. Notice that $M \circ \mathrm{Pf}_J(\varphi) \in k$ is the coefficient in $\mathrm{Pf}_J(\varphi)$ of the monomial corresponding to M.

Proposition 8.5. *Adopt the setting of Definition 8.1 with $d < n$ and $n + d$ odd. The ideal $C(\varphi)$ of T is generated by the elements f_M of (8.4.1) for a fixed i, where M ranges over all monomials in $k[y_1, \ldots, y_d]$ of degree $\frac{1}{2}(n - d - 1)$; it is also generated by the elements h_M of (8.4.2) for a fixed j, where M ranges over all monomials of degree $\frac{1}{2}(n - d + 1)$ that are divisible by y_j.*

Proof. We prove the first claim. From Proposition 8.4.(a) we know that $C(\varphi) = 1/T_i \cdot c_T(F_i)$, for $1 \le i \le n$. Expanding the Pfaffian F_i by maximal minors of B, one obtains

$$F_i = \pm \sum_J (-1)^{|J|} \mathrm{Pf}_J(\varphi_i) \cdot \Delta_J(B),$$

where, again, J ranges over all subsets of $\{1, \ldots, n\} \setminus \{i\}$ of cardinality d; see for instance [Iarrobino and Kanev 1999, Lemma B.1]. Regarded as polynomials in $T[x_1, \ldots, x_d]$, the elements $\Delta_J(B)$ are constants and F_i is homogeneous of degree $\frac{1}{2}(n - d - 1)$. Therefore, the content ideal $c_T(F_i)$ is generated by the elements

$$M \circ F_i = \pm \sum_J (-1)^{|J|}(M \circ \mathrm{Pf}_J(\varphi_i)) \cdot \Delta_J(B),$$

where M ranges over all monomials in $k[y_1, \ldots, y_d]$ of degree $\frac{1}{2}(n - d - 1)$.

To prove the second claim we recall that $C(\varphi) = c_T(F_{n+j})$, for $1 \le j \le d$; see Proposition 8.4.(b). Now expand the Pfaffian F_{n+j} along the last $d - 1$ rows and use the fact that the monomials in the support of $F_{n+j} \in T[x_1, \ldots, x_d]$ have degree $\frac{1}{2}(n - d + 1)$ and are divisible by x_j; see Lemma 8.2 and the discussion following it. □

We illustrate the propositions above with an application to the ideal $I_d(B) + C(\varphi)$ in the case $n = d + 1$; see [Johnson 1997, 2.10] for a similar result.

Example 8.6. Adopt the setting of Definition 8.1 with $n = d + 1$. Write Δ_i for the maximal minor of B obtained by deleting column i. The first part of Proposition 8.5 (or Proposition 8.4.(a) and the fact that $F_i = \pm \Delta_i$) show that $C(\varphi)$ is the principal ideal generated by Δ_i / T_i, for any i with $1 \le i \le n$. Therefore $I_d(B) + C(\varphi)$ is the principal ideal generated by Δ_i / T_i.

The next theorem is [Kustin et al. 2017b, 8.3].

Theorem 8.7. *Adopt the setting of Definition 8.1 with $3 \le d < n$ and n odd. If $n - d \le \operatorname{ht} I_d(B)$, then the following statements hold:*

(a) *The ring $T / (I_d(B) + C(\varphi))$ is Cohen–Macaulay on the punctured spectrum and*

$$\operatorname{depth}(T/(I_d(B) + C(\varphi))) = \begin{cases} 1 & \text{if } d \text{ is odd,} \\ 2 & \text{if } d \text{ is even and } d + 3 \le n, \\ n - 1 & \text{if } d + 1 = n. \end{cases}$$

(b) *The unmixed part of $I_d(B)$ is equal to $I_d(B)^{\mathrm{unm}} = I_d(B) + C(\varphi)$.*

(c) *If d is odd, then the ideal $I_d(B)$ is unmixed; furthermore, $I_d(B)$ has a linear free resolution and $\operatorname{reg}(T/I_d(B)) = d - 1$.*

(d) *The Hilbert series of $T / (I_d(B) + C(\varphi))$ is*

$$\frac{1}{(1-z)^d} \left(\sum_{\ell=0}^{d-2} \binom{\ell + n - d - 1}{n - d - 1} z^\ell + \sum_{\ell=0}^{n-d-2} (-1)^{\ell+d+1} \binom{\ell + d - 1}{d - 1} z^{\ell + 2d - n} \right)$$

$$+ (-1)^d \sum_{j \le \lceil \frac{n-d-3}{2} \rceil} \binom{j + d - 1}{d - 1} z^{2j + 2d - n}.$$

(e) *The multiplicity of $T / (I_d(B) + C(\varphi))$ is*

$$\sum_{i=0}^{\lfloor \frac{n-d}{2} \rfloor} \binom{n - 2 - 2i}{d - 2}.$$

Remark. The conclusions of assertion (c) do not hold when d is even: the ideal $I_d(B)$ can be mixed and the minimal homogeneous resolution of $I_d(B)^{\mathrm{unm}}$ may not be linear; see, for example, [Kustin et al. 2017b, (4.4.5)].

Corollary 8.8. *Adopt Data 3.1. Further assume that I is a linearly presented Gorenstein ideal of height three. Let φ be a minimal homogeneous alternating presentation matrix for $g_1, \ldots, g_n$, B be the Jacobian dual of φ in the sense of 3.6, and $C(\varphi)$ be the content ideal of Definition 8.1. Then*

$$I_d(B) + C(\varphi) \subset I_d(B) :_T (T_1, \ldots, T_n) \subset I(X).$$

In particular, $I_d(B) + C(\varphi)$ is contained in the defining ideal $\mathcal{J}$ of the Rees ring of I.

Proof. The first containment follows from Proposition 8.4.(c). Since I is linearly presented, $I_d(B)$ is an ideal of T and therefore $I_d(B) \subset I(X)$ according to (3.6.2) and (3.3.1). The second containment follows because $I(X)$ is a prime ideal containing $I_d(B)$ but not $(T_1, \ldots, T_n)$. □

9. Defining equations of blowup algebras of linearly presented height three Gorenstein ideals

In this section we assemble the proof of the main result of the paper, Theorem 9.1. We also provide a more detailed version of the theorem than the one stated in the Introduction; in particular, we express the relevant defining ideals as socles and iterated socles.

Theorem 9.1. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k, I be a linearly presented Gorenstein ideal of height three which is minimally generated by homogeneous forms $g_1, \ldots, g_n$, φ be a minimal homogeneous alternating presentation matrix for $g_1, \ldots, g_n$, B be the Jacobian dual of φ in the sense of 3.6, $C(\varphi)$ be the ideal of $T = k[T_1, \ldots, T_n]$ given in Definition 8.1, and $\mathcal{L}$ be the ideal of $S = R[T_1, \ldots, T_n]$ which defines $\mathrm{Sym}(I)$ as described in 3.6. If I satisfies G_d, then the following statements hold*:

(a) *The ideal of S defining the Rees ring of I is*

$$\mathcal{J} = \mathcal{L} + I_d(B)S + C(\varphi)S.$$

If d is odd, then $C(\varphi)$ is zero and $\mathcal{J}$ has the expected form.

(b) *The ideal of T defining the variety X parametrized by $g_1, \ldots, g_n$ is*

$$I(X) = I_d(B) + C(\varphi).$$

(c) *The two ideals $\mathcal{J}$ and $\mathcal{L} :_S (x_1, \ldots, x_d)(T_1, \ldots, T_n)$ of S are equal.*

(d) *The three ideals $I(X)$, $I_d(B) :_T (T_1, \ldots, T_n)$, and $I_d(B)^{\mathrm{unm}}$ of T are equal.*

Proof. From Corollary 8.8 we have the containment

$$I_d(B) + C(\varphi) \subset I(X). \tag{9.1.1}$$

If $n \leq d$, then $\mathcal{J} = \mathcal{L}$ by (3.7.2) and hence $I(X) = 0$ by (3.3.1), which also gives $I_d(B) + C(\varphi) = 0$. Therefore we may assume that $d < n$.

Corollary 6.4 guarantees that the ideal I is of fiber type, which means that $\mathcal{J} = \mathcal{L} + I(X)S$; see 3.4. Thus (a) follows from part (b) (and Remark 8.3.b).

According to Theorem 4.4 the homogeneous coordinate ring $A = T/I(X)$ of X has dimension d. Now Remark 7.4 and Theorem 7.2.(a) show that

$$\mathrm{ht}\, I_d(B) = \mathrm{ht}\, I(X) = n - d. \tag{9.1.2}$$

The hypothesis of Theorem 8.7 is satisfied; thus assertions (b) and (e) of Theorem 8.7 yield

$$I_d(B)^{\mathrm{unm}} = I_d(B) + C(\varphi), \tag{9.1.3}$$

and the multiplicity of $T/(I_d(B) + C(\varphi))$ is

$$\sum_{i=0}^{\lfloor \frac{n-d}{2} \rfloor} \binom{n-2-2i}{d-2}.$$

By Theorem 4.4, the coordinate ring $A = T/I(X)$ has the same multiplicity, by (9.1.1) and (9.1.2), the rings $T/(I_d(B) + C(\varphi))$ and $T/I(X)$ have the same dimension, and by (9.1.3) both rings are unmixed. Hence the containment $I_d(B) + C(\varphi) \subset I(X)$ of (9.1.1) is an equality, and (b) is also established.

Part (d) follows from Corollary 8.8, part (b), and (9.1.3). For the proof of (c) notice that parts (a), (b), and (d) give

$$\mathscr{J} = \mathscr{L} + (I_d(B) :_T (T_1, \ldots, T_n))S.$$

Therefore,

$$\mathscr{J} \subset (\mathscr{L} + I_d(B)S) :_S (T_1, \ldots, T_n).$$

Since $\mathscr{L} + I_d(B)S \subset \mathscr{L} :_S (x_1, \ldots, x_d)$ according to (3.6.2), we obtain

$$\mathscr{J} \subset (\mathscr{L} :_S (x_1, \ldots, x_d)) :_S (T_1, \ldots, T_n) = \mathscr{L} :_S (x_1, \ldots, x_d)(T_1, \ldots, T_n) \subset \mathscr{J}. \quad \square$$

Remark 9.2. Adopt Data 3.1. The assumption in Theorem 9.1 that I is linearly presented can be weakened to the condition that the entries of a minimal homogeneous presentation matrix φ of I generate a complete intersection ideal.

Indeed, let φ be a minimal homogeneous alternating presentation matrix of $g_1, \ldots, g_n$. By the symmetry of the minimal homogeneous R-resolution of R/I all entries of φ have the same degree D. Let $y_1, \ldots, y_s$ be a regular sequence of homogeneous forms of degree D that generate $I_1(\varphi)$. These forms are algebraically independent over k, and $R = k[x_1, \ldots, x_d]$ is flat over the polynomial ring $k[y_1, \ldots, y_s]$. The entries of φ are forms of degree D and are R-linear combinations of $y_1, \ldots, y_s$; hence these entries are linear forms in the ring $k[y_1, \ldots, y_s]$. Since $g_1, \ldots, g_n$ are signed submaximal Pfaffians of φ according to [Buchsbaum and Eisenbud 1977], these elements also belong to the ring $k[y_1, \ldots, y_s]$, and by flat descent they satisfy the assumptions of Theorem 9.1 as elements of this ring. Now apply Theorem 9.1. Flat base change then gives the statements about the Rees ring over R, with $y_1, \ldots, y_s$ in place of $x_1, \ldots, x_d$; the statements about the ideal $I(X)$ are independent of the ambient ring.

The paper [Kustin et al. 2017b] was written with the intention of understanding the ideals $I_d(B)^{\mathrm{unm}}$ in order to determine the equations defining the Rees algebra

and the special fiber ring of height three Gorenstein ideals, the ultimate goal of the present paper. In particular, in [Kustin et al. 2017b] the ideals $I_d(B)^{\mathrm{unm}}$ have been resolved. Now that we have proven that $I_d(B)^{\mathrm{unm}}$ defines the special fiber ring, we are able to harvest much information from the results of [Kustin et al. 2017b].

Corollary 9.3. *Adopt the notation and hypotheses of Theorem 9.1. The following statements hold*:

(a) *The variety X is Cohen–Macaulay.*

(b) *If $n \le d+1$, then the homogeneous coordinate ring A of X is Cohen–Macaulay. Otherwise*

$$\operatorname{depth} A = \begin{cases} 1 & \textit{if d is odd,} \\ 2 & \textit{if d is even.} \end{cases}$$

(c) *If d is odd and $d < n$, then $I(X)$ has a linear free resolution and $\operatorname{reg}(A) = d-1$.*

(d) *If $d < n$, then the Hilbert series of A is*

$$\frac{1}{(1-z)^d}\left(\sum_{\ell=0}^{d-2}\binom{\ell+n-d-1}{n-d-1}z^\ell + \sum_{\ell=0}^{n-d-2}(-1)^{\ell+d+1}\binom{\ell+d-1}{d-1}z^{\ell+2d-n} \right)$$
$$+ (-1)^d \sum_{j\le\lceil\frac{n-d-3}{2}\rceil}\binom{j+d-1}{d-1}z^{2j+2d-n}.$$

Proof. We may assume that $d < n$ since otherwise $I(X) = 0$; see (3.7.2) and (3.3.1). From Theorem 9.1.(b) we know that $I(X) = I_d(B) + C(\varphi)$, where $C(\varphi) = 0$ if d is odd. Thus the assertions are items (a), (c), and (d) of Theorem 8.7, which applies due to (9.1.2). □

Remark. The minimal homogeneous resolution of the ideal $I(X)$ may not be linear when d is even; see [Kustin et al. 2017b, (4.4.5)].

References

[Aberbach et al. 1995] I. M. Aberbach, S. Huckaba, and C. Huneke, "Reduction numbers, Rees algebras and Pfaffian ideals", *J. Pure Appl. Algebra* **102**:1 (1995), 1–15. MR Zbl

[Artin and Nagata 1972] M. Artin and M. Nagata, "Residual intersections in Cohen–Macaulay rings", *J. Math. Kyoto Univ.* **12** (1972), 307–323. MR Zbl

[Boffi and Sánchez 1992] G. Boffi and R. Sánchez, "On the resolutions of the powers of the Pfaffian ideal", *J. Algebra* **152**:2 (1992), 463–491. MR Zbl

[Boij et al. 2014] M. Boij, J. Migliore, R. M. Miró-Roig, U. Nagel, and F. Zanello, "On the weak Lefschetz property for Artinian Gorenstein algebras of codimension three", *J. Algebra* **403** (2014), 48–68. MR Zbl

[Boswell and Mukundan 2016] J. A. Boswell and V. Mukundan, "Rees algebras and almost linearly presented ideals", *J. Algebra* **460** (2016), 102–127. MR Zbl

[Bruns et al. 2013] W. Bruns, A. Conca, and M. Varbaro, "Relations between the minors of a generic matrix", *Adv. Math.* **244** (2013), 171–206. MR Zbl

[Bruns et al. 2015] W. Bruns, A. Conca, and M. Varbaro, "Maximal minors and linear powers", *J. Reine Angew. Math.* **702** (2015), 41–53. MR Zbl

[Buchsbaum and Eisenbud 1977] D. A. Buchsbaum and D. Eisenbud, "Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3", *Amer. J. Math.* **99**:3 (1977), 447–485. MR Zbl

[Burch 1968] L. Burch, "On ideals of finite homological dimension in local rings", *Proc. Cambridge Philos. Soc.* **64** (1968), 941–948. MR Zbl

[Busé 2009] L. Busé, "On the equations of the moving curve ideal of a rational algebraic plane curve", *J. Algebra* **321**:8 (2009), 2317–2344. MR Zbl

[Chardin et al. 2001] M. Chardin, D. Eisenbud, and B. Ulrich, "Hilbert functions, residual intersections, and residually S_2 ideals", *Compositio Math.* **125**:2 (2001), 193–219. MR Zbl

[Conca and Valla 2000] A. Conca and G. Valla, "Betti numbers and lifting of Gorenstein codimension three ideals", *Comm. Algebra* **28**:3 (2000), 1371–1386. MR Zbl

[Cortadellas Benítez and D'Andrea 2013] T. Cortadellas Benítez and C. D'Andrea, "Rational plane curves parameterizable by conics", *J. Algebra* **373** (2013), 453–480. MR Zbl

[Cortadellas Benítez and D'Andrea 2014] T. Cortadellas Benítez and C. D'Andrea, "Minimal generators of the defining ideal of the Rees algebra associated with a rational plane parametrization with $\mu = 2$", *Canad. J. Math.* **66**:6 (2014), 1225–1249. MR Zbl

[Cox et al. 2008] D. Cox, J. W. Hoffman, and H. Wang, "Syzygies and the Rees algebra", *J. Pure Appl. Algebra* **212**:7 (2008), 1787–1796. MR Zbl

[De Negri and Valla 1995] E. De Negri and G. Valla, "The h-vector of a Gorenstein codimension three domain", *Nagoya Math. J.* **138** (1995), 113–140. MR Zbl

[Diesel 1996] S. J. Diesel, "Irreducibility and dimension theorems for families of height 3 Gorenstein algebras", *Pacific J. Math.* **172**:2 (1996), 365–397. MR Zbl

[Eisenbud and Ulrich 2008] D. Eisenbud and B. Ulrich, "Row ideals and fibers of morphisms", *Michigan Math. J.* **57** (2008), 261–268. MR Zbl

[Eisenbud et al. 2006] D. Eisenbud, C. Huneke, and B. Ulrich, "The regularity of Tor and graded Betti numbers", *Amer. J. Math.* **128**:3 (2006), 573–605. MR Zbl

[Elias and Iarrobino 1987] J. Elias and A. A. Iarrobino, "The Hilbert function of a Cohen–Macaulay local algebra: extremal Gorenstein algebras", *J. Algebra* **110**:2 (1987), 344–356. MR Zbl

[Gaeta 1953] F. Gaeta, "Ricerche intorno alle varietà matriciali ed ai loro ideali", pp. 326–328 in *Atti del Quarto Congresso dell'Unione Matematica Italiana* (Taormina, 1951), vol. II, Casa Editrice Perrella, Roma, 1953. MR Zbl

[Geramita and Migliore 1997] A. V. Geramita and J. C. Migliore, "Reduced Gorenstein codimension three subschemes of projective space", *Proc. Amer. Math. Soc.* **125**:4 (1997), 943–950. MR Zbl

[Gruson et al. 1983] L. Gruson, R. Lazarsfeld, and C. Peskine, "On a theorem of Castelnuovo, and the equations defining space curves", *Invent. Math.* **72**:3 (1983), 491–506. MR Zbl

[Hartshorne 2004] R. Hartshorne, "Geometry of arithmetically Gorenstein curves in $\mathbb{P}^4$", *Collect. Math.* **55**:1 (2004), 97–111. MR Zbl

[Herzog et al. 1983] J. Herzog, A. Simis, and W. V. Vasconcelos, "Koszul homology and blowing-up rings", pp. 79–169 in *Commutative algebra* (Trento, 1981), edited by S. Greco and G. Valla, Lecture Notes in Pure and Appl. Math. **84**, Dekker, New York, 1983. MR Zbl

[Herzog et al. 1985] J. Herzog, W. V. Vasconcelos, and R. Villarreal, "Ideals with sliding depth", *Nagoya Math. J.* **99** (1985), 159–172. MR Zbl

[Hong et al. 2008] J. Hong, A. Simis, and W. V. Vasconcelos, "On the homology of two-dimensional elimination", *J. Symbolic Comput.* **43**:4 (2008), 275–292. MR Zbl

[Huneke 1982] C. Huneke, "Linkage and the Koszul homology of ideals", *Amer. J. Math.* **104**:5 (1982), 1043–1062. MR Zbl

[Huneke and Rossi 1986] C. Huneke and M. Rossi, "The dimension and components of symmetric algebras", *J. Algebra* **98**:1 (1986), 200–210. MR Zbl

[Iarrobino 1994] A. A. Iarrobino, "Associated graded algebra of a Gorenstein Artin algebra", pp. viii+115 Mem. Amer. Math. Soc. **514**, 1994. MR Zbl

[Iarrobino and Kanev 1999] A. Iarrobino and V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics **1721**, Springer, 1999. MR Zbl

[Johnson 1997] M. R. Johnson, "Second analytic deviation one ideals and their Rees algebras", *J. Pure Appl. Algebra* **119**:2 (1997), 171–183. MR Zbl

[Jouanolou 1996] J.-P. Jouanolou, "Résultant anisotrope, compléments et applications", *Electron. J. Combin.* **3**:2 (1996), art. id. #2. MR Zbl

[Jouanolou 1997] J. P. Jouanolou, "Formes d'inertie et résultant: un formulaire", *Adv. Math.* **126**:2 (1997), 119–250. MR Zbl

[Kimura and Terai 2015] K. Kimura and N. Terai, "Arithmetical rank of Gorenstein squarefree monomial ideals of height three", *J. Algebra* **422** (2015), 11–32. MR Zbl

[Kleppe and Miró-Roig 1998] J. O. Kleppe and R. M. Miró-Roig, "The dimension of the Hilbert scheme of Gorenstein codimension 3 subschemes", *J. Pure Appl. Algebra* **127**:1 (1998), 73–82. MR Zbl

[Kustin and Ulrich 1992] A. R. Kustin and B. Ulrich, *A family of complexes associated to an almost alternating map, with applications to residual intersections*, vol. 95, Mem. Amer. Math. Soc. **461**, 1992. MR Zbl

[Kustin et al. 2011] A. R. Kustin, C. Polini, and B. Ulrich, "Rational normal scrolls and the defining equations of Rees algebras", *J. Reine Angew. Math.* **650** (2011), 23–65. MR Zbl

[Kustin et al. 2015] A. R. Kustin, C. Polini, and B. Ulrich, "Degree bounds for local cohomology", preprint, 2015. arXiv

[Kustin et al. 2016a] A. R. Kustin, C. Polini, and B. Ulrich, "Blowups and fibers of morphisms", *Nagoya Math. J.* **224**:1 (2016), 168–201. MR

[Kustin et al. 2016b] A. R. Kustin, C. Polini, and B. Ulrich, "The Hilbert series of the ring associated to an almost alternating matrix", *Comm. Algebra* **44**:7 (2016), 3053–3068. MR Zbl

[Kustin et al. 2017a] A. R. Kustin, C. Polini, and B. Ulrich, "The bi-graded structure of symmetric algebras with applications to Rees rings", *J. Algebra* **469** (2017), 188–250. MR Zbl

[Kustin et al. 2017b] A. R. Kustin, C. Polini, and B. Ulrich, "A matrix of linear forms which is annihilated by a vector of indeterminates", *J. Algebra* **469** (2017), 120–187. MR Zbl

[Madsen 2015] J. Madsen, "Equations of Rees algebras of ideals in two variables", preprint, 2015. arXiv

[Migliore and Peterson 1997] J. C. Migliore and C. Peterson, "A construction of codimension three arithmetically Gorenstein subschemes of projective space", *Trans. Amer. Math. Soc.* **349**:9 (1997), 3803–3821. MR Zbl

[Morey 1996] S. Morey, "Equations of blowups of ideals of codimension two and three", *J. Pure Appl. Algebra* **109**:2 (1996), 197–211. MR Zbl

[Morey and Ulrich 1996] S. Morey and B. Ulrich, "Rees algebras of ideals with low codimension", *Proc. Amer. Math. Soc.* **124**:12 (1996), 3653–3661. MR Zbl

[Nguyen 2014] L. P. H. Nguyen, "On Rees algebras of linearly presented ideals", *J. Algebra* **420** (2014), 186–200. MR Zbl

[Polini and Ulrich 1999] C. Polini and B. Ulrich, "Necessary and sufficient conditions for the Cohen–Macaulayness of blowup algebras", *Compositio Math.* **119**:2 (1999), 185–207. MR Zbl

[Ulrich 1994] B. Ulrich, "Artin–Nagata properties and reductions of ideals", pp. 373–400 in *Commutative algebra: syzygies, multiplicities, and birational algebra* (South Hadley, MA, 1992), edited by W. J. Heinzer et al., Contemp. Math. **159**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Vasconcelos 1991] W. V. Vasconcelos, "On the equations of Rees algebras", *J. Reine Angew. Math.* **418** (1991), 189–218. MR Zbl

[Watanabe 1973] J. Watanabe, "A note on Gorenstein rings of embedding codimension three", *Nagoya Math. J.* **50** (1973), 227–232. MR Zbl

kustin@math.sc.edu   *Department of Mathematics, University of South Carolina, 1523 Greene Street, Columbia, SC 29208, United States*

cpolini@nd.edu   *Department of Mathematics, University of Notre Dame, 255 Hurley Hall, Notre Dame, IN 46556-4618, United States*

ulrich@math.purdue.edu   *Department of Mathematics, Purdue University, 150 N University Street, West Lafayette, IN 47907-2067, United States*

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