ERRATUM TO "EXTENDED EIGENVARIETIES FOR OVERCONVERGENT COHOMOLOGY"

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The statement of [JN19, Lemma 2.1.6] is incorrect — we thank Judith Ludwig for discussions that led us to discover this error. The purpose of this note is to correct this. Section 1 gives a corrected version of [JN19, Lemma 2.1.6]. Section 2 then gives alternative arguments for the parts of [JN19] that depend on that Lemma. We use the notation and terminology of [JN19] freely throughout this note.

1. Correction to [JN19, Lemma 2.1.6]

For concreteness, let us start by providing a counterexample to the statement of [JN19, Lemma 2.1.6]:

Example 1.1. Consider $R = \mathbb{Q}_p\langle p^{-1}T, p^2T^{-1}\rangle$, the ring of rigid analytic functions on the annulus $p^{-2} \leq |T| \leq p^{-1}$. Let $|-|_{sp}$ be the spectral norm on R in the sense of rigid analytic geometry. We see that $|T^{-1}|_{sp} = p^2$ and $|T|_{sp} = p^{-1}$, so T is a topologically nilpotent unit in R. In particular, we may define a second norm on R by

$$|f|_T = \inf\{p^{-n} \mid f \in T^n R^\circ,\, n \in \mathbb{Z}\},$$

equivalent to $|-|_{sp}$. Assume that there are constants $C, s > 0$ such that $|f|_{sp} \leq C|f|_T^s$ for all $f \in R$. Let $n > 0$ be an integer. If we set $f = T^n$, we obtain $p^{-n} \leq Cp^{-ns}$, so $s \leq 1$ since n is arbitrary. On the other hand, if we set $f = T^{-n}$, then we obtain $p^{2n} \leq Cp^{ns}$ which forces $s \geq 2$, so we have a contradiction. In particular, the two norms $|-|_{sp}$ and $|-|_T$ are equivalent but do not satisfy the statement of [JN19, Lemma 2.1.6].

The following corrected version of [JN19, Lemma 2.1.6] can be salvaged.

Lemma 1.2. *Let R be a complete Tate ring, and let $\varpi, \pi \in R$ be topologically nilpotent units. Assume that we have two equivalent norms $|-|_\varpi$ and $|-|_\pi$ on R (inducing the intrinsic topology) such that ϖ is multiplicative for $|-|_\varpi$ and π is multiplicative for $|-|_\pi$. Then we may find constants $C_1, C_2, s > 0$ such that, if $|a|_\pi < 1$,*

$$|a|_\varpi \leq C_1,$$

and if $|a|_\pi \geq 1$, then

$$|a|_\varpi \leq C_2|a|_\pi^s.$$

Proof. For the first inequality, first note that we can find a constant $D_1 < 1$ such that if $|a|_\pi \leq D_1$ then $|a|_\varpi \leq 1$ (since the norms induce the same topology). Choose an integer $m > 0$ such that $|\varpi^m|_\pi \leq D_1$ and set $C_1 = |\varpi|_\varpi^{-m}$. So, if $|a|_\pi \leq 1$, then $|\varpi^m a|_\pi \leq D_1$ and hence $|\varpi^m a|_\varpi \leq 1$. Since ϖ is multiplicative for $|-|_\varpi$, we deduce that $|a|_\varpi \leq |\varpi|_\varpi^{-m} = C_1$.

We now prove the second inequality. Assume that $|a|_\pi \geq 1$. Set

$$n = \left\lceil \frac{\log|a|_\pi}{\log|\varpi^m|_\pi^{-1}} \right\rceil;$$

then $n \geq 0$ by the assumption on a and the definition of m. By definition, we have that $|\varpi^m|_\pi^n|a|_\pi \leq 1$, and hence $|\varpi^{mn}a|_\pi \leq 1$. Therefore we have $|\varpi^{mn}a|_\varpi \leq C_1$. By multiplicativity of ϖ for $|-|_\varpi$ we get $|a|_\varpi \leq C_1|\varpi|_\varpi^{-mn} = C_1^{n+1}$. Then we have

$$|a|_\varpi \leq C_1^{n+1} \leq C_1^{\frac{\log|a|_\pi}{\log|\varpi^m|_\pi^{-1}}+2} = C_2|a|_\pi^s$$

where we have put $s = (\log_{C_1}|\varpi^m|_\pi^{-1})^{-1}$ and $C_2 = C_1^2$; note that $s > 0$. This finishes the proof. $\square$

Note that the conclusion of [JN19, Remark 2.1.7] still holds, and can be proved in a similar way to Lemma 1.2:

Lemma 1.3. *Let R be a complete Tate ring, and let $\varpi \in R$ be a topologically nilpotent unit. Assume that we have two equivalent norms $|-|_1$ and $|-|_2$ on R (inducing the intrinsic topology) such that ϖ is multiplicative for both $|-|_1$ and $|-|_2$. Then we may find constants $C_1, C_2 > 0$ such that*

$$C_1|a|_1^s \leq |a|_2 \leq C_2|a|_1^s$$

for all $a \in R$, where s is determined by $|\varpi|_2 = |\varpi|_1^s$.

Proof. We again fix a constant $D_1 < 1$ such that if $|a|_1 \leq D_1$ then $|a|_2 \leq 1$. Choose an integer $m > 0$ such that $|\varpi|_1^m \leq D_1$. For a non-zero a, we set

$$n = \left\lceil \frac{\log |a|_1}{\log |\varpi^m|_1^{-1}} \right\rceil.$$

Equivalently, n is defined so that $|\varpi^{-m(n-1)}|_1 < |a|_1 \leq |\varpi^{-mn}|_1$. So $|a\varpi^{mn}|_1 \leq 1$ and hence $|a|_2 \leq |\varpi|_2^{-m(n+1)}$. Let s be such that $|\varpi|_2 = |\varpi|_1^s$. Then $|a|_2 \leq |\varpi|_1^{-m(n+1)s} \leq C_2|a|_1^s$, where $C_2 = |\varpi|_1^{-2ms}$. Swapping $|-|_1$ and $|-|_2$ we get a similar inequality $|a|_1 \leq C_1'|a|_2^{1/s}$. □

2. Additional corrections

To begin, we remark that [JN19, Lemma 2.1.6] was used to justify that the topology on the ON-able modules $c_R(I)$ do not depend on the choice of norm on R. This can be seen by a more direct argument (as was already noted in *op. cit*). Perhaps better, the basic notions of functional analysis introduced in [JN19, §2.1], such as potential orthonormalisability, topologies coming from operator norms, and compactness, can be defined in purely topological terms (and this is not so difficult and certainly well known). We refer to [Gul19, §2] for a good summary in the context needed for [JN19]. We also remark that this independence is, strictly speaking, not needed for any of the main results or constructions in [JN19]. In fact, it turns out to be good enough to work only with bounded-equivalence throughout the functional analytic part and to rely only on the independence results presented below (Propositions 2.1 and 2.2).

Lemma 2.1.6 of [JN19] was explicitly used in the proofs of Propositions 3.2.6 and 4.1.4 of *op. cit.* We now move on to address these issues. Proposition 2.1 below is [JN19, Proposition 3.2.6]; we give a proof based on Lemma 1.2. Recall that we only consider norms for which there exists a multiplicative pseudo-uniformizer, and such that the natural map $\mathbb{Z}_p \to R$ is norm-decreasing.

Proposition 2.1. *$\mathscr{D}(G, R)$ is independent of the choice of norm on R.*

Proof. Choose a norm on R and a minimal set of topological generators for G. Recall that $\mathscr{D}(G, R) = \varprojlim_r \mathcal{D}^r(G, R)$ and that $\mathcal{D}^r(G, R)$ can be described as

$$\mathcal{D}^r(G, R) = \left\{ \sum_\alpha d_\alpha \mathbf{b}^\alpha \mid d_\alpha \in R,\ |d_\alpha| r^{|\alpha|} \to 0 \right\}.$$

As a consequence, $\mathscr{D}(G, R)$ consists of the formal sums $\sum_\alpha d_\alpha \mathbf{b}^\alpha$ for which $|d_\alpha| r^{|\alpha|} \to 0$ for all $r \in [1/p, 1)$. The topology is given by the family of norms $(||-||_r)_{r\in[1/p,1)}$, with $||\sum_\alpha d_\alpha \mathbf{b}^\alpha||_r = \sup_\alpha |d_\alpha| r^{|\alpha|}$. In particular, it is metrizable and translation-invariant, so the topology is determined by the set of sequences that tend to 0.

To prove independence, we start by proving that the condition $|d_\alpha| r^{|\alpha|} \to 0$ for all $r \in [1/p, 1)$ is independent of the choice of norm. Let $|-|_\pi$ and $|-|_\varpi$ be two equivalent norms on R with multiplicative pseudo-uniformizers π and ϖ, respectively. Choose $t \in [1/p, 1)$; by symmetry we need to show that if $|d_\alpha|_\pi r^{|\alpha|} \to 0$ for all $r \in [1/p, 1)$, then $|d_\alpha|_\varpi t^{|\alpha|} \to 0$. By Lemma 1.2, if $|d_\alpha|_\pi < 1$ then $|d_\alpha|_\varpi \leq C_1$ for some constant C_1. As $t < 1$, we see that it remains to show $|d_\alpha|_\varpi t^{|\alpha|} \to 0$ for the subsequence of α's for which $|d_\alpha|_\pi \geq 1$. But for those α, Lemma 1.2 gives us constants $C_2, s > 0$ such that

$$|d_\alpha|_\varpi t^{|\alpha|} \leq C_2 |d_\alpha|_\pi^s t^{|\alpha|} = C_2 \left(|d_\alpha|_\pi (t^{1/s})^{|\alpha|} \right)^s.$$

Since $t^{1/s} \in (0, 1)$, $|d_\alpha|_\pi (t^{1/s})^{|\alpha|} \to 0$ by assumption, so $|d_\alpha|_\varpi t^{|\alpha|} \to 0$ as desired.

This shows that the R-module $\mathscr{D}(G, R)$ is independent of the choice of norm, so we are left with checking independence of the topology, for which it suffices to check that convergence to 0 is independent.

Let $\mu_n = \sum_\alpha d_\alpha^{(n)} \mathbf{b}^\alpha$ be a sequence in $\mathscr{D}(G, R)$. Let $t \in [1/p, 1)$. By symmetry, we need to show that if $\sup_\alpha |d_\alpha^{(n)}|_\pi r^{|\alpha|} \to 0$ for all $r \in [1/p, 1)$, then $\sup_\alpha |d_\alpha^{(n)}|_\varpi t^{|\alpha|} \to 0$. Choose an arbitrary $\epsilon > 0$; we need to show that $\sup_\alpha |d_\alpha^{(n)}|_\varpi t^{|\alpha|} \leq \epsilon$ for large enough n.

Let $C_1, C_2, s > 0$ be the constants from Lemma 1.2. There is a finite subset S of α's such that if $\alpha \notin S$, then $C_1 t^{|\alpha|} \leq \epsilon$. Write

$$\sup_\alpha |d_\alpha^{(n)}|_\varpi t^{|\alpha|} = \max \left\{ \sup_{\alpha \in S} |d_\alpha^{(n)}|_\varpi t^{|\alpha|}, \sup_{\alpha \notin S} |d_\alpha^{(n)}|_\varpi t^{|\alpha|} \right\}.$$

Note that for a fixed α, $|d_\alpha^{(n)}|_\pi \to 0$, so by equivalence $|d_\alpha^{(n)}|_\varpi \to 0$. In particular, the term $\sup_{\alpha \in S} |d_\alpha^{(n)}|_\varpi t^{|\alpha|} \to 0$ since S is finite, so we can find N_1 such that $n \geq N_1$ implies that $\sup_{\alpha \in S} |d_\alpha^{(n)}|_\varpi t^{|\alpha|} \leq \epsilon$. It then remains to treat the term $\sup_{\alpha \notin S} |d_\alpha^{(n)}|_\varpi t^{|\alpha|}$. Let $\alpha \notin S$. If $|d_\alpha^{(n)}|_\pi < 1$, then

$$|d_\alpha^{(n)}|_\varpi t^{|\alpha|} \leq C_1 t^{|\alpha|} \leq \epsilon$$

independent of n. On the other hand, if $|d_\alpha^{(n)}|_\pi \geq 1$, then as above

$$|d_\alpha^{(n)}|_\varpi t^{|\alpha|} \leq C_2 \left(|d_\alpha^{(n)}|_\pi (t^{1/s})^{|\alpha|} \right)^s$$

and by assumption $\sup_\alpha |d_\alpha^{(n)}|_\pi (t^{1/s})^{|\alpha|} \to 0$ as $n \to \infty$, so we can find N_2 such $n \geq N_2$ implies $\sup_{\alpha \notin S} |d_\alpha^{(n)}|_\varpi t^{|\alpha|} \leq \epsilon$. Thus $\sup_\alpha |d_\alpha^{(n)}|_\varpi t^{|\alpha|} \leq \epsilon$ for $n \geq N = \max(N_1, N_2)$, which finishes the proof. $\square$

Next, we revise some parts of the eigenvariety construction from [JN19, §4]. The eigenvariety construction has two key steps: the construction of an auxiliary Fredholm series over weight space and the construction of a coherent sheaf on the corresponding Fredholm hypersurface. Both constructions are local and depend a priori on the choice of some auxiliary norms — one then has to check independence, which is done in [JN19, §4.1]. We indicate a more efficient argument which removes the use of the incorrect [JN19, Lemma 2.1.6].

Recall from [JN19, §4.1] that one fixes once an element $t \in \Sigma^{cpt}$. If $\kappa : T_0 \to R^\times$ is a weight, choose a Banach–Tate $\mathbb{Z}_p$-algebra norm $|-|$ on R which is adapted to κ, and let $\widetilde{U}_{\kappa,r} = \widetilde{U}_{t,\kappa,r}$ denote the corresponding Hecke operator on $C^\bullet(K, \mathcal{D}_\kappa^r)$ (here $r \geq r_\kappa$). This operator is compact and we let

$$F_\kappa^{r,|-|}(T) = \det \left(1 - T\widetilde{U}_{\kappa,r} \mid C^*(K, \mathcal{D}_\kappa^r) \right)$$

denote its Fredholm determinant. The following Proposition combines [JN19, Proposition 4.1.2, Proposition 4.1.4]. The simple proof given below is an observation of Daniel Gulotta; see the beginning of [Gul19, §4.4].

Proposition 2.2. *The Fredholm series $F_\kappa^{r,|-|}$ is independent of the choice of $r \geq r_\kappa$ and on the choice of norm $|-|$ on R.*

Proof. We assume $p > 2$ to simplify the notation, the proof for $p = 2$ is the same (except that one needs to write down the potential ON-basis below differently). Let $|-|_\pi$ and $|-|_\varpi$ be two adapted Banach–Tate $\mathbb{Z}_p$-algebra norms with multiplicative pseudouniformizers π and ϖ, respectively, and let $r, s \in [1/p, 1)$ be such that $|\kappa(t) - 1|_\pi \leq r$ and $|\kappa(t) - 1|_\varpi \leq s$ for all $t \in T_\epsilon$. Consider $\mathcal{D}_\kappa^r$ and $\mathcal{D}_\kappa^s$, where the former is formed using $|-|_\pi$ and the latter is formed using $|-|_\varpi$. Identify $\mathcal{D}_\kappa^r$ and $\mathcal{D}_\kappa^s$ with $\mathcal{D}^r(\overline{N}_1, R)$ and $\mathcal{D}^s(\overline{N}_1, R)$, respectively, and choose a minimal set of topological generators $n_1, \ldots, n_k$ of $\overline{N}_1$. Put $\mathbf{n}^\alpha = \prod_i (\delta_{n_i} - 1)^{\alpha_i}$. Then $\mathcal{D}^r(\overline{N}_1, R)$ has a potential ON-basis $(\pi^{-n(r,\pi,\alpha)} \mathbf{n}^\alpha)$ and $\mathcal{D}(\overline{N}_1, R)^s$ has a potential ON-basis $(\varpi^{-n(r,\varpi,\alpha)} \mathbf{n}^\alpha)$, cf. [JN19, Eqn. (3.2.1)]. In particular, these bases differ by scalars and produce potential ON-bases of $C^*(K, \mathcal{D}_\kappa^r)$ and $C^*(K, \mathcal{D}_\kappa^s)$ which also differ by scalars. It then follows that the infinite matrices describing the actions $\widetilde{U}_{\kappa,r}$ and $\widetilde{U}_{\kappa,s}$ differ by the conjugation of an infinite *diagonal* matrix. This then shows that $F_\kappa^{r,|-|_\pi} = F_\kappa^{s,|-|_\varpi}$. $\square$

The rest of the arguments of [JN19, §4.1] go through as before. We do remark that one needs independence of the choice of norm of the 'locally analytic' distribution module $\mathscr{D}_\kappa$ for the gluing statement of [JN19, Proposition 4.1.6] which is used to construct the complex of coherent sheaves in [JN19, Corollary 4.1.8]; this independence follows from Proposition 2.1.

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