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Tautological rings of Shimura varieties and cycle classes of Ekedahl–Oort strata

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We define the tautological ring as the subring of the Chow ring of a Shimura variety generated by all Chern classes of all automorphic bundles. We explain its structure for the special fiber of a good reduction of a Shimura variety of Hodge type and show that it is generated by the cycle classes of the Ekedahl–Oort strata as a vector space. We compute these cycle classes. As applications we get the triviality of ℓ-adic Chern classes of flat automorphic bundles in characteristic 0, an isomorphism of the tautological ring of smooth toroidal compactifications in positive characteristic with the rational cohomology ring of the compact dual of the hermitian domain given by the Shimura datum, and a new proof of Hirzebruch–Mumford proportionality for Shimura varieties of Hodge type.

Introduction

Tautological rings. The Chow ring $A^\bullet(S_K)$ (always with rational coefficients) of a Shimura variety S_K is still a very mysterious object. Here we study the subring generated by all Chern classes of all automorphic bundles on the Shimura variety or on a smooth toroidal compactification of the Shimura variety. In the Siegel case this subring was already studied by van der Geer and Ekedahl [1999; 2009]. Following

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their terminology, we call this subring the *tautological ring*[1] of the Shimura variety or of some toroidal compactification.

More precisely, let $(\boldsymbol{G}, \boldsymbol{X})$ be a Shimura datum and $K \subset \boldsymbol{G}(\mathbb{A}_f)$ a sufficiently small open compact subgroup. The Shimura datum defines a conjugacy class of cocharacters μ of $\boldsymbol{G}$ whose field of definition is the reflex field E of the attached Shimura variety. We denote the canonical model over E of this Shimura variety by $\boldsymbol{S}_K = \mathrm{Sh}_K(\boldsymbol{G}, \boldsymbol{X})$. To simplify the notation here in the introduction let us assume that $\boldsymbol{G}$ does not contain a $\mathbb{Q}$-anisotropic and $\mathbb{R}$-split torus in its center. This condition is automatic if $(\boldsymbol{G}, \boldsymbol{X})$ is of Hodge type. The Borel embedding of the hermitian space $\boldsymbol{X}$ into its compact dual $\boldsymbol{X}^\vee$ induces a morphism

$$\sigma : \boldsymbol{S}_K \to \mathtt{Hdg} := [\boldsymbol{G}\backslash \boldsymbol{X}^\vee]$$

of algebraic stacks over E [Milne 1990, III]. By definition, a vector bundle on $\boldsymbol{S}_K$ is automorphic[2] if it is isomorphic to the pullback of a vector bundle on $\mathtt{Hdg}$. Moreover, it is flat if it is obtained by pullback from a vector bundle on the classifying stack $[\boldsymbol{G}\backslash *]$, i.e., if it is induced by a finite-dimensional representation of $\boldsymbol{G}$ (see Section 5 for details). For a smooth toroidal compactification $\boldsymbol{S}_K^{\mathrm{tor}}$ of $\boldsymbol{S}_K$ given by the choice of a suitable polyhedral cone decomposition, the theory of canonical extensions of automorphic vector bundles shows that there is a canonical extension of σ to $\boldsymbol{S}_K^{\mathrm{tor}}$.

Definition 1 (Definition 5.7). The *tautological ring of* $\boldsymbol{S}_K$ (resp. of $\boldsymbol{S}_K^{\mathrm{tor}}$) is the image of the Chow ring of $\mathtt{Hdg}$ in the Chow ring of $\boldsymbol{S}_K$ (resp. of $\boldsymbol{S}_K^{\mathrm{tor}}$) under pullback via σ.

In the Siegel case, the tautological ring is the subring generated by all Chern classes of the Hodge bundle in the de Rham cohomology of the universal abelian scheme (Example 5.9), which is the definition of van der Geer in this case.

Ekedahl–Oort strata. From now on we assume that $(\boldsymbol{G}, \boldsymbol{X})$ is of Hodge type and that $p > 2$ is a prime of good reduction for the Shimura datum. Then the reductive group $\boldsymbol{G}_{\mathbb{Q}_p}$ has a reductive model $\mathscr{G}$ over $\mathbb{Z}_p$ and hence the algebraic stack $\mathtt{Hdg}$ has a good integral model over the ring of integers of the completion of E at a place above p. Denote by G the special fiber $\mathscr{G}$. Hence G is a reductive group over $\mathbb{F}_p$. Moreover, since the Shimura variety is of Hodge type, there are canonical smooth integral models $\mathscr{S}_K$ and $\mathscr{S}_K^{\mathrm{tor}}$ with special fibers S_K and S_K^{tor} by the work of Vasiu [1999], Kisin [2010], and Kim and Madapusi Pera [2016; 2019] such that the morphism σ extends. Hence we also obtain in characteristic p tautological rings of S_K and S_K^{tor} as images under pullback maps

$$\sigma^* : A^\bullet(\mathtt{Hdg}) \to A^\bullet(S_K^{\mathrm{tor}}), \tag{0.1}$$

where we again denote by $\mathtt{Hdg}$ the special fiber of the above integral model of $\mathtt{Hdg}$. In characteristic p the work of Viehmann and Wedhorn [2013] (for Shimura varieties of PEL type), of Zhang [2018]

[1]One could argue against this terminology: By analogy to the notion of tautological rings of moduli spaces of curves, the tautological ring should be the subring "generated by all interesting classes". But with our definition there are many interesting classes, for instance those of special subvarieties, that are in general not contained in the tautological ring.

[2]More precisely, it is the underlying vector bundle of an automorphic bundle since we ignore the actions by Hecke operators here.

and Wortmann [2013] (for Shimura varieties of Hodge type), and of W. Goldring and Koskivirta [2019a] (for toroidal compactifications of Shimura varieties of Hodge type) shows that the morphism σ factors into

$$\sigma : S_K^{\mathrm{tor}} \xrightarrow{\zeta^{\mathrm{tor}}} G\text{-}\mathtt{Zip}^{\mu} \xrightarrow{\beta} \mathtt{Hdg}, \tag{0.2}$$

where $G\text{-}\mathtt{Zip}^{\mu}$ is the stack of G-zips of type μ which was defined and studied in [Pink et al. 2011; 2015]. Here μ is as above, now considered as an element of the set of conjugacy classes of cocharacters of $G_{\overline{\mathbb{F}}_p}$. The stack $G\text{-}\mathtt{Zip}^{\mu}$ has a finite stratification by gerbes Z_w, where w runs through a certain subset ${}^I W$ of the Weyl group W of G (see Section 3 for a reminder on G-zips). We refer to the Z_w as the Ekedahl–Oort strata in $G\text{-}\mathtt{Zip}^{\mu}$. The locally closed subschemes

$$S_w := \zeta^{-1}(Z_w) \subseteq S_K \quad \text{and} \quad S_w^{\mathrm{tor}} := \zeta^{\mathrm{tor},-1}(Z_w) \subseteq S_K^{\mathrm{tor}}$$

are by definition the Ekedahl–Oort strata of S_K and S_K^{tor}. As ζ and ζ^{tor} are smooth by the work of Zhang [2018] and Andreatta [2023], many results proved for $Z_w \subseteq G\text{-}\mathtt{Zip}^{\mu}$ in [Pink et al. 2011], such as smoothness, a formula for its codimension, or closure relations of the strata, are known to hold also for the Ekedahl–Oort strata S_w and S_w^{tor}. Using a deep result on the existence of Hasse invariants ([Goldring and Koskivirta 2019a]; see also Boxer [2015] in the PEL case of type A and C) we can also prove the following connectedness result on Ekedahl–Oort strata. (From now on, we abbreviate Ekedahl–Oort strata to EO-strata.)

Theorem 2 (Theorem 6.15, Corollary 6.17). (1) *For all $j \geq 1$, the union of all EO-strata of dimension $\leq j$ is geometrically connected in each geometric connected component of the toroidal compactification S_K^{tor}.*

(2) *For Shimura varieties of PEL type, the union of all EO-strata of dimension ≤ 1 is geometrically connected in each geometric connected component of the Shimura variety S_K.*

The first assertion seems to be new even in the Siegel case. Assertion (2) was proved in the Siegel case by Oort [2001, Theorem 1.1].

The tautological ring and the Chow ring of the stack of G-zips. By (0.2), the pullback σ^* is a composition

$$\sigma^* : A^{\bullet}(\mathtt{Hdg}) \xrightarrow{\beta^*} A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu}) \xrightarrow{\zeta^{\mathrm{tor},*}} A^{\bullet}(S_K^{\mathrm{tor}}). \tag{0.3}$$

Brokemper [2018] has given two descriptions for $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$. From his multiplicative description (recalled in Proposition 4.8) we deduce:

Theorem 3 (Theorem 4.16, Lemma 4.2, Corollary 4.12). (1) *The map β^* is surjective and its kernel is generated by all Chern classes in degree > 0 of vector bundles attached to representations of the group G. In particular, the tautological ring of S_K (resp. S_K^{tor}) is equal to the image of ζ^* (resp. $\zeta^{\mathrm{tor},*}$).*

(2) *The graded $\mathbb{Q}$-algebra $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$ is isomorphic to the rational cohomology ring $H^{2\bullet}(\boldsymbol{X}^{\vee}, \mathbb{Q})$ of the complex manifold $\boldsymbol{X}^{\vee}$.*

As a consequence we obtain:

Corollary 4 (Theorem 7.1, Corollary 7.2, Theorem 7.19). *In characteristic $p > 0$, Chern classes of flat automorphic bundles are zero in degree > 0. In characteristic 0, the ℓ-adic Chern classes of flat automorphic bundles are "locally" zero in degree > 0.*

Esnault and Harris [2017] prove in characteristic 0 for compact Shimura varieties (not necessarily of Hodge type) the stronger result that the ℓ-adic Chern classes of flat automorphic bundles are even globally zero, i.e., in the ℓ-adic continuous cohomology with values in the number field over which the automorphic bundle is defined.

One particular important line bundle is the Hodge line bundle $\omega^\flat(\iota) \in \mathrm{Pic}(G\text{-}\mathtt{Zip}^\mu)$ associated to an embedding ι of $(\boldsymbol{G}, \boldsymbol{X})$ in the Siegel Shimura datum. Its pullback to the Shimura variety is the determinant line bundle of the Hodge filtration of the "universal" abelian scheme attached to ι. Combining Corollary 4 with a result of Goldring and Koskivirta [2018] one gets:

Corollary 5 (Proposition 7.5). *Suppose that the adjoint group of $\boldsymbol{G}$ is $\mathbb{Q}$-simple. Then $c_1(\omega^\flat(\iota)) \in A^1(G\text{-}\mathtt{Zip}^\mu)$ does not depend on ι, up to multiplication with positive rational numbers.*

The second description of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ by Brokemper (recalled in Proposition 4.14) shows that the classes $[\bar{Z}_w]$ of closures of EO-strata form a $\mathbb{Q}$-basis of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$. Hence the tautological rings in characteristic p are generated as a $\mathbb{Q}$-vector space by the classes of the closures of EO-strata, which are indexed by the subset ${}^I W$ of the geometric Weyl group W of $\boldsymbol{G}$.

In fact, it is also possible to define classes in $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ whose pullbacks to the Shimura variety S_K are the classes of the closures of the Newton strata or of central leaves in S_K. In particular, these classes are also contained in the tautological ring of S_K. This will be pursued in another paper.

The technical heart of the paper is to relate both descriptions of Brokemper:

Theorem 6 (Section 4D). *Let G be a reductive group over $\mathbb{F}_p$, where p is any prime ($p = 2$ included), and let μ be a cocharacter of G. There is a concrete algorithm to express, for $w \in {}^I W$, the cycle class $[\bar{Z}_w] \in A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ of the closure of an EO-stratum as a polynomial in Chern classes of vector bundles on* $\mathtt{Hdg}$.

We refer to Section 4D for the meaning of the phrase "there is a concrete algorithm". By pulling back to the Shimura variety or to a toroidal compactification (for $p > 2$) we get the same descriptions of cycle classes of EO-strata in the Chow rings of S_K and S_K^{tor}.

To obtain a description as in Theorem 6, we follow a strategy already used by Ekedahl and van der Geer [2009] in the Siegel case, albeit using a somewhat different language. Following [Goldring and Koskivirta 2019a], we construct a commutative diagram

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^\mu \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathtt{Brh}_G \arrow[d, "\gamma"] \\ G\text{-}\mathtt{Zip}^\mu \arrow[r, "\beta"] & \mathtt{Hdg} \end{tikzcd} \tag{0.4}$$

where $G\text{-}\mathtt{ZipFlag}^\mu$ is the algebraic stack of flagged G-zips defined by Goldring and Koskivirta [2019a; 2019b] and where $\mathtt{Brh}_G = [B\backslash G/B] = [B\backslash *] \times_{[G\backslash *]} [B\backslash *]$ is the Bruhat stack (which is called the

Schubert stack in the articles of Goldring and Koskivirta). Here $B \subseteq G$ is a Borel subgroup. Then we proceed in three steps.

1. Calculation of cycles of Schubert varieties: In $A^\bullet(\mathtt{Brh}_G)$ there are the classes $[\overline{\mathtt{Brh}}_w]$ of Schubert varieties for $w \in W$. They can be computed as follows. The cycle class of the smallest Schubert variety $[\mathtt{Brh}_e]$ is the class of the diagonal and can be computed by a result of Graham. Then one defines explicit operators δ_w such that $[\overline{\mathtt{Brh}}_w] = \delta_w([\mathtt{Brh}_e])$. This is certainly well known but to our surprise we found this only explained in the literature for classical groups (and sometimes only over the complex numbers). Hence we explain this for arbitrary split reductive groups over an arbitrary field in Section 2.

2. Pullback to $G\text{-}\mathtt{ZipFlag}^\mu$: One describes the pullback via ψ explicitly and obtains a description for the cycle classes in $A^\bullet(G\text{-}\mathtt{ZipFlag}^\mu)$ of the closures of $Z_w^\varnothing := \psi^{-1}(\mathtt{Brh}_w)$ (Sections 4A and 4B).

3. Push down to $G\text{-}\mathtt{Zip}^\mu$: By a result of Koskivirta [2018], π induces for $w \in {}^I W$ a finite étale map $Z_w^\varnothing \to Z_w$. If $\gamma(w)$ is its degree, we obtain
$$[\bar{Z}_w] = \gamma(w)\pi_*([\bar{Z}_w^\varnothing]).$$
Using a result of Brion [1996] one can describe π_* explicitly (Theorem 4.17). Moreover, we explain how to compute $\gamma(w)$ as the number of $\mathbb{F}_p$-rational points of the flag variety of an explicitly given form of a Levi subgroup of G (Section 3F).

We also introduce the flag space over the Shimura variety (Section 6C) that classifies — roughly speaking — refinements of the Hodge filtration. This generalizes a construction of Ekedahl and van der Geer [2009] and appeared already in Goldring and Koskivirta [2019a; 2019b]. It carries a stratification obtained by pullback from the stratification of the Bruhat stack. From the analogous properties of Schubert varieties, we deduce that the closure of these strata are normal, Cohen–Macaulay, and have only rational singularities. This also generalizes results from these works.

Structure of the tautological ring. By definition the tautological ring is a quotient of $A^\bullet(\mathtt{Hdg})$, and $A^\bullet(\mathtt{Hdg})$ can be described explicitly (Remark 5.8). There is the following conjecture about the tautological ring.

Conjecture 7. The tautological ring of a smooth toroidal compactification $\boldsymbol{S}_K^{\mathrm{tor}}$ (considered as a scheme over some splitting field of $\boldsymbol{G}$) in characteristic zero or S_K^{tor} in characteristic p is isomorphic to the rational cohomology ring of the compact dual $\boldsymbol{X}^\vee$.

By the work of van der Geer [1999] and Esnault and Viehweg [2002], Conjecture 7 is known in characteristic zero in the Siegel case. In Proposition 7.16, we show that this conjecture is equivalent to the property that all Chern classes of positive degree of flat automorphic bundles vanish in the Chow ring of $\boldsymbol{S}_K^{\mathrm{tor}}$ (resp. S_K^{tor}). This equivalence has also been shown in [Esnault and Harris 2017, 1.11] if the Shimura variety is compact. We show that the conjecture always holds in characteristic p:

Theorem 8 (Theorem 7.12). *The map of graded $\mathbb{Q}$-algebras $H^{2\bullet}(X^\vee) \cong A^\bullet(G\text{-}\mathtt{Zip}^\mu) \to A^\bullet(S_K^{\mathrm{tor}})$ is injective.*

Finally, as an immediate application we obtain a very strong form of Hirzebruch–Mumford proportionality in positive characteristic (Theorem 7.20). From this we deduce a new and purely algebraic proof of Hirzebruch–Mumford proportionality for Shimura varieties of Hodge type over $\mathbb{C}$ (Corollary 7.22).

Structure of the paper. The paper starts with a preliminary section in which we recall the notion of Chow groups of quotient stacks and some basic properties of these groups. Then the main body of the paper consists of two parts.

The first part (Sections 2–4) explains how to compute cycle classes of EO-strata in the Chow ring of the stack of G-zips of type μ. This is a purely group-theoretic part and everything is done for arbitrary reductive groups, arbitrary cocharacters, and in arbitrary positive characteristic $p \geq 2$.

In Section 2 we explain how to calculate the cycle classes of Schubert varieties in the Bruhat stack of a split reductive group. All of this is well documented in the literature for classical groups.

Section 3 recalls the stack of G-zips and of G-zips "endowed with a refinement of the Hodge filtration" and defines the commutative diagram (0.4). In Section 4 we explain what maps are induced from this diagram on Chow rings. This allows us to prove Theorem 3 and to give an algorithm for the determination of cycle classes of EO-strata in $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ (Section 4D). The section closes with stating some easy functoriality properties for maps of reductive groups inducing an isomorphism on adjoint groups.

In the second part of the paper (Sections 5–7) we apply the results from the first part to Shimura varieties of Hodge type. Here we have to make the assumption $p > 2$.

In Section 5 we define the tautological ring for arbitrary Shimura varieties in characteristic 0 and for Shimura varieties of Hodge type in characteristic p, where p is a prime of good reduction.

In Section 6 we recall the definition of EO-strata and prove Theorem 2. Here we also give the definition of the flag space over the Shimura variety and its stratification.

Then we prove in Section 7 the triviality of Chern classes of flat automorphic bundles, the uniqueness of the class of a Hodge line bundle (up to positive scalar), and our results on the structure of the tautological ring and on Hirzebruch–Mumford proportionality.

In the final Section 8 we illustrate our results in the special cases of the Siegel Shimura variety, the Hilbert–Blumenthal variety, and Shimura varieties of Spin type.

Notation. All algebraic spaces and algebraic stacks are assumed to be quasiseparated and of finite type over their respective base.

Notation on reductive groups. Throughout all reductive groups are assumed to be connected (following [SGA 3$_{\text{III}}$ 1970]). Let k be a field, and let k^s be a separable closure. Suppose that G is a reductive group over k and that $T \subseteq G$ is a maximal torus, defined over k. Then we denote by $W = (N_G(T)/T)(k^s)$ the Weyl group of (G, T). It carries an action of $\Gamma = \mathrm{Gal}(k^s/k)$ by group automorphisms. We will denote by $X^*(T)$ (resp. $X_*(T)$) the group of characters (resp. cocharacters) of T_{k^s}.

Now suppose that G is quasisplit over k. Then we can choose a Borel pair $T \subseteq B \subseteq G$ defined over k. The choice of B defines a subset $\Sigma \subset W$ of simple reflections and Γ acts by automorphisms of the Coxeter system (W, Σ). We denote by $\ell(\,\cdot\,)$ the length function and by $\leq$ the Bruhat order on the

Coxeter system (W, Σ). We choose representatives $\dot{w} \in G(k^s)$ of $w \in W$ such that $(w_1\dot{}w_2) = \dot{w}_1\dot{w}_2$ if $\ell(w_1w_2) = \ell(w_1)+\ell(w_2)$. We denote by $w_0 \in W$ the element of maximal length and by $e \in W$ the identity.

For any subset $K \subseteq \Sigma$, we denote by W_K the subgroup of W generated by K, and we set

$$^{K}W := \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in K\},$$

which is a system of representatives of $W_K \backslash W$. Let $w_{0,K}$ be the element of maximal length in W_K.

We denote by $\Phi \subset X^*(T)$ (resp. $\Phi^\vee \subset X_*(T)$) the set of roots (resp. coroots) of $(G, T)_{k^s}$ and by $\Phi^+ \subset \Phi$ the set of positive roots given by B (that is, a root α is in Φ^+ if and only if $U_\alpha \subset B$). The based root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee, \Phi^+)$ and the Coxeter system (W, Σ) do not depend on the choice of (T, B), up to unique isomorphism, and are called "the" based root datum of G and "the" Weyl group of G. For a set of simple reflections $K \subset \Sigma$, we denote by $\Phi_K \subset \Phi$ the set of roots that are in the $\mathbb{Z}$-span of the simple roots corresponding to K, and let $\Phi_K^+ := \Phi^+ \cap \Phi_K$.

Let $\mu : \mathbb{G}_{m,k^s} \to G_{k^s}$ be a cocharacter of G_{k^s}. It gives rise to a pair of opposite parabolic subgroups $(P_-(\mu), P_+(\mu))$ and a Levi subgroup $L := L(\mu) = P_-(\mu) \cap P_+(\mu)$ defined by the condition that $\mathrm{Lie}(P_-(\mu))$ $\big($resp. $\mathrm{Lie}(P_+(\mu))\big)$ is the sum of the nonpositive (resp. nonnegative) weight spaces of μ in $\mathrm{Lie}(G)$. On k^s-valued points we have

$$P_+(\mu) = \{g \in G \mid \lim_{t\to 0} \mu(t)g\mu(t)^{-1} \text{ exists}\}, \qquad P_-(\mu) = \{g \in G \mid \lim_{t\to\infty} \mu(t)g\mu(t)^{-1} \text{ exists}\},$$

and $L = \mathrm{Cent}_G(\mu)$.

We will also need to consider reductive groups over more general rings than a field. Hence let S be a scheme. To simplify the notation we assume that S is connected. Let G be a reductive group scheme over S, i.e., a smooth affine group scheme over S whose geometric fibers are reductive groups. The map that attaches to $s \in S$ the isomorphism class of the based root datum of the geometric fiber $G_{\bar{s}}$ is locally constant [SGA 3$_{\text{III}}$ 1970, Exp. XXII, Proposition 2.8] and hence constant because we assumed S to be connected. Hence we may again speak of "the" based root datum of G. Let (W, Σ) be the Weyl group together with its set of simple reflections of this based root datum. Fix $I \subseteq \Sigma$, and let Par_I be the scheme of parabolic subgroups of G of type I. It is defined étale locally on S because G is split étale locally on S [SGA 3$_{\text{III}}$ 1970, Exp. XXII, Corollaire 2.3].

If $\lambda : \mathbb{G}_{m,S'} \to G_{S'}$ is a cocharacter of G defined over some covering $S' \to S$ for the étale topology, then the constructions of the parabolic subgroups $P_+(\lambda)$ and $P_-(\lambda)$ over a field generalize to arbitrary schemes [Conrad 2014, 4.1.7] and yield parabolic subgroups of $G_{S'}$. If I is the type of $P_+(\lambda)$, we also write Par_λ instead of Par_I.

In other words, we say that a parabolic subgroup P of $G_{S'}$ is of type λ if it is locally for the étale topology conjugate to $P_+(\lambda)$. In fact, P is then already locally for the Zariski topology conjugate to $P_+(\lambda)$ by [SGA 3$_{\text{III}}$ 1970, Exp. XXVI, Corollaire 5.5].

1. Chow groups of quotient stacks

Let k be a field. All Chow groups in the following will have $\mathbb{Q}$-coefficients.

1A. ***Chow rings of smooth quotient stacks.*** By a quotient stack we will mean a stack of the form $[G\backslash X]$ where X is a quasiseparated algebraic space of finite type over $\mathrm{Spec}(k)$ and G is an affine group scheme of finite type over $\mathrm{Spec}(k)$ which acts on X from the left.

For such X and G, the equivariant Chow groups $A_i^G(X)$ are defined in [Edidin and Graham 1998] as follows: Let $n = \dim X$ and $g = \dim G$. There exists a representation of G on an ℓ-dimensional k-vector space V such that there exists an open subset U of V with complement of codimension strictly bigger than $n-i$ on which G acts freely. For such a U, the quotient $G\backslash(X \times U)$ by the diagonal action exists as an algebraic space and $A_i^G(X)$ is defined to be $A_{i+\ell-g}(G\backslash(X \times U))$. By [Edidin and Graham 1998], this group does not depend on the choice of U, and in fact, by Proposition 16 in that work, the group $A_i([G\backslash X]) := A_{i+g}^G(X)$ depends up to a canonical isomorphism only on the stack $[G\backslash X]$ and not on the chosen presentation of this stack.

A quotient stack is smooth if it admits a presentation as above with X smooth. Suppose that X is in addition separated and equidimensional of dimension n. In this case for $A^i([G\backslash X]) := A_{n-g-i}([G\backslash X])$ on the graded vector space $A^\bullet([G\backslash X]) := \bigoplus_{i \geq 0} A^i([G\backslash X])$ there is a naturally defined cup product turning this group into a graded $\mathbb{Q}$-algebra [Edidin and Graham 1998, Section 2.5]. This construction has been generalized to arbitrary smooth algebraic stacks of finite type over a field by Kresch [1999]. Here we will need only the case of smooth quotient stacks.

By [Edidin and Graham 1998, Proposition 3], the equivariant Chow groups have the same functoriality properties as the usual Chow groups for G-equivariant morphisms $X \to Y$. Every representable morphism $\mathcal{X} \to \mathcal{Y}$ of quotient stacks arises in this way: For a presentation $\mathcal{Y} = [G\backslash Y]$, take $X = \mathcal{X} \times_{\mathcal{Y}} Y$. This is a G-torsor over $\mathcal{X}$ so that $\mathcal{X} = [G\backslash X]$ and by assumption it is representable by an algebraic space. This shows that $A^\bullet(\_)$ is contravariantly functorial for representable morphisms of quotient stacks and covariantly functorial for proper representable morphisms of quotient stacks. In fact, by [Kresch 1999] it is also contravariantly functorial for flat (not necessarily representable) morphisms of smooth quotient stacks.

For an algebraic group H over k, we denote the quotient stack $[H\backslash \mathrm{Spec}(k)]$ by $[H\backslash *]$. This is a smooth algebraic stack over k of dimension $-\dim(H)$. In this paper we will mainly use the following types of morphisms between quotient stacks. For all of them, $A^\bullet(\_)$ is contravariantly functorial.

Example 1.1. Let G and X be as above.

(1) For a quasiseparated algebraic space Y of finite type over k, every morphism $Y \to [G\backslash X]$ is representable.

(2) Let $\mathscr{X}$ be any equidimensional algebraic stack over k. Then every morphism $\mathscr{X} \to [G\backslash *]$ is flat of constant relative dimension. In particular, if $\varphi : H \to G$ is a homomorphism of affine algebraic groups, the canonical morphism $[H\backslash *] \to [G\backslash *]$ is flat of relative dimension $\dim(G) - \dim(H)$.

Let $\varphi : G \to H$ be a map of algebraic groups over k. Let $f : X \to Y$ be a map of quasiseparated algebraic spaces of finite type over k. Suppose that G acts on X and that H acts on Y such that $f(gx) = \varphi(g)f(x)$ for $g \in G(R)$ and $x \in X(R)$ for any k-algebra R. Then f induces a morphism of algebraic stacks $[f] : [G\backslash X] \to [H\backslash Y]$.

Lemma 1.2. (1) *If f is flat, then $[f]$ is flat.*

(2) *If φ is a monomorphism, then $[f]$ is representable.*

Proof. The first assertion is clear, and the second is a very special case of [Stacks 2005–, Tag 04YY]. $\square$

Proposition 1.3. *Let $\mathscr{X} = [G\backslash X]$ be a smooth equidimensional quotient stack over k, and let k' be a Galois extension of k with Galois group Γ. Then the canonical homomorphism*

$$A^\bullet(\mathscr{X}) \to A^\bullet(\mathscr{X}_{k'})^\Gamma$$

is an isomorphism.

Proof. This is well known (e.g., [Brokemper 2018, 1.3.6]) if $\mathscr{X}$ is an algebraic space. In general let $n := \dim(X)$ and $g := \dim(G)$. For $i \geq 0$, choose an ℓ-dimensional representation V of G and an open subset U of V such that G acts freely on U and such that $V \setminus U$ has codimension $> i$. Then

$$A^i(\mathscr{X}) = A_{n-i+\ell-g}(G\backslash(X \times U)) \xrightarrow{\sim} A_{n-i+\ell-g}(G_{k'}\backslash(X_{k'} \times U_{k'}))^\Gamma = A^i(\mathscr{X}_{k'})^\Gamma. \qquad \square$$

We will also use the following result by Brokemper [2018, 1.4.7], which shows that $A^\bullet(\,\cdot\,)$ "ignores unipotent actions".

Proposition 1.4. *Let $0 \to U \to G \to H \to 0$ be a split exact sequence of linear algebraic groups over k, where U is a smooth connected unipotent group scheme over k. Choose a splitting $H \to G$. Let X be a smooth quasiprojective G-scheme over k and endow X with the H-action via the chosen splitting. Then the pullback map*

$$A^\bullet([G\backslash X]) \to A^\bullet([H\backslash X])$$

is an isomorphism of $\mathbb{Q}$-algebras.

1B. ***A variant of a result of Leray and Hirsch.*** We have the following Leray–Hirsch–type result from [Edidin and Graham 1997]:

Proposition 1.5. *Let $\mathcal{X} \to \mathcal{Y}$ be a representable morphism of smooth quotient stacks over k. Suppose that $\mathcal{Y}$ is connected. Assume that there exists a proper and smooth algebraic space F over k which admits a decomposition into locally closed algebraic subspaces isomorphic to $\mathbb{A}^m_k$ such that $\mathcal{X} \to \mathcal{Y}$ is a Zariski-locally trivial fibration with fiber F, i.e., such that $\mathcal{X} \to \mathcal{Y}$ is Zariski-locally on $\mathcal{Y}$ isomorphic to $\mathcal{Y} \times_k F \to \mathcal{Y}$.*

Then the following hold:

(i) *For any $i \geq 0$, a family of elements of $A^i(\mathcal{X})$ restricts to a basis of $A^i(F')$ for every geometric fiber F' of $\mathcal{X} \to \mathcal{Y}$ if it does so for a single such fiber.*

(ii) *For any $i \geq 0$, there exists a family of elements of $A^i(\mathcal{X})$ which restricts to a basis of $A^i(F')$ for every geometric fiber F' of $\mathcal{X} \to \mathcal{Y}$.*

(iii) *Let $(B_i \subset A^i(\mathcal{X}))_{i \geq 0}$ be a collection of families as in* (ii). *Then $A^\bullet(\mathcal{X})$ is a free module over $A^\bullet(\mathcal{Y})$ and $\bigcup_{i\geq 0} B_i$ is a basis of the $A^\bullet(\mathcal{Y})$-module $A^\bullet(\mathcal{X})$.*

Proof. By taking presentations $\mathcal{X} = [G\backslash X]$ and $\mathcal{Y} = [G\backslash Y]$ as well as suitable $U \subset V$ for G as above the claim reduces to an analogous claim for the morphism $G\backslash(X \times U) \to G\backslash(Y \times U)$ of algebraic spaces. Then the claim is given by [Edidin and Graham 1997, Proposition 6, its proof, and Lemma 1]. □

2. The Bruhat stack and cycle classes of Schubert varieties

From now on we fix a split reductive group scheme G over the field k, a Borel subgroup $B \subset G$ over k and a maximal torus $T \subset B$ over k which is split over k.

2A. ***Chow rings of classifying stacks.*** By [Edidin and Graham 1998, Section 3.2], the Chow rings $A^\bullet([T\backslash *])$, $A^\bullet([B\backslash *])$ and $A^\bullet([G\backslash *])$ are given as follows: Every $\chi \in X^*(T)$ induces a line bundle on $[T\backslash *]$, and we get a morphism $X^*(T) \to A^1([T\backslash *])$ sending χ to the Chern class of this line bundle; see [Edidin and Graham 1998, Section 2.4]. This extends to an isomorphism

$$\mathrm{Sym}(X^*(T)_{\mathbb{Q}}) \xrightarrow{\sim} S := A^\bullet([T\backslash *]).$$

This in fact holds even with $\mathbb{Z}$-coefficients. The canonical homomorphism $A^\bullet([B\backslash *]) \to S = A^\bullet([T\backslash *])$ is an isomorphism (Proposition 1.4). The action of the Weyl group W on T induces an action of W on the abelian group $X^*(T)$ by $(w, \chi) \mapsto \chi \circ \mathrm{int}(w^{-1})$. By functoriality we obtain an action of W on the graded $\mathbb{Q}$-algebra S. Then the natural homomorphism $A^\bullet([G\backslash *]) \to S$ yields an identification

$$A^\bullet([G\backslash *]) \xrightarrow{\sim} S^W \tag{2.1}$$

(recall that we consider rational coefficients).

Example 2.2. Let $G = \mathrm{GL}_n$. Let $T \subseteq G$ be the diagonal torus identified with $\mathbb{G}_m^n$. Then $X^*(T) = \mathbb{Z}^n$ and $A^\bullet([T\backslash *]) = S = \mathbb{Q}[t_1, \dots, t_n]$, where $(t_1, \dots, t_n)$ is the standard basis of $\mathbb{Q}^n = X^*(T)_{\mathbb{Q}}$. Moreover,

$$A^\bullet([G\backslash *]) = \mathbb{Q}[t_1, \dots, t_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n],$$

where σ_i is the elementary symmetric polynomial of degree i in $t_1, \dots, t_n$.

If $\mathscr{X}$ is a smooth quotient stack and $\mathscr{V}$ is a vector bundle of rank n on $\mathscr{X}$, then $\mathscr{V}$ corresponds to a flat morphism $\alpha_{\mathscr{V}} : \mathscr{X} \to [\mathrm{GL}_n \backslash *]$ of algebraic stacks and the i-th Chern class of $\mathscr{V}$ is given by

$$c_i(\mathscr{V}) = \alpha_{\mathscr{V}}^*(\sigma_i) \in A^\bullet(\mathscr{X}).$$

The determinant $\det : \mathrm{GL}_n \to \mathbb{G}_m$ induces a flat morphism $[\mathrm{GL}_n \backslash *] \to [\mathbb{G}_m\backslash *]$ and hence a pullback morphism of $\mathbb{Q}$-algebras

$$\det^* : A^\bullet([\mathbb{G}_m\backslash *]) = \mathbb{Q}[t_1] \to A^\bullet([\mathrm{GL}_n \backslash *]) = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$$

which sends t_1 to σ_1. In particular,

$$c_1(\mathscr{V}) = c_1(\det \mathscr{V}).$$

Proposition 2.3 [Demazure 1974, 4.6]. *The homomorphism $S \to A^\bullet(G/B)$ sending $\chi \in X^*(T)$ to the Chern class of the induced line bundle on G/B is surjective and its kernel is the ideal J of S generated by the homogeneous elements of S^W of degree > 0.*

2B. *The Chow ring of the Bruhat stack.* We consider the *Bruhat stack*

$$\mathtt{Brh} := \mathtt{Brh}_G := [B\backslash *] \times_{[G\backslash *]} [B\backslash *] \cong [B\backslash G/B],$$

together with its Bruhat decomposition into the locally closed substacks $\mathtt{Brh}_w = [B\backslash BwB/B]$. We are interested in the classes $[\overline{\mathtt{Brh}}_w]$ of the closures $\overline{\mathtt{Brh}}_w$ in $A^\bullet(\mathtt{Brh})$.

Proposition 2.4. (1) *For both natural homomorphisms $S = A^\bullet([B\backslash *]) \to A^\bullet(\mathtt{Brh})$, the module $A^\bullet(\mathtt{Brh})$ is free over S with a basis given by the classes $[\overline{\mathtt{Brh}}_w]$ for $w \in W$.*

(2) *The natural homomorphism $S \otimes_{S^W} S \to A^\bullet(\mathtt{Brh})$ is an isomorphism.*

Proof. Consider $\mathtt{Brh}$ as a G/B-fibration over $[B\backslash *]$ via, say, the first projection. Let $w_0 \in W$ be the longest element. The substack $\mathtt{Brh}_{w_0} \subset \mathtt{Brh}$ is open and given by the open Bruhat cell in G/B. Since the stabilizer of w_0 in $B \times B$ is isomorphic to T, the substack $\mathtt{Brh}_{w_0}$ can be identified with $[T\backslash *]$. Hence it has a natural structure as a U^--torsor over $[B\backslash *]$, where U^- is the unipotent radical of the unique Borel subgroup B^- of G such that $B^- \cap B = T$. The pushout along the open immersion $U^- \hookrightarrow G/B$, $u \mapsto uB$, is isomorphic to $\mathtt{Brh}$. Any U^--torsor is Zariski-locally trivial and hence so is $\mathtt{Brh} \to [B\backslash *]$. Thus $\mathtt{Brh} \to [B\backslash *]$ satisfies the conditions of Proposition 1.5. Hence (1) follows from Proposition 1.5 using the fact that the closures of the Bruhat strata on G/B give a basis of $A^\bullet(G/B)$ (see [Demazure 1974, Corollaire to Proposition 1]).

For (2), we can argue as follows: Consider $S \otimes_{S^W} S$ as an S module via the first factor. The ring S is free over S^W of rank $|W|$ and hence so is $S \otimes_{S^W} S$ over S. The S-module $A^\bullet(\mathtt{Brh})$ is free of rank $|W|$ by (1). Thus $S \otimes_{S^W} S \to A^\bullet(\mathtt{Brh})$ is a homomorphism of free S-modules of the same rank and it suffices to prove that it is surjective. By Proposition 2.3, we can take homogenous elements x_i in S which map to a basis of $A^\bullet(G/B)$. Then by Proposition 1.5 the images of $1 \otimes x_i$ in $A^\bullet(\mathtt{Brh})$ form a basis of $A^\bullet(\mathtt{Brh})$ over S. This proves surjectivity. □

To give a description of the class of $\overline{\mathtt{Brh}}_w$ in $S \otimes_{S^W} S$, we now proceed as follows. We first recall a formula for the class of the diagonal $\mathtt{Brh}_e$ in $\mathtt{Brh}$ by Graham. Then we define explicit operators δ_w on $A^\bullet(\mathtt{Brh})$ such that $[\overline{\mathtt{Brh}}_w] = \delta_w[\mathtt{Brh}_e]$.

2C. *The class of the diagonal.* In [Graham 1997, Theorem 1.1] the following formula for the class of the diagonal $\mathtt{Brh}_e$ in $\mathtt{Brh}$ is proved in the case $k = \mathbb{C}$. The proof given there can be readily adapted to arbitrary fields.

For $w \in W$, let $i_w : S \otimes_{S^W} S \to S$, $r \otimes r' \mapsto rw(r')$. The map $\prod_{w \in W} i_w : S \otimes_{S^W} S \to \prod_{w \in W} S$ is injective because $\mathrm{Spec}(S \otimes_{S^W} S) = \mathrm{Spec}(S) \times_{\mathrm{Spec}(S)/W} \mathrm{Spec}(S)$.

Theorem 2.5 (Graham). *The image of $[\mathtt{Brh}_e]$ under i_e is $\prod_{\alpha \in \Phi^+} \alpha \in S$. The image of $[\mathtt{Brh}_e]$ under i_w for $w \neq 1$ is zero.*

Example 2.6. We recall the results of Fulton on the class of the diagonal for classical groups. For the classical groups GL_n, SO_{2n+1}, Sp_{2n}, and SO_{2n}, we choose the standard maximal torus $T \cong \mathbb{G}_m^n$ and Borel subgroup giving rise to the Weyl group descriptions of page 279 of [Fulton 1996] to obtain $S = \mathrm{Sym}(X^*(T)_{\mathbb{Q}})$ and the roots in $X^*(T)$. We give elements $\widetilde{[\mathtt{Brh}_e]}$ in $S \otimes_{\mathbb{Q}} S = \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]$, where x_i and y_i represent the same $\mathbb{G}_m$-factor of T. The images of these elements in $S \otimes_{S^W} S$ are $[\mathtt{Brh}_e]$. As a reference we use [Fulton 1996], where y_j is denoted by y_{n+1-j} and the Schubert variety corresponding to $w = e$ is denoted by Ω_{w_0}.

We fix $n \in \mathbb{N}$ and introduce the following polynomials:

$$\Phi := \Phi_n := \prod_{1 \le i < j \le n} (x_i - y_j) \in S \otimes_{\mathbb{Q}} S,$$

$$\Gamma_k := \det((c_{k+1+j-2i})_{1 \le i,j \le k}) \in \mathbb{Q}[c_{-k+2}, c_{-k+3}, \dots, c_{2k+1}].$$

For instance,

$$\begin{aligned} \Phi_1 &= 1, \qquad \Phi_2 = x_1 - y_2, \\ \Gamma_1 &= c_1, \qquad \Gamma_2 = c_1 c_2 - c_0 c_3. \end{aligned} \tag{2.7}$$

We also let $\sigma_1, \dots, \sigma_n$ be the elementary symmetric polynomials in n variables with $\deg(\sigma_i) = i$. We also set $\sigma_0 := 1$.

(A_{n-1}) Let $n \ge 2$. Then

$$\widetilde{[\mathtt{Brh}_e]} = \Phi_n. \tag{2.8}$$

(B_n) Let $n \ge 2$. Then

$$\begin{aligned} \widetilde{[\mathtt{Brh}_e]} &= \Phi_n \Gamma_n, \\ c_i &:= \begin{cases} \frac{1}{2}(\sigma_i(x_1, \dots, x_n) + \sigma_i(y_1, \dots, y_n)) & \text{if } 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.9}$$

(C_n) Let $n \ge 2$. Then

$$\begin{aligned} \widetilde{[\mathtt{Brh}_e]} &= \Phi_n \Gamma_n, \\ c_i &:= \begin{cases} \sigma_i(x_1, \dots, x_n) + \sigma_i(y_1, \dots, y_n) & \text{if } 0 \le i \le n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.10}$$

(D_n) Let $n \ge 3$. Then

$$\begin{aligned} \widetilde{[\mathtt{Brh}_e]} &= \Phi_n \Gamma_{n-1}, \\ c_i &:= \begin{cases} \frac{1}{2}(\sigma_i(x_1, \dots, x_n) + \sigma_i(y_1, \dots, y_n)) & \text{if } 0 \le i \le n-1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.11}$$

2D. *The Chevalley formula.* For $(\lambda, \mu) \in X^*(T) \times X^*(T)$, we have a natural line bundle $\mathscr{L}_{\lambda,\mu}$ on $\mathtt{Brh}$ with Chern class $\lambda \otimes \mu$. The following gives a version of a classical formula of Chevalley in this context:

Theorem 2.12 [Goldring and Koskivirta 2019a, Theorem 5.2.2]. (1) *The line bundle $\mathscr{L}_{\lambda,\mu}$ has a global section on* $\mathtt{Brh}_w$ *if and only if* $\mu = w^{-1}\lambda$.

(2) *The space* $H^0(\mathtt{Brh}_w, \mathscr{L}_{\lambda,w^{-1}\lambda})$ *has dimension* 1.

(3) *For $w \in W$, set $E_w := \{\alpha \in \Phi^+ \mid ws_\alpha < w, \ell(ws_\alpha) = \ell(w) - 1\}$. The divisor of any nonzero section of $\mathscr{L}_{\lambda,w^{-1}\lambda}$ on $\mathtt{Brh}_w$ is equal to*

$$\sum_{\alpha\in E_w} \langle\lambda, \alpha^\vee\rangle[\overline{\mathtt{Brh}}_{ws_\alpha}].$$

Note that the formula in loc. cit. contains an additional minus sign because there the positive roots are defined by the opposite Borel subgroup.

For $w \in W$ and $\lambda \in X^*(T)$, this implies the following relation in $A^\bullet(\mathtt{Brh})$:

$$(\lambda \otimes w^{-1}(\lambda))[\overline{\mathtt{Brh}}_w] = \sum_{\alpha\in E_w} \langle\lambda, \alpha^\vee\rangle[\overline{\mathtt{Brh}}_{ws_\alpha}]. \tag{2.13}$$

2E. *The operators δ_w.* We use certain operators on S and $S\otimes_{S^W} S$: Let n be the semisimple rank of G and $\alpha_1, \ldots, \alpha_n$ be the simple roots with respect to T and B. For $1 \le i \le n$, let $s_i := s_{\alpha_i}$ be the simple reflection in W corresponding to α_i and $P_i := B \cup Bs_iB$ the "i-th minimal parabolic" with root system $\{\pm\alpha_i\}$.

Construction 2.14. Let $X_i := [B\backslash *] \times_{[P_i\backslash *]} [B\backslash *]$, and let $p_1, p_2 : X_i \to [B\backslash *]$ be the two projections, which are proper. Then we define $\delta_i : S \to S$ to be the correspondence $p_{1,*} \circ p_2^* : S \to S$.

Construction 2.15. Let $1 \le i \le n$. Consider S as the ring of polynomial functions on $X^*(T)_{\mathbb{Q}}$, that is, the ring of functions $X^*(T)_{\mathbb{Q}} \to \mathbb{Q}$ which, with respect to some (or equivalently any) basis of $X^*(T)$, can be written as a polynomial. For $f \in S$, the element $f - s_{\alpha_i}(f)$ of S vanishes on the hyperplane in $X^*(T)_{\mathbb{Q}}$ given by the vanishing of the coroot $\alpha_i^\vee$. Hence we obtain an element $\tilde{\delta}_i(f) := (f - s_{\alpha_i}(f))/\alpha_i^\vee \in S$. This defines a $\mathbb{Q}$-linear homomorphism $\tilde{\delta}_i : S \to S$.

Theorem 2.16. (1) *For each $1 \le i \le n$, we have $\delta_i = \tilde{\delta}_i$.*

(2) *For $w \in W$, one gets a well-defined operator δ_w on S by letting $\delta_w = \delta_{i_1} \cdots \delta_{i_k}$ for any decomposition $w = s_{i_1} \cdots s_{i_k}$ with $k = \ell(w)$.*

Proof. When $k = \mathbb{C}$ and G is semisimple and simply connected, this is proven in [Bernšteĭn et al. 1973]. See Theorem 5.7 in that work for (1) and Theorem 3.4 there for (2).

The general case can be deduced from this as follows: First, using the functoriality of the various constructions with respect to homomorphisms of reductive groups inducing an isomorphism on adjoint groups one can reduce to the case that G is semisimple and simply connected. Now let $\tilde{k}$ be another algebraically closed base field, $\widetilde{G}$ the reductive group over $\tilde{k}$ with the same root datum as G, with $\widetilde{T}$, $\widetilde{B}$, $\widetilde{\mathtt{Brh}} := \mathtt{Brh}_{\widetilde{G}}$, etc. the corresponding data for $\widetilde{G}$. We have a natural W-equivariant isomorphism $A^\bullet(*/T) \cong A^\bullet(*/\widetilde{T})$ which induces an isomorphism $A^\bullet(\mathtt{Brh}) \cong A^\bullet(\widetilde{\mathtt{Brh}})$. We claim that for $w \in W$, the classes of $\overline{\mathtt{Brh}}_w$ and $\overline{\widetilde{\mathtt{Brh}}}_w$ correspond to each other under this isomorphism. For $w = e$, this follows from Theorem 2.5. From this one deduces the claim by induction on $\ell(w)$ using (2.13).

By taking $\tilde{k} = \mathbb{C}$ this implies the claim. ∎

Remark 2.17. Construction 2.15 shows that one has the following Leibniz-type formula:

$$\begin{aligned}\delta_i(fg) &= \frac{fg - s_{\alpha_i}(f)s_{\alpha_i}(g)}{\alpha_i^\vee} = \frac{(f - s_{\alpha_i}(f))g + s_{\alpha_i}(f)(g - s_{\alpha_i}(g))}{\alpha_i^\vee} \\ &= \delta_i(f)g + s_{\alpha_i}(f)\delta_i(g).\end{aligned} \tag{2.18}$$

Now, for $w \in W$, we define an operator δ_w on $A^\bullet(\mathtt{Brh}) = S \otimes_{S^W} S$ by letting the δ_w just defined on S act on the first factor. For $1 \le i \le n$, the operator $\delta_i = \delta_{s_i}$ on $S \otimes_{S^W} S$ can also be described as follows: Let $\mathtt{Brh}_i$ be the following fiber product:

$$\begin{tikzcd} \mathtt{Brh}_i \arrow[r, "q_2"] \arrow[d, "q_1"] & \mathtt{Brh} \arrow[d] \\ \mathtt{Brh} \arrow[r] & {[P_i\backslash *] \times_{[G\backslash *]} [B\backslash *]} \end{tikzcd} \tag{2.19}$$

Then $\delta_{s_i} = q_{1,*} \circ q_2^* : S \otimes_{S^W} S \to S \otimes_{S^W} S$.

Theorem 2.20. *Let $w \in W$ and $1 \le i \le n$. Then $\delta_{s_i}[\overline{\mathtt{Brh}}_w] = [\overline{\mathtt{Brh}}_{s_i w}]$ if $\ell(s_i w) = \ell(w) + 1$ and $\delta_{s_i}[\overline{\mathtt{Brh}}_w] = 0$ otherwise.*

Proof. We let P_i act on P_i/B and G/B by multiplication from the left and on products of these varieties by the diagonal action. Then the P_i-equivariant diagram

$$\begin{tikzcd} P_i/B \times P_i/B \times G/B \arrow[r, "\pi_{13}"] \arrow[d, "\pi_{23}"] & P_i/B \times G/B \arrow[d, "\pi_2"] \\ P_i/B \times G/B \arrow[r, "\pi_2"] & G/B \end{tikzcd}$$

gives a presentation of (2.19). Here the quotient morphism $P_i/B \times G/B \to \mathtt{Brh} = [B\backslash G/B]$ sends (pB, gB) to $(Bp^{-1}gB)$ and the preimage of $\mathtt{Brh}_w$ is the P_i-orbit

$$O_w := \{(pB, gB) \in P_i/B \times G/B \mid Bp^{-1}gB = BwB\}$$

in $P_i/B \times G/B$. We prove the claim by showing the corresponding claim for the classes of the closed subvarieties $[\overline{O}_w]$ in $A^\bullet(P_i/B \times G/B)$.

The image $\pi_{23}(\pi_{13}^{-1}(O_w))$ is contained in $P_i/B \times P_i wB/B = P_i/B \times (Bs_i BwB/B \cup BwB/B)$. When $s_i w < w$, the latter set is contained in $P_i/B \times \overline{BwB}/B$. Since $\dim(O_w) = \dim(BwB/B) + \dim(P_i/B) > \dim(BwB/B)$, it is then of strictly smaller dimension than $\pi_{23}^{-1}(O_w) = P_i/B \times O_w$. This proves the claim in this case.

Now assume $s_i w > w$. We have $Bs_i BwB = Bs_i wB$ and

$$\pi_{23}(\pi_{13}^{-1}(O_w)) = P_i/B \times P_i wB/B = P_i/B \times (Bs_i wB/B \cup BwB/B).$$

This is a locally closed subset of $\overline{O}_{s_i w}$ of the same dimension, hence it is open in $\overline{O}_{s_i w}$. Thus it suffices to prove that $\pi_{23} : \pi_{13}^{-1}(O_w) \to \pi_{23}(\pi_{13}^{-1}(O_w))$ is an isomorphism. For this it suffices to prove that every fiber of π_{23} above a point in this image consists of a single point. For such a point (pB, gB), the fiber of π_{23}

is isomorphic to $\{qB \in P_i/B \mid Bq^{-1}gB = BwB\}$. When $gB \in BwB/B$, the identity $Bq^{-1}gB = BwB$ implies $q \in B$ and hence the fiber consists of a single point.

Now assume $gB \in Bs_iwB/B$ and let $qB \in P_i/B$ such that $Bq^{-1}gB = BwB$. Then necessarily $q \in Bs_iB$. Then (see [Springer 1998, Lemma 8.3.6]) we can write $q = us_i$ and $g = u's_ibw$ for elements u, u' in the root group U_{α_i} associated to α_i and an element $b \in B$. Then $q^{-1}g = s_iu^{-1}u's_ibw$ with $s_iu^{-1}u's_i \in Bs_iBs_iB = B \cup Bs_iB$. Since $q^{-1}g \in BwB$, we get $s_iu^{-1}u's_i \in B \cap U_{-\alpha_i} = \{e\}$. Thus $u = u'$, which proves that the fiber of π_{23} above (qB, gB) again consists of a single point. This finishes the proof. □

By induction on $\ell(w)$, we get:

Corollary 2.21. *Let $w \in W$. Then $[\overline{\mathtt{Brh}}_w] = \delta_w[\mathtt{Brh}_e]$.*

3. The stacks of G-zips and of flagged G-zips

3A. ***The stack of G-zips.*** We recall the construction of the moduli stack of G-zips as a quotient space; see [Pink et al. 2015, Theorem 1.5]. From now on let k be an algebraic closure of $\mathbb{F}_p$ and G a reductive group scheme over $\mathbb{F}_p$. If X is an object over some $\mathbb{F}_p$-algebra, then we denote by $X^{(p)}$ the pullback of X under the absolute Frobenius. For a scheme X over $\mathbb{F}_p$, we denote by $\varphi : X \to X^{(p)} = X$ its relative Frobenius.

The zip datum. Let $\mu : \mathbb{G}_{m,k} \to G_k$ be a cocharacter of G_k. It gives rise to a pair of opposite parabolic subgroups $(P_-(\mu), P_+(\mu))$ and a Levi subgroup $L := L(\mu) = P_-(\mu) \cap P_+(\mu)$ defined by the condition that $\mathrm{Lie}(P_-(\mu))$ $\big($resp. $\mathrm{Lie}(P_+(\mu))\big)$ is the sum of the nonpositive (resp. nonnegative) weight spaces of μ in $\mathrm{Lie}(G)$. On k-valued points we have

$$P_+(\mu) = \{g \in G \mid \lim_{t\to 0} \mu(t)g\mu(t)^{-1} \text{ exists}\}, \qquad P_-(\mu) = \{g \in G \mid \lim_{t\to\infty} \mu(t)g\mu(t)^{-1} \text{ exists}\},$$

and $L = \mathrm{Cent}_G(\mu)$. We set

$$P := P_-, \qquad Q := (P_+)^{(p)}, \qquad M := L^{(p)} = \mathrm{Cent}_G(\varphi \circ \mu).$$

Hence M is a Levi subgroup of Q.

The stack of G-zips of type μ. Denote the projections to the Levi components $P \to L$ and $Q \to M$ both by $x \mapsto \bar{x}$. The zip group E is defined as

$$E := \{(x, y) \in P \times Q \mid \varphi(\bar{x}) = \bar{y}\}. \tag{3.1}$$

We let $G \times G$ act on G by $(x, y) \cdot g := xgy^{-1}$. By restriction we obtain actions of $P \times Q$ and of E on G.

We denote by

$$G\text{-}\mathtt{Zip}^{\mu} := [E\backslash G]$$

the quotient stack. It is a smooth algebraic stack of dimension 0.

Every morphism $f : G \to G'$ of reductive groups over $\mathbb{F}_p$ yields a morphism of stacks $G\text{-}\mathtt{Zip}^{\mu} \to G'\text{-}\mathtt{Zip}^{f\circ\mu}$. In particular, if $\mu' = \mathrm{int}(h) \circ \mu$ for some $h \in G(k)$, then conjugation with h yields an isomorphism $G\text{-}\mathtt{Zip}^{\mu} \xrightarrow{\sim} G\text{-}\mathtt{Zip}^{\mu'}$. Let κ be the field of definition of the conjugation class of μ. As G is quasisplit, there exists an element in that conjugacy class that is defined over κ. Therefore it is harmless to assume that μ is defined over κ, the field of definition of its conjugacy class. We do assume this from now on. Then the stack $G\text{-}\mathtt{Zip}^{\mu}$ is defined over κ as well.

3B. *Choosing a frame.*

Lemma 3.2. *Let κ be a finite extension of $\mathbb{F}_p$. Let G be a reductive group defined over $\mathbb{F}_p$, let $Q \subseteq G_\kappa$ be a parabolic subgroup, and let $M \subseteq Q$ be a Levi subgroup that is also defined over κ. Then there exists $g \in G(\kappa)$ and a Borel pair $T \subseteq B \subseteq G$ that is already defined over $\mathbb{F}_p$ with $T \subseteq {}^gM$ and $B \subseteq {}^gQ$.*

Proof. As every reductive group over a finite field is quasisplit, we can choose a maximal torus T and a Borel subgroup $B \supseteq T$ defined over $\mathbb{F}_p$. By [SGA 3$_{\text{III}}$ 1970, Exp. XXVI, Lemme 3.8], there exists a parabolic subgroup Q' defined over κ with the same type as Q such that $B \subseteq Q'$. Let M' be the unique Levi subgroup of Q' that contains T. By [SGA 3$_{\text{III}}$ 1970, Exp. XXVI, Corollaire 5.5(iv)] there exists an element $g \in G(\kappa)$ with ${}^gQ = Q'$ and ${}^gM = M'$. □

After replacing μ by some conjugate cocharacter μ', we may (and do) assume by Lemma 3.2 that there exists a Borel pair $T \subseteq B \subseteq G$ defined over $\mathbb{F}_p$ with $B \subseteq Q$ and $T \subseteq M$. If μ is defined over some finite extension κ of $\mathbb{F}_p$, we may assume that its conjugate is also defined over κ. Then T is also a maximal torus of M and hence contains its center. Hence $\varphi \circ \mu$ factors through T. Because T is defined over $\mathbb{F}_p$, also μ itself factors through T. As $B \subseteq Q$, the cocharacter $\varphi \circ \mu$ is B-dominant. Hence μ is also B-dominant because B is defined over $\mathbb{F}_p$.

Recall that we denote by (W, Σ) the Coxeter system associated to (G, B, T). The Frobenius φ on G induces an automorphism of the Coxeter system (W, Σ), which is again denoted by φ (see also Section 3D below). Let $I, J \subseteq \Sigma$ be the set of simple reflections corresponding to the conjugacy classes of P and Q, respectively.

By [Pink et al. 2011, 3.7] (and its proof), we find $z \in G(k)$ with ${}^zT = T$ such that (B, T, z) is a frame for (G, P, L, Q, M, φ) in the sense of [Pink et al. 2011, 3.6], i.e., ${}^zB \subseteq P$ and $\varphi({}^zB \cap L) = B \cap M$. In fact we can and will choose z as follows.

Lemma 3.3. *Let $z \in \mathrm{Norm}_G(T)(k)$ be a lift of $\bar{z} := w_{0,I}w_0 \in W$. Then ${}^zB \subseteq P$ and $\varphi({}^zB \cap L) = B \cap M$.*

Proof. The first claim follows from the fact that ${}^{w_0}B = \varphi({}^{w_0}B) \subset P$. The second claim follows from

$$\varphi({}^zB \cap L) = {}^{\varphi(w_{0,I})}\varphi({}^{w_0}B \cap L) = B \cap M. \qquad \square$$

By [Pink et al. 2011, 3.11], the map

$$\varphi \circ \mathrm{int}(z) : (W_I, I) \xrightarrow{\sim} (W_J, J) \tag{3.4}$$

is an isomorphism of Coxeter systems and I, J, and $\varphi \circ \mathrm{int}(z)$ are independent of the choice of the frame.

3C. *Classification of G-zips.* For $w \in W$, let $G_w \subseteq G$ be the E-orbit of $\dot{w}z$. By [Pink et al. 2011, 7.5], there is a bijection

$$^I W \leftrightarrow \{E\text{-orbits in } G\}, \quad w \mapsto G_w, \tag{3.5}$$

and $\dim(G_w) = \ell(w) + \dim(P)$ for $w \in {}^I W$. We call the corresponding locally closed algebraic substack of $G\text{-}\mathtt{Zip}^\mu$,

$$Z_w := [E \backslash G_w] \subseteq G\text{-}\mathtt{Zip}^\mu, \tag{3.6}$$

the *zip stratum corresponding to* $w \in {}^I W$. One has $\operatorname{codim}_{G\text{-}\mathtt{Zip}^\mu}(Z_w) = \dim(G) - \dim(P) - \ell(w)$.

Let $\overline{G}_w$ be the closure of the E-orbit G_w. We set $\bar{Z}_w := [E \backslash \overline{G}_w]$. This is the unique reduced closed algebraic substack of $G\text{-}\mathtt{Zip}^\mu$ whose underlying topological space is the closure of the one-point topological space underlying Z_w. By [Pink et al. 2011, 6.2], we have

$$\bar{Z}_w = \bigcup_{w' \preceq w} Z_{w'} \tag{3.7}$$

for a partial order $\preceq$ on ${}^I W$ defined in [loc. cit., 6.2]. Here we will need only the following properties of this partial order (see [He 2007, §3]).

Lemma 3.8. (1) *There exists a unique minimal element in* ${}^I W$*, namely the neutral element* e*, and a unique maximal element in* ${}^I W$*, namely* $w_{0,I} w_0$*, where* w_0 *and* $w_{0,I}$ *are unique elements of maximal length in* W *and in* W_I*, respectively.*

(2) *The partial order* $\preceq$ *is at least as fine as the Bruhat order.*

(3) *Let* $w' \preceq w$*. Then* $\ell(w') \leq \ell(w)$ *and one has* $\ell(w') = \ell(w)$ *if and only if* $w' = w$*.*

(4) *If* $w' \prec w$ *and there exists no* $u \in {}^I W$ *with* $w' \prec u \prec w$*, then* $\ell(w') = \ell(w) - 1$*.*

3D. *The action of Frobenius.* Recall that we denote by $\varphi : G \to G$ the relative Frobenius. We also denote by $\sigma : k \to k$, $x \mapsto x^p$ the arithmetic Frobenius. As T and B are defined over $\mathbb{F}_p$, we can identify canonically $T^{(p)}$ with T and $B^{(p)}$ with B. Hence the relative Frobenius induces isogenies $\varphi : T \to T$ and $\varphi : B \to B$.

Set $\mathcal{W} := \operatorname{Norm}_G(T)/T = \pi_0(\operatorname{Norm}_G(T))$, which is a finite étale group scheme over $\mathbb{F}_p$. Then $W = \mathcal{W}(k)$ is the absolute Weyl group. As $\operatorname{Norm}_G(T)$ is also defined over $\mathbb{F}_p$, the relative Frobenius φ induces an automorphism of $\mathcal{W}$ and hence an automorphism φ of the finite group W. As B is defined over $\mathbb{F}_p$, this automorphism preserves the set Σ of simple reflections in W defined by B. By functoriality, σ also defines an automorphism $w \mapsto {}^\sigma w$ of $W = \mathcal{W}(k)$ and we have $\varphi(w) = {}^{\sigma^{-1}} w$ for all $w \in W$. If T is a split torus, then $\varphi = \mathrm{id}$ on W.

We denote by $X^*(T)$ the group of characters of $T \otimes_{\mathbb{F}_p} k$. For $\lambda \in X^*(T)$, we set $\varphi(\lambda) := \lambda \circ \varphi$, which defines an endomorphism φ on the abelian group $X^*(T)$. We denote by $\lambda \mapsto {}^\sigma\lambda$ the canonical action of σ on $X^*(T)$, i.e.,

$${}^\sigma\lambda := (\mathrm{id}_{\mathbb{G}_{m,\mathbb{F}_p}} \otimes \sigma) \circ \lambda \circ (\mathrm{id}_T \otimes \sigma^{-1}).$$

Then one has, for $\lambda \in X^*(T)$,

$$\varphi(\lambda) = p^{\sigma^{-1}}\lambda. \tag{3.9}$$

If T is a split torus, then ${}^{\sigma}\lambda = \lambda$ and $\varphi(\lambda) = p\lambda$ for all $\lambda \in X^*(T)$.

By functoriality, the actions of φ and σ on $X^*(T)$ also induce actions on the graded $\mathbb{Q}$-algebra $S = \mathrm{Sym}(X^*(T))_{\mathbb{Q}}$, and for $f \in S$ of degree d, we have

$$\varphi(f) = p^{d}\,{}^{\sigma^{-1}}f. \tag{3.10}$$

3E. ***The stack of flagged G-zips of type μ.*** We fix a subset $I_0 \subseteq I$ and let P_0 be the unique parabolic subgroup of G of type I_0 with ${}^{z}B \subseteq P_0 \subseteq P$. We let E act on $G \times P/P_0$ by

$$(x, y) \cdot (g, aP_0) := (xgy^{-1}, xaP_0)$$

and set

$$G\text{-}\mathtt{ZipFlag}^{\mu, I_0} := [E\backslash(G \times P/P_0)].$$

If $I_0 = \varnothing$, then $P_0 = {}^{z}B$ and we abbreviate $G\text{-}\mathtt{ZipFlag}^{\mu} := G\text{-}\mathtt{ZipFlag}^{\mu,\varnothing}$. Note that $G\text{-}\mathtt{ZipFlag}^{\mu,I} = G\text{-}\mathtt{Zip}^{\mu}$. For $I_0' \subseteq I_0$, there are canonical projection maps

$$G\text{-}\mathtt{ZipFlag}^{\mu, I_0'} \to G\text{-}\mathtt{ZipFlag}^{\mu, I_0}$$

that are P_0/P_0'-bundles, where P_0' is the unique parabolic subgroup of type I_0' with ${}^{z}B \subseteq P_0' \subseteq P$. In particular, these maps are proper, smooth, and representable. By taking $I_0' = \varnothing$ and $I_0 = I$, we obtain a projection map

$$\pi : G\text{-}\mathtt{ZipFlag}^{\mu} \to G\text{-}\mathtt{Zip}^{\mu}.$$

Let $L_0 \subset P_0$ be the unique Levi subgroup containing T. We set

$$M_0 := L_0^{(p)} \quad \text{and} \quad Q_0 := M_0 B.$$

Then Q_0 is a parabolic subgroup containing B of type

$$J_0 := \varphi({}^{z}I_0)$$

and M_0 is the unique Levi subgroup of Q_0 containing T. Then (B, T, z) is again a frame for $(G, P_0, L_0, Q_0, M_0, \varphi)$. By [Goldring and Koskivirta 2019b, (3.2.3)], the morphism $G \times P \to G$, $(g, x) \mapsto \bar{x} g \varphi(\bar{x})^{-1}$ induces a smooth representable morphism of algebraic stacks

$$\psi^{I_0} : G\text{-}\mathtt{ZipFlag}^{\mu, I_0} \to \mathtt{Brh}^{I_0} := [P_0\backslash G/Q_0]$$

with irreducible fibers. The maps ψ^{I_0} are compatible with passing to $I_0' \subseteq I_0$.

For $I_0 = \varnothing$, we have $P_0 = {}^{z}B$ and $Q_0 = B$. Therefore $g \mapsto z^{-1}g$ yields an isomorphism $\mathtt{Brh}^{\varnothing} \xrightarrow{\sim} \mathtt{Brh}_G$ and we denote by ψ the composition

$$\psi : G\text{-}\mathtt{ZipFlag}^{\mu} \xrightarrow{\psi^{\varnothing}} \mathtt{Brh}^{\varnothing} \xrightarrow{\sim} \mathtt{Brh}_G,$$

which is a smooth representable morphism with irreducible fibers. For $w \in W$, we write

$$Z_w^{\varnothing} := \psi^{-1}(\mathtt{Brh}_w) \subseteq G\text{-}\mathtt{ZipFlag}^{\mu}. \tag{3.11}$$

Since ψ is smooth, the $Z_w^{\varnothing}$ form a stratification of $G\text{-}\mathtt{ZipFlag}^{\mu}$ whose closure relation is given by the Bruhat order on W:

$$\overline{Z_w^{\varnothing}} = \bigcup_{w' \leq w} Z_{w'}^{\varnothing}.$$

Proposition 3.12. *The strata $Z_w^{\varnothing}$ are smooth and irreducible. Their closures $\overline{Z_w^{\varnothing}}$ are normal and with only rational singularities. In particular, they are Cohen–Macaulay.*

Proof. As ψ is smooth with irreducible fibers and $\mathtt{Brh}_w$ is smooth and irreducible, the first assertion holds. The smoothness of ψ also implies that $\overline{Z_w^{\varnothing}} = \psi^{-1}(\overline{\mathtt{Brh}_w})$. Hence all remaining assertions follow from the analogous properties for Schubert varieties [Brion and Kumar 2005, 3.2.2, 3.4.3]. □

By [Koskivirta 2018, 2.2.1], we have the following:

Proposition 3.13. *The projection $\pi : G\text{-}\mathtt{ZipFlag}^{\mu} \to G\text{-}\mathtt{Zip}^{\mu}$ induces for $w \in {}^I W$ representable finite étale maps*

$$\pi_w : Z_w^{\varnothing} \to Z_w.$$

Definition 3.14. We set $\gamma(w) := \deg(\pi_w)$.

In the next section we give a description of $\gamma(w)$.

Remark 3.15. Like their name suggests, the spaces $G\text{-}\mathtt{ZipFlag}^{\mu,I_0}$ admit a moduli description as a "flag space" over $G\text{-}\mathtt{Zip}^{\mu}$. Specifically, the stack $G\text{-}\mathtt{ZipFlag}^{\mu}$ is canonically isomorphic to the moduli stack of pairs consisting of a G-zip (I, I_+, I_-, ι) of type μ as in [Pink et al. 2015, Definition 3.1], together with a P_0-subtorsor of the P-torsor I_+. See [Goldring and Koskivirta 2019b, Section 3.1] for details on this construction.

3F. *Calculation of $\gamma(w)$.*

Fix $w \in {}^I W$.

The type of $w \in {}^I W$. We recall the following construction from [Pink et al. 2011, §5]. Fix $w \in {}^I W$. Let I_w be the largest subset of I such that

$$\varphi({}^z I_w) = {}^{w^{-1}} I_w$$

and call it the *type of* w. In other words,

$$I_w = \{s \in I \mid (\mathrm{int}(w) \circ \varphi \circ \mathrm{int}(z))^k(s) \in I \text{ for all } k \geq 1\}. \tag{3.16}$$

For instance, as $\varphi({}^z I) = J$, one has $I_e = I$ if and only if $I = J$. Let P_w be the unique parabolic subgroup of type I_w with ${}^z B \subseteq P_w$, and let L_w be the unique Levi subgroup of P_w with $L_w \supseteq T$. As for an arbitrary subset of I, we obtain

$$M_w := {}^{(zw)^{-1}} L_w = L_w^{(p)} \quad \text{and} \quad Q_w := M_w B.$$

Hence Q_w is the unique parabolic subgroup containing B of type J_w, where

$$J_w := {}^{w^{-1}}I_w = \varphi({}^z I_w),$$

and M_w is the unique Levi subgroup of Q_w containing T. Note that M_w (resp. J_w) is denoted by H_w (resp. K_w) in [Pink et al. 2011, §5].

Description of $\gamma(w)$ via flag varieties. Set

$$A_w := \{x \in L_w \mid {}^{zw}\varphi(x) = x\}.$$

Then we have, by [Koskivirta 2018, 2.2.1] and [Pink et al. 2011, 8.1],

$$\gamma(w) = \#(A_w/(A_w \cap {}^z B)). \tag{3.17}$$

Lemma 3.18. (1) $(\operatorname{int}(zw) \circ \varphi)(L_w) = L_w$.

(2) $(\operatorname{int}(zw) \circ \varphi)(L_w \cap {}^z B) = L_w \cap {}^z B$.

Proof. The first assertion follows from ${}^{zw}\varphi({}^z I_w) = {}^z I_w$. Let us show the second assertion. Both sides are Borel subgroups of L_w which contain T. Hence it suffices to show that they contain the same root subgroups. Let Φ be the set of roots for (G, T), and let Φ^+ be the set of positive roots with respect to B. For a set of simple reflections K, let Φ_K be the set of roots of the standard Levi subgroup L_K of type K. Then $\Phi_K^+ := \Phi_K \cap \Phi^+$ is the system of positive roots given by the Borel subgroup $L_K \cap B$ of L_K.

Because z normalizes T, we can consider its image in W, which we denote again by z. Then the set of roots corresponding to L_w is ${}^z\Phi_{I_w}$ and the set of roots corresponding to $L_w \cap {}^z B$ is ${}^z\Phi_{I_w}^+$. So we must show

$${}^w\varphi({}^z\Phi_{I_w}^+) = \Phi_{I_w}^+.$$

As both sides have the same cardinality, it suffices to show that the left side is contained in the right side. By definition of a frame, we have $\varphi({}^z B \cap L_w) \subseteq B \cap M_w$, and this shows

$${}^w\varphi({}^z\Phi_{I_w}^+) \subseteq {}^w\Phi_{J_w}^+ = \Phi_{I_w}^+,$$

because ${}^w J_w = I_w$. □

Hence $\operatorname{int} zw \circ \varphi$ defines a descent datum from k to $\mathbb{F}_p$ for the reductive group L_w together with its Borel subgroup ${}^z B \cap L_w$. We obtain a reductive group L'_w and a Borel subgroup B'_w defined over $\mathbb{F}_p$ and its full flag variety by $F\ell_w := L'_w/B'_w$. Then we have by (3.17) the following description of $\gamma(w)$.

Proposition 3.19. *For $w \in {}^I W$, one has*

$$\gamma(w) = L'_w(\mathbb{F}_p)/B'_w(\mathbb{F}_p) = F\ell_w(\mathbb{F}_p). \tag{3.20}$$

Here the second identity follows from $H^1(\mathbb{F}_p, B'_w) = 0$.

Remark 3.21. By definition, L'_w is a form defined over $\mathbb{F}_p$ of the standard Levi subgroup of G corresponding to the set of simple reflections I_w. It is split if and only if ${}^w\varphi({}^z s) = s$ for all $s \in I_w$. If the

Dynkin diagram of L_w has no automorphisms (e.g., if it is connected of type B_n, C_n, E_7, E_8, F_4, or G_2), then this is automatic.

If L'_w is split, one obtains, from the decomposition of the flag variety $F\ell_w$ into a disjoint union of Schubert cells, the formula

$$\gamma(w) = \sum_{w \in W_{I_w}} p^{\ell(w)}. \tag{3.22}$$

3G. ***The key diagram.*** The projection $E \to P$, $(x, y) \mapsto x$ is a surjective homomorphism of algebraic groups. We obtain a composition

$$\beta : G\text{-}\mathtt{Zip}^{\mu} = [E\backslash G] \to [E\backslash *] \to [P\backslash *]. \tag{3.23}$$

Finally, we have a morphism $\gamma : \mathtt{Brh}_G \to [P\backslash *]$ defined as the composition

$$\gamma : \mathtt{Brh}_G = [B\backslash *] \times_{[G\backslash *]} [B\backslash *] \xrightarrow{\mathrm{pr}_1} [B\backslash *] \xrightarrow{\sim} [{}^zB\backslash *] \to [P\backslash *], \tag{3.24}$$

where the second map is induced by the isomorphism $b \mapsto zbz^{-1}$ and where the third map is induced by the inclusion ${}^zB \to P$.

The following commutative diagram, where $\alpha := \beta \circ \pi$, will be our key diagram:

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^{\mu} \arrow[r, "\psi"] \arrow[d, "\pi"'] \arrow[dr, "\alpha"] & \mathtt{Brh}_G \arrow[d, "\gamma"] \\ G\text{-}\mathtt{Zip}^{\mu} \arrow[r, "\beta"] & {[P\backslash *]} \end{tikzcd} \tag{3.25}$$

All morphisms are flat of constant relative dimension. Moreover, π is a $P/{}^zB$-bundle. Note that $P/{}^zB = L/({}^zB \cap L)$ is the full flag variety for L. In particular, π is proper, smooth, and representable.

4. Induced maps of Chow rings

In this section we describe the maps induced by the key diagram (3.25) on Chow rings. If $\mathscr{X}$ is any smooth algebraic quotient stack defined over some subfield k_0 of k, we set $A^\bullet(\mathscr{X}) := A^\bullet(\mathscr{X} \otimes_{k_0} k)$.

4A. ***The Chow ring of G-$\mathtt{Zip}^{\mu}$ and G-$\mathtt{ZipFlag}^{\mu}$.*** We recall the description of Brokemper [2018] of the Chow ring of $A^\bullet(G\text{-}\mathtt{Zip}^{\mu})$:

Recall that $S := \mathrm{Sym}(X^*(T)_{\mathbb{Q}}) = A^\bullet([T\backslash *])$. This is a graded $\mathbb{Q}$-algebra carrying an action by the Weyl group W by graded automorphisms. We also denote by $S_+ := S_{\geq 1}$ the augmentation ideal of S. Let

$$\mathcal{I} := (f - \varphi(f) \mid f \in S_+^W) \subseteq S^W \tag{4.1}$$

be the ideal generated by $f - \varphi(f)$ for $f \in S_+^W$ in S^W. Because we work with rational coefficients, there is also a simpler description of $\mathcal{I}$ (see Remark 4.13 for why the definition in (4.1) is more natural in this context).

Lemma 4.2. $$\mathcal{I} = S_+^W.$$

Proof. We have to show that $S^W_+ \subseteq \mathcal{I}$. Let $f \in S^W$ be of degree $d \geq 1$, and let $s \geq 1$ be an integer such that $T \otimes_{\mathbb{F}_p} \mathbb{F}_{p^s}$ is split. Thus σ^s acts trivially on $X^*(T)$ and hence on S. Then $\varphi^s(f) = p^{ds} f$ by (3.10) and therefore

$$(1 - p^{ds}) f = f - \varphi^s(f) = \sum_{i=1}^{s} (\varphi^{i-1}(f) - \varphi^i(f)) \in \mathcal{I}. \qquad \square$$

For every set $K \subseteq \Sigma$ of simple reflections, S^{W_K} is a finite free S^W-algebra of rank $\#(W/W_K)$, and hence the canonical map

$$S^W/\mathcal{I} \to S^{W_K}/\mathcal{I}S^{W_K}$$

is finite and faithfully flat and, in particular, injective.

We keep the notation from Section 3A. For every type $K \subseteq \Sigma$ of a parabolic subgroup, we denote by K^{o} the opposite type. Then

$$I^{\mathrm{o}} = {}^z I = \varphi^{-1}(J) \tag{4.3}$$

is a set of simple reflections and $LB = {}^z P_I$ is the standard parabolic subgroup of type ${}^z I$.

For a subgroup H of G, we denote by $[H_\varphi \backslash G]$ the quotient stack for the action of H on G by φ-conjugation $(h, g) \mapsto hg\varphi^{-1}(h)$. The following description of the Chow ring of these stacks for $H = T$ and $H = L$ is given by [Brokemper 2018, 2.3.2; 2016, 1.1] and their proofs.

Proposition 4.4. (1) *Consider the homomorphism*

$$S \otimes_{S^W} S \cong A^\bullet([B\backslash G/B]) \cong A^\bullet([T\backslash G/T]) \to A^\bullet([T_\varphi\backslash G]) \tag{4.5}$$

induced by pullback along the quotient morphism $[T_\varphi\backslash G] \to [T\backslash G/T]$ and the homomorphism

$$S \to S \otimes_{S^W} S, \quad f \mapsto f \otimes 1.$$

The composition $S \to A^\bullet([T_\varphi\backslash G])$ of these homomorphisms factors through an isomorphism of graded $\mathbb{Q}$-algebras

$$S/\mathcal{I}S \cong A^\bullet([T_\varphi\backslash G]). \tag{4.6}$$

(2) *The homomorphism $S \otimes_{S^W} S \to S/\mathcal{I}S$ given by (4.5) and (4.6) sends $f \otimes g$ to the class of $f\varphi(g)$.*

(3) *The homomorphism*

$$S^{W_{I^{\mathrm{o}}}}/\mathcal{I}S^{W_{I^{\mathrm{o}}}} \to S/\mathcal{I}S$$

induced by the inclusion $S^{W_{I^{\mathrm{o}}}} \hookrightarrow S$ is injective and free of rank $|W_{I^{\mathrm{o}}}|$.

(4) *The homomorphism $A^\bullet([L_\varphi\backslash G]) \to A^\bullet([T_\varphi\backslash G])$ induced by the quotient morphism $[T_\varphi\backslash G] \to [L_\varphi\backslash G]$ is injective. Under (4.6) it gives an isomorphism of graded $\mathbb{Q}$-algebras*

$$A^\bullet([L_\varphi\backslash G]) \cong S^{W_{I^{\mathrm{o}}}}/\mathcal{I}S^{W_{I^{\mathrm{o}}}}. \tag{4.7}$$

Proposition 4.8 (see [Brokemper 2018, 2.4.4]). (1) *The homomorphism $E \to L$, $x \mapsto \bar{x}$ induces a morphism*

$$G\text{-}\mathtt{Zip}^\mu = [E\backslash G] \to [L_\varphi\backslash G].$$

Using (4.7), on Chow rings this morphism induces an isomorphism

$$S^{W_{I^o}}/\mathcal{I}S^{W_{I^o}} \cong A^\bullet([L_\varphi\backslash G]) \cong A^\bullet(G\text{-}\mathtt{Zip}^\mu) \tag{4.9}$$

of graded $\mathbb{Q}$-algebras.

(2) *For the group scheme $E' = E \cap ({}^zB \times G)$, we have a natural identification*

$$G\text{-}\mathtt{ZipFlag}^\mu = [E\backslash(G \times P/{}^zB)] = [E'\backslash G]. \tag{4.10}$$

Under this identification, the homomorphism $E' \to T$, $(x, y) \mapsto \bar{x}$ induces a morphism

$$G\text{-}\mathtt{ZipFlag}^\mu \to [T_\varphi\backslash G].$$

Using (4.6), on Chow rings this morphism induces an isomorphism

$$S/\mathcal{I}S \cong A^\bullet([T_\varphi\backslash G]) \cong A^\bullet(G\text{-}\mathtt{ZipFlag}^\mu) \tag{4.11}$$

of graded $\mathbb{Q}$-algebras.

(3) *Under the isomorphisms (4.9) and (4.11), the homomorphism $\pi^* : S^{W_{I^o}}/\mathcal{I}S^{W_{I^o}} \to S/\mathcal{I}S$ induced on Chow rings by the projection $\pi : G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$ is the one induced by the inclusion $S^{W_{I^o}} \hookrightarrow S$.*

Proof. The kernels of the surjective homomorphisms $E \to L$ and $E' \to T$ are unipotent. So (1) and (2) follow from Proposition 1.4. Then (3) follows from the compatibility of the various constructions. □

The above results allow us to give a noncanonical identification of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ with the rational cohomology ring of a certain flag variety. Let $\boldsymbol{G}_\mathbb{C}$ be the reductive group over $\mathbb{C}$ with the same based root datum as G_k, let $\boldsymbol{P}$ be a parabolic subgroup of type I of $\boldsymbol{G}_\mathbb{C}$, and set $\boldsymbol{X}^\vee := \boldsymbol{G}_\mathbb{C}/\boldsymbol{P}_I$. If (G, μ) is induced by a Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ (see Section 5 below), then $\boldsymbol{X}^\vee$ is the compact dual of $\boldsymbol{X}$. This explains the notation. Write

$$H^{2\bullet}(\boldsymbol{X}^\vee) := \bigoplus_{i=0}^{d} H^{2i}(\boldsymbol{X}^\vee(\mathbb{C}), \mathbb{Q})$$

to denote the cohomology ring of the complex manifold $\boldsymbol{X}^\vee(\mathbb{C})$ with rational coefficients. The multiplication is given by cup product. As the cohomology is concentrated in even degree, this is a commutative graded $\mathbb{Q}$-algebra.

Corollary 4.12. *There is an isomorphism of graded $\mathbb{Q}$-algebras $A^\bullet(G\text{-}\mathtt{Zip}^\mu) \cong H^{2\bullet}(\boldsymbol{X}^\vee)$.*

Proof. We use the description of $\mathcal{I}$ as in Lemma 4.2. Then we have isomorphisms of graded $\mathbb{Q}$-algebras

$$A^\bullet(G\text{-}\mathtt{Zip}^\mu) \cong S^{W_{I^o}}/\mathcal{I}S^{W_{I^o}} \cong S^{W_I}/\mathcal{I}S^{W_I} \cong H^{2\bullet}(\boldsymbol{X}^\vee),$$

where the first isomorphism is given by (4.9), the second isomorphism is given by conjugation with the longest element w_0 in the Weyl group, and the third isomorphism holds by [Borel 1953, Theorem 26.1], identifying $\boldsymbol{X}^\vee$ with a quotient of the real compact form of $\boldsymbol{G}_{\mathbb{C}}$. $\square$

Remark 4.13. Recall that we work with $\mathbb{Q}$-coefficients, hence we may use the description of $\mathcal{I}$ in Lemma 4.2. But from the results of Brokemper it follows that the results of Proposition 4.8 even hold with $\mathbb{Z}$-coefficients if one uses the description (4.1) of $\mathcal{I}$ and if the group G is special, i.e., every étale G-torsor is already Zariski-locally trivial. Examples for special groups are GL_n, SL_n, GSp_{2n}, or Sp_{2n}. A nonexample would be PGL_n for $n \geq 2$.

By [Brokemper 2018, 2.4.10, 2.4.11], we also have the following result.

Proposition 4.14. *The ring $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ is a finite $\mathbb{Q}$-algebra of dimension $\#^I W$. A $\mathbb{Q}$-basis is given by the classes $[\bar{Z}_w]$ of the closures of the E-orbits on G.*

Even for special groups this result cannot be strengthened to integral coefficients, as the more precise description of the integral Chow ring of $G\text{-}\mathtt{Zip}^\mu$ given in [Brokemper 2018, 2.4.12] for $G = \mathrm{GL}_n$ shows. The examples calculated in the Section 8 below suggest that the index of the abelian group generated by the classes $[\bar{Z}_w]$ in the integral Chow ring of $G\text{-}\mathtt{Zip}^\mu$ is of the form $f_{R,\mu}(p)$ for a polynomial $f_{R,\mu} \in \mathbb{Z}[T]$ that depends only on the based root datum R of G with its automorphism given by Frobenius and the cocharacter μ.

4B. ***Pullback maps for the key diagram.*** We now apply $A^\bullet(-)$ as a contravariant functor to the key diagram. Recall that $\mathcal{I} = S_+^W$ is the augmentation ideal of S^W. We have $A^\bullet([P\backslash *]) = A^\bullet([L\backslash *]) = S^{W_{I^\circ}}$ by Proposition 1.4 and (2.1), and

$$A^\bullet(\mathtt{Brh}_G) = S \otimes_{S^W} S = (S \otimes_{\mathbb{Q}} S)/(1 \otimes f - f \otimes 1 \mid f \in \mathcal{I})$$

by Proposition 2.4. Using this, (4.9) and (4.11), we obtain the following commutative diagram of graded $\mathbb{Q}$-algebras by applying $A^\bullet(-)$ to (3.25):

$$\begin{tikzcd} S/\mathcal{I}S & & S \otimes_{S^W} S \arrow[ll, "\psi^*"'] \\ S^{W_{I^\circ}}/\mathcal{I}S^{W_{I^\circ}} \arrow[u, hook, "\pi^*"'] & & S^{W_{I^\circ}} \arrow[ll, "\beta^*"] \arrow[u, "\gamma^*"'] \arrow[ull, "\alpha^*"'] \end{tikzcd} \tag{4.15}$$

Theorem 4.16. *The morphisms in (4.15) are as follows:*

(1) *The homomorphisms π^* and α^* are induced from the inclusion $S^{W_{I^\circ}} \hookrightarrow S$.*

(2) *The homomorphism β^* is the canonical projection.*

(3) *The homomorphism γ^* is the composition (using (4.3))*

$$\gamma^* : S^{W_{I^\circ}} = z(S^{W_I}) \xrightarrow{z^{-1}} S^{W_I} \xrightarrow{f \mapsto f \otimes 1} S \otimes_{S^W} S.$$

(4) *The homomorphism ψ^* is induced by*

$$f \otimes g \mapsto z(f)\varphi(g).$$

Proof. The description of π^* is given by Proposition 4.8. Since π^* is injective by Proposition 4.4, the descriptions of α^* and β^* will follow from those of ψ^* and γ^* since (4.15) commutes. The description of γ^* follows from the definition of γ and the construction of the isomorphism $A^\bullet(\mathtt{Brh}_G) \cong S \otimes_{S^W} S$.

To verify the description of ψ^*, we consider the following commutative diagram:

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^\mu = [E'\backslash G] \arrow[r, "\psi"] \arrow[d] & \mathtt{Brh}_G = [(B\times B)\backslash G] \arrow[d] \\ {[T_\varphi\backslash G]} \arrow[r] & {[(T\times T)\backslash G]} \end{tikzcd}$$

The morphisms in this diagram are given as follows: The morphism ψ is induced from $G \to G$, $g \mapsto z^{-1}g$ and $E' \to B \times B$, $(x, y) \mapsto ({}^{z^{-1}}x, x)$. Similarly the bottom horizontal morphism is induced from $G \to G$, $g \mapsto z^{-1}g$ and $T \to T \times T$, $t \mapsto ({}^{z^{-1}}t, \varphi(t))$. The left vertical morphism is the one from Proposition 4.8 and the right vertical one is induced from the identity on G and the projection $B \times B \to (B \times B)/\operatorname{rad}^u(B \times B) \cong T \times T$, where rad^u denotes the unipotent radical.

The two vertical morphisms induce isomorphisms on Chow rings. Using Proposition 4.4 one checks that the bottom horizontal morphism induces the morphism $S \otimes_{S^W} S \to S/IS$ which sends $f \otimes g$ to the class of $z(f)\varphi(g)$. This shows what we want. $\square$

4C. *Description of π_*.* The morphism $\pi : G\text{-}\mathtt{ZipFlag}^\mu \to G\text{-}\mathtt{Zip}^\mu$, being a $P/{}^zB$-bundle, is proper. Hence under (4.9) and (4.11) it induces a pushforward morphism

$$\pi_* : A^\bullet(G\text{-}\mathtt{ZipFlag}^\mu) \cong S/\mathcal{I}S \to A^\bullet(G\text{-}\mathtt{Zip}^\mu) \cong S^{W_{I^\circ}}/\mathcal{I}S^{W_{I^\circ}}.$$

As an application of a general pushforward formula of Brion [1996], we get the following description of π_*:

Theorem 4.17. *The pushforward $\pi_* : S/\mathcal{I}S \to S^{W_{I^\circ}}/\mathcal{I}S^{W_{I^\circ}}$ sends the class of $f \in S$ to the class of*

$$\frac{\sum_{w\in W_{I^\circ}}(-1)^{\ell(w)}w(f)}{\prod_{\alpha\in\Phi^+_{I^\circ}}\alpha} \in S^{W_{I^\circ}}.$$

Proof. Consider the following cartesian diagram:

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^\mu \arrow[r, "\psi^\varnothing"] \arrow[d, "\pi"'] & \mathtt{Brh}^\varnothing \cong [{}^zB\backslash G/B] \cong [*/{}^zB] \times_{[*/G]} [*/B] \arrow[r, "\mathrm{pr}_1"] & {[*/{}^zB]} \arrow[d, "\tilde\pi"] \\ G\text{-}\mathtt{Zip}^\mu \arrow[rr] & & {[*/P]} \end{tikzcd}$$

On Chow rings this induces the following maps:

$$\begin{tikzcd} S/\mathcal{I}S \arrow[d, bend left, "\pi_*"] & S \arrow[l] \arrow[d, bend left, "\tilde\pi_*"] \\ S^{W_{I^\circ}}/\mathcal{I}S^{W_{I^\circ}} \arrow[u, bend left, "\pi^*"] & S^{W_{I^\circ}} \arrow[l] \arrow[u, bend left, "\tilde\pi^*"] \end{tikzcd}$$

Hence it suffices to prove the corresponding formula for $\tilde{\pi}_* : A^\bullet([*/{}^zB]) \cong S \to A^\bullet([*/P]) \cong S^{W_{I^\circ}}$. Similarly, using the cartesian diagram

$$\begin{tikzcd} {[*/{}^zB \cap L]} \arrow[r] \arrow[d, "\tilde{\tilde{\pi}}"'] & {[*/{}^zB]} \arrow[d, "\tilde{\pi}"'] \\ {[*/L]} \arrow[r] & {[*/P]} \end{tikzcd}$$

whose horizontal morphisms induce isomorphisms on Chow groups, one reduces to proving the corresponding formula for $\tilde{\tilde{\pi}}_*$. This formula is given by [Brion 1996, Proposition 1.1]. □

The following gives an alternative way of computing the expression in Theorem 4.17:

Lemma 4.18 [Demazure 1973, Lemme 4]. *For $f \in S$, we have*

$$\delta_{w_{0,I^\circ}}(f) = \frac{\sum_{w \in W_{I^\circ}} (-1)^{\ell(w)} w(f)}{\prod_{\alpha \in \Phi^+_{I^\circ}} \alpha},$$

where $\delta_{w_{0,I^\circ}} : S \to S$ is the operator associated to the longest element w_{0,I° of W_{I° by Theorem 2.16.

So the following diagram is commutative:

$$\begin{tikzcd} S \arrow[r, "\delta_{w_{0,I^\circ}}"] \arrow[d] & S \arrow[d] \\ S/\mathcal{I}S \arrow[r, "\pi_*"'] & S^{W_{I^\circ}}/\mathcal{I}S^{W_{I^\circ}} \end{tikzcd}$$

From Proposition 3.13, we now get the following:

Proposition 4.19. *For $w \in {}^IW$, we have $[\bar{Z}_w] = \gamma(w)\pi_*([\bar{Z}^\varnothing_w])$ in $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$.*

4D. *Computing the cycle classes of the Ekedahl–Oort strata on G-$\mathtt{Zip}^\mu$.* By putting together the above results we get the following procedure for computing the classes $[\bar{Z}_w]$ in $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ for $w \in {}^IW$:

For computations, it is convenient to replace the rings appearing in the diagram (4.15) with certain simpler rings mapping surjectively onto them. For this we consider the following diagram of graded algebras, in which all rings are either polynomial rings or subrings of polynomial rings:

$$\begin{tikzcd} S & S \otimes_{\mathbb{Q}} S \arrow[l, "\tilde{\psi}^*"'] \\ S^{W_{I^\circ}} \arrow[u, hook, "\tilde{\pi}^*"'] & S^{W_{I^\circ}} \arrow[l, "\tilde{\beta}^*"] \arrow[u, "\tilde{\gamma}^*"'] \end{tikzcd} \tag{4.20}$$

Here we define the homomorphisms as follows:

(i) The homomorphism $\tilde{\pi}^*$ is the inclusion $S^{W_{I^\circ}} \hookrightarrow S$.

(ii) The homomorphism $\tilde{\beta}^*$ is the identity.

(iii) The homomorphism $\tilde{\gamma}^*$ is the composition

$$\tilde{\gamma}^* : S^{W_{I^\circ}} = z(S^{W_I}) \xrightarrow{z^{-1}} S^{W_I} \xrightarrow{f \mapsto f \otimes 1} S \otimes_{S^W} S.$$

(iv) The homomorphism $\tilde{\psi}^*$ is given by

$$f \otimes g \mapsto z(f)\varphi(g).$$

Using Theorem 4.16 one readily checks that under the canonical surjections from the objects in the diagram (4.20) to the corresponding objects in the diagram (4.15) these two diagrams are compatible. Similarly, using Theorem 4.17, one checks that the morphism $\pi_* : S/\mathcal{I}S \to S^{W_{I^\circ}}/\mathcal{I}S^{W_{I^\circ}}$ lifts to a morphism $\tilde{\pi}_* : S \to S^{W_{I^\circ}}$ given by the formula from Theorem 4.17.

In the following, for a class c in one of the algebras of (4.15), we will refer to a lift of c to the corresponding algebra in (4.20) as a formula for c. Then, for $w \in {}^I W$, we can compute a formula for $[\bar{Z}_w]$ as follows:

(i) Using the results from Section 2C one finds a formula for the class of the diagonal $\mathtt{Brh}_e$ in $S \otimes S$.

(ii) The operator δ_w on $S \otimes_{S^W} S$ from Section 2E lifts to an operator on $S \otimes S$ by letting the operator δ_w on S from Section 2E act on the first factor of $S \otimes S$. Then, by Corollary 2.21, by applying this operator δ_w to a formula for $[\mathtt{Brh}_e]$ one gets a formula for the class $[\overline{\mathtt{Brh}}_w]$.

(iii) By the definition of the subscheme $Z_w^\varnothing$ of G-$\mathtt{ZipFlag}^\mu$, the image of a formula for $[\overline{\mathtt{Brh}}_w]$ under the homomorphism $\tilde{\psi}^*$ gives a formula for the class $[\bar{Z}_w^\varnothing]$.

(iv) By applying $\tilde{\pi}_*$ to a formula for $[\bar{Z}_w^\varnothing]$ one gets a formula for $\pi_*([\bar{Z}_w^\varnothing])$.

(v) Using the results from Section 3F one computes the number $\gamma(w)$.

(vi) Using Proposition 3.13 by multiplying the results of the previous two steps we get a formula for $[\bar{Z}_w] = \gamma(w)\pi_*([\bar{Z}_w^\varnothing])$.

4E. ***Functoriality in the zip datum.*** To simplify notation it is often convenient for the computations in Section 4D to replace G by some other group $\widetilde{G}$. Here we explain that this is harmless as long as G and $\widetilde{G}$ have the same adjoint group.

Let (G, μ) and $(\widetilde{G}, \tilde{\mu})$ be two pairs consisting of a reductive group over $\mathbb{F}_p$ and a cocharacter defined over the algebraic closure k of $\mathbb{F}_p$. Let

$$f : G \to \widetilde{G}$$

be a map of algebraic groups over $\mathbb{F}_p$ with $f \circ \mu = \tilde{\mu}$. Let κ (resp. $\tilde{\kappa}$) be the field of definition of the conjugacy class of μ (resp. of $\tilde{\mu}$). Then $\tilde{\kappa} \subseteq \kappa$.

Let P and Q be the parabolics and E the zip group attached to (G, μ) as in Section 3A. Let $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{E}$ be the parabolics and zip group attached similarly to $(\widetilde{G}, \tilde{\mu})$. Then f induces maps $P \to \widetilde{P}$, $Q \to \widetilde{Q}$, and $E \to \widetilde{E}$ and hence a morphism

$$[f] : G\text{-}\mathtt{Zip}^\mu \to \widetilde{G}\text{-}\mathtt{Zip}^{\tilde{\mu}} \otimes_{\tilde{\kappa}} \kappa$$

of smooth algebraic quotient stacks over κ. Every map $f : G \to \widetilde{G}$ of algebraic groups can be factorized into the composition of a faithfully flat map $G \to G' = G/\operatorname{Ker}(f)$ and a closed embedding $G' \to \widetilde{G}$. If G is reductive, then G' is reductive. Therefore the following lemma implies, in particular, that the pullback $[f]^*$ on Chow rings exists.

Lemma 4.21. (1) *If f is flat, then $[f]$ is flat.*

(2) *If f is a monomorphism, then $[f]$ is representable.*

Proof. This follows from Lemma 1.2 because if f is a monomorphism, then the induced map $E \to \widetilde{E}$ is also a monomorphism. □

Lemma 4.22. *Suppose that f induces an isomorphism of adjoint groups $G^{\mathrm{ad}} \xrightarrow{\sim} \widetilde{G}^{\mathrm{ad}}$.*

(1) *Let $\widetilde{Z}$ be the radical of $\widetilde{G}$. Let (T, B, z) be a frame as in Section 3B for (G, μ). Set $\widetilde{T} := \widetilde{Z} f(T)$ and $\widetilde{B} := \widetilde{Z} f(B)$. Then $(\widetilde{T}, \widetilde{B}, f(z))$ is a frame for $(\widetilde{G}, \tilde{\mu})$.*

(2) *The map f induces an isomorphism $W \xrightarrow{\sim} \widetilde{W}$ of the Weyl groups with their set of simple reflections attached to (G, B, T) and $(\widetilde{G}, \widetilde{B}, \widetilde{T})$, respectively.*

(3) *The morphism $[f]$ of algebraic stacks induces a homeomorphism of the underlying topological spaces.*

Proof. The hypothesis on f means that $\operatorname{Ker}(f)$ is central and that $\operatorname{Cent}(\widetilde{G}) f(G) = \widetilde{G}$. As $\widetilde{Z} f(G)$ is of finite index in $\operatorname{Cent}(\widetilde{G}) f(G)$ and $\widetilde{G}$ is connected, this implies $\widetilde{Z} f(G) = \widetilde{G}$. As $\widetilde{Z}$ is a torus and clearly commutes with $f(T)$, $\widetilde{T} := \widetilde{Z} f(T)$ is a torus. Its dimension is the reductive rank of $\widetilde{G}$. Hence it is a maximal torus. By hypothesis, f induces a bijection between the roots of (G, T) and of $(\widetilde{G}, \widetilde{T})$. This shows that $\widetilde{Z} f(B)$ is a Borel subgroup and that f induces an isomorphism $W \xrightarrow{\sim} \widetilde{W}$. This implies all remaining assertions. □

We continue to assume that f induces an isomorphism of adjoint groups $G^{\mathrm{ad}} \xrightarrow{\sim} \widetilde{G}^{\mathrm{ad}}$ and use the notation of the lemma. We identify W with $\widetilde{W}$ via the isomorphism induced by f.

We define $\widetilde{G}\text{-}\mathtt{ZipFlag}^{\tilde{\mu}}$ and $\mathtt{Brh}_{\widetilde{G}}$ using $(\widetilde{G}, \widetilde{P}, \widetilde{Q}, \widetilde{B})$. Then using the description of $G\text{-}\mathtt{ZipFlag}^{\mu}$ given in (4.10) one sees that f also induces a map $[\tilde{f}]$ on stacks of flagged G-zips making the diagram

$$\begin{tikzcd} G\text{-}\mathtt{ZipFlag}^{\mu} \arrow[r, "{[\tilde{f}]}"] \arrow[d] & \widetilde{G}\text{-}\mathtt{ZipFlag}^{\tilde{\mu}} \otimes_{\tilde{\kappa}} \kappa \arrow[d] \\ G\text{-}\mathtt{Zip}^{\mu} \arrow[r, "{[f]}"] & \widetilde{G}\text{-}\mathtt{Zip}^{\tilde{\mu}} \otimes_{\tilde{\kappa}} \kappa \end{tikzcd}$$

commute. Moreover, the same arguments as above show that the pullback $[\tilde{f}]^*$ on Chow rings exists.

The key diagram (3.25) and the corresponding diagram of Chow rings (4.15) is functorial for f. The induced map of $\mathbb{Q}$-algebras $\widetilde{S} := \operatorname{Sym}(X^*(\widetilde{T})_{\mathbb{Q}}) \to S$ is equivariant for the action of W. More precisely, if we choose splittings of the exact sequences of tori

$$1 \to \operatorname{Ker}(f)^0 \to T \to T/\operatorname{Ker}(f)^0 \to 1 \quad \text{and} \quad 1 \to f(T) \to \widetilde{T} \to \widetilde{T}/f(T) \to 1,$$

then $\widetilde{S} \to S$ is of the form

$$\widetilde{S} \twoheadrightarrow \mathrm{Sym}(X^*(f(T))_{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Sym}(X^*(T/\operatorname{Ker}(f)^0)_{\mathbb{Q}}) \hookrightarrow S,$$

where the second map is an isomorphism of $\mathbb{Q}$-algebras with W-action. The map $\widetilde{S} \to S$ is also equivariant for the action of the Frobenius because f is defined over $\mathbb{F}_p$.

Proposition 4.23. *Let $f : G \to \widetilde{G}$ be a map of algebraic groups defined over $\mathbb{F}_p$ that induces an isomorphism on adjoint groups.*

(1) *One has a commutative diagram of $\mathbb{Q}$-linear maps*

$$\begin{tikzcd} A^\bullet(\widetilde{G}\text{-}\mathtt{ZipFlag}^{\tilde{\mu}}) \arrow[r] \arrow[d, "\pi_*"'] & A^\bullet(G\text{-}\mathtt{ZipFlag}^{\mu}) \arrow[d, "\pi_*"] \\ A^\bullet(\widetilde{G}\text{-}\mathtt{Zip}^{\tilde{\mu}}) \arrow[r, "\sim"] & A^\bullet(G\text{-}\mathtt{Zip}^{\mu}) \end{tikzcd} \tag{4.24}$$

where the horizontal maps are the maps of $\mathbb{Q}$-algebras induced by f. The lower horizontal map is an isomorphism.

(2) *For $w \in {}^I W$, the numbers $\gamma(w)$ defined in Definition 3.14 for (G, μ) coincide with those defined for $(\widetilde{G}, \tilde{\mu})$.*

Proof. Under the identifications (4.9) and (4.11) the horizontal maps are both induced by the W-equivariant map $\widetilde{S} \to S$. Hence the commutativity of (4.24) follows from the concrete description of π_* in Theorem 4.17. From Proposition 4.14 and Lemma 4.22(3), we also deduce that the lower horizontal map sends a $\mathbb{Q}$-basis to a $\mathbb{Q}$-basis. In particular, it is an isomorphism.

Let us show (2). The upper horizontal map sends for all $w \in W$ the cycle $[\bar{Z}_w^{\varnothing}]$ defined for $(\widetilde{G}, \tilde{\mu})$ to the cycle $[\bar{Z}_w^{\varnothing}]$ defined for (G, μ) because ψ^* is functorial for f. Hence (2) follows from (1) and Proposition 4.19. □

Remark 4.25. One can also show that the classes $[\bar{Z}_w^{\varnothing}]$ for $w \in W$ form a basis of $A^\bullet(G\text{-}\mathtt{ZipFlag}^{\mu})$, and that hence the upper horizontal map in (4.24) is an isomorphism as well.

5. The tautological ring of a Shimura variety

5A. ***Automorphic bundles and the tautological ring in characteristic zero.*** Let $(\boldsymbol{G}, \boldsymbol{X})$ be a Shimura datum. Recall that this means that $\boldsymbol{G}$ is a connected reductive group over $\mathbb{Q}$, that $\boldsymbol{X}$ is a $\boldsymbol{G}(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to \boldsymbol{G}_{\mathbb{R}}$ of real algebraic groups, where $\mathbb{S} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$ is $\mathbb{C}^\times$ viewed as a real algebraic group, and that the pair $(\boldsymbol{G}, \boldsymbol{X})$ satisfies a list of axioms [Deligne 1979, 2.1.1].

For $h \in \boldsymbol{X}$, let μ_h be the associated cocharacter of $\boldsymbol{G}_{\mathbb{C}}$, i.e., μ_h is the restriction of

$$h_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} = \prod_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$$

to the factor indexed by $\mathrm{id} \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. For each faithful finite-dimensional representation $\rho : \boldsymbol{G}_{\mathbb{R}} \to \mathrm{GL}(V)$

of $\boldsymbol{G}$ over $\mathbb{R}$, the Hodge filtration induced by $\rho\circ h$ on V has as stabilizer the parabolic subgroup $P_-(\rho\circ\mu_h)$ of $\mathrm{GL}(V_{\mathbb{C}})$ (here we follow the normalizations of [Deligne 1979] using negative μ_h-weights). The $\boldsymbol{G}(\mathbb{C})$-conjugacy class of μ_h has as field of definition a finite extension E of $\mathbb{Q}$, called the reflex field.

Let $\boldsymbol{X}^\vee$ be the compact dual of $\boldsymbol{X}$. Then $\boldsymbol{X}^\vee = \mathrm{Par}_{\boldsymbol{G}_{\mathbb{C}},\mu_h^{-1}}$ is the scheme of parabolic subgroups of type μ_h^{-1}. It is a projective homogeneous $\boldsymbol{G}$-space and it is defined over E.

For each neat open compact subgroup K of $\boldsymbol{G}(\mathbb{A}_f)$, we denote by $\boldsymbol{S}_K := \mathrm{Sh}_K(\boldsymbol{G},\boldsymbol{X})$ the canonical model of the attached Shimura variety at level K. This is a smooth quasiprojective scheme over E.

Denote by $\boldsymbol{G}^c$ the quotient of $\boldsymbol{G}$ by the maximal $\mathbb{Q}$-anisotropic $\mathbb{R}$-split torus in the center of $\boldsymbol{G}$. For instance, if $(\boldsymbol{G},\boldsymbol{X})$ is of Hodge type, then $\boldsymbol{G}=\boldsymbol{G}^c$ and this is the only case that we will use later. But for future reference we explain the following notions and results in full generality. And in general $\boldsymbol{G}\neq\boldsymbol{G}^c$, for instance, if $\boldsymbol{G}=\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$ for a nontrivial totally real extension F of $\mathbb{Q}$. The action of $\boldsymbol{G}_E$ on the E-scheme $\boldsymbol{X}^\vee$ factors through $\boldsymbol{G}^c_E$.

Milne [1990, III] constructs a diagram of schemes defined over E

$$\begin{tikzcd} & \widetilde{\boldsymbol{S}}_K \arrow[dl, "\pi"'] \arrow[dr, "\tilde{\sigma}"] & \\ \boldsymbol{S}_K & & \boldsymbol{X}^\vee \end{tikzcd} \tag{5.1}$$

where π is a $\boldsymbol{G}^c_E$-torsor and $\tilde{\sigma}$ is $\boldsymbol{G}_E$-equivariant. We set

$$\mathtt{Hdg}_E := [\boldsymbol{G}^c_E\backslash\boldsymbol{X}^\vee], \tag{5.2}$$

which is an algebraic stack over E. The diagram (5.1) corresponds to a morphism of algebraic stacks

$$\sigma:\boldsymbol{S}_K\to\mathtt{Hdg}_E \tag{5.3}$$

making

$$\begin{tikzcd} \widetilde{\boldsymbol{S}}_K \arrow[r, "\tilde{\sigma}"] \arrow[d, "\pi"'] & \boldsymbol{X}^\vee \arrow[d] \\ \boldsymbol{S}_K \arrow[r, "\sigma"] & \mathtt{Hdg}_E \end{tikzcd}$$

cartesian.

Let $\boldsymbol{S}^{\mathrm{tor}}_K$ be a smooth toroidal compactification of $\boldsymbol{S}_K$. Then by [Milne 1990, V, Theorem 6.1] the morphism σ canonically extends to a morphism

$$\sigma^{\mathrm{tor}}:\boldsymbol{S}_K\to\mathtt{Hdg}_E.$$

Note that a vector bundle on the quotient stack $\mathtt{Hdg}_E=[\boldsymbol{G}^c_E\backslash\boldsymbol{X}^\vee]$ is the same as a $\boldsymbol{G}^c$-equivariant vector bundle on $\boldsymbol{X}^\vee$.

Definition 5.4. Let E' be an extension of E. A vector bundle $\mathcal{E}$ on $\boldsymbol{S}_{K,E'}$ (resp. on $\boldsymbol{S}^{\mathrm{tor}}_{K,E'}$) is called an *automorphic bundle* if there exists a vector bundle $\mathscr{E}$ on $[\boldsymbol{G}^c_E\backslash\boldsymbol{X}^\vee]_{E'}$ such that $\mathcal{E}\cong\sigma^*(\mathscr{E})$ (resp. such that $\mathcal{E}\cong(\sigma^{\mathrm{tor}})^*(\mathscr{E})$). Moreover, $(\sigma^{\mathrm{tor}})^*(\mathscr{E})$ is called the *canonical extension* of $\sigma^*(\mathscr{E})$.

Remark 5.5. Suppose that $\boldsymbol{X}^\vee(E') \neq \varnothing$, i.e., there exists a parabolic subgroup $\boldsymbol{P}$ of $\boldsymbol{G}_{E'}$ of type μ^{-1} that is defined over E'. Then the choice of $\boldsymbol{P}$ yields isomorphisms $\boldsymbol{X}^\vee_{E'} \cong \boldsymbol{G}_{E'}/\boldsymbol{P} \cong \boldsymbol{G}^c_{E'}/\boldsymbol{P}^c$, where $\boldsymbol{P}^c$ is the image of $\boldsymbol{P}$ in $\boldsymbol{G}^c_{E'}$. We obtain an isomorphism

$$\mathtt{Hdg}_{E'} \cong [\boldsymbol{P}^c\backslash *].$$

Hence in this case a vector bundle $\mathscr{E}$ on $[\boldsymbol{G}^c_E\backslash \boldsymbol{X}^\vee]_{E'}$ is the same as a finite-dimensional representation (V,η) of $\boldsymbol{P}^c$ over E', and $\sigma^*(\mathscr{E})$ is the automorphic bundle attached to the representation η.

The structure morphism $\boldsymbol{X}^\vee \to \operatorname{Spec} E$ induces a morphism of algebraic stacks

$$\tau : \mathtt{Hdg}_E \to [\boldsymbol{G}^c_E\backslash *].$$

Definition 5.6. Let E' be an extension of E. An automorphic vector bundle on $\boldsymbol{S}_{K,E'}$ (resp. on $\boldsymbol{S}^{\mathrm{tor}}_{K,E'}$) is called *flat* if it is isomorphic to a vector bundle obtained by pullback via $\sigma\circ\tau$ (resp. via $\sigma^{\mathrm{tor}}\circ\tau$) from a vector bundle on $[\boldsymbol{G}^c_{E'}\backslash *]$.

In other words, flat automorphic bundles are those given by representations of G^c. They are endowed with a canonical integrable connection which in the Hodge case is the Gauss–Manin connection.

Definition 5.7. Let E' be a field extension of E. The images of $A^\bullet(\mathtt{Hdg}_{E'})$ in $A^\bullet(\boldsymbol{S}_{K,E'})$ and in $A^\bullet(\boldsymbol{S}^{\mathrm{tor}}_{K,E'})$ are called the *tautological rings* of $\boldsymbol{S}_{K,E'}$ and of $\boldsymbol{S}^{\mathrm{tor}}_{K,E'}$, respectively. They are denoted by $\mathcal{T}_{E'}$ and $\mathcal{T}^{\mathrm{tor}}_{E'}$, respectively.

Remark 5.8. Let E' be a field extension of E, and let E'' be a Galois extension of E' with Galois group Γ. Then Γ acts on $\mathcal{T}_{E''}$ and one has $(\mathcal{T}_{E''})^\Gamma = \mathcal{T}_{E'}$ by Proposition 1.3.

In particular, assume that the reductive group $\boldsymbol{G}$ splits over E''. Then we can choose $\boldsymbol{P} \in \boldsymbol{X}^\vee(E'')$ and

$$A^\bullet(\mathtt{Hdg}_{E''}) \cong A^\bullet([\boldsymbol{P}\backslash *]) = A^\bullet([\boldsymbol{L}\backslash *]) \cong \operatorname{Sym}(X^*(\boldsymbol{T})_{\mathbb{Q}})^{W_{\boldsymbol{L}}},$$

where $\boldsymbol{L}$ is the Levi quotient of $\boldsymbol{P}$, $\boldsymbol{T} \subseteq \boldsymbol{L}$ a maximal torus and $W_{\boldsymbol{L}}$ the Weyl group of $\boldsymbol{L}$. Hence $\mathcal{T}_{E'}$ is a quotient of $\operatorname{Sym}(X^*(\boldsymbol{T})_{\mathbb{Q}})^{\Gamma\ltimes W_{\boldsymbol{L}}}$.

Example 5.9. In the Siegel case, we have $\boldsymbol{G} = \boldsymbol{G}^c = \mathrm{GSp}_{2g}$ and $\boldsymbol{P}$ is a Siegel parabolic subgroup, i.e., the stabilizer of some Lagrangian subspace. We identify $\boldsymbol{S}_K$ with the moduli space of principally polarized abelian varieties of dimension g endowed with some sufficiently fine level structure. Let $f : \mathcal{A} \to \boldsymbol{S}_K$ be the universal abelian scheme over $\boldsymbol{S}_K$.

The Hodge stack $\mathtt{Hdg} = [\boldsymbol{P}\backslash *]$ parametrizes in this case vector bundles together with a symplectic pairing that has values in some line bundle and Lagrangian subbundles. The morphism σ is the classifying map of the de Rham cohomology of $\mathcal{A}$ and its Hodge filtration where the pairing is induced by the principal polarizations.

The projection of $\boldsymbol{P}$ onto its Levi quotient $\boldsymbol{L}$ yields an isomorphism $A^\bullet(\mathtt{Hdg}) \cong A^\bullet([\boldsymbol{L}\backslash *])$. There is an isomorphism $\boldsymbol{L} \cong \mathrm{GL}_g \times \mathbb{G}_m$ for which the projection $\mathrm{GL}_g \times \mathbb{G}_m \to \mathrm{GL}_g$ yields a vector bundle $\Omega^\flat$ on $[\boldsymbol{L}\backslash *]$ whose pullback to $\boldsymbol{S}_K$ is the Hodge filtration bundle $f_*\Omega^1_{\mathcal{A}/\boldsymbol{S}_K}$, and for which the projection $\boldsymbol{L} \to \mathbb{G}_m$ is the restriction of the multiplier character of G and hence the pullback of the corresponding line bundle to S_K is trivial. Therefore in this case the tautological ring is the $\mathbb{Q}$-subalgebra generated by the Chern classes of $f_*\Omega^1_{\mathcal{A}/S_K}$ and our notion agrees with the one introduced in [Ekedahl and van der Geer 2009].

5B. ***Stacks of filtered fiber functors.*** In Remark 5.5 we explained that $\mathtt{Hdg}_{E'}$ is the classifying stack of a certain parabolic subgroup $\boldsymbol{P}^c$ of $\boldsymbol{G}^c$ if such a subgroup can be defined over E'. In this case, $\mathtt{Hdg}_{E'}$ simply classifies $\boldsymbol{P}^c$-torsors. In this subsection we briefly digress to give a moduli-theoretic description of $\mathtt{Hdg}$ in general. This will not be needed in the rest of the article.

Hence, for the moment, let k be any field, let G be a reductive group over k, and let λ be a cocharacter of G defined over some field extension k' of k. Suppose that the conjugacy class of λ is defined over k or, equivalently, that the scheme Par_λ of parabolic subgroups of type λ is defined over k. The reductive group scheme G acts on Par_λ, and we consider the quotient stack

$$\mathtt{Hdg}_{G,\lambda} := [G\backslash \mathrm{Par}_\lambda].$$

Clearly, $\mathtt{Hdg}_{G,\lambda}$ is a smooth algebraic stack over k.

Denote by $\mathtt{Rep}(G)$ the k-linear abelian rigid $\otimes$-category of finite-dimensional representations of G over k. For any k-scheme T, we denote by $\mathtt{FilLF}(T)$ the exact rigid tensor category of filtered finite locally free $\mathscr{O}_T$-modules [Pink et al. 2015, 4C].

Proposition 5.10. *The stack $\mathtt{Hdg}_{G,\lambda}$ is canonically equivalent to the stack $\mathcal{F}_\lambda$ sending a k-scheme T to the groupoid $\mathcal{F}_\lambda(T)$ of exact k-linear $\otimes$-functors $\mathtt{Rep}(G) \to \mathtt{FilLF}(T)$ of type λ* (see [Pink et al. 2015, 5.3]).

Proof. First we construct a canonical morphism $\mathcal{F}_\lambda \to \mathtt{Hdg}_{G,\lambda}$ as follows: Let T be a k-scheme and $\varphi : \mathtt{Rep}(G) \to \mathtt{FilLF}(T)$ be an exact k-linear tensor functor of type λ. Similarly, let $\varphi_\lambda : \mathtt{Rep}(G) \to \mathtt{FilLF}(\mathrm{Spec}(k'))$ be the exact k-linear tensor functor induced by the cocharacter λ. Then, by definition, the fact that φ is of type λ means that there exists an fpqc covering T' of $T_{k'}$ over which the functors φ and φ_λ become isomorphic. The group of automorphisms of $\varphi_{\lambda,T'}$ is $P_+(\lambda)_{T'}$. Hence the sheaf $\underline{\mathrm{Isom}}^\otimes(\varphi_{\lambda,T'}, \varphi_{T'})$ of tensor isomorphisms $\varphi_{\lambda,T'} \to \varphi_{T'}$ is a right $P_+(\lambda)_{T'}$-torsor over T'. Thus under the canonical isomorphism $[P_+(\lambda)_{T'}\backslash *] \cong (\mathtt{Hdg}_{G,\lambda})_{T'}$ noted above we obtain an object $\mathcal{P}_\varphi$ of $\mathtt{Hdg}_{G,\lambda}(T')$. Since φ is defined over T, there is a canonical descent datum for T'/T on $\varphi_{T'}$. This descent datum induces an analogous descent datum on $\mathcal{P}_\varphi$, so that $\mathcal{P}_\varphi$ descends canonically to an object of $\mathtt{Hdg}_{G,\lambda}(T)$. Finally one checks that the assignment $\varphi \mapsto \mathcal{P}_\varphi$ naturally extends to a morphism of groupoids $\mathcal{F}_\lambda(T) \to \mathtt{Hdg}_{G,\lambda}(T)$ and that for varying T these morphisms are compatible with base change.

To check that the morphism $\mathcal{F}_\lambda \to \mathtt{Hdg}_{G,\lambda}$ is an isomorphism of stacks we may work fpqc-locally on $\mathrm{Spec}(k)$. Hence we may assume that λ is defined over k. Then, under the above isomorphism $\mathtt{Hdg}_{G,\lambda} \cong [P_+(\lambda)\backslash *]$, the claim is given by [Pink et al. 2015, Theorem 5.6]. □

5C. ***The tautological ring in positive characteristic.*** From now on we assume that the Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ is of Hodge type. Then $\boldsymbol{G} = \boldsymbol{G}^c$. Let p be a prime of good reduction, i.e., there exists a reductive group scheme $\mathscr{G}$ over $\mathbb{Z}_p$ such that $\mathscr{G}_{\mathbb{Q}_p} = \boldsymbol{G}_{\mathbb{Q}_p}$. We fix a neat level structure $K = K^pK_p \subseteq \boldsymbol{G}(\mathbb{A}_f)$ with $K^p \subseteq \boldsymbol{G}(\mathbb{A}_f^p)$ compact open and $K_p = \mathscr{G}(\mathbb{Z}_p) \subseteq \boldsymbol{G}(\mathbb{Q}_p)$ hyperspecial.

Integral models. We fix a place v of the reflex field E over p and denote by E_v the v-adic completion of E. As $\boldsymbol{G}_{\mathbb{Q}_p}$ has a reductive model over $\mathbb{Z}_p$, E_v is an unramified extension $\mathbb{Q}_p$. Let $\mathscr{S}_K$ be the canonical smooth integral model of $\boldsymbol{S}_K$ over the ring of integers O_{E_v} defined by Kisin [2010] and Vasiu [1999] for $p > 2$ and by Kim and Madapusi Pera [2016] for $p = 2$. Let $\breve{\mathbb{Q}}_p$ be the completion of a maximal unramified extension of E_v, and let k be the residue field of the ring of integers of $\breve{\mathbb{Q}}_p$. Let κ be the residue field of O_{E_v}. Then k is an algebraic closure of κ. Let S_K be the special fiber of $\mathscr{S}_K$ over κ, and let G be the special fiber of $\mathscr{G}$. Hence G is a reductive group over $\mathbb{F}_p$.

By definition, E is the field of definition of the conjugacy class of μ_h. As conjugacy classes of cocharacters depend only on the root datum of the reductive group, we can view the $\boldsymbol{G}(\mathbb{C})$-conjugacy class of μ_h also as a $\boldsymbol{G}(\breve{\mathbb{Q}}_p)$-conjugacy $[\mu_h]_{\breve{\mathbb{Q}}_p}$ of cocharacters of $\boldsymbol{G}_{\breve{\mathbb{Q}}_p}$ because $\boldsymbol{G}_{\mathbb{Q}_p}$ splits over an unramified extension. We may also view it as a $G(k)$-conjugacy class $[\mu_h]_k$ of cocharacters of G_k. The field of definition of $[\mu_h]_{\breve{\mathbb{Q}}_p}$ is E_v and the field of definition of $[\mu_h]_k$ is κ.

As $\mathscr{G}$ and G are quasisplit, we may choose an element in $[\mu_h]_{\breve{\mathbb{Q}}_p}$ which extends to a cocharacter μ of $\mathscr{G}$ defined over O_{E_v}. We also denote by μ its reduction modulo p, a cocharacter of G_κ. As μ_h is minuscule, so is μ.

Arithmetic compactifications. We recall some results on integral compactifications by Madapusi Pera [2019]. Let $\mathscr{S}_K^{\mathrm{tor}}$ be some smooth proper toroidal compactification of the integral model $\mathscr{S}_K$. It depends on the choice of a smooth, finite, admissible rational polyhedral cone decomposition. Moreover, let $\mathscr{S}_K^{\mathrm{min}}$ be the minimal compactification of $\mathscr{S}_K$, and let

$$\pi : \mathscr{S}_K^{\mathrm{tor}} \to \mathscr{S}_K^{\mathrm{min}}$$

be the canonical morphism. It is constructed as the Stein factorization of a certain proper morphism [Madapusi Pera 2019, 5.2.1]. In particular, π is proper and has geometrically connected fibers, and for every line bundle $\mathscr{L}$ on $\mathscr{S}_K^{\mathrm{min}}$, one has a canonical isomorphism

$$\mathscr{L} \xrightarrow{\sim} \pi_*\pi^*\mathscr{L}. \tag{5.11}$$

We denote the special fibers over κ of $\mathscr{S}_K^{\mathrm{tor}}$ and of $\mathscr{S}_K^{\mathrm{min}}$ by S_K^{tor} and S_K^{min}, respectively. The restriction of π to the special fibers is again denoted by π.

Recall that a morphism $f : X \to Y$ of finite type between noetherian schemes is called *normal* if it is flat and has geometrically normal fibers. This notion is stable under base change $Y' \to Y$ and if Y is normal and $X \to Y$ is normal, then X is normal.

Lemma 5.12. *The minimal compactification $\mathscr{S}_K^{\mathrm{min}}$ is normal over O_{E_v}.*

Proof. For Shimura varieties of PEL type this is shown in [Lan 2013, 7.2.4.3] using the description of completed local rings of $\mathscr{S}_K^{\mathrm{min}}$ in geometric points [Lan 2013, 7.2.3.17]. But the same description also holds for the minimal compactification for Shimura varieties of Hodge type by [Madapusi Pera 2019, 5.2.8]. □

The tautological ring in characteristic p. The cocharacter $\mu : \mathbb{G}_{m,O_{E_v}} \to \mathscr{G}_{O_{E_v}}$ defines a parabolic subgroup $\mathscr{P} := P_-(\mu)$ of $\mathscr{G}_{O_{E_v}}$, and we set

$$\mathtt{Hdg}_{O_{E_v}} := [\mathscr{P}\backslash *].$$

Then $\mathtt{Hdg}_{O_{E_v}} \otimes_{O_{E_v}} E_v = \mathtt{Hdg}_{E_v}$ by Remark 5.5. We denote by $P := \mathscr{P}_\kappa$ the special fiber of $\mathscr{P}$ which is a parabolic subgroup of G_κ. Then we have

$$\mathtt{Hdg}_\kappa := \mathtt{Hdg}_{O_{E_v}} \otimes_{O_{E_v}} \kappa = [P\backslash *].$$

By [Madapusi Pera 2019, 5.3], the morphisms σ and σ^{tor} extend to morphisms

$$\sigma : \mathscr{S}_K \to \mathtt{Hdg}_{O_{E_v}} \quad \text{and} \quad \sigma^{\mathrm{tor}} : \mathscr{S}_K^{\mathrm{tor}} \to \mathtt{Hdg}_{O_{E_v}} \tag{5.13}$$

of smooth algebraic stacks over O_{E_v}. Let O' be a local finite flat extension of O_{E_v}. As $\mathtt{Hdg}_{O_{E_v}} = [\mathscr{P}\backslash *]$, a vector bundle on $\mathtt{Hdg}_{O'}$ is given by an algebraic representation ρ of the group scheme $\mathscr{P}_{O'}$ on some finite free O'-module. The pullback of such a vector bundle to $\mathscr{S}_{K,O'}$ via σ (resp. to $\mathscr{S}_{K,O'}^{\mathrm{tor}}$ via σ^{tor}) is denoted by

$$\mathscr{V}(\rho) \quad (\text{resp. } \mathscr{V}(\rho)^{\mathrm{tor}}).$$

Again we define vector bundles on $\mathscr{S}_K$ of this form to be *automorphic vector bundles* and $\mathscr{V}(\rho)^{\mathrm{tor}}$ is the canonical extension of $\mathscr{V}(\rho)$ to the toroidal compactification $\mathscr{S}_K^{\mathrm{tor}}$.

The morphisms σ and σ^{tor} induce on special fibers morphisms

$$\sigma : S_K \to \mathtt{Hdg}_\kappa \quad \text{and} \quad \sigma^{\mathrm{tor}} : S_K^{\mathrm{tor}} \to \mathtt{Hdg}_\kappa \tag{5.14}$$

of smooth algebraic stacks over κ. Again we have the notion of an automorphic bundle on S_K and its canonical extension to S_K^{tor}.

We now define the tautological rings in positive characteristic as in characteristic 0.

Definition 5.15. For a field extension κ' of κ, we call the images of $A^\bullet(\mathtt{Hdg}_{\kappa'})$ in $A^\bullet(S_{K,\kappa'})$ and in $A^\bullet(S_{K,\kappa'}^{\mathrm{tor}})$ the *tautological rings* of $S_{K,\kappa'}$ and of $S_{K,\kappa'}^{\mathrm{tor}}$, respectively. We denote them by $\mathcal{T}_{\kappa'}$ and $\mathcal{T}_{\kappa'}^{\mathrm{tor}}$, respectively.

6. Cycle classes of Ekedahl–Oort strata

We continue to use the notation of Section 5C, i.e., $\boldsymbol{S}_K/E$ denotes the Shimura variety attached to a Shimura datum of Hodge type $(\boldsymbol{G}, \boldsymbol{X})$ and a neat open compact subgroup $K \subset \boldsymbol{G}(\mathbb{A}_f)$, $\mathscr{S}_K/O_{E_v}$ denotes its smooth integral model at a prime p of good reduction, S_K/κ its special fiber. We denote by $\mathscr{S}_K^{\mathrm{tor}}$ a fixed smooth proper toroidal compactification of $\mathscr{S}_K$ and by S_K^{tor} its special fiber. Moreover, $\mathscr{G}$ denotes the reductive model of $\boldsymbol{G}_{\mathbb{Q}_p}$ which is endowed with a cocharacter μ defined over O_{E_v}. We denote by (G, μ) the special fiber of $(\mathscr{G}, \mu)$.

From now on we assume that $p > 2$. This hypothesis is only needed for the existence of the smooth morphism ζ and the morphism ζ^{tor} defined below in (6.1) and (6.2). It seems probable that these morphisms also exist with the stated properties for $p = 2$, using ideas from [Kim and Madapusi Pera 2016].

6A. ***Ekedahl–Oort strata.*** From the reductive group G over $\mathbb{F}_p$ and the cocharacter $\mu : \mathbb{G}_{m,\kappa} \to G_\kappa$, we obtain the stack $G\text{-}\mathtt{Zip}^\mu$ recalled in Section 3. We use all notation introduced in Section 3 for this pair (G, μ). In particular, we define $P := P_-(\mu)$, a parabolic subgroup of G of type I which is defined over κ. The choice of P yields an isomorphism $\mathtt{Hdg}_\kappa \cong [P\backslash *]$. The morphism σ (5.14) is a morphism $\sigma : S_K \to [P\backslash *]$.

In a series of papers, Viehmann and Wedhorn [2013], Zhang [2018], and Wortmann [2013] defined (for $p > 2$) a smooth morphism

$$\zeta : S_K \to G\text{-}\mathtt{Zip}^\mu, \tag{6.1}$$

which has also been extended to toroidal compactification

$$\zeta^{\mathrm{tor}} : S_K^{\mathrm{tor}} \to G\text{-}\mathtt{Zip}^\mu \tag{6.2}$$

by Goldring and Koskivirta [2019a, Theorem 6.2.1]. By Andreatta [2023, Theorem 1.2], the morphism ζ^{tor} is smooth as well. Moreover, one has by construction

$$\beta \circ \zeta = \sigma \quad \text{and} \quad \beta \circ \zeta^{\mathrm{tor}} = \sigma^{\mathrm{tor}}, \tag{6.3}$$

where β is the morphism defined in (3.23).

Recall the definition of the zip strata $Z_w \subseteq G\text{-}\mathtt{Zip}^\mu$ (3.6). The *Ekedahl–Oort strata of* S_K (resp. of S_K^{tor}) are defined for $w \in {}^I W$ as

$$S_{K,w} := \zeta^{-1}(Z_w) \quad (\text{resp. } S_{K,w}^{\mathrm{tor}} := (\zeta^{\mathrm{tor}})^{-1}(Z_w)).$$

The smoothness of ζ^{tor} implies the following properties of the EO-strata.

(1) For all $w \in {}^I W$, the $S_{K,w}$ (resp. the $S_{K,w}^{\mathrm{tor}}$) are locally closed smooth subschemes of S_K (resp. S_K^{tor}). They are equidimensional of dimension $\ell(w)$ by [Pink et al. 2011, 5.11].

(2) By (3.7), one has

$$\zeta^{-1}(\bar{Z}_w) = \bar{S}_{K,w} = \bigcup_{w' \preceq w} S_{K,w'} \tag{6.4}$$

and

$$(\zeta^{\mathrm{tor}})^{-1}(\bar{Z}_w) = \overline{S_{K,w}^{\mathrm{tor}}} = \bigcup_{w' \preceq w} S_{K,w'}^{\mathrm{tor}}. \tag{6.5}$$

(3) The map $\zeta^* : A^\bullet(G\text{-}\mathtt{Zip}^\mu) \to A^\bullet(S_K)$ of $\mathbb{Q}$-algebras sends $[\bar{Z}_w]$ to $[\bar{S}_{K,w}]$. Analogously, the map $(\zeta^{\mathrm{tor}})^* : A^\bullet(G\text{-}\mathtt{Zip}^\mu) \to A^\bullet(S_K^{\mathrm{tor}})$ of $\mathbb{Q}$-algebras sends $[\bar{Z}_w]$ to $[\bar{S}_{K,w}^{\mathrm{tor}}]$.

Proposition 6.6. *The tautological rings $\mathcal{T}$ and $\mathcal{T}^{\mathrm{tor}}$ are finite-dimensional $\mathbb{Q}$-algebras that are generated as $\mathbb{Q}$-vector spaces by $[\bar{S}_{K,w}]$ and $[\bar{S}_{K,w}^{\mathrm{tor}}]$ for $w \in {}^I W$, respectively.*

Proof. By (6.3), the tautological rings are the images of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$. Hence the claim follows from Proposition 4.14. □

We also recall and complement some results from [Goldring and Koskivirta 2019a] about the EO-strata in the minimal compactification. Let $\pi : S_K^{\mathrm{tor}} \to S_K^{\min}$ be the projection, and set

$$S_{K,w}^{\min} := \pi(S_{K,w}^{\mathrm{tor}}), \qquad w \in {}^I W. \tag{6.7}$$

Then the $S_{K,w}^{\min}$ are pairwise disjoint — in other words, $\pi^{-1}(S_{K,w}^{\min}) = S_{K,w}^{\mathrm{tor}}$ — and locally closed in $S_K^{\min}$ by [Goldring and Koskivirta 2019a, 6.3.1]. We endow $S_{K,w}^{\min}$ with the reduced scheme structure.

We will also use the following result of Goldring and Koskivirta.

Theorem 6.8 [Goldring and Koskivirta 2019a]. *The EO-strata $S_{K,w}^{\min}$ in the minimal compactification are affine for all $w \in {}^I W$.*

Theorem 6.8 implies, in particular, that the EO-strata $S_{K,w}$ are quasiaffine for all $w \in {}^I W$, which was known before.

Because π is proper, the closure relation (6.5) implies

$$\overline{S_{K,w}^{\min}} = \bigcup_{w' \preceq w} S_{K,w'}^{\min}. \tag{6.9}$$

Below (Corollary 6.16) we will also show that $S_{K,w}^{\min}$ is equidimensional of dimension $\ell(w)$.

6B. *Connectedness of unions of Ekedahl–Oort strata.* In this subsection we show that the smoothness of ζ^{tor} allows us to deduce from [Goldring and Koskivirta 2019a] certain results on the connectedness of EO-strata. These are new even in the Siegel case.

Two lemmas on connectedness. For lack of a reference we collect two probably well-known lemmas.

For a topological space X, we denote by $\pi_0(X)$ the space of connected components of X. This defines a functor π_0 from the category of topological spaces to the category of totally disconnected topological spaces which is left adjoint to the inclusion functor. If X is a noetherian scheme, then $\pi_0(X)$ is a finite discrete space.

Lemma 6.10. *Let $f : X \to Y$ be a continuous map between topological spaces with connected* (and hence nonempty) *fibers. Suppose that the topology on Y is the quotient topology of the topology on X* (as occurs, e.g., if f is closed or open). *Then $\pi_0(f) : \pi_0(Y) \xrightarrow{\sim} \pi_0(X)$ is a homeomorphism.*

Proof. For topological spaces Z and Z', let $\mathcal{C}(Z, Z')$ be the set of continuous maps $Z \to Z'$. Let S be a totally disconnected space. We have functorial bijections

$$\begin{aligned}\mathcal{C}(\pi_0(Y), S) = \mathcal{C}(Y, S) &= \{g \in \mathcal{C}(X, S) \mid g_{|f^{-1}(y)} \text{ is constant for all } y \in Y\}\\ &= \mathcal{C}(X, S) = \mathcal{C}(\pi_0(X), S),\end{aligned}$$

where the first and last equalities hold by adjointness of π_0 and the inclusion functor, the second equality holds because Y carries the quotient topology of X, and the third equality holds because all fibers of f are connected. Therefore $\pi_0(f)$ is a homeomorphism by Yoneda's lemma. □

Let $l \geq 0$ be an integer. Recall that a noetherian scheme X is called *connected in dimension* $\geq l$ if $X \setminus Z$ is connected for every closed subset $Z \subseteq X$ of dimension $< l$. Hence X is connected if and only if X is connected in dimension ≥ 0. A scheme X of finite type over a field k is called *geometrically connected in dimension* $\geq l$ if $X \otimes_k k'$ is connected in dimension $\geq l$ for all field extensions k' of k.

We recall the following variant of a theorem of Grothendieck.

Proposition 6.11. *Let k be a field, let X be a proper k-scheme, and let $D \subseteq X$ be an effective ample divisor. Let $l \geq 1$ be an integer. Suppose that the irreducible components of X have dimension $\geq l+1$ and that X is geometrically connected in dimension $\geq l$. Then the irreducible components of D have dimension $\geq l$, and D is geometrically connected in dimension $\geq l-1$.*

Proof. Let D be the vanishing locus of a section s of an ample line bundle $\mathscr{L}$. Replacing $\mathscr{L}$ and s by some power, we may assume that $\mathscr{L}$ is very ample and hence that X is a closed subscheme of projective space $\mathbb{P}_k^N$ and that $D = X \cap H$ for some hyperplane H. Then the result follows from [SGA 2 2005, Exp. XIII, 2.3]. □

Inheritance of connectedness. For any subset $A \subseteq {}^I W$, we set $Z_A := \bigcup_{w \in A} Z_w$, considered as a subspace of the underlying topological space of $G\text{-}\mathtt{Zip}^\mu$. We also set $S_{K,A} := \bigcup_{w \in A} S_{K,w}$ and define similarly subsets $S_{K,A}^{\mathrm{tor}}$ and $S_{K,A}^{\mathrm{min}}$ of S_K^{tor} and S_K^{min}, respectively. Then $\zeta^{-1}(Z_A) = S_{K,A}$ and $(\zeta^{\mathrm{tor}})^{-1}(Z_A) = S_{K,A}^{\mathrm{tor}}$.

Now let $A \subseteq {}^I W$ be a closed subset, i.e., if $w \in A$ and $w' \in {}^I W$ with $w' \preceq w$, then $w' \in A$. Let A^0 be the set of maximal elements in A with respect to $\preceq$ and set $\partial A := A \setminus A^0$. Then Z_A is closed in $G\text{-}\mathtt{Zip}^\mu$ and Z_{A^0} is open and dense in Z_A. We consider Z_A, Z_{A^0}, and $Z_{\partial A}$ as reduced locally closed algebraic substacks of $G\text{-}\mathtt{Zip}^\mu$.

The subvariety $S_{K,A}$ is closed in S_K, and S_{K,A^0} is open and dense in $S_{K,A}$ by (6.4). Analogous assertions also hold for unions of EO-strata in S_K^{tor} and S_K^{min} by (6.5) and (6.9).

For brevity we say that a scheme X of finite type over a field is *l-gc* if all irreducible components of X have dimension $\geq l+1$, and X is geometrically connected in dimension $\geq l$.

Lemma 6.12. *Let $Y \subseteq S_K^{\mathrm{min}}$ be a closed subscheme, let $A \subseteq {}^I W$ be closed as above, and set $Y_A := Y \cap S_{K,A}^{\mathrm{min}}$. Let $l \geq 1$ be an integer. If Y_A is l-gc, then $Y_{\partial A}$ is $(l-1)$-gc.*

The proof relies heavily on results from [Goldring and Koskivirta 2019a], using Proposition 6.11 as an additional ingredient.

Proof. Let $Y_A^{\mathrm{tor}} := \pi^{-1}(Y_A)$, and let

$$Y_A^{\mathrm{tor}} \xrightarrow{\pi'} Y_A' \xrightarrow{f} Y_A$$

be the Stein factorization of $\pi : Y_A^{\mathrm{tor}} \to Y_A$. As π has geometrically connected fibers, the same holds for the finite morphism f. Hence f is a universal homeomorphism. Therefore Y_A' is l-gc and it suffices to show that $Y_{\partial A}' := f^{-1}(Y_{\partial A})$ is $(l-1)$-gc.

Let ω^{tor} be the Hodge line on S_K^{tor} obtained from some Siegel embedding of the Shimura datum. Let $\omega^{\mathrm{min}} := \pi_* \omega^{\mathrm{tor}}$. By [Madapusi Pera 2019, 5.2.11], ω^{min} extends the Hodge line bundle on S_K, it is ample, and $\pi^*(\omega^{\min}) \cong \omega^{\mathrm{tor}}$. The restrictions of ω^{tor} and $\omega^{\min}$ to Y_A^{tor} and Y_A, respectively, are denoted by $\omega_{Y_A}^{\mathrm{tor}}$ and $\omega_{Y_A}^{\min}$. Set $\omega'_{Y_A} := f^*\omega_{Y_A}^{\min}$. Then for all $N > 1$ one has

$$\pi'_*\omega_{Y_A}^{\mathrm{tor},\otimes N} = \pi'_*\pi'^*(\omega_{Y_A}'^{\otimes N}) = \omega_{Y_A}'^{\otimes N}, \tag{$*$}$$

where the second equality follows from $\pi'_*(\mathscr{O}_{Y_A^{\mathrm{tor}}}) = \mathscr{O}_{Y'_A}$.

In the special case $Y = S_K^{\min}$ and $A = {}^I W$ one has $\pi' = \pi$ and we also see

$$\pi_*\omega^{\mathrm{tor},\otimes N} \cong \omega^{\min,\otimes N}. \tag{6.13}$$

By [Goldring and Koskivirta 2019a, 6.2.2], there exists an $N \geq 1$ such that for all $w \in {}^I W$ there exist sections $h_w \in \Gamma(\overline{S_{K,w}^{\mathrm{tor}}}, \omega^{\mathrm{tor},\otimes N})$ whose nonvanishing locus is $S_{K,w}^{\mathrm{tor}}$. For $w, w' \in A^0$ with $w \neq w'$,

$$h_w|_{\overline{S_{K,w}^{\mathrm{tor}}} \cap \overline{S_{K,w'}^{\mathrm{tor}}}} = 0 = h_{w'}|_{\overline{S_{K,w}^{\mathrm{tor}}} \cap \overline{S_{K,w'}^{\mathrm{tor}}}},$$

so after passing to some power of N and of h_w, we can glue the sections h_w with $w \in A^0$ to a section $h_A \in \Gamma(S_{K,A}^{\mathrm{tor}}, \omega^{\otimes N})$ whose nonvanishing locus is S_{K,A^0}^{tor} [Goldring and Koskivirta 2019a, 5.2.1]. We denote the restriction of h_A to Y_A^{tor} again by h_A. Using $(*)$ we obtain a section

$$h_A \in \Gamma(Y_A^{\mathrm{tor}}, \omega_{Y_A}^{\mathrm{tor},\otimes N}) = \Gamma(Y'_A, \omega_{Y_A}'^{\otimes N})$$

whose vanishing locus in Y'_A is precisely $Y'_{\partial A}$. As $\omega_{Y_A}'^{\otimes N}$ is the pullback of $\omega^{\min,\otimes N}$ under a finite morphism $Y'_A \to S_K^{\min}$, it is ample and we conclude by Proposition 6.11. □

Connectedness of the length strata. Let $d := \dim S_K = \dim S_K^{\mathrm{tor}} = \langle 2\rho, \mu \rangle$, where ρ denotes as usual half of the sum of all positive roots on the root datum of G. For $j = 0, \ldots, d$, we set

$$S_{K,\leq j}^? := \bigcup_{\ell(w) \leq j} S_{K,w}^? \quad \text{and} \quad S_{K,j}^? := \bigcup_{\ell(w) = j} S_{K,w}^?$$

for $? \in \{\varnothing, \mathrm{tor}, \min\}$. Then $S_{K,\leq j}^?$ is closed in $S_K^?$ and $S_{K,j}^?$ is open and dense in $S_{K,\leq j}^?$ by (6.4), (6.5), and (6.9). We endow them with the reduced subscheme structure. The closed subschemes $S_{K,\leq j}^?$ of $S_K^?$ are called *closed length strata*.

Lemma 6.14. *The schemes $S_{K,j}$ and $S_{K,j}^{\mathrm{tor}}$ are smooth.*

Proof. By Lemma 3.8, no two elements of ${}^I W$ of the same length are comparable with respect to $\preceq$. Hence $S_{K,j}^?$ is the topological sum of the $S_{K,w}^?$ for $w \in {}^I W$ with $\ell(w) = j$. This shows the lemma because $S_{K,w}$ and $S_{K,w}^{\mathrm{tor}}$ are smooth for all $w \in {}^I W$. □

Let X be a scheme of finite type over a field k. By a *geometric connected component of X* we mean a connected component Y of $X_{\bar{k}}$, where $\bar{k}$ is an algebraic closure of k. Then Y is already defined over some finite extension of k.

Theorem 6.15. *Let Y be a geometric connected component of $S_K^{\min}$, and let $Y^{\mathrm{tor}} := \pi^{-1}(Y)$ be the corresponding* (Lemma 6.10) *geometric connected component of S_K^{tor}. Then, for all $j = 1, \ldots, d$, the length strata $S_{K,\leq j}^{\mathrm{tor}} \cap Y^{\mathrm{tor}}$ and $S_{K,\leq j}^{\min} \cap Y$ are geometrically connected and equidimensional of dimension j. In particular, they are nonempty.*

Proof. We already know that $S^{\mathrm{tor}}_{K,\leq j}$ is equidimensional of dimension j. This shows that $S^{\mathrm{tor}}_{K,\leq j}\cap Y^{\mathrm{tor}}$ is either empty or equidimensional of dimension j.

Next we show that $S^{\min}_{K,\leq j}\cap Y$ is geometrically connected in dimension $\geq j-1$ and equidimensional of dimension j by descending induction on j. We have $S^{\min}_{K,\leq d}=S^{\min}_K$. Because $S^{\min}_K$ is (geometrically) normal (Lemma 5.12), Y is irreducible of dimension d, and in particular, it is geometrically connected in dimension $\geq d-1$. Now let $A_j:=\{w\in {}^I W \mid \ell(w)\leq j\}$. Then $A^0_j=\{w\in {}^I W\mid \ell(w)=j\}$ and $\partial A=A_{j-1}$ by Lemma 3.8. Hence by induction we deduce from Lemma 6.12 that $S^{\min}_{K,\leq j}\cap Y$ is geometrically connected in dimension $\geq j-1$ and that every irreducible component of $S^{\min}_{K,\leq j}\cap Y$ has dimension $\geq j$. On the other hand we have $\dim(Y)\leq \dim(S^{\min}_{K,j})=\dim(\pi(S^{\mathrm{tor}}_{K,j}))=j$. Hence $S^{\min}_{K,j}$ is equidimensional of dimension j.

This shows, in particular, that $S^{\min}_{K,\leq j}\cap Y$ is nonempty, which implies that

$$S^{\mathrm{tor}}_{K,\leq j}\cap Y^{\mathrm{tor}}=\pi^{-1}(S^{\min}_{K,\leq j}\cap Y)$$

is nonempty. Moreover, $S^{\mathrm{tor}}_{K,\leq j}\cap Y^{\mathrm{tor}}$ is geometrically connected by Lemma 6.10. □

Corollary 6.16. *Each EO-stratum $S^{\min}_{K,w}$ in the minimal compactification is equidimensional of dimension $\ell(w)$.*

Proof. Let $Y\subseteq S^{\min}_{K,w}$ be an irreducible component, and let $\overline{Y}$ be its closure in $S^{\min}_K$. By (6.9), $\overline{Y}$ is an irreducible component of $S^{\min}_{K,\leq \ell(w)}$. Hence $\dim(Y)=\dim(\overline{Y})=\ell(w)$ by Theorem 6.15. □

Corollary 6.17. *Let $S^{\mathrm{tor}}_{K,e}$ be the 0-dimensional EO-stratum in S^{tor}_K. Suppose that $S^{\mathrm{tor}}_{K,e}$ is already contained in S_K. Let Y be a geometric connected component of S^{tor}_K. Then the length 1 stratum $S_{K,\leq 1}\cap Y$ in $S_K\cap Y$ is geometrically connected.*

The condition that $S^{\mathrm{tor}}_{K,e}$ is contained in S_K is satisfied for all Shimura varieties of PEL type [Goldring and Koskivirta 2019a, 6.4.1], and we expect it to hold in general.

Proof. Let Y and Y' be irreducible components of $S^{\mathrm{tor}}_{K,\leq 1}$. As $S^{\mathrm{tor}}_{K,\leq 1}\cap Y$ is geometrically connected by Theorem 6.15, it suffices to show that $Y\cap Y'\subseteq S^{\mathrm{tor}}_{K,e}=S_{K,e}$. But this is clear because $S^{\mathrm{tor}}_{K,1}$ is smooth (Lemma 6.14) and hence cannot contain intersection points of irreducible components. □

The following result generalizes [Ekedahl and van der Geer 2009, Proposition 6.1]:

Proposition 6.18. *Let $w\in {}^I W$ and Y an irreducible component of $\overline{S^{\min}_{K,w}}$.*

(i) *The variety Y has dimension $\ell(w)$ and is geometrically connected in dimension $\geq \ell(w)-1$.*

(ii) *The intersection $Y\cap S^{\min}_{K,e}$ is nonempty.*

Proof. (i) As in the proof of Corollary 6.16, the variety Y is an irreducible component of $S^{\min}_{K,\leq \ell(w)}$. Hence (i) follows from Theorem 6.15.

(ii) Let $Y^\circ := Y\cap S^{\min}_{K,w}$. Since Y° is an irreducible component of $S^{\min}_{K,w}$, it is affine by Theorem 6.8. Thus if Y° is closed in $S^{\min}_K$ it has dimension zero, which, using (i), implies $\ell(w)=0$ and hence $w=e$. Otherwise $Y\setminus Y^\circ$ is nonempty. Let Y' be an irreducible component of $Y\setminus Y^\circ = Y\cap\bigcup_{w'\prec w} S^{\min}_{K,w'}$.

Using (i), Lemma 6.12 yields $\dim Y' \geq \ell(w)-1$. On the other hand, the inclusion $Y' \subset \bigcup_{w' \prec w} S^{\min}_{K,w'}$ and Corollary 6.16 yield $\dim Y' \leq \ell(w)-1$. Thus $\dim Y' = \ell(w)-1$, which again by Corollary 6.16 implies that Y' must be an irreducible component of $\overline{S^{\min}_{K,w'}}$ for some $w' \prec w$ with $\ell(w') = \ell(w)-1$. Now we may conclude by induction on $\ell(w)$. □

6C. *The flag space over S^{tor}_K.* Let $\pi_K : \mathcal{F}^{\mathrm{tor}}_K \to S^{\mathrm{tor}}_K$ be defined by the following fiber product:

$$\begin{tikzcd} \mathcal{F}^{\mathrm{tor}}_K \arrow[r] \arrow[d, "\pi_K"'] & G\text{-}\mathtt{ZipFlag}^{\mu} \arrow[d, "\pi"] \\ S^{\mathrm{tor}}_K \arrow[r, "\zeta^{\mathrm{tor}}"] & G\text{-}\mathtt{Zip}^{\mu} \end{tikzcd}$$

Similarly, we let $\mathcal{F}_K$ be the restriction of $\mathcal{F}^{\mathrm{tor}}_K$ to S_K. As π is representable by schemes, smooth, and proper, $\mathcal{F}^{\mathrm{tor}}_K$ and $\mathcal{F}_K$ are schemes, and π_K is smooth and proper.

By pulling back the stratification $G\text{-}\mathtt{ZipFlag}^{\mu} = \bigcup_{w \in W} Z^{\varnothing}_w$ (3.11) to $\mathcal{F}_K$ and $\mathcal{F}^{\mathrm{tor}}_K$, we obtain stratifications

$$\mathcal{F}_K = \bigcup_{w \in W} \mathcal{F}_{K,w} \quad \text{and} \quad \mathcal{F}^{\mathrm{tor}}_K = \bigcup_{w \in W} \mathcal{F}^{\mathrm{tor}}_{K,w}.$$

Proposition 6.19. *The strata $\mathcal{F}_{K,w}$ and $\mathcal{F}^{\mathrm{tor}}_{K,w}$ are smooth and equidimensional of dimension $\ell(w)$. Their closures $\overline{\mathcal{F}_{K,w}}$ and $\overline{\mathcal{F}^{\mathrm{tor}}_{K,w}}$ are normal, Cohen–Macaulay, with only rational singularities.*

Proof. Since ζ^{tor} is smooth, this follows from Proposition 3.12. □

7. Applications

7A. *Triviality of Chern classes of flat automorphic bundles.* Let E' be an extension of E. By definition, an automorphic bundle over E' is a vector bundle on $\mathrm{Sh}_K(\boldsymbol{G},\boldsymbol{X})_{E'}$ that arises by pullback of a vector bundle on $\mathtt{Hdg}_{E'}$ via the map σ. Recall that such an automorphic bundle is called *flat* if it comes from a vector bundle on $[\boldsymbol{G}_{E'} \backslash *]$ by pullback via the composition

$$\mathrm{Sh}_K(\boldsymbol{G},\boldsymbol{X})_{E'} \xrightarrow{\sigma} \mathtt{Hdg}_{E'} \to [\boldsymbol{G}_{E'} \backslash *],$$

i.e., it is an automorphic bundle associated with a finite-dimensional representation of $\boldsymbol{G}_{E'}$. Similarly, we define what it means for an automorphic bundle (or their canonical extensions to the toroidal compactification) on the integral model or its special fiber to be flat.

Now the theory of Chow rings of G-zips allows us easily to show the following result for flat automorphic bundles on the special fiber.

Theorem 7.1. *Let κ' be an extension of κ, let $\mathscr{V}$ be a flat automorphic bundle on $S_{K,\kappa'}$, and let $\mathscr{V}^{\mathrm{tor}}$ be its canonical extension to $S^{\mathrm{tor}}_{K,\kappa'}$. Then for all $i \geq 1$ the i-th Chern class of $\mathscr{V}$ in $A^{\bullet}(S_{K,\kappa'})$ and of $\mathscr{V}^{\mathrm{tor}}$ in $A^{\bullet}(S^{\mathrm{tor}}_{K,\kappa'})$ are zero.*

Proof. As all automorphic bundles are defined over some finite extension of κ, we may assume that κ' is an algebraic extension of κ. By Proposition 1.3, we may assume that $\kappa' = k$ is an algebraic closure of κ. As σ and σ^{tor} both factor through $G\text{-}\mathtt{Zip}^{\mu}$, it suffices to show that under pullback via the composition

$$G\text{-}\mathtt{Zip}^{\mu} \xrightarrow{\beta} \mathtt{Hdg}_k \xrightarrow{\nu} [G_k\backslash *],$$

all elements of degree > 0 in $A^{\bullet}([G_k\backslash *])$ are sent to 0. Here ν is the canonical projection $\mathtt{Hdg}_K = [P_k\backslash *] \to [G_k\backslash *]$ which induces via pullback on Chow rings the inclusion $A^{\bullet}([G_k\backslash *]) = S^W \hookrightarrow S^{W_{I^o}} = A^{\bullet}(\mathtt{Hdg}_k)$. Hence the description of $\mathcal{I}$ in Lemma 4.2 and of β^* in Theorem 4.16 implies the claim. □

Using proper smooth base change we obtain a triviality result for étale Chern classes in characteristic 0 as follows. For a scheme of finite type over a field k, we denote by $H^i(X, \mathbb{Q}_\ell(d))$ the i-th continuous ℓ-adic cohomology with Tate twist defined by Jannsen [1988] or, equivalently, the pro-étale cohomology defined by Bhatt and Scholze [2015]. Here ℓ is a prime different from the characteristic of k. Recall that $\boldsymbol{S}_K$ denotes a Shimura variety of Hodge type in characteristic 0 and that $\boldsymbol{S}_K^{\mathrm{tor}}$ denotes a smooth proper toroidal compactification of $\boldsymbol{S}_K$.

Corollary 7.2. *Let E' be a finite extension of the reflex field E contained in the algebraic closure $\bar{E}$ of E in $\mathbb{C}$. Let $\mathscr{V}$ be a flat automorphic bundle over $\boldsymbol{S}_{K,E'}$, and let $\mathscr{V}^{\mathrm{tor}}$ be its canonical extension to $\boldsymbol{S}^{\mathrm{tor}}_{K,E'}$. Let $p \neq \ell$ be a prime of good reduction for the Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ and v' a place of E' above p. Then for all $i \geq 1$ the i-th étale Chern classes $c_i(\mathscr{V}) \in H^{2i}(\boldsymbol{S}_{K,E'_{v'}}, \mathbb{Q}_\ell(i))$ and $c_i(\mathscr{V}^{\mathrm{tor}}) \in H^{2i}(\boldsymbol{S}^{\mathrm{tor}}_{K,E'_{v'}}, \mathbb{Q}_\ell(i))$ are zero.*

A stronger version of this statement for continuous cohomology over E' instead of $E'_{v'}$ has been proved by Esnault and Harris [2017] for compact Shimura varieties.

First we note the following fact:

Lemma 7.3. *Let $\mathscr{G}$ be a flat affine group scheme over a Dedekind ring R with quotient field Q. Every representation of $\mathscr{G}_Q$ on a finite-dimensional Q-vector space V extends to a representation of $\mathscr{G}$ on a locally free R-module of finite type.*

Proof. Let $A := \Gamma(\mathscr{G}, \mathscr{O}_{\mathscr{G}})$ be the ring of functions of $\mathscr{G}$. If $\sigma : V \to V \otimes_Q A_Q$ is the comodule map corresponding to the representation in question, then we consider V as a comodule under A via $V \xrightarrow{\sigma} V \otimes_Q \rho_Q = V \otimes_R A$. Then, using the local finiteness of A-comodules, we find an A-sub-comodule $L \subset V$ which is finitely generated over R and which generates V as a Q-vector space. Such an L is torsion-free and hence projective because R is a Dedekind domain. It gives the desired extension. □

Now we prove Corollary 7.2:

Proof. Let v be the restriction of v' to E, and let $\mathscr{S}_K^{\mathrm{tor}}$ be a smooth proper toroidal compactification of $\mathscr{S}_K$ over O_{E_v} with generic fiber $\boldsymbol{S}^{\mathrm{tor}}_{K,E_v}$. Let κ' be the residue field of $O' := O_{E'_{v'}}$. We also use the notation of Section 6. In particular, we denote by $\mathscr{G}$ a reductive model of $\boldsymbol{G}_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. Let $\mathscr{V}$ be associated to a representation $\rho : \boldsymbol{G}_{E'} \to \mathrm{GL}((E')^n)$. By Lemma 7.3, we can extend the base change $\rho_{E'_{v'}}$ to a dualizable representation $\tilde{\rho}$ of $\mathscr{G}$ over O'.

The special fiber of $\tilde{\rho}$ is then a representation of the split reductive group $G_{\kappa'}$. Let $\mathscr{V}^{\mathrm{tor}}_{\kappa'}$ be the corresponding flat automorphic bundle on $S^{\mathrm{tor}}_{K,\kappa'}$. By construction it lifts to a flat automorphic bundle over $\mathscr{S}^{\mathrm{tor}}_{K,O'}$ whose generic fiber is $\mathscr{V}^{\mathrm{tor}}$. By Theorem 7.1, the i-th Chern class of $\mathscr{V}^{\mathrm{tor}}_{\kappa'}$ in $A^i(S^{\mathrm{tor}}_{K,\kappa'})$ is zero for $i \geq 1$. In particular, its étale cycle class vanishes in

$$H^{2i}(S^{\mathrm{tor}}_{\kappa'}, \mathbb{Q}_\ell(i)) = H^{2i}(\boldsymbol{S}^{\mathrm{tor}}_{K,E'_{v'}}, \mathbb{Q}_\ell(i)),$$

where the equality holds by smooth and proper base change. But this cycle class in the space on the right-hand side is the étale cycle class of $\mathscr{V}^{\mathrm{tor}}$ because the étale cycle class map from Chow groups to étale cohomology is compatible with specialization. By restriction this implies the result for $\boldsymbol{S}_{K,E'_{v'}}$ as well. □

7B. ***The Hodge half-line.*** As the Shimura datum is of Hodge type there exists a Siegel embedding of G, i.e., an embedding $\iota : G \hookrightarrow \mathrm{GSp}(V)$ of algebraic groups over $\mathbb{F}_p$ such that $\tilde{\mu} := \iota \circ \mu$ is minuscule and the parabolic $P_+(\tilde{\mu})$ is the stabilizer of a Lagrangian subspace $U \subseteq V$. Consider the character

$$\chi(\iota) := \det(V/U)^\vee \tag{7.4}$$

of P, which is defined over κ. It corresponds to a line bundle on the Hodge stack $\mathtt{Hdg}$ over κ. We denote its pullback to $G\text{-}\mathtt{Zip}^\mu$ by $\omega^\flat(\iota)$. We call a class in $A^1(G\text{-}\mathtt{Zip}^\mu)$ a *Hodge line bundle class* if it is the first Chern class of the line bundle $\omega^\flat(\iota)$ given by a symplectic embedding.

Such a Hodge line bundle class is essentially independent of the choice of the embedding by combining Theorem 7.1 with a result of Goldring and Koskivirta.

Proposition 7.5. *Suppose that $\boldsymbol{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple. If ι and ι' are two Siegel embeddings, then there exists $\rho \in \mathbb{Q}_{>0}$ such that*

$$c_1(\omega^\flat(\iota)) = \rho c_1(\omega^\flat(\iota')) \in A^1(G\text{-}\mathtt{Zip}^\mu).$$

Proof. By Theorem 7.1, it suffices to show that there exists a character λ of G and $m, n \in \mathbb{Z}_{>0}$ such that $m\chi(\iota) = \lambda + n\chi(\iota')$ as characters of P or, equivalently, of the Levi subgroup L. Let $\widetilde{L}$ be the connected component of the preimage of L in the simply connected cover of the derived group of G, and let $\tilde{\chi}$ and $\tilde{\chi}'$ be the characters obtained from $\chi(\iota)$ and $\chi(\iota')$, respectively, by composition with $\widetilde{L} \to L$. Then it suffices to show there exist $m, n \in \mathbb{Z}_{>0}$ such that $m\tilde{\chi} = n\tilde{\chi}'$. But this is shown in [Goldring and Koskivirta 2018, 1.4.5]. □

It is easy, as was explained to us by Goldring, to give examples where the assertions fail without the assumption that $\boldsymbol{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple. Indeed if $\boldsymbol{G} := \{(A, B) \in \mathrm{GL}_{2,\mathbb{Q}} \mid \det(A) = \det(B)\}$ and $\boldsymbol{X}$ is the $\boldsymbol{G}(\mathbb{R})$-conjugacy class of

$$\mathbb{C}^\times \to \boldsymbol{G}(\mathbb{R}), \quad x + iy \mapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right),$$

then $\mathrm{Sh}_K(\boldsymbol{G}, \boldsymbol{X})$ is the Shimura variety that classifies pairs of elliptic curves (with some level structure). Let GSp_6 be the group of symplectic similitudes over $\mathbb{Q}$ defined by the alternating form

$$\langle x, y \rangle := x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 + x_5y_6 - x_6y_5 \quad \text{for } x, y \in \mathbb{Q}^6.$$

The embeddings of Shimura data $\boldsymbol{G} \to \mathrm{GSp}_6$ given by

$$(A, B) \mapsto \begin{pmatrix} A & & \\ & A & \\ & & B \end{pmatrix} \quad \text{and} \quad (A, B) \mapsto \begin{pmatrix} A & & \\ & B & \\ & & B \end{pmatrix}$$

then yield the embeddings of $\mathrm{Sh}_K(\boldsymbol{G}, \boldsymbol{X})$ into the moduli space of principally polarized abelian threefolds given by

$$(E_1, E_2) \mapsto E_1^2 \times E_2 \quad \text{and} \quad (E_1, E_2) \mapsto E_1 \times E_2^2.$$

These embeddings then yield Hodge line bundle classes in $A^1(G\text{-}\mathtt{Zip}^{\mu})$ that are not multiples of each other.

Let $\mathcal{T}$ and $\mathcal{T}^{\mathrm{tor}}$ be the tautological rings of S_K and S_K^{tor}, respectively.

Definition 7.6. Suppose that $\boldsymbol{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple. We call the $\mathbb{Q}_{>0}$ half-line in $A^1(G\text{-}\mathtt{Zip}^{\mu})$ generated by $c_1(\omega^{\flat}(\iota))$ the *Hodge half-line of* G-$\mathtt{Zip}^{\mu}$. Its image in the tautological rings $\mathcal{T}_{\kappa}$ and $\mathcal{T}_{\kappa}^{\mathrm{tor}}$ is also called the *Hodge half-line*.

By [Madapusi Pera 2019, Theorem 5], we find that the pullback of a Hodge line bundle class to $\mathcal{T}_{\kappa}$ (resp. to $\mathcal{T}_{\kappa}^{\mathrm{tor}}$) is generated by the determinant of the sheaf of invariant differentials of the abelian scheme (resp. semiabelian scheme) that is obtained via pullback from the universal abelian (resp. semiabelian) scheme over the Siegel Shimura variety (resp. over a suitable toroidal compactification of the Siegel Shimura variety). In particular, the pullback of a Hodge line bundle class to $\mathcal{T}_{\kappa}$ is ample.

7C. *Powers of Hodge line bundle classes.* By Propositions 4.8 and 4.14 the Chow ring $A^{\bullet}(G\text{-}\mathtt{Zip}^{\mu})$ is a graded finite-dimensional $\mathbb{Q}$-algebra of dimension $\#^I W$. For $j = 0, \dots, d := \langle 2\rho, \mu\rangle$, the cycle classes $[\bar{Z}_w]$ with $w \in {}^I W$ such that $\ell(w) = d - j$ form a basis of the $\mathbb{Q}$-vector space $A^j(G\text{-}\mathtt{Zip}^{\mu})$. In particular, its top-degree part $A^d(G\text{-}\mathtt{Zip}^{\mu})$ is 1-dimensional and generated by the unique closed zip stratum $[Z_e]$, which we call the *superspecial stratum*.

Proposition 7.7. *Let $\lambda^{\flat} \in A^1(G\text{-}\mathtt{Zip}^{\mu})$ be a Hodge line bundle class. Then for all $j = 0, \dots, d$ one has*

$$(\lambda^{\flat})^{d-j} = \sum_{\substack{w \in {}^I W \\ \ell(w) = j}} \alpha_w [\bar{Z}_w],$$

with $\alpha_w \in \mathbb{Q}_{>0}$. In particular, there exists $\alpha_e \in \mathbb{Q}_{>0}$ such that

$$(\lambda^{\flat})^d = \alpha_e [Z_e]. \tag{7.8}$$

Proof. This follows by an easy induction from [Goldring and Koskivirta 2019a, 5.2.2]. □

Remark 7.9. Calculations of examples suggest that the coefficients α_w should be equal for $w \in {}^I W$ with $\ell(w) = j$ if $\boldsymbol{G}^{\mathrm{ad}}$ is $\mathbb{Q}$-simple. We cannot prove this.

By pullback we obtain:

Corollary 7.10. *Let $\lambda^\flat \in A^1(G\text{-}\mathtt{Zip}^\mu)$ be a Hodge line bundle class. Let $\lambda \in \mathcal{T}$ be its pullback. Then for all $j = 0, \dots, d$ one has*

$$\lambda^{d-j} = \sum_{\substack{w \in {}^I W \\ \ell(w) = j}} \alpha_w [\overline{S_{K,w}}],$$

with $\alpha_w \in \mathbb{Q}_{>0}$. In particular, there exists $\alpha_e \in \mathbb{Q}_{>0}$ such that $\lambda^d = \alpha_e [S_e]$.

7D. *Description of the tautological ring.* We now show the pullback map $\zeta^{\mathrm{tor},*} : A^\bullet(G\text{-}\mathtt{Zip}^\mu_k) \to A^\bullet(S^{\mathrm{tor}}_{K,k})$ is always injective. By Proposition 1.3, this also implies the injectivity of $\zeta^{\mathrm{tor},*} : A^\bullet(G\text{-}\mathtt{Zip}^\mu_{\kappa'}) \to A^\bullet(S^{\mathrm{tor}}_{K,\kappa'})$ for every algebraic extension κ' of κ. In particular, we obtain an isomorphism of the tautological ring $\mathcal{T}^{\mathrm{tor}}_{\kappa'}$ with $A^\bullet(G\text{-}\mathtt{Zip}^\mu_{\kappa'})$.

The tool for showing injectivity is the following lemma.

Lemma 7.11. *Let $\alpha : A^\bullet(G\text{-}\mathtt{Zip}^\mu_k) \to T$ be a map of graded $\mathbb{Q}$-algebras. Then α is injective if and only if $\alpha([Z_e]) \neq 0$.*

Proof. It suffices to show that any graded nonzero ideal of $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ contains $[Z_e]$. By Corollary 4.12, $A^\bullet(G\text{-}\mathtt{Zip}^\mu)$ is isomorphic to the rational cohomology ring of the flag space $\boldsymbol{X}^\vee$ over $\mathbb{C}$. In particular, multiplication yields, for all $j = 0, \dots, d = \dim \boldsymbol{S}_K$, a perfect pairing

$$A^j(G\text{-}\mathtt{Zip}^\mu) \times A^{d-j}(G\text{-}\mathtt{Zip}^\mu) \to A^d(G\text{-}\mathtt{Zip}^\mu) = \mathbb{Q}[Z_e].$$

This implies our claim. □

Theorem 7.12. *The map $\zeta^{\mathrm{tor},*}$ is injective. One has*

$$\mathcal{T}^{\mathrm{tor}}_k \cong A^\bullet(G\text{-}\mathtt{Zip}^\mu_k) \cong H^{2\bullet}(\boldsymbol{X}^\vee). \tag{7.13}$$

Proof. Let $\lambda^\flat \in A^1(G\text{-}\mathtt{Zip}^\mu)$ be a Hodge line bundle class, say the first Chern class of a line bundle $\omega^\flat$ on $G\text{-}\mathtt{Zip}^\mu$. Let $\omega^{\mathrm{tor}} := \zeta^{\mathrm{tor},*}(\omega^\flat)$. Let $\pi : S^{\mathrm{tor}}_K \to S^{\mathrm{min}}_K$ be the canonical proper birational map to the minimal compactification. By [Madapusi Pera 2019, 5.2.11], there exists an ample line bundle ω^{min} on S^{min}_K such that $\pi^*(\omega^{\mathrm{min}}) \cong \omega^{\mathrm{tor}}$.

By Lemma 7.11 and (7.8), we have to show that

$$\zeta^{\mathrm{tor},*}(c_1(\omega^\flat)^d \cap [G\text{-}\mathtt{Zip}^\mu_k]) = c_1(\omega^{\mathrm{tor}})^d \cap [S^{\mathrm{tor}}_K] \neq 0, \tag{$*$}$$

where the equality holds by [Fulton 1998, 6.6] because ζ^{tor} is a smooth morphism.

The projection formula shows

$$\pi_*(c_1(\omega^{\mathrm{tor}})^d \cap [S^{\mathrm{tor}}_K]) = c_1(\omega^{\mathrm{min}})^d \cap [S^{\mathrm{min}}_K],$$

which is nonzero because ω^{min} is ample and S^{min}_K is proper and of pure dimension d over κ. Hence the left-hand side of $(*)$ is nonzero.

The isomorphisms in (7.13) are then a consequence by using Corollary 4.12. □

It is conjectured that analogously the tautological ring of a smooth toroidal compactification of the Shimura variety in characteristic 0 should be isomorphic to the cohomology ring of the compact dual. Let E' be an algebraic extension of E_v, and let κ' be the residue field of the ring of integers of E'. There is a commutative diagram

$$\begin{tikzcd} A^\bullet(\mathtt{Hdg}_{E'}) \arrow[r, "\sigma^{\mathrm{tor},*}"] \arrow[d, "\cong"'] & A^\bullet(S^{\mathrm{tor}}_{K,E'}) \arrow[d] \\ A^\bullet(\mathtt{Hdg}_{\kappa'}) \arrow[r] & A^\bullet(S^{\mathrm{tor}}_{K,\kappa'}) \end{tikzcd} \tag{7.14}$$

where the vertical arrows are the specialization maps. For the Hodge stacks, one can show that the specialization map is an isomorphism. In particular, the right-hand side specialization map induces a surjective map of $\mathbb{Q}$-algebras

$$\mathrm{sp}^{\mathrm{tor}} : \mathcal{T}^{\mathrm{tor}}_{E'} \to \mathcal{T}^{\mathrm{tor}}_{\kappa'}. \tag{7.15}$$

The analogous diagram to (7.14) with the specialization of $A^\bullet(S_{K,E'}) \to A^\bullet(S_{K,\kappa'})$ as the right vertical arrow yields also a surjective map $\mathrm{sp} : \mathcal{T}_{E'} \to \mathcal{T}_{\kappa'}$.

Proposition 7.16. *Suppose that E' is chosen such that $\kappa' = k$ is algebraically closed. Then the following assertions are equivalent.*

(i) *The map $\mathrm{sp}^{\mathrm{tor}} : \mathcal{T}^{\mathrm{tor}}_{E'} \to \mathcal{T}^{\mathrm{tor}}_{k}$ is injective* (and hence yields an isomorphism $\mathcal{T}^{\mathrm{tor}}_{E'} \cong H^{2\bullet}(X^\vee)$ by (7.13)).

(ii) *The composition $A^\bullet([G_{E'}\backslash *]) \to A^\bullet(\mathtt{Hdg}_{E'}) \to A^\bullet(S^{\mathrm{tor}}_K)$ is zero in degree > 0.*

Proof. The commutative diagram (7.14) can be extended to a commutative diagram

$$\begin{tikzcd} A^\bullet([G_{E'}\backslash *]) \arrow[r, hook] \arrow[d, "\cong"'] & A^\bullet(\mathtt{Hdg}_{E'}) \arrow[rr, two heads, "\sigma^{\mathrm{tor},*}"] \arrow[d, "\cong"'] & & \mathcal{T}^{\mathrm{tor}}_{E'} \arrow[d] \\ A^\bullet([G_k\backslash *]) \arrow[r, hook] & A^\bullet(\mathtt{Hdg}_{\kappa'}) \arrow[r, two heads, "\beta^*"'] & A^\bullet(G\text{-}\mathtt{Zip}^\mu) \arrow[r, "\sim", "\zeta^{\mathrm{tor},*}"'] & \mathcal{T}^{\mathrm{tor}}_{k} \end{tikzcd} \tag{7.17}$$

Hence the equivalence follows as the kernel of β^* is generated by the image of $A^{>0}([G_k\backslash *])$ by Theorem 4.16. □

Although we cannot prove this description of the tautological ring in characteristic 0, we can reprove the following analogous statement for cohomology. This was previously known by Chern–Weil theory.

Theorem 7.18. *For the $\mathbb{Q}_\ell$-algebra $H^{2\bullet}(S^{\mathrm{tor}}_{K,\bar{E}}) := \bigoplus_i H^{2i}(S^{\mathrm{tor}}_{K,\bar{E}}, \mathbb{Q}_\ell(i))$, the composition*

$$A^\bullet(\mathtt{Hdg}_{\bar{E}}) \to A^\bullet(S^{\mathrm{tor}}_{K,\bar{E}}) \to H^{2\bullet}(S^{\mathrm{tor}}_{K,\bar{E}})$$

induces an injection $H^{2\bullet}(X^\vee) \hookrightarrow H^{2\bullet}(S^{\mathrm{tor}}_{K,\bar{E}})$.

Proof. The existence of the factorization $A^\bullet(\mathtt{Hdg}_{\bar{E}}) \to H^{2\bullet}(X^\vee) \to H^{2\bullet}(S^{\mathrm{tor}}_{K,\bar{E}})$ is given by Corollary 7.2. To prove injectivity, we may replace $\bar{E}$ by $\bar{\mathbb{Q}}_p$ for some $p \neq \ell$ at which the Shimura variety has good reduction. As in the proof of Corollary 7.2, one then reduces to proving that the morphism $H^{2\bullet}(\boldsymbol{X}^\vee) \cong A^\bullet(G\text{-}\mathtt{Zip}^\mu) \to A^\bullet(S_K^{\mathrm{tor}}) \to H^{2\bullet}(S_K^{\mathrm{tor}})$ is injective in characteristic p. This is given by Theorem 7.12. □

It is conjectured that Theorem 7.18 holds over E instead of $\bar{E}$. This is shown in [Esnault and Harris 2017] for compact Shimura varieties. The strongest statement on the Chern classes of automorphic vector bundles in continuous cohomology which we can obtain with our methods here is the following:

Theorem 7.19. *Let E' be a finite extension of the reflex field E contained in the algebraic closure $\bar{E}$ of E in $\mathbb{C}$. Let $\mathscr{V}$ be a flat automorphic bundle over $\mathbf{S}_{K,E'}$, and let $\mathscr{V}^{\mathrm{tor}}$ be its canonical extension to $\mathbf{S}_{K,E'}^{\mathrm{tor}}$. Let $U \subset \mathrm{Spec}(\mathcal{O}_E)$ be the locus of good reduction of S_K and $\mathscr{S}_{K,U}^{\mathrm{tor}}$ the canonical integral model of $\mathbf{S}_{K,E'}^{\mathrm{tor}}$ over U. Then, for each $i > 0$, the i-th Chern class of $\mathscr{V}^{\mathrm{tor}}$ in $H^{2i}(\mathbf{S}_{K,E'}^{\mathrm{tor}}, \mathbb{Q}_\ell(i))$ lies in the image of the natural map from*

$$\ker\Big(H^{2i}(\mathscr{S}_{K,U}^{\mathrm{tor}}, \mathbb{Q}_\ell(i)) \to \bigoplus_{v\in U} H^{2i}(\mathscr{S}_{K,\mathcal{O}_{v'}}^{\mathrm{tor}}, \mathbb{Q}_\ell(i))\Big)$$

to $H^{2i}(\mathbf{S}_{K,E'}^{\mathrm{tor}}, \mathbb{Q}_\ell(i))$. (Here $H^{2i}(\mathscr{S}_{K,U}^{\mathrm{tor}}, \mathbb{Q}_\ell(i))$ denotes the continuous or pro-étale cohomology of $\mathscr{S}_{K,U}^{\mathrm{tor}}$.)

Proof. This is proved in the same way as Corollary 7.2: First one uses Lemma 7.3 to extend $\mathscr{V}$ to U, and then proper base change to show that the Chern classes of such an extension lie in the given kernel. □

7E. *Hirzebruch–Mumford proportionality.* The above results immediately imply a very strong form of Hirzebruch–Mumford proportionality in positive characteristic and the usual form of Hirzebruch–Mumford proportionality in characteristic 0.

Recall that an automorphic bundle on $S_{K,k}^{\mathrm{tor}}$ is by definition a vector bundle of the form $\sigma^{\mathrm{tor},*}(\mathscr{E})$ for some vector bundle $\mathscr{E}$ on $\mathtt{Hdg}_k = [G_k \backslash G_k / P_k]$. Let $X^\vee := G_k/P_k$ be the characteristic p version of the compact dual $\boldsymbol{X}^\vee$, and let

$$\rho : X^\vee \to \mathtt{Hdg}_k$$

be the projection. If we consider $\mathscr{E}$ as a G_k-equivariant vector bundle on $X_k^\vee$, then $\rho^*(\mathscr{E})$ is the underlying vector bundle.

Theorem 7.20. *There is an isomorphism $u : A^\bullet(X^\vee) \xrightarrow{\sim} \mathcal{T}_k^{\mathrm{tor}}$ of graded $\mathbb{Q}$-algebras such that for every G_k-equivariant vector bundle $\mathscr{E}$ on $X_k^\vee$ the i-th Chern class of the underlying vector bundle on $X_k^\vee$ is sent by u to the i-th Chern class of the automorphic bundle $\sigma^{\mathrm{tor},*}(\mathscr{E})$.*

Proof. The kernel of the surjective map $\rho^* : A^\bullet(\mathtt{Hdg}_k) \to A^\bullet(X_k^\vee)$ is the same as the kernel of the surjective map $\beta^* : A^\bullet(\mathtt{Hdg}_k) \to A^\bullet(G\text{-}\mathtt{Zip}_k^\mu)$ by Lemma 4.2. Hence we obtain some isomorphism of graded $\mathbb{Q}$-algebras $A^\bullet(X^\vee) \xrightarrow{\sim} A^\bullet(G\text{-}\mathtt{Zip}_k^\mu)$. Composing it with $\zeta^{\mathrm{tor},*} : A^\bullet(G\text{-}\mathtt{Zip}_k^\mu) \to \mathcal{T}_k^{\mathrm{tor}}$, which is an isomorphism by Theorem 7.12, we obtain the desired isomorphism u. □

For a smooth proper equidimensional scheme X over k, we denote by $\int_X : A^{\dim X}(X) \to \mathbb{Q}$ the degree map. Let $\mathbb{Q}[c_1, \dots, c_d]$ be the graded polynomial ring with $\deg(c_i) = i$.

The isomorphism u from Theorem 7.20 induces, in particular, an isomorphism of the 1-dimensional top-degree parts $A^d(X_k^\vee)$ and $\mathcal{T}_k^{\mathrm{tor},d}$, where $d := \dim(X^\vee) = \dim(S_K^{\mathrm{tor}})$. From this we obtain:

Corollary 7.21. *There exists a rational number $R \in \mathbb{Q}^\times$ such that for all classes $\alpha \in A^d(\mathtt{Hdg}_k)$ one has*

$$\int_{X_k^\vee} \rho^*(\alpha) = R \int_{S^{\mathrm{tor}}_{K,k}} \sigma^{\mathrm{tor},*}(\alpha).$$

As specialization of cycles commutes with taking degrees we obtain a new and purely algebraic proof of Hirzebruch–Mumford proportionality in characteristic 0. The original proof of this result is given in [Hirzebruch 1958] and [Mumford 1977].

Corollary 7.22. *There exists a rational number $R \in \mathbb{Q}^\times$ such that for all homogenous $f \in \mathbb{Q}[c_1, \dots, c_d]$ of degree d and all $\boldsymbol{G}_{\mathbb{C}}$-equivariant vector bundles $\mathscr{E}$ on $\boldsymbol{X}^\vee$ one has*

$$\int_{\boldsymbol{X}^\vee} f\big(c_1(\rho^*(\mathscr{E})), \dots, c_d(\rho^*(\mathscr{E}))\big) = R \int_{\boldsymbol{S}^{\mathrm{tor}}_{K,\mathbb{C}}} f\big(c_1(\sigma^{\mathrm{tor},*}(\mathscr{E})), \dots, c_d(\sigma^{\mathrm{tor},*}(\mathscr{E}))\big).$$

Proof. All $\boldsymbol{G}_{\mathbb{C}}$-equivariant vector bundles $\mathscr{E}$ on $\boldsymbol{X}^\vee$ are already defined over some splitting field E' of G that we may assume to be a finite extension of the reflex field. We now choose p and v' as in the proof of Corollary 7.2: let p be a prime number of good reduction for the Shimura datum $(\boldsymbol{G}, \boldsymbol{X})$ such that there exists an unramified place v' of E' over p. Let v be the restriction of v' to E, and let $\mathscr{S}_K^{\mathrm{tor}}$ be a smooth proper toroidal compactification of $\mathscr{S}_K$ over O_{E_v} with generic fiber $\boldsymbol{S}^{\mathrm{tor}}_{K,E_v}$. Consider the commutative diagram

$$\begin{array}{ccccc} A^\bullet(\boldsymbol{X}^\vee_{E'}) & \xleftarrow{\rho^*} & A^\bullet(\mathtt{Hdg}_{E'}) & \xrightarrow{\sigma^{\mathrm{tor},*}} & A^\bullet(\boldsymbol{S}^{\mathrm{tor}}_{K,E'}) \\ \downarrow{\scriptstyle \mathrm{sp}} & & \downarrow{\scriptstyle \mathrm{sp}} & & \downarrow{\scriptstyle \mathrm{sp}} \\ A^\bullet(X^\vee_{\kappa'}) & \xleftarrow{\rho^*} & A^\bullet(\mathtt{Hdg}_{\kappa'}) & \xrightarrow{\sigma^{\mathrm{tor},*}} & A^\bullet(S^{\mathrm{tor}}_{K,\kappa'}) \end{array} \tag{7.23}$$

where the vertical maps are given by specialization. Then we have

$$\begin{aligned} \int_{\boldsymbol{X}^\vee} f\big(c_1(\rho^*(\mathscr{E})), \dots, \rho^*(c_d(\mathscr{E}))\big) &= \int_{X^\vee_{\kappa'}} \mathrm{sp}\big(f\big(c_1(\rho^*(\mathscr{E})), \dots, c_d(\rho^*(\mathscr{E}))\big)\big) \\ &= \int_{X^\vee_{\kappa'}} \rho^*\big(\mathrm{sp}\big(f(c_1(\mathscr{E}), \dots, c_d(\mathscr{E}))\big)\big) \\ &= R \int_{S^{\mathrm{tor}}_{K,\kappa'}} \sigma^{\mathrm{tor},*}\big(\mathrm{sp}\big(f(c_1(\mathscr{E}), \dots, c_d(\mathscr{E}))\big)\big) \\ &= R \int_{S^{\mathrm{tor}}_{K,\kappa'}} \mathrm{sp}\big(\sigma^{\mathrm{tor},*}\big(f(c_1(\mathscr{E}), \dots, c_d(\mathscr{E}))\big)\big) \\ &= R \int_{\boldsymbol{S}^{\mathrm{tor}}_{K,\mathbb{C}}} f\big(c_1(\sigma^{\mathrm{tor},*}(\mathscr{E})), \dots, c_d(\sigma^{\mathrm{tor},*}(\mathscr{E}))\big). \end{aligned}$$

Here the first and the last equality hold because taking the degree commutes with specialization [Fulton 1998, 20.3(a)], and the third equality is a special case of Corollary 7.21. □

The proof shows that the numbers R of Corollaries 7.21 and 7.22 coincide.

8. Examples

For a permutation $\pi \in S_n$, we also write $\pi = [\pi(1), \pi(2), \ldots, \pi(n)]$. We will always denote by $\tau_{i,j}$ the transposition of i and j. For any permutation σ, one has $\sigma\tau_{i,j}\sigma^{-1} = \tau_{\sigma(i),\sigma(j)}$.

8A. ***Siegel case.*** Fix $g \geq 1$. We consider the vector space $\mathbb{F}_p^{2g}$ with the symplectic pairing

$$((a_i)_i, (b_i)_i) \mapsto \sum_{1\leq i\leq g} a_i b_{2g+1-i} - \sum_{g+1\leq i\leq 2g} a_i b_{2g+1-i}.$$

We take G to be the resulting group of symplectic similitudes and let μ be the cocharacter of G with weights $(1, \ldots, 1, 0, \ldots, 0)$ (with each weight having multiplicity g) on the above representation $\mathbb{F}_p^{2g}$ of G.

Let T be the group of diagonal matrices in G. We use the description of the Weyl group W of (G, T) given in [Viehmann and Wedhorn 2013, Section A7], i.e.,

$$W = \{w \in S_{2g} \mid w(i) + w(i^\perp) = 2g + 1\},$$

where $i^\perp := 2g+1-i$. Its simple reflections are $s_i = \tau_{i,i+1}\tau_{2g-i,2g+1-i}$ for $i = 1, \ldots, g-1$ and $s_g = \tau_{g,g+1}$. Every element $w \in W$ is uniquely determined by $w(1), \ldots, w(g)$. As G is split over $\mathbb{F}_p$, the Frobenius φ acts trivially on W.

We get a frame for the resulting zip datum by taking T to be the above torus, B the group of upper triangular matrices in G, and z the canonical representative of the element $[1+g, \ldots, 2g, 1, \ldots, g]$ of W. The types I and J of P and Q are both equal to $\{s_1, \ldots, s_{g-1}\}$, and

$${}^I W = \{w \in W \mid w^{-1}(1) < \cdots < w^{-1}(g)\} = \{w \in W \mid w^{-1}(g+1) < \cdots < w^{-1}(2g)\}.$$

Elements of T have diagonal entries $(t_1, \ldots, t_g, t_g^{-1}, \ldots, t_1^{-1})$. Hence if for $1 \leq i \leq g$ we let $x_i \in X^*(T)$ be the character sending such an element to t_i, we obtain a basis $(x_1, \ldots, x_g)$ of $X^*(T)$ which induces an isomorphism $S \cong \mathbb{Q}[x_1, \ldots, x_g]$. The element $z \in W$ is given by $z(i) = g + i$ for all $i = 1, \ldots, g$. It acts on $X^*(T)$ via $x_i \mapsto -x_{g+1-i}$. We have

$$z s_i z^{-1} = s_{g-i} \quad \text{for all } i = 1, \ldots, g-1. \tag{8.1}$$

Computation of $\gamma(w)$. For $w \in {}^I W$, set $\sigma_w := \mathrm{int}(wz)$. Then we find

$$I_w = \bigcap_{m\geq 1} I_w^{(m)}, \quad I_w^{(m)} := \{s \in I \mid \sigma_w^k(s) \in I \text{ for all } k = 1, \ldots, m\}$$

by (3.16). For instance, $s_i \in I_w^{(1)}$ if and only if $w(g-i)$ and $w(g+1-i)$ are both $\leq g$ or both $\geq g+1$. In this case $w(g+1-i) = w(g-i) + 1$ and $\sigma_w(s)$ is $s_{w(g-i)}$ if $w(g-i) \leq g$ and it is $s_{w(g+1-i)^\perp}$ if $w(g-i) \geq g+1$.

We can consider I_w as a subset of vertices of the Dynkin diagram of G and get a subgraph with those edges in the Dynkin diagram of G that have vertices in I_w. As σ_w preserves angles between roots, it is an automorphism of the Dynkin diagram I_w. In particular, it permutes all connected components of I_w. Let $\mathfrak{c}$ be a $\sigma_w^{\mathbb{Z}}$-orbit of such connected components. Choose some connected component C in $\mathfrak{c}$.

Let $m(\mathfrak{c})$ be the number of vertices in C, and let $l(\mathfrak{c})$ be the minimal integer $n \geq 1$ such that $\sigma_w^n(C) = C$. We say that $\mathfrak{c}$ is of linear type if $\sigma^{l(\mathfrak{c})}(w) = w$ for all $w \in C$. Otherwise it is called of unitary type. Then $m(\mathfrak{c})$, $l(\mathfrak{c})$, and the type do not depend on the choice of C.

Now we can calculate $\gamma(w)$ by (3.20) as follows. If $\mathfrak{c}$ is of linear type, then we let $\gamma_{\mathfrak{c}}(w)$ be the number of $\mathbb{F}_p$-valued points in the full flag variety of the scalar restriction of $\mathrm{GL}_{m(\mathfrak{c})+1}$ over $\mathbb{F}_{p^{l(\mathfrak{c})}}$, i.e.,

$$\gamma_{\mathfrak{c}}(w) = \sum_{\pi \in S_{m(\mathfrak{c})+1}} p^{l(\mathfrak{c})\ell(\pi)} = \prod_{1 \leq j \leq m(\mathfrak{c})} \frac{q^{j+1}-1}{q-1},$$

where $q := p^{l(\mathfrak{c})}$. If $\mathfrak{c}$ is of unitary type, then we let $\gamma_{\mathfrak{c}}(w)$ be the number of $\mathbb{F}_p$-valued points in the full flag variety of the scalar restriction of a unitary group in $m+1$ variables over $\mathbb{F}_{p^{l(\mathfrak{c})}}$. To describe this concretely, we let τ be the conjugation with the longest element in the symmetric group $S_{m(\mathfrak{c})+1}$, an automorphism of Coxeter groups of order 2 (except if $m(\mathfrak{c}) = 1$). For $\pi \in S_{m(\mathfrak{c})+1}$, set $\delta(\pi) := p^{2l(\mathfrak{c})\ell(\pi)}$ if $\pi \neq \tau(\pi)$ and $\delta(\pi) := p^{l(\mathfrak{c})\ell(\pi)}$ if $\tau(\pi) = \pi$. Then

$$\gamma_{\mathfrak{c}}(w) = \sum_{\pi \in S_{m(\mathfrak{c})+1}/\tau} \delta(\pi).$$

Altogether we obtain

$$\gamma(w) = \prod_{\mathfrak{c}} \gamma_{\mathfrak{c}}(w),$$

where $\mathfrak{c}$ runs through all orbits of connected components of I_w.

For instance, fix $0 \leq f \leq g$, and let u_f be the permutation

$$u_f := [g+1,\ g+2,\ \ldots,\ g+f,\ 1,\ g+f+1,\ \ldots,\ 2g-1,\ 2,\ \ldots,\ g-f,\ 2g,\ g-f+1,\ \ldots,\ g].$$

Then $u_g = z$ and $\bar{Z}_{u_0}$ is the locus where the p-rank is 0. We have

$$I_{u_f} = I \setminus \{s_1, s_2, \ldots, s_{g-f}\},$$

and in particular, $I_{u_1} = I_{u_0} = \varnothing$. Moreover, I_{u_f} has only one connected component and it is of linear type. Therefore

$$\gamma(u_f) = \prod_{1 \leq j \leq f-1} \frac{p^{j+1}-1}{p-1}.$$

Cycle classes. By Example 2.6, we find that

$$[\mathtt{Brh}_e] = \prod_{1 \leq i < j \leq g} (x_i \otimes 1 - 1 \otimes x_j)\Gamma_g(c_1, \ldots, c_g), \tag{8.2}$$

where $c_i = \sigma_i(x_1, \ldots, x_g) \otimes 1 + 1 \otimes \sigma_i(x_1, \ldots, x_g)$ for $i = 1, \ldots, g$, and we set $c_0 = 2$ and $c_i = 0$ for all $i \notin \{0, \ldots, g\}$.

The operators δ_{s_i} act on S by

$$\delta_{s_i}(f) = \frac{f(x_1, \ldots, x_g) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_g)}{x_i - x_{i+1}}$$

for $i = 1, \dots, g-1$, and by

$$\delta_{s_g}(f) = \frac{f(x_1, \dots, x_g) - f(x_1, \dots, x_{g-1}, -x_g)}{2x_g}$$

for $i = g$.

The element z acts on S by $x_i \mapsto -x_{g+1-i}$. Since the torus T is split over $\mathbb{F}_p$, the Frobenius φ acts on S by $x_i \mapsto px_i$. Hence ψ^* sends $x_i \otimes 1$ to $-x_{g+1-i}$ and $1 \otimes x_i$ to px_i. Thus for $w \in W$ we find

$$[\overline{\mathtt{Brh}}_w] = \delta_w\Bigg(\prod_{1 \le i < j \le g} (x_i \otimes 1 - 1 \otimes x_j)\Gamma_g(c_1, \dots, c_g)\Bigg)$$

and

$$[\bar{Z}_w^{\varnothing}] = \psi^*([\overline{\mathtt{Brh}}_w]). \tag{8.3}$$

Such a formula is already given in [Ekedahl and van der Geer 2009, Theorem 12.1]. The formula in loc. cit. agrees with (8.3) if one takes the following into account: We believe that in loc. cit. there is a typo and the polynomial should be evaluated at $y_j = p\ell_{g+1-j}$ instead of $y_j = p\ell_j$. Then the formulas agree under the substitution $x_i = \ell_{g+1-i}$.

The case $g = 2$. As an example, let us consider the case $g = 2$. We let

$$\begin{aligned}
\Phi &:= x_1 \otimes 1 - 1 \otimes x_2, \\
\Gamma &:= c_1c_2 = ((x_1 + x_2) \otimes 1 + 1 \otimes (x_1 + x_2))(x_1x_2 \otimes 1 + 1 \otimes x_1x_2),
\end{aligned}$$

so that

$$[\mathtt{Brh}_e] = \Phi\Gamma.$$

The set ${}^IW$ consists of the elements $\{e, s_2, s_2s_1, s_2s_1s_2\}$. By applying the operators δ_w, we find

$$\begin{aligned}
[\mathtt{Brh}_{s_2}] &= \Phi(x_1 \otimes 1 + 1 \otimes x_1)(x_1 \otimes 1 + 1 \otimes x_2), \\
[\mathtt{Brh}_{s_2s_1}] &= (x_1 \otimes 1 + 1 \otimes x_1)(x_1 \otimes 1 + 1 \otimes x_2), \\
[\mathtt{Brh}_{s_2s_1s_2}] &= x_1 \otimes 1 + 1 \otimes x_1.
\end{aligned}$$

Applying ψ^* yields

$$\begin{aligned}
[\bar{Z}_e^{\varnothing}] &= -(p^4 - 1)(x_1 + x_2)x_1x_2^2, \\
[\bar{Z}_{s_2}^{\varnothing}] &= -(p^2 - 1)(px_1 - x_2)x_2^2, \\
[\bar{Z}_{s_2s_1}^{\varnothing}] &= (p - 1)(px_1 - x_2)x_2, \\
[\bar{Z}_{s_2s_1s_2}^{\varnothing}] &= px_1 - x_2.
\end{aligned} \tag{8.4}$$

We have $I = I^{\circ} = \{s_1\}$. Hence, by Theorem 4.17, $\pi_* = \delta_{s_1}$. Since I has only a single element, we see that for $w \in {}^IW$ either $\sigma_w(s_1) = s_1$ and hence $I_w = I$ or $I_w = \varnothing$. In the first case we find $\gamma(w) = p + 1$, in the second $\gamma(w) = 1$. Using this we find

$$\gamma(e) = \gamma(s_2s_1s_2) = p + 1 \quad \text{and} \quad \gamma(s_2) = \gamma(s_2s_1) = 1. \tag{8.5}$$

Altogether we obtain the following formulas for the classes of the EO-strata:

$$\begin{aligned}
[\bar{Z}_e] &= (p+1)(p^4-1)(x_1+x_2)x_1x_2,\\
[\bar{Z}_{s_2}] &= (p^2-1)((p-1)x_1x_2-x_1^2-x_2^2),\\
[\bar{Z}_{s_2s_1}] &= (p-1)(x_1+x_2),\\
[\bar{Z}_{s_2s_1s_2}] &= (p+1)^2.
\end{aligned} \tag{8.6}$$

These formulas agree with the ones given in [Ekedahl and van der Geer 2009, 12.2] with x_i corresponding to ℓ_{g+1-i}, except that it appears that in loc. cit. the rows for s_1s_2 and s_2s_1 should be switched and the entry for $\pi_*([\overline{U}_{s_2}])$ is incorrect.

8B. ***Hilbert–Blumenthal case.*** Fix $d \geq 1$, and let $\widetilde{G} := \mathrm{Res}_{\mathbb{F}_{p^d}/\mathbb{F}_p} \mathrm{GL}_2$. Define G by the cartesian diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & \mathbb{G}_{m,\mathbb{F}_p} \arrow[d] \\
\widetilde{G} \arrow[r, "\det"] & \mathrm{Res}_{\mathbb{F}_{p^d}/\mathbb{F}_p} \mathbb{G}_m
\end{tikzcd}$$

where the right vertical map is the canonical embedding. Let Σ be the set of embeddings $\mathbb{F}_{p^d} \hookrightarrow k$. We fix an embedding ι_0 and identify the set $\mathbb{Z}/d\mathbb{Z}$ with Σ via $i \mapsto \sigma^{-i}\iota_0$, where $\sigma : x \to x^p$ is the arithmetic Frobenius. Let $\tilde{\mu}$ be the cocharacter of $\widetilde{G}_k = \prod_\Sigma \mathrm{GL}_2$ given by $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in each component. Then $\tilde{\mu}$ factors through a cocharacter μ of G. Let $\widetilde{T}$ be the standard torus of $\widetilde{G}$, i.e., $\widetilde{T}_k$ is the product of the diagonal tori. For $i \in \mathbb{Z}/d\mathbb{Z}$ and $j = 1, 2$, let $x_j^{(i)}$ be the character

$$\left(\begin{pmatrix} t_1^{(i)} & 0 \\ 0 & t_2^{(i)} \end{pmatrix}\right)_{i\in\mathbb{Z}/d\mathbb{Z}} \mapsto t_j^{(i)}$$

of $\widetilde{T}_k$. Then $\widetilde{S} = \mathrm{Sym}(X^*(\widetilde{T})_\mathbb{Q}) = \mathbb{Q}[x_1^{(i)}, x_2^{(i)}; i \in \mathbb{Z}/d\mathbb{Z}]$. The intersection $T = \widetilde{T} \cap G$ is a maximal torus of G and $S = \mathrm{Sym}(X^*(T)_\mathbb{Q})$ identifies with the quotient of $\widetilde{S}$ by the ideal generated by $(x_1^{(i)} + x_2^{(i)}) - (x_1^{(i+1)} + x_2^{(i+1)})$ for $i \in \mathbb{Z}/d\mathbb{Z}$. We will compute all cycle classes of EO-strata for $(\widetilde{G}, \tilde{\mu})$. This yields then also the corresponding cycle classes for (G, μ) by Section 4E.

Let $\widetilde{B}$ be the Borel subgroup of $\widetilde{G}$ such that $\widetilde{B}_k$ is the product of groups of upper triangular matrices in GL_2. The Weyl group is $W = \{\pm 1\}^{\mathbb{Z}/d\mathbb{Z}}$ and we have $I = J = \varnothing$. Thus ${}^IW = W$. As a frame for $(\widetilde{G}, \tilde{\mu})$ we choose $(\widetilde{T}, \widetilde{B}, z)$ with z a representative of $(-1, \ldots, -1) \in W$.

By (2.8), we have

$$[\mathtt{Brh}_e] = \prod_{i\in\mathbb{Z}/d\mathbb{Z}} (x_1^{(i)} \otimes 1 - 1 \otimes x_2^{(i)}) \in A^\bullet(\mathtt{Brh}_{\widetilde{G}}).$$

Let $w = (\epsilon_i)_{i\in\mathbb{Z}/d\mathbb{Z}} \in W$. Then $\ell(w) = \#\{i \in \mathbb{Z}/d\mathbb{Z} \mid \epsilon_i = -1\}$. We have

$$[\overline{\mathtt{Brh}}_w] = \prod_{\substack{i\in\mathbb{Z}/d\mathbb{Z}\\ \epsilon_i=1}} (x_1^{(i)} \otimes 1 - 1 \otimes x_2^{(i)}),$$

and hence

$$[\bar{Z}_w^{\varnothing}] = \psi^*([\overline{\mathtt{Brh}}_w]) = \prod_{\substack{i \in \mathbb{Z}/d\mathbb{Z} \\ \epsilon_i = 1}} (x_2^{(i)} - p x_2^{(i+1)}).$$

With the notation of Section 3F, we find $I_w = \varnothing$ and $L_w = T$. Hence $\gamma(w) = 1$ for all $w \in W$. Also, π is an isomorphism. Therefore we have isomorphisms

$$A^\bullet(\widetilde{G}\text{-}\mathtt{ZipFlag}^{\tilde{\mu}}) \cong A^\bullet(\widetilde{G}\text{-}\mathtt{Zip}^{\tilde{\mu}}) \cong A^\bullet(G\text{-}\mathtt{Zip}^{\mu}) \cong A^\bullet(G\text{-}\mathtt{ZipFlag}^{\mu})$$

in this case. From the description of $A^\bullet(G\text{-}\mathtt{ZipFlag}^{\mu})$ in Proposition 4.8(2) one deduces easily that $x_2^{(i)} \mapsto z_i$ yields an isomorphism of graded $\mathbb{Q}$-algebras

$$A^\bullet(G\text{-}\mathtt{Zip}^{\mu}) \cong \mathbb{Q}[z_0, \dots, z_{d-1}]/(z_0^2, \dots, z_{d-1}^2).$$

Via this isomorphism we get, for the cycle classes of the $\bar{Z}_w$,

$$[\bar{Z}_w] = \prod_{\substack{i \in \mathbb{Z}/d\mathbb{Z} \\ \epsilon_i = 1}} (z_i - p z_{i+1}) \in A^\bullet(G\text{-}\mathtt{Zip}^{\mu}).$$

To describe the Hodge half-line in $A^\bullet(G\text{-}\mathtt{Zip}^{\mu})$ we use the notation from Section 7B. The restriction of the standard embedding ι of G into GSp_{2d} to the maximal torus is given by

$$\left(\begin{pmatrix} t_1^{(i)} & 0 \\ 0 & t_2^{(i)} \end{pmatrix}\right)_{i \in \mathbb{Z}/d\mathbb{Z}} \mapsto \mathrm{diag}(t_1^{(0)}, t_1^{(1)}, \dots, t_1^{(d-1)}, t_2^{(d-1)}, \dots, t_2^{(0)}).$$

Therefore the character $\chi(\iota)$ (see (7.4)) is given by

$$\left(\begin{pmatrix} t_1^{(i)} & 0 \\ 0 & t_2^{(i)} \end{pmatrix}\right)_{i \in \mathbb{Z}/d\mathbb{Z}} \mapsto \prod_{i \in \mathbb{Z}/d\mathbb{Z}} (t_2^{(i)})^{-1},$$

and the Hodge half-line consists of all $\mathbb{Q}_{>0}$-multiples of the class of

$$\lambda := -\sum_{i \in \mathbb{Z}/d\mathbb{Z}} x_2^{(i)} = -(z_0 + \dots + z_{d-1}) \in A^\bullet(G\text{-}\mathtt{Zip}^{\mu}).$$

Hence (as an illustration of Proposition 7.7) we see that

$$[Z_{\le d-1}] = \sum_{i \in \mathbb{Z}/d\mathbb{Z}} (z_i - p z_{i+1}) = (p-1)\lambda \quad \text{and} \quad [Z_e] = (1 + (-1)^d p^d) \prod_{i \in \mathbb{Z}/d\mathbb{Z}} z_i = \frac{p^d + (-1)^d}{d!} \lambda^d.$$

8C. ***The odd spin case.*** We assume that $p > 2$. Let (V, Q) be a quadratic space over $\mathbb{F}_p$ of odd dimension $2n + 1 \ge 3$. We denote by $C(V) = C^+(V) \oplus C^-(V)$ its Clifford algebra. It is a $\mathbb{Z}/2\mathbb{Z}$-graded (noncommutative) $\mathbb{F}_p$-algebra of dimension 2^{2n+1} generated as an algebra by the image of the canonical injective $\mathbb{F}_p$-linear map $V \hookrightarrow C(V)$. It is endowed with an involution * uniquely determined by $(v_1 \cdots v_r)^* = v_r \cdots v_1$ for $v_1, \dots, v_r \in V$.

The *spinor similitude group* is the reductive group $G = \mathrm{GSpin}(V)$ over $\mathbb{F}_p$ defined by

$$G(R) = \{g \in C^+(V_R)^\times \mid gV_Rg^{-1} = V_R, g^*g \in R^\times\}.$$

Then $g \mapsto (v \mapsto g \bullet v := gvg^{-1})$ defines a surjective map of reductive groups $G \to \widetilde{G} := \mathrm{SO}(V)$ with kernel $\mathbb{G}_m$. The groups G and $\widetilde{G}$ are both of Dynkin type B_n.

We now assume that we can find an $\mathbb{F}_p$-basis $(v_0, v_1, \dots, v_{2n})$ such that the matrix of the bilinear form attached to Q with respect to this basis is given by

$$\begin{pmatrix} 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & \iddots & & \\ & & 1 & & & \\ & 1 & & & & \end{pmatrix}.$$

Although there are two isomorphism classes of quadratic spaces over $\mathbb{F}_p$ of dimension $2n+1$, the associated groups GSpin and SO are isomorphic. Hence our assumption is harmless.

We define the cocharacter $\mu : \mathbb{G}_m \to G$ by

$$\mu(t) = tv_1v_{2n} + v_{2n}v_1.$$

The composition of μ with $G \to \mathrm{SO}(V)$ yields the cocharacter

$$\tilde{\mu} : \mathbb{G}_m \to \mathrm{SO}(V), \quad t \mapsto \mathrm{diag}(1, t, 1, \dots, 1, t^{-1}).$$

We will compute the cycle classes of EO-strata for $(\widetilde{G}, \tilde{\mu})$. Again by Section 4E this yields then also the corresponding cycle classes for (G, μ).

As a maximal torus $\widetilde{T}$ for $\widetilde{G} = \mathrm{SO}(V)$ we choose

$$\widetilde{T} = \{\mathrm{diag}(1, t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \mid t_i \in \mathbb{G}_m\}.$$

For $i = 1, \dots, n$, let x_i be the character $\mathrm{diag}(1, t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}) \mapsto t_i$. Then $\widetilde{S} = \mathbb{Q}[x_1, \dots, x_n]$. The Weyl group W is the group

$$W = \{w \in S_{2n} \mid w(i) + w(2n+1-i) = 2n+1 \text{ for all } i\}$$

acting on T in the standard way via the last $2n$ coordinates. The roots of $(\widetilde{G}, \widetilde{T})$ are given by $\pm x_i \pm x_j$ for $1 \le i \ne j \le n$ and $\pm x_i$ for $i = 1, \dots, n$. Let $\widetilde{B}$ be the Borel subgroup such that the corresponding simple roots are given by $x_1 - x_2, \dots, x_{n-1} - x_n, x_n$. Then the set Σ of simple reflections in W corresponding to B is given by $s_1, \dots, s_{n-1}, s_n$, where s_i is the transposition $\tau_{i,i+1}\tau_{2n-i,2n+1-i}$ for $i = 1, \dots, n-1$ and $s_n = \tau_{n,n+1}$.

Let $\tilde{z} \in \widetilde{G}$ be a lift of the element in W with $1 \mapsto 2n$, $2n \mapsto 1$ and $i \mapsto i$ for all $i = 2, \dots, 2n-1$. Then $(\widetilde{B}, \widetilde{T}, \tilde{z})$ is a frame by Lemma 3.3.

As I is the set of simple reflections corresponding to simple roots α with $\langle \mu, \alpha \rangle = 0$, we find $I = \{s_2, \dots, s_n\}$. Hence we find a bijection

$$ {}^I W = \{w^{-1} \in W \mid w(2) < w(3) < \cdots < w(2n-1)\} \xrightarrow{\sim} \{1, \dots, 2n\}, \quad w \mapsto w^{-1}(1). \tag{8.7}$$

Moreover, $\ell(w) = w^{-1}(1) - 1$ for $w \in {}^I W$. By parts (2) and (4) of Lemma 3.8, this implies that the order $\preceq$ coincides with the Bruhat order on ${}^I W$ and that (8.7) is an isomorphism of ordered sets. There is a concrete reduced expression of $w \in {}^I W$ as a product of simple reflections:

$$ w = \begin{cases} s_1 s_2 \cdots s_{\ell(w)} & \text{if } \ell(w) \le n; \\ s_1 s_2 \cdots s_n s_{n-1} \cdots s_{2n-\ell(w)} & \text{if } \ell(w) > n. \end{cases} \tag{8.8}$$

For all $s \in I$, one has ${}^z s = s$. As $\widetilde{G}$ is split over $\mathbb{F}_p$, the Frobenius φ acts trivially on W. Therefore, for $w \in {}^I W$, the subset $I_w \subseteq I$ defined in Section 3F is the largest subset such that ${}^w I_w = I_w$. Hence

$$ \begin{gathered} I_e = I, \quad I_{s_1} = I \setminus \{s_2\}, \quad \dots \quad I_{s_1 \cdots s_{n-1}} = \varnothing, \quad I_{s_1 \cdots s_n} = \varnothing, \\ I_{s_1 \cdots s_n s_{n-1}} = \varnothing, \quad I_{s_1 \cdots s_n s_{n-1} s_{n-2}} = \{s_n\}, \quad \dots \quad I_{s_1 \cdots s_n \cdots s_1} = I \setminus \{s_2\}. \end{gathered} \tag{8.9}$$

Hence $F\ell_w$ is the flag variety of a split group over $\mathbb{F}_p$ of Dynkin type B_k, where

$$ k = \begin{cases} n - 1 - \ell(w) & \text{if } \ell(w) \le n-1, \\ 0 & \text{if } \ell(w) = n, \\ \ell(w) - n + 1 & \text{if } \ell(w) \ge n+1. \end{cases}$$

By Example 2.6, we find that

$$ [\mathtt{Brh}_e] = \prod_{1 \le i < j \le n} (x_i \otimes 1 - 1 \otimes x_j) \Gamma_n(c_1, \dots, c_n), \tag{8.10}$$

where $c_i = \frac{1}{2}(\sigma_i(x_1, \dots, x_n) \otimes 1 + 1 \otimes \sigma_i(x_1, \dots, x_n))$ for $i = 1, \dots, n$, and we set $c_0 = 1$ and $c_i = 0$ for all $i \notin \{0, \dots, n\}$. For instance, if $n = 2, 3$, we find

$$ [\mathtt{Brh}_e] = \begin{cases} (x_1 \otimes 1 - 1 \otimes x_2) c_1 c_2 & \text{if } n = 2, \\ \prod_{1 \le i < j \le 3} (x_i \otimes 1 - 1 \otimes x_j) c_3 (c_1 c_2 - c_3) & \text{if } n = 3. \end{cases} \tag{8.11}$$

The operators δ_{s_i} from Section 2E act on $\widetilde{S}$ by

$$ \delta_{s_i}(f) = \frac{f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n)}{x_i - x_{i+1}}$$

for $1 \le i \le n-1$ and by

$$ \delta_{s_n}(f) = \frac{f(x_1, \dots, x_n) - f(x_1, \dots, x_{n-1}, -x_n)}{x_n}$$

for $i = n$.

The homomorphism $\tilde{\psi}^* : \widetilde{S} \otimes \widetilde{S} \to \widetilde{S}$ is given by $x_1 \otimes 1 \mapsto -x_1,\ x_i \otimes 1 \mapsto x_i$ for $i = 2, \dots, n$ and $1 \otimes x_i \mapsto p x_i$ for $i = 1, \dots, n$.

Finally, by Lemma 4.18, the operator $\tilde{\pi}_*$ is given by $\delta_{w_{0,I^\circ}}$, where w_{0,I° is the longest element of the Weyl group $W_{I^\circ} = W_{\{s_2, \dots, s_n\}}$ of type B_{n-1}.

The case $n=2$. For $i=1,\dots,4$, we denote the element of length $i-1$ by $w_i \in {}^IW$. By (8.9), we find

$$\gamma(w_1)=\gamma(w_4)=\#\mathbb{P}^1(\mathbb{F}_p)=p+1 \quad \text{and} \quad \gamma(w_2)=\gamma(w_3)=1. \tag{8.12}$$

Using the above we find the following formulas for the classes of the closures of the EO-strata:

$$\begin{aligned} [\bar{Z}_{w_1}] &= (p+1)\tfrac{1}{2}(1-p^2)((p^2+p)x_2^2+(p-1)x_1^2)x_1, \\ [\bar{Z}_{w_2}] &= \tfrac{1}{2}(p^2-1)(p-1)x_1^2, \\ [\bar{Z}_{w_3}] &= (p-1)x_1, \\ [\bar{Z}_{w_4}] &= (p+1)^2. \end{aligned} \tag{8.13}$$

Since the Dynkin diagrams of type B_2 and C_2 are isomorphic, by Proposition 4.23, the Chow rings of the associated moduli spaces of G-zips are isomorphic. Indeed one can check that the formulas in (8.13) match those in (8.6) (up to terms in S_+^W) under the isomorphism

$$\mathbb{Q}[x_1,x_2]\to\mathbb{Q}[x_1,x_2], \qquad x_1\mapsto x_1+x_2, \quad x_2\mapsto x_1-x_2.$$

The case $n=3$. For $i=1,\dots,6$, we denote, as above, the element of length $i-1$ by $w_i \in {}^IW$. By (8.9),

$$\gamma(w_1)=\gamma(w_6)=p^3+2p^2+2p+1, \quad \gamma(w_2)=\gamma(w_5)=\#\mathbb{P}^1(\mathbb{F}_p)=p+1, \quad \gamma(w_3)=\gamma(w_4)=1. \tag{8.14}$$

Using the above we find the following formulas for the classes of the closures of the EO-strata:

$$\begin{aligned} [\bar{Z}_{w_1}] &= \tfrac{1}{2}(p^3+2p^2+2p+1)(p^2+p+1)(p+1)^2(p-1) \\ &\qquad \cdot (p^4x_2^2x_3^2+p^3x_1^2x_2^2+p^3x_1^2x_3^2+p^2x_1^4+p^2x_2^2x_3^2-2px_1^4-px_1^2x_2^2-px_1^2x_3^2+x_1^4)x_1, \\ [\bar{Z}_{w_2}] &= -\tfrac{1}{2}(p+1)^3(p-1)^2(p^2x_2^2+p^2x_3^2+px_1^2-x_1^2)x_1^2, \\ [\bar{Z}_{w_3}] &= \tfrac{1}{2}(p^2+p+1)(p+1)(p-1)^3x_1^3, \\ [\bar{Z}_{w_4}] &= (p+1)(p-1)^2x_1^2, \\ [\bar{Z}_{w_5}] &= (p+1)^2(p-1)x_1, \\ [\bar{Z}_{w_6}] &= (p^3+2p^2+2p+1)(p^2+1)(p+1)^2. \end{aligned}$$

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