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Spectral reciprocity via integral representations

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We prove a spectral reciprocity formula for automorphic forms on GL(2) over a number field that is reminiscent of one found by Blomer and Khan. Our approach uses period representations of L-functions and the language of automorphic representations.

1. Introduction

In the past few years, some attention has been given to spectral reciprocity formulae. By this we mean identities of the shape

$$\sum_{\pi\in\mathcal{F}} \mathcal{L}(\pi)\mathcal{H}(\pi) = \sum_{\pi\in\widetilde{\mathcal{F}}} \widetilde{\mathcal{L}}(\pi)\widetilde{\mathcal{H}}(\pi), \tag{1}$$

where $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are families of automorphic representations, $\mathcal{L}(\pi)$ and $\widetilde{\mathcal{L}}(\pi)$ are certain L-values associated to π, and $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are some weight functions.

The term *spectral reciprocity* first appeared in this context in a paper by Blomer, Li and Miller [Blomer et al. 2019] but such identities have been around at least since Motohashi's formula [1993] connecting the fourth moment of the Riemann zeta-function to the cubic moment of L-functions of cusp forms for GL(2).

The more recent results concern the cases where the families $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are the same or nearly the same. Most commonly, these families are taken to be formed by automorphic representations of GL(2).

There are at least two reasons that help understand the appeal of such formulae. The first one is that they give a somewhat conceptual way of summarizing a technique often used in dealing with problems on families of GL(2) L-functions in which one uses the Kuznetsov formula on both directions in order to estimate a moment of L-values. The second one comes from their satisfying intrinsic nature relating objects that have no a priori reason to be linked.

The first versions of these GL(2) spectral reciprocity formulae [Blomer and Khan 2019a; 2019b; Andersen and Kıral 2018] used classical techniques such as the Voronoi summation formula and the Kuznetsov formula. Starting from [Zacharias 2021], it became clear that an adelic approach could be of interest. Not only does this render generalization to number fields almost immediate, it can also avoid some of the combinatorial difficulties that arise when applying the Voronoi formula.

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Blomer and Khan [2019a] have shown a reciprocity formula which is the main inspiration for the present work: Let Π be a fixed automorphic representation of GL(3) over $\mathbb{Q}$. Let q and ℓ be coprime integers. We write

$$\mathcal{M}(q,\ell;h) := \frac{1}{q}\sum_{\mathrm{cond}(\pi)=q} \frac{L\big(\frac{1}{2},\Pi\times\pi\big)L\big(\frac{1}{2},\pi\big)}{L(1,\mathrm{Ad},\pi)}\frac{\lambda_\pi(\ell)}{\ell^{1/2}}h(t_\pi)+(\cdots),$$

where

- π runs over cuspidal automorphic representations of PGL(2),
- $\lambda_\pi(\ell)$ is the eigenvalue of the Hecke operator T_ℓ on π,
- t_π is the spectral parameter,
- h is a *fairly general* smooth function, and
- $(\cdots)$ denotes the contribution of the Eisenstein part, the terms of lower conductor and some degenerate terms.

Blomer and Khan have showed that

$$\mathcal{M}(q,\ell,h) = \mathcal{M}(\ell,q,\check{h}),$$

where $h\mapsto\check{h}$ is given by an explicit integral transformation. When Π corresponds to an Eisenstein series, this has an application to subconvexity: Let π be a cuspidal automorphic representation for GL(2) over $\mathbb{Q}$ of *squarefree* conductor. Then

$$L\big(\tfrac{1}{2},\pi\big) \ll_\epsilon (\mathrm{cond}(\pi))^{\frac{1}{4}-\frac{1}{24}(1-2\vartheta)+\epsilon}, \tag{2}$$

where ϑ is an admissible exponent towards the Ramanujan conjecture (we know that $\frac{7}{64}$ is admissible and $\vartheta=0$ corresponds to the conjecture). This was then the best-known bound of its kind but it was later superseded by the one in [Blomer et al. 2020].

In this article we use the theory of adelic automorphic representations and integral representations of Rankin–Selberg L-functions to deduce a result on number fields of similar flavor to that of [Blomer and Khan 2019a, Theorem 1].

With respect to Blomer and Khan's result, our result has the advantage of being valid for any number field. On the other hand we need to make some technical restrictions that prevent us from having a full generalization of their reciprocity formula. For the moment our results only work when the fixed GL(3) form is cuspidal and our formula only contemplates forms that are spherical at every infinity place. The first restriction is made for analytic reasons and is due to the fact that unlike cusp forms, the Eisenstein series are not of rapid decay. This can probably be resolved by means of a suitable notion of regularized integrals. As for the second restriction, this seems to be of a more representation-theoretic nature. It requires showing analyticity of certain local factors for nonunitary representations of GL(2). We hope to address both of these technical issues in future work.

1A. ***Statement of results.*** Let F be a number field, with ring of integers $\mathfrak{o}_F$. Let Π be a cuspidal automorphic representation of GL(3) over F. For each automorphic representation π of GL(2), we consider the completed L-functions

$$\Lambda(s,\pi),\quad \Lambda(s,\mathrm{Ad},\pi)\quad\text{and}\quad \Lambda(s,\Pi\times\pi).$$

These are, respectively, the Hecke L-function and the adjoint L-function of π, and the Rankin–Selberg L-function of $\Pi\times\pi$, where for the Rankin–Selberg L-functions we take the naive definition (20). These coincide with the local L-functions à la Langlands at all the unramified places but might differ at the ramified ones. Notice that this might also affect the values of $L(s,\mathrm{Ad},\pi)$.

Let ξ_F denote the completed Dedekind zeta function of F, and let $\xi_F^*(1)$ denote its residue at 1. Let $\Phi\simeq\bigotimes_v\Phi_v$ be a vector in the representation space of Π. Let s and w be complex numbers, and let H denote the weight function given by (29). We consider the sums

$$\mathcal{C}_{s,w}(\Phi):=\sum_{\pi\in C(S)}\frac{\Lambda(s,\Pi\times\pi)\Lambda(w,\pi)}{\Lambda(1,\mathrm{Ad},\pi)}H(\pi)$$

and

$$\mathcal{E}_{s,w}(\Phi):=\sum_{\omega\in\Xi(S)}\int_{-\infty}^{\infty}\frac{\Lambda(s,\Pi\times\pi(\omega,it))\Lambda(w,\pi(\omega,it))}{\Lambda^*(1,\mathrm{Ad},\pi(\omega,it))}H(\pi(\omega,it))\frac{\mathrm{d}t}{2\pi},\tag{3}$$

where S is any finite set of places containing all the archimedean ones and those for which Φ_v is ramified, $C(S)$ (resp. $\Xi(S)$) denotes the collection of cuspidal automorphic representations of GL(2) (resp. unitary normalized idele characters) over F that are unramified everywhere outside S. Finally, $\pi(\omega,it)$ denotes a normalized induced representation as in Section 3A1 and $\Lambda^*(1,\mathrm{Ad},\pi)$ denotes the first nonzero Laurent coefficient of $\Lambda(s,\mathrm{Ad},\pi)$ at $s=1$. The main object of study in this work is the following "moment":

$$\mathcal{M}_{s,w}(\Phi):=\mathcal{C}_{s,w}(\Phi)+\mathcal{E}_{s,w}(\Phi).\tag{4}$$

We remark that the values of $\Lambda(s,\Pi\times\pi(\omega,it))$, $\Lambda(w,\pi(\omega,it))$ and $\Lambda^*(1,\mathrm{Ad},\pi(\omega,it))$ can be given in terms of simpler L-functions as follows:

$$\begin{aligned}\Lambda(s,\Pi\times\pi(\omega,it))&=\Lambda(s+it,\Pi\times\omega)\Lambda(s-it,\Pi\times\bar{\omega}),\\ \Lambda(w,\pi(\omega,it))&=\Lambda(w+it,\omega)\Lambda(w-it,\bar{\omega}),\\ \Lambda^*(1,\mathrm{Ad},\pi(\omega,it))&=\mathrm{Res}_{s=1}[\Lambda(s+2it,\omega^2)\Lambda(s-2it,\bar{\omega}^2)\xi_F(s)]\\ &=\Lambda(1+2it,\omega^2)\Lambda(1-2it,\bar{\omega}^2)\xi_F^*(1)\qquad (t\neq 0),\end{aligned}$$

where $\Lambda(s,\Pi\times\omega)$ and $\Lambda(s,\omega)$ are the (completed) Rankin–Selberg L-function of $\Pi\times\omega$ and Dirichlet L-function of ω, respectively.

We start with the following result which can be seen as a preliminary reciprocity formula.

Theorem 1.1. *Let $s,w\in\mathbb{C}$ and define*

$$(s',w'):=\left(\tfrac{1}{2}(1+w-s),\tfrac{1}{2}(3s+w-1)\right).\tag{5}$$

Let H be as in (29) *and $\check{H}$ be given by* (31). *Suppose the real parts of s, w, s' and w' are sufficiently large. Then we have the relation*

$$\mathcal{M}_{s,w}(\Phi)+\mathcal{D}_{s,w}(\Phi)=\mathcal{M}_{s',w'}(\check{\Phi})+\mathcal{D}_{s',w'}(\check{\Phi}),$$

where $\mathcal{D}_{s,w}(\Phi)$ is given by (30).

Theorem 1.1 is a completely symmetrical formula but only holds when the real part of the parameters s, w, s' and w' are sufficiently large. In order to obtain a formula that also holds at the central point $s=w=s'=w'=\frac{1}{2}$, we need to analytically continue the term $\mathcal{E}_{s,w}(\Phi)$. This is done in Section 9 under a technical condition enclosed in Hypothesis 1.

Spectral reciprocity at the central point. Let Π be an everywhere unramified cuspidal automorphic representation for GL(3) over F. This means that $\Pi\simeq\bigotimes_v'\Pi_v$, where for each v, Π_v is isomorphic to the isobaric sum

$$|\cdot|_v^{it_{1,v}}\boxplus|\cdot|_v^{it_{2,v}}\boxplus|\cdot|_v^{it_{3,v}}.$$

We say that Π is θ-tempered if for all v and $i=1,2,3$, we have $|\mathrm{Re}(t_{i,v})|\le\theta$. It follows from a result of Luo, Rudnick and Sarnak [Luo et al. 1999] that every automorphic representation of GL(n) is θ-tempered for some $\theta<\frac{1}{2}$. Therefore we can, and will, let $\theta=\theta(\Pi)<\frac{1}{2}$ be such that Π is θ-tempered.

Suppose that Φ_v is spherical for every archimedean place v. The reason for this restriction is twofold. The first and main reason is that this leads to weight functions satisfying Hypothesis 1. The second is that this trivializes the transformation $H_v\to\check{H}_v$ on the local archimedean weights. It would be very interesting to have a better understanding of this transformation. In particular it would be interesting to have an understanding of $\check{H}_v$ when H_v is taken to be a bump function selecting spectral parameters of a certain size.

Let $s,w\in\mathbb{C}$, let $\mathfrak{q}$ and $\mathfrak{l}$ be coprime ideals with absolute norms q and ℓ, respectively, and write $U_\infty:=\prod_{v|\infty}\{y\in F_v^\times:|y_v|=1\}$. Suppose that $\Phi=\Phi^{\mathfrak{q},\mathfrak{l}}$ is the vector given in Section 7. It follows from (34) and Propositions 7.1 and 7.2 that

$$H(\pi)=\delta_\infty(\pi)\frac{\widehat{\lambda}_\pi(\mathfrak{l},w)}{(N\mathfrak{l})^w}\frac{\varphi(q)}{q^2}h_{\mathfrak{q}}(s,w;\Pi,\pi),$$

where $\delta_\infty(\pi)$ is the characteristic function of representations that are unramified at every archimedean place, $\widehat{\lambda}_\pi(\mathfrak{l},w)$ are modified Hecke eigenvalues given by

$$\widehat{\lambda}_\pi(\mathfrak{l},w):=\prod_{\substack{\mathfrak{p}^{n_\mathfrak{p}}\|\mathfrak{l}\\ n_\mathfrak{p}\ge1}}\left(\lambda_\pi(\mathfrak{p}^{n_\mathfrak{p}})-\frac{\lambda_\pi(\mathfrak{p}^{n_\mathfrak{p}-1})}{(N\mathfrak{p})^w}\right),\tag{6}$$

φ is the Euler function, and finally, $h_{\mathfrak{q}}(s,w;\Pi,\pi)=1$ if $\mathrm{cond}(\pi)=\mathfrak{q}$, $h_{\mathfrak{q}}(s,w;\Pi,\pi)\ll q^{\theta+\epsilon}$ if $\mathrm{cond}(\pi)\mid\mathfrak{q}$, and it vanishes otherwise. Thus, choosing Φ as above, we get

$$\mathcal{M}_{s,w}(\Phi)=\mathcal{M}_0(\Pi,s,w,\mathfrak{q},\mathfrak{l}),$$

where

$$\mathcal{M}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) := \mathcal{C}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) + \mathcal{E}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}),$$

with

$$\mathcal{C}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) = \frac{\varphi(q)}{q^2} \sum_{\substack{\pi \text{ cusp}^0 \\ \text{cond}(\pi)|\mathfrak{q}}} \frac{\Lambda(s, \Pi \times \pi)\Lambda(w, \pi)}{\Lambda(1, \text{Ad}, \pi)} \frac{\widehat{\lambda}_\pi(\mathfrak{l}, w)}{\ell^w} h_{\mathfrak{q}}(s, w; \Pi, \pi)$$

and

$$\mathcal{E}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) = \frac{\varphi(q)}{q^2} \sum_{\substack{\omega \in \widehat{F^\times U_\infty \backslash \mathbb{A}^\times_{(1)}} \\ \text{cond}(\omega)^2|\mathfrak{q}}} \int_{-\infty}^{\infty} \frac{\Lambda(s, \Pi \times \pi(\omega, it))\Lambda(w, \pi(\omega, it))}{\Lambda^*(1, \text{Ad}, \pi(\omega, it))} \times \frac{\widehat{\lambda}_{\pi(\omega,it)}(\mathfrak{l}, w)}{\ell^w} h_{\mathfrak{q}}(s, w; \Pi, \pi(\omega, it)) \frac{dt}{2\pi}.$$

The notation cusp^0 denotes that we are restricting to forms that are unramified at every archimedean place and the analogous role in the Eisenstein part is played by quotienting by U_∞. Finally, we let

$$\mathcal{N}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) := \mathcal{D}_{s',w'}(\check{\Phi}) + \mathcal{R}_{s',w'}(\check{\Phi}) - \mathcal{D}_{s,w}(\Phi) - \mathcal{R}_{s,w}(\Phi),$$

where $\mathcal{D}$ is given by (30) and $\mathcal{R}$ is given by (53).

Theorem 1.2. *Let* Π *be an everywhere unramified cuspidal automorphic representation of* GL(3) *over* F. *Suppose* $\mathfrak{q}$ *and* $\mathfrak{l}$ *are coprime ideals with absolute norms* q *and* ℓ, *respectively, and that* $\frac{1}{2} \le \text{Re}(s) \le \text{Re}(w) < \frac{3}{4}$. *Then we have*

$$\mathcal{M}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) = \mathcal{N}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) + \mathcal{M}_0(\Pi, s', w', \mathfrak{l}, \mathfrak{q}),$$

where s' *and* w' *are as in* (5). *Moreover, in this same region,* $\mathcal{N}_0$ *satisfies*

$$\mathcal{N}_0(\Pi, s, w, \mathfrak{q}, \mathfrak{l}) \ll_{s,w,\epsilon} \min(q, \ell)^{\theta - 1 + \epsilon}. \tag{7}$$

As an application, we may deduce a nonvanishing result which is similar in spirit to [Khan 2012, Theorem 1.2]: we prove an asymptotic formula for a family of forms of prime level $\mathfrak{p}$, and let $N\mathfrak{p}$ tend to infinity. It may be worth mentioning that although the results are similar, Khan's result concerns modular forms of sufficiently large weight k for GL(2) over the field of rationals, while our result holds for everywhere unramified forms for GL(2) over an arbitrary number field.

Corollary 1.3. *Let* Π *be an unramified cuspidal automorphic representation of* GL(3) *over* F, *and let* $\mathfrak{p}$ *be a prime ideal of* $\mathfrak{o}_F$ *with absolute norm* q. *Then, for every* $\epsilon > 0$,

$$\frac{\varphi(q)}{q} \sum_{\substack{\pi \text{ cusp}^0 \\ \text{cond}(\pi)=\mathfrak{p}}} \frac{\Lambda\left(\frac{1}{2}, \Pi \times \pi\right)\Lambda\left(\frac{1}{2}, \pi\right)}{\Lambda(1, \text{Ad}, \pi)} = \frac{4\Lambda(1, \Pi)\Lambda(0, \Pi)}{\xi_F(2)} + O_\epsilon(q^{\vartheta - \frac{1}{2} + \epsilon}).$$

In particular, for q *sufficiently large, there is at least one automorphic representation* π *of conductor* $\mathfrak{p}$, *unramified for every archimedean place and such that* $\Lambda\left(\frac{1}{2}, \Pi \times \pi\right)$ *and* $\Lambda\left(\frac{1}{2}, \pi\right)$ *are both nonzero.*

Plan of the paper. In Section 2 we lay down our first conventions on number fields and local fields. In Section 3 we recall the notion of automorphic representations for $\mathrm{GL}(n)$ over F and some of its properties. Special attention is given to the case $n = 2$ where, in particular, we recall the construction of Eisenstein series and write down an explicit spectral decomposition. In Section 4, we introduce the Whittaker models and their relation to periods of Rankin–Selberg L-functions for $\mathrm{GL}(n+1)\times\mathrm{GL}(n)$. We work in complete generality but only use the results in the cases $n = 1$ and $n = 2$.

In Section 5 we prove an identity between periods which we call abstract reciprocity. This is connected to the actual reciprocity via a spectral decomposition which is performed in Section 6. In Section 7 we make some explicit computation for the local weights. Section 8 is dedicated to analyzing the degenerate term $\mathcal{D}_{s,w}(\Phi)$ and we show the meromorphic continuation of the spectral moment in Section 9, thus introducing the term $\mathcal{R}_{s,w}(\Phi)$. Theorem 1.1 only uses the results up to Section 6 and a few observations from Section 8. On the other hand, Theorem 1.2 requires the full power of the results in Sections 7, 8 and 9 and its proof is given in Section 10 along with that of Corollary 1.3.

2. Notation

Number fields and completions. Throughout the paper, F will denote a fixed number field with ring of integers $\mathfrak{o}_F$ and discriminant d_F. For v a place of F, we let F_v be the completion of F at the place v. If v is nonarchimedean, we write $\mathfrak{o}_v$ for the ring of integers in F_v, $\mathfrak{m}_v$ for its maximal ideal and ϖ_v for its uniformizer. The adele ring of F is denoted by $\mathbb{A}$, its unit group is denoted by $\mathbb{A}^\times$, and finally, $\mathbb{A}^\times_{(1)}$ denotes the ideles of norm 1. We also fix, once and for all, an isomorphism $\mathbb{A}^\times \simeq \mathbb{A}^\times_{(1)} \times \mathbb{R}_{>0}$.

Additive characters. We let $\psi = \bigotimes_v \psi_v$ be the additive character $\psi = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{F/\mathbb{Q}}$, where $\mathrm{Tr}_{F/\mathbb{Q}}$ is the trace map and $\psi_{\mathbb{Q}}$ is the additive character on $\mathbb{A}_{\mathbb{Q}}$ which is trivial on $\mathbb{Q}$ and such that $\psi(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$. Let d_v be the conductor of ψ_v, i.e., the smallest nonnegative integer such that ψ_v is trivial on $\mathfrak{m}_v^{-d_v}$. Notice that $d_v = 0$ for every finite place not dividing the discriminant and we have the relation $d_F = \prod_v p_v^{d_v}$, where $p_v := |\mathfrak{o}_v/\mathfrak{m}_v|$.

Measures. In the group $\mathbb{A}$ we use a product measure $\mathrm{d}x = \prod_v \mathrm{d}x_v$, where for real v, $\mathrm{d}x_v$ is the Lebesgue measure on $\mathbb{R}$, for complex v, $\mathrm{d}x_v$ is twice the Lebesgue measure on $\mathbb{C}$ and for each finite v, $\mathrm{d}x_v$ is a Haar measure on F_v giving measure $p_v^{-\frac{1}{2}d_v}$ to the compact subgroup $\mathfrak{o}_v$. As for the multiplicative group $\mathbb{A}^\times$, we also take a product measure $\mathrm{d}^\times x = \prod \mathrm{d}^\times x$, where $\mathrm{d}^\times x_v = \zeta_v(1)(\mathrm{d}x_v/|x_v|)$ for infinite or unramified v and we take $\mathrm{d}^\times x_v := p_v^{\frac{1}{2}d_v}\xi_{F_v}(1)(\mathrm{d}x_v/|x_v|)$ for ramified v so that for any finite v, we are giving measure 1 to $\mathfrak{o}_v^\times$. Such measures can naturally give rise to measures on the quotient spaces $F\backslash\mathbb{A}$ and $F^\times\backslash\mathbb{A}^\times_{(1)}$ such that

$$\mathrm{vol}(F\backslash\mathbb{A}) = 1 \quad \text{and} \quad \mathrm{vol}(F^\times\backslash\mathbb{A}^\times_{(1)}) = d_F^{\frac{1}{2}}\xi_F^*(1).$$

The first can be found in Tate's thesis [1950] and the second is [Lang 1994, Proposition XIV.13] (the factor $d_F^{\frac{1}{2}}$ comes from our different normalization of the multiplicative measure).

3. Preliminaries on automorphic representations

In the course of studying automorphic forms in $\mathrm{GL}(n)$, it will be important to distinguish a few of its subgroups. For any unitary ring R with group of invertible elements given by $R^\times$, we let $Z_n(R)$ denote the group of central matrices (i.e., nontrivial multiples of the identity) and $N_n(R)$ denote the maximal unipotent group formed by matrices with entries 1 on the diagonal and 0 below the diagonal, and we let $A_n(R)$ denote the diagonal matrices with lower-right entry 1.

We extend our additives character to N_n in the following way: If $n = (x_{i,j})_{1\le i,j\le n} \in \mathrm{N}_n(\mathbb{A})$, then $\psi(n) := \psi(x_{1,2} + \cdots + x_{n-1,n})$ and similarly for ψ_v. We can extend the measures on the local fields F_v and their unit groups $F_v^\times$ to measures on the groups $Z_n(F_v)$, $N_n(F_v)$ and $A_n(F_v)$ using the obvious isomorphisms $Z_n(R) \simeq R^\times$, $N_n(R) \simeq R^{\frac{1}{2}n(n-1)}$ and $A_n(R) \simeq (R^\times)^{n-1}$.

Moreover, let K_v denote a maximal compact subgroup of $\mathrm{GL}_n(F_v)$ given by

$$K_v := \begin{cases} O(n) & \text{if } F_v = \mathbb{R}, \\ U(n) & \text{if } F_v = \mathbb{C}, \\ \mathrm{GL}_n(\mathfrak{o}_v) & \text{for } v < \infty. \end{cases}$$

We can now define a Haar measure on $\mathrm{GL}_n(F_v)$ by appealing to the Iwasawa decomposition. Let $\mathrm{d}k$ be a Haar probability measure on K_v and consider the surjective map

$$Z_n(F_v) \times N_n(F_v) \times A_n(F_v) \times K_v \to \mathrm{GL}_n(F_v), \quad (z,n,a,k) \mapsto znak,$$

and let $\mathrm{d}g_v$ be the pullback by this map of the measure

$$\Delta(a)^{-1} \prod_{k=1}^{n-1} y_k^{-k(n-k)} \times \mathrm{d}z \times \mathrm{d}n \times \mathrm{d}a \times \mathrm{d}k,$$

where

$$\Delta\begin{pmatrix} y_1 & & & \\ & \ddots & & \\ & & y_{n-1} & \\ & & & 1 \end{pmatrix} = \prod_{j=1}^{n-1} |y_j|^{n+1-2j}.$$

In particular, for GL_2,

$$\int_{\mathrm{GL}_2(F_v)} f(g_v)\,\mathrm{d}g_v = \int_{K_v}\int_{F_v^\times}\int_{F_v}\int_{F_v^\times} f\bigl(z(u)n(x)a(y)k\bigr)\,\mathrm{d}^\times u\,\mathrm{d}x\,\frac{\mathrm{d}^\times y}{|y|_v}\,\mathrm{d}k,$$

where

$$z(u) = \begin{pmatrix} u & \\ & u \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad a(y) = \begin{pmatrix} y & \\ & 1 \end{pmatrix}.$$

Similarly, we shall consider measures on the quotients

$$N_n(F_v)\backslash \mathrm{GL}_n(F_v) \quad \text{and} \quad \mathrm{PGL}_n(F_v) := Z_n(F_v)\backslash \mathrm{GL}_n(F_v)$$

by omitting the terms $\mathrm{d}n$ and $\mathrm{d}z$ respectively. Now, given a group G for which we have attached Haar measures $\mathrm{d}g_v$ to $G(F_v)$, we attach to $G(\mathbb{A})$ the product measure $\mathrm{d}g = \prod_v \mathrm{d}g_v$. Since $\mathrm{PGL}_2(F) \hookrightarrow \mathrm{PGL}_2(\mathbb{A})$ discretely, we may use the measure of $\mathrm{PGL}_2(\mathbb{A})$ to define one on

$$X := \mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A}) = Z_2(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}),$$

which turns out to have finite total measure $\mathrm{vol}(X) < +\infty$.

3A. ***Automorphic representations for* GL(2).** Consider the Hilbert space $L^2(X)$ with an action of $\mathrm{GL}_2(\mathbb{A})$ given by right multiplication and a $\mathrm{GL}_2(\mathbb{A})$-invariant inner product given by

$$\langle \phi_1, \phi_2 \rangle_{L^2(X)} = \int_X \phi_1(g)\overline{\phi_2(g)}\,\mathrm{d}g. \tag{8}$$

It is well known that this space decomposes as

$$L^2(X) = L^2_{\mathrm{cusp}}(X) \oplus L^2_{\mathrm{res}}(X) \oplus L^2_{\mathrm{cont}}(X), \tag{9}$$

where $L^2_{\mathrm{cusp}}(X)$ denotes the closed subspace of cuspidal functions given by the functions satisfying the relation

$$\int_{N_2(F)\backslash N_2(\mathbb{A})} \phi(ng)\,\mathrm{d}n = 0,$$

$L^2_{\mathrm{res}}(X)$ is the residual spectrum consisting of all the one-dimensional subrepresentations of $L^2(X)$, and $L^2_{\mathrm{cont}}(X)$ is expressed in terms of Eisenstein series which we discuss further below. Moreover, $L^2_{\mathrm{cusp}}(X)$ decomposes as a direct sum of irreducible representations, which are called the cuspidal automorphic representations.

3A1. *Induced representations and Eisenstein series.* Given a character $\omega : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ (not necessarily unitary), we denote by $\pi(\omega)$ the isobaric sum $\omega \boxplus \omega^{-1}$, i.e., the space of *measurable* functions f on $\mathrm{GL}_2(\mathbb{A})$ such that

$$f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = |a/d|^{\frac{1}{2}}\omega(a)\omega^{-1}(d)f(g), \quad \langle f, f\rangle_{\mathrm{Ind}} < +\infty,$$

where $|\cdot|$ denotes the adelic norm, $K := \prod_v K_v$, and $\langle f_1, f_2\rangle_{\mathrm{Ind}} < \infty$, where we put

$$\begin{aligned}\langle f_1, f_2\rangle_{\mathrm{Ind}} &:= \int_{F^\times\backslash\mathbb{A}^\times_{(1)}\times K} f_1(a(y)k)\overline{f_2(a(y)k)}\,\mathrm{d}^\times y\,\mathrm{d}k \\ &= \mathrm{vol}(F^\times\backslash\mathbb{A}^\times_{(1)}) \int_K f_1(k)\overline{f_2(k)}\,\mathrm{d}k.\end{aligned} \tag{10}$$

Given such ω and $f \in \pi(\omega)$, we define an Eisenstein series by a process of analytic continuation. It is given by the following series, as long as it converges:

$$\mathrm{Eis}(f)(g) := \sum_{\gamma\in B_2(F)\backslash\mathrm{GL}_2(F)} f(\gamma g),$$

where for a ring R,

$$B_2(R) := \left\{\begin{pmatrix} a & b \\ & d \end{pmatrix} : a, d \in R^\times, b \in R\right\}.$$

Suppose $\omega \neq 1$. We define further the *normalized Eisenstein series* by taking

$$\widetilde{\mathrm{Eis}}(f) := L(1,\omega^2)\,\mathrm{Eis}(f).$$

It will be convenient to define the inner product of two normalized Eisenstein series in terms of the inner product in the induced model of the functions used for generating it. In other words, for $f_1, f_2 \in \pi(\omega)$ and $\phi_i = \widetilde{\mathrm{Eis}}(f_i)$, where $i = 1, 2$, we write

$$\langle \phi_1, \phi_2 \rangle_{\widetilde{\mathrm{Eis}}} := |L(1,\omega^2)|^2 \langle f_1, f_2 \rangle_{\mathrm{Ind}} = d_F^{\frac{1}{2}} \Lambda^*(1, \mathrm{Ad}\,\pi(\omega)) \int_K f_1(k)\overline{f_2(k)}\,\mathrm{d}k. \tag{11}$$

Finally, for a complex parameter s, we use the notation $\pi(\omega, s) := \pi(\omega \times |\cdot|^s)$. For a character ω_v of F_v we can similarly define the induced representation $\pi_v(\omega_v, s)$ so that if $\omega \simeq \bigotimes_v' \omega_v$, we have $\pi(\omega, s) \simeq \bigotimes_v' \pi_v(\omega_v, s)$.

3A2. *Spectral decomposition for smooth functions.* We already encountered a decomposition of $L^2(\boldsymbol{X})$ in (9), but in practice we will encounter functions in $L^2(\boldsymbol{X})$ which are right-invariant by a large compact subgroup $K_0 \subset \mathrm{GL}_2(\mathbb{A})$ and moreover we will need more uniformity than simply L^2-convergence. In the following, we write down a more precise form of this decomposition for functions in $\mathcal{C}^\infty(\boldsymbol{X}/K^S)$, where for a finite set S of places of F containing the archimedean ones, K^S is the compact group given by

$$K^S := \prod_{v \notin S} K_v.$$

The only intervening representations are those that are unramified outside S. That means $\pi \in C(S)$, $\pi = \pi(\omega, it)$ for $\omega \in \Xi(S)$ or $\pi = \omega \circ \det$ for $\omega \in \Xi(S)$. For each cuspidal automorphic representation π, we let $\mathcal{B}_c(\pi)$ denote an orthonormal basis of the realization of π in $L^2(\boldsymbol{X})$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(\boldsymbol{X})}$. Similarly, for an induced representation $\pi = \pi(\omega)$, we define $\mathcal{B}_e(\pi)$ to be a basis of normalized Eisenstein series (not vectors in the induced models!) with respect to $\langle \cdot, \cdot \rangle_{\widetilde{\mathrm{Eis}}}$. We may therefore state the following version of the spectral theorem:

Proposition 3.1. *Let $F \in \mathcal{C}^\infty(\boldsymbol{X}/K^S)$ be of rapid decay. Then*

$$F(g) = \sum_{\pi \in C(S)} \sum_{\phi \in \mathcal{B}_c(\pi)} \langle F, \phi \rangle \phi(g) + \mathrm{vol}(\boldsymbol{X})^{-1} \sum_{\substack{\omega \in \Xi(S) \\ \omega^2 = 1}} \langle F, \omega \circ \det \rangle \omega(\det g)$$
$$+ \frac{1}{4\pi} \sum_{\omega \in \Xi(S)} \int_{-\infty}^{\infty} \sum_{\phi \in \mathcal{B}_e(\pi(\omega, it))} \langle F, \phi \rangle \phi(g)\,\mathrm{d}t,$$

where $\langle \cdot, \cdot \rangle$ denotes the same integral as in the definition of $\langle \cdot, \cdot \rangle_{L^2(\boldsymbol{X})}$ and convergence is absolute and uniform for g on any compact subset of $\boldsymbol{X}$.

The result, for pseudo-Eisenstein series, follows from (4.21) and (4.25) in [Gelbart and Jacquet 1979] and by extending the inner product $(a_1(iy), a_2(iy))$ with respect to an orthogonal basis of $L^2(F^\times \backslash \mathbb{A}^\times_{(1)} \times K)$. The general result is a consequence of the fact that the space of cusp forms decomposes discretely and spans the orthogonal complement to the space of pseudo-Eisenstein series.

4. Whittaker models and periods

In this section, we consider irreducible automorphic representations π of $\mathrm{GL}_n(\mathbb{A})$ and the period integrals related to some Rankin–Selberg L-functions. We will only be concerned with the generic representations, which are those admitting a Whittaker model. This is done for arbitrary n but only the cases $n = 2$ and $n = 1$ are used in the sequel. Finally, we shall not distinguish between a representation π and its space of smooth vectors V_π^∞. An automorphic form ϕ will always denote a smooth vector in an irreducible automorphic representation.

4A. ***Whittaker functions.*** Let π be a generic automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, and let $\phi \in \pi$ be an automorphic form. Let $W_\phi : \mathrm{GL}_n(\mathbb{A}) \to \mathbb{C}$ be the Whittaker function of ϕ given by

$$W_\phi(g) = \int_{N_n(F)\backslash N_n(\mathbb{A})} \phi(ng)\overline{\psi(n)}\,\mathrm{d}n. \tag{12}$$

It satisfies $W_\phi(ng) = \psi(n)W_\phi(g)$ for all $n \in \mathrm{N}_n(\mathbb{A})$.

Given a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, we might write down an isomorphism $\pi \simeq \bigotimes_v' \pi_v$ where for each v, π_v is a local generic admissible representation of $\mathrm{GL}_n(F_v)$, and we might define Whittaker functions for each local representation such that for every $\phi \in \pi$ with $\phi = \bigotimes_v' \phi_v$ through the above isomorphism, we have

$$W_\phi(g) = \prod_v W_{\phi_v}(g_v), \quad g = (g_v)_v \in \mathrm{GL}_n(\mathbb{A}). \tag{13}$$

In fact, the map $\phi \mapsto W_\phi$ is an intertwiner between π and its image, denoted by $\mathcal{W}(\pi, \psi)$, the so-called Whittaker model of π. We similarly define the local Whittaker models $\mathcal{W}(\pi_v, \psi_v)$. Later on, it will be convenient to exchange freely between a representation and its associated Whittaker model. The importance of the latter comes from its close relation to local Rankin–Selberg L-functions, as we will see in Section 4B.

There is a similar story for noncuspidal forms but in this case it is better to work with *normalized* Eisenstein series. As we will only need this for $n = 2$, we shall restrict to this case. Let $f \in \pi(\omega)$ and suppose that f is factorable, i.e., $f = \bigotimes_v' f_v$ with $f_v \in \pi_v(\omega_v)$. Then it follows by analytic continuation and Bruhat decomposition that

$$W_{\widetilde{\mathrm{Eis}}(f)}(g) = L(1, \omega^2) \int_{N_2(\mathbb{A})} f(wng)\overline{\psi(n)}\,\mathrm{d}n = \prod_v W_{f_v}^J(g_v),$$

where $W_{f_v}^J$ is the normalized Jacquet integral, given by

$$W_{f_v}^J(g_v) = L(1, \omega_v^2) \int_{N_2(F_v)} f_v(wng)\overline{\psi(n)}\,\mathrm{d}n.$$

By putting $\phi = \widetilde{\mathrm{Eis}}(f)$ and $W_{\phi_v} := W_{f_v}^J$, we see that (13) also holds in this case.

It is also important to consider Whittaker functions with respect to the inverse character $\psi' = \overline{\psi}$, so we analogously define W'_ϕ and W'_{ϕ_v} by replacing ψ_v by $\psi'_v = \overline{\psi}_v$ and ψ by $\psi' = \prod_v \psi'_v$ in all the previous definitions. It follows from uniqueness of local Whittaker functions that we may take

$$W'_{\phi_v} = \overline{W_{\phi_v}} \quad \text{for all places } v \text{ of } F. \tag{14}$$

If a local generic admissible representation π_v is unramified for some finite place v, this mean that in π_v there exists a vector which is right-invariant by the action of $\mathrm{GL}_n(\mathfrak{o}_v)$. Such a vector is called *spherical* and spherical vectors are unique up to multiplication by scalars. Among the spherical vectors we shall distinguish a certain one which we call *normalized spherical*. If v is unramified, the normalized spherical vector will be the one for which $W_{\phi_v}(e) = 1$, where $e \in \mathrm{GL}_n(F_v)$ denotes the identity element. For the finite ramified places we simply define it to be the newform (defined in Section 4C). This avoids repetition and is justified by the fact that the two notions also coincide for unramified primes.

4B. ***Integral representations of $\mathbf{GL}_{n+1} \times \mathbf{GL}_n$ L-functions.*** This theory is an outgrowth of Hecke's theory of L-functions for GL_2 and has been developed by Jacquet, Piatetski-Shapiro and Shalika. We start with Π and π irreducible automorphic representations of $\mathrm{GL}_{n+1}(\mathbb{A})$ and $\mathrm{GL}_n(\mathbb{A})$, respectively, and let $\Phi \in \Pi$ and $\phi \in \pi$ be automorphic forms. Suppose momentarily that Φ is a cusp form and hence rapidly decreasing. We can thus consider for every $s \in \mathbb{C}$ the integral

$$I(s, \Phi, \phi) := \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})} \Phi\begin{pmatrix} h & \\ & 1 \end{pmatrix} \phi(h) |\det h|^{s-\frac{1}{2}} \, \mathrm{d}h.$$

It follows from the Whittaker decomposition of cusp forms (see [Cogdell 2007, Theorem 1.1]) that if Φ is a cuspidal function, then

$$I(s, \Phi, \phi) = \Psi(s, W_\Phi, W'_\phi) \qquad (\mathrm{Re}(s) \gg 1), \tag{15}$$

where

$$\Psi(s, W, W') := \int_{N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} W\begin{pmatrix} h & \\ & 1 \end{pmatrix} W'(h) |\det h|^{s-\frac{1}{2}} \, \mathrm{d}h. \tag{16}$$

Our next result gives some of the good properties of $\Psi(s, W, W')$, namely, convergence and the fact that it factors into local integrals whenever Φ and ϕ also factor.

Proposition 4.1. *Let Π and π be automorphic representations of $\mathrm{GL}(n+1)$ and $\mathrm{GL}(n)$ over F, respectively. Let $\Phi = \bigotimes'_v \Phi_v \in \Pi$ and $\phi = \bigotimes'_v \phi_v \in \pi$ be automorphic forms. Let W_{Φ_v} and W'_{ϕ_v} be as in Section 4A. Then, for $\mathrm{Re}(s) \gg 1$, $\Psi(s, W_\Phi, W'_\phi)$ converges and we have the factorization*

$$\Psi(s, W_\Phi, W'_\phi) = \prod_v \Psi_v(s, W_{\Phi_v}, W'_{\phi_v}),$$

where

$$\Psi_v(s, W, W') := \int_{N_n(F_v)\backslash \mathrm{GL}_n(F_v)} W\begin{pmatrix} h_v & \\ & 1 \end{pmatrix} W'(h_v) |\det h_v|^{s-\frac{1}{2}} \, \mathrm{d}h_v. \tag{17}$$

Moreover, if v is finite and both Π_v and π_v are unramified and Φ_v and ϕ_v are normalized spherical,

$$\Psi_v(s, W_{\Phi_v}, W'_{\phi_v}) = p_v^{d_v l_n(s)} L(s, \Pi_v \times \pi_v), \quad \textit{where } l_n(s) = \tfrac{1}{2}n(n+1)s - \tfrac{1}{12}n(n+1)(2n+1).$$

Proof. The first part follows from gauge estimates for Whittaker functions (see [Jacquet et al. 1979, §2]). It is an important fact that this part does not require the representations to be cuspidal. The reason is that, in some sense, the integral representation using Whittaker functions only sees the nonconstant terms. For the local computation this is well known when p_v is unramified (see, e.g., [Cogdell 2007, Theorem 3.3]). In general we may restrict to the unramified situation by following the computation in the proof of [Cogdell and Piatetski-Shapiro 1994, Lemma 2.1]. □

Remark. When $n = 1$, we write $I(s,\phi)$, $\Psi(s, W_\phi)$ and $\Psi_v(s, W_{\phi_v})$ instead of $I(s,\phi,\mathbf{1})$, $\Psi(s, W_\phi, W_{\mathbf{1}_v})$ and $\Psi_v(s, W_{\phi_v}, W_{\mathbf{1}})$, where $\mathbf{1}$ and $\mathbf{1}_v$ denote the constant functions on $\mathrm{GL}_1(\mathbb{A})$ and $\mathrm{GL}_1(F_v)$ respectively.

4C. ***Newforms and ramified L-factors.*** For a finite place v and any admissible irreducible generic representation of $\mathrm{GL}_n(F_v)$, not necessarily unramified, we define a distinguished vector in its Whittaker model, called *newform*. This was first introduced by Casselman [1973] when $n = 2$ by translating the results of Atkin and Lehner to the representation-theoretic language. This was later generalized by Jacquet, Piatetski-Shapiro and Shalika [Jacquet et al. 1981] for general n by requiring that they are good test vectors for representing L-functions via Rankin–Selberg periods as in Section 4B. Moreover, when π_v is unramified, these coincide with normalized spherical vectors.

The fact that these newvectors are test vectors for Rankin–Selberg L-functions can be rephrased by relating their values to the Langlands parameters of the representation. This was carefully carried out in [Miyauchi 2014]. In order to quote these results we introduce the following notation: for $\underline{\nu} = (\nu_1, \dots, \nu_{n-1}) \in \mathbb{Z}^{n-1}$, let $s(\underline{\nu}) = \sum_{i=1}^{n-1} \frac{1}{2} i(n-i)\nu_i$, and for $y \in F_v^\times$, we write

$$a(\underline{\nu}) := \begin{pmatrix} \varpi_v^{\nu_1+\cdots+\nu_{n-1}} & & & \\ & \varpi_v^{\nu_2+\cdots+\nu_{n-1}} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

The main result of [Miyauchi 2014] states that if ψ_v is unramified, then $W_{\pi_v}(a(\underline{\nu})) = p^{s(\nu)}\lambda_\pi(\underline{\nu})$, where

$$\lambda_{\pi_v}(\underline{\nu}) = 0 \quad \text{unless } \nu_1, \dots, \nu_{n-1} \geq 0, \tag{18}$$

and the $\lambda_{\pi_v}(\underline{\nu})$ are in general given by Schur polynomials evaluated on the Langlands parameters of π_v (see [Miyauchi 2014] for details).

When ψ_v is ramified, this has to be modified. First, we write $\psi_v(x) = \psi_{F_v}(\lambda x)$ for some $\lambda \in F_v^\times$, where ψ_{F_v} is an unramified additive character of F_v, and let $d = v(\lambda)$. We then define the *newvector* by taking

$$W_{\pi_v}(g) = W_{\pi_v}^{\mathrm{unr}}(a(\iota_n(d))g),$$

where $\iota_n(d) = (d, d, \dots, d) \in \mathbb{Z}^{n-1}$ and $W_{\pi_v}^{\mathrm{unr}}$ denotes the newvector for the unramified character ψ_{F_v}. The term $a(\iota_n(d))$ is responsible for the change in the additive character.

In addition to Proposition 4.1, we shall also need to compute L-functions for certain ramified local representations. In particular, we require the following computation that appears, for instance, in the work of Booker, Krishnamurthy and Lee [Booker et al. 2020, proof of Lemma 3.1]: Let $n > m$, and let Π_v (resp. π_v) be an irreducible admissible generic representation of $\mathrm{GL}_n(F_v)$ (resp. $\mathrm{GL}_m(F_v)$) with Langlands parameters $(\gamma^{(i)}_{\Pi_v})_{i=1}^n$ (resp. $(\gamma^{(j)}_{\pi_v})_{j=1}^m$). Supposing further that π_v is ramified, one then has

$$\sum_{\substack{\underline{\nu}\in\mathbb{Z}^{m-1}\\ \nu_1,\dots,\nu_{m-1}\geq 0}} \frac{\lambda_{\Pi_v}(\underline{\nu},0,\dots,0)\lambda_{\pi_v}(\underline{\nu})}{p_v^{(\nu_1+2\nu_2+\cdots+(m-1)\nu_{m-1})s}} = L(s,\Pi_v\times\pi_v), \tag{19}$$

where

$$L(s,\Pi_v\times\pi_v) := \prod_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} (1-\gamma^{(i)}_{\Pi_v}\gamma^{(j)}_{\pi_v} p_v^{-s})^{-1}. \tag{20}$$

This coincides with Langlands local L-function when Π_v is unramified, which we shall suppose.

4D. *Relation between inner products on* GL(2). Let π be a generic automorphic representation of $\mathrm{GL}(2)$ over F with trivial central character. We define a $\mathrm{GL}_2(\mathbb{A})$-invariant inner product on the representation space of π as follows: If π is cuspidal, then we may see V_π embedded in $L^2(\boldsymbol{X})$ and therefore π may inherit the inner product from $L^2(\boldsymbol{X})$ given by (8). If π is Eisenstein we cannot see the representation space of π inside $L^2(\boldsymbol{X})$ and hence we equip it with the inner product given by (11).

There is however another way of defining an inner product for factorable vectors in these representations which is independent of whether π is cuspidal or Eisenstein. This is done by using the Whittaker model as follows: For each place v, we have a $\mathrm{GL}_2(F_v)$-invariant inner form on $\mathcal{W}(\pi_v,\psi_v)$ by letting

$$\vartheta_v(W_1,W_2) = \frac{\int_{F_v^\times} W_1(a(y_v))\overline{W_2(a(y_v))}\,\mathrm{d}^\times y_v,}{\zeta_v(1)L_v(1,\mathrm{Ad}\,\pi)/\zeta_v(2)}. \tag{21}$$

The fact that the numerator of (21) is indeed right $\mathrm{GL}_2(F_v)$-invariant follows from the theory of the Kirillov model and the inclusion of the denominator is to ensure the following property: Whenever π_v and ψ_v are unramified and W is normalized spherical, we have $\vartheta_v(W,W)=1$. Finally, letting $\phi_1=\bigotimes\phi_{1,v}$ and $\phi_2=\bigotimes\phi_{2,v}$ be either cusp forms or normalized Eisenstein series, we define the *canonical* inner product by the formula

$$\langle\phi_1,\phi_2\rangle_{\mathrm{can}} := 2d_F^{\frac{1}{2}}\Lambda^*(1,\mathrm{Ad}\,\pi)\times\prod_v \vartheta_v(W_{\phi_{1,v}},W_{\phi_{2,v}}). \tag{22}$$

Since every two $\mathrm{GL}_2(\mathbb{A})$-invariant inner products in π must be equal up to multiplication by some scalar, it follows that we can compare the canonical inner product with the ones introduced earlier for cuspidal and Eisenstein representations. Indeed, Rankin–Selberg theory in the cuspidal case and a direct computation in the Eisenstein case gives us the following relation:

$$\langle\phi_1,\phi_2\rangle_{\mathrm{can}} = \begin{cases} \langle\phi_1,\phi_2\rangle_{L^2(\boldsymbol{X})} & \text{if } \pi \text{ is cuspidal,}\\ 2\langle\phi_1,\phi_2\rangle_{\mathrm{Eis}} & \text{if } \pi \text{ is Eisenstein.}\end{cases} \tag{23}$$

The computation in the Eisenstein case follows from [Wu 2014, Lemma 2.8]. For the cusp forms we combine the proof of [Wu 2014, Proposition 2.13] with the value of the residue of an Eisenstein series computed in [Michel and Venkatesh 2010, (4.6)].[1]

5. Abstract reciprocity

In this section we show an identity between two periods. At this point we make no attempt to relate them to moments of L-functions. The proof is a rather simple matrix computation.

Suppose $\Phi \in \mathcal{C}^\infty\big(Z_3(\mathbb{A})\mathrm{GL}_3(F)\backslash\mathrm{GL}_3(\mathbb{A})\big)$ is such that for every $h \in \mathrm{GL}_2(\mathbb{A})$, the integral

$$\mathcal{A}_s\Phi(h) := |\det h|^{s-\frac{1}{2}} \int_{F^\times\backslash\mathbb{A}^\times} \Phi\begin{pmatrix} z(u)h & \\ & 1\end{pmatrix} |u|^{2s-1} d^\times u \tag{24}$$

converges and such that $y \mapsto \mathcal{A}_s\Phi\big(\big(\begin{smallmatrix} y & \\ & 1\end{smallmatrix}\big)h\big)$ is of rapid decay as $|y| \to 0$ or $+\infty$.

Proposition 5.1. *Let Φ be as above, and let $I(w,\cdot)$ be as in the remark on page* 1392. *Then, for every $s, w \in \mathbb{C}$, we have the reciprocity relation*

$$I(w, \mathcal{A}_s\Phi) = I(w', \mathcal{A}_{s'}\check{\Phi}),$$

where (s', w') are as in (5) *and*

$$\check{\Phi}(g) := \Phi(g w_{23}), \quad w_{23} = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}. \tag{25}$$

Proof. By definition,

$$I(w, \mathcal{A}_s\Phi) = \int_{F^\times\backslash\mathbb{A}^\times}\int_{F^\times\backslash\mathbb{A}^\times} \Phi\begin{pmatrix} z(u)a(y) & \\ & 1\end{pmatrix} |u|^{2s-1}|y|^{s+w-1}\, \mathrm{d}^\times u\, \mathrm{d}^\times y. \tag{26}$$

Now, since Φ is left-invariant by $Z_3(\mathbb{A})\mathrm{GL}_3(F)$, we see that for every $u, y \in \mathbb{A}^\times$, one has

$$\Phi\begin{pmatrix} uy & & \\ & u & \\ & & 1\end{pmatrix} = \Phi\left(\begin{pmatrix} u & & \\ & u & \\ & & u\end{pmatrix} w_{23}\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1\end{pmatrix} w_{23}\right) = \Phi\left(\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1\end{pmatrix} w_{23}\right) = \check{\Phi}\begin{pmatrix} y & & \\ & u^{-1} & \\ & & 1\end{pmatrix}.$$

Applying this to (26) and making the change of variables $(u, y) = (u'^{-1}, u'y')$ gives the result. ☐

6. Spectral expansion of the period

In this section we will give a spectral decomposition of the period $I(w, \mathcal{A}_s\Phi)$. Let Π be an automorphic cuspidal representation for GL(3) over F, and let $\Phi = \bigotimes_v \Phi_v \in \Pi$ be a cusp form. Let S be a finite set of places containing all archimedean places and all the places for which Φ is not normalized spherical. Since Φ is of rapid decay, then the same holds for $\mathcal{A}_s\Phi$. More precisely this follows by combining the

[1] In [Wu 2014], a factor $d_F^{\frac{1}{2}}$ seems to be missing in the computation of this residue.

rapid decay of Whittaker functions with the action of the Weyl group of GL(3). We can thus spectrally decompose it as in Proposition 3.1:

$$\mathcal{A}_s\Phi(h) = \sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_c(\pi)}\langle\mathcal{A}_s\Phi,\phi\rangle\phi(h) + \mathrm{vol}(X)^{-1}\sum_{\substack{\omega\in\Xi(S)\\ \omega^2=1}}\langle\mathcal{A}_s\Phi,\omega\circ\det\rangle\omega(\det g) + \frac{1}{4\pi}\sum_{\omega\in\Xi(S)}\int_{-\infty}^{\infty}\sum_{\phi\in\mathcal{B}_e(\pi(\omega,it))}\langle\mathcal{A}_s\Phi,\phi\rangle\phi(h)\,\mathrm{d}t.$$

By integrating both sides of the above expression against an additive character and over the compact set $N_2(F)\backslash N_2(\mathbb{A})$, we get the following relation for Whittaker functions:

$$W_{\mathcal{A}_s\Phi}(h) = \sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_c(\pi)}\langle\mathcal{A}_s\Phi,\phi\rangle W_\phi(h) + \frac{1}{4\pi}\sum_{\pi\in\Xi(S)}\int_{-\infty}^{\infty}\sum_{\phi\in\mathcal{B}_e(\pi(\omega,it))}\langle\mathcal{A}_s\Phi,\phi\rangle W_\phi(h)\,\mathrm{d}t.$$

Notice that since the one-dimensional representations are not generic, they do not contribute to the above expression. Now, because of rapid decay of the Whittaker functions W_ϕ as $|y|\to+\infty$, if we take $\mathrm{Re}(w)$ sufficiently large, we get

$$\Psi(w,\mathcal{A}_s\Phi) = \sum_{\pi\in C(S)}\sum_{\phi\in\mathcal{B}_c(\pi)}\langle\mathcal{A}_s\Phi,\phi\rangle\Psi(w,W_\phi) + \frac{1}{4\pi}\sum_{\omega\in\Xi(S)}\int_{-\infty}^{\infty}\sum_{\phi\in\mathcal{B}_e(\pi(\omega,it))}\langle\mathcal{A}_s\Phi,\phi\rangle\Psi(w,W_\phi)\,\mathrm{d}t. \quad (27)$$

By using the Fourier decomposition of $\mathcal{A}_s\Phi$, we see that

$$\int_{\mathbb{A}^\times} W_{\mathcal{A}_s\Phi}(a(y))|y|^{w-\frac{1}{2}}\,\mathrm{d}^\times y = I(w,\mathcal{A}_s\Phi) - I(w,(\mathcal{A}_s\Phi)_0),$$

where for any ϕ on $\mathcal{C}^\infty(X)$, ϕ_0 is given by

$$\phi_0(h) := \int_{F\backslash\mathbb{A}}\phi(n(x)h)\,\mathrm{d}x.$$

The next step is to realize the terms $\langle\mathcal{A}_s\Phi,\phi\rangle$ and $\Psi(w,W_\phi)$ as a product of local integrals. First, it follows from Proposition 4.1 that if $\phi=\bigotimes_v\phi_v$ is decomposable,

$$\Psi(w,W_\phi) = \prod_v\Psi_v(w,W_{\phi_v}).$$

Moreover, from the definition of $\mathcal{A}_s\Phi$, we deduce, after changing variables, that

$$\langle\mathcal{A}_s\Phi,\phi\rangle = I(s,\Phi,\bar{\phi}).$$

Since Φ is a cusp form on GL(3), it follows from (15) and Proposition 4.1 that for $\mathrm{Re}(s)$ sufficiently large and factorable ϕ,

$$I(s,\Phi,\bar{\phi}) = \Psi(s,W_\Phi,W'_{\bar{\phi}}) = \prod_v\Psi_v(s,W_{\Phi_v},W'_{\bar{\phi}_v}),$$

where $\Psi_v(s, W, W')$ is given by (17). As a consequence, we have

$$I(w, W_{\mathcal{A}_s\Phi}) = I(w, (\mathcal{A}_s\Phi)_0) + \sum_{\pi\in C(S)} \sum_{\phi\in\mathcal{B}_c(\pi)} \prod_v \Psi_v(s, W_{\Phi_v}, W'_{\bar{\phi}_v}) \prod_v \Psi_v(s, W_{\phi_v}) + \frac{1}{4\pi} \sum_{\omega\in\Xi(S)} \int_{-\infty}^{\infty} \sum_{\phi\in\mathcal{B}_e(\pi(\omega,it))} \prod_v \Psi_v(s, W_{\Phi_v}, W'_{\bar{\phi}_v}) \prod_v \Psi_v(s, W_{\phi_v})\, \mathrm{d}t. \quad (28)$$

For each generic automorphic representation π we will now construct an orthonormal basis for V_π which is formed exclusively by factorable vectors: We start by choosing for each place v, an orthogonal basis $\mathcal{B}^W(\pi_v)$ of the space $\mathcal{W}(\pi_v, \psi_v)$. Consider now the elements $\phi = \bigotimes_v \phi_v$ such that for every finite v, W_{ϕ_v} lies in $\mathcal{B}^W(\pi_v)$, and for all but finitely many v, W_{ϕ_v} is normalized spherical. This provides us with an *orthogonal* basis for V_π. In order to get an *orthonormal* basis we multiply these vectors by the correcting factors coming from (23). Applying these steps to (28) leads to the following (the slightly awkward normalization is justified by the last part of Proposition 4.1):

Proposition 6.1. *Let Π be a cuspidal automorphic representation, and let $\Phi = \bigotimes_v \Phi_v \in \Pi$ be a cusp form. Then, for complex numbers s and w with sufficiently large real parts, we have*

$$2d_F^{\frac{7}{2}-3s-w} I(w, \mathcal{A}_s\Phi) = \mathcal{M}_{s,w}(\Phi) + \mathcal{D}_{s,w}(\Phi),$$

where

$$H(\pi) = \prod_v H_v(\pi_v), \quad H_v(\pi_v) := p_v^{d_v(3-3s-w)} \sum_{W\in\mathcal{B}^W(\pi_v)} \frac{\Psi_v(s, W_{\Phi_v}, \overline{W})\Psi_v(w, W)}{L(s, \Pi_v\times\pi_v)L(w, \pi_v)}, \quad (29)$$

$\mathcal{M}_{s,w}(\Phi)$ is as in (4), *and*

$$\mathcal{D}_{s,w}(\Phi) := 2d_F^{\frac{7}{2}-3s-w} \int_{F^\times\backslash\mathbb{A}^\times} \int_{F\backslash\mathbb{A}} \int_{F^\times\backslash\mathbb{A}^\times} \Phi\begin{pmatrix} z(u)n(x)a(y) & \\ & 1 \end{pmatrix} |u|^{2s-1} |y|^{s+w-1}\, \mathrm{d}^\times u\, \mathrm{d}x\, \mathrm{d}^\times y. \quad (30)$$

We will refer to the function H given by (29) where $\Phi = \bigotimes \Phi_v \in \Pi$ as the (s, w)-weight function of kernel Φ. If s and w and Φ are clear from the context, we shall refer to it simply as the weight function of kernel Φ.

Finally, given $s, w \in \mathbb{C}$, if H is the (s, w)-weight function with kernel Φ, we let $\check{H}$ be the (s', w')-weight function associated to $\check{\Phi}$, where s' and w' are as in (5) and $\check{\Phi}$ is as in (25). In other words,

$$\check{H}(\pi) = \prod_v \check{H}_v(\pi_v), \quad \check{H}_v(\pi) := p_v^{d_v(3-3s'-w')} \sum_{W\in\mathcal{B}^W(\pi_v)} \frac{\Psi_v(s', W_{\check{\Phi}_v}, \overline{W})\Psi_v(w', W)}{L(s', \Pi_v\times\pi_v)L(w', \pi_v)}. \quad (31)$$

7. Local computations

Let Π be an unramified cuspidal automorphic representation of PGL(3) over F. For all v, we let Φ_v^0 correspond to the normalized spherical vector in the Whittaker model, that is, $W_{\Phi_v^0} = W_{\Pi_v}$. Let $\mathfrak{q}$ and $\mathfrak{l}$

be two coprime integral ideals of F. Finally, let $\Phi^{\mathfrak{q},\mathfrak{l}} = \bigotimes_v \Phi_v^{\mathfrak{q},\mathfrak{l}}$, where, for all $v \nmid \mathfrak{q}\mathfrak{l}$, we put $\Phi_v^{\mathfrak{q},\mathfrak{l}} = \Phi_v^0$, for $v \mid \mathfrak{q}$,

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^n} \sum_{\beta \in \mathfrak{m}_v^{-n}/\mathfrak{o}_v} \Phi_v^0\left(g \begin{pmatrix} 1 & & \beta \\ & 1 & \\ & & 1 \end{pmatrix}\right), \tag{32}$$

where $n = v(\mathfrak{q})$, and, for $v \mid \mathfrak{l}$,

$$\Phi_v^{\mathfrak{q},\mathfrak{l}}(g) := \frac{1}{p_v^m} \sum_{\beta \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v} \Phi_v^0\left(g \begin{pmatrix} 1 & \beta & \\ & 1 & \\ & & 1 \end{pmatrix}\right),$$

with $m = v(\mathfrak{l})$.

We will now proceed to the calculation of H_v for $\Phi = \Phi^{\mathfrak{q},\mathfrak{l}}$. First notice that if, for some compact group K_v' of $\mathrm{GL}_2(F_v)$, we have that Φ_v is invariant on the right by matrices of the shape $\begin{pmatrix} k & \\ & 1 \end{pmatrix}$, where $k \in K_v'$, then we may restrict the sum over the basis $\mathcal{B}^W(\pi_v)$ to a sum over a basis of the right K_v'-invariant vectors. In particular, if $v < \infty$ and $v \nmid \mathfrak{q}\mathfrak{l}$, this basis will have only one element, which can be taken to be normalized spherical. Thus, by Proposition 4.1, we see that $H_v(\pi_v) = 1$ in those cases. We divide the remaining cases in three subcategories: $v \mid \mathfrak{l}$, $v \mid \mathfrak{q}$ and $v \mid \infty$ and treat them in that order.

7A. *Nonarchimedean case I: $v \mid \mathfrak{l}$.* Even though this is not obvious at first glance, we will show that H_v vanishes unless π_v is unramified. First, notice that by right GL(2)-invariance of the Whittaker norm, we have that for every orthonormal basis $\mathcal{B}$ of $\mathcal{W}(\pi_v, \psi_v)$, one may construct another one by taking $\mathcal{B}' := \{\pi_v(h)W, W \in \mathcal{B}\}$. Applying this for $h = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$ for $\beta \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v$, changing variables in the $\mathrm{GL}(3) \times \mathrm{GL}(2)$ Rankin–Selberg integral and summing over β, we deduce that

$$H_v(\pi_v) = p_v^{d_v(3-3s-w)} \sum_{W \in \mathcal{B}^W(\pi_v)} \frac{\Psi_v(s, W_{\Pi_v}, \overline{W})\Psi_v(w, W^{(m)})}{L(s, \Pi_v \times \pi_v)L(w, \pi_v)},$$

where

$$W^{(m)}(h) := p^{-m} \sum_{\beta \in \mathfrak{m}_v^{-m}/\mathfrak{o}_v} W\left(h \begin{pmatrix} 1 & -\beta \\ & 1 \end{pmatrix}\right).$$

Now, since W_{Π_v} is spherical, we may restrict the sum over $\mathcal{B}^W(\pi_v)$ to only one term for which W is the normalized spherical vector. Now, by Proposition 4.1, $\Psi_v(s, W_{\Pi_v}, \overline{W}_{\pi_v}) = p_v^{d_v(3s-\frac{5}{2})} L(s, \Pi_v \times \pi_v)$ and

$$\begin{aligned} \Psi_v(w, W_{\pi_v}^{(m)}) &= \int_{F_v^\times} \delta_{v(y) \geq m-d} W_{\pi_v}\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{w-\frac{1}{2}} \,\mathrm{d}^\times y, \\ &= p_v^{d_v(w-\frac{1}{2})} \sum_{\mu \geq m} \frac{\lambda_{\pi_v}(\mu)}{p_v^{\mu w}}, = p_v^{-mw}\left(\lambda_{\pi_v}(m) - \frac{\lambda_{\pi_v}(m-1)}{p_v^w}\right) p_v^{d_v(w-\frac{1}{2})} L(w, \pi_v). \end{aligned} \tag{33}$$

Hence we have that

$$H_v(\pi_v) = p_v^{-mw}\left(\lambda_{\pi_v}(m) - \frac{\lambda_{\pi_v}(m-1)}{p_v^w}\right). \tag{34}$$

7B. ***Nonarchimedean case II: v | q.*** We will show that $H_v(\pi_v)$ vanishes unless $c(\pi_v) \leq n$ and that $H_v(\pi_v) \ll_\epsilon p_v^{n(\theta-1+\epsilon)}$, and if $c(\pi_v) = n$, then $H_v(\pi_v) = \varphi(p_v^n) p_v^{-2n}$.

We first notice that by a result of Casselman [1973], if we let $W_0 = W_{\pi_v}$ be the newvector and for each $j \geq 0$, we let

$$W_j := \pi_v \begin{pmatrix} 1 & \\ & \varpi_v^j \end{pmatrix} W_0. \tag{35}$$

Then, for each $j \geq 0$, $\{W_0, W_1, \dots, W_j\}$ is a basis for the $K_v[n_0+j]$-invariant vectors in $\mathcal{W}(\pi_v, \psi_v)$, where $n_0 = c(\pi_v)$. We now construct an *orthonormal* basis by employing the Gram–Schmidt process. This is the local counterpart of the method in [Blomer and Milićević 2015].

Let $\lambda_{\pi_v} = \lambda_{\pi_v}(1)$ be as in Section 4C and $\delta_{\pi_v} = \delta_{n_0=0}$, and take $\alpha_{\pi_v} := \lambda_{\pi_v}/(\sqrt{p_v}(1 + \delta_{\pi_v}/p_v))$. We put

$$\xi_{\pi_v}(0,0) = 1, \quad \xi_{\pi_v}(1,1) = \frac{1}{\sqrt{1-\alpha_{\pi_v}^2}}, \quad \xi_{\pi_v}(1,0) = -\alpha_{\pi_v} p_v^{\frac{1}{2}} \xi_{\pi_v}(1,1),$$

and

$$\xi_{\pi_v}(j,j) = \frac{1}{\sqrt{1-\alpha_{\pi_v}^2}\sqrt{1-\delta_{\pi_v}/p_v^2}}, \quad \xi_{\pi_v}(j,j-1) = -\lambda_{\pi_v}\xi_{\pi_v}(j,j), \quad \xi_{\pi_v}(j,j-2) = \delta_{\pi_v}\xi_{\pi_v}(j,j),$$

and $\xi_{\pi_v}(j,k) = 0$ for $k \leq j-2$. If one assumes any nontrivial bound towards the Ramanujan conjecture $\lambda_{\pi_v} \ll p_v^{\vartheta}$, with $\vartheta < \frac{1}{2}$, one has that $|\alpha_{\pi_v}|$ is uniformly bounded by some constant $C_\vartheta < 1$ and therefore

$$\xi_{\pi_v}(j,k) \ll p_v^{j\epsilon} p_v^{(j-k)\vartheta}. \tag{36}$$

More importantly, for $j \geq 0$, $\{\widetilde{W_0}, \widetilde{W_1}, \dots, \widetilde{W_j}\}$ is an orthonormal basis for the space of $K_v[n_0+j]$-vectors in $\mathcal{W}(\pi_v, \psi_v)$, where

$$\widetilde{W_j} := \frac{1}{\langle W_0, W_0\rangle^{1/2}} \sum_{k=1}^{j} \xi_{\pi_v}(j,k) p_v^{\frac{1}{2}(k-j)} W_k. \tag{37}$$

To see this we first compute $\langle W_{k_1}, W_{k_2}\rangle$, which, by (35) and the definition of λ_{π_j}, equals

$$p^{-\frac{1}{2}|k_2-k_1|} S_{|k_2-k_1|},$$

where, for $t \geq 0$,

$$S_t = \frac{\zeta_v(2)}{L_v(1, \pi_v \times \bar{\pi}_v)} \sum_{\nu \geq 0} \frac{\lambda_{\pi_v}(\nu)\lambda_{\pi_v}(\nu+t)}{p_v^\nu}.$$

It follows from the Hecke relations for $\lambda_{\pi_v}(\nu)$ that

$$S_t = \lambda_{\pi_v} S_{t-1} - \delta_{\pi_v} S_{t-2} \quad \text{for } t \geq 2 \qquad \text{and} \qquad S_1 = \alpha_{\pi_v} p_v^{\frac{1}{2}} S_0,$$

from which the claim follows.

By definition, we have

$$W_{\Phi_v}\begin{pmatrix} a & b & \\ c & d & \\ & & 1\end{pmatrix} = \frac{1}{p_v^n}\sum_{\beta\in\mathfrak{m}_v^{-n}/\mathfrak{o}_v}\psi(\beta c)W_{\Pi_v}\begin{pmatrix} a & b & \\ c & d & \\ & & 1\end{pmatrix} = \delta_{v(c)\geq n-d_v}W_{\Pi_v}\begin{pmatrix} a & b & \\ c & d & \\ & & 1\end{pmatrix}.$$

Hence, if we write $h=z(u)n(x)a(y)k$, with $x\in F_v$, $u,y\in F_v^\times$ and $k=(k_{ij})\in K_v$, then W_{Φ_v} vanishes unless $v(uk_{21})\geq n-d_v$. Letting $d_1:=\min(n,v(u)+d_v)$ and $d_2:=n-d_1$, we see that this is equivalent to k belonging to $K_v[d_2]$. This allows us to write

$$\Psi_v(s,W_{\Phi_v},\overline{W}) = p_v^{d_v(2s-1)}\sum_{d_1+d_2=n}\ \sum_{\min(\nu_1,n)=d_1} p_v^{-2\nu_1(s-\frac{1}{2})}\Psi_{\nu_1,d_2}(W), \tag{38}$$

where

$$\Psi_{\nu_1,d_2}(W) = \int_{F_v^\times}\int_{K_v[d_2]} W_{\Pi_v}\big(z(\varpi_v^{\nu_1-d_v})a(y)\big)\overline{W}(a(y)k)|y|^{s-\frac{3}{2}}\,\mathrm{d}^\times y\,\mathrm{d}k.$$

Now, if $W=\widetilde{W_j}$ is an element of our basis, given by (37), then it follows that

$$\int_{K_v[f]}\widetilde{W_j}(hk)\,\mathrm{d}k = \begin{cases}\mathrm{vol}(K_v[f])\widetilde{W_j}(h) & \text{if } j+n_0\leq f,\\ 0 & \text{otherwise.}\end{cases} \tag{39}$$

We reason as follows: On the one hand, for every j, $\widetilde{W_j}$ is $K_v[n_0+j]$-invariant and is orthogonal to $\mathcal{W}(\pi_v,\psi_v)^{K_v[n_0+j-1]}$. On the other, the operator

$$W\mapsto\frac{1}{\mathrm{vol}(K_v[f])}\int_{K_v[f]}\pi_v(k)W\,\mathrm{d}k$$

is the orthogonal projection into the space of $K_v[f]$-invariant vectors.

Applying (38) and (39) to the definition of $H_v(\pi_v)$ and changing order of summation, we are led to

$$H_v(\pi_v) = \frac{p_v^{d_v(2-s-w)}}{L(s,\Pi_v\times\pi_v)L(w,\pi_v)}\sum_{d_1+d_2=n}\mathrm{vol}(K_v[d_2])\sum_{j\leq d_2-n_0}\ \sum_{\min(\nu_1,n)=d_1}p_v^{-2\nu_1(s-\frac{1}{2})}$$
$$\times\int_{F_v^\times}W_{\Pi_v}\begin{pmatrix} z(\varpi_v^{\nu_1-d_v})a(y) & \\ & 1\end{pmatrix}\overline{\widetilde{W_j}}(a(y))|y|^{s-\frac{3}{2}}\,\mathrm{d}^\times y\,\Psi_v(w,\widetilde{W_j}). \tag{40}$$

By letting $\lambda_{\pi_v,j}(\mu)=\widetilde{W_j}(a(\varpi_v^{\mu-d_v}))p^{\frac{1}{2}\mu}$ and using (18), we see that

$$\int_{F_v^\times}W_{\Pi_v}\begin{pmatrix} z(\varpi_v^{\nu_1-d_v})a(y) & \\ & 1\end{pmatrix}\overline{\widetilde{W_j}}(a(y))|y|^{s-\frac{3}{2}}\,\mathrm{d}^\times y$$
$$= p_v^{d_v(s-\frac{3}{2})}\sum_{\nu_2\geq 0}\lambda_{\pi_v}(\nu_2,\nu_1)\overline{\lambda_{\pi_v,j}(\nu_2)}p_v^{-\nu_1}p_v^{-\nu_2 s} \tag{41}$$

and also

$$\Psi_v(w,\widetilde{W_j}) = p_v^{d_v(w-\frac{1}{2})}\sum_{\mu\geq 0}\lambda_{\pi_v,j}(\mu)p_v^{-\mu w}. \tag{42}$$

Inserting (41) and (42) in (40), we deduce that

$$H_v(\pi_v)=\frac{1}{L(s,\Pi_v\times\pi_v)L(w,\pi_v)}\times\sum_{d_1+d_2=n}\mathrm{vol}(K_v[d_2])\sum_{j\le d_2-n_0}\sum_{\min(\nu_1,n)=d_1}\sum_{\nu_2\ge 0}\sum_{\mu\ge 0}\frac{\lambda_{\Pi_v}(\nu_2,\nu_1)\overline{\lambda_{\pi_v,j}(\nu_2)}\lambda_{\pi_v,j}(\mu)}{p_v^{(2\nu_1+\nu_2)s}p_v^{\mu w}}.$$

Combining (35) and (37), we get

$$\lambda_{\pi_v,j}(\nu)=\langle W_0,W_0\rangle^{-\frac12}\sum_{k=1}^{\min(j,k)}\xi_{\pi_v}(j,k)p_v^{k-\frac12 j}\lambda_{\pi_v}(\nu-k)\delta_{\nu\ge k}.$$

As a consequence, we deduce, after changing variables, that

$$H_v(\pi_v)=\frac{\langle W_0,W_0\rangle^{-1}}{L(s,\Pi_v\times\pi_v)L(w,\pi_v)}\times\sum_{d_1+d_2=n}\mathrm{vol}(K_v[d_2])\sum_{j\le d_2-n_0}p_v^{-j}\sum_{k_1,k_2\le j}\xi_{\pi_v}(j,k_1)\xi_{\pi_v}(j,k_2)p_v^{k_1(1-w)}p_v^{k_2(1-s)}p_v^{-2d_1 s}\times\sum_{\min(\nu_1,d_2)=0}\sum_{\nu_2\ge 0}\frac{\lambda_{\Pi_v}(\nu_2+k_2,\nu_1+d_1)\overline{\lambda_{\pi_v}(\nu_2)}}{p_v^{(2\nu_1+\nu_2)s}}\sum_{\mu\ge 0}\frac{\lambda_{\pi_v}(\mu)}{p_v^{\mu w}}.$$

We recognize the last sum as $L(w,\pi_v)$, so that

$$H_v(\pi_v)=\frac{\langle W_0,W_0\rangle^{-1}}{L(s,\Pi_v\times\pi_v)}\sum_{d_1+d_2=n}\mathrm{vol}(K_v[d_2])\sum_{j\le d_2-n_0}p_v^{-j}\sum_{k_1,k_2\le j}\xi_{\pi_v}(j,k_1)\xi_{\pi_v}(j,k_2)\times p_v^{k_1(1-w)}p_v^{k_2(1-s)}p_v^{-2d_1 s}\sum_{\min(\nu_1,d_2)=0}\sum_{\nu_2\ge 0}\frac{\lambda_{\Pi_v}(\nu_2+k_2,\nu_1+d_1)\overline{\lambda_{\pi_v}(\nu_2)}}{p_v^{(2\nu_1+\nu_2)s}}. \tag{43}$$

We are now ready to prove the following

Proposition 7.1. *Let $\Phi_v=\Phi_v^{\mathfrak{q},\mathfrak{l}}$ be as in* (32), *and let $n=v(\mathfrak{q})$. Then for every $\epsilon>0$ there exists $\delta>0$ such that*

(i) *$H_v(\pi_v)$ vanishes if $c(\pi_v)>n$,*
(ii) *$H_v(\pi_v)=\varphi(p_v^n)p_v^{-2n}$ if $c(\pi_v)=n$,*
(iii) *$H_v(\pi_v)\ll_\epsilon p_v^{n(\theta-1+\epsilon)}$ in general for $\mathrm{Re}(s),\mathrm{Re}(w)>\frac12-\delta$ and $\delta>0$ sufficiently small.*

The first assertion follows by observing that if $n_0=c(\pi_v)>n$ then the sum over j in (43) will vanish independently of the value of d_2.

The second one holds because if $n_0=n$, we automatically have $d_2=n$ and $d_1=j=k_1=k_2=0$ and moreover

$$\langle W_0,W_0\rangle=\frac{\zeta_v(2)}{L_v(1,\pi_v\times\overline{\pi}_v)}\sum_{n\ge 1}\frac{|\lambda_{\pi_v}(n)|^2}{p_v^n}=\begin{cases}1 & \text{if } n_0=0,\\ \zeta_v(2) & \text{otherwise.}\end{cases}$$

Hence,

$$H_v(\pi_v) = \frac{\varphi(p_v^n)p_v^{-2n}}{L(s, \Pi_v \times \pi_v)} \sum_{\nu_2 \geq 0} \lambda_{\Pi_v}(\nu_2, 0)\lambda_{\pi_v}(\nu_2),$$

and we conclude by (19).

Finally, in order to show (iii), we apply the estimate in (36) and the bounds

$$\lambda_{\Pi_v}(\nu_1, \nu_2) \ll p_v^{(\nu_1+\nu_2)\theta}, \quad \lambda_{\pi_v}(\nu) \ll p_v^{\nu\vartheta}$$

to (43), which gives

$$H_v(\pi_v) \ll_\epsilon p_v^{n(-1+\epsilon)} \sum_{d_1+d_2=n} p^{2(d_1+j)\delta} \sum_{j=0}^{d_2-n_0} \sum_{k_1,k_2=0}^{j} p_v^{(k_1+k_2-2j)(\frac{1}{2}-\vartheta)} p_v^{(d_1+k_2)\theta}$$

for $\Re(s) > \frac{1}{2}\theta, \theta + \vartheta$ and it follows from the results in [Luo et al. 1999] and the Kim–Sarnak bound [2003, Appendix 2] that one has $\theta + \vartheta < \frac{1}{2}$. We conclude by taking δ sufficiently small.

7C. *Local computations, the archimedean case.* The analysis of the archimedean weight functions is of a somewhat different nature from the nonarchimedean case. For those places, we make the simplest choice imaginable. Namely we impose that Π_v is unramified and Φ_v is normalized spherical for every archimedean place v. As a consequence it easily follows that $H_v(\pi_v)$ vanishes unless π_v is itself unramified, in which case we may choose a basis of $B^W(\pi_v)$ such that each term corresponds to a different K-type, and then there will be at most one element W of $\mathcal{B}^W(\pi_v)$ for which the period $\Psi_v(s, W_{\Phi_v}, W)$ is nonvanishing, and it must be a spherical vector for π_v. Moreover, it follows from Stade's formula [2001, Theorem 3.4] that

$$\Psi_v(s, W_{\Phi_v}, \overline{W}) = L_v(s, \Pi_v \times \pi_v), \quad \Psi_v(w, W) = L_v(w, \pi_v).$$

where $W = W_{\pi_v} \in \mathcal{W}(\pi_v, \psi_v)$ is spherical and such that $\vartheta_v(W, W) = 1$. In particular, the following holds:

Proposition 7.2. *Let v be an archimedean place of F. Let Π_v be an irreducible admissible generic representation for $\mathrm{GL}_3(F_v)$. Then there exists a vector $\Phi_v \in \Pi_v$ such that for every irreducible admissible generic representation for $\mathrm{GL}_2(F_v)$, we have*

$$H(\pi_v) = \begin{cases} 1 & \text{if } \pi_v \text{ is unramified,} \\ 0 & \text{otherwise.} \end{cases}$$

7D. *Meromorphic continuation with respect to the spectral parameter.* Let $\Phi^{\mathfrak{q},\mathfrak{l}}$ be as in (32), and let H be the (s, w)-weight function of kernel $\Phi^{\mathfrak{q},\mathfrak{l}}$. Our goal in this section is to find that for any unitary character ω of $F_v^\times$ there is a domain of $\mathbb{C}^3$ on which the function

$$(s, w, t) \mapsto H_v(\pi_v)$$

is meromorphic with respect to all three variables with only finitely many polar divisors, where $\pi_v = \pi_v(\omega_v, it)$ (see Section 3A1).

From our computations so far, we know that $H_v(\pi_v) = 1$ unless $v \mid \mathfrak{l}$ or $v \mid \mathfrak{q}$. Moreover, in the first of these cases, we saw that

$$H_v(\pi_v) = p_v^{-mw}\left(\lambda_{\pi_v}(m) - \frac{\lambda_{\pi_v}(m-1)}{p_v^w}\right), \quad m = v(\mathfrak{l}),$$

which is clearly an entire function with respect to s, w and t, since it is a combination of terms of the shape $p_v^{\alpha w+\beta t}$, where $\alpha, \beta \in \mathbb{C}$.

We are now left with the case where $v \mid \mathfrak{q}$. It follows from the proof of Proposition 7.1 that given $\eta < \frac{1}{2}$, there exists $\delta > 0$ such that the right-hand side of (43) converges in the region

$$|\mathrm{Im}(t)| < \eta, \quad \mathrm{Re}(s), \mathrm{Re}(w) > \tfrac{1}{2} - \delta,$$

and defines in it a holomorphic function in the variables s and w. We observe that $H_v(\pi_v)$ is a linear combination of terms of the shape

$$L_{\omega,k_2,d_1,d_2}(s,t) := \frac{1}{L(s, \Pi_v \times \pi_v)} \sum_{\min(\nu_1,d_2)=0} \sum_{\nu_2 \geq 0} \frac{\lambda_{\Pi_v}(\nu_2 + k_2, \nu_1 + d_1)\overline{\lambda_{\pi_v}(\nu_2)}}{p_v^{(2\nu_1+\nu_2)s}},$$

with coefficients given by meromorphic functions in the variables s, w and t. The only possible polar divisors occur for t satisfying $\omega(\varpi_v)^2 p_v^{2it} = p_v^{\pm 1}$, due to the term $(1-\alpha_{\pi_v}^2)^{-1}$ appearing as a factor of $\xi_{\pi_v}(j,k_1)\xi_{\pi_v}(j,k_2)$. Moreover, it follows from [Blomer and Khan 2019a, Lemma 14], applied to the tuple $(M, d, g_1, g_2, q) = (p_v^{k_2}, p_v^{d_1}, 1, p_v^{d_2}, p_v^{n})$, that for any $\epsilon > 0$, there exists $\delta > 0$ such that $L_{\omega,k_2,d_1,d_2}(s,t)$ admits a holomorphic continuation to the region

$$\mathrm{Re}(s) > \tfrac{1}{4} - \delta, \quad \mathrm{Re}(s) \pm \mathrm{Im}(t) > -\delta. \tag{44}$$

Moreover, using again the Ramanujan bound for $\lambda_{\Pi_v}(\nu_2, \nu_2)$ and recalling that $\pi_v = \pi_v(\omega_v, it)$, so that $\lambda_{\pi_v}(\nu) \ll p_v^{\nu|\mathrm{Im}(t)|+\epsilon}$, we see that in the region (44) we have

$$L_{\omega,k_2,d_1,d_2}(s,w,t) \ll p_v^{(d_1+k_2)(\theta+\epsilon)}. \tag{45}$$

As a consequence, $H_v(\pi_v)$ admits meromorphic continuation to (44). Now, suppose $\omega = \mathbf{1}$ is the trivial character, and let

$$D_v(s,w) := H_v(\pi_v)|_{t=(1-w)/i}. \tag{46}$$

From what we have just seen, $D_v(s,w) = 1$ unless $v \mid \mathfrak{l}$ or $v \mid \mathfrak{q}$. In the first case, it is clear that $D_v(s,w)$ is entire with respect to both s and w. Also, when $\frac{1}{2} \leq \mathrm{Re}(s), \mathrm{Re}(w) < 1$, we have $\lambda_{\pi_v}(m) \ll p_v^{m(1-\mathrm{Re}(w))}$, and thus by (34), we see that

$$D_v(s,w) \ll 1.$$

Finally, if $v \mid \mathfrak{q}$, then for sufficiently small $\delta > 0$, D_v is meromorphic in the region

$$\tfrac{1}{2} - \delta < \mathrm{Re}(s), \mathrm{Re}(w) \leq 1, \tag{47}$$

where the only possible polar divisors are at the values of w such that $p_v^{2-2w} = p_v$. We will now show that such poles cannot occur. To see this, let

$$E_{j,k_2}(w) := \xi_{\pi_v}(j,k_2) \sum_{k_1=0}^{j} \xi_{\pi_v}(j,k_1) p_v^{k_1(1-w)}.$$

An easy computation shows that

$$E_{j,k_2}(w) = \begin{cases} 1 & \text{if } j = 0, \\ t_{j,k_2}(w)/(1-p_v^{2w-3}) & \text{if } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where t_{j,k_2} is an entire function. This and the fact that $L_{\omega,k,d_1,d_2}(s,w,(1-w)/i)$ is holomorphic in (47) are enough to guarantee that $H_v(\pi_v)$ is holomorphic in the same region. Moreover, we may argue analogously to Proposition 7.1(iii), appealing to (45), to deduce that for $\frac{1}{2} \le \mathrm{Re}(s), \mathrm{Re}(w) < 1$, we have the inequality

$$D_v(s,w) \ll_{s,w,\epsilon} p_v^{n(-1+\theta+\epsilon)} \sum_{d_1+d_2=n} \sum_{j=0}^{d_2-n_0} p_v^{j(1-2\,\mathrm{Re}(w))} + p_v^{j(1-\mathrm{Re}(s)-\mathrm{Re}(w))} \ll_\epsilon p_v^{n(-1+\theta+\epsilon)}.$$

We now summarize what we obtained in this subsection as follows:

Proposition 7.3. *Let $\omega = \bigotimes_v' \omega_v$ be an unitary character of $F^\times \backslash \mathbb{A}^\times$, and let H_v be given by* (29) *with $\Phi_v = \Phi_v^{\mathfrak{q},\mathfrak{l}}$, with $\Phi_v^{\mathfrak{q},\mathfrak{l}}$ given by* (32). *Then $(s,w,t) \mapsto H_v(\pi_v(\omega_v, it))$ admits meromorphic continuation to the region* (44) *with possible polar divisor of the form $t = t_0$, where t_0 is a solution to $\omega(\varpi_v)^2 p_v^{2it_0} = p_v^{\pm 1}$. Moreover, if D_v is given by* (46), *then it admits a holomorphic continuation to the region* (47) *and if $\frac{1}{2} \le \mathrm{Re}(s), \mathrm{Re}(w) < 1$, it satisfies $D_v(s,w) = 1$ unless $v \mid \mathfrak{q}\mathfrak{l}$, in which case*

$$D_v(s,w) \ll_{s,w,\epsilon} \begin{cases} p_v^{m\epsilon} & \text{if } v \mid \mathfrak{l}, \\ p_v^{n(-1+\theta+\epsilon)} & \text{if } v \mid \mathfrak{q}. \end{cases}$$

8. The degenerate term

In this section we study the term $\mathcal{D}_{s,w}(\Phi)$ given by (30) and its companion $\mathcal{D}_{s',w'}(\check{\Phi})$. First, by rapid decay of Whittaker functions and the action of the Weyl group of GL(3), we may see that both converge for any values of $s, w \in \mathbb{C}$. This is all that is needed to know with respect to these terms for Theorem 1.1.

Let us now turn to their use in Theorem 1.2. Here we make the specialization to $\Phi = \Phi^{\mathfrak{q},\mathfrak{l}}$. It turns out that is easier to study first the term $\mathcal{D}_{s',w'}(\check{\Phi})$, so we start with this one and later deduce an analogous result for the other by using their symmetry. First, we recall that

$$\mathcal{D}_{s',w'}(\check{\Phi}) = 2d_F^{\frac{7}{2}-3s'-w'} \int_{F^\times\backslash\mathbb{A}^\times} \int_{F\backslash\mathbb{A}} \int_{F^\times\backslash\mathbb{A}^\times} \check{\Phi}\begin{pmatrix} z(u)n(x)a(y) & \\ & 1 \end{pmatrix} |u|^{2s'-1} |y|^{s'+w'-1} \,\mathrm{d}^\times u \,\mathrm{d}x \,\mathrm{d}^\times y. \quad (48)$$

We will show that, in the region

$$\mathrm{Re}(3s+w) > 1, \quad \mathrm{Re}(s+w), \quad \mathrm{Re}(2s) > \theta, \quad (49)$$

$\mathcal{D}_{s',w'}(\check{\Phi}) \ll_{s,w,\epsilon} \ell^{\theta-\mathrm{Re}(s)-\mathrm{Re}(w)+\epsilon}$, where ℓ is the absolute norm of $\mathfrak{l}$.

We begin by noticing that, using the definition of $\check{\Phi}$, reversing the change of variables used in the proof of Proposition 5.1 and changing the order of summation, we see that the integral in (48) equals

$$\int_{(F^\times\backslash\mathbb{A}^\times)^2}\left(\int_{F\backslash\mathbb{A}}\Phi\left(\begin{pmatrix}1&&x\\&1&\\&&1\end{pmatrix}\begin{pmatrix}z(u)a(y)&\\&1\end{pmatrix}\right)\mathrm{d}x\right)|u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y.$$

By the Whittaker expansion of Φ, the inner integral is

$$\int_{F\backslash\mathbb{A}}\sum_{\gamma\in N_2(F)\backslash\mathrm{GL}_2(F)}W_\Phi\left(\begin{pmatrix}\gamma&\\&1\end{pmatrix}\begin{pmatrix}1&&x\\&1&\\&&1\end{pmatrix}\begin{pmatrix}z(u)a(y)&\\&1\end{pmatrix}\right)\mathrm{d}x,$$

which, by elementary manipulations and changing the order of summation and integration, becomes

$$\sum_{\gamma\in N_2(F)\backslash\mathrm{GL}_2(F)}W_\Phi\begin{pmatrix}\gamma z(u)a(y)&\\&1\end{pmatrix}\int_{F\backslash\mathbb{A}}\psi(\gamma_{21}x)\,\mathrm{d}x,$$

where γ_{21} is the lower left entry of γ. Since $\gamma_{21}\in F$, the inner integral vanishes unless $\gamma_{21}=0$, in which case, it equals one. In other words, we may change the sum over $N_2(F)\backslash\mathrm{GL}_2(F)$ into a sum over $N_2(F)\backslash B_2(F)$, which can be parametrized by $Z_2(F)A_2(F)$. Altogether, this implies that

$$\begin{aligned}\mathcal{D}_{s',w'}(\check{\Phi})&=2d_F^{\frac{7}{2}-3s'-w'}\int_{(F^\times\backslash\mathbb{A}^\times)^2}\sum_{\gamma\in Z_2(F)A_2(F)}W_\Phi\begin{pmatrix}\gamma z(u)a(y)&\\&1\end{pmatrix}|u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y\\&=2d_F^{\frac{7}{2}-3s'-w'}\int_{(\mathbb{A}^\times)^2}W_\Phi\begin{pmatrix}z(u)a(y)&\\&1\end{pmatrix}|u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y.\end{aligned}\tag{50}$$

Suppose that $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$ are sufficiently large. We are now in a fairly advantageous position, as the integral above can be factored into local ones. These local integrals are

$$\mathcal{J}_v=\int_{(F_v^\times)^2}W_{\Phi_v}\begin{pmatrix}z(u)a(y)&\\&1\end{pmatrix}|u|^{2s-1}|y|^{s+w-1}\,\mathrm{d}^\times u\,\mathrm{d}^\times y.$$

We notice that for a finite place v for which Π_v is unramified, this equals

$$p_v^{d_v(3s+w-2)}\sum_{\nu_1,\nu_2\geq 0}\frac{\lambda_{\Pi_v}(\nu_1,\nu_2)}{p_v^{\nu_1(s+w)+2\nu_2 s}},$$

whose inner sum we recognize as being the local factor of Bump's double Dirichlet series (see, e.g., [Goldfeld 2006, §6.6]). In particular, it follows that for $\mathrm{Re}(s+w),\mathrm{Re}(2s)>\theta$ (recall the bound $\lambda_{\Pi_v}(\nu_1,\nu_2)\ll p_v^{(\nu_1+\nu_2)\theta}$) the above equals

$$\mathcal{J}_v^0:=p_v^{d_v(3s+w-2)}\frac{L(s+w,\Pi_v)L(2s,\overline{\Pi}_v)}{\zeta_v(3s+w)}\asymp_{s,w}1.\tag{51}$$

As for the remaining places, we first observe that for $v\mid\mathfrak{q}$, the unipotent averaging has no effect on the values of the Whittaker function at diagonal element. Thus, it follows that the local integral $\mathcal{J}_v$ will also coincide with (51). Furthermore, (51) also holds for archimedean v. For real places this is done in [Bump 1984] and for the complex places this is [Bump and Friedberg 1989, Theorem 1]. Finally, for $v \mid \mathfrak{l}$,

$$W_{\Phi_v}\begin{pmatrix} z(u)a(y) & \\ & 1 \end{pmatrix} = \delta_{v(y)\geq m-d_v} W_{\Pi_v}\begin{pmatrix} z(u)a(y) & \\ & 1 \end{pmatrix},$$

where $m = v(\mathfrak{l})$. Hence, in this case, the local factor is

$$\mathcal{J}_v = p_v^{d_v(3s+w-2)} \sum_{\nu_1 \geq m, \nu_2 \geq 0} \frac{\lambda_{\Pi_v}(\nu_1,\nu_2)}{p_v^{\nu_1(s+w)+2\nu_2 s}},$$

which, by using yet again the Ramanujan bound for $\lambda_{\Pi_v}(\nu_1,\nu_2)$, we may see converges in the region $\mathrm{Re}(s+w), \mathrm{Re}(2s) > \theta$, where it satisfies

$$\mathcal{J}_v \ll_\epsilon p_v^{m(\theta-\mathrm{Re}(s)-\mathrm{Re}(w)+\epsilon)}.$$

In particular, if $\mathfrak{l} = 1$,

$$\mathcal{D}_{\frac{1}{2},\frac{1}{2}}(\check{\Phi}) = 2d_F^{\frac{3}{2}} \frac{\Lambda(1,\Pi)\Lambda(1,\overline{\Pi})}{\xi_F(2)}. \tag{52}$$

Now, notice that $\mathcal{D}_{s,w}(\Phi)$ is the same as $\mathcal{D}_{s',w'}(\check{\Phi})$ but with $(\mathfrak{q},\mathfrak{l},s,w)$ replaced by $(\mathfrak{l},\mathfrak{q},s',w')$. This allows us to immediately reuse our efforts in this section to study the latter function as well. We record the results for both these functions in a weaker form in the following proposition.

Proposition 8.1. *Let $\mathcal{D}_{s,w}(\Phi)$ and $\mathcal{D}_{s,w}(\Phi)$ be as defined in* (30) *with $\Phi = \Phi^{\mathfrak{q},\mathfrak{l}}$ and $\check{\Phi}$ given by* (25). *Then they are entire functions of s and w, and in the region*

$$\mathrm{Re}(s), \mathrm{Re}(w), \mathrm{Re}(s'), \mathrm{Re}(w') > \tfrac{1}{4},$$

they satisfy

$$\mathcal{D}_{s,w}(\Phi) \ll_{s,w,\epsilon} q^{\theta-\mathrm{Re}(s+w)+\epsilon} \quad \text{and} \quad \mathcal{D}_{s',w'}(\check{\Phi}) \ll_{s,w,\epsilon} \ell^{\theta-\mathrm{Re}(s+w)+\epsilon}.$$

9. Analytic continuation of the Eisenstein part

The conclusion of our next proposition will be subject to the following hypothesis, whose verification when $\Phi = \Phi^{\mathfrak{q},\mathfrak{l}}$ follows from the main results of Section 7:

Hypothesis 1. There exists $\delta > 0$ such that for every idele character ω, the function

$$(s,w,t) \mapsto H(\pi(\omega,it))$$

is holomorphic in the region

$$\mathrm{Re}(s), \mathrm{Re}(w) > \tfrac{1}{2} - \delta, \quad |\mathrm{Im}(t)| < \delta.$$

Moreover, $H(\pi(\omega,(1-w))$ admits a holomorphic continuation to the region

$$\tfrac{1}{2} - \delta < \mathrm{Re}(s), \mathrm{Re}(w) < 1.$$

We will show that the term $\mathcal{E}_{s,w}(\Phi)$ admits meromorphic continuation for values of s and w with real parts smaller than 1. The proof follows the same lines as those of Blomer and Khan [2019a, Lemma 16; 2019b, Lemma 3].

Proposition 9.1. *Suppose that* Π *is a cuspidal automorphic representation, and let* $\Phi \in \Pi$ *be an automorphic form such that the associated weight function* H *satisfies Hypothesis 1 for some* $\delta > 0$. *Let* $\mathcal{E}_{s,w}(\Phi)$ *be given by* (3), *defined initially for* $\mathrm{Re}(s), \mathrm{Re}(w) \gg 1$. *It admits a meromorphic continuation to* $\mathrm{Re}(s), \mathrm{Re}(w) \geq \frac{1}{2} - \epsilon$ *for some* $\epsilon > 0$ *with at most finitely many polar divisors. If* $\frac{1}{2} \leq \mathrm{Re}(s), \mathrm{Re}(w) < 1$, *its analytic continuation is given by* $\mathcal{E}_{s,w}(\Phi) + \mathcal{R}_{s,w}(\Phi)$, *where*

$$\mathcal{R}_{s,w}(\Phi) = \sum_{\pm} \operatorname*{res}_{t=\pm(1-w)/i} (\pm i) \frac{\Lambda(s+it,\Pi)\Lambda(s-it,\Pi)\xi_F(w+it)\xi_F(w-it)}{\xi_F^*(1)\xi_F(1+2it)\xi_F(1-2it)} H(\pi(\mathbf{1},it)). \quad (53)$$

Proof. Let $\delta > 0$ to be chosen later. We use nonvanishing of completed Dirichlet L-functions $\Lambda(s,\omega)$ at $\mathrm{Re}(s) = 1$ and continuity to define a continuous function $\sigma : \mathbb{R} \mapsto (0,\delta)$ so that neither $\Lambda(1-2\sigma-2it,\omega^2)$ nor $H(\pi(\omega, it+\sigma))$ have poles for $0 \leq \sigma < \sigma(t)$.

We start by noticing that we can *analytically* continue $\mathcal{E}_{s,w}(\Phi)$ to $\mathrm{Re}(s), \mathrm{Re}(w) > 1$, since in that region, one does not encounter any poles of $\Lambda(w, \pi(\mathbf{1}, it))$. Now, suppose that

$$1 < \mathrm{Re}(s) < 1 + \sigma(\mathrm{Im}(s)) \quad \text{and} \quad 1 < \mathrm{Re}(w) < 1 + \sigma(\mathrm{Im}(w)).$$

We shift the contour of the integral defining $\mathcal{E}_{s,w}(\Phi)$ down to $\mathrm{Im}\, t = -\sigma(\mathrm{Re}(t))$. We pick up a pole of $\Lambda(w-it,\omega)$ when ω is the trivial character and $w - it = 1$.

We observe that in view of our choice for σ, the resulting integral defines a holomorphic function in the region

$$\begin{cases} 1-\sigma(\mathrm{Im}(s)) < \mathrm{Re}(s) < 1+\sigma(\mathrm{Im}(s)), \\ 1-\sigma(\mathrm{Im}(t)) < \mathrm{Re}(w) < 1+\sigma(\mathrm{Im}(w)). \end{cases}$$

Take now s and w satisfying $1-\sigma(\mathrm{Im}(s)) < \mathrm{Re}(s) < 1$ and $1-\sigma(\mathrm{Im}(t)) < \mathrm{Re}(w) < 1$. We may shift the contour back to the real line and pick a new pole when ω is trivial and at $w + it = 1$. This proves the desired formula for $1-\sigma(\mathrm{Im}(s)) < \mathrm{Re}(s) < 1$ and $1-\sigma(\mathrm{Im}(t)) < \mathrm{Re}(w)$ and it follows in general by analytic continuation to all s, w such that $\frac{1}{2} - \delta < \mathrm{Re}(s), \mathrm{Re}(w) < 1$ by Proposition 7.3. ☐

10. Conclusion

In this section we put together the results of the last three sections and deduce Theorem 1.2. We have seen in Proposition 6.1 that for sufficiently large values of $\mathrm{Re}(s)$ and $\mathrm{Re}(w)$, we have the relation

$$2d_F^{\frac{7}{2}-3s-w} I(w, \mathcal{A}_s\Phi) = \mathcal{M}_{s,w}(\Phi) + \mathcal{D}_{s,w}(\Phi).$$

If we assume that H satisfies Hypothesis 1, then we may apply Proposition 9.1, and deduce that, for $\frac{1}{2} - \delta < \mathrm{Re}(s), \mathrm{Re}(w) < 1$,

$$2d_F^{\frac{7}{2}-3s-w} I(w, \mathcal{A}_s\Phi) = \mathcal{M}_{s,w}(\Phi) + \mathcal{D}_{s,w}(\Phi) + \mathcal{R}_{s,w}(\Phi). \quad (54)$$

Now suppose that $\breve{H}$ also satisfies Hypothesis 1 and that $\frac{1}{2} < \mathrm{Re}(s) \le \mathrm{Re}(w) \le \frac{3}{4}$. The last assertion implies that

$$\tfrac{1}{2} \le \mathrm{Re}(s'), \mathrm{Re}(w') < 1.$$

Thus, we may deduce that (54) also holds with H, s and w replaced by $\breve{H}$, s' and w', respectively. The main equality in Theorem 1.2 is now a direct consequence of Proposition 5.1 and the description of the local weights $H_v(\pi_v)$ from Section 7. In particular, we showed in Section 7D that the weight function associated to $\Phi^{\mathfrak{q},\mathfrak{l}}$ satisfies Hypothesis 1. As for the inequality (7), it follows from (53) and Propositions 7.3 and 8.1.

10A. ***Proof of Corollary 1.3.*** We use Theorem 1.2 with $s = w = \frac{1}{2}$, $\mathfrak{l} = \mathfrak{o}_F$ and $\mathfrak{q} = \mathfrak{p}$, a prime ideal. We obtain that

$$\mathcal{M}(\Phi) = \mathcal{D}(\breve{\Phi}) + \mathcal{R}(\breve{\Phi}) - \mathcal{D}(\Phi) - \mathcal{R}(\Phi) + \mathcal{M}(\breve{\Phi}),$$

where we dropped the $\frac{1}{2}, \frac{1}{2}$ from the index for brevity. It follows from Proposition 9.1 and the fact that $\widehat{\lambda}_{\pi(\mathbf{1},\pm\frac{1}{2})}\big(\frac{1}{2}, \mathfrak{q}\big) = 1$ for any ideal $\mathfrak{q}$ that

$$\mathcal{R}(\breve{\Phi}) = 2\frac{\Lambda(1,\Pi)\Lambda(0,\Pi)}{\xi_F(2)}.$$

Furthermore, we have from (52) that

$$\mathcal{D}(\breve{\Phi}) = 2d_F^{\frac{3}{2}}\frac{\Lambda(1,\Pi)\Lambda(1,\overline{\Pi})}{\xi_F(2)} = 2\frac{\Lambda(1,\Pi)\Lambda(0,\Pi)}{\xi_F(2)}.$$

Moreover, in view of Propositions 7.1 and 8.1, we may obtain that

$$\mathcal{R}(\Phi), \mathcal{D}(\Phi) \ll (N\mathfrak{p})^{\theta-1+\epsilon}$$

and

$$\mathcal{M}(\Phi) = \frac{\varphi(q)}{q^2} \sum_{\substack{\pi \text{ cusp}^0 \\ \mathrm{cond}(\pi)=\mathfrak{p}}} \frac{\Lambda\big(\frac{1}{2}, \Pi\times\pi\big)\Lambda\big(\frac{1}{2},\pi\big)}{\Lambda(1,\mathrm{Ad},\pi)} + O(q^{\theta-1}\mathcal{M}^*),$$

where

$$\mathcal{M}^* := \sum_{\substack{\pi \text{ cusp}^0 \\ \mathrm{cond}(\pi)=\mathfrak{o}_F}} \frac{\big|\Lambda\big(\frac{1}{2}, \Pi\times\pi\big)\Lambda\big(\frac{1}{2},\pi\big)\big|}{|\Lambda(1,\mathrm{Ad},\pi)|} + \sum_{\substack{\omega\in\widehat{F^\times U_\infty\backslash\mathbb{A}^\times_{(1)}} \\ \mathrm{cond}(\omega)=\mathfrak{o}_F}} \int_{-\infty}^{\infty} \frac{\big|\Lambda\big(\frac{1}{2}, \Pi\times\pi(\omega,it)\big)\Lambda\big(\frac{1}{2},\pi(\omega,it)\big)\big|}{|\Lambda^*(1,\mathrm{Ad},\pi(\omega,it))|}\frac{\mathrm{d}t}{2\pi}.$$

Finally, it is easy to see that we also have the bound

$$\mathcal{M}(\breve{\Phi}) \ll q^{\vartheta-\frac{1}{2}}\mathcal{M}^*.$$

Corollary 1.3 will follow provided that one is able to show $\mathcal{M}^* \ll 1$. In other words, we just need to ensure that it converges since it is clearly independent of $\mathfrak{p}$. To see that, we notice that the finite part of $\Lambda\left(\frac{1}{2}, \Pi \times \pi\right)\Lambda\left(\frac{1}{2}, \pi\right)/\Lambda(1, \mathrm{Ad}, \pi)$ is bounded polynomially in terms of the eigenvalues of π_v for archimedean v and that, by Stirling's formula, we have, for archimedean v,

$$\frac{L\left(\frac{1}{2}, \Pi_v \times \pi_v\right)L\left(\frac{1}{2}, \pi_v\right)}{L(1, \mathrm{Ad}, \pi_v)} \ll |t_{\pi_v}|^C e^{-2c_{F_v}\pi|t_{\pi_v}|}$$

for $\pi_v = \pi_v(\mathbf{1}, i t_{\pi_v})$, where

$$c_{F_v} = \begin{cases} 1 & \text{if } F_v = \mathbb{R}, \\ 2 & \text{if } F_v = \mathbb{C}. \end{cases}$$

This implies that the factor $\Lambda\left(\frac{1}{2}, \Pi \times \pi\right)\Lambda\left(\frac{1}{2}, \pi\right)/\Lambda(1, \mathrm{Ad}, \pi)$ decays exponentially as the t_{π_v} grow and the convergence of $\mathcal{M}^*$ follows by appealing to the Weyl law for GL(2) over number fields (see [Palm 2012, Theorem 3.2.1]).

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Volume 17 No. 8 2023

On the first nontrivial strand of syzygies of projective schemes and condition ND(ℓ) 1359
JEAMAN AHN, KANGJIN HAN and SIJONG KWAK

Spectral reciprocity via integral representations 1381
RAMON M. NUNES

Quadratic points on intersections of two quadrics 1411
BRENDAN CREUTZ and BIANCA VIRAY

A p-adic Simpson correspondence for rigid analytic varieties 1453
YUPENG WANG

On moment map and bigness of tangent bundles of G-varieties 1501
JIE LIU